Fixed Point Results for (ϕ, ψ)-Weak Contraction in Fuzzy Metric Spaces

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Abstract. In the present work, a fixed point result for generalized weakly contractive mapping in fuzzy metric space has been established. An example is cited to illustrate the obtained result.

Keywords: Weak contraction · Fuzzy metric · Fixed points

1 Introduction and Preliminaries

The concept of fuzzy metric spaces have been introduced in different ways by many authors. Among which, KM-fuzzy metric space, introduced by Kramosil and Michalek [2] and GV-fuzzy metric space, introduced by George and Veeramani [3], are two most widely used fuzzy metric spaces. KM-fuzzy metric space is similar to generalized Menger space [4]. George and Veeramani imposed a strong condition on the definition of Kramosil and Michalek for topological reasons. Several fixed point results in these fuzzy metric spaces can be found in [5,7,8,10,11].

Alber et al. extended the concept of Banach contraction to the weak contraction and established a fixed point result in Hilbert space [1]. There after B.E. Rhoades investigated this result in metric space [6]. Fixed point problem for weak contraction mapping have been investigated by many authors [12–15,17–24]. In [9] Dutta et al. extended the results of Rhoades. Motivated by the works of [9,16,25], in the present work, a fixed point result in fuzzy metric space, introduced by George and Veeramani, is obtained and an example is added in the support of main result.

Definition 1.1 [4]. *A continuous t-norm $*$ is a binary operation on $[0,1]$, which satisfies the following conditions:*

(i) $*$ *is associative and commutative,*
(ii) $x * 1 = x$, *for all* $x \in [0,1]$,
(iii) $x * y \leq u * v$, *whenever* $x \leq u$ *and* $y \leq v$, *for all* $x, y, u, v \in [0,1]$,
(iv) $*$ *is continuous.*

For example: (a) The minimum t-norm, $*_M$, defined by $x *_M y = \min\{x, y\}$; (b) The product t-norm, $*_P$, defined by $x *_P y = x.y$, are two basic t-norms.

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Definition 1.2 [3]. *The triplet $(X, M, *)$ is called fuzzy metric space if X is a non-empty set, $*$ is continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:*

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ *if and only if* $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
(v) $M(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous,*

for all $t, s \in (0, \infty)$ *and* $x, y, z \in X$.

In this paper, we use the notion of fuzzy metric space introduced by George and Veeramani.

Definition 1.3 [3]. *Let $(X, M, *)$ be a fuzzy metric space. Then*

(i) *A sequence $\{x_n\} \subseteq X$ is said to converge to a point $x \in X$ if* $\lim_{n\to\infty} M(x_n, x, t) = 1$, *for all* $t > 0$.
(ii) *A sequence $\{x_n\} \subseteq X$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists an $N \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for each $m, n \geq N$.*
(iii) *A fuzzy metric space is called complete if every Cauchy sequence in this space is convergent.*

Lemma 1.1 [5]. *Let $(X, M, *)$ be a fuzzy metric space. Then $(X, M, .)$ is non-decreasing for all $x, y \in X$.*

Lemma 1.2 [25]. *If $*$ is a continuous t-norm, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences such that $\alpha_n \to \alpha, \gamma_n \to \gamma$ as $n \to \infty$, then $\overline{\lim_{k\to\infty}}(\alpha_k * \beta_k * \gamma_k) = \alpha * \overline{\lim_{k\to\infty}} \beta_k * \gamma$ and $\underline{\lim_{k\to\infty}}(\alpha_k * \beta_k * \gamma_k) = \alpha * \underline{\lim_{k\to\infty}} \beta_k * \gamma$.*

Lemma 1.3 [25]. *Let $\{f(k, .) : (0, \infty) \to (0, 1], k = 0, 1, 2, ...\}$ be a sequence of functions such that $f(k, .)$ is continuous and monotone increasing for each $k \geq 0$. Then $\overline{\lim_{k\to\infty}} f(k, t)$ is a left continuous function in t and $\underline{\lim_{k\to\infty}} f(k, t)$ is a right continuous function in t.*

2 Main Results

Theorem 2.1. *Let $(X, M, *)$ be a complete fuzzy metric space with an arbitrary continuous t-norm $'*'$ and let $T : X \to X$ be a self mapping satisfying the following condition:*

$$\psi(M(Tx, Ty, t)) \leq \psi(\min(M(x, y, t), M(x, Tx, t), M(y, Ty, t))) - \phi(\min(M(x, y, t), M(y, Ty, t))), \tag{2.1}$$

where $\psi, \phi : (0, 1] \to [0, \infty)$ are two functions such that:

(i) *ψ is continuous and monotone decreasing function with $\psi(t) = 0$ if and only if $t = 1$,*
(ii) *ϕ is lower semi continuous function with $\phi(t) = 0$ if and only if $t = 1$.*

Then T has a unique fixed point.

