Chapter 2
Analysis for Inerter-Based Vibration System

Abstract This chapter investigates the influence of inerter on the natural frequencies of vibration systems. First of all, the natural frequencies of a single-degree-of-freedom (SDOF) system and a two-degree-of-freedom (TDOF) system are derived algebraically and the fact that inerter can reduce the natural frequencies of these systems is demonstrated. Then, to further investigate the influence of inerter in a general vibration system, a multi-degree-of-freedom system (MDOF) is considered. Sensitivity analysis is performed on the natural frequencies and mode shapes to demonstrate that the natural frequencies of the MDOF system can always be reduced by increasing the inertance of any inerter. The condition for a general MDOF system of which the natural frequencies can be reduced by an inerter is also derived. Finally, the influence of inerter position on the natural frequencies is investigated and the efficiency of inerter in reducing the largest natural frequencies is verified by simulating a six-degree-of-freedom system, where a reduction of more than 47% is obtained by employing only five inerters.

Keywords Natural frequency · Single-degree-of-freedom system · Two-degree-of-freedom system · Multi-degree-of-freedom system · Sensitivity analysis

2.1 Introduction

Inerter has been applied in various mechanical systems. However, among these applications, inerter always appears in some mechanical networks which possess more complex structures than the conventional networks consisting of only springs and dampers. The networks with inerters will surely be better than or at least equal to the conventional networks consisting of only springs and dampers as they can always reduce to the conventional ones when the values of element coefficients (spring stiffness, damping coefficient, or inertance) become zero or infinity (Chen et al. 2012). It is true that inerter can provide extra flexibility in structure, but the basic functionality of inerter in vibration systems has not yet been clearly understood and demonstrated.

It is well known that in a vibration system, spring can store energy, provide static support, and determine the natural frequencies, while viscous damper can dissipate

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energy, limit the amplitude of oscillation at resonance, and slightly decrease the natural frequencies if the damping is small (Tomson 1993). As shown in Smith (2002), inerter can store energy. However, for the other inherent properties of vibration systems such as natural frequencies, the influence of inerter has not been investigated before.

The objective of this chapter is to study the fundamental influence of inerter on the natural frequencies of vibration systems. The fact that inerter can reduce the natural frequencies of vibration systems is theoretically demonstrated in this chapter and the question that how to efficiently use inerter to reduce the natural frequencies is also addressed.

2.2 Preliminary

It is well known that all systems containing mass and elasticity are capable of free vibration, that is, the vibration occurring without external excitation (Tomson 1993). Natural frequency of vibration is of primary interest for such systems. For a single-degree-of-freedom spring–mass system shown in Fig. 2.1, the motion of equation can be written as

$$m\ddot{x} + c\dot{x} + kx = 0.$$

In another form,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \tag{2.1}$$

where

$$\omega_n = \sqrt{\frac{k}{m}},\ \zeta = \frac{c}{2\sqrt{mk}}.$$

Here, ω_n is called *natural frequency* and ζ is the mode damping coefficient.

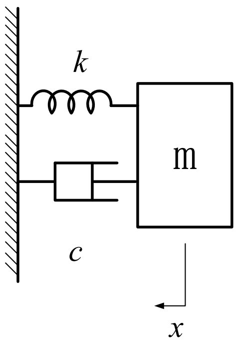

Fig. 2.1 A single-degree-of-freedom spring–mass system

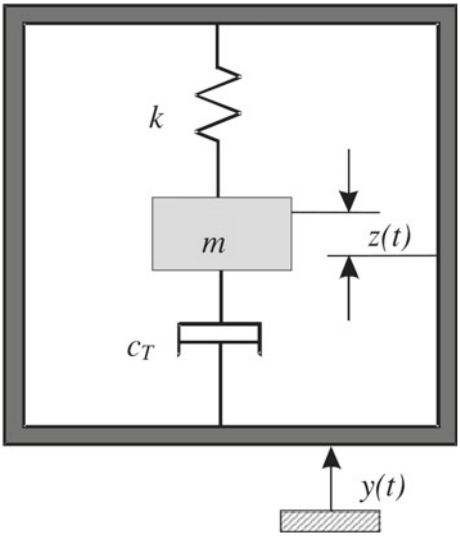

Fig. 2.2 Model of a vibration-based self-powered system

Since the influence of damping on natural frequencies is well known, only the undamped conservative systems are considered for simplicity. For the undamped system, i.e., $\zeta = 0$, the solution of (2.1) is

$$x(t) = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t,$$

where $\dot{x}(0)$ and $x(0)$ are the initial velocity and displacement. This implies that the system harmonically vibrates at the natural frequency.

For forced vibration cases, when the frequency of the excitation is equal to one of the natural frequencies, there may occur a phenomenon known as *resonance*, which may lead to excessive deflections and failure (Tse et al. 1979). In practice, it is always desirable to adjust the natural frequencies of a vibration system to avoid or induce resonance where appropriate. For example, for vibration-based self-powered systems (Beeby et al. 2006) (as shown in Fig. 2.2), the natural frequency of an embedded spring–mass system should be consistent with the environment to obtain maximum vibration power by utilizing resonance, while for the engine mounting systems (Yu et al. 2001), the natural frequency should be below the engine disturbance frequency of the engine idle speed to avoid excitation of mounting system resonance.

The traditional methods to reduce the natural frequencies of an elastic system are either decreasing the elastic stiffness or increasing the mass of the vibration system. However, this may be problematic; for example, the stiffness values of an engine mount that are too low will lead to large static and quasi-static engine displacements and damage of some engine components (Yu et al. 2001). It will be shown below that other than these two methods, a parallel-connected inerter can also effectively reduce natural frequencies.

2.3 Single-Degree-of-Freedom System

A SDOF system with an inerter is shown in Fig. 2.3. The equation of motion for free vibration of this system is

$$(m + b)\ddot{x} + kx = 0. \tag{2.2}$$

Transformation of the above equation into the standard form for vibration analysis yields

$$\ddot{x} + \omega_n^2 x = 0,$$

where $\omega_n = \sqrt{\frac{k}{m+b}}$ is called the natural frequency of the undamped system.

Proposition 1 *The natural frequency ω_n of an SDOF system is a decreasing function of the inertance b. Thus, inerter can reduce the natural frequency of an SDOF system.*

Remark 2.1 Note that in Smith (2002), one application of inerter is to simulate the mass by connecting a terminal of an inerter to the mechanical ground. Observing (2.2), one concludes that the inerter with one terminal connected to ground can effectively enlarge the mass which is connected at the other terminal.

2.4 Two-Degree-of-Freedom System

To investigate the general influence of inerter on the natural frequencies of a vibration system, a TDOF system, shown in Fig. 2.4, is investigated in this section.

