On the Relationship Between *L*-fuzzy Closure Spaces and *L*-fuzzy Rough Sets

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Abstract. This work is towards the establishment of bijective correspondence between the family of all L-fuzzy reflexive/tolerance approximation spaces and the family of all quasi-discrete L-fuzzy closure spaces satisfying a certain condition.

Keywords: L-fuzzy closure space
L-fuzzy reflexive approximation space
L-fuzzy tolerance approximation space

1 Introduction

Rough sets, firstly introduced by Pawlak [11] has been advanced notably with worthy of attention due to its widespread applications in both mathematics and computer sciences for the study of intelligent systems having insufficient, imprecise, uncertain and incomplete information. The partition or equivalence (indiscernibility) relations were the fundamental and abstract tools of the rough set theory introduced by Pawlak. Researchers have made several generalizations of rough sets using an arbitrary relation in place of an equivalence relation (cf., [4,7,21,22]). Dubois and Prade [3], proposed fuzzy version of rough sets in which fuzzy relations play a key roll instead of crisp relations. The fuzzy rough sets and their relationship with fuzzy topological spaces were described in detail by several authors (e.g., cf., [2,6,10,12–14,16,17,19,20]). Moreover, in [6,10,17], the set of all L-fuzzy preorder approximation spaces together with the set of all saturated L-fuzzy topological spaces were center of interest, and it was shown that under a certain extra condition there exists a bijective correspondence between them. The silence on such relationship between the set of other generalized approximation spaces (such as L-fuzzy reflexive approximation space and L-fuzzy tolerance approximation spaces) and the set of some L-fuzzy topological structures, in the cited work, attract our attention and lead us an

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attempt to establish such relationships by using the concept of L-fuzzy closure spaces. Finally, we have established the similar result for the set of all, L-fuzzy preorder approximation spaces and L-fuzzy closure spaces, respectively.

2 Preliminaries

We begin by recalling the following concept of a residuated lattice from [1].

Definition 1. *An algebra $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ define a* ***residuated lattice****, if $(L, \wedge, \vee, 0, 1)$ is a lattice having 0 and 1 as least and greatest element, respectively, $(L, *, 1)$ is a commutative monoid having unit 1, and $*$ and $\rightarrow$ form an adjoint pair, i.e., $\forall\ x, y, z \in L$, $x * y \leq z \Leftrightarrow x \leq y \rightarrow z$. Also, L is said to be a* **complete residuated lattice** *if lattice $(L, \vee, \wedge, 0, 1)$ is complete.*

Definition 2. *The* **precomplement** *on L is a map $\neg: L \longrightarrow L$ such that $\neg\, x = x \rightarrow 0, \forall x \in L$.*

Throughout, L denotes the complete residuated lattice. For a nonempty set X, L^X denote the collection of all L-fuzzy sets in X, for $\alpha \in L$, $\bar{\alpha}$ denotes the constant L-fuzzy set.

Definition 3. *A complete residuated lattice L is called* **regular** *if $\neg\,(\neg\, a) = a$, $\forall a \in L$.*

The basic properties of a complete regular residuated lattice, which we use in subsequent sections are listed in following proposition.

Proposition 1. *For all $a, b, a_i \in L$, $i \in J$ an index set, we have*

(i) $a * b = \neg\,(a \rightarrow (\neg\, b))$,
(ii) $a \rightarrow b = \neg\,(a * (\neg\, b))$,
(iii) $\neg\,(\wedge\{a_i\}) = \vee\{\neg\, a_i\}$,
(iv) $\neg\,(\vee\{a_i\}) = \wedge\{\neg\, a_i\}$.

Definition 4 [5]. *Let X be a nonempty set, then L-fuzzy relation on X is a map $R : X \times X \rightarrow L$.*
*For, properties of an L-fuzzy relation we refer to [5,10,15]. However, for completeness we emphasize from [10,15] that an L-fuzzy reflexive and L-fuzzy symmetric relation R is known as L-***fuzzy tolerance relation** *and, if R is L-fuzzy reflexive as well as L-fuzzy transitive then it is called L-***fuzzy preorder***.*

Definition 5 [6,10,15,17]. *Let R be an L-fuzzy relation on a nonempty set X, then an L-***fuzzy approximation space** *is a pair (X, R), which is further known as L-***fuzzy reflexive/tolerance/preorder approximation space***, respectively, according as underlying L-fuzzy relation R is an reflexive, tolerance or preorder.*

Throughout, set of all L-fuzzy approximation space over a nonempty set X is denoted by Ω.

