Global Stability of a Delayed Eco-Epidemiological Model with Holling Type III Functional Response

Hongfang Bai and Rui Xu

Abstract In this paper, we consider an eco-epidemiological model with Holling type III functional response and a time delay representing the gestation period of the predator. In the model, it is assumed that the predator population suffers a transmissible disease. By means of Lyapunov functionals and Laselle's invariance principle, sufficient conditions are obtained for the global stability of the endemic coexistence of the system.

Keywords Eco-epidemiological model · Delay · Laselle's invariance principle
Global stability

1 Introduction

Epidemiological models have received considerate attention in the literature to explain the spread and control of infectious disease [1–4]. Most of these models descend from the pioneering work of Kermack and Mckendrick [5], who proposed the classical SIR model . Seeing that species do not exist alone in the nature world, so it is very important to study the system of two or more interacting species subjected to disease [6].

Recently, great attention has been paid to study the relationships between demographic processes among different populations and diseases (see, e.g., [7–11]). Such as, Zhang et al. [7] studied the following eco-epidemiological model with Holling type I response function

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$$\dot{x}(t) = rx(t) - a_{11}x^2(t) - a_{12}x(t)S(t),$$
$$\dot{S}(t) = a_{21}x(t-\tau)S(t-\tau) - r_1S(t) - \beta S(t)I(t), \tag{1.1}$$
$$\dot{I}(t) = \beta S(t)I(t) - r_2I(t),$$

where $x(t), S(t), I(t)$ denote the densities of the prey, the susceptible predator, and the infected predator population, respectively.

In system (1.1), it assumes that the per capita rate of predation depends on the prey numbers only. But Holling found that each predator increased its consumption rate when exposed to a higher prey density, and also predator density increased with increasing prey density [12, 13]. So he suggested the following three kinds of functional responses referring to the number of prey eaten per predator per unit time.

$$(1)\ p_1(x) = ax, \quad (2)\ p_2(x) = \frac{ax}{m+x}, \quad (3)\ p_3(x) = \frac{ax^2}{m+x^2},$$

where x denotes the density of prey, $a > 0$ is the search rate of the predator, $m > 0$ is half-saturation constant, $p_1(x), p_2(x)$, and $p_3(x)$ represent Holling type I, II, and III functional responses, respectively.

Holling type III functional response reveals that the risk of being preyed upon is small at low prey density but increases up to a certain point as prey density increases, which is in accordance with some phenomena of natural world. Also, we know that many factors contribute to a type III functional response such as prey refuge, predator learning, and the presence of alternative prey [14].

Motivated by the works of Holling [14] and Zhang et al. [7], in this paper, we consider a delayed eco-epidemiological model with Holling type III functional response, which suffers a transmissible disease. Thus, we study the following eco-epidemiological model:

$$\dot{x}(t) = rx(t) - a_{11}x^2(t) - \frac{a_{12}x^2(t)S(t)}{1+mx^2(t)} - \frac{a_{13}x^2(t)I(t)}{1+mx^2(t)},$$
$$\dot{S}(t) = k\frac{a_{12}x^2(t-\tau)S(t-\tau)}{1+mx^2(t-\tau)} - r_1S(t) - \beta S(t)I(t), \tag{1.2}$$
$$\dot{I}(t) = \beta S(t)I(t) + k\frac{a_{13}x^2(t-\tau)I(t-\tau)}{1+mx^2(t-\tau)} - r_2I(t).$$

where $x(t), S(t)$, and $I(t)$ represent the densities of the prey, the susceptible predator, and the infected predator population, respectively. r is the intrinsic growth rate of prey population without disease, r/a_{11} is the environmental carrying capacity, a_{12} is the capturing rate of the susceptible predators. The infected predator also can catch the prey; here, a_{13} denotes the capturing rate of the infected predator. k is the conversion rate of nutrients into the reproduction of predators by consuming prey, β is the disease transmission coefficient, r_1 is the natural death rate of the susceptible predators, r_2 is the natural and disease-related mortality rate of the infected predator. Here, $r_1 < r_2$. τ is a time delay representing a duration of τ time units elapses

when an individual prey is killed and the moment when the corresponding addition is made to the predator population. All the parameters are positive.

