Chapter 8
Impact-Induced Internal Resonance Phenomena in Nonlinear Doubly Curved Shallow Shells with Rectangular Base

Yury A. Rossikhin (†), Marina V. Shitikova, and Mohammed Salih Khalid

Abstract The problem of the low-velocity impact of an elastic sphere upon a nonlinear doubly curved shallow shell with a rectangular platform is investigated. The approach utilized in the present paper is based on the fact that during impact only the modes strongly coupled by some internal resonance condition are initiated. Such an approach differs from the Galerkin method, wherein resonance phenomena are not involved. Since is it assumed that shell's displacements are finite, then the local bearing of the shell and impactor's materials is neglected with respect to the shell deflection in the contact region. In other words, the Hertz's theory, which is traditionally in hand for solving impact problems, is not used in the present study; instead, the method of multiple time scales is adopted, which is used with much success for investigating vibrations of nonlinear systems subjected to the conditions of the internal resonance. The influence of impactor's mass on the phenomenon of the impact-induced internal resonance is revealed.

8.1 Introduction

Doubly curved panels are widely used in aeronautics, aerospace and civil engineering and are subjected to dynamic loads that can cause vibration amplitude of the order of the shell thickness, giving rise to significant non-linear phenomena (Amabili, 2005; Alijani and Amabili, 2012; Leissa and Kadi, 1971; Volmir, 1972). A review of the literature devoted to dynamic behavior of curved panels and shells could be found in Amabili and Paidoussis (2003); Amabili (2005), wherein it has been emphasized that free vibrations of doubly curved shallow shells were studied in the majority of papers either utilizing a slightly modified version of the Donnell's theory taking into

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H. Altenbach et al. (eds.), *Analysis and Modelling of Advanced Structures and Smart Systems*, Advanced Structured Materials 81,
https://doi.org/10.1007/978-981-10-6895-9_8

account the double curvature (Chia, 1988; Leissa and Kadi, 1971) or the nonlinear first-order theory of shells (Abe et al, 2000; Kobayashi and Leissa, 1995).

Large-amplitude vibrations of doubly curved shallow shells with rectangular base, simply supported at the four edges and subjected to harmonic excitation were investigated in Amabili (2005), while chaotic vibrations were analyzed in Alijani and Amabili (2012). It has been revealed that such an important nonlinear phenomenon as the occurrence of internal resonances in the problems considered in Amabili (2005); Alijani and Amabili (2012) is of fundamental importance in the study of curved shells.

In spite of the fact that the impact theory is substantially developed, there is a limited number of papers devoted to the problem of impact over geometrically nonlinear shells. Literature review on this subject could be found in Kistler and Waas (1998a,b). An analysis to predict the transient response of a thin, curved laminated plate subjected to low velocity transverse impact by a rigid object was carried out by Ramkumar and Thakar (1987), in so doing the contact force history due to the impact phenomenon was assumed to be a known linear-dependent input to the analysis. The coupled governing equations, in terms of the Airy stress function and shell deformation, were solved using Fourier series expansions for the variables.

A methodology for the stability analysis of doubly curved orthotropic shells with simply supported boundary condition and under impact load from the viewpoint of nonlinear dynamics was suggested in Zhang et al (2001). The nonlinear governing differential equations were derived based on a Donnell-type shallow shell theory, and the displacement was expanded in terms of the eigenfunctions of the linear operator of the motion equation. To analyze the influence of each single mode on the response to impact loading, only one term composed of two half-waves was used in developing the governing equation, whereas the contact force was proposed to be a sine function during the contact duration.

The review of papers dealing with the impact response of curved panels and shells shows that a finite element method and such commercial finite element software as ABAQUS or LS-DYNA and its modifications are the main numerical tools adopted by many researchers, among them: Chandrashekhara and Schoeder (1995); Liu and Swaddiwudhipong (1997); Cho et al (2000); Fu et al (2008); Fu and Mao (2008); Fu et al (2010); Gong et al (1995); Goswami (1998); Antoine and Batra (2015).

Thus, the nonlinear impact response of laminated composite cylindrical and doubly curved shells was analyzed using a modified Hertzian contact law in Chandrashekhara and Schoeder (1995) via a finite element model, which was developed based on Sander's shell theory involving shear deformation effects and nonlinearity due to large deflection. The nonlinear time dependent equations were solved using an iterative scheme and Newmark's method. Numerical results for the contact force and center deflection histories were presented for various impactor conditions, shell geometry and boundary conditions.

Later large deflection dynamic responses of laminated composite cylindrical shells under impact have been analyzed in Cho et al (2000) by the geometrically nonlinear finite element method based on a generalized Sander's shell theory with the first order shear deformation and the von Kármán large deflection assumption.

Nonlinear dynamic response for shallow spherical moderate thick shells with damage under low velocity impact has been studied in Fu and Mao (2008) by using the orthogonal collocation point method and the Newmark method to discrete the unknown variable function in space and in time domain, respectively, and the whole problem is solved by the iterative method. Further this approach was generalized for investigating dynamic response of elasto-plastic laminated composite shallow spherical shell under low velocity impact (Fu et al, 2010), and for functionally graded shallow spherical shell under low velocity impact in thermal environment (Mao et al, 2011).

The nonlinear transient response of laminated composite shell panels subjected to low velocity impact in hygrothermal environments was investigated in Swamy Naidu and Sinha (2005) using finite element method considering doubly curved thick shells involving large deformations with Green-Lagrange strains. The analysis was carried out using quadratic eight-noded isoparametric element. A modified Hertzian contact law was incorporated into the finite element program to evaluate the impact force. The nonlinear equation was solved using the Newmark average acceleration method in conjunction with an incremental modified Newton-Raphson scheme. A parametric study was carried out to investigate the effects of the curvature and side to thickness ratios of simply supported composite cylindrical and spherical shell panels.

The impact behavior and the impact-induced damage in laminated composite cylindrical shell subjected to transverse impact by a foreign object were studied in Kumar et al (2007); Kumar (2010) using three-dimensional non-linear transient dynamic finite element formulation. Non-linear system of equations resulting from non-linear strain displacement relation and non-linear contact loading was solved using Newton-Raphson incremental-iterative method. Some example problems of graphite/epoxy cylindrical shell panels were considered with variation of impactor and laminate parameters and influence of geometrical non-linear effect on the impact response and the resulting damage was investigated.

The Sander's shallow shell theory in conjunction with the Reissner-Mindlin shear deformation theory was employed in Maiti and Sinha (1996a) to develop a finite element analysis procedure to study the impact response of doubly curved laminated composite shells, in so doing the nine-noded quadratic isoparametric elements of Lagrangian family were utilized. Modified Hertzian contact law is used to calculate the contact force. Numerical results were obtained for cylindrical and spherical shells to investigate the effects of various parameters, such as radius to span ratio, span to thickness ratio, boundary condition and stacking sequence on the impact behavior of the target structure (Maiti and Sinha, 1996c,b).

A 4-noded 48 degree-of-freedom doubly curved quadrilateral shell finite element based on Kirchhoff-Love shell theory was used in Ganapathy and Rao (1998) for the nonlinear finite element analysis to predict the damage of laminated composite cylindrical and spherical shell panels subjected to low velocity impact. The large displacement stiffness matrix was formed using Green's strain tensor based on total Lagrangian approach with further utilization of an iterative scheme for solving resulting nonlinear algebraic equation by Newton-Raphson method. The load due to low velocity impact was treated as an equivalent quasi-static load and Hertzian law of contact was used for finding the peak contact force.

Recently a new approach has been proposed for the analysis of the impact interactions of nonlinear doubly curved shallow shells with rectangular base under the low-velocity impact by an elastic sphere (Rossikhin et al, 2014). It has been assumed that the shell is simply supported and partial differential equations have been obtained in terms of shell's transverse displacement and Airy's stress function. The local bearing of the shell and impactor's materials has been neglected with respect to the shell deflection in the contact region. The equations of motion have been reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements. Assuming that only two natural modes of vibrations dominate during the process of impact and applying the method of multiple time scales, the set of equations has been obtained, which allows one to find the time dependence of the contact force and to determine the contact duration and the maximal contact force.

In the present paper, the approach proposed by Rossikhin et al (2014) has been generalized for studying the influence of the impact-induced internal resonances on the low velocity impact response of a nonlinear doubly curved shallow shell with rectangular platform. Such an additional nonlinear phenomenon as the internal resonance could be examined only via analytical treatment, since any of existing numerical procedures could not catch this subtle phenomenon. Impact-induced internal resonance phenomena should be studied as their initiation during impact interaction may lead to the fact that the impacted shell could occur under extreme loading conditions resulting in its invisible and/or visible damage and even failure.

8.2 Problem Formulation and Governing Equations

In this section, first of all we recall the problem formulation following reasoning presented in Rossikhin et al (2014, 2015). Assume that a sphere (or a body of arbitrary shape but with a rounded end) of mass M moves along the z-axis towards a thin-walled doubly curved shell with thickness h, curvilinear lengths a and b, principle curvatures k_x and k_y and rectangular base, as shown in Fig. 8.1. Impact occurs at the moment $t = 0$ with the low velocity εV_0 at the point N with Cartesian coordinates x_0, y_0, where ε is a small dimensionless parameter.

According to the Donnell-Mushtari nonlinear shallow shell theory, the equations of motion could be obtained in terms of lateral deflection w and Airy's stress function ϕ (Mushtari and Galimov, 1957)

$$\frac{D}{h}\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \phi}{\partial x \partial y} + k_y\frac{\partial^2 \phi}{\partial x^2} + k_x\frac{\partial^2 \phi}{\partial y^2} + \frac{F}{h} - \rho \ddot{w}, \tag{8.1}$$

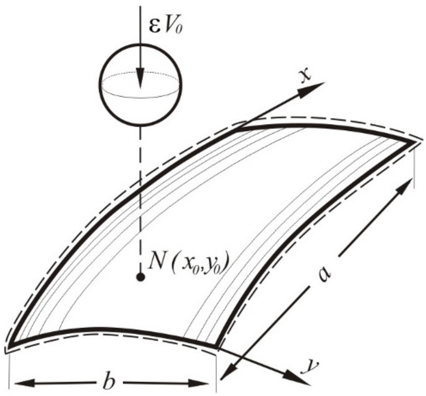

Fig. 8.1 Geometry of a doubly curved shallow shell.

$$\frac{1}{E}\left(\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}\right)=-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-k_y\frac{\partial^2 w}{\partial x^2}-k_x\frac{\partial^2 w}{\partial y^2}, \tag{8.2}$$

where $D=\frac{Eh^3}{12(1-\nu^2)}$ is the cylindrical rigidity, ρ is the density, E and ν are the elastic modulus and Poisson's ratio, respectively, t is time, $F=P(t)\delta(x-x_0)\delta(y-y_0)$ is the contact force, $P(t)$ is yet unknown function, δ is the Dirac delta function, x and y are Cartesian coordinates, overdots denote time-derivatives, $\phi(x,y)$ is the stress function which is the potential of the in-plane force resultants

$$N_x=h\frac{\partial^2\phi}{\partial y^2},\;\; N_y=h\frac{\partial^2\phi}{\partial x^2},\;\; N_{xy}=-h\frac{\partial^2\phi}{\partial x\partial y}. \tag{8.3}$$

The equation of motion of the sphere is written as

$$M\ddot{z}=-P(t) \tag{8.4}$$

subjected to the initial conditions

$$z(0)=0,\quad \dot{z}(0)=\varepsilon V_0, \tag{8.5}$$

where $z(t)$ is the displacement of the sphere, in so doing

$$z(t)=w(x_0,y_0,t). \tag{8.6}$$

Considering a simply supported shell with movable edges, the following conditions should be imposed at each edge:
at $x=0,\ a$

$$w = 0, \quad \int_0^b N_{xy} dy = 0, \quad N_x = 0, \quad M_x = 0, \tag{8.7}$$

and at $y = 0,\ b$

$$w = 0, \quad \int_0^a N_{xy} dx = 0, \quad N_y = 0, \quad M_y = 0, \tag{8.8}$$

where M_x and M_y are the moment resultants.

The suitable trial function that satisfies the geometric boundary conditions is

$$w(x,y,t) = \sum_{p=1}^{\tilde{p}} \sum_{q=1}^{\tilde{q}} \xi_{pq}(t)\ \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right), \tag{8.9}$$

where p and q are the number of half-waves in x and y directions, respectively, and $\xi_{pq}(t)$ are the generalized coordinates. Moreover, $\tilde{p}$ and $\tilde{q}$ are integers indicating the number of terms in the expansion. Substituting (8.9) in (8.6) and using (8.4), we obtain

$$P(t) = -M \sum_{p=1}^{\tilde{p}} \sum_{q=1}^{\tilde{q}} \ddot{\xi}_{pq}(t) \sin\left(\frac{p\pi x_0}{a}\right) \sin\left(\frac{q\pi y_0}{b}\right). \tag{8.10}$$

In order to find the solution of the set of Eqs. (8.1) and (8.2), it is necessary first to obtain the solution of Eq. (8.2). For this purpose, let us substitute (8.9) in the right-hand side of Eq. (8.2) and seek the solution of the equation obtained in the form

$$\phi(x,y,t) = \sum_{m=1}^{\tilde{m}} \sum_{n=1}^{\tilde{n}} A_{mn}(t) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \tag{8.11}$$

where $A_{mn}(t)$ are yet unknown functions.

Substituting (8.9) and (8.11) in Eq. (8.2) and using the orthogonality conditions of sines within the segments $0 \le x \le a$ and $0 \le y \le b$, we have

$$A_{mn}(t) = \frac{E}{\pi^2} K_{mn} \xi_{mn}(t) + \frac{4E}{a^3 b^3} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2} \sum_k \sum_l \sum_p \sum_q B_{pqklmn} \xi_{pq}(t) \xi_{kl}(t), \tag{8.12}$$

where

$$B_{pqklmn} = pqkl B^{(2)}_{pqklmn} - p^2 l^2 B^{(1)}_{pqklmn},$$

$$B^{(1)}_{pqklmn} = \int_0^a \int_0^b \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right) \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{l\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dxdy,$$

$$B^{(2)}_{pqklmn} = \int_0^a \int_0^b \cos\left(\frac{p\pi x}{a}\right) \cos\left(\frac{q\pi y}{b}\right) \cos\left(\frac{k\pi x}{a}\right) \cos\left(\frac{l\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dxdy,$$

$$K_{mn} = \left(k_y \frac{m^2}{a^2} + k_x \frac{n^2}{b^2}\right)^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{-2}.$$

Substituting then (8.9)–(8.12) in Eq. (8.1) and using the orthogonality condition of sines within the segments $0 \le x \le a$ and $0 \le y \le b$, we obtain an infinite set of coupled nonlinear ordinary differential equations of the second order in time for defining the generalized coordinates

$$\begin{aligned}\ddot{\xi}_{mn}(t) + \Omega_{mn}^2 \xi_{mn}(t) &+ \frac{8\pi^2 E}{a^3 b^3 \rho} \sum_p \sum_q \sum_k \sum_l B_{pqklmn}\left(K_{kl} - \frac{1}{2} K_{mn}\right) \xi_{pq}(t) \xi_{kl}(t) \\ &+ \frac{32\pi^4 E}{a^6 b^6 \rho} \sum_r \sum_s \sum_i \sum_j \sum_k \sum_l \sum_p \sum_q B_{rsijmn} B_{pqklij} \xi_{rs}(t) \xi_{pq}(t) \xi_{kl}(t) \\ &+ \frac{4M}{ab\rho h} \sin\left(\frac{m\pi x_0}{a}\right) \sin\left(\frac{n\pi y_0}{b}\right) \sum_p \sum_q \ddot{\xi}_{pq}(t) \sin\left(\frac{p\pi x_0}{a}\right) \sin\left(\frac{q\pi y_0}{b}\right) = 0, \end{aligned} \quad (8.13)$$

where Ω_{mn} is the natural frequency of the mnth mode of the shell vibration defined as

$$\Omega_{mn}^2 = \frac{E}{\rho}\left[\frac{\pi^4 h^2}{12(1-\nu^2)} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + K_{mn}\right]. \quad (8.14)$$

The last term in each equation from (8.13) describes the influence of the coupled impact interaction of the target with the impactor of the mass M applied at the point with the coordinates x_0, y_0.