The equations of motion for free vibration of this system are

$$m_1\ddot{x}_1 + k_1(x_1 - x_2) + b_1(\ddot{x}_1 - \ddot{x}_2) = 0,$$
$$m_2\ddot{x}_2 - k_1(x_1 - x_2) - b_1(\ddot{x}_1 - \ddot{x}_2) + k_2x_2 + b_2\ddot{x}_2 = 0,$$

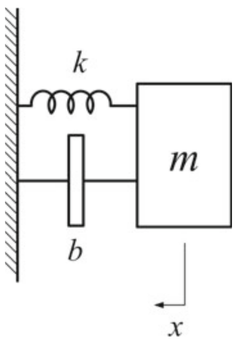

Fig. 2.3 SDOF system with an inerter

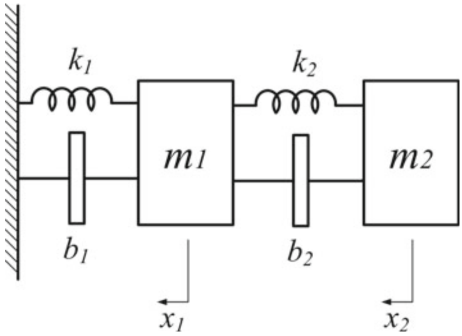

Fig. 2.4 TDOF system with two inerters

or, in a compact form,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0,$$

where **M** is called the inertia matrix and **K** is the stiffness matrix (Tse et al. 1979), and

$$\mathbf{M} = \begin{bmatrix} m_1 + b_1 & -b_1 \\ -b_1 & m_2 + b_1 + b_2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix}.$$

Note that the inertances b_1 and b_2 only exist in the inertia matrix **M**, but the positions of b_1 and b_2 are different as b_1 exists in all the elements of **M** while b_2 only appears in the last element of **M**. Since one terminal of b_2 is connected to the ground, b_2 effectively enlarges the mass m_2, which is consistent with the conclusion made in Remark 2.1.

The two natural frequencies can be obtained by solving the characteristic equation (Tse et al. 1979)

$$\begin{aligned} \Delta(\omega) &= \left|\mathbf{K} - \mathbf{M}\omega^2\right| \\ &= (m_1 m_2 + m_1(b_1 + b_2) + m_2 b_1 + b_1 b_2)\omega^4 - ((m_1 + m_2)k_1 + m_1 k_2 + \\ &\quad k_1 b_2 + b_1 k_2)\omega^2 + k_1 k_2 = 0, \end{aligned} \tag{2.3}$$

which yields

$$\omega_{n1} = \sqrt{\frac{k_1 k_2 (f_1 + f_2 - \sqrt{(f_1 - f_2)^2 + 4d_0})}{2(f_1 f_2 - d_0)}}, \tag{2.4}$$

$$\omega_{n2} = \sqrt{\frac{k_1 k_2 (f_1 + f_2 + \sqrt{(f_1 - f_2)^2 + 4d_0})}{2(f_1 f_2 - d_0)}}, \tag{2.5}$$

where $f_1 = (m_1 + m_2 + b_2)k_1$, $f_2 = (m_1 + b_1)k_2$, and $d_0 = k_1 k_2 m_1^2$.

Proposition 2 *For a TDOF system with two inerters, both natural frequencies ω_{n1} and ω_{n2} are decreasing functions of the inertances b_1 and b_2.*

Proof The monotonicity of ω_{n1} and ω_{n2} can be proven by checking the signs of the first-order derivatives of ω_{n1}^2 and ω_{n2}^2 in terms of f_1 and f_2, respectively.

$$\frac{\partial \omega_{n1}^2}{\partial f_1} = -\frac{k_1 k_2 (q_1 - q_2)}{2(d_0 - f_1 f_2)^2 \sqrt{(f_1 - f_2)^2 + 4d_0}},$$
$$\frac{\partial \omega_{n2}^2}{\partial f_1} = -\frac{k_1 k_2 (q_1 + q_2)}{2(d_0 - f_1 f_2)^2 \sqrt{(f_1 - f_2)^2 + 4d_0}},$$

where $q_1 = (d_0 + f_2^2)\sqrt{(f_1 - f_2)^2 + 4d_0}$ and $q_2 = f_1(d_0 - f_2^2) + 3f_2 d_0 + f_2^3$.

Note that $q_1 > 0$ and

$$q_1^2 - q_2^2 = 4d_0 f_2^2 (f_1 - d_0/f_2)^2,$$

so one obtains $|q_1| > |q_2|$, which implies $\frac{\partial \omega_{n1}^2}{\partial f_1} < 0$ and $\frac{\partial \omega_{n2}^2}{\partial f_1} < 0$, that is, both ω_{n1} and ω_{n2} are decreasing functions of inertance b_2.

Similarly,

$$\frac{\partial \omega_{n1}^2}{\partial f_2} = -\frac{k_1 k_2 (q_3 - q_4)}{2(d_0 - f_1 f_2)^2 \sqrt{(f_1 - f_2)^2 + 4d_0}},$$
$$\frac{\partial \omega_{n2}^2}{\partial f_2} = -\frac{k_1 k_2 (q_3 + q_4)}{2(d_0 - f_1 f_2)^2 \sqrt{(f_1 - f_2)^2 + 4d_0}},$$

where $q_3 = (d_0 + f_1^2)\sqrt{(f_1 - f_2)^2 + 4d_0}$ and $q_4 = f_2(d_0 - f_1^2) + 3f_1 d_0 + f_1^3$.

Since $q_3 > 0$ and $q_3^2 - q_4^2 = 4d_0 f_1^2 (f_2 - d_0/f_1)^2 > 0$, one has $|q_3| > |q_4|$, $\frac{\partial \omega_{n1}^2}{\partial f_2} < 0$, and $\frac{\partial \omega_{n2}^2}{\partial f_2} < 0$, that is, both ω_{n1} and ω_{n2} are decreasing functions of inertance b_1. □

2.5 Multi-degree-of-Freedom System

From the previous two sections, one sees that inerter can reduce the natural frequencies of both SDOF and TDOF systems. To find out whether this holds for any vibration system, a general MDOF system, shown in Fig. 2.5, is investigated in this section.

