Chapter 9
An Integral Formula Adapted to Different Boundary Conditions for Arbitrarily High-Dimensional Nonlinear Klein–Gordon Equations

This chapter is concerned with the initial-boundary value problem for arbitrarily high-dimensional Klein–Gordon equations, posed on a bounded domain $\Omega \subset \mathbb{R}^d$ for $d \geq 1$ and subject to suitable boundary conditions. We derive and analyse an integral formula which proves to be adapted to different boundary conditions for general Klein–Gordon equations in arbitrarily high-dimensional spaces. The formula gives a closed-form solution to arbitrarily high-dimensional homogeneous linear Klein–Gordon equations, which is totally different from the well-known D'Alembert, Poisson and Kirchhoff formulas.

9.1 Introduction

Nonlinear phenomena appear in many areas of scientific and engineering applications such as solid state physics, plasma physics, fluid dynamics, gas dynamics, wave mechanics, mathematical biology and chemical kinetics, which can be modelled by partial differential equations (PDEs). For the past four decades, there has been broad interest in a class of nonlinear evolution equations that admits extremely stable solutions termed solitons (see, e.g. [1, 2, 5, 6, 8, 16, 21, 27]). An important and typical example of such equations is the Klein–Gordon equation which can be expressed in the form:

$$\begin{cases} U_{tt}(X,t) - a^2 \Delta U(X,t) = g\big(U(X,t)\big), \quad X \in \Omega,\ t_0 < t \leq T, \\ U(X,t_0) = U_0(X), \\ U_t(X,t_0) = U_1(X), \end{cases} \tag{9.1}$$

where g is a function of U, $U : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ with $d \geq 1$, representing the wave displacement at position $X \in \mathbb{R}^d$ and time t, and

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$$g\big(U(X,t)\big) = -G'(U) = -dG(U),$$

for some smooth function $G(U)$. The general Klein–Gordon equation can be written as

$$\begin{cases} U_{tt}(X,t) - a^2 \Delta U(X,t) = -G'\big(U(X,t)\big), \quad X \in \Omega,\ t_0 < t \le T, \\ U(X,t_0) = U_0(X), \\ U_t(X,t_0) = U_1(X). \end{cases} \tag{9.2}$$

The Klein–Gordon equation was derived in 1928 as a relativistic version of the Schrödinger equation describing free particles. However, the Klein–Gordon equation was named after the physicists Oskar Klein and Walter Gordon, and proposed in 1926. The model describes relativistic electrons and correctly represents the spinless pion, a composite particle [17]. Here, it is assumed that (9.1) is subject to the given boundary conditions, such as Dirichlet boundary conditions, or Neumann boundary conditions, or Robin boundary conditions. Equation (9.1) is a natural generalization of the linear wave equation (see, e.g. [16]). A simple model of (9.1) with $d = 1$ and $g = 0$ is the homogeneous one-dimensional undamped wave equation,

$$\begin{cases} U_{tt} - a^2 U_{xx} = 0, \quad x_l < x < x_r,\ t_0 < t \le T, \\ U(x,t_0) = u_0(x), \\ U_t(x,t_0) = u_1(x), \end{cases} \tag{9.3}$$

subject to the Dirichlet boundary conditions

$$U(x_l,t) = \alpha(t),\ U(x_r,t) = \beta(t),\ t_0 \le t \le T,$$

where a means the horizontal propagation speed of the wave motion.

In the numerical simulation it is well known that the method of lines is an effective approach to solving partial differential equations such as nonlinear wave equations. Using the method of lines [20], the semidiscretisation of (9.1) in space in the one-dimensional case suggests semi-discrete differential equations, namely, a system of second-order ordinary differential equations in time. Using this approach, each one-dimensional nonlinear wave equation can be converted into a system of second-order ordinary differential equations in time:

$$\begin{cases} q''(t) + Mq(t) = \tilde{g}\big(q(t), q'(t)\big), \qquad t \in [t_0, t_{\text{end}}], \\ q(t_0) = q_0, \quad q'(t_0) = q_0', \end{cases} \tag{9.4}$$

where $\tilde{g} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is assumed to be continuous and M is a $m \times m$ positive semi-definite constant matrix. The solution of system (9.4) is a nonlinear multi-frequency oscillator. Such an oscillatory system has received a great deal of attention in the last few years (see, e.g. [3, 10, 12, 14, 24, 31]).

With regard to the exact solution of the system (9.4) and its derivative, the authors in [29, 32] established the following matrix-variation-of-constants formula which in fact is a semi-analytical expression of the solution of (9.4), or an integral formula for the oscillatory system (9.4).

Theorem 9.1 *If $\tilde{g} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous in* (9.4)*, then the solution of* (9.4) *and its derivative satisfy*

$$\begin{cases} q(t) = \phi_0\big((t-t_0)^2 M\big)q_0 + (t-t_0)\phi_1\big((t-t_0)^2 M\big)q_0' \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} (t-\zeta)\phi_1((t-\zeta)^2 M)\tilde{g}\big(q(\zeta), q'(\zeta)\big)\mathrm{d}\zeta, \\ q'(t) = -(t-t_0)M\phi_1\big((t-t_0)^2 M\big)q_0 + \phi_0\big((t-t_0)^2 M\big)q_0' \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} \phi_0\big((t-\zeta)^2 M\big)\tilde{g}\big(q(\zeta), q'(\zeta)\big)\mathrm{d}\zeta, \end{cases} \tag{9.5}$$

for $t_0, t \in (-\infty, +\infty)$, where the unconditionally convergent matrix-valued functions are defined by

$$\phi_j(M) := \sum_{k=0}^{\infty} \frac{(-1)^k M^k}{(2k+j)!}, \qquad j = 0, 1, \ldots. \tag{9.6}$$

Much attention has been paid to the matrix-variation-of-constants formula to develop new integrators such as ARKN (Adapted Runge–Kutta Nyström) methods, ERKN (Extended Runge–Kutta Nyström) methods, Gautschi-type methods, and trigonometric Fourier collocation methods for solving (9.4) (see, e.g. [9–13, 15, 22, 23, 25, 26, 29–32, 34]).

In practice, there exists a very small class of nonlinear PDEs that can be solved exactly by analytical methods. One such method is the well-known inverse scattering method (see, e.g. [4]), also called the inverse spectral transform, which is, for nonlinear PDEs, a direct generalization of the Fourier transform for linear PDEs. Regrettably, the inverse scattering method can solve the initial value problems for a very small class of nonlinear PDEs (see, e.g. [8]) with the requirement that U and various of its derivatives tend to zero as $\|X\| \to \infty$. For this reason, one therefore might think that the set of solvable nonlinear PDEs has "measure zero", and that linear PDEs and solvable nonlinear PDEs could be considered as belonging to a class in which solutions can be added in some function spaces (see, e.g. [16]). On the other hand, it is known that a formal solution to arbitrarily high-dimensional Klein–Gordon equations may be valuable in understanding new nonlinear physical phenomena and investigating novel numerical integrators for the simulation of nonlinear phenomena.

As stated above, nonlinear PDEs in general cannot be solved explicitly. Fortunately, however, we note that the mathematical structure of (9.1) is similar to (9.4), observing the fact that $-M$ in (9.4) can be regarded as a discrete operator of the Laplacian Δ in the one-dimensional case of the nonlinear wave equation based on the method of lines. This observation motivates us to derive and analyse an integral formula for the general Klein–Gordon equation (9.1) posed on a bounded domain $\Omega \subset \mathbb{R}^d$ for $d \geq 1$ equipped with the requirement of suitable boundary conditions.

The outline of this chapter is as follows. In Sect. 9.2, we analyse and derive an integral formula for (9.1). In Sect. 9.3, for the one-dimensional Klein–Gordon equations, we show in detail the consistency of the integral formula with the corresponding Dirichlet boundary conditions and Neumann boundary conditions, respectively. In Sect. 9.4, for arbitrarily high-dimensional Klein–Gordon equations, we prove the consistency of the formula with the underlying Dirichlet boundary conditions, and Neumann boundary conditions, respectively. To show the applications of the formula, illustrative examples are presented in Sect. 9.5. The last section is devoted to conclusions.

9.2 An Integral Formula for Arbitrarily High-Dimensional Klein–Gordon Equations

9.2.1 General Case

It is known that Δ is an unbounded operator which is not defined for all $v \in L^2(\Omega)$. In order to model boundary conditions, we restrict ourselves to the case where Δ is defined on a domain $D(\Delta) \subset L^2(\Omega)$, such that the underlying boundary condition is satisfied. For example, we will consider the one-dimensional Klein–Gordon equation of the form

$$\begin{cases} u_{tt} - a^2 \Delta u = f(u), \quad x \in [0, \Gamma], t \geq t_0, \\ u(x, t_0) = \varphi_1(x), \ u_t(x, t_0) = \varphi_2(x), x \in [0, \Gamma], \end{cases} \tag{9.7}$$

subject to the periodic boundary condition

$$u(0, t) = u(\Gamma, t),$$

where Γ is a fundamental period with respect to x, where $\Delta = \partial_x^2$ and $f(u) = -V'(u)$ is the negative derivative of a potential function $V(u)$. We then have

$$D(\Delta) = \{v(x) : \forall v \in L^2([0, \Gamma]) \text{ and } v(0) = v(\Gamma)\}.$$

The functions in $D(\Delta)$ are continuously differentiable and satisfy the underlying boundary condition.

In what follows, we will present an integral formula for the arbitrarily high-dimensional Klein–Gordon equation (9.1). To this end, we first define the formal operator-argument functions as follows:

$$\phi_j(\Delta) := \sum_{k=0}^{\infty} \frac{\Delta^k}{(2k+j)!}, \qquad j = 0, 1, \ldots, \tag{9.8}$$

where Δ is an operator defined on a normed space, such as the Laplacian defined on a subspace $D(\Delta)$ of $L^2(\Omega)$, and in this case, the operator-argument functions $\phi_j(\Delta)$ for $j = 0, 1, \ldots$ defined by (9.8) are bounded. Accordingly, $\phi_j(\Delta)$ in (9.8) can be called Laplacian-argument functions defined on $D(\Delta)$. Besides, Δ can also be a linear transformation such as a matrix and in the particular case of $\Delta = -M$, where M is a positive semi-definite constant matrix, (9.8) reduces to the matrix-valued functions (9.6) which have been widely used in the study of ARKN methods and ERKN methods for solving oscillatory or highly oscillatory differential equations (see, e.g. [32]).

It can be observed that (9.8) is obtained from replacing $-x$ by Δ in

$$\phi_j(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{(2k+j)!}, \ \ j = 0, 1, 2, \ldots,$$

and all $\phi_j(x)$ are bounded for any $x \geq 0$. Each of these operators has a complete system of orthogonal eigenfunctions in the complex Hilbert space $L^2(\Omega)$. Because of the isomorphism between L^2 and ℓ^2, the operator Δ on $L^2(\Omega)$ induces a corresponding operator on ℓ^2. *An elementary analysis which is similar to that for the exponential differential operator presented by Hochbruck and Ostermann in* [15] *can make sure that the Laplacian-argument functions defined on $D(\Delta)$ depending on different boundary conditions are bounded operators with respect to the norm* $\|\cdot\|_{L^2(\Omega)\leftarrow L^2(\Omega)}$, where Ω is the space region under consideration. The details can be found in [18]. It is noted that the exponential differential operator has the properties of a semigroup which are required for analysis. However, the operators defined by (9.8) do not have the semigroup property, but this is not needed in our analysis here.

