A New Analytical Technique for Solving Nonlinear Non-smooth Oscillators Based on the Rational Harmonic Balance Method

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Abstract In the present paper, a new analytical technique based on the rational harmonic balance method (RHBM) has been introduced to determine approximate periodic solutions for the nonlinear non-smooth oscillator. A frequency–amplitude relationship has also been obtained by a novel analytical way. The standard rational harmonic balance method (SRHBM) cannot be used directly; it is possible if we rewrite the nonlinear differential equations (NDEs). To overcome this previously stated issue, we offered a modified rational harmonic balance method (MRHBM). It is noticed that a MRHBM works very well for the whole range of initial amplitudes and the excellent agreement of the approximate frequencies as well as the corresponding periodic solutions with its exact ones. The method is basically illustrated by the nonlinear non-smooth oscillators, but it is additionally useful for other nonlinear oscillatory problems with mixed parity arising in recent development of nonlinear sciences and engineering.

Keywords Analytical technique · Approximate angular frequencies
Nonlinear non-smooth oscillator · Power series solution · Rational harmonic balance method

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1 Introduction

Along with the rapid progress of nonlinear sciences, an intensifying interest among scientists and researchers has been emerged in the field of nonlinear oscillating systems with the nonlinear non-smooth oscillators because this issue is very applicable in dynamics of structures which is stated in Chopra 1995. Nowadays, obtaining exact solutions of the nonlinear oscillatory problems is one of the biggest challenges. In general, it is often more difficult to obtain an analytic approximation than a numerical one. A few nonlinear systems can be solved explicitly, and numerical methods, especially the most popular Runge-Kutta fourth-order method, are frequently used to calculate approximate solutions. However, numerical schemes do not always give accurate results, especially the class of stiff differential equations, chaotic differential equation, which present a more serious challenge to numerical analysis. And also, the frequency-amplitude relationship cannot be obtained. The popular method for solving nonlinear differential equations (NDEs) associated with oscillatory systems is perturbation method (Nayfeh 1973; Azad et al. 2012) which is the most versatile tools available in nonlinear analysis of engineering problems, and they are constantly being developed and applied to even more complex problems. However, for the strongly nonlinear regime perturbation method cannot yield desired results.

As a result, due to conquering these weak points, in recent past, numerous researchers have devoted their time and effort to find potent approaches for investigating the nonlinear phenomena. As the earliest effort, they developed a large variety of approximate methods commonly used for strongly nonlinear oscillators including homotopy perturbation method (Belendez 2009; Ozis and Akci 2011), modified He's homotopy perturbation method (Belendez et al. 2007), He's modified Lindsted-Poincare method (Ozis and Yildirim 2007), max–min approach method (Ganji and Azimi 2012), global residue harmonic balance method (Peijun 2015), energy balance method (Hosen 2016, 2017), He's energy balance method (Askari et al. 2014), rational energy balance method (Daeichin et al. 2013), iteration method (Ikramul et al. 2013; Mickens 2006), harmonic balance method (Mickens 2010; Hosen et al. 2012; Cveticanin 2009; Lim et al. 2005; Gottlieb 2003), and so on. However, the results obtained by most of the mentioned methods only first-order approximation has been considered which leads insufficient accuracy. Furthermore, the solution procedures are tremendously difficult task and cumbersome, especially for obtaining higher-order approximation. In this situation, we will see that the rational harmonic balance method (RHBM) considered in this paper can be applied to nonlinear non-smooth oscillator. The RHBM discussed by Mickens and Semwogerere (1996), for instance, has rarely been applied to the determination of periodic solutions of the nonlinear problems. In fact, to the best of our knowledge, recently Belendez et al. (2008) and Yamgoue et al. (2010) used it to solve a simple-term oscillator equation of plasma physics in a completely analytic fashion. Generally, a set of complicated nonlinear algebraic equations are found when RHBM is applied. Sometimes analytical solutions of these algebraic equations fail, especially for large amplitude. In the present study, this limitation is removed. The nonlinear algebraic equations have been approximated using power series solution (a new small parameter). Consider the interesting issue that the proposed technique provides accurate results and it is more convenient and efficient for solving more complex nonlinear problems.