The equations of motion of the MDOF system shown in Fig. 2.5 are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0,$$

where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$, and

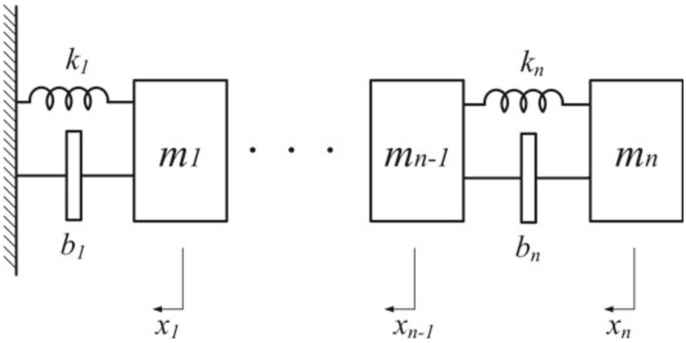

Fig. 2.5 MDOF system with inerters

$$\mathbf{M} = \begin{bmatrix} m_1 + b_1 & -b_1 & & \\ -b_1 & m_2 + b_1 + b_2 & -b_2 & \\ & \ddots & \ddots & \ddots \\ & & -b_{n-1} & m_n + b_{n-1} + b_n \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} k_1 & -k_1 & & \\ -k_1 & k_1 + k_2 & -k_2 & \\ & \ddots & \ddots & \ddots \\ & & -k_{n-1} & k_{n-1} + k_n \end{bmatrix}.$$

It is well known that the free vibration of the MDOF system can be described by the eigenvalue problem as follows (Tomson 1993; Zhao and DeWolf 1999)

$$(\mathbf{K} - \mathbf{M}\lambda_j)\boldsymbol{\varphi_j} = \mathbf{0}, \tag{2.6}$$

where $j = 1, \ldots, n$, $\omega_{ni} = \sqrt{\lambda_j}$ are the natural frequencies of this system, and $\boldsymbol{\varphi_j}$ is the jth mode shape corresponding to natural frequency ω_{nj} and is normalized to be unit-mass mode shapes, i.e., $\boldsymbol{\varphi_j}^T\mathbf{M}\boldsymbol{\varphi_j} = 1$.

Sensitivity analysis is performed on the eigenvalues and eigenvectors with respect to each inertance and the following proposition is derived.

Proposition 3 *Consider the MDOF system shown in Fig. 2.5. For an arbitrary eigenvalue λ_j, $j = 1, \ldots, n$, and an arbitrary inertance b_i, $i = 1, \ldots, n$, the following equations hold:*

$$\frac{\partial \lambda_j}{\partial b_i} = -\lambda_j \Phi_{ij}, \tag{2.7}$$

$$\frac{\partial \Phi_{ij}}{\partial b_i} = 2\Phi_{ij}\left(-\frac{1}{2}\Phi_{ij} + \sum_{l=1, l\neq j}^{n} \frac{\lambda_j}{\lambda_l - \lambda_j}\Phi_{il}\right), \tag{2.8}$$

$$\frac{\partial^2 \lambda_j}{\partial b_i^2} = 2\lambda_j \Phi_{ij} \left(\Phi_{ij} - \sum_{l=1, l \neq j}^{n} \frac{\lambda_j}{\lambda_l - \lambda_j} \Phi_{il} \right), \tag{2.9}$$

where Φ_{ij}, $j = 1, \ldots, n$, *is defined as*

$$\Phi_{ij} = \boldsymbol{\varphi_j}^T \frac{\partial \mathbf{M}}{\partial b_i} \boldsymbol{\varphi_j} = \begin{cases} \left(\boldsymbol{\varphi}_j^{(i)} - \boldsymbol{\varphi}_j^{(i+1)} \right)^2, & i \neq n \\ \left(\boldsymbol{\varphi}_j^{(n)} \right)^2, & i = n \end{cases}$$

Proof The proof is inspired by the sensitivity analysis on natural frequencies (eigenvalues) and model shapes (eigenvectors) with respect to structure parameters in Zhao and DeWolf (1999), Lin and Parker (1999), Lee and Kim (1999).

Sensitivity analysis on natural frequencies:
Considering the influence of the ith inertance b_i on the jth natural frequency ω_{nj}, the derivative of (2.6) with respect to b_i is

$$\left(\frac{\partial \mathbf{K}}{\partial b_i} - \frac{\partial \lambda_j}{\partial b_i} \mathbf{M} - \lambda_j \frac{\partial \mathbf{M}}{\partial b_i} \right) \boldsymbol{\varphi_j} + (\mathbf{K} - \lambda_j \mathbf{M}) \frac{\partial \boldsymbol{\varphi_j}}{\partial b_i} = 0. \tag{2.10}$$

Premultiplying both sides of (2.10) by $\boldsymbol{\varphi_j}^T$ and considering the relations that $\frac{\partial \mathbf{K}}{\partial b_i} = 0$ ($\mathbf{K}$ is independent of b_i), $\boldsymbol{\varphi_j}^T (\mathbf{K} - \lambda_j \mathbf{M}) = 0$, and $\boldsymbol{\varphi_j}^T \mathbf{M} \boldsymbol{\varphi_j} = 1$, one obtains

$$\frac{\partial \lambda_j}{\partial b_i} = -\lambda_j \boldsymbol{\varphi_j}^T \frac{\partial \mathbf{M}}{\partial b_i} \boldsymbol{\varphi_j} = 0. \tag{2.11}$$

Note that

$$\frac{\partial \mathbf{M}}{\partial b_i} = \begin{cases} \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 1 & -1 & & \\ & & -1 & 1 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}, & i \neq n \\ \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{bmatrix}, & i = n \end{cases} \tag{2.12}$$

where the nonzero elements for the case $i \neq n$ locate on the ith, $i + 1$th rows and ith, $i + 1$th columns.

Thus, one obtains

$$\frac{\partial \lambda_j}{\partial b_i} = \begin{cases} -\lambda_j \left(\boldsymbol{\varphi_j}^{(i)} - \boldsymbol{\varphi_j}^{(i+1)} \right)^2, & i \neq n \\ -\lambda_j \left(\boldsymbol{\varphi_j}^{(n)} \right)^2, & i = n \end{cases} \tag{2.13}$$

where $\boldsymbol{\varphi_j}^{(i)}$, $i = 1, \ldots, n$, denotes the ith element of $\boldsymbol{\varphi_j}$.

Denoting

$$\Phi_{ij} = \boldsymbol{\varphi_j}^T \frac{\partial \mathbf{M}}{\partial b_i} \boldsymbol{\varphi_j} = \begin{cases} \left(\boldsymbol{\varphi_j^{(i)}} - \boldsymbol{\varphi_j^{(i+1)}} \right)^2, & i \neq n \\ \left(\boldsymbol{\varphi_j^{(n)}} \right)^2, & i = n \end{cases}$$

where $j = 1, \ldots, n$, one obtains (2.7). □

It is clearly shown in (2.7) that

$$\frac{\partial \lambda_j}{\partial b_i} \leq 0,$$

and the equality is achieved if $\boldsymbol{\varphi_j}^{(i)} = \boldsymbol{\varphi_j}^{(i+1)}$ for $i \neq n$ or $\boldsymbol{\varphi_j}^{(n)} = 0$ for $i = n$. Since j and i are arbitrarily selected, (2.7) holds for any natural frequency with respect to any inertance b_i, which means that the natural frequencies of the MDOF system can always be reduced by increasing the inertance of any inerter.