The initial conditions for system (1.2) are

$$
\begin{aligned}
&x(\theta) = \phi_1(\theta), S(\theta) = \phi_2(\theta), I(\theta) = \phi_3(\theta), \theta \in [-\tau, 0], \\
&\phi_i \in C([-\tau, 0], R_+^3), \quad \phi_i > 0, \quad i = 1, 2, 3,
\end{aligned}
\tag{1.3}
$$

where $R_+^3 = (x_1, x_2, x_3) : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

The organization of this paper is as follows. In Sect. 2, the positivity and the equilibria of system (1.2) are presented. In Sect. 3, we consider about the permanence of system (1.2) by using the persistence theory on infinite dimensional systems developed by Hale and Waltman [15]. In Sect. 4, we establish sufficient conditions for the global asymptotic stability of the endemic-coexistence equilibrium of system (1.2) by constructing suitable Lyapunov functionals and adopting Lasalle's invariance principle. Finally, we discuss the biological meaning of the result obtained in this paper.

2 Preliminaries

In this section, we consider the positivity of solutions and the equilibria of system (1.2).

2.1 Positivity of Solutions

Theorem 2.1 *Suppose that $(x(t), S(t), I(t))$ is a solution of system (1.2) with initial conditions (1.3). Then, $x(t) \geq 0$, $S(t) \geq 0$, and $I(t) \geq 0$ for all $t \geq 0$.*

Proof From the first equation of system (1.2), we have

$$
x(t) = x(0) \exp \left\{ \int_0^t \left[r - a_{11}x(u) - a_{12}x(u)S(u)/(1 + mx^2(u)) - a_{13}x(u)I(u)/(1 + mx^2(u)) \right] du \right\} > 0.
$$

Hence, $x(t)$ is positive.

In order to prove that $S(t)$ is positive on $[0, \infty]$, suppose that there exists $t_1 > 0$ such that $S(t_1) = 0$, and $S(t) > 0$ for $t \in [0, t_1]$. Then, $\dot{S}(t_1) \leq 0$. From the second equation of (1.2), we have

$$\dot{S}(t_1) = k\frac{a_{12}x^2(t_1 - \tau)S(t_1 - \tau)}{1 + mx^2(t_1 - \tau)} - r_1S(t_1) - \beta S(t_1)I(t_1)$$

$$= k\frac{a_{12}x^2(t_1 - \tau)S(t_1 - \tau)}{1 + mx^2(t_1 - \tau)} > 0,$$

which is a contradiction.

In order to show that $I(t)$ is positive on $[0, \infty]$, suppose that there exists $t_2 > 0$ such that $I(t_2) = 0$, and $I(t) > 0$ for $t \in [0, t_2]$. Then, $\dot{I}(t_2) \leq 0$. From the third equation of (1.2), we have

$$\dot{I}(t_2) = \beta S(t_2)I(t_2) + k\frac{a_{13}x^2(t_2 - \tau)I(t_2 - \tau)}{1 + mx^2(t_2 - \tau)} - r_2I(t_2)$$

$$= k\frac{a_{13}x^2(t_2 - \tau)I(t_2 - \tau)}{1 + mx^2(t_2 - \tau)} > 0,$$

which is a contradiction.   □

2.2 Equilibria

System (1.2) possesses the following equilibria in general.

  (i) The trivial equilibrium $E_0 = (0, 0, 0)$.
 (ii) The predator-extinction equilibrium $E_1 = (r/a_{11}, 0, 0)$.
(iii) The disease-free equilibrium $E_2 = (x_2, S_2, 0)$, where

$$x_2 = \sqrt{\frac{r_1}{ka_{12} - r_1m}},$$
$$S_2 = \frac{k}{\sqrt{r_1(ka_{12} - r_1m)}}\left(r - a_{11}\sqrt{\frac{r_1}{ka_{12} - r_1m}}\right). \tag{2.1}$$

We denote an ecological threshold parameter by $\mathfrak{R}_1 = \dfrac{k}{r_1}\dfrac{r^2a_{12}}{a_{11}^2 + mr^2}$. It is easy to show that if $\mathfrak{R}_1 > 1$, then $x_2 > 0$, $I_2 > 0$.

(iv) The planar equilibrium $E_3 = (x_3, 0, I_3)$, where

$$x_3 = \sqrt{\frac{r_2}{ka_{13} - r_2m}},$$
$$I_3 = \frac{k}{\sqrt{r_2(ka_{13} - r_2m)}}\left(r - a_{11}\sqrt{\frac{r_2}{ka_{13} - r_2m}}\right). \tag{2.2}$$

Similar, we denote $\mathfrak{R}_2 = \dfrac{k}{r_2} \dfrac{r^2 a_{13}}{a_{11}^2 + mr^2}$. It is easy to show that if $\mathfrak{R}_2 > 1$, then $x_3 > 0, I_3 > 0$.