Note that at $M = 0$ Eqs. (8.13) are reduced to the equations describing free vibrations of shallow shells with a rectangular platform, which have been proposed in Kobayashi and Leissa (1995); Leissa and Kadi (1971) and wherein curvature effects on shallow shell vibrations, and in particular on natural frequencies (8.14), have been studied.

It is known (Anderson et al, 1994; Nayfeh, 1973a) that during nonstationary excitation of thin bodies not all possible modes of vibration would be excited. Moreover, the modes which are strongly coupled by any of the so-called internal resonance conditions are initiated and dominate in the process of vibration, in so doing the types of modes to be excited are dependent on the character of the external excitation.

Thus, in order to study the additional nonlinear phenomenon induced by the coupled impact interaction due to Eq. (8.13), we suppose that only two natural modes of vibrations are excited during the process of impact, namely, $\Omega_{\alpha\beta}$ and $\Omega_{\gamma\delta}$.

Then the set of Eqs. (8.13) is reduced to the following two nonlinear differential equations (Rossikhin et al, 2014):

$$p_{11}\ddot{\xi}_{\alpha\beta} + p_{12}\ddot{\xi}_{\gamma\delta} + \Omega_{\alpha\beta}^2 \xi_{\alpha\beta} + p_{13}\xi_{\alpha\beta}^2 + p_{14}\xi_{\gamma\delta}^2 + p_{15}\xi_{\alpha\beta}\xi_{\gamma\delta} + p_{16}\xi_{\alpha\beta}^3 + p_{17}\xi_{\alpha\beta}\xi_{\gamma\delta}^2 = 0, \quad (8.15)$$

$$p_{21}\ddot{\xi}_{\alpha\beta} + p_{22}\ddot{\xi}_{\gamma\delta} + \Omega^2_{\gamma\delta}\xi_{\gamma\delta} + p_{23}\xi^2_{\gamma\delta} + p_{24}\xi^2_{\alpha\beta} + p_{25}\xi_{\alpha\beta}\xi_{\gamma\delta} + p_{26}\xi^3_{\gamma\delta} + p_{27}\xi^2_{\alpha\beta}\xi_{\gamma\delta} = 0, \tag{8.16}$$

where

$$p_{11} = 1 + \frac{4M}{\rho hab}\, s_1^2, \quad p_{22} = 1 + \frac{4M}{\rho hab}\, s_2^2, \quad p_{12} = p_{21} = \frac{4M}{\rho hab}\, s_1 s_2,$$

$$s_1 = \sin\left(\frac{\alpha\pi x_0}{a}\right)\sin\left(\frac{\beta\pi y_0}{b}\right), \quad s_2 = \sin\left(\frac{\gamma\pi x_0}{a}\right)\sin\left(\frac{\delta\pi y_0}{b}\right),$$

$$p_{13} = \frac{8\pi^2 E}{a^3 b^3 \rho}\, B_{\alpha\beta\alpha\beta\alpha\beta}\, \frac{1}{2}\, K_{\alpha\beta}, \quad p_{14} = \frac{8\pi^2 E}{a^3 b^3 \rho}\, B_{\gamma\delta\gamma\delta\alpha\beta}\left(K_{\gamma\delta} - \frac{1}{2}\, K_{\alpha\beta}\right),$$

$$p_{15} = \frac{8\pi^2 E}{a^3 b^3 \rho}\left[B_{\gamma\delta\alpha\beta\alpha\beta}\, \frac{1}{2}\, K_{\alpha\beta} + B_{\alpha\beta\gamma\delta\alpha\beta}\left(K_{\gamma\delta} - \frac{1}{2}\, K_{\alpha\beta}\right)\right],$$

$$p_{23} = \frac{8\pi^2 E}{a^3 b^3 \rho}\, B_{\alpha\beta\alpha\beta\gamma\delta}\left(K_{\alpha\beta} - \frac{1}{2}\, K_{\gamma\delta}\right), \quad p_{24} = \frac{8\pi^2 E}{a^3 b^3 \rho}\, B_{\gamma\delta\gamma\delta\gamma\delta}\, \frac{1}{2}\, K_{\gamma\delta},$$

$$p_{25} = \frac{8\pi^2 E}{a^3 b^3 \rho}\left[B_{\alpha\beta\gamma\delta\gamma\delta}\, \frac{1}{2}\, K_{\gamma\delta} + B_{\gamma\delta\alpha\beta\gamma\delta}\left(K_{\alpha\beta} - \frac{1}{2}\, K_{\gamma\delta}\right)\right],$$

$$p_{16} = \frac{32\pi^2 E}{a^3 b^3 \rho}\sum_i\sum_j B_{\alpha\beta ij\alpha\beta} B_{\alpha\beta\alpha\beta ij}, \quad p_{26} = \frac{32\pi^2 E}{a^3 b^3 \rho}\sum_i\sum_j B_{\gamma\delta ij\gamma\delta} B_{\gamma\delta\gamma\delta ij},$$

$$p_{17} = \frac{32\pi^2 E}{a^3 b^3 \rho}\sum_i\sum_j \left(B_{\alpha\beta ij\alpha\beta} B_{\gamma\delta\gamma\delta ij} + B_{\gamma\delta ij\alpha\beta} B_{\alpha\beta\gamma\delta ij} + B_{\gamma\delta ij\alpha\beta} B_{\gamma\delta\alpha\beta ij}\right),$$

$$p_{27} = \frac{32\pi^2 E}{a^3 b^3 \rho}\sum_i\sum_j \left(B_{\alpha\beta ij\gamma\delta} B_{\gamma\delta\gamma\delta ij} + B_{\gamma\delta ij\gamma\delta} B_{\alpha\beta\gamma\delta ij} + B_{\gamma\delta ij\gamma\delta} B_{\gamma\delta\alpha\beta ij}\right).$$

8.3 Method of Solution

In order to solve a set of two nonlinear Eqs. (8.15) and (8.16), we apply the method of multiple time scales (Nayfeh, 1973b) via the following expansions:

$$\xi_{ij}(t) = \varepsilon X^1_{ij}(T_0, T_1, T_2) + \varepsilon^2 X^2_{ij}(T_0, T_1, T_2) + \varepsilon^3 X^3_{ij}(T_0, T_1, T_2), \tag{8.17}$$

where $ij = \alpha\beta$ or $\gamma\delta$, $T_n = \varepsilon^n t$ are new independent variables, among them: $T_0 = t$ is a fast scale characterizing motions with the natural frequencies, and $T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$ are slow scales characterizing the modulation of the amplitudes and phases of the modes with nonlinearity.

Considering that

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2\left(D_1^2 + 2D_0 D_2\right),$$

where $ij = \alpha\beta$ or $\gamma\delta$, and $D_i^n = \partial^n/\partial T_i^n$ $(n = 1,2,\ \ i = 0,1)$, and substituting the proposed solution (8.17) in (8.15) and (8.16), after equating the coefficients at like powers of ε to zero, we are led to a set of recurrence equations to various orders:
to order ε

$$p_{11}D_0^2X_1^1 + p_{12}D_0^2X_2^1 + \Omega_1^2X_1^1 = 0, \tag{8.18}$$

$$p_{21}D_0^2X_1^1 + p_{22}D_0^2X_2^1 + \Omega_2^2X_2^1 = 0; \tag{8.19}$$

to order ε^2

$$\begin{aligned} p_{11}D_0^2X_1^2 \ + \ p_{12}D_0^2X_2^2 + \Omega_1^2X_1^2 &= -2p_{11}D_0D_1X_1^1 - 2p_{12}D_0D_1X_2^1 \\ &- \ p_{13}(X_1^1)^2 - p_{14}(X_2^1)^2 - p_{15}X_1^1X_2^1, \end{aligned} \tag{8.20}$$

$$\begin{aligned} p_{21}D_0^2X_1^2 \ + \ p_{22}D_0^2X_2^2 + \Omega_2^2X_2^2 &= -2p_{21}D_0D_1X_1^1 - 2p_{22}D_0D_1X_2^1 \\ &- \ p_{23}(X_1^1)^2 - p_{24}(X_2^1)^2 - p_{25}X_1^1X_2^1, \end{aligned} \tag{8.21}$$

to order ε^3

$$\begin{aligned} p_{11}D_0^2X_1^3 \ + \ p_{12}D_0^2X_2^3 + \Omega_1^2X_1^3 &= -2p_{11}D_0D_1X_1^2 - 2p_{12}D_0D_1X_2^2 \\ &- \ p_{11}\left(D_1^2 + 2D_0D_2\right)X_1^1 - p_{12}\left(D_1^2 + 2D_0D_2\right)X_2^1 - 2p_{13}X_1^1X_1^2 \\ &- \ 2p_{14}X_2^1X_2^2 - p_{15}\left(X_1^1X_2^2 + X_1^2X_2^1\right) - p_{16}\left(X_1^1\right)^3 - p_{17}X_1^1\left(X_2^1\right)^2, \end{aligned} \tag{8.22}$$

$$\begin{aligned} p_{21}D_0^2X_1^3 \ + \ p_{22}D_0^2X_2^3 + \Omega_2^2X_2^3 &= -2p_{21}D_0D_1X_1^2 - 2p_{22}D_0D_1X_2^2 \\ &- \ p_{21}\left(D_1^2 + 2D_0D_2\right)X_1^1 - p_{22}\left(D_1^2 + 2D_0D_2\right)X_2^1 - 2p_{23}X_2^1X_2^2 \\ &- \ 2p_{24}X_1^1X_1^2 - p_{25}\left(X_1^1X_2^2 + X_1^2X_2^1\right) - p_{26}\left(X_2^1\right)^3 - p_{27}\left(X_1^1\right)^2X_2^1, \end{aligned} \tag{8.23}$$

where for simplicity is it denoted $X_1^1 = X_{\alpha\beta}^1$, $X_2^1 = X_{\gamma\delta}^1$, $X_1^2 = X_{\alpha\beta}^2$, $X_2^2 = X_{\gamma\delta}^2$, $\Omega_1 = \Omega_{\alpha\beta}$, and $\Omega_2 = \Omega_{\gamma\delta}$.

8.3.1 Solution of Equations at Order of ε

We seek the solution of (8.18) and (8.19) in the form:

$$X_1^1 = A_1\left(T_1,T_2\right)e^{i\omega_1T_0} + A_2\left(T_1,T_2\right)e^{i\omega_2T_0} + \text{cc}, \tag{8.24}$$

$$X_2^1 = \alpha_1A_1\left(T_1,T_2\right)e^{i\omega_1T_0} + \alpha_2A_2\left(T_1,T_2\right)e^{i\omega_2T_0} + \text{cc}, \tag{8.25}$$

where $A_1(T_1,T_2)$ and $A_2(T_1,T_2)$ are unknown complex functions, cc is the complex conjugate part to the preceding terms, and $\bar{A}_1(T_1,T_2)$ and $\bar{A}_2(T_1,T_2)$ are their complex conjugates, ω_1 and ω_2 are unknown frequencies of the coupled process of impact interaction of the impactor and the target, and α_1 and α_2 are yet unknown coefficients.

Substituting (8.24) and (8.25) in (8.18) and (8.19) and gathering the terms with $e^{i\omega_1 T_0}$ and $e^{i\omega_2 T_0}$ yield

$$\left(-p_{11}\omega_1^2 - p_{12}\alpha_1\omega_1^2 + \Omega_1^2\right)A_1 e^{i\omega_1 T_0} + \left(-p_{11}\omega_2^2 - p_{12}\alpha_2\omega_2^2 + \Omega_1^2\right)A_2 e^{i\omega_2 T_0} + \text{cc} = 0, \tag{8.26}$$

$$\left(-p_{21}\omega_1^2 - p_{22}\alpha_1\omega_1^2 + \alpha_1\Omega_2^2\right)A_1 e^{i\omega_1 T_0} + \left(-p_{21}\omega_2^2 - p_{22}\alpha_2\omega_2^2 + \Omega_2^2\alpha_2\right)A_2 e^{i\omega_2 T_0} + \text{cc} = 0. \tag{8.27}$$

In order to satisfy Eqss (8.26) and (8.27), it is a need to vanish to zero each bracket in these equations. As a result, from four different brackets we have

$$\alpha_1 = -\frac{p_{11}\omega_1^2 - \Omega_1^2}{p_{12}\omega_1^2}, \tag{8.28}$$

$$\alpha_1 = -\frac{p_{21}\omega_1^2}{p_{22}\omega_1^2 - \Omega_2^2}, \tag{8.29}$$

$$\alpha_2 = -\frac{p_{11}\omega_2^2 - \Omega_1^2}{p_{12}\omega_2^2}, \tag{8.30}$$

$$\alpha_2 = -\frac{p_{21}\omega_2^2}{p_{22}\omega_2^2 - \Omega_2^2}. \tag{8.31}$$

Since the left-hand side parts of relationships (8.28) and (8.29), as well as (8.30) and (8.31) are equal, then their right-hand side parts should be equal as well. Now equating the corresponding right-hand side parts of (8.28), (8.29) and (8.30), (8.31), we are led to one and the same characteristic equation for determining the frequencies ω_1 and ω_2:

$$\left(\Omega_1^2 - p_{11}\omega^2\right)\left(\Omega_2^2 - p_{22}\omega^2\right) - p_{12}^2\omega^4 = 0, \tag{8.32}$$

hence it follows that

$$\omega_{1,2}^2 = \frac{\left(p_{22}\Omega_1^2 + p_{11}\Omega_2^2\right) \pm \sqrt{\left(p_{22}\Omega_1^2 - p_{11}\Omega_2^2\right)^2 + 4\Omega_1^2\Omega_2^2 p_{12}^2}}{2\left(p_{11}p_{22} - p_{12}^2\right)}. \tag{8.33}$$

Reference to relationships (8.33) shows that the frequencies of the mechanical system "target+impactor", ω_1 and ω_2, depend on the natural frequencies of the target, Ω_1 and Ω_2, and coefficients p_{11}, p_{12} and p_{22}, which in their turn depend on the impactor's mass M and coordinates of the point of impact. Therefore, as the impactor mass $M \to 0$, the frequencies ω_1 and ω_2 tend to the natural frequencies of the shell vibrations Ω_1 and Ω_2, respectively. Coefficients s_1 and s_2 depend on the numbers of the natural modes involved in the process of impact interaction, $\alpha\beta$ and $\gamma\delta$, and on the coordinates of the contact force application x_0, y_0, resulting in the fact that their particular combinations could vanish coefficients s_1 and s_2 and, thus, coefficients $p_{12} = p_{21} = 0$.