Note that for a discrete vibration system, $\lambda_j > 0$, $j = 1, \ldots, n$ always holds (if $\lambda_j = 0$, the vibration system reduces to a lower degree-of-freedom system), then the necessary and sufficient condition for $\frac{\partial \lambda_j}{\partial b_i} \leq 0$ is

$$\frac{\partial \mathbf{M}}{\partial b_i} \geq 0. \tag{2.14}$$

Thus, one obtains the following proposition:

Proposition 4 *1. The natural frequencies of the MDOF system shown in Fig. 2.5 can always be reduced by increasing the inertance of any inerter.*

2. The natural frequencies of any MDOF system can be reduced by an inerter if the inertial matrix satisfies (2.14).

Remark 2.2 The second conclusion in Proposition 4 means that the vibration systems of which the natural frequencies can be reduced by using an inerter are not restricted to the "uni-axial" MDOF system shown in Fig. 2.5, but any MDOF system satisfying (2.14), such as full-car suspension systems (Smith and Wang 2004), train suspension systems (Wang and Liao 2009; Wang et al. 2011; Jiang et al. 2012), buildings (Wang et al. 2010), etc.

Remark 2.3 Proposition 4 is easy to interpret physically. For a small increment of inertance ε_{b_i} of a particular inerter b_i, one obtains

$$\mathbf{M} = \mathbf{M}_0 + \varepsilon_{b_i} \frac{\partial \mathbf{M}}{\partial b_i}, \tag{2.15}$$

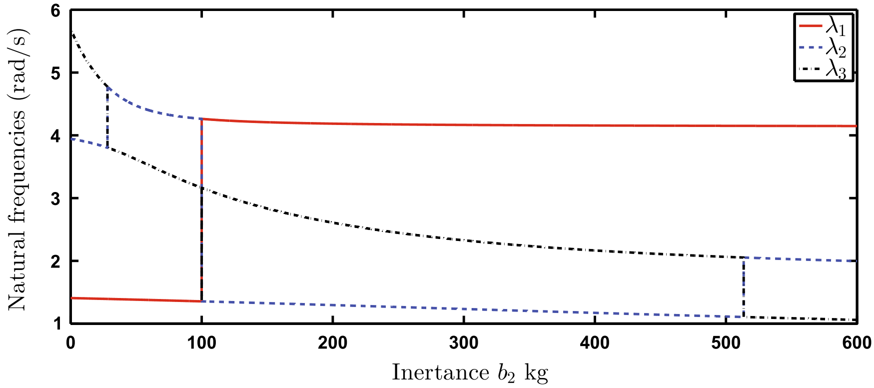

Fig. 2.6 The permutation of natural frequencies of a three-degree-of-freedom system with $m_i = 100$ kg, $k_i = 1000$ N/m, $i = 1, 2, 3$ and $b_1 = b_3 = 0$ kg, $b_2 \in [0, 600]$ kg

where $\mathbf{M_0}$ is the original inertial matrix. Sine $\frac{\partial \mathbf{M}}{\partial b_i}$ is positive semidefine, (2.15) can be interpreted as increasing the mass of the whole system, which will surely result in the reduction of natural frequencies.

Note that from Proposition 4, it seems that any natural frequency of an MDOF system will be reduced if an inerter with a relatively large value of inertance is inserted since the added inertance can always be viewed as an integration of small increments. However, this is not always true since there exist permutations of two particular natural frequencies if the divergence between two eigenvalues of the original system is not large enough or the increment of inertance ε_{b_i} is not small enough. Figure 2.6 shows the permutation of the natural frequencies of a three-degree-of-freedom system. As shown in Fig. 2.6, if one denotes the eigenvalues in the order of $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ all the time, the λ_i, $i = 1, \ldots, n$, will always decrease when the inertance increases. Hence, in the following sections, the eigenvalues are always sorted in a descending order unless otherwise stated.

Remark 2.4 Note that the equality sign can be achieved for some natural frequencies of a particular system. This means that for some particular system, it is possible to reduce part of natural frequencies while maintaining others unchanged. This fact can be demonstrated by using a Two DOF system as shown in Fig. 2.7. If $m_1 = m_2 = m$, $k_1 = k_3 = k$, $b_1 = b_3 = b$, then the natural frequencies of the system are

$$\omega_{n1} = \sqrt{\frac{k}{m + b}}, \tag{2.16}$$

$$\omega_{n2} = \sqrt{\frac{k + 2k_2}{m + b + 2b_2}}. \tag{2.17}$$

It is clear that increasing b_2 can reduce ω_{n2} but cannot reduce ω_{n1}.

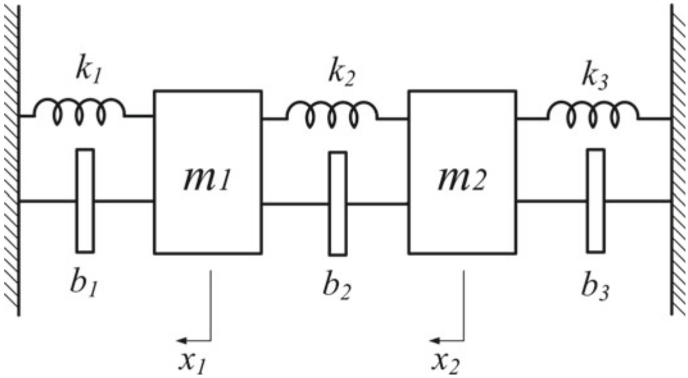

Fig. 2.7 A special TDOF system

2.6 Influence of the Inerter Position on the Natural Frequencies

The fact that inerter can reduce the natural frequencies of any MDOF system satisfying (2.14) has been demonstrated. However, for an MDOF system such as the "uni-axial" MDOF system shown in Fig. 2.5, the influence of inerter position on a specific natural frequency is still unknown. In particular, a practical problem is: for a specific natural frequency such as the largest natural frequency, where is the most efficient position to insert an inerter so that the largest reduction will be achieved? A TDOF system shown in Fig. 2.4 will be investigated in detail and analytical solutions will be derived for the TDOF system.

Considering (2.13) with $n = 2$, one obtains

$$\frac{\partial \lambda_j}{\partial b_1} = -\lambda_j \left(\boldsymbol{\varphi}_j^{(1)} - \boldsymbol{\varphi}_j^{(2)}\right)^2, \tag{2.18}$$

$$\frac{\partial \lambda_j}{\partial b_2} = -\lambda_j \left(\boldsymbol{\varphi}_j^{(2)}\right)^2, \tag{2.19}$$

where $j = 1, 2$.

For a small increment of inertance, to compare the efficiency of reducing natural frequencies in terms of b_1 and b_2, it is equivalent to compare the absolute values of the derivatives in (2.18) and (2.19). Then, the following proposition can be derived.