Some useful properties of Laplacian-argument functions (9.8) are established in the next two theorems.

Theorem 9.2 *Suppose that Δ is the Laplacian defined on a subspace $D(\Delta)$ of $L^2(\Omega)$. The Laplacian-argument functions ϕ_0 and ϕ_1 defined by* (9.8) *satisfy:*

$$\begin{cases} \dfrac{d}{d\zeta}\big[\phi_0\big((t-\zeta)^2 a^2 \Delta\big)\big] = -(t-\zeta) a^2 \Delta \phi_1\big((t-\zeta)^2 a^2 \Delta\big), \\ \dfrac{d}{d\zeta}\big[(t-\zeta)\phi_1\big((t-\zeta)^2 a^2 \Delta\big)\big] = -\phi_0\big((t-\zeta)^2 a^2 \Delta\big), \qquad t,\ \zeta \in \mathbb{R}. \end{cases} \tag{9.9}$$

Proof

$$
\begin{aligned}
\frac{d}{d\zeta}\big[\phi_0\big((t-\zeta)^2a^2\Delta\big)\big] &= \frac{d}{d\zeta}\sum_{k=0}^{\infty}\frac{(t-\zeta)^{2k}a^{2k}\Delta^k}{(2k)!}\\
&= -\sum_{k=1}^{\infty}\frac{(t-\zeta)^{2k-1}a^{2k}\Delta^k}{(2k-1)!}\\
&= -\sum_{k=0}^{\infty}\frac{(t-\zeta)^{2k+1}a^{2k+2}\Delta^{k+1}}{(2k+1)!}\\
&= -(t-\zeta)a^2\Delta\phi_1\big((t-\zeta)^2a^2\Delta\big).
\end{aligned}
$$

The second formula of (9.9) can be proved in a similar way. □

Theorem 9.3 *For a symmetric negative (semi-) definite operator Δ, the ϕ-functions defined by* (9.8) *satisfy:*

(i)

$$
\begin{cases}
\phi_0(a^2\Delta) = \displaystyle\sum_{k=0}^{\infty}\frac{a^{2k}\Delta^k}{(2k)!} = \sum_{k=0}^{\infty}\frac{(a\sqrt{-\Delta})^{2k}(-1)^k}{(2k)!} = \cos(a\sqrt{-\Delta}),\\
\phi_1(a^2\Delta) = \displaystyle\sum_{k=0}^{\infty}\frac{a^{2k}\Delta^k}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(a\sqrt{-\Delta})^{2k}(-1)^k}{(2k+1)!} = \frac{1}{a\sqrt{-\Delta}}\sin(a\sqrt{-\Delta}),\ a\neq 0.
\end{cases}
\tag{9.10}
$$

(ii)

$$\phi_0^2(a^2\Delta) - a^2\Delta\phi_1^2(a^2\Delta) = I, \tag{9.11}$$

$$\phi_0(a^2\Delta) - I = a^2\Delta\phi_2(a^2\Delta). \tag{9.12}$$

(iii)

$$
\begin{cases}
\phi_1^2(a^2\Delta) - \phi_0(a^2\Delta)\phi_2(a^2\Delta) = \phi_2(a^2\Delta),\\
\phi_0(a^2\Delta)\phi_1(a^2\Delta) - a^2\Delta\phi_1(a^2\Delta)\phi_2(a^2\Delta) = \phi_1(a^2\Delta),\\
\dfrac{1}{2}\big(\phi_1^2(a^2\Delta) - a^2\Delta\phi_2^2(a^2\Delta)\big) = \phi_2(a^2\Delta).
\end{cases}
\tag{9.13}
$$

(iv)

$$
\begin{aligned}
&\int_0^1\frac{(1-\xi)\phi_1\big(a^2(1-\xi)^2\Delta\big)\xi^j}{j!}\mathrm{d}\xi = \phi_{j+2}(a^2\Delta),\\
&\int_0^1\frac{\phi_0\big(a^2(1-\xi)^2\Delta\big)\xi^j}{j!}\mathrm{d}\xi = \phi_{j+1}(a^2\Delta).
\end{aligned}
\tag{9.14}
$$

Proof These results can be derived straightforwardly and we omit the details of the proof for the sake of brevity. □

We are now in a position to present an integral formula for the initial-value problem of the general arbitrarily high-dimensional Klein–Gordon equation (9.1).

Theorem 9.4 *If Δ is a Laplacian defined on a subspace $D(\Delta)$ of $L^2(\Omega)$ and $g(U)$ in (9.1) is continuous, then the exact solution of (9.1) and its derivative satisfy*

$$\begin{cases} U(X,t) = \phi_0\big((t-t_0)^2 a^2\Delta\big)U(X,t_0) + (t-t_0)\phi_1\big((t-t_0)^2 a^2\Delta\big)U_t(X,t_0) \\ \qquad + \displaystyle\int_{t_0}^{t} (t-\xi)\phi_1\big((t-\xi)^2 a^2\Delta\big)\tilde{f}(\xi)\mathrm{d}\xi, \\ U'(X,t) = (t-t_0)a^2\Delta\phi_1\big((t-t_0)^2 a^2\Delta\big)U(X,t_0) + \phi_0\big((t-t_0)^2 a^2\Delta\big)U_t(X,t_0) \\ \qquad + \displaystyle\int_{t_0}^{t} \phi_0\big((t-\xi)^2 a^2\Delta\big)\tilde{f}(\xi)\mathrm{d}\xi \end{cases} \tag{9.15}$$

for $t_0, t \in (-\infty, +\infty)$, where $\tilde{f}(\xi) = g\big(U(X,\xi)\big)$, and the Laplacian-argument functions ϕ_0 and ϕ_1 are defined by (9.8).

Proof We first let

$$\begin{aligned} Y(X,t) &= \big(U(X,t), U_t(X,t)\big)^{\intercal}, \\ Y_0(X) &= \big(U_0(X), U_1(X)\big)^{\intercal}, \\ F\big(Y(X,t)\big) &= \big(0, g(U(X,t))\big)^{\intercal}, \end{aligned}$$

and

$$W = \begin{pmatrix} 0 & I \\ a^2\Delta & 0 \end{pmatrix}.$$

Then the initial value problem (9.1) can be rewritten in a more compact form

$$\begin{cases} Y_t(X,t) = WY(X,t) + F\big(Y(X,t)\big), \\ Y(X,t_0) = Y_0(X), \quad t \ge t_0. \end{cases} \tag{9.16}$$

From the well-known result on inhomogeneous linear differential equations, the solution at $t \ge t_0$ of the system (9.16) has the form

$$Y(X,t) = \exp\big((t-t_0)W\big)Y_0(X) + \int_{t_0}^{t} \exp\big((t-\xi)W\big)F\big(Y(X,t-\xi)\big)\mathrm{d}\xi. \tag{9.17}$$

It follows from a careful calculation that

$$\begin{aligned} &W^2 = \begin{pmatrix} a^2\Delta & 0 \\ 0 & a^2\Delta \end{pmatrix}, \quad W^3 = \begin{pmatrix} 0 & a^2\Delta \\ a^4\Delta^2 & 0 \end{pmatrix}, \quad W^4 = \begin{pmatrix} a^4\Delta^2 & 0 \\ 0 & a^4\Delta^2 \end{pmatrix}, \\ &W^5 = \begin{pmatrix} 0 & a^4\Delta^2 \\ a^6\Delta^3 & 0 \end{pmatrix}, \quad W^6 = \begin{pmatrix} a^6\Delta^3 & 0 \\ 0 & a^6\Delta^3 \end{pmatrix}, \quad W^7 = \begin{pmatrix} 0 & a^6\Delta^3 \\ a^8\Delta^4 & 0 \end{pmatrix}, \\ &\ldots. \end{aligned}$$

An argument by induction leads to the result that, for each nonnegative integer k, we have

$$W^k = \begin{pmatrix} \frac{1+(-1)^k}{2}(a^2\Delta)^{\lfloor k/2\rfloor} & \frac{1-(-1)^k}{2}(a^2\Delta)^{\lfloor k/2\rfloor} \\ \frac{1-(-1)^k}{2}(a^2\Delta)^{\lfloor k/2\rfloor+1} & \frac{1+(-1)^k}{2}(a^2\Delta)^{\lfloor k/2\rfloor} \end{pmatrix},$$

where $\lfloor k/2\rfloor$ denotes the integer part of $k/2$, and then we have

$$\begin{aligned} \exp\big((t-t_0)W\big) &= \sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}W^k \\ &= \begin{pmatrix} I + \frac{(t-t_0)^2}{2!}a^2\Delta + \frac{(t-t_0)^4}{4!}(a^2\Delta)^2 + \ldots & (t-t_0)I + \frac{(t-t_0)^3}{3!}a^2\Delta + \ldots \\ (t-t_0)a^2\Delta + \frac{(t-t_0)^3}{3!}(a^2\Delta)^2 + \ldots & I + \frac{(t-t_0)^2}{2!}a^2\Delta + \frac{(t-t_0)^4}{4!}(a^2\Delta)^2 + \ldots \end{pmatrix} \\ &= \begin{pmatrix} \phi_0\big((t-t_0)^2a^2\Delta\big) & (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big) \\ (t-t_0)a^2\Delta\phi_1\big((t-t_0)^2a^2\Delta\big) & \phi_0\big((t-t_0)^2a^2\Delta\big) \end{pmatrix}. \end{aligned} \tag{9.18}$$

Inserting the result of (9.18) into Eq. (9.17) yields

$$\begin{aligned} \begin{pmatrix} U(X,t) \\ U_t(X,t) \end{pmatrix} &= \begin{pmatrix} \phi_0\big((t-t_0)^2a^2\Delta\big) & (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big) \\ (t-t_0)a^2\Delta\phi_1\big((t-t_0)^2a^2\Delta\big) & \phi_0\big((t-t_0)^2a^2\Delta\big) \end{pmatrix}\begin{pmatrix} U(X,t_0) \\ U_t(X,t_0) \end{pmatrix} \\ &+ \int_{t_0}^{t}\begin{pmatrix} \phi_0\big((t-\xi)^2a^2\Delta\big) & (t-\xi)\phi_1\big((t-\xi)^2a^2\Delta\big) \\ (t-\xi)a^2\Delta\phi_1\big((t-\xi)^2a^2\Delta\big) & \phi_0\big((t-\xi)^2a^2\Delta\big) \end{pmatrix}\begin{pmatrix} 0 \\ g\big(U(X,\xi)\big) \end{pmatrix}\mathrm{d}\xi \\ &= \begin{pmatrix} \phi_0\big((t-t_0)^2a^2\Delta\big)u(x,t_0) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)U_t(X,t_0) \\ (t-t_0)a^2\Delta\phi_1\big((t-t_0)^2a^2\Delta\big)U(X,t_0) + \phi_0\big((t-t_0)^2a^2\Delta\big)U_t(X,t_0) \end{pmatrix} \\ &+ \begin{pmatrix} \int_{t_0}^{t}(t-\xi)\phi_1\big((t-\xi)^2a^2\Delta\big)\tilde{f}(\xi)\mathrm{d}\xi \\ \int_{t_0}^{t}\phi_0\big((t-\xi)^2a^2\Delta\big)\tilde{f}(\xi)\mathrm{d}\xi \end{pmatrix}. \end{aligned}$$

This gives the form of (9.15) exactly and completes the proof. □

Let $U^n(X) = U(X,t_n)$ and $U_t^n(X) = U_t(X,t_n)$ represent the exact solution of (9.7) and its derivative with respect to t at $t = t_n$. It follows immediately from (9.15) with the change of variable $\xi = t_n + hz$ that

$$\begin{cases} U^{n+1}(X) = \phi_0\big(h^2a^2\Delta\big)U^n(X) + h\phi_1\big(h^2a^2\Delta\big)U_t^n(X) \\ \qquad\qquad + h^2\displaystyle\int_0^1(1-z)\phi_1\big((1-z)^2h^2a^2\Delta\big)\tilde{f}(z)\mathrm{d}z, \\ U_t^{n+1}(X) = ha^2\Delta\phi_1\big(h^2a^2\Delta\big)U^n(X) + \phi_0\big(h^2a^2\Delta\big)U_t^n(X) \\ \qquad\qquad + h\displaystyle\int_0^1\phi_0\big((1-z)^2h^2a^2\Delta\big)\tilde{f}(z)\mathrm{d}z, \end{cases} \tag{9.19}$$

where h is the temporal stepsize.