2 Solution Procedure by the Standard Rational Harmonic Balance Method

Consider a general second-order nonlinear differential equation with mixed parity which is of the following form as

$$\ddot{x} = -\varepsilon x^{\frac{1}{2n+1}} \text{ and the initial condition } x(0) = a_0,\ \dot{x}(0) = 0, \tag{1}$$

where $x^{\frac{1}{2n+1}}$, $n = 1, 2, 3, \cdots$ is a fractional-order nonlinear function and ε is a constant.

The nth-order periodic solution of Eq. 1 can be considered as

$$x(t) = \frac{A_1 \cos\varphi + A_3 \cos 3\varphi + A_5 \cos 5\varphi + \cdots}{1 + u \cos 2\varphi + v \cos 4\varphi + w \cos 6\varphi + \cdots}. \tag{2}$$

where $\varphi = \omega t$ and A_1, A_3, A_5, u, v, w are unknown constants. The solution of Eq. 2 does not satisfied of Eq. 1 directly; it is possible if we rewrite the Eq. 1. Then applying Eq. 2 into the rewritten Eq. 1, it can be transformed into

$$\begin{aligned} &\mathrm{A}_1^{2n+1}\omega^{4n+2}[(1+u^2+\cdots)\cos(\omega t) + (1-u+\cdots)\cos 3\varphi + \cdots] \\ &\quad = -\varepsilon[F_1(A_1, u, \cdots)\cos\varphi + F_3(A_3, u, \cdots)\cos 3\varphi + \cdots] \end{aligned} \tag{3}$$

By comparing the coefficients of equal harmonic terms of Eq. 3, one could obtain as

$$\begin{aligned} A_1^{2n+1}\omega^{4n+2}(1+u^2+\cdots) &= -\varepsilon F_1(A_1, u, \cdots) \\ A_1^{2n+1}\omega^{4n+2}(1-u+\cdots) &= -\varepsilon F_3(A_3, u, \cdots), \\ A_1^{2n+1}\omega^{4n+2}(1-v+\cdots) &= -\varepsilon[F_5(A_5, u, \cdots) \end{aligned} \tag{4}$$

With help of the first equation, ω^{4n+2} is eliminated from all the remaining equations of Eq. 4. Thus, second and third equations of Eq. 4 can be expressed as

$$u = G_1(\varepsilon, a_0, u, v, \cdots,), \quad v = G_2(\varepsilon, a_0, u, v, \cdots,), \cdots, \tag{5}$$

where $G_1,\ G_2,\ \cdots$ exclude, respectively, the linear terms of $u,\ v,\ \cdots$.

Whatever the values of ε and a_0, there exists a parameter $\lambda_0(\varepsilon, a_0) \ll 1$, such that $u,\ v,\ \cdots$ are expandable in following series

$$u = U_1\lambda_0 + U_2\lambda_0^2 + \cdots, \quad v = V_1\lambda_0 + V_2\lambda_0^2 + \cdots, \quad \cdots \tag{6}$$

where $U_1,\ U_2, \cdots, V_1,\ V_2, \cdots$ are constants.

Finally, substituting the values of $u, v, \cdots$ from Eq. 6 into the first equation of Eq. 4, the unknown angular frequency ω is determined. This completes the determination of all related functions for the proposed periodic solution as given in Eq. 2.

3 Solution Procedure by the Modified Rational Harmonic Balance Method

Here, solution Eq. 2 is applied into Eq. 1 directly, if we expand the fractional nonlinear terms $x^{\frac{1}{2n+1}}$ in a Fourier series as

$$x^{\frac{1}{2n+1}} = \sum_{n=0}^{\infty} b_{2n+1} f(x) = b_1 \cos(\omega t) + b_3 \cos(3\omega t) + \cdots. \tag{7}$$

where b_1 , b_3 , $\cdots$ will be calculated by using the following integration

$$b_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} x^{\frac{1}{2n+1}} \cos[(2n+1)\varphi] d\varphi; \quad n = 0, 1, 2, 3, \cdots \tag{8}$$

where $\varphi = \omega t$.