Definition 6 [10,15,17]. *Consider an* $(X, R) \in \Omega$ *and* $A \in L^X$*. The* **lower approximation** $\underline{apr}_R(A)$ *of* A *and the* **upper approximation** $\overline{apr}_R(A)$ *of* A *in* (X, R) *are respectively defined as follows:*

$$\underline{apr}_R(A)(x) = \wedge\{R(x, y) \rightarrow A(y) : y \in X\}, and$$
$$\overline{apr}_R(A)(x) = \vee\{R(x, y) * A(y) : y \in X\}.$$

For an $(X, R) \in \Omega$ *and* $A \in L^X$*, we called the pair* $(\underline{apr}_R(A), \overline{apr}_R(A))$ *an* L-**fuzzy rough set***.*

Proposition 2 [17]. *Consider an* $(X, R) \in \Omega$*, where* L *is regular as well, then* $forall A \in L^X$*,*

(i) $\underline{apr}_R(A) = \rightharpoondown \overline{apr}_R(\rightharpoondown A)$*, and*
(ii) $\overline{apr}_R(A) = \rightharpoondown \underline{apr}_R(\rightharpoondown A)$*.*

Proposition 3 [6,15,17]. *Consider an* $(X, R) \in \Omega$*, then* $\forall A_i \in L^X, i \in J$ *and* $\alpha \in L$*,*

(i) $\overline{apr}_R(\vee\{A_i : i \in J\}) = \vee\overline{apr}_R\{A_i : i \in J\}$*,*
(ii) $\underline{apr}_R(\wedge\{A_i : i \in J\}) = \wedge\underline{apr}_R\{A_i : i \in J\}$*, and*
(iii) $\overline{apr}_R(A * \bar{\alpha}) = \overline{apr}_R(A) * \bar{\alpha}$*.*

Proposition 4 [17]. *Consider an* $(X, R) \in \Omega$*, which is reflexive and* $A \in L^X$*, then*

(i) $\underline{apr}_R(A) \leq A$*, and*
(ii) $A \leq \overline{apr}_R(A)$*.*

Proposition 5 [17]. *Consider an* $(X, R) \in \Omega$ *and* $A \in L^X$*, then* R *is an* L*-fuzzy transitive relation on* X *iff* $\overline{apr}_R(\overline{apr}_R(A)) \leq \overline{apr}_R(A)$*.*

Proposition 6. *Let* $(X, R), (X, S) \in \Omega$*, then* $R \leq S$ *iff* $\overline{apr}_R(A) \leq \overline{apr}_S(A)$*,* $\forall A \in L^X$*.*

Proof. Let $\overline{apr}_R(A) \leq \overline{apr}_S(A)$, $\forall A \in L^X$, i.e., $\vee\{R(x, y) * A(y)\} \leq \vee\{S(x, y) * A(y)\}$, $\forall A \in L^X$. Thus $R \leq S$, $\forall x, y \in X$.
Conversely, let $R \leq S$ and $x \in X$. Then $\overline{apr}_R(A)(x) = \vee\{R(x, y) * A(y) : y \in X\} \leq \vee\{S(x, y) * A(y) : y \in X\} = \overline{apr}_S(A)(x)$. Thus $\overline{apr}_R(A) \leq \overline{apr}_S(A)$.

The L-fuzzy topological concepts, we use here, are fairly standard and based on [8].

Definition 7. *An* L-**fuzzy topology** τ *over a nonempty set* X *is a subset of* L^X *closed under arbitrary suprema and finite infima and which contains all constant* L*-fuzzy sets.*

The pair (X, τ) is called an L-bffuzzy topological space. As usual, the member of τ are called L-**fuzzy** τ-**open sets**.

Definition 8. *A* **Kuratowski L-fuzzy closure operator** *over a nonempty set X is a map $k : L^X \to L^X$, whit property that $\forall A, \in L^X$ and $\forall \alpha \in L$,*

(i) $k(\bar{\alpha}) = \bar{\alpha}$,
(ii) $A \leq k(A)$,
(iii) $k(A \vee B) = k(A) \vee k(B)$, *and*
(iv) $k(k(A)) = k(A)$.