Remark 9.1 In comparison with the matrix-variation-of-constants formula (9.5) for (9.4) based on the method of lines for solving one-dimensional nonlinear wave equations, the formula (9.15) is a formal solution to the Klein–Gordon equation (9.1), whereas the matrix-variation-of-constants formula (9.5) is a formal solution to (9.4) but not a formal solution to (9.1). Thus, significant progress has been made on integral representations of solutions of the arbitrarily high-dimensional Klein–Gordon equation (9.1).

9.2.2 Homogeneous Case

We now turn to the special and important homogeneous case.

If $g(U) = 0$, then (9.1) reduces to the following homogeneous linear Klein–Gordon equation:

$$\begin{cases} U_{tt} - a^2 \Delta U = 0, \\ U(X, t_0) = U_0(X), \\ U_t(X, t_0) = U_1(X), \end{cases} \tag{9.20}$$

and then (9.15) becomes

$$\begin{cases} U(X,t) = \phi_0\big((t-t_0)^2 a^2 \Delta\big) U_0(X) + (t-t_0)\phi_1\big((t-t_0)^2 a^2 \Delta\big) U_1(X), \\ U'(X,t) = (t-t_0) a^2 \Delta \phi_1\big((t-t_0)^2 a^2 \Delta\big) U_0(X) + \phi_0\big((t-t_0)^2 a^2 \Delta\big) U_1(X), \end{cases} \tag{9.21}$$

which integrates (9.20) exactly. This means that (9.21) expresses a closed-form solution to the arbitrarily high-dimensional homogeneous linear Klein–Gordon equation (9.20). This fact shows that (9.21) possesses the additional advantage of energy preservation and quadratic invariant preservation for the homogeneous linear Klein–Gordon equation (9.20). Another key point is that, compared with the seminal D'Alembert, Poisson and Kirchhoff formulas, *the formula* (9.21) *doesn't depend on the evaluation of complicated integrals, whereas the evaluation of integrals is required by the D'Alembert, Poisson and Kirchhoff formulas*.

9.2.3 Towards Numerical Simulations

For the purpose of numerical simulations, we rewrite the Klein–Gordon equation (9.1) as

$$\begin{cases} U_{tt}(X,t) = g\big(U(X,t)\big) + a^2 \Delta U(X,t), \qquad X \in \Omega \subseteq \mathbb{R}^d,\ t > t_0 \\ U(X,t_0) = \varphi_1(X),\ U_t(X,t_0) = \varphi_2(X), \quad X \in \Omega \cup \partial\Omega, \end{cases} \tag{9.22}$$

where

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

It follows from Theorem 9.4 that the solution to (9.22) is given by

$$\begin{cases} U(X,t) = U(X,t_0) + (t-t_0)U_t(X,t_0) + \int_{t_0}^{t} (t-\zeta)\hat{g}\big(U(X,\zeta)\big)\mathrm{d}\zeta, \\ U_t(X,t) = U_t(X,t_0) + \int_{t_0}^{t} \hat{g}\big(U(X,\zeta)\big)\mathrm{d}\zeta, \\ X \in \Omega \cup \partial\Omega, \end{cases} \tag{9.23}$$

i.e.,

$$\begin{cases} U(X,t) = \varphi_1(X) + (t-t_0)\varphi_2(x) + \int_{t_0}^{t} (t-\zeta)\hat{g}\big(U(X,\zeta)\big)\mathrm{d}\zeta, \\ U_t(X,t) = \varphi_2(X) + \int_{t_0}^{t} \hat{g}\big(U(X,\zeta)\big)\mathrm{d}\zeta, \\ X \in \Omega \cup \partial\Omega, \end{cases} \tag{9.24}$$

or

$$\begin{cases} U^{n+1}(X) = U^n(X) + hU_t^n(x) + h^2 \int_0^1 (1-z)\hat{g}\big(U(X,t_n+zh)\big)\mathrm{d}z, \\ U_t^{n+1}(X) = U_t^n(X) + h \int_0^1 \hat{g}\big(U(X,t_n+zh)\big)\mathrm{d}z, \\ X \in \Omega \cup \partial\Omega, \end{cases} \tag{9.25}$$

where $U^n(X) = U(X,t_n)$ and

$$\hat{g}\big(U(X,\zeta)\big) = g\big(U(X,\zeta)\big) + a^2 \Delta U(X,\zeta).$$

Then, for each fixed $X \in \Omega \cup \partial\Omega$, approximating the integrals in (9.25) by using a quadrature formula yields a semi-analytical explicit RKN-type integrator.

Applying the modified midpoint rule (replacing $\hat{g}\big(U^{n+\frac{1}{2}}(X)\big)$ by $\hat{g}(\tilde{U}^n(X) + \frac{h}{2}\tilde{U}_t^n(X)))$ in the integrals in (9.25), we obtain

$$\begin{cases} \tilde{U}^{n+1}(X) = \tilde{U}^n(X) + h\tilde{U}_t^n(X) + \frac{h^2}{2}\hat{g}(\tilde{U}^n(X) + \frac{h}{2}\tilde{U}_t^n(X)), \\ \tilde{U}_t^{n+1}(X) = \tilde{U}_t^n(X) + h\hat{g}(\tilde{U}^n(X) + \frac{h}{2}\tilde{U}_t^n(X)), \end{cases} \tag{9.26}$$

where $\tilde{U}^n(X) \approx U^n(X) = U(X, t_n)$. This is the well-known Störmer–Verlet formula, and we call (9.26) the *SV-scheme* for (9.22). Hence, the SV-scheme is a symplectic integrator of order two.

In applications, (9.1) is defined on bounded domains on the boundary of which some physical conditions must be prescribed. These boundary conditions can be of different sorts. We will consider the most classical ones: Dirichlet boundary conditions, Neumann boundary conditions, and Robin boundary conditions. In what follows, we pay attention to the consistency of the formula (9.15) with the corresponding boundary conditions under suitable assumptions.

9.3 The Consistency of the Boundary Conditions for One-dimensional Klein–Gordon Equations

We now consider the initial problem in the one-dimensional case with $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\begin{cases} u_{tt} - a^2 \Delta u = f(u), \quad x_l < x < x_r, t > t_0, \\ u(x, t_0) = \varphi_1(x), u_t(x, t_0) = \varphi_2(x), \quad x_l \leq x \leq x_r, \end{cases} \tag{9.27}$$

where $f\big(u(x, t)\big) = -G'(u)$ for some smooth function $G(u)$. From Theorem 9.4, the solution of (9.27) satisfies

$$\begin{cases} u(x, t) = \phi_0\big((t - t_0)^2 a^2 \Delta\big)\varphi_1(x) + (t - t_0)\phi_1\big((t - t_0)^2 a^2 \Delta\big)\varphi_2(x) \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} (t - \zeta)\phi_1((t - \zeta)^2 a^2 \Delta)\tilde{f}\big(\zeta\big)\mathrm{d}\zeta, \\ u'(x, t) = (t - t_0)a^2 \Delta\phi_1\big((t - t_0)^2 a^2 \Delta\big)\varphi_1(x) + \phi_0\big((t - t_0)^2 a^2 \Delta\big)\varphi_2(x) \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} \phi_0\big((t - \zeta)^2 a^2 \Delta\big)\tilde{f}\big(\zeta\big)\mathrm{d}\zeta, \end{cases} \tag{9.28}$$

where $\Delta = \dfrac{\partial^2}{\partial x^2}$ and $\tilde{f}(\zeta) = f\big(u(x, \zeta)\big)$.

9.3.1 Dirichlet Boundary Conditions

Firstly, we consider the nonlinear wave equation (9.27) with the Dirichlet boundary conditions:

$$u(x_l, t) = \alpha(t), \quad u(x_r, t) = \beta(t), \quad t \geq t_0. \tag{9.29}$$

The next theorem shows the consistency of the formula (9.28) with the Dirichlet boundary conditions (9.29), i.e.,

$$\alpha(t) = \Big[\phi_0\big((t-t_0)^2 a^2 \Delta\big)\varphi_1(x) + (t-t_0)\phi_1\big((t-t_0)^2 a^2 \Delta\big)\varphi_2(x) \\ + \int_{t_0}^{t} (t-\zeta)\phi_1\big((t-\zeta)^2 a^2 \Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta\Big]\Big|_{x=x_l},$$

$$\beta(t) = \Big[\phi_0\big((t-t_0)^2 a^2 \Delta\big)\varphi_1(x) + (t-t_0)\phi_1\big((t-t_0)^2 a^2 \Delta\big)\varphi_2(x) \\ + \int_{t_0}^{t} (t-\zeta)\phi_1\big((t-\zeta)^2 a^2 \Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta\Big]\Big|_{x=x_r}.$$

Theorem 9.5 *Assume that $\alpha(t)$, $\beta(t)$, and $f\big(u(x,t)\big)$ are sufficiently differentiable with respect to t. Then the formula (9.28) is consistent with the Dirichlet boundary conditions (9.29).*

Proof Using the initial conditions, we obtain

$$\alpha(t_0) = \varphi_1(x_l), \quad \alpha'(t_0) = \varphi_2(x_l), \quad \beta(t_0) = \varphi_1(x_r), \quad \beta'(t_0) = \varphi_2(x_r).$$