8.3.2 Solution of Equations at Order of ε^2

Now substituting (8.24) and (8.25) in (8.20) and (8.21), we obtain

$$\begin{aligned}
&p_{11}D_0^2X_1^2 + p_{12}D_0^2X_2^2 + \Omega_1^2X_1^2 = -2i\omega_1(p_{11}+\alpha_1 p_{12})e^{i\omega_1 T_0}D_1A_1\\
&-2i\omega_2(p_{11}+\alpha_2 p_{12})e^{i\omega_2 T_0}D_1A_2 - (p_{13}+\alpha_1^2 p_{14}+\alpha_1 p_{15})A_1\left[A_1e^{2i\omega_1 T_0}+\bar{A}_1\right]\\
&-(p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15})A_2\left[A_2e^{2i\omega_2 T_0}+\bar{A}_2\right]\\
&-2\left[p_{13}+\alpha_1\alpha_2 p_{14}+(\alpha_1+\alpha_2)p_{15}\right]A_1\left[A_2e^{i(\omega_1+\omega_2)T_0}+\bar{A}_2e^{i(\omega_1-\omega_2)T_0}\right]+\text{cc},
\end{aligned} \tag{8.34}$$

$$\begin{aligned}
&p_{21}D_0^2X_1^2 + p_{22}D_0^2X_2^2 + \Omega_2^2X_2^2 = -2i\omega_1(p_{21}+\alpha_1 p_{22})e^{i\omega_1 T_0}D_1A_1\\
&-2i\omega_2(p_{21}+\alpha_2 p_{22})e^{i\omega_2 T_0}D_1A_2 - (p_{23}+\alpha_1^2 p_{24}+\alpha_1 p_{25})A_1\left[A_1e^{2i\omega_1 T_0}+\bar{A}_1\right]\\
&-(p_{23}+\alpha_2^2 p_{24}+\alpha_2 p_{25})A_2\left[A_2e^{2i\omega_2 T_0}+\bar{A}_2\right]\\
&-2\left[p_{23}+\alpha_1\alpha_2 p_{24}+(\alpha_1+\alpha_2)p_{25}\right]A_1\left[A_2e^{i(\omega_1+\omega_2)T_0}+\bar{A}_2e^{i(\omega_1-\omega_2)T_0}\right]+\text{cc}.
\end{aligned} \tag{8.35}$$

Reference to Eqs. (8.34) and (8.35) shows that the following two-to-one internal resonance could occur:

$$\omega_1 = 2\omega_2. \tag{8.36}$$

8.3.3 Impact-induced internal resonance $\omega_1 = 2\omega_2$

Suppose that, when the frequencies ω_1 and ω_2 are coupled by the two-to-one internal resonance (8.36), the functions A_1 and A_2 depend only on the time T_1. Then Eqs. (8.34) and (8.35) could be rewritten in the following form:

$$p_{11}D_0^2X_1^2 + p_{12}D_0^2X_2^2 + \Omega_1^2X_1^2 = B_1\exp(i\omega_1 T_0) + B_2\exp(i\omega_2 T_0) + \text{Reg} + \text{cc}, \tag{8.37}$$

$$p_{21}D_0^2X_1^2 + p_{22}D_0^2X_2^2 + \Omega_2^2X_2^2 = B_3\exp(i\omega_1 T_0) + B_4\exp(i\omega_2 T_0) + \text{Reg} + \text{cc}, \tag{8.38}$$

where all regular terms are designated by Reg, and

$$B_1 = -2i\Omega_1^2\omega_1^{-1}D_1A_1 - \left(p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15}\right)A_2^2,$$

$$B_2 = -2i\Omega_1^2\omega_2^{-1}D_1A_2 - 2[p_{13} + \alpha_1\alpha_2 p_{14} + (\alpha_1 + \alpha_2)\, p_{15}]A_1\bar{A}_2,$$
$$B_3 = -2i\Omega_2^2\omega_1^{-1}\alpha_1 D_1 A_1 - \left(p_{23} + \alpha_2^2 p_{24} + \alpha_2 p_{25}\right)A_2^2,$$
$$B_4 = -2i\Omega_2^2\omega_2^{-1}\alpha_2 D_1 A_2 - 2[p_{23} + \alpha_1\alpha_2 p_{24} + (\alpha_1 + \alpha_2)\, p_{25}]A_1\bar{A}_2.$$

Let us show that the terms with the exponents $\exp(\pm i\omega_i T_0)$ $(i = 1,2)$ produce circular terms. For this purpose we choose a particular solution in the form

$$\begin{aligned} X_{1\,p}^2 &= C_1 \exp(i\omega_1 T_0) + \text{cc},\\ X_{2\,p}^2 &= C_2 \exp(i\omega_1 T_0) + \text{cc}, \end{aligned} \tag{8.39}$$

or

$$\begin{aligned} X_{1\,p}^2 &= C_1' \exp(i\omega_2 T_0) + \text{cc},\\ X_{2\,p}^2 &= C_2' \exp(i\omega_2 T_0) + \text{cc}, \end{aligned} \tag{8.40}$$

where C_1, C_2 and C_1', C_2' are arbitrary constants.

Substituting the proposed solution (8.39) or (8.40) in (8.37) and (8.38), we are led to the following sets of equations, respectively:

$$\begin{cases} p_{12}\omega_1^2(\alpha_1 C_1 - C_2) = B_1,\\ p_{21}\omega_1^2\left(-C_1 + \frac{1}{\alpha_1}\, C_2\right) = B_3, \end{cases} \tag{8.41}$$

or

$$\begin{cases} p_{12}\omega_2^2\left(\alpha_2 C_1' - C_2'\right) = B_2,\\ p_{21}\omega_2^2\left(-C_1' + \frac{1}{\alpha_2}\, C_2'\right) = B_4. \end{cases} \tag{8.42}$$

From the sets of Eqs. (8.41) and (8.42) it is evident that the determinants comprised from the coefficients standing at C_1, C_2 and C_1', C_2' are equal to zero, therefore, it is impossible to determine the arbitrary constants C_1, C_2 and C_1', C_2' of the particular solutions (8.39) and (8.40), what proves the above proposition concerning the circular terms.

In order to eliminate the circular terms, the terms proportional to $e^{i\omega_1 T_0}$ and $e^{i\omega_2 T_0}$ should be vanished to zero putting $B_i = 0$ $(i = 1,2,3,4)$. So we obtain four equations for defining two unknown amplitudes $A_1(t)$ and $A_2(t)$. However, it is possible to show that not all of these four equations are linear independent from each other. For this purpose, let us first apply the operators $(p_{22}D_0^2 + \Omega_2^2)$ and $(-p_{12}D_0^2)$ to (8.37) and (8.38), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_2^2. If we apply the operators $(-p_{12}D_0^2)$ and $(p_{11}D_0^2 + \Omega_1^2)$ to (8.37) and (8.38), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_1^2. Thus, we obtain

$$\begin{aligned} &\left[(p_{11}p_{22} - p_{12}^2)D_0^4 + (p_{11}\Omega_2^2 + p_{22}\Omega_1^2)D_0^2 + \Omega_1^2\Omega_2^2\right]X_1^2\\ &\quad = \left[(p_{22}D_0^2 + \Omega_2^2)B_1 - p_{12}D_0^2 B_3\right]\exp(i\omega_1 T_0)\\ &\quad + \left[(p_{22}D_0^2 + \Omega_2^2)B_2 - p_{12}D_0^2 B_4\right]\exp(i\omega_2 T_0) + \text{Reg} + \text{cc}, \end{aligned} \tag{8.43}$$

$$\begin{aligned}&\left[(p_{11}p_{22}-p_{12}^2)D_0^4+(p_{11}\Omega_2^2+p_{22}\Omega_1^2)D_0^2+\Omega_1^2\Omega_2^2\right]X_2^2\\&=\left[-p_{12}D_0^2B_1+(p_{11}D_0^2+\Omega_1^2)B_3\right]\exp(i\omega_1T_0)\\&+\left[-p_{12}D_0^2B_2+(p_{11}D_0^2+\Omega_1^2)B_4\right]\exp(i\omega_2T_0)+\text{Reg}+\text{cc}.\end{aligned}\tag{8.44}$$

To eliminate the circular terms from Eqs. (8.43) and (8.44), it is necessary to vanish to zero the terms in each square bracket. As a result we obtain

$$\begin{cases}(\Omega_2^2-p_{22}\omega_1^2)B_1+p_{12}\omega_1^2B_3=0\\p_{12}\omega_1^2B_1+(\Omega_1^2-p_{11}\omega_1^2)B_3=0\end{cases}\tag{8.45}$$

and

$$\begin{cases}(\Omega_2^2-p_{22}\omega_2^2)B_2+p_{12}\omega_2^2B_4=0\\p_{12}\omega_2^2B_2+(\Omega_1^2-p_{11}\omega_2^2)B_4=0\end{cases}\tag{8.46}$$

From Eqs. (8.45) and (8.46) it is evident that the determinant of each set of equations is reduced to the characteristic Eq. (8.32), whence it follows that each pair of equations is linear dependent, therefore for further treatment we should take only one equation from each pair in order that these two chosen equations are to be linear independent. Thus, for example, taking the first equations from each pair and considering relationships (8.29) and (8.31), we have

$$B_1+\alpha_1B_3=0,\tag{8.47}$$

$$B_3+\alpha_2B_4=0.\tag{8.48}$$

Substituting values of B_1-B_4 in (8.47) and (8.48), we obtain the following solvability equations:

$$2i\omega_1k_1D_1A_1+b_1A_2^2=0,\tag{8.49}$$

$$2i\omega_2k_2D_1A_2+b_2A_1\bar{A}_2=0,\tag{8.50}$$

where

$$k_i=\frac{\Omega_1^2+\alpha_i^2\Omega_2^2}{\omega_i^2}\quad(i=1,2),\qquad b_1=p_{13}+\alpha_2^2p_{14}+\alpha_2p_{15}+\alpha_1(p_{23}+\alpha_2^2p_{24}+\alpha_2p_{25}),$$

$$b_2=2\{p_{13}+\alpha_1\alpha_2p_{14}+(\alpha_1+\alpha_2)p_{15}+\alpha_2[p_{23}+\alpha_1\alpha_2p_{24}+(\alpha_1+\alpha_2)p_{25}]\}.$$

Let us multiply Eqs. (8.49) and (8.50) by $\bar{A}_1$ and $\bar{A}_2$, respectively, and find their complex conjugates. After adding every pair of the mutually adjoint equations with each other and subtracting one from another, as a result we obtain

$$2i\omega_1\left(\bar{A}_1D_1A_1-A_1D_1\bar{A}_1\right)+\frac{b_1}{k_1}\left(A_2^2\bar{A}_1+\bar{A}_2^2A_1\right)=0,\tag{8.51}$$

$$2i\omega_1\left(\bar{A}_1 D_1 A_1 + A_1 D_1 \bar{A}_1\right) + \frac{b_1}{k_1}\left(A_2^2\bar{A}_1 - \bar{A}_2^2 A_1\right) = 0, \tag{8.52}$$

$$2i\omega_2\left(\bar{A}_2 D_1 A_2 - A_2 D_1 \bar{A}_2\right) + \frac{b_2}{k_2}\left(A_1\bar{A}_2^2 + \bar{A}_1 A_2^2\right) = 0, \tag{8.53}$$

$$2i\omega_2\left(\bar{A}_2 D_1 A_2 + A_2 D_1 \bar{A}_2\right) + \frac{b_2}{k_2}\left(A_1\bar{A}_2^2 - \bar{A}_1 A_2^2\right) = 0. \tag{8.54}$$

Representing $A_1(T_1)$ and $A_2(T_1)$ in Eqs. (8.51)–(8.54) in the polar form

$$A_i(T_1) = a_i(T_1)e^{i\varphi_i(T_1)} \quad (i = 1,2), \tag{8.55}$$

we are led to the system of four nonlinear differential equations in $a_1(T_1)$, $a_2(T_1)$, $\varphi_1(T_1)$, and $\varphi_2(T_1)$

$$(a_1^2)^{\cdot} = -\frac{b_1}{k_1\omega_1}\, a_1 a_2^2 \sin\delta, \tag{8.56}$$

$$\dot{\varphi}_1 - \frac{b_1}{2k_1\omega_1}\, a_1^{-1} a_2^2 \cos\delta = 0, \tag{8.57}$$

$$(a_2^2)^{\cdot} = \frac{b_2}{k_2\omega_2}\, a_1 a_2^2 \sin\delta, \tag{8.58}$$

$$\dot{\varphi}_2 - \frac{b_2}{2k_2\omega_2}\, a_1 \cos\delta = 0, \tag{8.59}$$

where $\delta = 2\varphi_2 - \varphi_1$, and a dot denotes differentiation with respect to T_1.

From Eqs. (8.56) and (8.58) we could find that

$$\frac{b_2}{k_2\omega_2}(a_1^2)^{\cdot} + \frac{b_1}{k_1\omega_1}(a_2^2)^{\cdot} = 0. \tag{8.60}$$

Multiplying Eq. (8.60) by MV_0 and integrating over T_1, we obtain the first integral of the set of Eqs. (8.56)–(8.59), which is the law of conservation of energy,

$$MV_0\left(\frac{b_2}{k_2\omega_2}\, a_1^2 + \frac{b_1}{k_1\omega_1}\, a_2^2\right) = K_0, \tag{8.61}$$

where K_0 is the initial energy. Considering that $K_0 = \frac{1}{2} MV_0^2$, Eq. (8.61) is reduced to the following form:

$$\frac{b_2}{k_2\omega_2}\, a_1^2 + \frac{b_1}{k_1\omega_1}\, a_2^2 = \frac{V_0}{2}. \tag{8.62}$$

Let us introduce into consideration a new function $\xi(T_1)$ in the following form:

$$a_1^2 = \frac{k_2\omega_2}{b_2}\, E_0\xi(T_1), \quad a_2^2 = \frac{k_1\omega_1}{b_1}\, E_0\left[1 - \xi(T_1)\right], \tag{8.63}$$

where $E_0 = V_0/2$.

It is easy to verify by the direct substitution that Eqs. (8.63) satisfy Eq. (8.62), while the value $\xi_0 = \xi(0)$ $(0 \le \xi(0) \le 1)$ governs the energy distribution between two subsystems, X_1^1 and X_2^1, at the moment of impact. Substituting (8.63) in (8.56) yields

$$\dot{\xi} = -b\sqrt{\xi}(1-\xi)\sin\delta, \tag{8.64}$$

where

$$b = \sqrt{\frac{b_2}{k_2\omega_2}}\sqrt{E_0}.$$

Subtracting Eq. (8.57) from the doubled Eq. (8.59), we have

$$\dot{\delta} = -b\,\frac{1-3\xi}{2\sqrt{\xi}}\cos\delta. \tag{8.65}$$

Equation (8.65) could be rewritten in another form considering that

$$\dot{\delta} = \frac{d\delta}{d\xi}\dot{\xi},$$

or with due account for (8.64)

$$\dot{\delta} = -\frac{d\delta}{d\xi}b\sqrt{\xi}(1-\xi)\sin\delta. \tag{8.66}$$

Substituting (8.66) in Eq. (8.65) yields

$$\sqrt{\xi}(1-\xi)\frac{d\cos\delta}{d\xi} + \frac{1-3\xi}{2\sqrt{\xi}}\cos\delta = 0. \tag{8.67}$$

Integrating (8.67), we have

$$\cos\delta = \frac{G_0}{\sqrt{\xi}(1-\xi)}, \tag{8.68}$$

where G_0 is a constant of integration to be determined from the initial conditions.

Based on relationship (8.68), it is possible to introduce into consideration the stream function $G(\delta,\xi)$ of the phase fluid on the plane $\delta\xi$ such that

$$G(\delta,\xi) = \sqrt{\xi}(1-\xi)\cos\delta = G_0, \tag{8.69}$$

which is one more first integral of the set of Eqs. (8.56)–(8.59). It is easy to verify that the function (8.69) is really a stream function, since

$$v_\delta = \dot{\delta} = -b\frac{\partial G}{\partial \xi}, \qquad v_\xi = \dot{\xi} = b\frac{\partial G}{\partial \delta}. \tag{8.70}$$

It is interesting to note that the stream function $G(\delta,\xi)$ (8.69) obtained for the doubly curved shallow shell being under conditions of the two-to-one internal resonance coincides with that for a suspension bridge subjected to the two-to-one internal resonance analyzed in Rossikhin and Shitikova (1995).