Proposition 7 [6]. *Consider an $(X, R) \in \Omega$, where R be an L-fuzzy reflexive relation, then $\tau_R = \{A \in L^X : \underline{apr}_R(A) = A\}$ is an L-fuzzy topology.*
One can easily verify that τ_R is a saturated[1] *L-fuzzy topology over X.*

Proposition 8 [17]. *Let k be as defined in Definition 8, then $\exists$ an L-fuzzy preorder S_k over X for which $\overline{apr}_{S_k}(A) = k(A)$ iff (i) $\forall i \in J$ an indexed set $k(\vee\{A_i\}) = \vee\{k(A_i)\}$, $\forall A_i \in L^X$ and (ii) $k(A * \bar{\alpha}) = k(A) * \bar{\alpha}$, $\forall A \in L^X$, $\forall \alpha \in L$.*

The concept of fuzzy closure spaces was proposed in (cf., [9]). Further, the concepts of subspace of a fuzzy closure space, sum of a family of pairwise disjoint fuzzy closure spaces and product of a family of fuzzy closure spaces were studied in [18]. Now, we introduce here the following concept of an L-fuzzy closure space as a generalization of the concept of a fuzzy closure space studied in [9,18].

Definition 9. *An L-fuzzy closure space over a nonempty set X is a pair (X, c), where the map $c : L^X \to L^X$ is such that $\forall A, B \in L^X$ and $\forall \alpha \in L$,*

(i) $c(\bar{\alpha}) = \bar{\alpha}$,
(ii) $A \leq c(A)$, *and*
(iii) $c(A \vee B) = c(A) \vee c(B)$.

Definition 10. *An L-fuzzy closure space (X, c) is called*

(i) ***quasi-discrete*** *if* $c\{\vee\{A_i : i \in J\}\} = \vee\{c(A_i) : i \in J\}$, $\forall A_i \in L^X$,
(ii) ***symmetric*** *if* $c(1_y)(x) = c(1_x)(y)$, $\forall x, y \in X$, *and*
(iii) $A \in L^X$ *is called* L-**fuzzy closed** *if* $c(A) = A$.

Proposition 9. *Let (X, c) be as in 9, then*

(i) for $A, B \in L^X$ *if* $A \leq B$ *then* $c(A) \leq c(B)$,
(ii) $c\{\wedge\{A_i : i \in J\}\} \leq \wedge\{c(A_i) : i \in J\}$, $\forall A_i \in L^X, i \in J$.

Proof. Follows obviously.

Proposition 10. *Consider L-fuzzy closure space (X, c), $H \in L^X$ and $\bar{c} : L^X \to L^X$ be a map such that $\bar{c}(H) = \wedge\{K \in L^X : H \leq K$ and $c(K) = K\}$. Then $\bar{c}$ is a Kuratowski L-fuzzy closure operator on X.*

[1] In the sense that arbitrary infimum of L-fuzzy τ_R-open sets is also, an L-fuzzy τ_R-open.

Proof. Obviously $\forall \alpha \in L, c(\bar{\alpha}) = \bar{\alpha}$ and $\forall H \in L^X, H \leq \bar{c}(H)$. Now, let $H, K \in X$. Then $\bar{c}(H \vee K) = \wedge\{G \in L^X : (H \vee K) \leq G \text{ and } c(G) = G\}$. Thus $\bar{c}(H \vee K) = \wedge\{G \in L^X : H \leq G, K \leq G \text{ and } c(G) = G\} = \{\wedge\{G \in L^X : H \leq G \text{ and } c(G) = G\} \vee \{\wedge(G \in L^X : K \leq G \text{ and } c(G) = G)\}\} = \bar{c}(H) \vee \bar{c}(K)$. Finally, $\bar{c}(\bar{c}(H)) = \bar{c}\{\{\wedge\{K \in L^X : H \leq K \text{ and } c(K) = K\}\} \leq \wedge\{\bar{c}(K) : H \leq K, c(K) = K\} = \wedge\{\wedge\{G : K \leq G, c(G) = G\} : H \leq K, c(K) = K\} = \wedge\{G : H \leq G, c(G) = G\} = \bar{c}(H)$.

Thus $\bar{c}$ induces an L-fuzzy topology, say, $\tau_{\bar{c}}$ and is given by $\tau_{\bar{c}} = \{H \in L^X : \bar{c}(\neg H) = \neg H\}$.

Proposition 11. *Let (X, c) be an L-fuzzy closure space. Then $\forall H \in L^X$,*

(i) $c(\bar{c}(H)) = \bar{c}(H)$, *i.e.,* $\bar{c}(H)$ *is L-fuzzy closed.*
(ii) $c(H) \leq \bar{c}(H)$,
(iii) $c(H) = H$ *iff* $\bar{c}(H) = H$.