Substituting Eqs. 2, 7–8 into Eq. 1 and then Eq. 1 can be transformed into an algebraic identity as

$$\begin{aligned} &A_1^{2n+1}\omega^{4n+2}[(1 + u^2 + \cdots)\cos(\omega t) + (1 - u \cdots)\cos 3\varphi + \cdots] \\ &\quad = -\varepsilon[F_1(A_1, u, \cdots)\cos\varphi + F_3(A_3, u, \cdots)\cos 3\varphi + \cdots] \end{aligned} \tag{9}$$

By comparing the coefficients of equal harmonics of Eq. 9, the following nonlinear algebraic equations can be found as

$$\begin{aligned} A_1^{2n+1}\omega^{4n+2}(1 + u^2 + \cdots) &= -\varepsilon F_1 \\ A_1^{2n+1}\omega^{4n+2}(1 - u + \cdots) &= -\varepsilon F_3(A_3, u, \cdots), \\ A_1^{2n+1}\omega^{4n+2}(1 - v + \cdots) &= -\varepsilon F_5(A_5, u, \cdots) \end{aligned} \tag{10}$$

With help of the first equation, ω^{4n+2} is eliminated from all the remaining equations of Eq. 10. Thus, second and third equations of Eq. 10 can be expressed into the following form as

$$u = G_1(\varepsilon, a_0, u, v, \cdots,), \quad v = G_2(\varepsilon, a_0, u, v, \cdots,), \ \cdots \tag{11}$$

where $G_1, \ G_2, \ \cdots$ exclude, respectively, the linear terms of $u, \ v, \ \cdots$.

Whatever the values of ε and a_0, there exists a parameter $\lambda_0(\varepsilon, a_0) \ll 1$, such that $u, \ v, \ \cdots$ are expandable in following series as

$$u = U_1\lambda_0 + U_2\lambda_0^2 + \cdots, \quad v = V_1\lambda_0 + V_2\lambda_0^2 + \cdots, \quad \cdots \tag{12}$$

where $U_1, \ U_2, \cdots, V_1, \ V_2, \cdots$ are constants.

Finally, substituting the values of $u, \ v, \cdots$ from Eq. 12 into the first equation of Eq. 10, the unknown angular frequency ω is determined. This completes the determination of all related functions for the proposed periodic solution as given in Eq. 2.

4 Application of the Standard Rational Harmonic Balance Method (SRHBM)

Consider $n = \varepsilon = 1$ into Eq. 1, the nonlinear non-smooth oscillator (Belendez 2009; Ozis and Yildirim 2007; Mickens 2006, 2010) can be written as

$$\ddot{x} + x^{1/3} = 0, \quad x(0) = a_0, \quad \dot{x}(0) = 0. \tag{13}$$

This is a conservative system, and the solution to Eq. 13 is periodic. We observe that in Eq. 13 direct application of SRHBM does not work. To apply the SRHBM, we rewrite the Eq. 13 as

$$\ddot{x}^3 + x = 0. \tag{14}$$

Now, the solution Eq. 2 can be expressed by Eq. 13. From Eq. 2, the second-order approximation solution of Eq. 14 can be supposed as

$$x(t) = \frac{A_1 \cos\varphi}{1 + u\,\cos 2\varphi} = \frac{A_1 \cos(\omega t)}{1 + u\,\cos(2\omega t)}. \tag{15}$$

Now using Eq. 15 in the Eq. 14 and then setting the coefficients of $\cos(\omega t)$ and $\cos(3\omega t)$ equal to zero, the following nonlinear algebraic equations can be obtained as

$$A_1^2[\omega^6(3/4 + 9u^2/16 - 6u^3 - 699u^4/32 + \cdots)] = 1 + 4u + 14u^2 + 21u^3 + \cdots, \tag{16}$$

$$A_1^2[\omega^6(1/4 - 15u/4 - 141u^2/16 + 91u^3/4 + \cdots)] = 4u + 7u^2 + 21u^3 + \cdots. \tag{17}$$

After simplification, Eq. 16 can be written as

$$\omega^6 = \frac{(1 + 4u + 14u^2 + 21u^3 + 105u^4/4 + 35u^5/2 + 35u^6/4 + \cdots)}{A_1^2(3/4 + 9u^2/16 - 6u^3 - 699u^4/32 + 105u^5 - 30645u^6/256)} \tag{18}$$

By elimination of ω^6 from Eq. 17 with the help of Eq. 18, the equation of u can be written as

$$u = \lambda_0\left(1 - \frac{409u^2}{4} - 311u^3 - \frac{637u^4}{8} + \frac{17577u^5}{16} + \frac{119113u^6}{64} + \frac{3179u^7}{8} + \cdots\right), \tag{19}$$

where $\lambda_0 = \frac{1}{23}$.