Proposition 5 *For a small increment of inertance and for a specific* λ_j, $j = 1, 2$, *it is more efficient to increase* b_1 *than* b_2 *if*

$$\frac{k_1}{2m_1 + b_1} < \lambda_{j0} < \frac{k_1}{b_1}, \tag{2.20}$$

or

$$\lambda_{j0} > \frac{k_2}{m_2 + b_2}, \ or\ \lambda_{j0} < \frac{k_2}{m_2 + b_2 + 2m_1}. \tag{2.21}$$

It is more efficient to increase b_2 than b_1 if

$$\lambda_{j0} > \frac{k_1}{b_1}, \ or\ \lambda_{j0} < \frac{k_1}{b_1 + 2m_1}, \tag{2.22}$$

or

$$\frac{k_2}{m_2 + b_2 + 2m_1} < \lambda_{j0} < \frac{k_2}{m_2 + b_2}, \tag{2.23}$$

where λ_{j0}, $j = 1, 2$ denote the eigenvalues of the original system.

Proof Considering (2.6), one obtains

$$\boldsymbol{\varphi_j}^{(1)} - \boldsymbol{\varphi_j}^{(2)} = \frac{\lambda_j m_1}{k_1 - \lambda_j (m_1 + b_1)} \boldsymbol{\varphi_j}^{(2)}, \tag{2.24}$$

$$= \frac{k_2 - \lambda_j (m_1 + m_2 + b_2)}{\lambda_j m_1} \boldsymbol{\varphi_j}^{(2)}, \tag{2.25}$$

where $j = 1, 2$, and (2.24) is obtained by checking the first row of (2.6) and (2.25) is obtained by summing the first and second rows of (2.6).

Note that

$$\left| \frac{\partial \lambda_j}{\partial b_1} \right| - \left| \frac{\partial \lambda_j}{\partial b_2} \right| = \lambda_j \left((\boldsymbol{\varphi_j}^{(1)} - \boldsymbol{\varphi_j}^{(2)})^2 - (\boldsymbol{\varphi_j}^{(2)})^2 \right).$$

Substituting (2.24) and (2.25), separately, one obtains the conditions in Proposition 5. □

Note that (2.20) and (2.21), (2.22) and (2.23) are equivalent, because (2.24) and (2.25) are equivalent. Proposition 5 is only applied to the case that the increment of inertance is small, as it is obtained by comparing the slopes of the tangent lines as shown in the proof of Proposition 5. If large increments of inertance are allowed for a given system that can be modeled as Fig. 2.4 and no inerter is employed in the original system, the question that which is more efficient in terms of b_1 and b_2 will be investigated as follows.

To answer this question, one needs to check two situations, where $b_2 = 0$ or $b_1 = 0$, respectively. If $b_2 = 0$, $b_1 = b$, from (2.4) and (2.5), one has

$$\omega_{n1} = \sqrt{\frac{(m_1 + m_2)k_1 + m_1 k_2 + k_2 b_1 - \sqrt{((m_1 + m_2)k_1 - m_1 k_2 - b_1 k_2)^2 + 4k_1 k_2 m_1^2}}{2(m_1 m_2 + (m_1 + m_2) b_1)}},$$

$$\omega_{n2} = \sqrt{\frac{(m_1 + m_2)k_1 + m_1 k_2 + k_2 b_1 + \sqrt{((m_1 + m_2)k_1 - m_1 k_2 - b_1 k_2)^2 + 4k_1 k_2 m_1^2}}{2(m_1 m_2 + (m_1 + m_2) b_1)}}.$$

If $b_1 = 0$, $b_2 = b$, one has

$$\omega'_{n1} = \sqrt{\frac{(m_1+m_2)k_1 + m_1k_2 + k_1b_2 - \sqrt{((m_1+m_2)k_1 - m_1k_2 + b_2k_1)^2 + 4k_1k_2m_1^2}}{2(m_1m_2 + m_1b_2)}},$$
$$\omega'_{n2} = \sqrt{\frac{(m_1+m_2)k_1 + m_1k_2 + k_1b_2 + \sqrt{((m_1+m_2)k_1 - m_1k_2 + b_2k_1)^2 + 4k_1k_2m_1^2}}{2(m_1m_2 + m_1b_2)}}.$$

The above question can be answered by comparing ω_{n1} and ω_{n2} with ω'_{n1} and ω'_{n2}, respectively. Thus, one has the following proposition.

Proposition 6 *Denote*

$$b_0 = \frac{k_1m_2(2m_1k_2 - (2m_1+m_2)k_1)}{(k_2-k_1)(m_1k_2 - (m_1+m_2)k_1)}.$$

For the larger natural frequency ω_{n2}:
If $k_2 \le (1 + \frac{m_2}{m_1})k_1$, b_1 is more efficient than b_2;
If $k_2 > (1 + \frac{m_2}{m_1})k_1$, b_1 is more efficient in $[0, b_0]$; b_2 is more efficient in $[b_0, +\infty)$.
For the smaller natural frequency ω_{n1}:
If $k_2 > (1 + \frac{m_2}{2m_1})k_1$, b_1 is more efficient than b_2;
If $k_1 \le k_2 \le (1 + \frac{m_2}{2m_1})k_1$, b_2 is more efficient in $[0, b_0]$; b_1 is more efficient in $[b_0, +\infty)$;
If $k_2 < k_1$, b_2 is more efficient than b_1.

Proof Denote $b_1 = b_2 = b$,

$$\begin{aligned}
d_1 &= 2(m_1m_2 + m_1b),\\
d_2 &= 2(m_1m_2 + (m_1+m_2)b),\\
d_3 &= (m_1+m_2)k_1 + m_1k_2 + k_2b,\\
d_4 &= (m_1+m_2)k_1 + m_1k_2 + k_1b,\\
d_5 &= \sqrt{(bk_2 + m_1k_2 - (m_1+m_2)k_1)^2 + 4k_1k_2m_1^2},\\
d_6 &= \sqrt{(bk_1 - m_1k_2 + (m_1+m_2)k_1)^2 + 4k_1k_2m_1^2},
\end{aligned}$$

and

$$\begin{aligned}
F_1(b) &= \omega_{n1}^2 - \omega'^2_{n1} = \frac{d_1d_3 - d_2d_4 - d_1d_5 + d_2d_6}{d_1d_2},\\
F_2(b) &= \omega_{n2}^2 - \omega'^2_{n2} = \frac{d_1d_3 - d_2d_4 + d_1d_5 - d_2d_6}{d_1d_2}.
\end{aligned}$$

Also denote

$$b_0 = \frac{k_1 m_2(2m_1k_2 - (2m_1 + m_2)k_1)}{(k_2 - k_1)(m_1k_2 - (m_1 + m_2)k_1)}.$$

By direct calculation, it can be easily verified that both $F_1(b) = 0$ and $F_2(b) = 0$ have solutions at 0 and b_0. However, note that $F_1(b)$ and $F_2(b)$ cannot be zero at the same time if $b \neq 0$, thus $F_1(b_0) = 0$ and $F_2(b_0) = 0$ cannot hold simultaneously. Particularly, since $b > 0$, one is more interested in the cases that $k_2 \in [k_1, (1 + m_2/(2m_1))k_1]$ and $k_2 \in [(1 + m_2/m_2)k_1, \infty)$, where $b_0 \geq 0$.