Proof. (i) Let $H \in L^X$. Then from Proposition 9, $c(\bar{c}(H)) = c(\wedge\{K : H \leq K$ and $c(K) = K\}) \leq \wedge\{c(K) : H \leq K$ and $c(K) = K\} = \wedge\{K : H \leq K$ and $c(K) = K\} = \bar{c}(H)$.
(ii) $H \leq \bar{c}(H) \Rightarrow c(H) \leq c(\bar{c}(H)) = \bar{c}(H)$.
(iii) Let $c(H) = H$, $\forall H \in L^X$. Then H is L-fuzzy closed. Therefore $\bar{c}(H) \leq H$ (cf., Proposition 10). This together with $H \leq \bar{c}(H)$ shows that $\bar{c}(H) = H$.
Conversely, let $\bar{c}(H) = H$. Then from (ii), $H \leq c(H) \leq \bar{c}(H) = H$. Thus $\bar{c}(H) = H$, whereby $c(H) = H$.

Proposition 12. *Let (X, c) be an L-fuzzy closure space. Then $\forall H \in L^X$, $c(H) = \bar{c}(H)$ iff $c(c(H)) = c(H)$.*

Proof. Let $c(H) = \bar{c}(H)$, $H \in L^X$. Then $c(c(H)) = c(\bar{c}(H)) = \bar{c}(H) = c(H)$.
Conversely, let $c(c(H)) = c(H)$. Then $c(H)$ is L-fuzzy closed. Hence from Proposition 11 (iii), $c(H) = \bar{c}(H)$.

Proposition 13. *Let (X, c) be a quasi-discrete L-fuzzy closure space. Then the L-fuzzy topology $\tau_{\bar{c}}$ on X is a saturated L-fuzzy topology.*

Proof. Follows from Definition 10 and Propositions 10 and 12.

3 L-fuzzy Closure Spaces and L-fuzzy Approximation Spaces

The existence of a bijective correspondence between the set of all L-fuzzy reflexive approximation spaces and the set of all quasi-discrete L-fuzzy closure spaces under a certain extra condition is established here. The similar relationship between the set of all L-fuzzy tolerance approximation spaces and the set of all symmetric quasi-discrete L-fuzzy closure spaces satisfying a certain extra condition is also demonstrated.
We begin with the following.

Proposition 14. *Consider an $(X, R) \in \Omega$, where R is L-fuzzy reflexive relation then $(X, \overline{apr}_R)$ is a quasi-discrete L-fuzzy closure space such that $\overline{apr}_R(A * \bar{\alpha}) = \overline{apr}_R(A) * \bar{\alpha}$, $\forall A \in L^X$ and $\forall \alpha \in L$.*

Proof. Follows from Propositions 3 and 4.

Definition 11. *For $y \in X$ and $\alpha \in L$, the L-fuzzy subset $1_y * \bar{\alpha}$ of X is called an L-**fuzzy point** in X, and is denoted as y_α.*

Proposition 15. *Let (X, c) be a quasi-discrete L-fuzzy closure space such that $c(A * \bar{\alpha}) = c(A) * \bar{\alpha}$, $\forall A \in L^X$ and $\forall \alpha \in L$. Then $\exists$ a L-fuzzy reflexive relation R_c over X which is unique and satisfy $\overline{apr}_{R_c}(A) = c(A)$, $\forall A \in L^X$.*

Proof. Let (X, c) be a quasi-discrete L-fuzzy closure space such that $c(A * \bar{\alpha}) = c(A) * \bar{\alpha}$, $\forall A \in L^X$ and $\forall \alpha \in L$. Also, let $R_c(x, t) = c(1_t)(x)$, $\forall x, t \in X$. Then R_c is an L-fuzzy relation on X such that $1 = 1_x(x) \leq c(1_x)(x)$. Thus $c(1_x)(x) = 1$, whereby R_c is an L-fuzzy reflexive relation over X. Now, let $A \in L^X$, $\alpha \in L$ and $x \in X$. Then

$$\begin{aligned}
\overline{apr}_{R_c}(A)(x) &= \overline{apr}_{R_c}(\vee\{t_\alpha : t \in X\})(x), \text{ where } \alpha = A(t) \\
&= \vee\{\vee\{R_c(x, r) * t_\alpha(r) : r \in X\} : t \in X\} \\
&= \vee\{\vee\{R_c(x, r) * t_\alpha(r) : r \in X, r \neq t\}, \\
&\qquad \vee\{R_c(x, t) * t_\alpha(r) : r \in X,\ r = t\} : t \in X\} \\
&= \vee\{0 \vee (R_c(x, t) * \alpha) : t \in X\} \\
&= \vee\{R_c(x, t) * \alpha : t \in X\} \\
&= \vee\{c(1_t)(x) * \alpha : t \in X\} \\
&= \vee\{c\{(1_t) * \bar{\alpha}\}(x) : t \in X\} \\
&= c\{\vee\{1_t * \bar{\alpha} : t \in X\}(x)\} \\
&= c(A).
\end{aligned}$$