It follows from (9.27) that

$$u_{tt} = a^2 \Delta u + f(u)$$
$$\Rightarrow \begin{cases} \alpha''(t_0) = a^2\Delta\varphi_1(x_l) + f\big(u(x_l,t_0)\big), \\ \beta''(t_0) = a^2\Delta\varphi_1(x_r) + f\big(u(x_r,t_0)\big), \end{cases}$$

$$u_t^{(3)} = a^2 \Delta u_t + f_t'(u)$$
$$\Rightarrow \begin{cases} \alpha^{(3)}(t_0) = a^2\Delta\varphi_2(x_l) + f_t'\big(u(x_l,t_0)\big), \\ \beta^{(3)}(t_0) = a^2\Delta\varphi_2(x_r) + f_t'\big(u(x_r,t_0)\big), \end{cases}$$

$$u_t^{(4)} = a^4\Delta^2 u + a^2\Delta f(u) + f_t^{(2)}(u)$$
$$\Rightarrow \begin{cases} \alpha^{(4)}(t_0) = a^4\Delta^2\varphi_1(x_l) + a^2\Delta f\big(u(x_l,t_0)\big) + f_t^{(2)}\big(u(x_l,t_0)\big), \\ \beta^{(4)}(t_0) = a^4\Delta^2\varphi_1(x_r) + a^2\Delta f\big(u(x_r,t_0)\big) + f_t^{(2)}\big(u(x_r,t_0)\big), \end{cases}$$

$$u_t^{(5)} = a^4\Delta^2 u_t + a^2\Delta f_t'(u) + f_t^{(3)}(u)$$
$$\Rightarrow \begin{cases} \alpha^{(5)}(t_0) = a^4\Delta^2\varphi_2(x_l) + a^2\Delta f_t'\big(u(x_l,t_0)\big) + f_t^{(3)}\big(u(x_l,t_0)\big), \\ \beta^{(5)}(t_0) = a^4\Delta^2\varphi_2(x_r) + a^2\Delta f_t'\big(u(x_r,t_0)\big) + f_t^{(3)}\big(u(x_r,t_0)\big), \end{cases}$$

$$u_t^{(6)} = a^6\Delta^3 u + a^4\Delta^2 f(u) + a^2\Delta f_t^{(2)}(u) + f_t^{(4)}(u)$$
$$\Rightarrow \begin{cases} \alpha^{(6)}(t_0) = a^6\Delta^3\varphi_1(x_l) + a^4\Delta^2 f\big(u(x_l,t_0)\big) \\ \qquad\qquad + a^2\Delta f_t^{(2)}\big(u(x_l,t_0)\big) + f_t^{(4)}\big(u(x_l,t_0)\big), \\ \beta^{(6)}(t_0) = a^6\Delta^3\varphi_1(x_r) + a^4\Delta^2 f\big(u(x_r,t_0)\big) \\ \qquad\qquad + a^2\Delta f_t^{(2)}\big(u(x_r,t_0)\big) + f_t^{(4)}\big(u(x_r,t_0)\big), \end{cases}$$

$$u_t^{(7)} = a^6 \Delta^3 u_t + a^4 \Delta^2 f_t'(u) + a^2 \Delta f_t^{(3)}(u) + f_t^{(5)}(u)$$
$$\Rightarrow \begin{cases} \alpha^{(7)}(t_0) = a^6 \Delta^3 \varphi_2(x_l) + a^4 \Delta^2 f_t'\big(u(x_l, t_0)\big) \\ \qquad\qquad +a^2 \Delta f_t^{(3)}\big(u(x_l, t_0)\big) + f_t^{(5)}\big(u(x_l, t_0)\big), \\ \beta^{(7)}(t_0) = a^6 \Delta^3 \varphi_2(x_r) + a^4 \Delta^2 f_t'\big(u(x_r, t_0)\big) \\ \qquad\qquad +a^2 \Delta f_t^{(3)}\big(u(x_r, t_0)\big) + f_t^{(5)}\big(u(x_r, t_0)\big). \end{cases}$$
$$\cdots\cdot$$

An argument by induction leads to the results

$$\begin{aligned} \alpha^{(2k)}(t_0) &= a^{2k} \Delta^k \varphi_1(x_l) + \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-2)}\big(u(x_l, t_0)\big), \\ \alpha^{(2k+1)}(t_0) &= a^{2k} \Delta^k \varphi_2(x_l) + \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-1)}\big(u(x_l, t_0)\big), \end{aligned} \tag{9.30}$$

and

$$\begin{aligned} \beta^{(2k)}(t_0) &= a^{2k} \Delta^k \varphi_1(x_r) + \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-2)}\big(u(x_r, t_0)\big) \\ \beta^{(2k+1)}(t_0) &= a^{2k} \Delta^k \varphi_2(x_r) + \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-1)}\big(u(x_r, t_0)\big), \end{aligned} \tag{9.31}$$

for $k = 1, 2, \ldots$.

The Taylor expansion of $\alpha(t)$ and $\beta(t)$ at the point t_0 gives

$$\begin{aligned} \alpha(t) &= \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \alpha^{(k)}(t_0) = \sum_{k=0}^{\infty} \frac{(t-t_0)^{2k}}{(2k)!} \alpha^{(2k)}(t_0) + \sum_{k=0}^{\infty} \frac{(t-t_0)^{2k+1}}{(2k+1)!} \alpha^{(2k+1)}(t_0), \\ \beta(t) &= \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \beta^{(k)}(t_0) = \sum_{k=0}^{\infty} \frac{(t-t_0)^{2k}}{(2k)!} \beta^{(2k)}(t_0) + \sum_{k=0}^{\infty} \frac{(t-t_0)^{2k+1}}{(2k+1)!} \beta^{(2k+1)}(t_0). \end{aligned} \tag{9.32}$$

Inserting the results of (9.30) and (9.31) into (9.32) yields

$$\begin{aligned} \alpha(t) = \Big\{ &\phi_0\big((t-t_0)^2 a^2 \Delta\big)\varphi_1(x) + (t-t_0)\phi_1\big((t-t_0)^2 a^2 \Delta\big)\varphi_2(x) \\ &+ \sum_{k=1}^{\infty} \Big[\frac{(t-t_0)^{2k}}{(2k)!} \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-2)}\big(u(x, t_0)\big) \\ &+ \frac{(t-t_0)^{2k+1}}{(2k+1)!} \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-1)}\big(u(x, t_0)\big) \Big] \Big\} \Big|_{x=x_l}, \end{aligned} \tag{9.33}$$

and

$$\beta(t) = \Big\{\phi_0\big((t-t_0)^2a^2\Delta\big)\varphi_1(x) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\varphi_2(x) \\ + \sum_{k=1}^{\infty}\Big[\frac{(t-t_0)^{2k}}{(2k)!}\sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j} f_t^{(2j-2)}\big(u(x,t_0)\big) \\ + \frac{(t-t_0)^{2k+1}}{(2k+1)!}\sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j} f_t^{(2j-1)}\big(u(x,t_0)\big)\Big]\Big\}\Big|_{x=x_r}. \tag{9.34}$$

Let

$$F(x,t) \triangleq \int_{t_0}^{t} (t-\zeta)\phi_1\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(\zeta\big)\mathrm{d}\zeta.$$

It is easy to see $F(x,t_0)=0$, and a careful calculation gives

$$\begin{aligned}
F_t'(x,t) &= \int_{t_0}^{t} \phi_0\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(\zeta\big)\mathrm{d}\zeta \\
&\Rightarrow F_t'(x,t_0) = 0, \\
F_t^{(2)}(x,t) &= f\big(u(x,t)\big) + \int_{t_0}^{t} (t-\zeta)a^2\Delta\phi_1\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(\zeta\big)\mathrm{d}\zeta \\
&\Rightarrow F_t^{(2)}(x,t_0) = f\big(u(x,t_0)\big), \\
F_t^{(3)}(x,t) &= f_t'\big(u(x,t)\big) + \int_{t_0}^{t} a^2\Delta\phi_0\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(\zeta\big)\mathrm{d}\zeta \\
&\Rightarrow F_t^{(3)}(x,t_0) = f_t'\big(u(x,t_0)\big), \\
F_t^{(4)}(x,t) &= f_t^{(2)}\big(u(x,t)\big) + a^2\Delta f\big(u(x,t)\big) + \int_{t_0}^{t} (t-\zeta)a^4\Delta^2\phi_1\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(\zeta\big)\big)\mathrm{d}\zeta \\
&\Rightarrow F_t^{(4)}(x,t_0) = f_t^{(2)}\big(u(x,t_0)\big) + a^2\Delta f\big(u(x,t_0)\big), \\
F_t^{(5)}(x,t) &= f_t^{(3)}\big(u(x,t)\big) + a^2\Delta f_t'\big(u(x,t)\big) + \int_{t_0}^{t} a^4\Delta^2\phi_0\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(\zeta\big)\mathrm{d}\zeta \\
&\Rightarrow F_t^{(5)}(x,t_0) = f_t^{(3)}\big(u(x,t_0)\big) + a^2\Delta f_t'\big(u(x,t_0)\big), \\
F_t^{(6)}(x,t) &= f_t^{(4)}\big(u(x,t)\big) + a^2\Delta f_t^{(2)}\big(u(x,t)\big) + a^4\Delta^2 f\big(u(x,t)\big) \\
&\quad + \int_{t_0}^{t} (t-\zeta)a^6\Delta^3\phi_1\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(\zeta\big)\mathrm{d}\zeta \\
&\Rightarrow F_t^{(6)}(x,t_0) = f_t^{(4)}\big(u(x,t_0)\big) + a^2\Delta f_t^{(2)}\big(u(x,t_0)\big) + a^4\Delta^2 f\big(u(x,t_0)\big)\big), \\
F_t^{(7)}(x,t) &= f_t^{(5)}\big(u(x,t)\big) + a^2\Delta f_t^{(3)}\big(u(x,t)\big) + a^4\Delta^2 f_t'\big(u(x,t)\big) \\
&\quad + \int_{t_0}^{t} a^6\Delta^3\phi_0\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(\zeta\big)\mathrm{d}\zeta \\
&\Rightarrow F_t^{(7)}(x,t_0) = f_t^{(5)}\big(u(x,t_0)\big) + a^2\Delta f_t^{(3)}\big(u(x,t_0)\big) + a^4\Delta^2 f_t'\big(u(x,t_0)\big), \\
&\ldots.
\end{aligned}$$

An argument by induction then gives

$$F_t^{(2k)}(x, t_0) = \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-2)}\big(u(x, t_0)\big)$$
$$F_t^{(2k+1)}(x, t_0) = \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-1)}\big(u(x, t_0)\big), \quad k = 1, 2, \dots .$$

The Taylor expansion of $F(x, t)$ at $t = t_0$ is

$$\begin{aligned} F(x,t) &= \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} F_t^{(k)}(x, t_0) = \sum_{k=2}^{\infty} \frac{(t-t_0)^k}{k!} F_t^{(k)}(x, t_0) \\ &= \sum_{k=1}^{\infty} \frac{(t-t_0)^{2k}}{(2k)!} F_t^{(2k)}(x, t_0) + \sum_{k=1}^{\infty} \frac{(t-t_0)^{2k+1}}{(2k+1)!} F_t^{(2k+1)}(x, t_0) \\ &= \sum_{k=1}^{\infty} \Big[\frac{(t-t_0)^{2k}}{(2k)!} \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-2)}\big(u(x,t_0)\big) \\ &\quad + \frac{(t-t_0)^{2k+1}}{(2k+1)!} \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-1)}\big(u(x,t_0)\big) \Big]. \end{aligned} \tag{9.35}$$

