8

Special Topics: A Short Course on Group Theory

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These lectures provide a brief introduction to group theory, largely focussing on finite groups. After giving the basic definition of groups, we start with a discussion on abelian groups followed by a discussion on non-abelian groups in the next section. Along the way, we define normal subgroups and conjugacy classes and discuss the commutator subgroup and abelianization. In the final sections, we discuss the examples of the Quaternionic group, as well as two examples of continuous groups- the rotation group, in particular its connection with the group of special unitary matrices in two dimensions as well as the conformal group.

8.1 Groups and Physics

Symmetry plays an important role in Physics. In classical theory, Noether's theorem relates symmetries of the action to conservation laws. So for instance, if the action, or Hamiltonian in the phase space formulation, is invariant under rotations, then the system described by the action, has total angular momentum conserved in time, just as invariance of the action under translation in space implies the conservation of total momentum in time. The set of such symmetry transformations which leaves something invariant (the action in this case), forms what is mathematically called a "group". The action of these symmetry transformations on a physical system are described by matrices- called group representations. Taking products of such matrices, corresponds to doing successive symmetry transformations of the system.

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The ideas of symmetry transformations and groups are even more powerful in quantum theory. For instance in quantum mechanics, if a system has some symmetry, then the corresponding group-matrices commute with the Hamiltonian. This fact along with a well known theorem called the Schur's lemma, can help in solving the problem of finding the energy levels of a system in some cases.

Group theory helps in classifying the various particles found in nature, since the various fundamental particles of nature, as described by the standard model of particle physics, correspond to different representations of the corresponding symmetry group.

It is sometimes the case that the ground state is invariant under a smaller set of symmetries than the Hamiltonian. This is known as ' spontaneous symmetry breaking' and plays a crucial role in very important and diverse physical phenomenon, like superconductivity and the 'Higgs mechanism' which gives mass to various fundamental particles in nature. Most phases of matter can be classified by the amount of symmetry that they break. As an example, in crystals, the underlying invariance of the Hamiltonian under continuous translations is broken to a set of discrete translations in the crystalline phase.

So, in short, the study of Group theory is well motivated in Physics. Before discussing some properties and examples of groups, we begin with the basic definition of groups in the next section.

8.2 Basic definition

A Group G is a set of elements $(g_1, g_2, ...)$, equipped with a composition rule, that basically tells us how composing any two elements of this set gives rise to a third $g_i \circ g_j = g_k$. This set along with the composition rule, has to satisfy the following conditions.

- Associativity: $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$
- Existence of an identity element denoted by e such that $e \circ g_i = g_i \; \forall \; i$.
- Existence of an inverse element for every element g_i for which we use the notation by g_i^{-1} such that $g_i^{-1} \circ g_i = e$

As a corollary of these properties, it follows that e and g_i^{-1} are unique and that $g \circ e = g$ and $g \circ g^{-1} = e$.

If the elements of the set G can be labelled by an integer, then the group is called *discrete*, while if its labelled by continuous numbers, then the group is called *continuous*. If a discrete group has a finite number of elements, then the number of elements is called the *order* of the group and is usually denoted by the symbol $|G|$.

In general, the composition rule for the group elements is not commutative. However if in special cases, the composition rule is commutative for all elements, then such a group is called an *abelian* group, else its called *non-abelian*.

The condition that two elements g_i, g_j commute, can be recast as $g_i^{-1} \circ g_j^{-1} \circ g_i \circ g_j = e$. An element of a group $g \in G$ is called a commutator if it can be expressed as $g = a^{-1} \circ b^{-1} \circ a \circ b \equiv [a, b]$. Where a, b are any two elements of G. Stated in terms of the commutator, a group is *abelian* iff the only commutator element of the group is the identity.

Some simple examples of Groups:

- The set of integers under addition ($\mathbb{Z}$).
- The set of continuous rotations of a rigid body
- Permutation of n objects
- The set of unitary $n \times n$ Matrices.

In the next section, we will first discuss the *abelian* case, and then in later sections discuss *non-abelian* groups.

8.3 Abelian Groups

For the *abelian* group, as is standard, we will borrow the notation from $\mathbb{Z}$. We therefore denote the composition rule by the addition symbol (+), the identity element by 0 and the inverse by $-g$. So that $g_i + g_j = g_k$ and $g + (-g) \equiv g - g = 0$.