Hence $\overline{apr}_{R_c}(A) = c(A)$. To show the uniqueness of L-fuzzy relation R_c, let R' be another L-fuzzy reflexive relation on X such that $\overline{apr}_{R'}(A) = c(A), \forall A \in L^X$. Then $R_c(x, t) = c(1_t)(x) = \overline{apr}_{R'}(1_t)(x) = \vee\{R'(x, r) * 1_t(r) : r \in X\} = R'(x, t)$. Thus $R_c = R'$. Hence the L-fuzzy relation R_c on X is unique.

Now, Propositions 14 and 15 lead us to the following.

Proposition 16. *Let $\mathcal{F}$ be the set of all L-fuzzy reflexive approximation spaces and $\mathcal{T}$ be the set of all quasi-discrete L-fuzzy closure spaces satisfying $c(A * \bar{\alpha}) = c(A) * \bar{\alpha}$, $\forall A \in L^X$ and $\forall \alpha \in L$. Then there exists a bijective correspondence between $\mathcal{F}$ and $\mathcal{T}$.*

Remark 1. In [6], it has been pointed out that for $A \in L^X$, $\underline{apr}_R(A)$ and $\overline{apr}_R(A)$ are not dual to each other. Therefore $\tau_{R_c} \neq \tau_{\bar{c}}$. The next proposition says that the equality holds if L is regular.

Proposition 17. *Let L be regular and (X, c) be a quasi-discrete satisfying $c(A * \bar{\alpha}) = c(A) * \bar{\alpha}, \forall A \in L^X, \forall \alpha \in L$. Then $\tau_{R_c} = \tau_{\bar{c}}$, where R_c is an L-fuzzy reflexive relation on X induced by c.*

Proof. Let $A \in \tau_{\bar{c}}$. Then $\bar{c}(\neg A) = \neg A$. As from Proposition 11, $c(A) \leq \bar{c}(A)$, $\forall A \in L^X$, $c(\neg A) \leq \bar{c}(\neg A)$, or that $A \leq \neg c(\neg A)$.

$$\begin{aligned}
\text{Now,} \quad \neg c(\neg A) &= \neg \overline{apr}_{R_c}(\neg A) \\
&= \neg \{\vee\{R_c(w,t) * (\neg A(t))\} : t \in X\} \\
&= \neg \{\vee\{\neg \{R_c(w,t) \rightarrow (\neg (\neg A(t)))\}\} : t \in X\} \\
&= \neg \{\vee\{\neg \{R_c(w,t) \rightarrow A(t)\}\} : t \in X\} \\
&= \wedge\{\neg\neg \{R_c(w,t) \rightarrow A(t)\} : t \in X\} \\
&= \wedge\{R_c(w,t) \rightarrow A(t)\} \\
&= \underline{apr}_{R_c}(A).
\end{aligned}$$

Thus $A \leq \underline{apr}_{R_c}(A)$. Also, $\underline{apr}_{R_c} \leq A$, whereby $\underline{apr}_{R_c} = A$. Hence $\tau_{\bar{c}} \leq \tau_{R_c}$.

Conversely, let $A \in \tau_{R_c}$. Then $\underline{apr}_{R_c}(A) = A$, or that $\wedge\{R_c(w,t) \rightarrow A(t) : t \in X\} = A$, i.e., $\wedge\{\neg \{R_c(w,t) * (\neg A(t)) : t \in X\}\} = A$, or that $\neg \{\vee\{R_c(w,t) * (\neg A(t)) : t \in X\}\} = A$, i.e., $\vee\{R_c(w,t) * (\neg A(t)) : t \in X\} = \neg A$, or that $\overline{apr}_{R_c}(\neg A) = \neg A$, whereby $c(\neg A) = \neg A$. Thus from Proposition 11, $\bar{c}(\neg A) = \neg A$, whereby $A \in \tau_{\bar{c}}$, or that $\tau_{R_c} \leq \tau_{\bar{c}}$. Hence $\tau_{R_c} = \tau_{\bar{c}}$.