In order to find the T_1-dependence of ξ, it is necessary to express $\sin\delta$ in terms of ξ in Eq. (8.64) with a help of relationship (8.68). As a result we obtain

$$\dot{\xi} = -b\sqrt{\xi(1-\xi)^2 - G_0^2},$$

or

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi^3 - 2\xi^2 + \xi - G_0^2}} = -bT_1, \tag{8.71}$$

where ξ_0 is the initial magnitude of the function $\xi = \xi(T_1)$. In other words, the calculation of the T_1-dependence of ξ is reduced to the calculation of the incomplete elliptic integral in the left hand-side of (8.71).

For the case of two-to-one internal resonance (8.36), the stream-function $G(\xi,\delta)$ is constructed according to (8.69), and its phase portrait showing the stream-lines of the phase fluid in the phase plane $\xi - \delta$ is presented in Fig. 8.2, which for the first time was presented in Rossikhin and Shitikova (1995) for the two-to-one internal resonance during nonlinear vibrations of suspension bridges. Magnitudes of G are

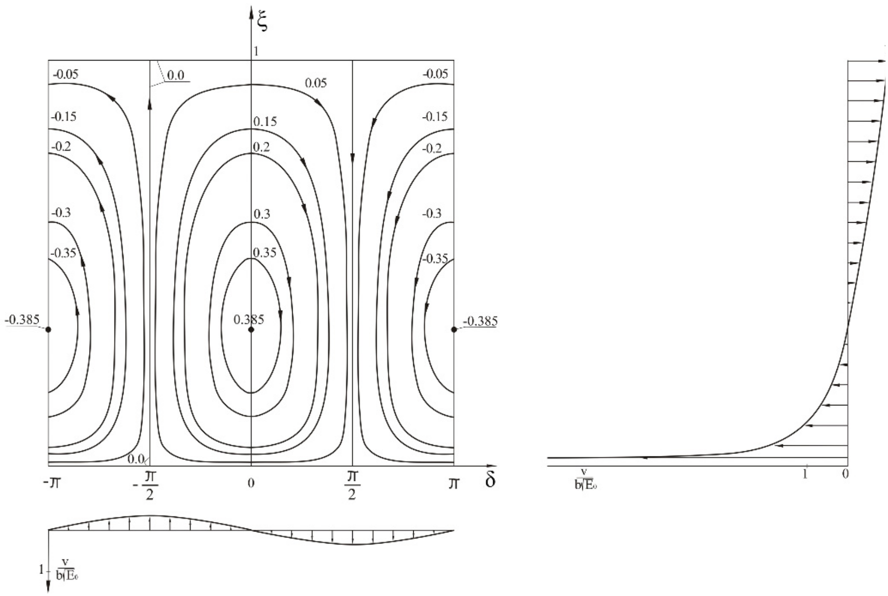

Fig. 8.2: Phase portrait: $\omega_1 = 2\omega_2$.

indicated by digits near the curves which correspond to the stream-lines; the flow direction of the phase fluid elements are shown by arrows on the stream-lines.

Figure 8.2 shows that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1$, and $\delta = \pm(\pi/2) \pm 2\pi n$ ($n = 0, 1, 2, ...$). As this takes place, the flow in each such rectangle becomes isolated. On all four rectangle sides $G = 0$ and inside it, the value G preserves its sign. The function G attains its extreme magnitudes at the points with the coordinates $\xi = \frac{1}{3}$, $\delta = \pm\pi n$ ($n = 0, 1, 2, ...$). Stream-lines give a pictorial estimate of the connection of G with all types of the energy-exchange mechanism. Thus, the points with the coordinates $\xi_0 = \frac{1}{3}$, $\delta_0 = \pm\pi n$ ($n = 0, 1, 2, ...$) correspond to the stationary regime, since $\dot{\delta} = 0$ and $\dot{\xi} = 0$ according to (8.64) and (8.65). The stationary points $\xi_0 = \frac{1}{3}$, $\delta_0 = \pm\pi n$ are centers, as with a small deviation from a center, a phase element begins to move around the stationary point along a closed trajectory. Closed stream-lines correspond to the periodic change of both amplitudes and phases.

Along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ ($n = 0, 1, 2, ...$) pure amplitude modulated aperiodic motions are realized, since with an increase in time t from 0 to ∞ the value ξ increases from ξ_0 to 1 (along the line $\delta = -\pi/2$) or decreases from ξ_0 to 0 (along the line $\delta = \pi/2$), and from Eq. (8.64) it follows that

$$\xi = \left[\frac{1 + \sqrt{\xi_0} - (1 - \sqrt{\xi_0}) \exp(-b\sqrt{E_0}\, T_1)}{1 + \sqrt{\xi_0} + (1 - \sqrt{\xi_0}) \exp(-b\sqrt{E_0}\, T_1)} \right]^2, \tag{8.72}$$

$$\delta(T_1) = \delta_0 = \frac{\pi}{2} \pm \pi n, \quad n = 0, 1, 2, ...$$

Along the line $\xi = 1$ only phase modulated motions are realized, because when $\xi = \xi_0 = 1$ the amplitudes $a_1 = \text{const}$ and $a_2 = 0$, and from (8.64) and (8.65) we could find that

$$b\sqrt{E_0}\, T_1 = \ln \frac{\tan\left(\frac{\delta}{2} + \frac{\pi}{4}\right)}{\tan\left(\frac{\delta_0}{2} + \frac{\pi}{4}\right)}, \quad \xi(T_1) = \xi_0 = 1. \tag{8.73}$$

The transition of fluid elements from the points $\xi = 0$, $\delta = \pi/2 \pm 2\pi n$ to the points $\xi = 0$, $\delta = -\pi/2 \pm 2\pi n$ proceeds instantly, because according to the distribution of the phase velocity (8.70) along the section $\delta = 0$ the magnitude of $\mathbf{v}$ tends to infinity as $\xi \to 0$. The distribution of the velocity along the vertical lines $\delta = \pm\pi n$ ($n = 0, 1, 2, ...$) has the aperiodic character, while in the vicinity of the line $\xi = 1/3$ it possesses the periodic character.

8.3.3.1 Initial Conditions

In order to construct the final solution of the problem under consideration, i.e. to solve the set of Eqs. (8.56)–(8.59) involving the functions $a_1(T_1)$, $a_2(T_1)$, or $\xi(T_1)$, as well as $\varphi_1(T_1)$, and $\varphi_2(T_1)$, or $\delta(T_1)$, it is necessary to use the initial conditions

$$w(x, y, 0) = 0, \tag{8.74}$$

$$\dot{w}(x_0, y_0, 0) = \varepsilon V_0, \tag{8.75}$$

$$\frac{b_2}{k_2\omega_2}\, a_1^2(0) + \frac{b_1}{k_1\omega_1}\, a_2^2(0) = E_0. \tag{8.76}$$

The two-term relationship for the displacement w (8.9) within an accuracy of ε according to (8.17) has the form

$$\begin{aligned} w(x,y,t) = \epsilon\Big[& X^1_{\alpha\beta}(T_0,T_1)\sin\left(\tfrac{\alpha\pi x}{a}\right)\sin\left(\tfrac{\beta\pi y}{b}\right) \\ & + X^1_{\gamma\delta}(T_0,T_1)\sin\left(\tfrac{\gamma\pi x}{a}\right)\sin\left(\tfrac{\delta\pi y}{b}\right)\Big] + O(\varepsilon^2). \end{aligned} \tag{8.77}$$

Substituting (8.24) and (8.25) in (8.77) with due account for (8.55) yields

$$\begin{aligned} w(x,y,t) = {} & 2\epsilon\Big\{a_1(\varepsilon t)\cos\left[\omega_1 t + \varphi_1(\varepsilon t)\right] \\ & + a_2(\varepsilon t)\cos\left[\omega_2 t + \varphi_2(\varepsilon t)\right]\Big\}\sin\left(\tfrac{\alpha\pi x}{a}\right)\sin\left(\tfrac{\beta\pi y}{b}\right) \\ & + 2\epsilon\Big\{\alpha_1 a_1(\varepsilon t)\cos\left[\omega_1 t + \varphi_1(\varepsilon t)\right] \\ & + \alpha_2 a_2(\varepsilon t)\cos\left[\omega_2 t + \varphi_2(\varepsilon t)\right]\Big\}\sin\left(\tfrac{\gamma\pi x}{a}\right)\sin\left(\tfrac{\delta\pi y}{b}\right)\Big] + O(\varepsilon^2). \end{aligned} \tag{8.78}$$

Differentiating (8.78) with respect to time t and limiting ourselves by the terms of the order of ε, we could find the velocity of the shell at the point of impact as follows

$$\begin{aligned} \dot{w}(x_0,y_0,t) = {} & -2\epsilon\Big\{\omega_1 a_1(\varepsilon t)\sin\left[\omega_1 t + \varphi_1(\varepsilon t)\right] + \omega_2 a_2(\varepsilon t)\sin\left[\omega_2 t + \varphi_2(\varepsilon t)\right]\Big\} s_1 \\ & -2\epsilon\Big\{\alpha_1\omega_1 a_1(\varepsilon t)\sin\left[\omega_1 t + \varphi_1(\varepsilon t)\right] + \alpha_2\omega_2 a_2(\varepsilon t)\sin\left[\omega_2 t + \varphi_2(\varepsilon t)\right]\Big\} s_2 + O(\varepsilon^2). \end{aligned} \tag{8.79}$$

Substituting (8.78) in the first initial condition (8.74) yields

$$a_1(0)\cos\varphi_1(0) + a_2(0)\cos\varphi_2(0) = 0, \tag{8.80}$$

$$\alpha_1 a_1(0)\cos\varphi_1(0) + \alpha_2 a_2(0)\cos\varphi_2(0) = 0. \tag{8.81}$$

From Eqs. (8.80) and (8.81) we find that

$$\cos\varphi_1(0) = 0, \quad \cos\varphi_2(0) = 0, \tag{8.82}$$

whence it follows that

$$\varphi_1(0) = \pm\frac{\pi}{2}, \quad \varphi_2(0) = \pm\frac{\pi}{2}, \tag{8.83}$$

and

$$\cos\delta_0 = \cos\left[2\varphi_2(0) - \varphi_1(0)\right] = 0, \tag{8.84}$$

i.e.,

$$\delta_0 = \pm\frac{\pi}{2} \pm 2\pi n. \tag{8.85}$$

The signs in (8.83) should be chosen considering the fact that the initial amplitudes are positive values, i.e. $a_1(0) > 0$ and $a_2(0) > 0$. Assume for definiteness that

$$\varphi_1(0) = -\frac{\pi}{2}, \quad \varphi_2(0) = -\frac{\pi}{2}. \tag{8.86}$$

Substituting now (8.79) in the second initial condition (8.75) with due account for (8.86), we obtain

$$\omega_1(s_1 + \alpha_1 s_2)a_1(0) + \omega_2(s_1 + \alpha_2 s_2)a_2(0) = E_0. \tag{8.87}$$

From Eqs. (8.76) and (8.87) we could determine the initial amplitudes

$$a_2(0) = \frac{E_0}{\omega_2(s_1 + \alpha_2 s_2)} - \frac{\omega_1(s_1 + \alpha_1 s_2)}{\omega_2(s_1 + \alpha_2 s_2)}\, a_1(0), \tag{8.88}$$

$$c_1 a_1^2(0) + c_2 a_1(0) + c_3 = 0, \tag{8.89}$$

where

$$c_1 = 1 + \frac{b_1 k_2 \omega_1 (s_1 + \alpha_1 s_2)^2}{b_2 k_1 \omega_2 (s_1 + \alpha_2 s_2)^2}, \quad c_2 = -\frac{b_1 k_2 (s_1 + \alpha_1 s_2) 2E_0}{b_2 k_1 \omega_2 (s_1 + \alpha_2 s_2)^2},$$

$$c_3 = \frac{b_1 k_2 E_0^2}{b_2 k_1 \omega_1 \omega_2 (s_1 + \alpha_2 s_2)^2} - \frac{k_2 \omega_2 E_0}{b_2}.$$

From Eqs. (8.88) and (8.89) it is evident that the initial magnitudes depend on the mass and the initial velocity of the impactor, on the coordinates of the point of impact, as well as on the numbers of the two modes induced by the impact.

Considering (8.84), from (8.68) we find the value of constant G_0

$$G_0 = 0. \tag{8.90}$$

Reference to (8.69) shows that G_0 could be zero in three cases: at $\xi_0 = 0$, $\xi_0 = 1$, or when $\cos\delta_0 = 0$. The above analysis of the phase portrait has revealed that the case $\xi_0 = 0$ is not realized. As for the case $\xi_0 = 1$, then the solution for the phase modulated motion takes the form of (8.73). However, for the found magnitudes of the initial phase difference δ_0 (8.85), the value of $\tan\left(\frac{\delta_0}{2} + \frac{\pi}{4}\right)$ in (8.73) is either equal to zero or to infinity, what means that this case could not be realized as well. That is why in further treatment we will analyze only the third case, resulting in the amplitude modulated motion (8.72) with

$$\delta(T_1) = \delta_0 = \text{const}. \tag{8.91}$$

Thus, we have determined all necessary constants from the initial conditions, therefore we could proceed to the construction of the solution for the contact force.

8.3.3.2 Contact Force and Shell's Deflection at the Point of Impact

Substituting relationship (8.79) differentiated one time with respect to time t in (8.4), we could obtain the contact force $P(t)$

$$\begin{aligned} P(t) &= 2\epsilon M\Big\{\omega_1^2 a_1(\varepsilon t)\cos\left[\omega_1 t+\varphi_1(\varepsilon t)\right]+\omega_2^2 a_2(\varepsilon t)\cos\left[\omega_2 t+\varphi_2(\varepsilon t)\right]\Big\}s_1 \\ &+2\epsilon M\Big\{\alpha_1\omega_1^2 a_1(\varepsilon t)\cos\left[\omega_1 t+\varphi_1(\varepsilon t)\right]+\alpha_2\omega_2^2 a_2(\varepsilon t)\cos\left[\omega_2 t+\varphi_2(\varepsilon t)\right]\Big\}s_2+O(\varepsilon^2). \end{aligned} \tag{8.92}$$

From Eqs. (8.57) and (8.59) with due account for (8.91) it follows that

$$\varphi_1(T_1)=\text{const}=\varphi_1(0), \qquad \varphi_2(T_1)=\text{const}=\varphi_2(0). \tag{8.93}$$

Considering (8.93) and (8.86), Eq. (8.92) is reduced to

$$P(t)=2\epsilon M\omega_2^2\Big\{8(s_1+\alpha_1 s_2)a_1(\varepsilon t)\cos\omega_2 t+(s_1+\alpha_2 s_2)a_2(\varepsilon t)\Big\}\sin\omega_2 t. \tag{8.94}$$

Substituting (8.63) in (8.94), we finally obtained

$$\begin{aligned} P(t) &= 2\epsilon M\omega_2^2\sqrt{E_0}\Big\{8(s_1+\alpha_1 s_2)\sqrt{\tfrac{k_2\omega_2}{b_2}}\sqrt{\xi(\varepsilon t)}\cos\omega_2 t \\ &+(s_1+\alpha_2 s_2)\sqrt{\tfrac{k_1\omega_1}{b_1}}\sqrt{1-\xi(\varepsilon t)}\Big\}\sin\omega_2 t, \end{aligned} \tag{8.95}$$

where the function $\xi(\varepsilon t)$ is defined by (8.72).