(v) The endemic-coexistence equilibrium $E^* = (x^*, S^*, I^*)$, where

$$
\begin{aligned}
I^* &= \frac{ka_{12}x^{*2}}{\beta(1 + mx^{*2})} - \frac{r_1}{\beta}, \\
S^* &= \frac{r_2}{\beta} - \frac{ka_{13}x^{*2}}{\beta(1 + mx^{*2})},
\end{aligned}
\tag{2.3}
$$

in which x^* is a positive real root of the following cubic equation:

$$
m\beta a_{11}x^3 - mr\beta x^2 + (a_{11}\beta + r_2 a_{12} - a_{13}r_1)x - r\beta = 0. \tag{2.4}
$$

It can be seen that if

(H1) $r_2(ka_{12} - r_1 m) > r_1(ka_{13} - r_2 m)$,

then system (1.2) has a endemic-coexistence equilibrium E^*.

3 Permanence

In this section, we study the permanence of system (1.2). Before starting our theorem, we give some basic concepts and corresponding theory.

Definition 3.1 System (1.2) is said to be permanent (uniformly persistent) if there are positive m_i and $M_i(i = 1, 2, 3)$ such that each positive solution $(x(t), S(t), I(t))$ of system (1.2) satisfies

$$
\begin{aligned}
m_1 &\leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq M_1, \\
m_2 &\leq \liminf_{t \to +\infty} S(t) \leq \limsup_{t \to +\infty} S(t) \leq M_2, \\
m_3 &\leq \liminf_{t \to +\infty} I(t) \leq \limsup_{t \to +\infty} I(t) \leq M_3.
\end{aligned}
$$

Definition 3.2 System (1.2) is said to be permanent if there exists a compact region $\Omega_0 \in \text{int}\,\Omega$ such that every solution of Eqs. (1.2) with initial condition (1.3) will eventually enter and remain in region Ω_0.

It is easy to see that for a dissipative system, uniform persistence is equivalent to permanence. For the sake of convenience, we present the uniform persistence theory for infinite dimensional systems.

Let X be a complete metric space with metric d. Suppose that T is a continuous semiflow on X, that is, a continuous mapping $T : [0, +\infty] \times X \to X$ with the following properties

$$T_t \circ T_s = T_{t+s}, \quad t, s \geq 0, \quad T_0(x) = x, x \in X,$$

where T_t denotes the mapping from X to X given by $T_t(x) = T(t, x)$.

The distance $d(x, Y)$ of a point $x \in X$ from a subset Y of X is defined by

$$d(x, Y) = \inf_{y \in Y} d(x, y).$$

Recall that the positive orbit $\gamma^+(x)$ through x is defined as $\gamma^+(x) = \cup_{t \geq 0} \{T(t)x\}$, and its $\omega-$ limit set is $\omega(x) = \cap_{s \geq 0} \overline{\cup_{t \geq s} \{T(t)x\}}$. Define $W^s(A)$ the strong stable set of a compact invariant set A as

$$W^s(A) = \{x : x \in X, \omega(x) \neq \emptyset, \omega(x) \subset A\}.$$

Suppose that X^0 is open and dense in X and $X^0 \cup X_0 = X$, $X^0 \cap X_0 = \emptyset$. Moreover, the C^0-semigroup $T(t)$ on X satisfies

$$T(t) : X^0 \to X^0, T(t) : X_0 \to X_0. \tag{3.1}$$

Let $T_b(t) = T(t) \mid_{X_0}$ and A_b be the global attractor for $T_b(t)$.

Lemma 3.1 (Hale and Waltman [15]) *Suppose that $T(t)$ satisfies (3.1). If the following hold*

(i)   there is a $t_0 \geq 0$ such that $T(t)$ is compact for $t > t_0$;
(ii) $T(t)$ is point dissipative in X; and
(iii) $\bar{A}_b = \cup_{x \in A_b} \omega(x)$ is isolated and has an acyclic covering $\hat{M}_t$, where

$$\hat{M}_t = \{\tilde{M}_1, \tilde{M}_2, \dots, \tilde{M}_n\};$$

(iv) $W^s(\tilde{M}_i) \cap X^0 = \emptyset$ for $i = 1, 2, \dots, n$.

Then, X_0 is a uniform repeller with respect to X^0; that is, there is an $\epsilon > 0$ such that for any $x \in X^0$, $\liminf_{t \to +\infty} d(T(t)x, X_0) \geq \epsilon$.

We also need the following result to study the permanence of system (1.2).