Inserting the result of (9.35) into (9.33) and (9.34) yields

$$\begin{aligned} \alpha(t) &= \Big[\phi_0\big((t-t_0)^2 a^2 \Delta\big) \varphi_1(x) + (t-t_0)\phi_1\big((t-t_0)^2 a^2 \Delta\big)\varphi_2(x) \\ &+ \int_{t_0}^{t} (t-\zeta)\phi_1\big((t-\zeta)^2 a^2 \Delta\big) \tilde{f}(\zeta) \mathrm{d}\zeta \Big]\Big|_{x=x_l} \\ \beta(t) &= \Big[\phi_0\big((t-t_0)^2 a^2 \Delta\big) \varphi_1(x) + (t-t_0)\phi_1\big((t-t_0)^2 a^2 \Delta\big)\varphi_2(x) \\ &+ \int_{t_0}^{t} (t-\zeta)\phi_1\big((t-\zeta)^2 a^2 \Delta\big) \tilde{f}(\zeta) \mathrm{d}\zeta \Big]\Big|_{x=x_r}. \end{aligned}$$

The proof is complete. □

9.3.2 Neumann Boundary Conditions

We next consider the nonlinear wave equation (9.27) with the Neumann boundary conditions

$$\frac{\partial u}{\partial x}\Big|_{x_l} = \gamma(t), \quad \frac{\partial u}{\partial x}\Big|_{x_r} = \delta(t). \tag{9.36}$$

Theorem 9.6 *Assume that $\gamma(t)$, $\delta(t)$, and $f\big(u(x,t)\big)$ are sufficiently differentiable with respect to t. Then the formula (9.28) is consistent with the Neumann boundary conditions (9.36).*

Proof From the initial conditions,we have

$$\gamma(t_0) = \varphi_1'(x_l), \quad \gamma'(t_0) = \varphi_2'(x_l), \quad \delta(t_0) = \varphi_1'(x_r), \quad \delta'(t_0) = \varphi_2'(x_r).$$

Calculating the derivative of u with respect to x in (9.27) gives

$$\begin{cases} \Big(\dfrac{\partial u}{\partial x}\Big)_{tt} = a^2\Delta\Big(\dfrac{\partial u}{\partial x}\Big) + \dfrac{\partial}{\partial x}\big(f(u)\big), & x_l < x < x_r, t > t_0, \\ \dfrac{\partial u}{\partial x}(x, t_0) = \varphi_1'(x), \dfrac{\partial u_t}{\partial x}(x, t_0) = \varphi_2'(x), & x_l \le x \le x_r. \end{cases} \tag{9.37}$$

Let $v = \dfrac{\partial u}{\partial x}$. We then have the following initial-boundary problem

$$\begin{cases} v_{tt} = a^2\Delta v + \tilde{f}(u), \quad x_l < x < x_r, \ t \ge t_0, \\ v(x, t_0) = \varphi_1'(x), \ v_t(x, t_0) = \varphi_2'(x), \quad x_l \le x \le x_r, \\ v(x_l, t) = \gamma(t), \ v(x_r, t) = \delta(t), \quad t \ge t_0, \end{cases} \tag{9.38}$$

where

$$\tilde{f}(u) = f_x'\big(u(x,t)\big) = \frac{\partial}{\partial x} f\big(u(x,t)\big).$$

For the transformed initial-boundary value problem (9.38), after an analysis similarly to that in Sect. 9.3.1, we conclude that

$$\begin{aligned} \gamma(t) = &\Big[\phi_0\big((t-t_0)^2a^2\Delta\big)\varphi_1'(x) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\varphi_2'(x) \\ &+ \int_{t_0}^t (t-\zeta)\phi_1\big((t-\zeta)^2a^2\Delta\big)\hat{f}(\zeta)\mathrm{d}\zeta\Big]\Big|_{x=x_l}, \\ \delta(t) = &\Big[\phi_0\big((t-t_0)^2a^2\Delta\big)\varphi_1'(x) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\varphi_2'(x) \\ &+ \int_{t_0}^t (t-\zeta)\phi_1\big((t-\zeta)^2a^2\Delta\big)\hat{f}(\zeta)\mathrm{d}\zeta\Big]\Big|_{x=x_r}, \end{aligned}$$

where $\hat{f}(\zeta) = f_x'\big(u(x,\zeta)\big)$.

The proof is complete. □

Another direct proof can be found in Appendix 1 of this chapter.

9.4 Towards Arbitrarily High-Dimensional Klein–Gordon Equations

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. We next consider the initial valued problem of the arbitrarily high-dimensional nonlinear Klein–Gordon equations

$$\begin{cases} U_{tt} - a^2 \Delta U = f\big(U\big), \quad X \in \Omega,\ t > t_0, \\ U(X, t_0) = \varphi_1(X),\ U_t(X, t_0) = \varphi_2(X), \quad X \in \Omega \cup \partial\Omega. \end{cases} \tag{9.39}$$

The integral formula for (9.39) is given by

$$\begin{cases} U(X,t) = \phi_0\big((t-t_0)^2 a^2 \Delta\big)\varphi_1(X) + (t-t_0)\phi_1\big((t-t_0)^2 a^2 \Delta\big)\varphi_2(X) \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} (t-\zeta)\phi_1((t-\zeta)^2 a^2 \Delta)\tilde{f}(\zeta)\mathrm{d}\zeta, \\ U'(X,t) = (t-t_0)a^2\Delta\phi_1\big((t-t_0)^2 a^2 \Delta\big)\varphi_1(X) + \phi_0\big((t-t_0)^2 a^2 \Delta\big)\varphi_2(X) \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} \phi_0\big((t-\zeta)^2 a^2 \Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta, \end{cases} \tag{9.40}$$

where $\tilde{f}(\zeta) = f\big(U(X,\zeta)\big)$.

9.4.1 *Dirichlet Boundary Conditions*

Firstly, we consider the arbitrarily high-dimensional nonlinear Klein–Gordon equation (9.39) with the Dirichlet boundary condition:

$$U(X,t) = \alpha(X,t), \quad X \in \partial\Omega,\ t \geq t_0. \tag{9.41}$$

Theorem 9.7 *Assume that $\alpha(X,t)$ and $f\big(U(X,t)\big)$ are sufficiently differentiable with respect to t. Then, formula (9.40) is consistent with the Dirichlet boundary condition (9.41).*

Proof From the initial conditions, we obtain

$$\alpha(X,t_0) = \varphi_1(X), \quad \alpha'_t(X,t_0) = \varphi_2(X), \quad X \in \partial\Omega.$$

It follows from (9.39) that

$$
\begin{aligned}
U_{tt} &= a^2 \Delta U + f(U) \\
&\Rightarrow \alpha_t^{''}(X, t_0) = \left[a^2 \Delta \varphi_1(X) + f\big(U(X, t_0)\big)\right]\Big|_{\partial\Omega}, \\
U_t^{(3)} &= a^2 \Delta U_t + f_t'(U) \\
&\Rightarrow \alpha_t^{(3)}(X, t_0) = \left[a^2 \Delta \varphi_2(X) + f_t'\big(U(X, t_0)\big)\right]\Big|_{\partial\Omega}, \\
U_t^{(4)} &= a^4 \Delta^2 U + a^2 \Delta f(U) + f_t^{(2)}(U) \\
&\Rightarrow \alpha_t^{(4)}(X, t_0) = \left[a^4 \Delta^2 \varphi_1(X) + a^2 \Delta f\big(U(X, t_0)\big) + f_t^{(2)}\big(U(X, t_0)\big)\right]\Big|_{\partial\Omega}, \\
U_t^{(5)} &= a^4 \Delta^2 U_t + a^2 \Delta f_t'(U) + f_t^{(3)}(U) \\
&\Rightarrow \alpha_t^{(5)}(X, t_0) = \left[a^4 \Delta^2 \varphi_2(X) + a^2 \Delta f_t'\big(U(X, t_0)\big) + f_t^{(3)}\big(U(X, t_0)\big)\right]\Big|_{\partial\Omega}, \\
U_t^{(6)} &= a^6 \Delta^3 U + a^4 \Delta^2 f(U) + a^2 \Delta f_t^{(2)}(U) + f_t^{(4)}(U) \\
&\Rightarrow \alpha_t^{(6)}(X, t_0) = \left[a^6 \Delta^3 \varphi_1(X) + a^4 \Delta^2 f\big(U(X, t_0)\big) + a^2 \Delta f_t^{(2)}\big(U(X, t_0)\big)\right. \\
&\qquad\qquad \left. + f_t^{(4)}\big(U(X, t_0)\big)\right]\Big|_{\partial\Omega}, \\
U_t^{(7)} &= a^6 \Delta^3 U_t + a^4 \Delta^2 f_t'(U) + a^2 \Delta f_t^{(3)}(U) + f_t^{(5)}(U) \\
&\Rightarrow \alpha_t^{(7)}(X, t_0) = \left[a^6 \Delta^3 \varphi_2(X) + a^4 \Delta^2 f_t'\big(U(X, t_0)\big) + a^2 \Delta f_t^{(3)}\big(U(X, t_0)\big)\right. \\
&\qquad\qquad \left. + f_t^{(5)}\big(U(X, t_0)\big)\right]\Big|_{\partial\Omega}, \\
&\cdots.
\end{aligned}
$$

An argument by induction leads to the results

$$
\begin{aligned}
\alpha^{(2k)}(X, t_0) &= a^{2k} \Delta^k \varphi_1(X) + \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-2)}\big(U(X, t_0)\big) \\
\alpha^{(2k+1)}(X, t_0) &= a^{2k} \Delta^k \varphi_2(X) + \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-1)}\big(U(X, t_0)\big),
\end{aligned}
\tag{9.42}
$$

for $k = 1, 2, \ldots,$ and $\forall X \in \partial\Omega$.

Inserting the results of (9.42) into the Taylor expansion of $\alpha(X, t)$ with respect to t at the point t_0 gives

$$
\begin{aligned}
\alpha(X, t) = \Big\{ &\phi_0\big((t - t_0)^2 a^2 \Delta\big)\varphi_1(X) + (t - t_0)\phi_1\big((t - t_0)^2 a^2 \Delta\big)\varphi_2(X) \\
&+ \sum_{k=1}^{\infty} \Big[\frac{(t - t_0)^{2k}}{(2k)!} \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-2)}\big(U(X, t_0)\big) \\
&+ \frac{(t - t_0)^{2k+1}}{(2k+1)!} \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-1)}\big(U(X, t_0)\big) \Big] \Big\} \Big|_{\partial\Omega}.
\end{aligned}
\tag{9.43}
$$

Let

$$F(X,t) \triangleq \int_{t_0}^{t} (t-\zeta)\phi_1\big((t-\zeta)^2 a^2 \Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta.$$