For a given quasi-discrete L-fuzzy closure space (X, c) satisfying $c(A * \bar{\alpha}) = c(A) * \bar{\alpha}$, $\forall A \in L^X$, $\forall \alpha \in L$ and its associated Kuratowski L-fuzzy closure operator $\bar{c}$, $(X, \bar{c})$ is obviously a quasi-discrete L-fuzzy closure space such that $\bar{c}(A * \bar{\alpha}) = \bar{c}(A) * \bar{\alpha}$, $\forall A \in L^X$, $\forall \alpha \in L$. Hence from Proposition 15, there exists an L-fuzzy reflexive relation, say, $S_{\bar{c}}$ on X, given by $S_{\bar{c}}(w,t) = \bar{c}(1_t)(w), \forall w, t \in X$.

Before stating next, we introduce the following.

Definition 12. *Let R and T be two L-fuzzy relations on X. Then T is called L-**fuzzy transitive closure** of R if T is the smallest L-fuzzy transitive relation containing R.*

Now, we have the following.

Proposition 18. *Let (X, c) be a quasi-discrete L-fuzzy closure space such that $c(H * \bar{\alpha}) = c(H) * \bar{\alpha}$, $\forall H \in L^X$, $\forall \alpha \in L$ and $\bar{c}$ be the associated Kuratowski L-fuzzy closure operator. Then the L-fuzzy relation $S_{\bar{c}}$ is L-fuzzy transitive closure of L-fuzzy relation R_c.*

Proof. Let $S_{\bar{c}} = \bar{c}(1_y)(x), \forall x, y \in X$.Transitivity of $S_{\bar{c}}$ follows from Propositions 5 and 15. Also, $R_c \leq S_{\bar{c}}$ follows from Proposition 11. To show the relation $S_{\bar{c}}$ is an L-fuzzy transitive closure of L-fuzzy relation R_c, it only remains to show that $S_{\bar{c}}$ is the smallest L-fuzzy reflexive and transitive relation containing R_c. For this, let T be another L-fuzzy reflexive and transitive relation on X such that $R_c \leq T$. Then from the reflexivity of T, $(X, \overline{apr}_T)$ is quasi-discrete L-fuzzy closure space. Now, from transitivity of T and Proposition 12 followed by Proposition 10, we have $\overline{apr}_T(H) = \wedge\{K \in L^X : H \leq K, \overline{apr}_T(K) = K\}, \forall H \in L^X$. Also, $S_{\bar{c}}$ being L-fuzzy reflexive and L-fuzzy transitive relation associated with Kuratowski L-fuzzy closure operator $\bar{c}$, from Proposition 8 $\overline{apr}_{S_{\bar{c}}}(H) = \bar{c}(H), \forall H \in L^X$ and $\bar{c}$ being Kuratowski L-fuzzy closure operator associated with quasi-discrete L-fuzzy closure space (X, c), $\forall H \in L^X$, it follows from Proposition 15 that $\bar{c}(H) = \wedge\{K \in L^X : H \leq K, c(K) = K\} = \wedge\{K \in L^X : H \leq K, \overline{apr}_{R_c}(K) = K\}$. Thus from Proposition 6, $\overline{apr}_{S_{\bar{c}}}(H) = \wedge\{K \in L^X : H \leq K, \overline{apr}_{R_c}(K) = K\} \leq \wedge\{K \in L^X : H \leq K, \overline{apr}_T(K) = K\} = \overline{apr}_T(H)$, whereby $\overline{apr}_{S_{\bar{c}}}(H) \leq \overline{apr}_T(H)$, showing that $S_{\bar{c}} \leq T$.

Now, we show that there is a bijective correspondence between the set of all L-fuzzy tolerance approximation spaces and the set of all symmetric quasi-discrete L-fuzzy closure spaces satisfying an extra condition.

Proposition 19. *Let (X, R) be an L-fuzzy tolerance approximation space. Then $(X, \overline{apr}_R)$ is a symmetric quasi-discrete L-fuzzy closure space such that $\overline{apr}_R(H * \bar{\alpha}) = \overline{apr}_R(H) * \bar{\alpha}$, $\forall H \in L^X$ and $\forall \alpha \in L$.*

Proof. From Propositions 3 and 4 it follows that $(X, \overline{apr}_R)$ is an L-fuzzy closure space and quasi-discrete. Now, $\forall x, y \in X, \overline{apr}_R(1_y)(x) = \vee\{R(x,t) * 1_y(t) : t \in X\} = \vee\{R(y,t) * 1_x(t) : t \in X\} = \overline{apr}_R(1_x)(y)$, showing that $(X, \overline{apr}_R)$ is symmetric. Also, for all $H \in L^X$ and $\alpha \in L, \overline{apr}_R(H * \bar{\alpha}) = \overline{apr}_R(H) * \bar{\alpha}$ follows from Proposition 3.