Lemma 3.2 *There are positive constants M_1 and M_2 such that for any positive solution $(x(t), S(t), I(t))$ of system (1.2) with initial conditions (1.3),*

$$\limsup_{t \to +\infty} x(t) < M_2, \quad \limsup_{t \to +\infty} S(t) < M_1, \quad \limsup_{t \to +\infty} I(t) < M_1. \tag{3.2}$$

Proof Let $(x(t), S(t), I(t))$ be any solution of system (1.2) with initial conditions (1.3). Consider the function

$$V(t) = kx(t) + S(t + \tau) + I(t + \tau).$$

From system (1.2), we get

$$\dot{V}(t) = krx(t) - ka_{11}x^2(t) - r_1 S(t + \tau) - r_2 I(t + \tau)$$
$$= k(r + r_1)x(t) - ka_{11}x^2(t) - r_1 V(t) + (r_1 - r_2)I(t + \tau)$$
$$\leq M_1 - r_1 V(t),$$

where $M_1 = \dfrac{k(r + r_1)^2}{4a_{11}}$. Which yields $\limsup_{t \to +\infty} V(t) \leq M_1$. If we choose $M_2 = M_1/k$, then (3.2) follows. This complete the proof. $\qquad\square$

In the following, we investigate the permanence of system (1.2).

Theorem 3.1 *If $\beta S_2 > r_2$ holds, then system (1.2) is permanent.*

Proof Let $C^+([-\tau, 0], \mathbb{R}_+^3)$ denote the space of continuous functions mapping $[-\tau, 0]$ into $\mathbb{R}_+^3$. Define

$$C_1 = \left\{ (\phi_1, \phi_2, \phi_3) \in C^+([-\tau, 0], \mathbb{R}_+^3) : \phi_1(\theta) \neq 0, \phi_2(\theta) = \phi_3(\theta) = 0, \theta \in [\tau, 0] \right\},$$
$$C_2 = \left\{ (\phi_1, \phi_2, \phi_3) \in C^+([-\tau, 0], \mathbb{R}_+^3) : \phi_1(\theta)\phi_2(\theta) \neq 0, \phi_3(\theta) = 0, \theta \in [\tau, 0] \right\}.$$

Denote $C_0 = C_1 \cup C_2$, $X = C^+([-\tau, 0], \mathbb{R}_+^3)$, and $C^0 = \text{int}C^+([-\tau, 0], \mathbb{R}_+^3)$.

We verify below that the conditions in Lemma 3.1 are satisfied. By the definition of C^0 and C_0, it is easy to know that C^0 and C_0 are positively invariant. Moreover, the conditions (i) and (ii) in Lemma 3.1 are clearly satisfied. Thus, we need only to verify that the conditions (iii) and (iv) hold. System (1.2) has two constant solutions in C_0: $\bar{E}_1 \in C_1, \bar{E}_2 \in C_2$ corresponding, respectively, to $x(t) = r/a_{11}, S(t) = 0, I(t) = 0$ and $x(t) = x_2, S(t) = S_2, I(t) = 0$.

Firstly, we verify the condition (iii) of Lemma 3.1. If $(x(t), S(t), I(t))$ is a solution of system (1.2) initiating from C_1, then $\dot{x}(t) = rx(t) - a_{11}x^2(t)$, which yields $x(t) \to r/a_{11}$ as $t \to +\infty$. If $(x(t), S(t), I(t))$ is a solution of system (1.2) initiating from C_2 with $\phi_1(\theta) > 0$ and $\phi_2(\theta) > 0$, then we have

$$\dot{x}(t) = rx(t) - a_{11}x^2(t) - \frac{a_{12}x^2(t)S(t)}{1 + mx^2(t)},$$
$$\dot{S}(t) = ka_{12}\frac{x^2(t - \tau)S(t - \tau)}{1 + mx^2(t - \tau)} - r_1 S(t). \tag{3.3}$$

It is obvious that if $\beta S_2/r_2 > 1$, then $\Re_1 > 1$. Using Lemmas 3.1 and 3.2, it is easy to prove that if $\Re_1 > 1$ holds, then system (3.3) is uniformly persistent. Noting that $C_1 \cap C_2 = \emptyset$, this shows that the invariant sets $\bar{E}_1$ and $\bar{E}_2$ are isolated. Hence, $\left\{ \bar{E}_1, \bar{E}_2 \right\}$ is isolated and is an acyclic covering.