Next, it is shown that the positive value of b_0 in $k_2 \in [(1 + m_2/m_2)k_1, \infty)$ belongs to $F_2(b) = 0$ and the other one belongs to $F_1(b) = 0$. Denote

$$\Delta_2 = m_1k_2 - (m_1 + m_2)k_1,$$
$$\Delta_1^2 = \Delta_2^2 + 4k_1k_2m_1^2.$$

Then

$$d_5 = \sqrt{bk_2^2 + 2\Delta_2k_2b + \Delta_1^2} = k_2b + \Delta_2 + \frac{2k_1m_1^2}{b} + O\left(\frac{1}{b^2}\right),$$
$$d_6 = \sqrt{bk_1^2 - 2\Delta_2k_1b + \Delta_1^2} = k_1b - \Delta_2 + \frac{2k_2m_1^2}{b} + O\left(\frac{1}{b^2}\right).$$

Hence, one has

$$\begin{aligned} F_2(b) &= \frac{d_1d_3 - d_2d_4 + d_1d_5 - d_2d_6}{d_1d_2} \\ &= \frac{\Delta_2(4b^2 + 4(m_1 + m_2)b + 4m_1m_2)}{d_1d_2} - \\ &\quad \frac{4m_1(m_2k_1 - m_1(k_1m_1 - k_2(m_1 + m_2)))}{d_1d_2} + O\left(\frac{1}{b}\right). \end{aligned}$$

Note that if $\Delta_2 < 0$ and $k_2 > k_1$, or $k_1 < k_2 < (1 + m_2/m_1)k_1$, $F_2(b)$ is always negative by omitting the higher order item $O\left(\frac{1}{b}\right)$. This indicates that if $k_2 < (1 + m_2/m_1)k_1$, then $F_2(b) = 0$ only has the trivial solution 0, while if $k_2 \geq (1 + m_2/m_1)k_1$, then $F_2(b) = 0$ has solutions at 0 and b_0. Consequently, if $k_2 < (1 + m_2/m_1)k_1$, then $F_1(b) = 0$ has roots at 0 and b_0, while if $k_2 \geq (1 + m_2/m_1)k_1$, then $F_1(b) = 0$ only has a trivial solution 0.

Besides, since

$$\begin{aligned} F_1(b) &= \frac{d_1d_3 - d_2d_4 - d_1d_5 + d_2d_6}{d_1d_2}, \\ &= \frac{4m_1(m_1 + m_2)(k_1 - k_2)b - 4m_1(m_1^2(k_1 - k_2) - m_2k_1(m_1 + m_2)) - O\left(\frac{1}{b}\right)}{d_1d_2}, \end{aligned}$$

by the relationship of the coefficients and the roots of $F_1(b)$ and $F_2(b)$, one has
If $k_2 > (1 + m_2/m_1)k_1$, $F_1(b) \leq 0$ and $F_2(b) \leq 0$ for $b \in [0, b_0]$, $F_2(b) > 0$ for $b \in (b_0, \infty)$;
If $(1 + m_2/(2m_1))k_1 \leq k_2 \leq (1 + m_2/m_1)k_1$, $F_1(b) < 0$ and $F_2(b) < 0$;
If $k_1 \leq k_1 < (1 + m_2/(2m_1))k_1$, $F_1(b) \geq 0$ for $b \in [0, b_0]$, $F_1(b) < 0$ for $b \in (b_0, \infty)$, and $F_2(b) < 0$;
If $k_2 < k_1$, $F_1(b) > 0$ and $F_2(b) < 0$.

Thus, Proposition 6 and the four cases shown in Fig. 2.8 have been proved. □

Proposition 6 has addressed four cases, which are $k_2 > (1 + m_2/m_1)k_1$, $(1 + m_2/(2m_1))k_1 \leq k_2 \leq (1 + m_2/m_1)k_1$, $k_1 \leq k_2 < (1 + m_2/(2m_1))k_1$, $k_2 \leq k_1$. A numerical example is performed with $m_1 = m_2 = 100$ kg, $k_1 = 1000$ N/m and k_2 chosen as 2500, 1800, 1300, 500 N/m corresponding to the four cases in Proposition 6. The results are shown in Fig. 2.8, where one sees that in terms of the larger natural frequency, although for small increment of inertance (about 0–250 kg) b_1 is more efficient than b_2, for large increment of inertance, b_2 tends to be more efficient than b_1.

Note that the above discussion is based on TDOF systems. For a general MDOF system, a similar argument as in Proposition 5 can be employed to determine the efficiency of the position of inerter by comparing the absolute values of the derivatives. For example, consider a six-degree-of-freedom system with $m_i = 100$ kg, $i = 1, \ldots, 6$, and $k_1 = 1000$ N/m, $k_2 = 1000$ N/m, $k_3 = 2000$ N/m, $k_4 = 2000$ N/m, $k_5 = 3000$ N/m, $k_6 = 3000$ N/m. The objective is to find out the most efficient position to insert an inerter so that largest reduction of the largest natural frequency will be achieved. By direct calculation, one obtains $\left|\frac{\partial \lambda_1}{\partial b_i}\right|$, $i = 1, \ldots, 6$ as 2.759×10^{-4}, 0.0134, 0.1559, 0.8571, 1.5999, 0.4043, respectively. Note that $\left|\frac{\partial \lambda_1}{\partial b_5}\right|$ possesses the largest value. Hence, the position between m_5 and m_6 would be the most efficient position to insert an inerter, which is consistent with the simulation shown in Fig. 2.9. Another method to find the most efficient position is by using Gershgorin's Theorem (Horn and Johnson 1988), which shows that the largest absolute row sums is an upper bound of the largest eigenvalue. Hence, an efficient way to reduce the largest natural frequency is to insert the inerter between the mass m_j and m_{j+1} or m_{j-1} and m_j, where the jth absolute row sum of $\mathbf{M}^{-1}\mathbf{K}$ is the largest absolute row sum of $\mathbf{M}^{-1}\mathbf{K}$. Taking the same six-degree-of-freedom system as an example, one obtains

$$\mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} 10 & -10 & 0 & 0 & 0 & 0 \\ -10 & 20 & -10 & 0 & 0 & 0 \\ 0 & -10 & 30 & -20 & 0 & 0 \\ 0 & 0 & -20 & 40 & -20 & 0 \\ 0 & 0 & 0 & -20 & 50 & -30 \\ 0 & 0 & 0 & 0 & -30 & 60 \end{bmatrix}.$$