Since the duration of contact is a small value, what is evident from experimental data (Kistler and Waas, 1998a; Kunukkasseril and Palaninathan, 1975; Rossikhin and Shitikova, 2007), then $P(t)$ could be calculated via an approximate formula, which is obtained from (8.94) at $\varepsilon t \approx 0$

$$P(t)\approx 16\epsilon M\omega_2^2\Big(\cos\omega_2 t+\frac{1}{8}\kappa\Big)(s_1+\alpha_1 s_2)a_1(0)\sin\omega_2 t+O(\varepsilon^2), \tag{8.96}$$

where the dimensionless parameter κ

$$\kappa=\frac{(s_1+\alpha_2 s_2)}{(s_1+\alpha_1 s_2)}\frac{a_2(0)}{a_1(0)} \tag{8.97}$$

is defined by the parameters of two impact-induced modes coupled by the two-to-one internal resonance (8.36), as well as by the coordinates of the point of impact and the initial velocity of impact. The deflection of the shell at the point of impact could be determined from (8.78) with due account for the found initial values of the phases

$$w(x_0,y_0,t)\approx 4\epsilon\Big(\cos\omega_2 t+\frac{1}{2}\kappa\Big)(s_1+\alpha_1 s_2)a_1(0)\sin\omega_2 t+O(\varepsilon^2). \tag{8.98}$$

The dimensionless time $\tau=\omega_2 t$ dependence of the dimensionless contact force P^*

$$P^*(\tau) \approx \left(\cos\tau + \frac{1}{8}\,\kappa\right)\sin\tau, \tag{8.99}$$

where

$$P^*(t) = \frac{P(t)}{16\varepsilon M\omega_2^2(s_1+\alpha_1 s_2)a_1(0)},$$

and the dimensionless deflection of the target at the point of impact

$$w^*(\tau) = \left(\cos\tau + \frac{1}{2}\,\kappa\right)\sin\tau, \tag{8.100}$$

where

$$w^*(t) = \frac{w(x_0,y_0,t)}{4\varepsilon(s_1+\alpha_1 s_2)a_1(0)},$$

are shown, respectively, in Figs. 8.3 (a) and (b) for the different magnitudes of the parameter κ: 0, 2, 4, and 8. Figures 8.3 (a) and (b) show that the increase in the parameter κ results in the increase of the maximal contact force, the duration of contact, and the maximal deflection of the target at the point of impact. In other words, from Figure 3 it is evident that the peak contact force, the duration of contact and shell's deflection depend essentially upon the parameters of two impact-induced modes coupled by the two-to-one internal resonance (8.36).

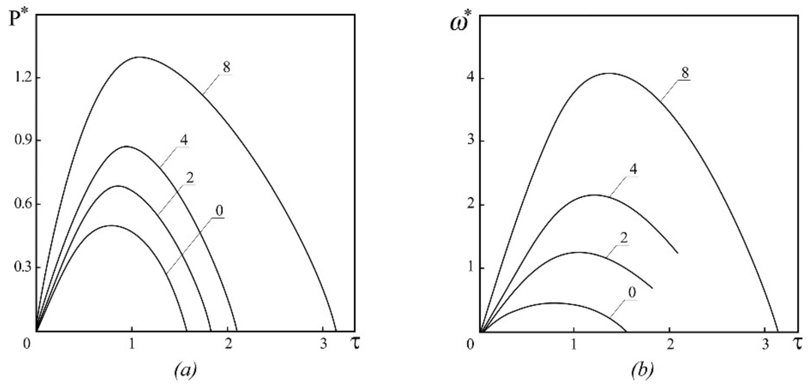

Fig. 8.3: Dimensionless time dependence of (a) the dimensionless contact force and (b) the target deflection at the point of impact for the case of the internal resonance $\omega_1 = 2\omega_2$.

8.3.4 Solution of Equations at Order of ε^3

In order to study internal resonances of the order of ε^3, in further treatment we assume that $\omega_1 \neq 2\omega_2$. In this case in order to eliminate secular terms in Eqss (8.34) and (8.35), it is sufficient to fulfill the following equations:

$$D_1 A_1 = 0, \qquad D_1 A_2 = 0, \tag{8.101}$$

whence it follows that the functions A_1 and A_2 are T_1-independent, i.e.,

$$A_1 = A_1(T_2), \qquad A_2 = A_2(T_2). \tag{8.102}$$

To solve Eqs. (8.34) and (8.35) with due account for (8.101) and (8.102), let us first apply the operators $(p_{22}D_0^2 + \Omega_2^2)$ and $(-p_{12}D_0^2)$ to (8.34) and (8.35), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_2^2. If we apply the operators $(-p_{12}D_0^2)$ and $(p_{11}D_0^2 + \Omega_1^2)$ to (8.34) and (8.35), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_1^2. Thus, we obtain

$$\begin{aligned}
&\left[(p_{11}p_{22} - p_{12}^2)D_0^4 + (p_{11}\Omega_2^2 + p_{22}\Omega_1^2)D_0^2 + \Omega_1^2\Omega_2^2\right] X_1^2 \\
&= -\Big[(p_{13} + \alpha_1^2 p_{14} + \alpha_1 p_{15})(p_{22}D_0^2 + \Omega_2^2) \\
&-(p_{23} + \alpha_1^2 p_{24} + \alpha_1 p_{25})p_{12}D_0^2\Big] A_1 \left[A_1 e^{2i\omega_1 T_0} + \bar{A}_1\right] \\
&-\left[(p_{13} + \alpha_2^2 p_{14} + \alpha_2 p_{15})(p_{22}D_0^2 + \Omega_2^2) - (p_{23} + \alpha_2^2 p_{24} + \alpha_2 p_{25})p_{12}D_0^2\right] \\
&\times A_2 \left[A_2 e^{2i\omega_2 T_0} + \bar{A}_2\right] - 2\{[p_{13} + \alpha_1\alpha_2 p_{14} + (\alpha_1 + \alpha_2)p_{15}]\,(p_{22}D_0^2 + \Omega_2^2) \\
&- [p_{23} + \alpha_1\alpha_2 p_{24} + (\alpha_1 + \alpha_2)p_{25}]\,p_{12}D_0^2\Big\} \\
&\times A_1 \left[A_2 e^{i(\omega_1+\omega_2)T_0} + \bar{A}_2 e^{i(\omega_1-\omega_2)T_0}\right] + \text{cc},
\end{aligned} \tag{8.103}$$

$$\begin{aligned}
&\left[(p_{11}p_{22} - p_{12}^2)D_0^4 + (p_{11}\Omega_2^2 + p_{22}\Omega_1^2)D_0^2 + \Omega_1^2\Omega_2^2\right] X_2^2 \\
&= -\Big[(p_{23} + \alpha_1^2 p_{24} + \alpha_1 p_{25})(p_{11}D_0^2 + \Omega_1^2) \\
&-(p_{13} + \alpha_1^2 p_{14} + \alpha_1 p_{15})p_{12}D_0^2\Big] A_1 \left[A_1 e^{2i\omega_1 T_0} + \bar{A}_1\right] \\
&-\left[(p_{23} + \alpha_2^2 p_{24} + \alpha_2 p_{25})(p_{11}D_0^2 + \Omega_1^2) - (p_{13} + \alpha_2^2 p_{14} + \alpha_2 p_{15})p_{12}D_0^2\right] \\
&\times A_2 \left[A_2 e^{2i\omega_2 T_0} + \bar{A}_2\right] - 2\{[p_{23} + \alpha_1\alpha_2 p_{24} + (\alpha_1 + \alpha_2)p_{25}]\,(p_{11}D_0^2 + \Omega_1^2) \\
&- [p_{13} + \alpha_1\alpha_2 p_{14} + (\alpha_1 + \alpha_2)p_{15}]\,p_{12}D_0^2\Big\} \\
&\times A_1 \left[A_2 e^{i(\omega_1+\omega_2)T_0} + \bar{A}_2 e^{i(\omega_1-\omega_2)T_0}\right] + \text{cc}.
\end{aligned} \tag{8.104}$$

The solution of (8.103) and (8.104) has the form

$$X_1^2 = F_1(T_2)\, e^{i\omega_1 T_0} + F_2(T_2) e^{i\omega_2 T_0} + N_1 A_1^2 e^{2i\omega_1 T_0} + N_2 A_2^2 e^{2i\omega_2 T_0} + N_3 A_1 \bar{A}_1 + N_4 A_2 \bar{A}_2 + N_5 A_1 A_2 e^{i(\omega_1+\omega_2)T_0} + N_6 A_1 \bar{A}_2 e^{i(\omega_1-\omega_2)T_0} + \text{cc}, \tag{8.105}$$

$$X_2^2 = \alpha_1 F_1(T_2)\, e^{i\omega_1 T_0} + \alpha_2 F_2(T_2) e^{i\omega_2 T_0} + E_1 A_1^2 e^{2i\omega_1 T_0} + E_2 A_2^2 e^{2i\omega_2 T_0} + E_3 A_1 \bar{A}_1 + E_4 A_2 \bar{A}_2 + E_5 A_1 A_2 e^{i(\omega_1+\omega_2)T_0} + E_6 A_1 \bar{A}_2 e^{i(\omega_1-\omega_2)T_0} + \text{cc}, \tag{8.106}$$

where $F_1(T_2)$ and $F_2(T_2)$ are unknown complex functions, and coefficients N_i and E_i $(i = 1, 2, ..., 6)$ are presented in Appendix.

Now substituting (8.24), (8.25), (8.105), and (8.106) in (8.20) and (8.21), we obtain (Rossikhin et al, 2015)

$$\begin{aligned} p_{11} D_0^2 X_1^3 + p_{12} D_0^2 X_2^3 + \Omega_1^2 X_1^3 = &-\left[2i\omega_1(p_{11} + \alpha_1 p_{12}) D_2 A_1 + K_1 A_1^2 \bar{A}_1 + K_2 A_1 A_2 \bar{A}_2\right] e^{i\omega_1 T_0} \\ &- \left[2i\omega_2(p_{11} + \alpha_2 p_{12}) D_2 A_2 + L_1 A_2^2 \bar{A}_2 + L_2 A_1 \bar{A}_1 A_2\right] e^{i\omega_2 T_0} \\ &- \left\{M_1 A_2^3 e^{3i\omega_2 T_0} + M_2 A_1 \bar{A}_2^2 e^{i(\omega_1 - 2\omega_2)T_0}\right\} \\ &- R_1 A_1^2 \bar{A}_2 e^{i(2\omega_1 - \omega_2)T_0} + \text{Reg} + \text{cc}, \end{aligned} \tag{8.107}$$

$$\begin{aligned} p_{21} D_0^2 X_1^3 + p_{22} D_0^2 X_2^3 + \Omega_2^2 X_2^3 = &-\left[2i\omega_1(p_{21} + \alpha_1 p_{22}) D_2 A_1 + K_3 A_1^2 \bar{A}_1 + K_4 A_1 A_2 \bar{A}_2\right] e^{i\omega_1 T_0} \\ &- \left[2i\omega_2(p_{21} + \alpha_2 p_{22}) D_2 A_2 + L_3 A_2^2 \bar{A}_2 + L_4 A_1 \bar{A}_1 A_2\right] e^{i\omega_2 T_0} \\ &- \left\{M_3 A_2^3 e^{3i\omega_2 T_0} + M_4 A_1 \bar{A}_2^2 e^{i(\omega_1 - 2\omega_2)T_0}\right\} \\ &- R_2 A_1^2 \bar{A}_2 e^{i(2\omega_1 - \omega_2)T_0} + \text{Reg} + \text{cc}, \end{aligned} \tag{8.108}$$

where all regular terms are designated by Reg, and coefficients K_i, L_i, M_i, and R_i $(i = 1, 2, 3, 4)$ are given in Appendix.

Reference to Eqs. (8.107) and (8.108) shows that the following three-to-one internal resonance could occur on this step:

$$\omega_1 = 3\omega_2. \tag{8.109}$$

8.3.4.1 Impact-Induced Three-to-One Internal Resonance

Suppose that $\omega_1 \approx 3\omega_2$ (8.109). Then Eqs. (8.107) and (8.108) could be rewritten in the following form:

$$p_{11} D_0^2 X_1^3 + p_{12} D_0^2 X_2^3 + \Omega_1^2 X_1^3 = B_1 \exp(i\omega_1 T_0) + B_2 \exp(i\omega_2 T_0) + \text{Reg} + \text{cc}, \tag{8.110}$$

$$p_{21} D_0^2 X_1^3 + p_{22} D_0^2 X_2^3 + \Omega_2^2 X_2^3 = B_3 \exp(i\omega_1 T_0) + B_4 \exp(i\omega_2 T_0) + \text{Reg} + \text{cc}, \tag{8.111}$$

where

$$\begin{aligned}
B_1 &= -2i\omega_1(p_{11}+\alpha_1 p_{12})D_2A_1 - K_1A_1^2\bar{A}_1 - K_2A_1A_2\bar{A}_2 - M_1A_2^3,\\
B_2 &= -2i\omega_2(p_{11}+\alpha_2 p_{12})D_2A_2 - L_1A_2^2\bar{A}_2 - L_2A_1\bar{A}_1A_2 - M_2A_1\bar{A}_2^2,\\
B_3 &= -2i\omega_1(p_{21}+\alpha_1 p_{22})D_2A_1 - K_3A_1^2\bar{A}_1 - K_4A_1A_2\bar{A}_2 - M_3A_2^3,\\
B_4 &= -2i\omega_2(p_{21}+\alpha_2 p_{22})D_2A_2 - L_3A_2^2\bar{A}_2 - L_4A_1\bar{A}_1A_2 - M_4A_1\bar{A}_2^2.
\end{aligned}$$

It could be shown in the same manner, as it has been done above for the case of the two-to-one internal resonance, that the terms with the exponents $\exp(\pm i\omega_i T_0)$ $(i=1,2)$ produce circular terms in Eqs. (8.110) and (8.111).