Secondly, we show that $W^s(\tilde{E}_i) \cap C^0 = \emptyset (i = 1, 2)$. Here, we restrict out attention to show $W^s(\tilde{E}_2) \cap C^0 = \emptyset$ holds because the proof of $W^s(\tilde{E}_1) \cap C^0 = \emptyset$ is simple. Assuming the contrary, namely $W^s(\tilde{E}_2) \cap C^0 \neq \emptyset$. Then, there exists a positive solution $(x(t), S(t), I(t))$ satisfying $\lim_{t \to +\infty}(x(t), S(t), I(t)) = (x_2, S_2, 0)$.

Since $\beta S_2 > r_2$, we can choose $\varepsilon > 0$ small enough such that

$$\beta(S_2 - \varepsilon) > r_2. \tag{3.4}$$

Noting that $\lim_{t\to+\infty} S(t) = S_2$, for $\varepsilon > 0$ sufficiently small satisfying (3.3), there is a $t_0 > 0$ such that if $t > t_0$, $S_2 - \varepsilon < S(t) < S_2 + \varepsilon$. For $\varepsilon > 0$ sufficiently small satisfying (3.4), it follows from the third equation of system (1.2) that for $t > t_0 + \tau$, $\dot{I}(t) > \beta(S_2 - \varepsilon)I(t) - r_2 I(t)$, which, follows from (3.4), yields $\lim_{t\to+\infty} I(t) = +\infty$. This is contradicts Lemma 3.2. Thus, we have $W^s(\tilde{E}_2) \bigcap C^0 = \emptyset$. By Lemma 3.1, we conclude that C_0 repels positive solutions of system (1.2) uniformly, and therefore, system (1.2) is permanent. The proof is complete. $\square$

4 Global Stability

Theorem 4.1 *If the endemic-coexistence equilibrium E^* of system (1.2) exists, then E^* is globally asymptotically stable provided that*
(H2): $\underline{x} \geq r/(2a_{11})$.
Here, $\underline{x}$ is the persistency constant for x satisfying $\liminf_{t\to+\infty} x \geq \underline{x}$.

Proof Assume that $(x(t), S(t), I(t))$ is any positive solution of system (1.2) with initial conditions (1.3). Denote $\phi(x(t)) = \dfrac{x^2(t)}{1 + mx^2(t)}$. Define

$$V_{11}(t) = k\left(x(t) - x^* - \int_{x^*}^x \frac{\phi(x^*)}{\phi(x(u))}du\right) + S(t) - S^* - S^* \ln\frac{S(t)}{S^*}$$
$$+ I(t) - I^* - I^* \ln\frac{I(t)}{I^*}. \tag{4.1}$$

Calculating the derivative of $V_{11}(t)$ along positive solutions of system (1.2), it follows that

$$\frac{d}{dt}V_{11}(t) = k\left(1 - \frac{\phi(x^*)}{\phi(x(t))}\right)\left[rx(t) - a_{11}x^2(t) - a_{12}\phi(x(t))S(t) - a_{13}\phi(x(t))I(t)\right]$$
$$+ \left(1 - \frac{S^*}{S(t)}\right)\left(ka_{12}\phi(x(t-\tau))S(t-\tau) - r_1 S(t) - \beta S(t)I(t)\right)$$
$$+ \left(1 - \frac{I^*}{I(t)}\right)\left(\beta S(t)I(t) + ka_{13}\phi(x(t-\tau))I(t-\tau) - r_2 I(t)\right). \tag{4.2}$$

On substituting $rx^* - a_{11}x^{*2} - a_{12}\phi(x^*)S^* - a_{13}\phi(x^*)I^* = 0$, $ka_{12}\phi(x^*)$ $S^* - r_1 S^* - \beta S^* I^* = 0$, and $\beta S^* I^* + ka_{13}\phi(x^*)I^* - r_2 I^* = 0$ into Eq. (4.2), we derive that

$$\frac{d}{dt} V_{11}(t) = k \left(1 - \frac{\phi(x^*)}{\phi(x(t))} \right) \left[rx(t) - rx^* - a_{11}(x^2(t) - x^{*2}) + a_{12}\phi(x^*)S^* + a_{13}\phi(x^*)I^* \right]$$
$$- ka_{12}\phi(x(t))S(t) + ka_{12}\phi(x(t-\tau))S(t-\tau) - ka_{13}\phi(x(t))I(t) + ka_{12}\phi(x(t-\tau))I(t-\tau)$$
$$- ka_{12}S^*\phi(x^*)\frac{\phi(x(t-\tau))S(t-\tau)}{S(t)\phi(x^*)} + ka_{12}S^*\phi(x^*)$$
$$- ka_{13}I^*\phi(x^*)\frac{\phi(x(t-\tau))I(t-\tau)}{I(t)\phi(x^*)} + ka_{13}I^*\phi(x^*)$$
$$+ ka_{12}\phi(x^*)S + ka_{13}\phi(x^*)I - r_1 S(t) + \beta S^*I - \beta I^*S(t) - r_2 I(t).$$