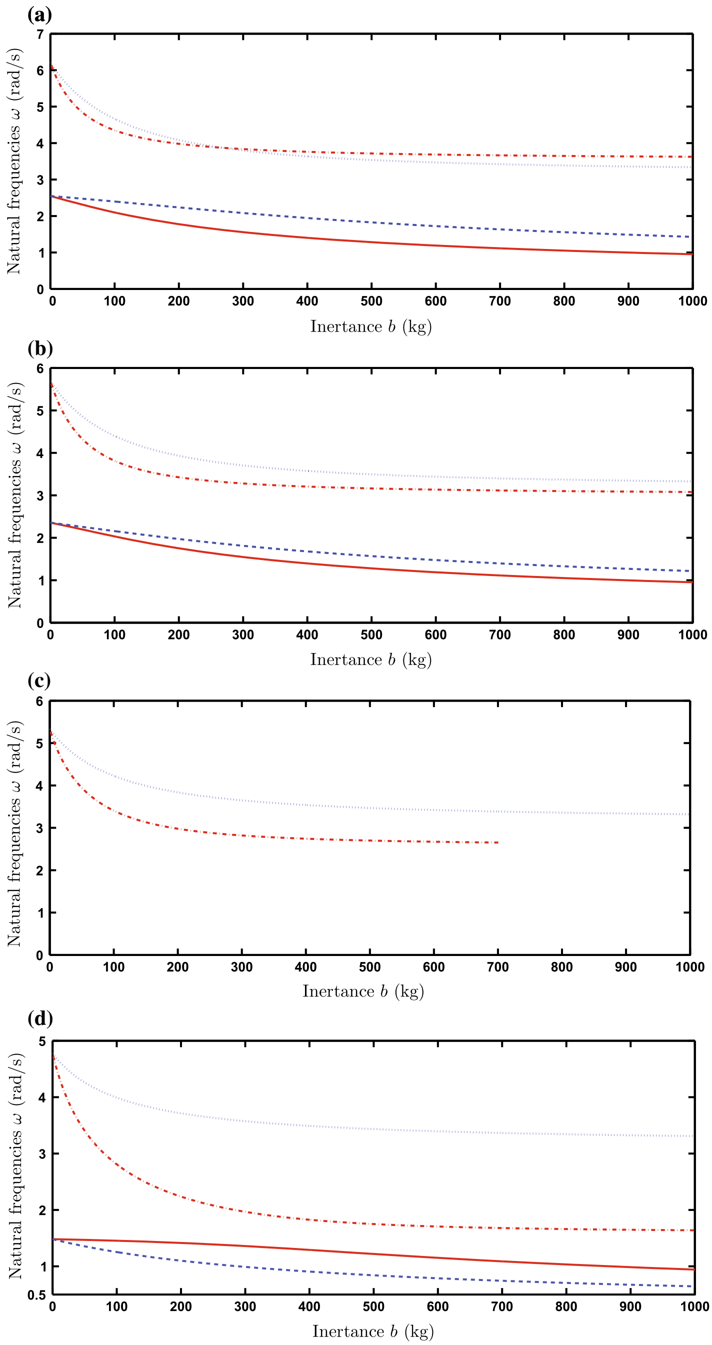

Fig. 2.8 The natural frequencies of the TDOF system. **a** $k_2 > (1 + m_2/m_1)k_1$; **b** $(1 + m_2/(2m_1))k_1 \le k_2 \le (1 + m_2/m_1)k_1$; **c** $k_1 \le k_2 < (1 + m_2/(2m_1))k_1$; **d** $k_2 \le k_1$. The red solid line: ω_{n1}; the blue dashed line: ω'_{n1}; the red dash-dot line: ω_{n2}; the blue dotted line: ω'_{n2}

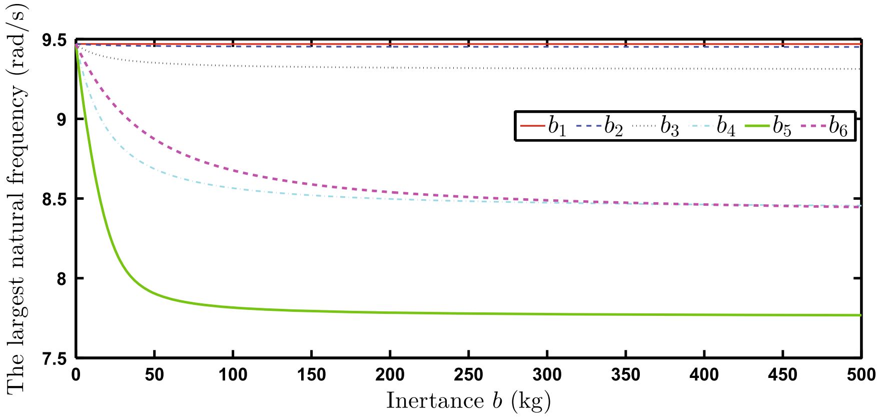

Fig. 2.9 The largest natural frequency of a six-degree-of-freedom system

The absolute row sums of $\mathbf{M}^{-1}\mathbf{K}$ are 20, 40, 60, 80, 100, and 90. Thus, one concludes that the optimal way is to insert an inerter between m_5 and m_6, which is consistent with the simulation shown in Fig. 2.9 as well.

2.7 Design Procedure and Numerical Example

The problem of reducing the largest natural frequency of a vibration system is considered in this section, where the efficiency of inerter in reducing natural frequencies will be quantitatively shown.

For the largest natural frequency, considering (2.8) and (2.9), one obtains

$$\frac{\partial \Phi_{ij}}{\partial b_i} \leq 0, \text{ and } \frac{\partial^2 \lambda_j}{\partial b_i^2} \geq 0.$$

Table 2.1 Structure model parameters

Floor masses (kg)	Stiffness coefficients (kN/m)
$m_1 = 5897$	$k_1 = 19059$
$m_2 = 5897$	$k_2 = 24954$
$m_3 = 5897$	$k_3 = 28621$
$m_4 = 5897$	$k_4 = 29093$
$m_5 = 5897$	$k_5 = 33732$
$m_6 = 6800$	$k_6 = 232$