Proposition 20. *Let (X, c) be a symmetric quasi-discrete L-fuzzy closure space such that $c(H * \bar{\alpha}) = c(H) * \bar{\alpha}, \forall H \in L^X$ and $\forall \alpha \in L$. Then $\exists$ a L-fuzzy tolerance relation R_c over X which is unique and satisfy $\overline{apr}_{R_c}(H) = c(H)$, $\forall H \in L^X$.*

Proof. Let (X, c) be a quasi-discrete and such that $c(H * \bar{\alpha}) = c(H) * \bar{\alpha}, \forall H \in L^X, \forall \alpha \in L$. Let L-fuzzy relation R_c on X be such that $R_c(x, y) = c(1_y)(x), \forall x, y \in X$. Then $1 = 1_x(x) \leq c(1_x)(x)$. Thus $c(1_x)(x) = 1$. Hence R_c is an L-fuzzy reflexive relation on X. Also, (X, c) being an L-fuzzy symmetric closure space, the L-fuzzy relation R_c is symmetric and $\overline{apr}_{R_c}(H) = c(H)$ (cf., Proposition 15). To show the uniqueness of L-fuzzy relation R_c, let R' be another L-fuzzy tolerance relation on X such that $\overline{apr}_{R'}(H) = c(H)$, $\forall H \in L^X$. Then $R_c(x, y) = c(1_y)(x) = \overline{apr}_{R'}(1_y)(x) = \vee\{R'(x,t) * 1_y(t) : t \in X\} = R'(x, y)$. Thus $R_c = R'$. Hence the L-fuzzy relation R_c on X is unique.

Proposition 21. *Let $\mathcal{F}$ be the set of all L-fuzzy tolerance approximation spaces and $\mathcal{T}$ be the set of all symmetric quasi-discrete L-fuzzy closure spaces satisfying $c(H * \bar{\alpha}) = c(H) * \bar{\alpha}$, $\forall H \in L^X$ and $\forall$ $\alpha \in L$, then $\exists$ a bijective correspondence between $\mathcal{F}$ and $\mathcal{T}$.*

Proof. Follows from Propositions 19 and 20.

Proposition 22. *Let (X, c) be a symmetric quasi-discrete L-fuzzy closure space such that $c(H * \bar{\alpha}) = c(H) * \bar{\alpha}$, $\forall H \in L^X$, $\forall \alpha \in L$ and $\bar{c}$ be the associated Kuratowski L-fuzzy closure operator. Then the L-fuzzy relation $S_{\bar{c}}$ is an L-fuzzy transitive closure of L-fuzzy relation R_c.*

Proof. Similar to that of Proposition 18.

Proposition 23. *Let (X, R) be an L-fuzzy preorder approximation space. Then $(X, \overline{apr}_R)$ is a quasi-discrete L-fuzzy closure space such that (i) $\overline{apr}_R(\overline{apr}_R(H)) = H$ and (ii) $\overline{apr}_R(H * \bar{\alpha}) = \overline{apr}_R(H) * \bar{\alpha}$, $\forall H \in L^X$, $\forall \alpha \in L$.*

Proof. Follows from Propositions 5 and 14.

Proposition 24. *Let (X, c) be a quasi-discrete L-fuzzy closure space such that (i) $c(c(H)) = c(H)$ and (ii) $c(H * \bar{\alpha}) = c(H) * \bar{\alpha}$, $\forall H \in L^X$, $\forall \alpha \in L$. Then there exists an unique L-fuzzy preorder R_c on X such that $\overline{apr}_{R_c}(H) = c(H)$, $\forall H \in L^X$.*

Proof. Follows from Propositions 8, 12 and 15.

Finally, the following is an equivalent characterization of the result regarding the bijective correspondence between the set of all L-fuzzy preorder approximation spaces and the set of all saturated L-fuzzy topological spaces observed in [6,10,17].

Proposition 25. *Let $\mathcal{F}$ be the set of all L-fuzzy preorder approximation spaces and $\mathcal{T}$ be the set of all quasi-discrete L-fuzzy closure spaces satisfying (i) $c(c(H)) = c(H)$ and (ii) $c(H * \bar{\alpha}) = c(H) * \bar{\alpha}$, $\forall H \in L^X$, $\forall \alpha \in L$. Then there exists a bijective correspondence between $\mathcal{F}$ and $\mathcal{T}$.*

Proof. Follows from Propositions 23 and 24.