The power series solution of Eq. 19 can be derived in terms of λ_0 as

$$u = \lambda_0 - \frac{409}{4}\lambda_0^3 - 311\lambda_0^4 + \frac{41661}{2}\lambda_0^5 + \frac{2561557}{16}\lambda_0^6 - \frac{40034213}{8}\lambda_0^7 + \cdots \tag{20}$$

Substituting the value of u from Eq. 20 into Eq. 18 and using $A_1 = a_0(1 + u)$, the approximate angular frequency can be determined as

$$\omega(a_0) = \sqrt[6]{\frac{(1 + 4u + 14u^2 + 21u^3 + 105u^4/4 + 35u^5/2 + 35u^6/4 + \cdots)}{A_1^2(3/4 + 9u^2/16 - 6u^3 - 699u^4/32 + 105u^5 - 30645u^6/256)}} = \frac{1.063575}{a_0^{1/3}} \tag{21}$$

Thus, the approximation solution of Eq. 13 is $x(t) = \frac{A_1 \cos(\omega t)}{1 + u\cos(2\omega t)}$ where u and ω are, respectively, given by Eqs. 20-21.

5 Application of the Modified Rational Harmonic Balance Method (MRHBM)

We can apply the MRHBM directly in Eq. 13. The second term of Eq. 13 *i.e.* $x^{1/3}$ can be expanded in a Fourier series as

$$x^{1/3} = \sum_{n=0}^{\infty} b_{2n+1}x^{1/3} = b_1\cos(\omega t) + b_3\cos(3\omega t) + \cdots. \tag{22}$$

Herein $b_1, b_3, \cdots$ are calculated by the following integration as

$$b_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} x^{1/3}\,\cos[(2n+1)\varphi]d\varphi, \tag{23}$$

setting $\varphi = \omega t$.

Now substituting Eq. 15, 22–23 into the Eq. 13 and then equating the coefficients of $\cos(\omega t)$ and $\cos(3\omega t)$, the following nonlinear algebraic equations are obtained as

$$-(1+u-11u^2/2)A_1\omega^2 + \frac{A_1^{1/3}(17820-2376u+1881u^2-938u^3)I_0}{5940\sqrt{\pi}} = 0, \tag{24}$$

$$(3u-9u^2/4)A_1\omega^2 - \frac{A_1^{1/3}(3564+3267u-531u^2+1211u^3)I_0}{5940\,\sqrt{\pi}} = 0, \tag{25}$$

where $b_1, b_3, \cdots$ are determined as

$$b_1 = \frac{A_1^{1/3}(17820-2376u+1881u^2-938u^3)\,I_0}{5940\sqrt{\pi}}, \tag{26}$$

$$b_3 = -\frac{A_1^{1/3}(3564+3267u-531u^2+1211u^3)I_0}{5940\sqrt{\pi}}, \text{where } I_0 = \frac{Gamma\left[\frac{7}{6}\right]}{Gamma\left[\frac{2}{3}\right]} \tag{27}$$

and so on.

After disentanglement, Eq. 24 can be written into another form as

$$\omega^2 = \frac{A_1^{1/3}(17820-2376u+1881u^2-938u^3)\,I_0}{5940(1+u-11u^2/2)\,A_1\sqrt{\pi}} \tag{28}$$

By omitting ω^2 from Eq. 25 with the help of Eq. 28 and then some modification, one could obtain the following nonlinear algebraic equation of u as

$$u = \lambda_0\left(1 + \frac{3373\,u^2}{396} - \frac{56555\,u^3}{7128} + \frac{44711\,u^4}{14256} - \frac{8771\,u^5}{3564}\right), \text{where } \lambda_0 = \frac{12}{157} \tag{29}$$

The power series solution of Eq. 29 in terms of λ_0 is

$$u = \lambda_0 + \frac{3373\lambda_0^3}{396} - \frac{56555\lambda_0^4}{7128} + \frac{7748693\lambda_0^5}{52272} - \frac{960746707\lambda_0^6}{2822688} + \cdots. \tag{30}$$

Now substituting the value of u from Eq. 30 into Eq. 28 and using $A_1 = a_0(1+u)$, the approximate angular frequency can be obtained as

$$\omega(a_0) = \sqrt{\frac{A_1^{1/3}(17820 - 2376u + 1881u^2 - 938u^3)I_0}{5940(1 + u - 11u^2/2)A_1\sqrt{\pi}}} = \frac{1.077845}{a_0^{1/3}} \tag{31}$$

Therefore, the modified approximate solution of Eq. 13 is $x(t) = \frac{A_1 \cos(\omega t)}{1 + u \cos(2\omega t)}$ where u and ω are, respectively, given by Eqs. (30)–(31).