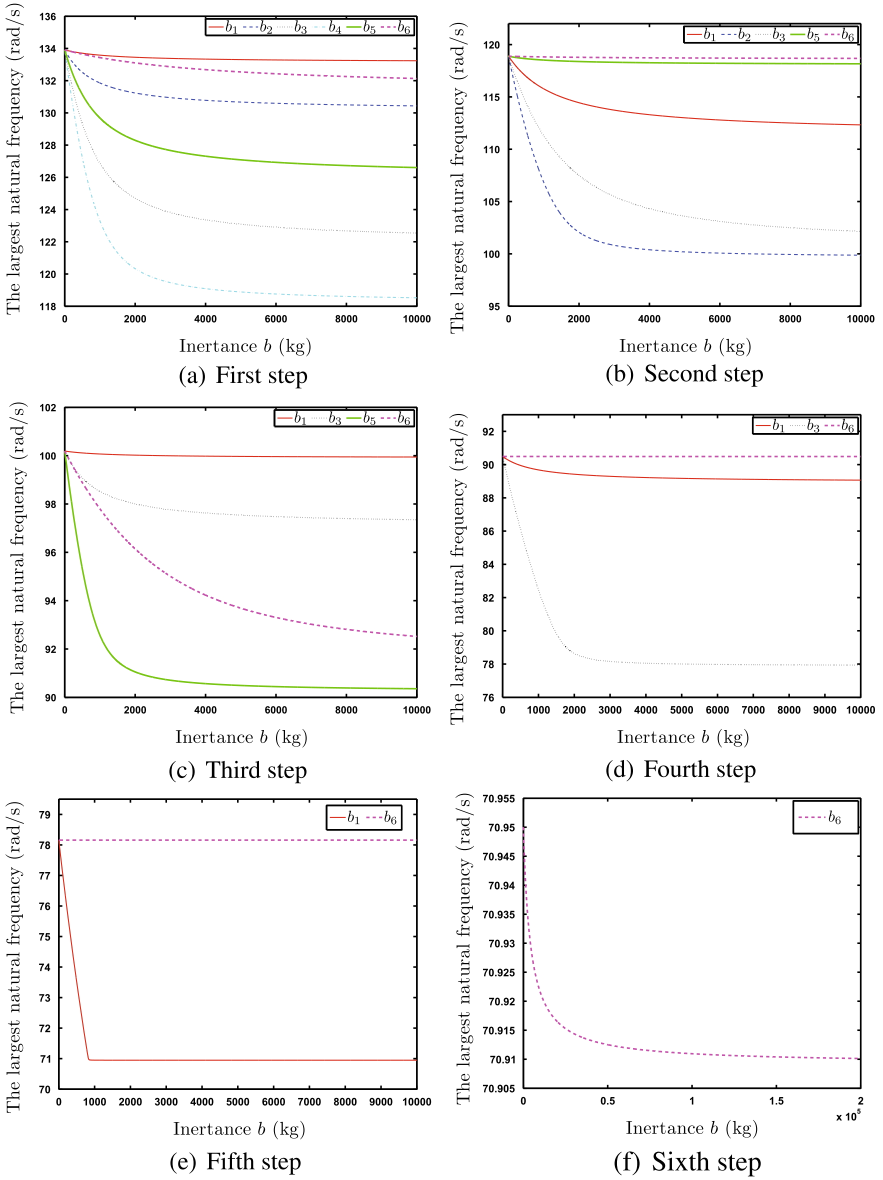

Fig. 2.10 Procedures. **a** first step; **b** second step: $b_4 = 5000$ kg; **c** third step: $b_4 = 5000$ kg, $b_2 = 5000$ kg; **d** fourth step: $b_4 = 5000$ kg, $b_2 = 5000$ kg, $b_5 = 5000$ kg; **e** fifth step: $b_4 = 5000$ kg, $b_2 = 5000$ kg, $b_5 = 5000$ kg, $b_3 = 3000$; **f** sixth step: $b_4 = 5000$ kg, $b_2 = 5000$ kg, $b_5 = 5000$ kg, $b_3 = 3000$, $b_1 = 1000$ kg

Table 2.2 Procedures and results

Steps	Inertance (kg)	ω_{max} (rad/s)	Percentages (%)
1st	$b_4 = 5000$	118.89	(11.22)
2th	$b_4 = 5000$ $b_2 = 5000$	100.19	(25.18)
3th	$b_4 = 5000$ $b_2 = 5000$ $b_5 = 5000$	90.49	(32.43)
4th	$b_4 = 5000$ $b_2 = 5000$ $b_5 = 5000$ $b_3 = 3000$	78.15	(41.64)
5th	$b_4 = 5000$ $b_2 = 5000$ $b_5 = 5000$ $b_3 = 3000$ $b_1 = 1000$	70.95	(47.02)
6th	$b_4 = 5000$ $b_2 = 5000$ $b_5 = 5000$ $b_3 = 3000$ $b_1 = 1000$ $b_6 = 1 \times 10^5$	70.91	(47.05)

Note that $\Phi_{ij} \geq 0$ and the equality is achieved with $\boldsymbol{\varphi}_j^{(i)} = \boldsymbol{\varphi}_j^{(i+1)}$ when $i \neq n$, or $\boldsymbol{\varphi}_j^{(n)} = 0$ when $i = n$, which means that for a specific inerter b_i, $i = 1, \ldots, n$, the largest natural frequency will always be reduced by increasing the inertance until the two masses connected by inerter b_i are rigidly connected.

In what follows, an intuitive and simple approach to lowering the largest natural frequency for a given structure is illustrated by inserting the inerters one by one, where the inerter in each step is placed at the most efficient position. Here, a procedure is presented to reduce the largest natural frequency of a structure discussed in Kelly et al. (1987), Ramallo et al. (2002) with parameters given in Table 2.1. Note that the largest natural frequency ω_{max} of this structure is 133.91 rad/s. The procedure to reduce ω_{max} is shown in Fig. 2.10 and Table 2.2.

Procedure description:

Step 1 Figure 2.10a shows that b_4 is the most efficient regarding the original system and for $b_4 > 5000$ kg, ω_{max} decreases slightly, and hence $b_4 = 5000$ kg is selected;

Step 2 Figure 2.10b shows that b_2 is the most efficient regarding the original system and b_4 and $b_2 > 5000$ kg, ω_{max} decreases slightly, and hence $b_2 = 5000$ kg is selected;

Step 3–Step 6 Similarly, from Fig. 2.10c to f, $b_5 = 5000$ kg, $b_3 = 3000$ kg, $b_1 = 1000$ kg, and $b_6 = 1 \times 10^5$ kg are selected, respectively.

Note that the above-illustrated approach is not optimal as the natural frequencies of a system can always be reduced by enlarging the inertance until the inertial matrix **M** became singular, where all the natural frequencies become zero. However, the efficiency of inerter in reducing natural frequencies can be clearly demonstrated by this approach. As shown in Table 2.2, attenuation about 47.05% has been obtained. It is worth pointing out that the required inertance for b_6 is 1×10^5 kg, which is quite large. However, the reduction of largest natural frequency is only improved by 0.03%. If the cost factor is considered in practice, b_6 can be omitted. In this way, only five inerters are employed.

2.8 Conclusions

This chapter has investigated the influence of inerter on the natural frequencies of vibration systems. By algebraically deriving the natural frequencies of an SDOF system and a TDOF system, the fact that inerter can reduce the natural frequencies of these systems has been clearly demonstrated. To reveal the influence of inerter on the natural frequencies of a general system, an MDOF system has been considered. Sensitivity analysis has been performed on the natural frequencies and mode shapes to demonstrate that any increment of the inertance of any inerter in an MDOF system results in the reduction of the natural frequencies. To that end, the effectiveness of inerter in reducing natural frequencies of a general vibration system has been clearly demonstrated. Finally, the influence of the inerter position has been investigated and a simple design procedure has been proposed to verify the efficiency of inerter in reducing the largest natural frequencies of vibration systems. The simulation result has shown that more than 47% reduction can be obtained with only five inerters employed in a six-degree-of-freedom vibration system.

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