Proof: Let $x_0 \in X$. We define the sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, for each $n \geq 0$. If there exists a positive integer k such that $x_k = x_{k+1}$, then x_k is a fixed point of T. Hence, we shall assume that $x_n \neq x_{n+1}$, for all $n \geq 0$. Now, from (2.1)

$$\begin{aligned}\psi(M(x_{n+1}, x_{n+2}, t)) &= \psi(M(Tx_n, Tx_{n+1}, t)) \\ &\leq \psi(\min\{M(x_n, x_{n+1}, t), M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t)\}) \\ &\quad - \phi(\min\{M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, t)\}). \qquad (2.2)\end{aligned}$$

Suppose that $M(x_n, x_{n+1}, t) > M(x_{n+1}, x_{n+2}, t)$, for some positive integer n. Then from (2.2), we have
$\psi(M(x_{n+1}, x_{n+2}, t)) \leq \psi(M(x_{n+1}, x_{n+2}, t)) - \phi(M(x_{n+1}, x_{n+2}, t))$, that is, $\phi(M(x_{n+1}, x_{n+2}, t)) \leq 0$, which implies that $M(x_{n+1}, x_{n+2}, t) = 1$. This gives that $x_{n+1} = x_{n+2}$, which is a contradiction.

Therefore, $M(x_{n+1}, x_{n+2}, t) \leq M(x_n, x_{n+1}, t)$ for all $n \geq 0$, and $\{M(x_n, x_{n+1}, t)\}$ is a monotone increasing sequence of non-negative real numbers. Hence, there exists an $r > 0$ such that $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = r$.

In view of the above facts, from (2.2), we have

$\psi(M(x_{n+1}, x_{n+2}, t)) \leq \psi(M(x_n, x_{n+1}, t)) - \phi(M(x_n, x_{n+1}, t))$, for all $n \geq 0$,

Taking the limit as $n \to \infty$ in the above inequality and using the continuities of ϕ and ψ we have $\psi(r) \leq \psi(r) - \phi(r)$, which is a contradiction unless $r = 1$. Hence

$$M(x_n, x_{n+1}, t) \to 1 \quad \text{as } n \to \infty. \qquad (2.3)$$

Next, we show that $\{x_n\}$ is Cauchy sequence. If otherwise, there exist λ, $\epsilon > 0$ with $\lambda \in (0, 1)$ such that for each integer k, there are two integers $l(k)$ and $m(k)$ such that $m(k) > l(k) \geq k$ and

$$M(x_{l(k)}, x_{m(k)}, \epsilon) \leq 1 - \lambda, \quad \text{for all } k > 0. \qquad (2.4)$$

By choosing $m(k)$ to be the smallest integer exceeding $l(k)$ for which (2.4) holds, then for all $k > 0$, we have

$$M(x_{l(k)}, x_{m(k)-1}, \epsilon) > 1 - \lambda.$$

Now, by triangle inequality, for any s with $0 < s < \frac{\epsilon}{2}$, for all $k > 0$, we have

$$\begin{aligned}1 - \lambda &\geq M(x_{l(k)}, x_{m(k)}, \epsilon) \\ &\geq M(x_{l(k)}, x_{l(k)+1}, s) * M(x_{l(k)+1}, x_{m(k)+1}, \epsilon - 2s) * M(x_{m(k)+1}, x_{m(k)}, s). \qquad (2.5)\end{aligned}$$

For $t > 0$, we define the function $h_1(t) = \overline{\lim_{n\to\infty}} M\left(x_{l(k)+1}, x_{m(k)+1}, t\right)$.

Taking lim sup on both the sides of (2.5), using (2.3) and the continuity property of $*$, by Lemma (1.2), we conclude that

$$\begin{aligned} 1-\lambda &\geq 1 * \overline{\lim_{k\to\infty}} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon - 2s) * 1 \\ &= \overline{\lim_{k\to\infty}} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon - 2s) \\ &= h_1(\epsilon - 2s). \end{aligned}$$

By an application of Lemma (1.3), h_1 is left continuous.

Letting limit as $s \to 0$ in the above inequality, we obtain

$$h_1(\epsilon) = \overline{\lim_{k\to\infty}} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon) \leq 1 - \lambda. \tag{2.6}$$

Next, for all $t > 0$, we define the function

$h_2(t) = \underline{\lim}_{k\to\infty} M\left(x_{l(k)+1}, x_{m(k)+1}, t\right)$.