In order to eliminate them, the terms proportional to $e^{i\omega_1 T_0}$ and $e^{i\omega_2 T_0}$ should be vanished to zero putting $B_i=0$ $(i=1,2,3,4)$. So we obtain four equations for defining two unknown amplitudes $A_1(t)$ and $A_2(t)$. However, it is possible to show, once again similarly to the above case of the two-to-one internal resonance, that not all of these four equations are linear independent from each other, and therefore to obtain the following solvability equations:

$$2i\omega_1D_2A_1 + p_1A_1^2\bar{A}_1 + p_2A_1A_2\bar{A}_2 + p_3A_2^3 = 0, \tag{8.112}$$

$$2i\omega_2D_2A_2 + p_4A_2^2\bar{A}_2 + p_5A_1\bar{A}_1A_2 + p_6A_1\bar{A}_2^2 = 0, \tag{8.113}$$

where

$$p_1 = \frac{K_1+\alpha_1K_3}{k_1},\quad p_2 = \frac{K_2+\alpha_1K_4}{k_1},\quad p_3 = \frac{M_1+\alpha_1M_3}{k_1},\quad p_4 = \frac{L_1+\alpha_2L_3}{k_2},$$

$$p_5 = \frac{L_2+\alpha_2L_4}{k_2},\quad p_6 = \frac{M_2+\alpha_2M_4}{k_2},\quad k_1 = \frac{\Omega_1^2+\alpha_1\Omega_2^2}{\omega_1^2},\quad k_2 = \frac{\Omega_1^2+\alpha_2\Omega_2^2}{\omega_2^2}.$$

Let us multiply Eqs. (8.112) and (8.113) by $\bar{A}_1$ and $\bar{A}_2$, respectively, and find their complex conjugates. After adding every pair of the mutually adjoint equations with each other and subtracting one from another, as a result we obtain

$$2i\omega_1\left(\bar{A}_1D_2A_1 - A_1D_2\bar{A}_1\right) + 2p_1A_1^2\bar{A}_1^2 + 2p_2A_1\bar{A}_1A_2\bar{A}_2 + p_3\left(\bar{A}_1A_2^3 + A_1\bar{A}_2^3\right) = 0, \tag{8.114}$$

$$2i\omega_1\left(\bar{A}_1D_2A_1 + A_1D_2\bar{A}_1\right) + p_3\left(\bar{A}_1A_2^3 - A_1\bar{A}_2^3\right) = 0, \tag{8.115}$$

$$2i\omega_2\left(\bar{A}_2D_2A_2 - A_2D_2\bar{A}_2\right) + 2p_4A_2^2\bar{A}_2^2 + 2p_5A_1\bar{A}_1A_2\bar{A}_2 + p_6\left(A_1\bar{A}_2^3 + \bar{A}_1A_2^3\right) = 0, \tag{8.116}$$

$$2i\omega_2\left(\bar{A}_2D_2A_2 + A_2D_2\bar{A}_2\right) + p_6\left(A_1\bar{A}_2^3 - \bar{A}_1A_2^3\right) = 0. \tag{8.117}$$

Representing $A_1(T_2)$ and $A_2(T_2)$ in Eqs. (8.114)–(8.117) in the polar form

$$A_i(T_2) = a_i(T_2)e^{i\varphi_i(T_2)} \quad (i = 1,2), \tag{8.118}$$

we are led to the system of four nonlinear differential equations in $a_1(T_2)$, $a_2(T_2)$, $\varphi_1(T_2)$, and $\varphi_2(T_2)$

$$(a_1^2)^{\cdot} = -\frac{p_3}{\omega_1}\, a_1 a_2^3 \sin\delta, \tag{8.119}$$

$$2\dot{\varphi}_1 - \frac{p_1}{\omega_1}\, a_1^2 - \frac{p_2}{\omega_1}\, a_2^2 - \frac{p_3}{\omega_1}\, a_1^{-1} a_2^3 \cos\delta = 0, \tag{8.120}$$

$$(a_2^2)^{\cdot} = \frac{p_6}{\omega_2}\, a_1 a_2^3 \sin\delta, \tag{8.121}$$

$$2\dot{\varphi}_2 - \frac{p_5}{\omega_2}\, a_1^2 - \frac{p_4}{\omega_2}\, a_2^2 - \frac{p_6}{\omega_2}\, a_1 a_2 \cos\delta = 0, \tag{8.122}$$

From Eqs. (8.119) and (8.121) we could find that

$$\frac{p_6}{\omega_2}(a_1^2)^{\cdot} + \frac{p_3}{\omega_1}(a_2^2)^{\cdot} = 0 \tag{8.123}$$

Multiplying (8.123) by MV_0 and integrating over T_2, we obtain the first integral of the set of Eqs. (8.119)–(8.122), which is the law of conservation of energy,

$$MV_0\left(\frac{p_6}{\omega_2}\, a_1^2 + \frac{p_3}{\omega_1}\, a_2^2\right) = K_0, \tag{8.124}$$

where K_0 is the initial energy. Considering that $K_0 = \frac{1}{2} MV_0^2$, Eq. (8.124) is reduced to the following form:

$$\frac{p_6}{\omega_2}\, a_1^2 + \frac{p_3}{\omega_1}\, a_2^2 = \frac{V_0}{2}. \tag{8.125}$$

Let us introduce into consideration a new function $\xi(T_2)$ in the following form:

$$a_1^2 = \frac{\omega_2}{p_6}\, E_0\xi(T_2), \quad a_2^2 = \frac{\omega_1}{p_3}\, E_0\,[1-\xi(T_2)]. \tag{8.126}$$

It is easy to verify by the direct substitution that formulas (8.126) satisfy Eq. (8.125), while the value $\xi(0)$ $(0 \le \xi(0) \le 1)$ governs the energy distribution between two subsystems, X_1^1 and X_2^1, at the moment of impact.

Substituting (8.126) in (8.119) yields

$$\dot{\xi} = -bE_0(1-\xi)\sqrt{\xi(1-\xi)}\,\sin\delta, \tag{8.127}$$

where

$$b = \sqrt{\frac{\omega_1 p_6}{\omega_2 p_3}}.$$

Subtracting Eq. (8.120) from the triple Eq. (8.122), we have

$$\dot{\delta} = bE_0\left(\frac{3}{2}\xi - \frac{1}{2}(1-\xi)\right)\sqrt{\frac{1-\xi}{\xi}}\cos\delta + E_0\left(\frac{3p_5}{2\omega_2} - \frac{p_1}{2\omega_1}\right)\frac{\omega_2}{p_6}\xi$$
$$+E_0\left(\frac{3p_4}{2\omega_2} - \frac{p_2}{2\omega_1}\right)\frac{\omega_1}{p_3}(1-\xi). \qquad (8.128)$$

Equation (8.128) could be rewritten in another form considering that

$$\dot{\delta} = \frac{d\delta}{d\xi}\dot{\xi},$$

or with due account for (8.127)

$$\dot{\delta} = -bE_0(1-\xi)\sqrt{\xi(1-\xi)}\,\frac{d\delta}{d\xi}\sin\delta. \qquad (8.129)$$

Substituting (8.129) in Eq. (8.128) yields

$$\frac{d\cos\delta}{d\xi} + \frac{1-4\xi}{2\xi(1-\xi)}\cos\delta - \frac{\Gamma_1}{\sqrt{\xi(1-\xi)}} - \frac{\Gamma_2}{1-\xi}\sqrt{\frac{\xi}{1-\xi}} = 0, \qquad (8.130)$$

where

$$\Gamma_1 = \frac{1}{b}\left(\frac{3p_4}{2\omega_2} - \frac{p_2}{2\omega_1}\right)\frac{\omega_1}{p_3}, \quad \Gamma_2 = \frac{1}{b}\left(\frac{3p_5}{2\omega_2} - \frac{p_1}{2\omega_1}\right)\frac{\omega_2}{p_6}.$$

Integrating (8.130), we have

$$\cos\delta = \frac{G_0}{(1-\xi)\sqrt{\xi(1-\xi)}} - \frac{\Gamma_1}{2}\sqrt{\frac{1-\xi}{\xi}} + \frac{\Gamma_2}{2}\frac{\xi}{(1-\xi)}\sqrt{\frac{\xi}{1-\xi}}, \qquad (8.131)$$

where G_0 is a constant of integration to be determined from the initial conditions. Based on relationship (8.131), it is possible to introduce into consideration the stream function $G(\delta,\xi)$ of the phase fluid on the plane $\delta\xi$ such that

$$G(\delta,\xi) = (1-\xi)\sqrt{\xi(1-\xi)}\cos\delta + \frac{\Gamma_1}{2}(1-\xi)^2 - \frac{\Gamma_2}{2}\xi^2 = G_0, \qquad (8.132)$$

which is one more first integral of the set of Eqs. (8.119)–(8.122).

It is easy to verify that the function (8.132) is really a stream function, since

$$v_\delta = \dot{\delta} = -bE_0\frac{\partial G}{\partial\xi}, \qquad v_\xi = \dot{\xi} = bE_0\frac{\partial G}{\partial\delta}. \qquad (8.133)$$

In order to find the T_2-dependence of ξ, it is necessary to express $\sin\delta$ in terms of ξ in Eq. (8.127) with a help of relationship (8.131). As a result we obtain

$$\dot{\xi} = -bE_0\sqrt{\xi(1-\xi)^3 - \left[G_0 - \frac{\Gamma_1}{2}(1-\xi)^2 + \frac{\Gamma_2}{2}\xi^2\right]^2}$$

or

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(1-\xi)^3 - \left[G_0 - \frac{\Gamma_1}{2}(1-\xi)^2 + \frac{\Gamma_2}{2}\xi^2\right]^2}} = -bE_0T_2, \tag{8.134}$$

where ξ_0 is the initial magnitude of the function $\xi = \xi(T_2)$. In other words, the calculation of the T_2-dependence of ξ is reduced to the calculation of the incomplete elliptic integral in the left hand-side of (8.134).

8.3.4.2 Phase Portraits

The qualitative analysis of the case of the three-to-one internal resonance (8.109) could be carried out with the be constructed according to (8.132) depends essentially on the magnitudes of the coefficients Γ_1 and Γ_2. Let us carry out the phenomenological analysis of the phase portraits constructing them at different magnitudes of the system parameters.

- The case when $\Gamma_1 = \Gamma_2 = 0$.
  Let us first consider the case when $\Gamma_1 = \Gamma_2 = 0$. Then (8.132) is reduced to

$$G(\delta,\xi) = (1-\xi)\sqrt{\xi(1-\xi)}\cos\delta = G_0, \tag{8.135}$$

  and the stream-lines of the phase fluid in the phase plane $\xi - \delta$ for this particular case are presented in Fig. 8.4. Magnitudes of G are indicated by digits near the curves which correspond to the stream-lines; the flow direction of the phase fluid elements are shown by arrows on the stream-lines.
  Reference to Fig. 8.4 shows that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1$, and $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0,1,2,...)$. As this takes place, the flow in each such rectangle becomes isolated. On all four rectangle sides $G = 0$ and inside it the value G preserves its sign. The function G attains its extreme magnitudes at the points with the coordinates $\xi = \frac{1}{4}$, $\delta = \pm\pi n$ $(n = 0,1,2,...)$.
  Along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0,1,2,...)$ the solution could be written as

$$\xi = \left[1 + \frac{1}{[c_0 + f(T_2)]^2}\right]^{-1}, \quad \delta(T_2) = \delta_0 = \frac{\pi}{2} \pm \pi n, \ \ n = 0,1,2,...$$

  where

$$f(T_2) = -bE_0T_2, \quad c_0 = \sqrt{\frac{\xi_0}{1-\xi_0}}.$$

  Along the line $\xi = 1$ the stationary boundary regime is realized, because when $\xi = \xi_0 = 1$ the amplitudes $a_1 = \text{const}$ and $a_2 = 0$, and from (8.127) and (8.129) it follows that $\dot{\xi} = \dot{\delta} = 0$. The transition of fluid elements from the points $\xi = 0$, $\delta = \pi/2 \pm 2\pi n$ to the points $\xi = 0$, $\delta = -\pi/2 \pm 2\pi n$ $(n = 0,1,2...)$ proceeds instantly, because according to the distribution of the phase velocity along the section $\delta = 0$

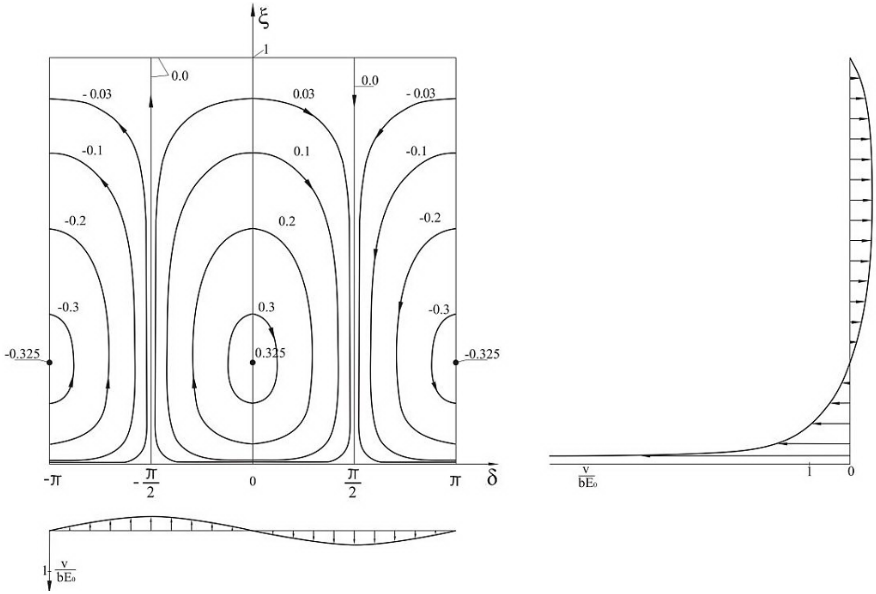

Fig. 8.4: Phase portrait for the case of 1:3 internal resonance at $\Gamma_1 = \Gamma_2 = 0$.

(see Fig. 8.4) the magnitude of **v** tends to infinity as $\xi \to 0$. The distribution of the velocity along the vertical lines $\delta = \pm\pi n$ has the aperiodic character, while in the vicinity of the line $\xi = 1/4$ it possesses the periodic character.

- The case when $\Gamma_1 = 0$ and $\Gamma_2/2 = 1$.
  In this case, the stream-function is defined as

$$G(\xi,\delta) = \xi^{1/2}(1-\xi)^{3/2}\cos\delta + (1-\xi)^2 = G(\xi_0,\delta_0),$$

  and Fig. 8.5 shows the streamlines of the phase fluid in the phase plane.
  As in the previous case, the phase fluid flows in an infinitely long channel, the boundaries of which are the straight lines $\xi = 0$ and $\xi = 1$, corresponding to the phase modulated motions. In one part the streamlines are non-closed, what corresponds to the periodic change of amplitudes and the aperiodic change of phases; in another part they are closed, what corresponds to the periodic change of both amplitudes and phases. The aperiodic regime lines are the boundaries of the closed and unclosed streamline areas. From the phase portrait in Figure 5 it is seen that the circulation zones are located in a staggered arrangement by the right and left channel sides (this configuration resembles that of von Kármán staggered vortex tracks).
  Each zone by the side $\xi = 1$ is surrounded by a line with the value $G = 0$. This line consists of two parts connected with each other at the branch points with the coordinates $\xi = 1$, $\delta = \pi/2 \pm \pi n$ $(n = 0,1,2...)$. One branch of this line corresponds to the phase-modulated regime $\xi = 1$, and the other to the aperiodic regime,

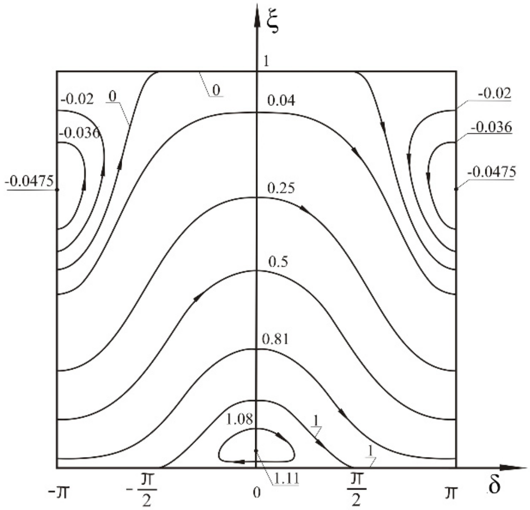

Fig. 8.5 Phase portrait for the case of 1:3 internal resonance at $\Gamma_1 = 0$, and $\Gamma_2/2 = 1$.

wherein ξ varies from $\xi_{\min} = 0.5$ to $\xi_{\max} = 1$. At the branch point itself, the phase fluid flow velocity is equal to zero. Along the separatrix, the analytic solution can be constructed in the following form:

$$\frac{2\sqrt{2}}{1-\xi}\sqrt{(1-\xi)(2-\xi)}\Big|_{\xi_0}^{\xi} = -bE_0T_2, \quad \cos\delta = -\sqrt{\frac{1-\xi}{\xi}}.$$

The circulation zones by the side $\xi = 0$ are surrounded by the line with the value $G = 1$. However, only those parts of the line $G = 1$ which bound these zones from above and come closer to the side $\xi = 0$ at the points $\xi = 0$, $\delta = \pi/2 \pm \pi n$ belong to the domain of the fluid flow. The transition of fluid elements from the points $\xi = 0$, $\delta = (\pi/2) \pm \pi n$ to the points $\xi = 0$, $\delta = (3\pi/2) \pm \pi n$ proceeds instantly. The line $G = 1$ conforms to the periodic change of the amplitudes and the aperiodic change of the phase. The separatrix $G = 1$ is defined by the following equations:

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(1-7\xi+7\xi^2-2\xi^3)}} = \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(0.170515-\xi)(2\xi^2-6.659\xi+5.865)}}$$

$$= -bE_0T_2, \qquad \cos\delta = \frac{2-\xi}{1-\xi}\sqrt{\frac{\xi}{1-\xi}},$$

wherein ξ varies from $\xi_{\min} = 0$ to $\xi_{\max} = 0.170515$.