If a subset of elements of a group, itself forms a group under the same composition law, then its called a subgroup. For example, the group $\mathbb{Z}$ has a subgroup which is simply obtained by multiplying each element by any specific integer say N, that is: $N\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. Given an abelian group G and some subgroup H, we can form a set of *equivalence classes*, called *coset*, where the elements of each class consists of all such elements, (g, x) which are related as: $\{g, x \subset G;$ such that $g = x + h;$ where $h \subset H\}$. It is easy to check that the above relation is an equivalence relation (which we denote as: $(g \sim x)$), because the relation is reflexive, transitive and associative.

- (a) reflexivity: $g \sim g$, because $g = g + 0$
- (b) transitivity: $g \sim x \Rightarrow x \sim g$, because $g = x + h \Rightarrow x = g - h$
- (c) associativity: if $g \sim x$ and $x \sim y \Rightarrow g \sim y$, because if $g = x + h_1$ and $x = y + h_2$, then $g = y + h_1 + h_2$

Each such coset class, is denoted by $[g]$, where g is any representative element of that class. For abelian groups the coset , which is the set of all such distinct equivalence classes, also has a group structure and hence is also called the *Quotient Group*. The notation used for such a Quotient group formed out of G and H is G/H. The identity element of this group denoted by [0] is simply the set of all elements of the type $(0 + h)$ which is just the full subgroup H.

In particular, for $G = \mathbb{Z}$ and $H = N\mathbb{Z}$, the elements of G/H are the equivalence classes $([0], [1], ...[N-1])$. For example with $N = 2$, we have the elements [0] and [1]. Where $[0] = (0, 2, -2, 4, -4, 6, -6, ...)$ and $[1] =$

$(1, -1, 3, -3, 5, -5, 7, -7, ...)$. Its easy to see that, $[0] + [1] = [1]$, $[1] + [1] \equiv 2[1] = [0]$. This Group is therefore in one - one correspondence with the group of two elements $(0, 1)$ under the composition law *addition modulo two*, $(a+b = c \; mod \; 2)$. This group is denoted as $\mathbb{Z}_2$. Similarly for general N, $N[1] = 0$ and the quotient group $\mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}_N$, which is the group of N integers-$(0, 1, ...N-1)$ under the composition rule *addition modulo* N, $(a + b = c \; mod \; N)$

In a general abelian group, the element obtained by adding the same element many times must be again an element of the group. If the group is finite, then it must be the case that for some positive integer N, $g+g+g...(Ntimes) \equiv Ng = 0$. Then, N is called the *order of the element* g. The set of all elements of finite order, in an abelian group forms a sub group, called the *torsion* subgroup and denoted by $t(G)$.

Problem 8.3.1: Show that $t(G)$ is a subgroup.

If all elements of a group is generated by a set of elements $(g_1, g_2, ...g_r)$, their inverse and the identity, the group is called a *finitely generated abelian group* $(FGAG)$. That is, $g = \sum_{i=1}^{r} N_i g_i; \; \forall \; g \in G$. Here, $N_i \in \mathbb{Z}$. If these generators are linearly independent, that is: $0 = \sum_{i=1}^{r} N_i g_i; \; \Rightarrow N_i = 0 \; \forall i$, then the $FGAG$ is called *free*. A free $FGAG$ with r independent generators is said to have rank r. A group generated by one element is called *cyclic*. The group $\mathbb{Z}$ is a cyclic group of order infinity, while $\mathbb{Z}_N$ is a cyclic group of finite order N.

Definition 8.3.1 (Group Homomorphisms). Given two groups (not necessarily abelian) G and H, if there exists a map $f(G)$, which maps elements of G into elements of H, then such a map is called a *Homomorphism*.

1. $f(g) \in H; \; \forall g \in G$
2. $f(g_1 \circ g_2) = f(g_1) \bullet f(g_2); \; \forall g_i \in G$

Here the $\circ$ and the $\bullet$ denote the composition laws in G and H respectively.

The subset of elements (x) of the group G, which maps to the identity in H is called *ker f*. $(x \in kerf; \; f(x) = e)$. While $Im \; f$ is the subset of elements y in H which have been mapped from some element of G. ie: $y = f(g)$.

Problem 8.3.2: Show that both $ker \; f$ and $Im \; f$ are subgroups of G and H respectively.

It is clear that the map from G to $Im \; f$ is not in general an *isomorphism*. That is, the map need not be *one-one onto*. This is so because, more than one element in G, can map to same element in H. $f(g_1) = f(g_2)$. This means that $f(g_1) \bullet f^{-1}(g_2) = e \Rightarrow f(g_1 \circ g_2^{-1}) = e \Rightarrow g_1 \circ g_2^{-1} \in kerf$.