In above similar process, we can prove that

$$h_2(\epsilon) = \underline{\lim_{k\to\infty}} M\left(x_{l(k)+1}, x_{m(k)+1}, \epsilon\right) \geq 1 - \lambda. \tag{2.7}$$

Combining (2.6) and (2.7), we get

$\overline{\lim}_{k\to\infty} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon) \leq 1 - \lambda \leq \underline{\lim}_{k\to\infty} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon)$.

This implies that

$$\lim_{n\to\infty} M(x_{l(k)+1}, x_{m(k)+1}, t) = 1 - \lambda. \tag{2.8}$$

Again by (2.6),

$$\overline{\lim_{k\to\infty}} M(x_{l(k)}, x_{m(k)}, \epsilon) \leq 1 - \lambda.$$

For $t > 0$, we define the function

$$h_3(t) = \underline{\lim_{k\to\infty}} M(x_{l(k)}, x_{m(k)}, \epsilon). \tag{2.9}$$

Now for $s > 0$,

$M(x_{l(k)}, x_{m(k)}, \epsilon + 2s) \geq M(x_{l(k)}, x_{l(k)+1}, s) * M(x_{l(k)+1}, x_{m(k)+1}, \epsilon) * M(x_{m(k)+1}, x_{m(k)}, s)$.

Taking lim inf both the sides, we have

$\underline{\lim}_{k\to\infty} M(x_{l(k)}, x_{m(k)}, \epsilon + 2s) \geq 1 * \underline{\lim}_{k\to\infty} M(x_{l(k)+1}, x_{m(k)+1}, \epsilon) * 1 = 1 - \lambda$.

Thus,

$$h_3(\epsilon + 2s) \geq 1 - \lambda. \tag{2.10}$$

Taking limit as $s \to 0$, we get $h_3(\epsilon) \geq 1 - \lambda$. Combining (2.9) and (2.10) we obtain

$$\lim_{n\to\infty} M(x_{l(k)}, x_{m(k)}, \epsilon) = 1.$$

Now,

$$\psi(M(x_{l(k)+1},x_{m(k)+1},\epsilon)) \leq \psi(\min(M(x_{l(k)},x_{m(k)},\epsilon), M(x_{l(k)},x_{l(k)+1},\epsilon)), M(x_{m(k)},x_{m(k)+1},\epsilon)) -\phi(\min(M(x_{l(k)},x_{m(k)},\epsilon), M(x_{m(k)},x_{m(k)+1},\epsilon))).$$

Taking limit as $k \to \infty$, we get
$\psi(1-\lambda) \leq \psi(1-\lambda) - \phi(1-\lambda)$, which is a contradiction.

Thus, $\{x_n\}$ is Cauchy sequence. Since X is complete, there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. Now,

$$\begin{aligned}\psi(M(x_{n+1},Tp,t)) &= \psi(M(Tx_n,Tp,t)) \\ &\leq \psi(\min\{M(x_n,p,t), M(x_n,x_{n+1},t), M(p,Tp,t)\}) \\ &\quad -\phi(\min\{M(x_n,p,t), M(p,Tp,t)\}).\end{aligned}$$

Taking limit as $n \to \infty$, we get
$\psi(M(p,Tp,t)) \leq \psi(M(p,Tp,t)) - \phi(M(p,Tp,t))$,
which implies that $\phi(M(p,Tp,t)) = 0$, that is,
$M(p,Tp,t) = 1$ or $p = Tp$.

We next establish that fixed point is unique. Let p and q be two fixed points of T.

Putting $x = p$ and $y = q$ in (2.1),
$\psi(M(Tp,Tq,t)) \leq \psi(\min M(p,q,t), M(p,Tp,t), M(q,Tq,t)) - \phi(\min M(p,q,t), M(q,Tq,t))$ or, $\psi(M(p,q,t)) \leq \psi(\min M(p,q,t), M(p,p,t), M(q,q,t)) - \phi(\min M(p,q,t), M(q,q,t))$ or, $\psi(M(p,q,t)) \leq \psi(M(p,q,t)) - \phi(M(p,q,t))$ or, $\phi(M(p,q,t)) \leq 0$, or, equivalently, $M(p,q,t) = 1$, that is, $p = q$.

The following example is in support of Theorem 2.1.