It is easy to see

$$F(X,t_0)=0,$$

and

$$\begin{aligned}
&F_t'(X,t)=\int_{t_0}^{t}\phi_0\big((t-\zeta)^2a^2\Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta\\
&\qquad\Rightarrow F_t'(X,t_0)=0,\\
&F_t^{(2)}(X,t)=f\big(U(X,t)\big)+\int_{t_0}^{t}(t-\zeta)a^2\Delta\phi_1\big((t-\zeta)^2a^2\Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta\\
&\qquad\Rightarrow F_t^{(2)}(X,t_0)=f\big(U(X,t_0)\big),\\
&F_t^{(3)}(X,t)=f_t'\big(U(X,t)\big)+\int_{t_0}^{t}a^2\Delta\phi_0\big((t-\zeta)^2a^2\Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta\\
&\qquad\Rightarrow F_t^{(3)}(X,t_0)=f_t'\big(U(X,t_0)\big),\\
&F_t^{(4)}(X,t)=f_t^{(2)}\big(U(X,t)\big)+a^2\Delta f\big(U(X,t)\big)+\int_{t_0}^{t}(t-\zeta)a^4\Delta^2\phi_1\big((t-\zeta)^2a^2\Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta\\
&\qquad\Rightarrow F_t^{(4)}(X,t_0)=f_t^{(2)}\big(U(X,t_0)\big)+a^2\Delta f\big(U(X,t_0)\big),\\
&F_t^{(5)}(X,t)=f_t^{(3)}\big(U(X,t)\big)+a^2\Delta f_t'\big(U(X,t)\big)+\int_{t_0}^{t}a^4\Delta^2\phi_0\big((t-\zeta)^2a^2\Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta\\
&\qquad\Rightarrow F_t^{(5)}(X,t_0)=f_t^{(3)}\big(U(X,t_0)\big)+a^2\Delta f_t'\big(U(X,t_0)\big),\\
&F_t^{(6)}(X,t)=f_t^{(4)}\big(U(X,t)\big)+a^2\Delta f_t^{(2)}\big(U(X,t)\big)+a^4\Delta^2 f\big(U(X,t)\big)\\
&\qquad+\int_{t_0}^{t}(t-\zeta)a^6\Delta^3\phi_1\big((t-\zeta)^2a^2\Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta\\
&\qquad\Rightarrow F_t^{(6)}(X,t_0)=f_t^{(4)}\big(U(X,t_0)\big)+a^2\Delta f_t^{(2)}\big(U(X,t_0)\big)+a^4\Delta^2 f\big(U(X,t_0)\big),\\
&F_t^{(7)}(X,t)=f_t^{(5)}\big(U(X,t)\big)+a^2\Delta f_t^{(3)}\big(U(X,t)\big)+a^4\Delta^2 f_t'\big(U(X,t)\big)\\
&\qquad+\int_{t_0}^{t}a^6\Delta^3\phi_0\big((t-\zeta)^2a^2\Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta\\
&\qquad\Rightarrow F_t^{(7)}(X,t_0)=f_t^{(5)}\big(U(X,t_0)\big)+a^2\Delta f_t^{(3)}\big(U(X,t_0)\big)+a^4\Delta^2 f_t'\big(U(X,t_0)\big),\\
&\ldots.
\end{aligned}$$

Likewise, an argument by induction yields the following results

$$F_t^{(2k)}(X, t_0) = \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-2)}\big(U(X, t_0)\big)$$

$$F_t^{(2k+1)}(X, t_0) = \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-1)}\big(U(X, t_0)\big), \quad k = 1, 2, \ldots .$$

The Taylor expansion of $F(X, t)$ at $t = t_0$ is

$$\begin{aligned} F(X, t) &= \sum_{k=1}^{\infty} \frac{(t-t_0)^{2k}}{(2k)!} F_t^{(2k)}(X, t_0) + \sum_{k=1}^{\infty} \frac{(t-t_0)^{2k+1}}{(2k+1)!} F_t^{(2k+1)}(X, t_0) \\ &= \sum_{k=1}^{\infty} \Big[\frac{(t-t_0)^{2k}}{(2k)!} \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-2)}\big(U(X, t_0)\big) \\ &\quad + \frac{(t-t_0)^{2k+1}}{(2k+1)!} \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} f_t^{(2j-1)}\big(U(X, t_0)\big) \Big]. \end{aligned} \tag{9.44}$$

Inserting the result of (9.44) into (9.43) gives

$$\alpha(X, t) = \phi_0\big((t-t_0)^2 a^2 \Delta\big)\varphi_1(X) + (t-t_0)\phi_1\big((t-t_0)^2 a^2 \Delta\big)\varphi_2(X) + \int_{t_0}^{t} (t-\zeta)\phi_1\big((t-\zeta)^2 a^2 \Delta\big)\tilde{f}(\zeta)\mathrm{d}\zeta, \ X \in \partial\Omega.$$

The proof is complete. □

9.4.2 *Neumann Boundary Conditions*

We next consider the arbitrarily high-dimensional nonlinear wave equation (9.39) with the following Neumann boundary condition:

$$\nabla U \cdot \mathbf{n} = \gamma(X, t), \quad X \in \partial\Omega, \tag{9.45}$$

where $\mathbf{n}$ is the unit outward normal vectors on the boundary $\partial\Omega$.

Theorem 9.8 *Assume that $\gamma(X, t)$ and $f\big(U(X, t)\big)$ are sufficiently differentiable with respect to t. Then the formula* (9.40) *is consistent with the Neumann boundary conditions* (9.45).

Proof Using the initial condition, we have

$$\gamma(X, t_0) = \nabla\varphi_1(X) \cdot \mathbf{n} \triangleq \tilde{\varphi}_1(X), \quad \gamma_t'(X, t_0) = \nabla\varphi_2(X) \cdot \mathbf{n} \triangleq \tilde{\varphi}_2(X), \quad \forall X \in \partial\Omega.$$

Calculating the directional derivative of U with respect to X in (9.39) yields

$$\begin{cases} \big(\nabla U\cdot\mathbf{n}\big)_{tt} = a^2\Delta\big(\nabla U\cdot\mathbf{n}\big) + \tilde{f}\big(U(X,t)\big), \quad X\in\Omega,\ t>t_0, \\ \big(\nabla U\cdot\mathbf{n}\big)(X,t_0) = \tilde{\varphi}_1(X),\ \big(\nabla U_t\cdot\mathbf{n}\big)(X,t_0) = \tilde{\varphi}_2(X), \quad X\in\Omega\cup\partial\Omega, \end{cases} \tag{9.46}$$

where $\tilde{f}\big(U(X,t)\big) = \nabla f\big(U(X,t)\big)\cdot\mathbf{n}$.

It follows from (9.46) that

$$\begin{aligned}
&\big(\nabla U\cdot\mathbf{n}\big)_{tt} = a^2\Delta\big(\nabla U\cdot\mathbf{n}\big) + \tilde{f}\big(U\big)\\
&\qquad\Rightarrow \gamma_t''(X,t_0) = \big[a^2\Delta\tilde{\varphi}_1(X) + \tilde{f}\big(U(X,t_0)\big)\big]\big|_{\partial\Omega},\\
&\big(\nabla U\cdot\mathbf{n}\big)_t^{(3)} = a^2\Delta\big(\nabla U\cdot\mathbf{n}\big)_t' + \tilde{f}_t'\big(U\big)\\
&\qquad\Rightarrow \gamma_t^{(3)}(X,t_0) = \big[a^2\Delta\tilde{\varphi}_2(X) + \tilde{f}_t'\big(U(X,t_0)\big)\big]\big|_{\partial\Omega},\\
&\big(\nabla U\cdot\mathbf{n}\big)_t^{(4)} = a^4\Delta^2\big(\nabla U\cdot\mathbf{n}\big) + a^2\Delta\tilde{f}\big(U\big) + \tilde{f}_t^{(2)}\big(U\big)\\
&\qquad\Rightarrow \gamma_t^{(4)}(X,t_0) = \big[a^4\Delta^2\tilde{\varphi}_1(X) + a^2\Delta\tilde{f}\big(U(X,t_0)\big) + \tilde{f}_t^{(2)}\big(U(X,t_0)\big)\big]\big|_{\partial\Omega},\\
&\big(\nabla U\cdot\mathbf{n}\big)_t^{(5)} = a^4\Delta^2\big(\nabla U\cdot\mathbf{n}\big)_t' + a^2\Delta\tilde{f}_t'\big(U\big) + \tilde{f}_t^{(3)}\big(U\big)\\
&\qquad\Rightarrow \gamma_t^{(5)}(X,t_0) = \big[a^4\Delta^2\tilde{\varphi}_2(X) + a^2\Delta\tilde{f}_t'\big(U(X,t_0)\big) + \tilde{f}_t^{(3)}\big(U(X,t_0)\big)\big]\big|_{\partial\Omega},\\
&\big(\nabla U\cdot\mathbf{n}\big)_t^{(6)} = a^6\Delta^3\big(\nabla U\cdot\mathbf{n}\big) + a^4\Delta^2\tilde{f}\big(U\big) + a^2\Delta\tilde{f}_t^{(2)}\big(U\big) + \tilde{f}_t^{(4)}\big(U\big)\\
&\qquad\Rightarrow \gamma_t^{(6)}(X,t_0) = \big[a^6\Delta^3\tilde{\varphi}_1(X) + a^4\Delta^2\tilde{f}\big(U(X,t_0)\big) + a^2\Delta\tilde{f}_t^{(2)}\big(U(X,t_0)\big)\\
&\qquad\qquad + \tilde{f}_t^{(4)}\big(U(X,t_0)\big)\big]\big|_{\partial\Omega},\\
&\big(\nabla U\cdot\mathbf{n}\big)_t^{(7)} = a^6\Delta^3\big(\nabla U\cdot\mathbf{n}\big)_t' + a^4\Delta^2\tilde{f}_t'\big(U\big) + a^2\Delta\tilde{f}_t^{(3)}\big(U\big) + \tilde{f}_t^{(5)}\big(U\big)\\
&\qquad\Rightarrow \gamma_t^{(7)}(X,t_0) = \big[a^6\Delta^3\tilde{\varphi}_2(X) + a^4\Delta^2\tilde{f}_t'\big(U(X,t_0)\big) + a^2\Delta\tilde{f}_t^{(3)}\big(U(X,t_0)\big)\\
&\qquad\qquad + \tilde{f}_t^{(5)}\big(U(X,t_0)\big)\big]\big|_{\partial\Omega},\\
&\cdots.
\end{aligned}$$

This leads to the results

$$\begin{aligned}
\gamma_t^{(2k)}(X,t_0) &= a^{2k}\Delta^k\tilde{\varphi}_1(X) + \sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-2)}\big(U(X,t_0)\big),\\
\gamma_t^{(2k+1)}(X,t_0) &= a^{2k}\Delta^k\tilde{\varphi}_2(X) + \sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-1)}\big(U(X,t_0)\big),
\end{aligned} \tag{9.47}$$

for $k = 1, 2, \ldots,$ and $\forall X\in\partial\Omega$.