$$(4.3)$$

Define

$$V_{12}(t) = ka_{12} \int_{t-\tau}^{t} \left[\phi(x(u))S(u) - \phi(x^*)S^* - \phi(x^*)S^* \ln \frac{\phi(x(u))S(u)}{\phi(x^*)S^*} \right] du,$$
$$V_{13}(t) = ka_{13} \int_{t-\tau}^{t} \left[\phi(x(u))I(u) - \phi(x^*)I^* - \phi(x^*)I^* \ln \frac{\phi(x(u))I(u)}{\phi(x^*)I^*} \right] du.$$

$$(4.4)$$

Then,

$$\frac{d}{dt} V_{12}(t) = ka_{12} \left(\phi(x(t))S(t) - \phi(x(t-\tau))S(t-\tau) + \phi(x^*)S^* \ln \frac{\phi(x(t-\tau))S(t-\tau)}{\phi(x(t))S(t)} \right),$$
$$\frac{d}{dt} V_{13}(t) = ka_{13} \left(\phi(x(t))I(t) - \phi(x(t-\tau))I(t-\tau) + \phi(x^*)I^* \ln \frac{\phi(x(t-\tau))I(t-\tau)}{\phi(x(t))I(t)} \right).$$

$$(4.5)$$

Set $V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t)$. It follows from (4.1) (4.4), and (4.5) that

$$\frac{d}{dt} V_1(t) = k \left(1 - \frac{\phi(x^*)}{\phi(x(t))} \right) \left[rx(t) - rx^* - a_{11}(x^2(t) - x^{*2}) + a_{12}\phi(x^*)S^* + a_{13}\phi(x^*)I^* \right]$$
$$+ ka_{12}\phi(x^*)S^* \ln \frac{\phi(x(t-\tau))S(t-\tau)}{\phi(x(t))S(t)} + ka_{13}\phi(x^*)I^* \ln \frac{\phi(x(t-\tau))I(t-\tau)}{\phi(x(t))I(t)}$$
$$- ka_{12}S^*\phi(x^*)\frac{\phi(x(t-\tau))S(t-\tau)}{S(t)\phi(x^*)} + ka_{12}S^*\phi(x^*)$$
$$- ka_{13}I^*\phi(x^*)\frac{\phi(x(t-\tau))I(t-\tau)}{I(t)\phi(x^*)} + ka_{13}I^*\phi(x^*)$$
$$+ ka_{12}\phi(x^*)S + ka_{13}\phi(x^*)I - r_1 S(t) + \beta S^*I - \beta I^*S(t) - r_2 I(t).$$

$$(4.6)$$

Noting that

$$\ln \frac{\phi(x(t-\tau))S(t-\tau)}{\phi(x(t))S(t)} = \ln \frac{\phi(x(t-\tau))S(t-\tau)}{S(t)\phi(x^*)} + \ln \frac{\phi(x^*)}{\phi(x(t))},$$
$$\ln \frac{\phi(x(t-\tau))I(t-\tau)}{\phi(x(t))I(t)} = \ln \frac{\phi(x(t-\tau))I(t-\tau)}{I(t)\phi(x^*)} + \ln \frac{\phi(x^*)}{\phi(x(t))},$$

$$(4.7)$$

we derive from (4.7) that

$$
\begin{aligned}
\frac{d}{dt}V_1(t) =& k\left(1 - \frac{\phi(x^*)}{\phi(x(t))}\right)\left[rx(t) - rx^* - a_{11}(x^2(t) - x^{*2})\right] \\
& -ka_{12}\phi(x^*)S^*\left[\frac{\phi(x^*)}{\phi(x(t))} - 1 - \ln\frac{\phi(x^*)}{\phi(x(t))}\right] - ka_{13}\phi(x^*)I^*\left[\frac{\phi(x^*)}{\phi(x(t))} - 1 - \ln\frac{\phi(x^*)}{\phi(x(t))}\right] \\
& -ka_{12}S^*\phi(x^*)\left[\frac{\phi(x(t-\tau))S(t-\tau)}{S(t)\phi(x^*)} - 1 - \ln\frac{\phi(x(t-\tau))S(t-\tau)}{S(t)\phi(x^*)}\right] \\
& -ka_{13}I^*\phi(x^*)\left[\frac{\phi(x(t-\tau))I(t-\tau)}{I(t)\phi(x^*)} - 1 - \ln\frac{\phi(x(t-\tau))I(t-\tau)}{I(t)\phi(x^*)}\right] \\
& +ka_{12}\phi(x^*)S + ka_{13}\phi(x^*)I - r_1 S(t) + \beta S^* I - \beta I^* S(t) - r_2 I(t).
\end{aligned}
\tag{4.8}
$$