Example 2.1. *Let* $X = [0,1]$. *Let*

$$M(x,y,t) = e^{-\frac{|x-y|}{t}},$$

for all $x,y \in X$ *and* $t > 0$, *then* $(X,M,*)$ *is a complete fuzzy metric space, where* $'*'$ *is product t-norm. Let* $\psi,\phi : (0,1] \to [0,\infty)$ *be defined by* $\psi(s) = \frac{1}{s} - 1$ *and* $\phi(s) = \frac{1}{s} - \frac{1}{\sqrt{s}}$. *Then* ψ *and* ϕ *satisfy all the conditions of Theorem (2.1). Let the mapping* $T : X \to X$ *be defined by* $Tx = \frac{x}{2}$, *for all* $x \in X$.
Now, we will show that

$$\psi(M(Tx,Ty,t)) \leq \psi(M(x,y)) - \phi(N(x,y)), \tag{2.11}$$

where $M(x,y) = \min\{M(x,y,t),\ M(x,Tx,t),\ M(y,Ty,t)\}$ *and* $N(x,y) = \phi(\min\{M(x,y,t),\ M(y,Ty,t)\})$. *Herein;*

$$\max\{|x-y|,\ \frac{x}{2},\ \frac{y}{2}\} = \begin{cases} x-y & 0 \leq y \leq \frac{x}{2} \\ \frac{x}{2} & \frac{x}{2} < y \leq x \\ \frac{y}{2} & x < y \leq 2x \\ y-x & 2x < y \leq 1 \end{cases}$$

and

$$\max\{|x-y|, \frac{y}{2}\} = \begin{cases} x-y & 0 \le y \le \frac{2x}{3} \\ \frac{y}{2} & \frac{2x}{3} < y \le 2x \\ y-x & 2x < y \le 1. \end{cases}$$

Case (1): When $0 \le y \le \frac{x}{2}$ *or* $2x < y \le 1$, *then*

$$\psi(M(Tx, Ty, t)) = \psi(e^{-|\frac{x-y}{2t}|}) = e^{|\frac{x-y}{2t}|} - 1$$

and

$$\psi(M(x,y)) - \phi(N(x,y)) = \psi(e^{-\frac{|x-y|}{t}}) - \phi(e^{-\frac{|x-y|}{t}}) = e^{|\frac{x-y}{2t}|} - 1.$$

Obviously, in this case, (2.11) is satisfied.

Case (2): When $\frac{x}{2} < y \le \frac{2x}{3}$, *then*

$$\psi(M(Tx, Ty, t)) = \psi(e^{-\frac{x-y}{2t}}) = e^{\frac{x-y}{2t}} - 1$$

and

$$\psi(M(x,y)) - \phi(N(x,y)) = \psi(e^{-\frac{x}{2t}}) - \phi(e^{-\frac{x-y}{t}}) = e^{\frac{x}{2t}} - 1 - e^{\frac{x-y}{t}} + e^{\frac{x-y}{2t}}.$$

In this case, $\frac{x}{2} \ge x - y$ *and exponential function is an increasing function. Therefore,* $e^{\frac{x-y}{2t}} \le e^{\frac{x}{2t}} - e^{\frac{x-y}{t}} + e^{\frac{x-y}{2t}}$ *and hence (2.11) is satisfied.*

Case (3): When $\frac{2x}{3} < y \le x$, *then*

$$\psi(M(Tx, Ty, t)) = \psi(e^{-\frac{x-y}{2t}}) = e^{\frac{x-y}{2t}} - 1$$

and

$$\psi(M(x,y)) - \phi(N(x,y)) = \psi(e^{-\frac{x}{2t}}) - \phi(e^{-\frac{y}{2t}}) = e^{\frac{x}{2t}} - 1 - e^{\frac{y}{2t}} + e^{\frac{y}{4t}}.$$

Since, in this case, $\frac{x-y}{2} \le \frac{y}{4}$ *and* $\frac{x}{2} \ge \frac{y}{2}$, *(2.11) is satisfied.*

Case (4): $x < y \le 2x$, *then*

$$\psi(M(Tx, Ty, t)) = \psi(e^{-\frac{y-x}{2t}}) = e^{\frac{y-x}{2t}} - 1$$

and

$$\psi(M(x,y)) - \phi(N(x,y)) = \psi(e^{-\frac{y}{2t}}) - \phi(e^{-\frac{y}{2t}}) = e^{\frac{x}{4t}} - 1.$$

Since, in this case, $\frac{y}{2} \ge y - x$, *(2.11) is satisfied. Hence, all the conditions of Theorem (2.1) are satisfied. Thus,* 0 *is the unique fixed point of* T.

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