Inside the both circulation zones there are points with the extreme values of the stream-function: maximal $G_{\max} = 1.11$ and minimal $G_{\min} = -0.0475$, respectively.

These points are the centers corresponding to the stable stationary regimes $\xi = \xi_0 = 0.0443$, $\delta = \delta_0 = \pm 2\pi n$ and $\xi = \xi_0 = 0.7057$, $\delta = \delta_0 = \pi \pm 2\pi n$, respectively. Between the lines corresponding to $G = 0$ and $G = 1$, unclosed streamlines are located which are in accordance with the periodic change of the amplitudes and the aperiodic change of the phase difference.

- The case when $\Gamma_1/2 = \Gamma_2/2 = 1$.
  In this case, the stream-function is defined as

$$G(\xi,\delta)=\xi^{1/2}(1-\xi)^{3/2}\cos\delta-\xi^2+(1-\xi)^2=G(\xi_0,\delta_0),$$

and Fig. 8.6 shows the streamlines of the phase fluid in the phase plane.
From Fig. 8.6 it is seen that, unlike the previous case presented in Fig. 8.6, the circulation zones by the side $\xi = 1$ and the aperiodic regime disappear. If $\xi \to 1$, then the streamlines level off and tend to the line $\xi = 1$ where $G = -1$. If $\xi \to 0$, then the streamlines tend to the piecewise continuous line $G = 1$ determined on the segments $[-(\pi/2) \pm 2\pi n,\ (\pi/2) \pm 2\pi n]$. The transition of fluid elements from the points $\xi = 0$, $\delta = (\pi/2) \pm 2\pi n$ to the points $\xi = 0$, $\delta = (3\pi/2) \pm 2\pi n$ proceeds instantly. The line $G = 1$ conforms to the periodic change of the amplitudes and the aperiodic change of the phase difference. The separatrix $G = 1$ is defined by the following equations:

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(1-7\xi+3\xi^2-\xi^3)}} = \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(0.1523-\xi)(\xi^2-2.8477\xi+6.5663)}}$$

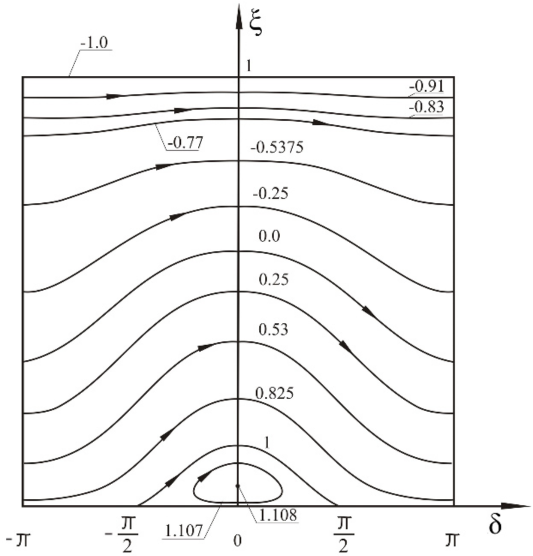

Fig. 8.6 Phase portrait for the case of 1:3 internal resonance at $\Gamma_1/2 = \Gamma_2/2 = 1$.

$$= -bE_0T_2, \qquad \cos\delta = \frac{2}{1-\xi}\sqrt{\frac{\xi}{1-\xi}},$$

wherein ξ varies from $\xi_{\min} = 0$ to $\xi_{\max} = 0.1523$.
Inside each circulation zone there is a point with the maximal value of the stream-function $G_{\max} = 1.108$. These points are the centers corresponding to the stable stationary regimes $\xi = \xi_0 = 0.04$, $\delta = \delta_0 = \pm 2\pi n$.
Between the lines corresponding to $G = -1$ and $G = 1$, unclosed streamlines are located which are in accordance with the periodic change of the amplitudes and the aperiodic change of the phase difference.

- The case when $\Gamma_1 = -21.84$ and $= \Gamma_2 = 0.01$.
  In this case, the stream-function is defined as

$$G(\xi,\delta) = \xi^{1/2}(1-\xi)^{3/2}\cos\delta - 0.005\xi^2 - 10.92(1-\xi)^2 = G(\xi_0,\delta_0),$$

and Fig. 8.7 shows the streamlines of the phase fluid in the phase plane. Figure 8.7 illustrates the phase portrait with only unclosed phase fluid streamlines along which the fluid flows in the direction of an increase in δ. With $\xi \to 0$ and $\xi \to 1$, the streamlines level off and tend, respectively, to the lines $\xi = 0$ with $G = G_{\min} = \Gamma_1/2 = -10.92$ and $\xi = 1$ with $G = G_{\max} = -\Gamma_2/2 = -0.005$.

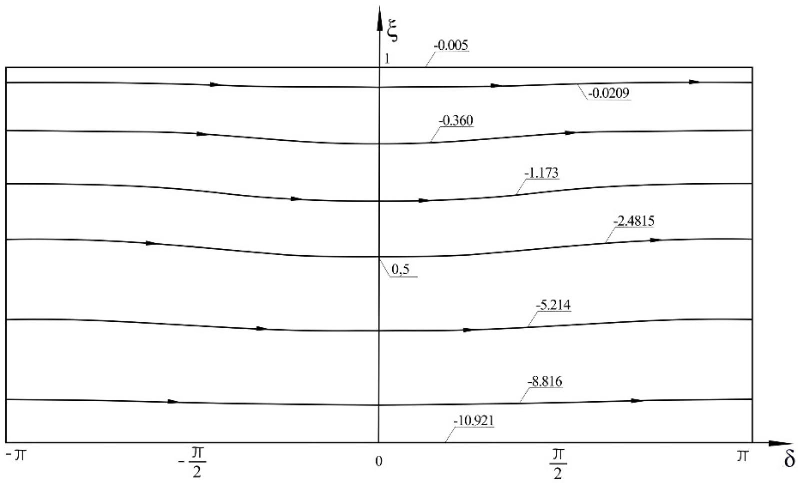

Fig. 8.7: Phase portrait for the case of 1:3 internal resonance at $\Gamma_1 = -21.84$ and $\Gamma_2 = 0.01$.

8.3.4.3 Initial Conditions

In order to construct the final solution of the problem under consideration, i.e. to solve the set of Eqs. (8.119)-(8.122) involving the functions $a_1(T_2)$, $a_2(T_2)$, or $\xi(T_2)$, as well as $\varphi_1(T_2)$, and $\varphi_2(T_2)$, or $\delta(T_2)$, it is necessary to use the initial conditions

$$w(x,y,0) = 0, \tag{8.136}$$

$$\dot{w}(x_0,y_0,0) = \varepsilon V_0, \tag{8.137}$$

$$\frac{p_6}{\omega_2}\, a_1^2(0) + \frac{p_3}{\omega_1}\, a_2^2(0) = E_0. \tag{8.138}$$

The two-term relationship for the displacement w (8.9) within an accuracy of ε according to (8.17) has the form

$$\begin{aligned} w(x,y,t) = \epsilon\Big[X^1_{\alpha\beta}(T_0,T_2)\sin\left(\tfrac{\alpha\pi x}{a}\right)\sin\left(\tfrac{\beta\pi y}{b}\right) \\ +X^1_{\gamma\delta}(T_0,T_2)\sin\left(\tfrac{\gamma\pi x}{a}\right)\sin\left(\tfrac{\delta\pi y}{b}\right)\Big] + O(\varepsilon^3). \end{aligned} \tag{8.139}$$

Substituting (8.24) and (8.25) in (8.139) with due account for (8.118) yields

$$\begin{aligned} w(x,y,t) = 2\epsilon\Big\{a_1(\varepsilon^2 t)\cos\left[\omega_1 t + \varphi_1(\varepsilon^2 t)\right] \\ +a_2(\varepsilon^2 t)\cos\left[\omega_2 t + \varphi_2(\varepsilon^2 t)\right]\Big\}\sin\left(\tfrac{\alpha\pi x}{a}\right)\sin\left(\tfrac{\beta\pi y}{b}\right) \\ +2\epsilon\Big\{\alpha_1 a_1(\varepsilon^2 t)\cos\left[\omega_1 t + \varphi_1(\varepsilon^2 t)\right] + \alpha_2 a_2(\varepsilon^2 t) \\ \times\cos\left[\omega_2 t + \varphi_2(\varepsilon^2 t)\right]\Big\}\sin\left(\tfrac{\gamma\pi x}{a}\right)\sin\left(\tfrac{\delta\pi y}{b}\right) + O(\varepsilon^3). \end{aligned} \tag{8.140}$$

Differentiating (8.140) with respect to time t and limiting ourselves by the terms of the order of ε, we could find the velocity of the shell at the point of impact as follows

$$\begin{aligned} \dot{w}(x_0,y_0,t) = -2\epsilon\Big\{\omega_1(s_1+\alpha_1 s_2)a_1(\varepsilon^2 t)\sin\left[\omega_1 t + \varphi_1(\varepsilon^2 t)\right] + \omega_2(s_1+\alpha_2 s_2) \\ \times a_2(\varepsilon^2 t)\sin\left[\omega_2 t + \varphi_2(\varepsilon^2 t)\right]\Big\} + O(\varepsilon^3). \end{aligned} \tag{8.141}$$

Substituting (8.140) in the first initial condition (8.136) and assuming that $a_1(0) > 0$ and $a_2(0) > 0$, we have

$$\cos\varphi_1(0) = 0, \quad \cos\varphi_2(0) = 0, \tag{8.142}$$

whence it follows that

$$\varphi_1(0) = \pm\frac{\pi}{2}, \quad \varphi_2(0) = \pm\frac{\pi}{2}, \tag{8.143}$$

and

$$\cos\delta_0 = \cos\left[3\varphi_2(0) - \varphi_1(0)\right] = \mp 1, \tag{8.144}$$

i.e.,

$$\delta_0 = \pm\pi(n+1) \quad (n = 0,1,2,...). \tag{8.145}$$

The signs in (8.143) should be chosen considering the fact that the initial amplitudes are positive values, i.e. $a_1(0) > 0$ and $a_2(0) > 0$. Assume for definiteness that

$$\varphi_1(0) = -\frac{\pi}{2}, \quad \varphi_2(0) = -\frac{\pi}{2}. \tag{8.146}$$

Substituting now (8.141) in the second initial condition (8.137) with due account for (8.146), we obtain

$$\omega_1(s_1+\alpha_1 s_2)a_1(0)+\omega_2(s_1+\alpha_2 s_2)a_2(0) = E_0. \tag{8.147}$$

From Eqs. (8.138) and (8.147) we could determine the initial amplitudes

$$a_2(0) = \frac{E_0}{\omega_2(s_1+\alpha_2 s_2)} - \frac{\omega_1(s_1+\alpha_1 s_2)}{\omega_2(s_1+\alpha_2 s_2)}\, a_1(0), \tag{8.148}$$

$$d_1 a_1^2(0) + d_2 a_1(0) + d_3 = 0, \tag{8.149}$$

where

$$d_1 = 1 + \frac{\omega_1^2(s_1+\alpha_1 s_2)^2}{b^2\omega_2^2(s_1+\alpha_2 s_2)^2}, \quad d_2 = -\frac{2E_0\omega_1(s_1+\alpha_1 s_2)}{b^2\omega_2^2(s_1+\alpha_2 s_2)^2},$$

$$d_3 = \frac{E_0^2}{b^2\omega_2^2(s_1+\alpha_2 s_2)^2} - \frac{E_0\omega_2}{p_6}.$$

It should be noted that the initial amplitudes depend not only on the initial velocity of the impactor, but according to (8.148) and (8.149) they are defined also by the parameters of two impact-induced modes coupled by the three-to-one internal resonance (8.109).

Considering (8.144), from (8.132) we find the value of constant G_0, which defines the trajectory of a point on the phase plane

$$G_0 = \frac{4}{V_0^2}\left[\pm\frac{p_3}{\omega_1}\sqrt{\frac{p_3 p_6}{\omega_1\omega_2}}\, a_1(0)a_2^3(0) + \frac{\Gamma_1 p_3^2}{2\omega_1^2}\, a_2^4(0) - \frac{\Gamma_2 p_6^2}{2\omega_2^2}\, a_1^4(0)\right]. \tag{8.150}$$

Thus, we have determined all necessary constants from the initial conditions, therefore we could proceed to the construction of the solution for the contact force.

8.3.4.4 The Contact Force and Shell's Deflection at the Point of Contact

Now knowing $a_1(0)$, $a_2(0)$, $\varphi_1(0)$, and $\varphi_2(0)$, it is possible to calculate the value $P(t)$, which within an accuracy of ε has the form:

$$P(t) = -\varepsilon M\left[\ddot{X}_1^1(t)s_1 + \ddot{X}_2^1(t)s_2\right] + O(\varepsilon^3), \tag{8.151}$$

or with due account for (8.141)

$$P(t) = 2\epsilon M\Big\{\omega_1^2(s_1+\alpha_1 s_2)a_1(\varepsilon^2 t)\cos\big[\omega_1 t+\varphi_1(\varepsilon^2 t)\big] + \omega_2^2(s_1+\alpha_2 s_2)a_2(\varepsilon^2 t)\cos\big[\omega_2 t+\varphi_2(\varepsilon^2 t)\big]\Big\} + O(\varepsilon^3). \tag{8.152}$$

Considering (8.146) and (8.109), Eq. (8.152) is reduced to

$$\begin{aligned} P(t) &= 2\varepsilon M\omega_2^2[9a_1(0)(s_1+\alpha_1 s_2)\sin 3\omega_2 t + a_2(0)(s_1+\alpha_2 s_2)\sin\omega_2 t] \\ &= 18M\varepsilon(s_1+\alpha_1 s_2)\omega_2^2 a_1(0)\sin\omega_2 t\left(3-4\sin^2\omega_2 t+\frac{1}{9}\kappa\right)+O(\varepsilon^3), \end{aligned} \tag{8.153}$$

where the dimensionless coefficient κ is calculated according to (8.97) and is defined by the parameters of two impact-induced modes coupled by the three-to-one internal resonance (8.109), as well as by the coordinates of the point of impact and the initial velocity of impact.

The deflection of the shell at the point of impact could be determined from (8.140) with due account for the found initial values of the phases

$$w(x_0,y_0,t) \approx 2\epsilon\big(\sin\omega_1 t+\kappa\sin\omega_2 t\big)(s_1+\alpha_1 s_2)a_1(0)+O(\varepsilon^2). \tag{8.154}$$

The contact force in the dimensionless form could be written as

$$P^*(\tau) = \left(\frac{3}{4}+\frac{1}{36}\kappa-\sin^2\tau\right)\sin\tau, \tag{8.155}$$

where

$$P^*(t) = \frac{P(t)}{72\varepsilon M\omega_2^2(s_1+\alpha_1 s_2)a_1(0)},$$

while the dimensional deflection for the case of the three-to-one internal resonance has the form

$$w^*(\tau) = \left(\frac{3}{4}+\frac{1}{4}\kappa-\sin^2\tau\right)\sin\tau, \tag{8.156}$$

where

$$w^*(t) = \frac{w(x_0,y_0,t)}{2\varepsilon(s_1+\alpha_1 s_2)a_1(0)}.$$

The dimensionless time $\tau=\omega_2 t$ dependence of the dimensionless contact force P^* defined by (8.155) and of the dimensionless deflection of the target at the point of impact w^* governed by (8.156) are shown, respectively, in Fig. 8.8 (a) and (b) for the different magnitudes of the parameter κ: 0, 3, 6, and 9.