Inserting the results of (9.47) into the Taylor expansion of $\gamma(X,t)$ with respect to t at the point $t = t_0$ gives

$$\begin{aligned}\gamma(X,t) &= \sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\gamma_t^{(k)}(X,t_0) = \sum_{k=0}^{\infty}\frac{(t-t_0)^{(2k)}}{(2k)!}\gamma_t^{(2k)}(X,t_0)\\ &\quad+\sum_{k=0}^{\infty}\frac{(t-t_0)^{(2k+1)}}{(2k+1)!}\gamma_t^{(2k+1)}(X,t_0)\\ &=\Big\{\phi_0\big((t-t_0)^2a^2\Delta\big)\tilde{\varphi}_1(x)+(t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\tilde{\varphi}_2(x)\\ &\quad+\sum_{k=1}^{\infty}\Big[\frac{(t-t_0)^{2k}}{(2k)!}\sum_{j=1}^{k}a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-2)}\big(U(x,t_0)\big)\\ &\quad+\frac{(t-t_0)^{2k+1}}{(2k+1)!}\sum_{j=1}^{k}a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-1)}\big(U(x,t_0)\big)\Big]\Big\}\Big|_{\partial\Omega}.\end{aligned}\tag{9.48}$$

Let

$$\tilde{F}(X,t)\triangleq\int_{t_0}^{t}(t-\zeta)\phi_1\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(U(X,\zeta)\big)\mathrm{d}\zeta.$$

Similarly to the case of Dirichlet boundary conditions, we can obtain

$$\tilde{F}(X,t_0)=0,\quad \tilde{F}_t'(X,t_0)=0,$$

and

$$\begin{aligned}\tilde{F}_t^{(2k)}(X,t_0)&=\sum_{j=1}^{k}a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-2)}\big(U(X,t_0)\big),\\ \tilde{F}_t^{(2k+1)}(X,t_0)&=\sum_{j=1}^{k}a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-1)}\big(U(X,t_0)\big),\quad k=1,2,\ldots.\end{aligned}$$

The Taylor expansion of $\tilde{F}(X,t)$ at the point t_0 with respect to t is

$$\begin{aligned}\tilde{F}(X,t)&=\sum_{k=1}^{\infty}\frac{(t-t_0)^{2k}}{(2k)!}\tilde{F}_t^{(2k)}(X,t_0)+\sum_{k=1}^{\infty}\frac{(t-t_0)^{2k+1}}{(2k+1)!}\tilde{F}_t^{(2k+1)}(X,t_0)\\ &=\sum_{k=1}^{\infty}\Big[\frac{(t-t_0)^{2k}}{(2k)!}\sum_{j=1}^{k}a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-2)}\big(U(x,t_0)\big)\\ &\quad+\frac{(t-t_0)^{2k+1}}{(2k+1)!}\sum_{j=1}^{k}a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-1)}\big(U(x,t_0)\big)\Big]\Big\}.\end{aligned}\tag{9.49}$$

Comparing the result of (9.48) with (9.49), we obtain

$$
\begin{aligned}
\gamma(X,t) &= \phi_0\big((t-t_0)^2a^2\Delta\big)\tilde{\varphi}_1(X) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\tilde{\varphi}_2(X) \\
&\quad + \int_{t_0}^{t}(t-\zeta)\phi_1\big((t-\zeta)^2a^2\Delta\big)\tilde{f}\big(U(X,\zeta)\big)\mathrm{d}\zeta \\
&= \phi_0\big((t-t_0)^2a^2\Delta\big)\big(\nabla\varphi_1(X)\cdot\mathbf{n}\big) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\big(\nabla\varphi_2(X)\cdot\mathbf{n}\big) \\
&\quad + \int_{t_0}^{t}(t-\zeta)\phi_1\big((t-\zeta)^2a^2\Delta\big)\big(\nabla f\big(U(X,\zeta)\big)\cdot\mathbf{n}\big)\mathrm{d}\zeta \\
&= \nabla U(X,t)\cdot\mathbf{n}, \quad \forall X\in\partial\Omega.
\end{aligned}
$$

This proves the theorem. □

9.4.3 *Robin Boundary Condition*

In what follows we consider the arbitrarily high-dimensional nonlinear wave equation (9.39) with the following Robin boundary condition:

$$
\nabla U\cdot\mathbf{n} + \lambda U = \beta(X,t), \quad X\in\partial\Omega, \tag{9.50}
$$

where $\mathbf{n}$ is the unit outward normal vector on the boundary $\partial\Omega$ and λ is a constant.

Theorem 9.9 *Assume that $\beta(X,t)$ and $f\big(U(X,t)\big)$ are sufficiently differentiable with respect to t. Then, the formula* (9.40) *is consistent with the Robin boundary condition given by* (9.50).

The proof of Theorem 9.9 is similar to that in the recent paper [33] and we omit the details here.

Remark 9.2 As stated in Sects. 9.3 and 9.4, one need not care about the boundary conditions when the formula (9.15) is used directly since the formula is adapted to Dirichlet boundary conditions, Neumann boundary conditions, and Robin boundary conditions, respectively. In fact, (9.15) presents an exact analytical formal of the true solution to the initial-value problem of arbitrarily high-dimensional Klein–Gordon equations subject to the given boundary conditions, under the appropriate assumptions.

9.5 Illustrative Examples

To show applications of the integral formula presented in this chapter, we next present some illustrative examples.

Problem 9.1 We first consider the following two-dimensional equation

$$\begin{cases} u_{tt} - (u_{xx} + u_{yy}) = \omega^2 \sin(\omega(x-t)) \sin(\omega y), \\ u|_{t=0} = \sin(\omega x)\sin(\omega y), \;\; u_t|_{t=0} = -\omega \cos(\omega x)\sin(\omega y). \end{cases} \tag{9.51}$$

Applying (9.15) to Problem 9.1 yields

$$\begin{cases} u(x,y,t) = \phi_0(t^2\Delta)\sin(\omega x)\sin(\omega y) - \omega t\phi_1(t^2\Delta)\cos(\omega x)\sin(\omega y) \\ \qquad + \omega^2 \int_0^t (t-\zeta)\phi_1((t-\zeta)^2\Delta)\sin(\omega(x-\zeta))\sin(\omega y)\mathrm{d}\zeta, \\ u_t(x,y,t) = t\Delta\phi_1(t^2\Delta)\sin(\omega x)\sin(\omega y) - \omega\phi_0(t^2\Delta)\cos(\omega x)\sin(\omega y) \\ \qquad + \omega^2 \int_0^t \phi_0((t-\zeta)^2\Delta)\sin(\omega(x-\zeta))\sin(\omega y)\mathrm{d}\zeta. \end{cases} \tag{9.52}$$

It follows from a careful calculation that

$$\begin{aligned} &\phi_0(t^2\Delta)\sin(\omega x)\sin(\omega y) = \sin(\omega x)\sin(\omega y)\cos(\sqrt{2}\omega t), \\ &-\omega t\phi_1(t^2\Delta)\cos(\omega x)\sin(\omega y) = -\frac{1}{\sqrt{2}}\cos(\omega x)\sin(\omega y)\sin(\sqrt{2}\omega t), \\ &\omega^2 \int_0^t (t-\zeta)\phi_1((t-\zeta)^2\Delta)\sin(\omega(x-\zeta))\sin(\omega y)\mathrm{d}\zeta \\ &= \frac{\omega}{\sqrt{2}} \int_0^t \sin(\sqrt{2}\omega(t-\zeta))\sin(\omega(x-\zeta))\sin(\omega y)\mathrm{d}\zeta. \end{aligned}$$

We then have

$$\begin{aligned} u(x,y,t) &= \sin(\omega x)\sin(\omega y)\cos(\sqrt{2}\omega t) - \frac{1}{\sqrt{2}}\cos(\omega x)\sin(\omega y)\sin(\sqrt{2}\omega t) \\ &\quad + \frac{\omega}{\sqrt{2}}\int_0^t \sin(\sqrt{2}\omega(t-\zeta))\sin(\omega(x-\zeta))\sin(\omega y)\mathrm{d}\zeta \\ &= \sin(\omega(x-t))\sin(\omega y). \end{aligned} \tag{9.53}$$

Likewise, we can obtain

$$u_t(x,y,t) = -\omega\cos(\omega(x-t))\sin(\omega y). \tag{9.54}$$

Problem 9.2 We next consider the three-dimensional linear homogeneous equation:

$$\begin{cases} u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}), \\ u|_{t=0} = x^3 + yz, \;\; u_t|_{t=0} = 0. \end{cases} \tag{9.55}$$

Applying (9.15) to Problem 9.2, we can obtain the analytical solution straightforwardly:

$$\begin{cases} u(x,y,z,t) = \phi_0(t^2a^2\Delta)(x^3+yz) = x^3+yz+3a^2t^2x, \\ u_t(x,y,z,t) = ta^2\Delta\phi_1(t^2a^2\Delta)(x^3+yz) = 6a^2tx. \end{cases} \tag{9.56}$$

We note that from Poisson's or Kirchhoff's formula (see, e.g. [7]) the solution to (9.55) can be expressed in the form

$$u(x,y,z,t) = \frac{1}{4\pi a^2 t}\frac{\partial}{\partial t}\iint_S (x^3+yz)\mathrm{d}S, \tag{9.57}$$

where S is the sphere of radius a centered at (x_0, y_0, z_0). The calculation of the integral in (9.57) is quite complicated.

Problem 9.3 We next consider the following the initial valued problem of one dimensional linear Klein–Gordon equation (see, e.g. [19])

$$\begin{cases} u_{tt} - u_{xx} = -9u, \quad -\dfrac{5\pi}{8} < x < \dfrac{5\pi}{8}, \; t > 0, \\ u(x,0) = \cos(4x), \quad u_t(x,0) = 5\cos(4x), \quad -\dfrac{5\pi}{8} \le x \le \dfrac{5\pi}{8}, \end{cases} \tag{9.58}$$

subject to the Dirichlet boundary conditions

$$u(-\frac{5\pi}{8}, t) = 0, \quad u(\frac{5\pi}{8}, t) = 0. \tag{9.59}$$

The exact solution of the initial-boundary valued problem (9.58) and (9.59) is given by

$$u(x,t) = \cos(4x)\left(\cos(5t) + \sin(5t)\right). \tag{9.60}$$

Applying the *SV-scheme* (9.26) with the stepsize $h = 0.001$ to this initial-boundary valued problem, we obtain the numerical results, together with the true solution and the global error, which are shown in Fig. 9.1. It can be observed from Fig. 9.1 that

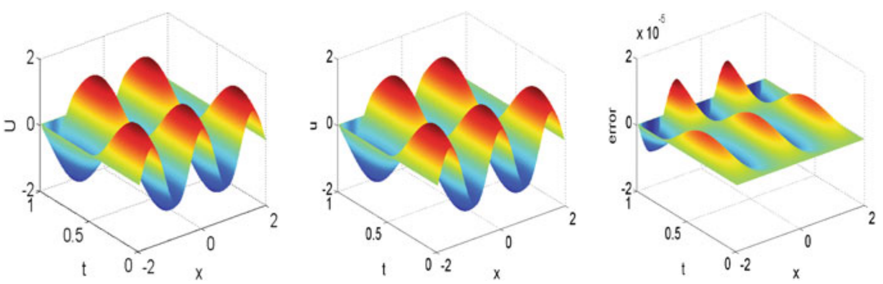

Fig. 9.1 The exact solution (left), the numerical solution (middle) and the global error (right) obtained by *SV-scheme* (9.26) with the stepsize $h = 0.001$, for Problem 9.3

the results show second-order behaviour of the *SV-scheme* (9.26). This indicates that the integral formula (9.15) is also helpful in gaining insight into developing efficient numerical integrators for Klein–Gordon equations.