On substituting $ka_{12}\phi(x^*) = r_1 + \beta I^*$ and $ka_{13}\phi(x^*) = r_2 - \beta S^*$ into Eq. (4.8), we derive that

$$
\begin{aligned}
\frac{d}{dt}V_1(t) =& k\left(1 - \frac{\phi(x^*)}{\phi(x(t))}\right)\left[rx(t) - rx^* - a_{11}(x^2(t) - x^{*2})\right] \\
& -ka_{12}\phi(x^*)S^*\left[\frac{\phi(x^*)}{\phi(x(t))} - 1 - \ln\frac{\phi(x^*)}{\phi(x(t))}\right] - ka_{13}\phi(x^*)I^*\left[\frac{\phi(x^*)}{\phi(x(t))} - 1 - \ln\frac{\phi(x^*)}{\phi(x(t))}\right] \\
& -ka_{12}S^*\phi(x^*)\left[\frac{\phi(x(t-\tau))S(t-\tau)}{S(t)\phi(x^*)} - 1 - \ln\frac{\phi(x(t-\tau))S(t-\tau)}{S(t)\phi(x^*)}\right] \\
& -ka_{13}I^*\phi(x^*)\left[\frac{\phi(x(t-\tau))I(t-\tau)}{I(t)\phi(x^*)} - 1 - \ln\frac{\phi(x(t-\tau))I(t-\tau)}{I(t)\phi(x^*)}\right].
\end{aligned}
\tag{4.9}
$$

Noting that $\phi(x^*) = \dfrac{x^{*2}(t)}{1 + mx^{*2}(t)}$ and $\phi(x) = \dfrac{x^2(t)}{1 + mx^2(t)}$, we derive from (4.9) that

$$
\begin{aligned}
\frac{d}{dt}V_1(t) =& k\frac{(x + x^*)(x(t) - x^*)^2}{x^2(t)(1 + mx^{*2})}\left[r - a_{11}(x(t) + x^*))\right] \\
& -ka_{12}\phi(x^*)S^*\left[\frac{\phi(x^*)}{\phi(x(t))} - 1 - \ln\frac{\phi(x^*)}{\phi(x(t))}\right] - ka_{13}\phi(x^*)I^*\left[\frac{\phi(x^*)}{\phi(x(t))} - 1 - \ln\frac{\phi(x^*)}{\phi(x(t))}\right] \\
& -ka_{12}S^*\phi(x^*)\left[\frac{\phi(x(t-\tau))S(t-\tau)}{S(t)\phi(x^*)} - 1 - \ln\frac{\phi(x(t-\tau))S(t-\tau)}{S(t)\phi(x^*)}\right] \\
& -ka_{13}I^*\phi(x^*)\left[\frac{\phi(x(t-\tau))I(t-\tau)}{I(t)\phi(x^*)} - 1 - \ln\frac{\phi(x(t-\tau))I(t-\tau)}{I(t)\phi(x^*)}\right].
\end{aligned}
\tag{4.10}
$$

Since (H2) holds, there exists a constant $T > 0$ such that if $t \geq T$, $x(t) > r/(2a_{11})$. In this case, we have that, for $t \geq T$,

$$
\frac{(x + x^*)(x(t) - x^*)^2}{x^2(t)(1 + mx^{*2})}\left[r - a_{11}(x(t) + x^*))\right] \leq 0,
\tag{4.11}
$$

with equality if and only if $x = x^*$. Seeing that the function $f(x) = x - 1 - \ln x$ is always nonnegative for any $x > 0$, and $f(x) = 0$ if and only if $x = 1$, therefor, if $t \geq T$, $\dot{V}_1(t) \leq 0$, which equality if and only if $x = x^*$, $S(t) = S(t - \tau)$, $I(t) = I(t - \tau)$. We now look for the invariant subset M within the set

$$M = \{(x, S, I) : x = x^*, S(t) = S(t - \tau), I(t) = I(t - \tau)\}. \qquad (4.12)$$