Reference to Fig. 8.8 shows that the increase in the parameter κ results in the increase of the maximal contact force, the duration of contact, as well as the peak of the shell deflection. In other words, from Figure 8 it is evident that the peak contact force and the duration of contact depend essentially upon the parameters of two impact-induced modes coupled by the three-to-one internal resonance (8.109).

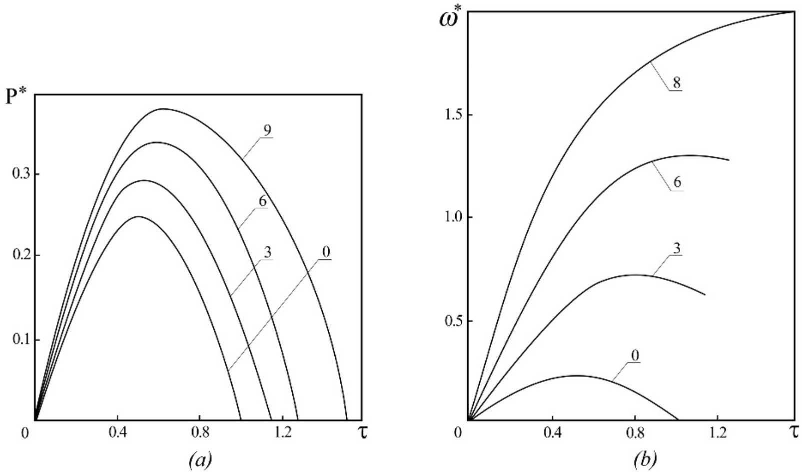

Fig. 8.8: Dimensionless time dependence of (a) the dimensionless contact force and (b) deflection of the target at the point of impact for the case of the internal resonance $\omega_1 = 3\omega_2$.

8.4 Conclusion

In the present paper, a new approach has been proposed for the analysis of the impact interactions of nonlinear doubly curved shallow shells with rectangular base under the low-velocity impact by an elastic sphere. It has been assumed that the shell is simply supported and partial differential equations have been obtained in terms of shell's transverse displacement and Airy's stress function. The equations of motion have been reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements.

The approach utilized in the present paper is based on the fact that during impact only two modes strongly coupled by the two-to-one or three-to-one internal resonance are initiated by the impactor. The influence of impactor's mass on the phenomenon of the impact-induced internal resonance is revealed.

Such an approach differs from the Galerkin method, wherein resonance phenomena are not involved (Zhang et al, 2001). Since it is assumed that shell's displacements are finite, then the local bearing of the shell and impactor's materials is neglected with respect to the shell deflection in the contact region. In other words, the Hertz's theory, which is traditionally in hand for solving impact problems, was not used in the present study; instead, the method of multiple time scales has been adopted, which is used with much success for investigating vibrations of nonlinear systems subjected to the conditions of the internal resonance, as well as to find the time dependence of the contact force.

It has been shown that the time dependence of the contact force depends essentially on the position of the point of impact and the parameters of two impact-induced modes coupled by the internal resonance. Besides, the contact force depends essentially on the magnitude of the initial energy of the impactor. This value governs the place on the phase plane, where a mechanical system locates at the moment of impact, and the phase trajectory, along which it moves during the process of impact.

It is shown that the intricate $P(t)$ dependence at impact-induced internal resonance (8.92) gives way to rather simple sine dependence, what is an accordance with a priori assumption of some researchers about a sine character of the contact force with time (Goldsmith, 1960; Gong et al, 1995; Kunukkasseril and Palaninathan, 1975; Lennertz, 1937; Zhang et al, 2001).

Table 8.1 summarizes the assumptions and principles which are the basis of the theory of impact on thin nonlinear bodies proposed in the present paper. It shows also its distinctive features in comparison with the traditional impact theory for linear thin bodies (a comprehensive review of papers in the field could be found in Rossikhin and Shitikova (2007)), which is based on the principles suggested by Timoshenko in his classical paper on the impact of an elastic sphere upon an elastic beam (Timoshenko, 1913).

Table 8.1: Comparison of main assumptions and principles used in the theories of low-velocity impact upon linear and nonlinear thin bodies

Linear thin body (target)	Nonlinear thin body (target)
1. Displacement of an impactor during the process of impact is the sum of two displacements: displacement of a target at the point of impact and local bearing of impactor and target's materials, i.e. impactor's indentation into the target	1. Displacement of an impactor during the process of impact coincides with target's displacement at the point of impact; local bearing is ignored, since it is assumed that target displacement is much larger than local bearing
2. Local bearing is defined via the Hertzian theory	2. Local bearing is equal to zero
3. It is assumed that all natural modes of vibrations are generated during impact, and therefore target's displacement is expanded over all modes	3. Under nonstationary excitation, only those modes, the natural frequencies of which satisfy certain resonant relationships (conditions of internal resonances), are generated and dominate, resulting in energy interchange between coupled modes
4. In order to obtain the solution, the method of expansion in terms of eigen functions and Hertz contact theory are employed	4. In order to obtain the solution, the method of multiple time scales in combination with different conditions of internal resonances is utilized
5. Contact force and local bearing of the impactor and target's materials are determined from nonlinear integro-differential equations	5. Contact force and target displacement at the place of contact are defined by a set of nonlinear algebraic equations

The procedure suggested in the present paper could be generalized for the analysis of impact response of plates and shells when their motions are described by three or five nonlinear differential equations.

Acknowledgements This research has been supported by the Ministry of Education and Science of the Russian Federation, project No. 9.5138.2017/8.9, and the first author was supported by the Ministry of Education and Science of the Russian Federation, project No. 1.4907.2017/6.7.

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Appendix

$$N_1 = -\left[(p_{13}+\alpha_1^2 p_{14}+\alpha_1 p_{15})(\Omega_2^2-4\omega_1^2 p_{22})+(p_{23}+\alpha_1^2 p_{24}+\alpha_1 p_{25})4\omega_1^2 p_{12}\right] \times\left[16\omega_1^4(p_{11}p_{22}-p_{12}^2)-4\omega_1^2(p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}, \tag{8.157}$$

$$N_2 = -\left[(p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15})(\Omega_2^2-4\omega_2^2 p_{22})+(p_{23}+\alpha_2^2 p_{24}+\alpha_2 p_{25})4\omega_2^2 p_{12}\right] \times\left[16\omega_2^4(p_{11}p_{22}-p_{12}^2)-4\omega_2^2(p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}, \tag{8.158}$$

$$N_3 = -\frac{p_{13}+\alpha_1^2 p_{14}+\alpha_1 p_{15}}{\Omega_1^2}, \qquad N_4 = -\frac{p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15}}{\Omega_1^2}, \tag{8.159}$$

$$N_5 = -2\{[p_{13}+\alpha_1\alpha_2 p_{14}+(\alpha_1+\alpha_2)p_{15}]\left[\Omega_2^2-p_{22}(\omega_1+\omega_2)^2\right]+p_{12}(\omega_1+\omega_2)^2 \times[p_{23}+\alpha_1\alpha_2 p_{24}+(\alpha_1+\alpha_2)p_{25}]\} \times\left[(\omega_1+\omega_2)^4(p_{11}p_{22}-p_{12}^2)-(\omega_1+\omega_2)^2\,(p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}, \tag{8.160}$$

$$N_6 = -2\{[p_{13}+\alpha_1\alpha_2 p_{14}+(\alpha_1+\alpha_2)p_{15}]\left[\Omega_2^2-p_{22}(\omega_1-\omega_2)^2\right]+p_{12}(\omega_1-\omega_2)^2 \times[p_{23}+\alpha_1\alpha_2 p_{24}+(\alpha_1+\alpha_2)p_{25}]\} \times\left[(\omega_1-\omega_2)^4(p_{11}p_{22}-p_{12}^2)-(\omega_1-\omega_2)^2\,(p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}, \tag{8.161}$$

$$E_1 = -\left[(p_{23}+\alpha_1^2 p_{24}+\alpha_1 p_{25})(\Omega_1^2-4\omega_1^2 p_{11})+(p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15})4\omega_1^2 p_{12}\right] \times\left[16\omega_1^4(p_{11}p_{22}-p_{12}^2)-4\omega_1^2(p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}, \tag{8.162}$$

$$E_2 = -\left[(p_{23}+\alpha_1^2 p_{24}+\alpha_1 p_{25})(\Omega_1^2-4\omega_2^2 p_{11})+(p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15})4\omega_2^2 p_{12}\right] \times\left[16\omega_2^4(p_{11}p_{22}-p_{12}^2)-4\omega_2^2(p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}, \tag{8.163}$$

$$E_3 = -\frac{p_{23}+\alpha_1^2 p_{24}+\alpha_1 p_{25}}{\Omega_2^2}, \qquad E_4 = -\frac{p_{23}+\alpha_2^2 p_{24}+\alpha_2 p_{25}}{\Omega_2^2}, \tag{8.164}$$

$$
\begin{aligned}
E_5 = &-2\{[p_{23}+\alpha_1\alpha_2 p_{24}+(\alpha_1+\alpha_2)p_{25}]\left[\Omega_1^2-p_{11}(\omega_1+\omega_2)^2\right] \\
&+p_{12}(\omega_1+\omega_2)^2\left[p_{13}+\alpha_1\alpha_2 p_{14}+(\alpha_1+\alpha_2)p_{15}\right]\} \\
&\times\left[(\omega_1+\omega_2)^4(p_{11}p_{22}-p_{12}^2)-(\omega_1+\omega_2)^2\,(p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1},
\end{aligned} \tag{8.165}
$$

$$
\begin{aligned}
E_6 = &-2\{[p_{23}+\alpha_1\alpha_2 p_{24}+(\alpha_1+\alpha_2)p_{25}]\left[\Omega_1^2-p_{11}(\omega_1-\omega_2)^2\right] \\
&+p_{12}(\omega_1-\omega_2)^2\left[p_{13}+\alpha_1\alpha_2 p_{14}+(\alpha_1+\alpha_2)p_{15}\right]\} \\
&\times\left[(\omega_1-\omega_2)^4(p_{11}p_{22}-p_{12}^2)-(\omega_1-\omega_2)^2\,(p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1},
\end{aligned} \tag{8.166}
$$

$$
K_1 = 3\left(p_{16}+\alpha_1^2 p_{17}\right)+(2p_{13}+\alpha_1 p_{15})(D_1+2D_3)+(2\alpha_1 p_{14}+p_{15})(E_1+2E_3), \tag{8.167}
$$

$$
\begin{aligned}
K_2 = &\,6p_{16}+2\alpha_2(2\alpha_1+\alpha_2)p_{17}+(2p_{13}+\alpha_1 p_{15})2D_4+(2p_{13}+\alpha_2 p_{15})(D_5+D_6) \\
&+(2\alpha_1 p_{14}+p_{15})2E_4+(2\alpha_2 p_{14}+p_{15})(E_5+E_6),
\end{aligned} \tag{8.168}
$$

$$
K_3 = 3\alpha_1\left(\alpha_1^2 p_{26}+p_{27}\right)+(2\alpha_1 p_{23}+p_{25})(E_1+2E_3)+(2p_{24}+\alpha_1 p_{25})(D_1+2D_3), \tag{8.169}
$$

$$
\begin{aligned}
K_4 = &\,6\alpha_1\alpha_2^2 p_{26}+2p_{27}(\alpha_1+2\alpha_2)+(2p_{24}+\alpha_1 p_{25})2D_4+(2p_{24}+\alpha_2 p_{25})(D_5+D_6) \\
&+(2\alpha_1 p_{23}+p_{25})2E_4+(2\alpha_2 p_{23}+p_{25})(E_5+E_6),
\end{aligned} \tag{8.170}
$$

$$
L_1 = 3\left(p_{16}+\alpha_2^2 p_{17}\right)+(2p_{13}+\alpha_2 p_{15})(D_2+2D_4)+(2\alpha_2 p_{14}+p_{15})(E_2+2E_4), \tag{8.171}
$$

$$
\begin{aligned}
L_2 = &\,6p_{16}+2\alpha_1(\alpha_1+2\alpha_2)p_{17}+(2p_{13}+\alpha_2 p_{15})2D_3+(2p_{13}+\alpha_1 p_{15})(D_5+D_6) \\
&+(2\alpha_2 p_{14}+p_{15})2E_3+(2\alpha_1 p_{14}+p_{15})(E_5+E_6),
\end{aligned} \tag{8.172}
$$

$$
L_3 = 3\alpha_2\left(\alpha_2^2 p_{26}+p_{27}\right)+(2p_{24}+\alpha_2 p_{25})(D_2+2D_4)+(2\alpha_2 p_{23}+p_{25})(E_2+2E_4), \tag{8.173}
$$

$$
\begin{aligned}
L_4 = &\,6\alpha_1^2\alpha_2 p_{26}+2p_{27}(2\alpha_1+\alpha_2)+(2p_{24}+\alpha_2 p_{25})2D_3+(2p_{24}+\alpha_1 p_{25})(D_5+D_6) \\
&+(2\alpha_2 p_{23}+p_{25})2E_3+(2\alpha_1 p_{23}+p_{25})(E_5+E_6),
\end{aligned} \tag{8.174}
$$

$$M_1 = p_{16} + \alpha_2^2 p_{17} + (2p_{13} + \alpha_2 p_{15})D_2 + (2\alpha_2 p_{14} + p_{15})E_2, \tag{8.175}$$

$$\begin{aligned} M_2 = {} & 3p_{16} + \alpha_2\,(2\alpha_1 + \alpha_2)\,p_{17} + (2p_{13} + \alpha_1 p_{15})D_2 + (2\alpha_1 p_{14} + p_{15})E_2 \\ & + (2p_{13} + \alpha_2 p_{15})D_6 + (2\alpha_2 p_{14} + p_{15})E_6, \end{aligned} \tag{8.176}$$

$$M_3 = \alpha_2\left(\alpha_2^2 p_{26} + p_{27}\right) + (2p_{24} + \alpha_2 p_{25})D_2 + (2\alpha_2 p_{23} + p_{25})E_2, \tag{8.177}$$

$$\begin{aligned} M_4 = {} & 3\alpha_1\alpha_2^2 p_{26} + p_{27}(\alpha_1 + 2\alpha_2) + (2p_{24} + \alpha_1 p_{25})D_2 + (2\alpha_1 p_{23} + p_{25})E_2 \\ & + (2p_{24} + \alpha_2 p_{25})D_6 + (2\alpha_2 p_{23} + p_{25})E_6. \end{aligned} \tag{8.178}$$

$$\begin{aligned} R_1 = {} & 3p_{16} + \alpha_1^2 p_{17} + (2p_{13} + \alpha_2 p_{15})D_1 + (2\alpha_2 p_{14} + p_{15})E_1 \\ & + (2p_{13} + \alpha_1 p_{15})D_6 + (2\alpha_1 p_{14} + p_{15})E_6, \end{aligned} \tag{8.179}$$

$$\begin{aligned} R_2 = {} & 3\alpha_1^2\alpha_2 p_{26} + p_{27}(2\alpha_1 + \alpha_2) + (2p_{24} + \alpha_2 p_{25})D_1 + (p_{25} + 2\alpha_2 p_{23})E_1 \\ & + (2p_{24} + \alpha_1 p_{25})D_6 + (p_{25} + 2\alpha_1 p_{23})E_6. \end{aligned} \tag{8.180}$$