6 Results and Discussions

The approximate angular frequencies have been obtained by standard rational harmonic balance method and modified harmonic balance method for the nonlinear non-smooth oscillators. For this nonlinear problem, the exact value of the frequency is

$$\omega_{ex}(a_0) = \frac{1.070451}{a_0^{1/3}},$$

which is stated in (Gottlieb 2003). The approximated angular frequencies have been plotted in Figs. 1 and 2. It is highly remarkable that the approximated results show a good agreement with the corresponding exact frequency. Moreover, the solution

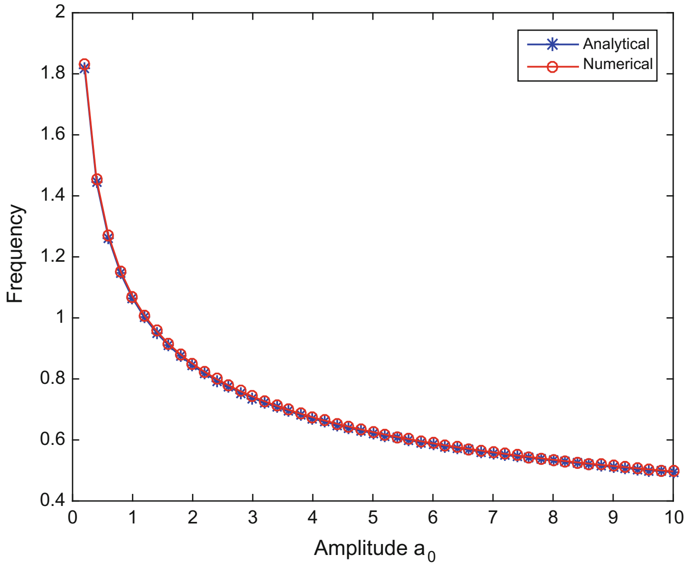

Fig. 1 Employed standard rational harmonic balance method

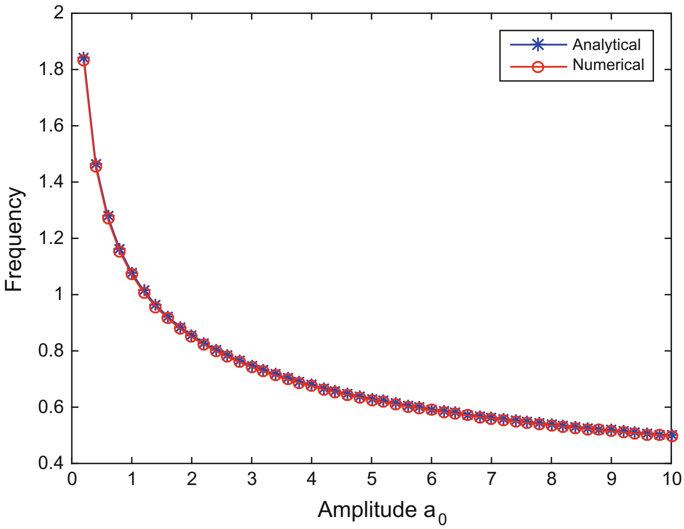

Fig. 2 Modified harmonic balance method

procedure of the proposed method is simple, straightforward, and quite easy. The advantages of this method include its analytical simplicity and computational efficiency, and the ability to objectively find better results.

7 Conclusion

A new analytical technique based on the rational harmonic balance method (RHBM) has been investigated to obtain approximate angular frequencies for the nonlinear non-smooth oscillators. The approximated angular frequencies give almost similar as compared to its exact ones. Moreover, in comparison with previously published methods the determination procedure of approximate solutions is straightforward and simple. The high accuracy and validity of the approximate frequencies assured about the results and reveal this method can be used easily for nonlinear non-smooth oscillators. To entirety up, we can say that the technique offered in this study for solving nonlinear non-smooth oscillators can be considered as powerful, an efficient alternative of the previously existing methods.

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