4 Conclusion

The present paper established an association between L-fuzzy rough sets and L-fuzzy closure spaces. In literature, the bijective correspondence between the set of all L-fuzzy preorder approximation spaces and the set of all L-fuzzy topological spaces of certain type is well known (cf., [6,10]). But the work done in this paper shows that actual theory for such bijective correspondence begins from the notion of L-fuzzy closure spaces. In future we will try to associate L-fuzzy approximation spaces and L-fuzzy topological spaces in categorical point of view.

References

1. Blount, K., Tsinakis, C.: The structure of residuated lattices. Int. J. Algebra Comput. **13**, 437–461 (2003)
2. Boixader, D., Jacas, J., Recasens, J.: Upper and lower approximations of fuzzy sets. Int. J. Gen. Syst. **29**, 555–568 (2000)
3. Dubois, D., Prade, H.: Rough fuzzy set and fuzzy rough set. Int. J. Gen. Syst. **17**, 191–209 (1990)
4. Gautam, V., Yadav, V.K., Singh, A.K., Tiwari, S.P.: On the topological structure of rough soft sets. In: RSKT 2014, LNAI, vol. 8818, pp. 39-48 (2014)
5. Goguen, J.A.: L-fuzzy sets. J. Math. Anal. Appl. **18**, 145–174 (1967)
6. Hao, J., Li, Q.: The relationship between L-fuzzy rough set and L-topology. Fuzzy Sets Syst. **178**, 74–83 (2011)
7. Kondo, M.: On the structure of generalized rough sets. Inf. Sci. **176**, 586–600 (2006)
8. Lowen, R.: Fuzzy topological space and fuzzy compactness. J. Math. Anal. Appl. **56**, 621–633 (1976)
9. Mashhour, A.S., Ghanim, M.H.: Fuzzy closure spaces. J. Math. Anal. Appl. **106**, 154–170 (1985)
10. Ma, Z.M., Hu, B.Q.: Topological and lattice structures of L-fuzzy rough sets determined by lower and upper sets. Inf. Sci. **218**, 194–204 (2013)
11. Pawlak, Z.: Rough sets. Int. J. Comput. Inf. Sci. **11**, 341–356 (1982)
12. Qin, K., Pei, Z.: On the topological properties of fuzzy rough sets. Fuzzy Sets Syst. **151**, 601–613 (2005)
13. Qin, K., Yang, J., Pei, Z.: Generalized rough sets based on reflexive and transitive relations. Inf. Sci. **178**, 4138–4141 (2008)
14. Ramadan, A.A., Elkordy, E.H., El-Dardery, M.: L-fuzzy approximation space and L-fuzzy topological spaces. Iran. J. Fuzzy Syst. **13**(1), 115–129 (2016)
15. Radzikowska, A.M., Kerre, E.E.: Fuzzy rough sets based on residuated lattices. In: Peters, J.F., Skowron, A., Dubois, D., Grzymała-Busse, J.W., Inuiguchi, M., Polkowski, L. (eds.) Transactions on Rough Sets II. LNCS, vol. 3135, pp. 278–296. Springer, Heidelberg (2004). https://doi.org/10.1007/978-3-540-27778-1_14
16. Sharan, S., Tiwari, S.P., Yadav, V.K.: Interval type-2 fuzzy rough sets and interval type-2 fuzzy closure spaces. Iran. J. Fuzzy Syst. **12**, 127–135 (2015)
17. She, Y.H., Wang, G.J.: An axiomatic approach of fuzzy rough sets based on residuated lattices. Comput. Math. Appl. **58**, 189–201 (2009)
18. Srivastava, R., Srivastava, M.: On T_0- and T_1-fuzzy closure spaces. Fuzzy sets Syst. **109**, 263–269 (2000)
19. Tiwari, S.P., Sharan, S., Yadav, V.K.: Fuzzy closure spaces vs. fuzzy rough sets. Fuzzy Inf. Eng. **6**, 93–100 (2014)
20. Wu, W.-Z.: A study on relationship between fuzzy rough approximation operators and fuzzy topological spaces. In: Wang, L., Jin, Y. (eds.) FSKD 2005. LNCS (LNAI), vol. 3613, pp. 167–174. Springer, Heidelberg (2005). https://doi.org/10.1007/11539506_21
21. Yao, Y.Y.: Constructive and algebraic methods of the theory of rough sets. Inf. Sci. **109**, 21–47 (1998)
22. Zhu, W.: Generalized rough sets based on relations. Inf. Sci. **177**(22), 4997–5011 (2007)