Since $x = x^*$, $S(t) = S(t - \tau)$, $I(t) = I(t - \tau)$ on M, it follows from the system (1.2) that

$$0 = \dot{x}(t) = rx^* - a_{11}x^{*2} - \frac{a_{12}x^{*2}S(t)}{1 + mx^{*2}} - \frac{a_{13}x^{*2}I(t)}{1 + mx^{*2}},$$

$$0 = \dot{S}(t) = \left[k\frac{a_{12}x^{*2}}{1 + mx^{*2}} - r_1 - \beta I(t)\right] S(t), \qquad (4.13)$$

$$0 = \dot{I}(t) = \left[\beta S(t) + k\frac{a_{13}x^{*2}}{1 + mx^{*2}} - r_2\right] I(t),$$

which yields $S = S^*$ and $I = I^*$. Hence, the only invariant set in M is $\mathbb{M} = (x^*, S^*, I^*)$. Therefore, the global asymptotic stability of E^* follows from Lasalle's invariance principle for delay differential systems [16]. This completes the proof. $\qquad \square$

5 Discussion

In this paper, we have proposed and analyzed an eco-epidemiological system with time delay due to the gestation of the predator. We assumed that a transmissible disease spreading among the predator population, meanwhile, both the susceptible predator and the infected predator can catch the prey. Specially, system (1.2) has no intraspecific competition terms in the second and the third equations. In this case, under what conditions will the global stability of a feasible equilibrium of system (1.2) persists independent of the time delay? We established global asymptotic stability of the endemic-coexistence equilibrium of the system by means of Lyapunov functionals and Laselle's invariance principle. According to Theorem 4.1, we can see that the endemic-coexistence equilibrium of system (1.2) is globally asymptotically stable when the prey population is abundant enough.

References

1. Beretta, E., Hara, T., Ma, W., Takeuchi, Y.: Global asymptotic stability of an SIR epidemic model with distributed time delay. Nonlinear Anal. **47**, 4017–4115 (2001)
2. Gakkhar, S., Negi, K.: Pulse vaccination in SIRS epidemic model with non-monotonic incidence rate. Chaos Solitions Fractals **35**, 626–638 (2008)
3. Xu, R., Ma, Z.E., Wang, Z.P.: Global stability of a delayed SIRS epidemic model with saturation incidence and temporary immunity. Comput. Math. Appl. **59**, 3211–3221 (2010)
4. Xu, R.: Global dynamics of an SEIRI epidemioligical model with time delay. Appl. Math. Comput. **232**, 436–444 (2014)
5. Kermack, W.Q., Mckendrick, A.G.: Contributions to the mathematical theory of epidemics (Part I). Proc. R. Soc. A **115**, 700–721 (1927)

 6. Anderson, R.M., May, R.M.: Regulation stability of host-parasite population interactions: I. Regulatory processes. J. Anim. Ecol. **47**, 219–267 (1978)
 7. Zhang, J., Li, W., Yan, X.: Hopf bifurcation and stability of periodic solutions in a delayed eco-epidemiological system. Appl. Math. Comput. **198**, 865–876 (2008)
 8. Debasis, M.: Hopf bifurcation in an eco-epidemic model. Appl. Math. Comput. **217**, 2118–2124 (2010)
 9. Bairagi, N.: Direction and stability of bifurcating periodic solution in a delay-induced eco-epidemiological system. Int. J. Differ. Equ. 1–25 (2011)
10. Xu, R., Tian, X.H.: Global dynamics of a delayed eco-epidemiological model with Holling type-III functional response. Math. Method Appl. Sci. **37**, 2120–2134 (2014)
11. Sahoo, B.: Role of additional food in eco-epidemiological system with disease in the prey. Appl. Math. Comput. **259**, 61–79 (2015)
12. Holling, C.S.: The components of predation as revealed by a study of small mammal predation of the European pine sawfly. Can. Entomol. **91**, 293–320 (1959)
13. Holling, C.S.: Some characteristics of simple types of predation and parasitism. Can. Entomol. **91**, 385–398 (1959)
14. Holling, C.S.: The functional response of predators to prey density and its role in mimicry and population regulation. Mem. Entomol. Soc. Can. **45**, 3–60 (1965)
15. Hale, J., Waltman, P.: Persistence in infinite-dimensional systems. SIAM J. Math. Anal. **20**, 383–395 (1989)
16. Haddock, J.R., Terjéki, J.: Liapunov-Razumikhin functions and an invariance principle for functional-differential equations. J. Differ. equ. **48**, 95–122 (1983)