9.6 Conclusions and Discussions

In this chapter, we considered the initial-boundary value problem of arbitrarily high-dimensional Klein–Gordon equations (9.1), posed on a bounded domain $\Omega \subset \mathbb{R}^d$ for $d \geq 1$ and equipped with various boundary conditions. We first defined the bounded operator-argument functions (9.8) which are restricted in a subspace $D(\Delta)$ of $L^2(\Omega)$, and then established an integral formula (9.15) for the Klein–Gordon equation in arbitrarily high-dimensional spaces. Thus, this chapter has made progress in research on integral representations of solutions of the arbitrarily high-dimensional Klein–Gordon equation (9.1). Another key aspect is that we showed in detail the consistency of the integral formula with Dirichlet boundary conditions, Neumann boundary conditions, and Robin boundary conditions, respectively. In other words, the integral formula (9.15) for (9.1) is completely adapted to the underlying boundary conditions under appropriate assumptions. If $g(U) = 0$, then (9.1) reduces to the arbitrarily high-dimensional homogeneous Klein–Gordon equation (9.20). Then, the integral formula (9.15) becomes (9.21), which integrates (9.20) exactly. In comparison with the seminal D'Alembert, Poisson and Kirchhoff formulas, formula (9.21) doesn't depend on the evaluation of complicated integrals, whereas the evaluation of integrals is required by the D'Alembert, Poisson and Kirchhoff formulas. To show the applications of the integral formula, some illustrative examples were also presented in Sect. 9.5.

Before the end of this chapter, we make a comment on the operator-variation-constants formula for PDEs. Once the operator-variation-constants formula is established for the underling PDEs, some structure-preserving schemes can be derived and analysed based on the formula. For example, Chaps. 10 and 11 will show the details for nonlinear wave equations. It is also believed that this approach to dealing with nonlinear wave PDEs can be extended to other PDEs, such as Maxwell's equations (see Yang et al. [35]). Further work on this research is in progress.

The material of this chapter is based on the work by Wu and Liu [28].

Appendix 1. A Direct Proof of Theorem 9.6

Proof It follows from (9.37) that

$$\left(\frac{\partial u}{\partial x}\right)_{tt} = a^2 \Delta\left(\frac{\partial u}{\partial x}\right) + \tilde{f}(u)$$

$$\Rightarrow \begin{cases} \gamma''(t_0) = a^2 \Delta\varphi_1'(x_l) + \tilde{f}\big(u\big), \\ \delta''(t_0) = \ a^2 \Delta\varphi_1'(x_r) + \tilde{f}\big(u\big), \end{cases}$$

$$\left(\frac{\partial u}{\partial x}\right)_t^{(3)} = a^2 \Delta\left(\frac{\partial u}{\partial x}\right)_t + \tilde{f}_t'(u)$$

$$\Rightarrow \begin{cases} \gamma^{(3)}(t_0) = a^2 \Delta\varphi_2'(x_l) + \tilde{f}_t'\big(u\big), \\ \delta^{(3)}(t_0) = \ a^2 \Delta\varphi_2'(x_r) + \tilde{f}_t'\big(u\big), \end{cases}$$

$$\left(\frac{\partial u}{\partial x}\right)_t^{(4)} = a^4 \Delta^2\left(\frac{\partial u}{\partial x}\right) + a^2 \Delta \tilde{f}(u) + \tilde{f}_t^{(2)}(u)$$

$$\Rightarrow \begin{cases} \gamma^{(4)}(t_0) = a^4 \Delta^2\varphi_1'(x_l) + a^2 \Delta \tilde{f}\big(u\big) + \tilde{f}_t^{(2)}\big(u\big), \\ \delta^{(4)}(t_0) = \ a^4 \Delta^2\varphi_1'(x_r) + a^2 \Delta \tilde{f}\big(u\big) + \tilde{f}_t^{(2)}\big(u\big), \end{cases}$$

$$\left(\frac{\partial u}{\partial x}\right)_t^{(5)} = a^4 \Delta^2\left(\frac{\partial u}{\partial x}\right)_t + a^2 \Delta \tilde{f}_t'(u) + \tilde{f}_t^{(3)}(u)$$

$$\Rightarrow \begin{cases} \gamma^{(5)}(t_0) = a^4 \Delta^2\varphi_2'(x_l) + a^2 \Delta \tilde{f}_t'\big(u\big) + \tilde{f}_t^{(3)}\big(u\big), \\ \delta^{(5)}(t_0) = \ a^4 \Delta^2\varphi_2'(x_r) + a^2 \Delta \tilde{f}_t'\big(u\big) + \tilde{f}_t^{(3)}\big(u\big), \end{cases}$$

$$\left(\frac{\partial u}{\partial x}\right)_t^{(6)} = a^6 \Delta^3\left(\frac{\partial u}{\partial x}\right) + a^4 \Delta^2 \tilde{f}(u) + a^2 \Delta \tilde{f}_t^{(2)}(u) + \tilde{f}_t^{(4)}(u)$$

$$\Rightarrow \begin{cases} \gamma^{(6)}(t_0) = a^6 \Delta^3\varphi_1'(x_l) + a^4 \Delta^2 \tilde{f}\big(u\big) \\ \qquad\qquad + a^2 \Delta \tilde{f}_t^{(2)}\big(u\big) + \tilde{f}_t^{(4)}\big(u\big), \\ \delta^{(6)}(t_0) = \ a^6 \Delta^3\varphi_1'(x_r) + a^4 \Delta^2 \tilde{f}\big(u\big) \\ \qquad\qquad + a^2 \Delta \tilde{f}_t^{(2)}\big(u\big) + \tilde{f}_t^{(4)}\big(u\big), \end{cases}$$

$$\left(\frac{\partial u}{\partial x}\right)_t^{(7)} = a^6 \Delta^3\left(\frac{\partial u}{\partial x}\right)_t + a^4 \Delta^2 \tilde{f}_t'(u) + a^2 \Delta \tilde{f}_t^{(3)}(u) + \tilde{f}_t^{(5)}(u)$$

$$\Rightarrow \begin{cases} \gamma^{(7)}(t_0) = a^6 \Delta^3\varphi_2'(x_l) + a^4 \Delta^2 \tilde{f}_t'\big(u\big) \\ \qquad\qquad + a^2 \Delta \tilde{f}_t^{(3)}\big(u\big) + \tilde{f}_t^{(5)}\big(u\big), \\ \delta^{(7)}(t_0) = \ a^6 \Delta^3\varphi_2'(x_r) + a^4 \Delta^2 \tilde{f}_t'\big(u\big) \\ \qquad\qquad + a^2 \Delta \tilde{f}_t^{(3)}\big(u\big) + \tilde{f}_t^{(5)}\big(u\big), \end{cases}$$

$\cdots.$

After an argument by induction we obtain the following results

$$\begin{aligned} \gamma^{(2k)} &= a^{2k} \Delta^k \varphi_1'(x_l) + \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} \tilde{f}_t^{(2j-2)}\big(u\big) \\ \gamma^{(2k+1)} &= a^{2k} \Delta^k \varphi_2'(x_l) + \sum_{j=1}^{k} a^{2(k-j)} \Delta^{k-j} \tilde{f}_t^{(2j-1)}\big(u\big), \quad k = 1, 2, \ldots. \end{aligned} \tag{9.61}$$

and

$$\begin{aligned}\delta^{(2k)}(t_0) &= a^{2k}\Delta^k\varphi_1'(x_r) + \sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-2)}\big(u\big)\\ \delta^{(2k+1)}(t_0) &= a^{2k}\Delta^k\varphi_2'(x_r) + \sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-1)}\big(u\big), \quad k = 1, 2, \ldots.\end{aligned} \tag{9.62}$$

Inserting the results of (9.61) and (9.62) into the Taylor expansion of $\gamma(t)$ and $\delta(t)$ at point t_0 yields

$$\begin{aligned}\gamma(t) = \Big\{&\phi_0\big((t-t_0)^2a^2\Delta\big)\varphi_1'(x) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\varphi_2'(x)\\ &+ \sum_{k=1}^{\infty}\Big[\frac{(t-t_0)^{2k}}{(2k)!}\sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-2)}\big(u\big)\\ &+ \frac{(t-t_0)^{2k+1}}{(2k+1)!}\sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-1)}\big(u\big)\Big]\Big\}\Big|_{x=x_l},\end{aligned} \tag{9.63}$$

and

$$\begin{aligned}\delta(t) = \Big\{&\phi_0\big((t-t_0)^2a^2\Delta\big)\varphi_1'(x) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\varphi_2'(x)\\ &+ \sum_{k=1}^{\infty}\Big[\frac{(t-t_0)^{2k}}{(2k)!}\sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-2)}\big(u\big)\\ &+ \frac{(t-t_0)^{2k+1}}{(2k+1)!}\sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-1)}\big(u\big)\Big]\Big\}\Big|_{x=x_r}.\end{aligned} \tag{9.64}$$

Let

$$\tilde{F}(x,t) \triangleq \int_{t_0}^{t}(t-\zeta)\phi_1\big((t-t_0)^2a^2\Delta\big)\hat{f}\big(\zeta\big)\mathrm{d}\zeta.$$

As deduced for the Dirichlet boundary conditions in Sect. 9.4.1, it can be shown that

$$\begin{aligned}\tilde{F}_t^{(2k)} &= \sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-2)}\big(u\big)\\ \tilde{F}_t^{(2k+1)} &= \sum_{j=1}^{k} a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-1)}\big(u\big), \quad k = 1, 2, \ldots.\end{aligned} \tag{9.65}$$

Inserting (9.65) into the Taylor expansion of $\tilde{F}(x,t)$ at the point $t = t_0$ gives

$$\begin{aligned}
\tilde{F}(x,t) &= \sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\tilde{F}_t^{(k)} = \sum_{k=2}^{\infty}\frac{(t-t_0)^k}{k!}\tilde{F}_t^{(k)} \\
&= \sum_{k=1}^{\infty}\frac{(t-t_0)^{2k}}{(2k)!}\tilde{F}_t^{(2k)} + \sum_{k=1}^{\infty}\frac{(t-t_0)^{2k+1}}{(2k+1)!}\tilde{F}_t^{(2k+1)} \\
&= \sum_{k=1}^{\infty}\Big[\frac{(t-t_0)^{2k}}{(2k)!}\sum_{j=1}^{k}a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-2)}(u) \\
&\quad + \frac{(t-t_0)^{2k+1}}{(2k+1)!}\sum_{j=1}^{k}a^{2(k-j)}\Delta^{k-j}\tilde{f}_t^{(2j-1)}(u)\Big].
\end{aligned} \tag{9.66}$$

Comparing the results of (9.66) with (9.63) and (9.64) yields

$$\begin{aligned}
\gamma(t) &= \Big[\phi_0\big((t-t_0)^2a^2\Delta\big)\varphi_1'(x) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\varphi_2'(x) \\
&+ \int_{t_0}^{t}(t-\zeta)\phi_1\big((t-\zeta)^2a^2\Delta\big)\hat{f}(\zeta)\mathrm{d}\zeta\Big]\Big|_{x=x_l}, \\
\delta(t) &= \Big[\phi_0\big((t-t_0)^2a^2\Delta\big)\varphi_1'(x) + (t-t_0)\phi_1\big((t-t_0)^2a^2\Delta\big)\varphi_2'(x) \\
&+ \int_{t_0}^{t}(t-\zeta)\phi_1\big((t-\zeta)^2a^2\Delta\big)\hat{f}(\zeta)\mathrm{d}\zeta\Big]\Big|_{x=x_r}.
\end{aligned}$$

This finishes the direct proof. □

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