Returning to the case of abelian groups, this means that g_1 - $g_2 = h$, $h \in kerf$. This implies that any two elements which map to the same element in H belong to the same equivalence class, and therefore is a single element of the *Quotient* group $G/kerf$. This *Quotient* group has now elements in one to one correspondence with the elements of $Im\ f$, so that the two groups are isomorphic. ie: $G/kerf \cong Imf$

A few applications of this result are given below.

1. Consider the homomorphism between the group $\mathbb{Z}$ and $\mathbb{Z}_2$. $f(2n) = 0$ and $f(2n+1) = 1$. The kernel of this map $f(x) = 0$ is simply all even integers: ie: $kerf = 2\mathbb{Z}$, so that we recover: $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$.

2. Consider any group H which is a free $FGAG$ of rank r generated by the set $(g_1, ..., g_r)$. This can be thought of as the image of the homomorphism from the group $G = \bigoplus_{i=1}^{r} \mathbb{Z}$. The map being simply: $(N_1, ...N_r) \in G;\ f(g) = \sum_{i=1}^{r} N_i g_i \in H$. Since the group H which is the image of f is a free $FGAG$, the $kerf = \{0\}$. Then it follows that any free $FGAG$ of rank r, $H \cong \bigoplus_{i=1}^{r} \mathbb{Z}$.

3. By a similar argument, any cyclic group G with generator g of finite order N, $(x \in G;\ x = ng;\ Ng = 0)$ is isomorphic to $\mathbb{Z}_N$. To get this result, we simply start from the group $\mathbb{Z}$ and then take $f(n) = ng;\ n \in \mathbb{Z}$. Since $Ng = 0$, it follows that $kerf = N\mathbb{Z}$. Therefore, any cyclic group $G \cong \mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}_N$.

4. More generally, any $FGAG$ with r generators $(g_1, ...g_r)$ not necessarily free, is isomorphic to $\mathbb{Z} \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}...(m\ times) \bigoplus \mathbb{Z}_{k_1} \bigoplus \mathbb{Z}_{k_2} ... \bigoplus \mathbb{Z}_{k_n}$, where the set of integers $(m, k_1, ...k_n)$ are fixed for a given $FGAG$, and $m + n = r$. m is known as the *rank* of the $FGAG$.

We sketch a proof of the last statement below.

Problem 8.3.3: For a $free\ FGAG$ of $rank = r$ with generators $(g_1, ..., g_r)$, the subset of elements generated by $(k_1 g_{i_1}, ...k_p g_{i_p})$ is always a subgroup. Where $(g_{i_i} ....g_{i_p})$ are any p generators from the set of r generators of the *free* $FGAG$ and $(k_1, ..., k_p)$ are all integers.

In fact, it turns out that *all* subgroups of a $FGAG$ can be generated this way. This means, that any subgroup H of a free $FGAG$ is isomorphic to $k_1\mathbb{Z} \bigoplus k_2\mathbb{Z} \bigoplus ... \bigoplus k_p\mathbb{Z}$.

We can now construct a homomorphism, as before, from the group $G = \bigoplus_{i=1}^{r} \mathbb{Z}$ to $H = FGAG$. The map being, again as before, $f(G) = f(N_1, ...N_r) = \sum_{i=1}^{r} N_i g_i$. Then the kernel of this map, being a subgroup of G,

- $kerf \cong k_1\mathbb{Z} \bigoplus k_2\mathbb{Z} \bigoplus ... \bigoplus k_p\mathbb{Z}$, for some set of positive integers $(p, k_1, ...k_p)$.

Then the desired result follows.

- $H = Imf \cong G/kerf = \mathbb{Z} \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}...(m\ times) \bigoplus \mathbb{Z}_{k_1} \bigoplus \mathbb{Z}_{k_2} ... \bigoplus \mathbb{Z}_{k_p}$, with $m + p = r$

8.4 Nonabelian cases: Conjugacy class, cosets

In this section, we will start discussing the *non-abelian* case. We will introduce some definitions, some of which we have already seen in the abelian case, and some which are interesting and non-trivial only when the group is non-abelian. For the non abelian groups, we will use the notation gh for denoting the composition $g \circ h$.

Definition 8.4.1 (*conjugacy classes*)**.** All the elements of a group can be placed in various conjugacy classes. Two elements $(x,\ y)$ of a group G are said to be conjugate to each other, if there exists some element g of the group G, such that $gxg^{-1} = y$. The conjugation relation is an equivalence relation.

Problem 8.4.1: Show that conjugation is an equivalence relation

All elements which are so related belong to the same conjugacy class. It is trivial to see that for abelian groups, there are no non-trivial conjugacy classes. Every element is conjugate only to itself.

- For the group of rotation matrices in three dimensions, all rotations by the same angle but about different axis fall into the same conjugacy class.
- For the group of two dimensional special unitary matrices, all matrices having the same trace are conjugate to each other.

Definition 8.4.2 (*coset*)**.** Given any element (g) in G, and a subgroup H, one can form a set denoted as $[g]$, which has elements $(g, gh_1, gh_2, ..)$; $\forall h_i \in H$. This subset of elements is denoted as $gH \equiv [g]$. A coset is a set whose elements are all such distinct classes $[g_1], [g_2]$.... Its usually denoted as G/H.

Problem 8.4.2: Show that if there are two such classes $[g_1]$ and $[g_2]$, then either they share all elements or none.

The number of elements in each such class $[g]$ is equal to $|H|$- the order of H. Therefore $|G|/|H|$ = number of distinct coset classes. So it follows that for any subgroup H of G, $|H|$ factorizes $|G|$.

Cosets have been introduced in the context of abelian groups. Unlike in the abelian case however, cosets do not form a group in general. This is because the coset classes do not satisfy the composition rule for groups. ie: All elements of $[g_1][g_2] \neq [g_1 g_2]$. This is so because due to the non-abelian nature of the group, $g_1 h_1 g_2 h_2 \neq g_1 g_2 h_3$ for any $g_i \in G$ and $h_i \in H$. Another way of expressing this is $g_1 H g_2 H \neq g_1 g_2 H$, A coset would have been a Group, iff $(g_i H g_i^{-1} = H\ \forall\ i)$. This brings us to the definition of a *normal subgroup*.

Definition 8.4.3 (*Normal Subgroup* and *Quoient Groups*)**.** A subgroup which satisfies the property that $gHg^{-1} = H$; $\forall g \in G$ is called a normal subgroup.

This means that the conjugate of any element of H is also in H. If we construct a coset out of a normal subgroup, then the coset so formed is a Group, called the Quotient Group. For the abelian case, all subgroups are trivially normal.

Problem 8.4.3: Show that $kerf$, where f is a homomorphism between two groups G and H is a normal subgroup of G

8.5 Commutator subgroup and abelianization

As an example of a normal subgroup, we consider the *commutator* sub group. Given a group G, the commutator subgroup $[G, G]$ is the subgroup which is generated by all the commutator elements $[a, b]$. So, the elements of the subgroup $[G, G]$ are $(x \in [G, G];\ x = [a_1, b_1]^{n_1}[a_2, b_2]^{n_2}...[a_i, b_i]^{n_i}...)$. This automatically implies that the product of two commutator elements is also a commutator element. All such elements clearly form a group. In fact it forms a Normal subgroup. This follows from the following two properties.

- $[a, b]^{-1} = b^{-1}a^{-1}ba \equiv [b, a]$
- $g[a, b]g^{-1} = [a_c, b_c]; \quad a_c = gag^{-1}$ and $b_c = gbg^{-1}$

This further means that

1. $\forall\, x \in [G, G],\ x^{-1} \in [G, G]$
2. $\forall\, x \in [G, G]\ and\ \forall g \in G,\ gxg^{-1} \in [G, G]$

- The first property along with the fact that the identity is also a commutator element implies that $[G, G]$ is a subgroup.
- From the second property, it follows that $[G, G]$ is a normal subgroup.

The Quotient group $G/[G, G]$ is always *abelian*. This means that $g_1[G, G]g_2[G, G] = g_2[G, G]g_1[G, G]$. As a check of this statement, do the following problem.

Problem 8.5.1: Show $g_1[a_1, b_1]g_2[a_2, b_2] = g_2[a_1^c, b_1^c]g_1[g_1, g_2][a_2, b_2]$ where $a^c = g_2^{-1}g_1ag_1^{-1}g_2$ and similarly for b^c.

This process of constructing an abelian group from a non-abelian group is called *abelianization*.

8.6 Examples of Groups

In this and the following section, we will consider various examples of groups.

8.6.1 The Quaternionic Group

The Quaternionic group is the set of elements denoted as $(1, -1, \mathbf{i}, \mathbf{j}, \mathbf{k}, -\mathbf{i}, -\mathbf{j}, -\mathbf{k})$, with the composition rule: $\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. $(1,\ -1)$ commute with all elements. In particular, 1 is the identity element and -1 has order two. (**-i**, **-j**, **-k**) are the inverse elements of (**i**, **j**, **k**) respectively

One representation of these elements are as 2×2 σ matrices and the identity matrix.

Problem 8.6.1: Check that the following representations of the elements of the Quaternionic group, indeed satisfy the group composition rules.
$\mathbf{i} \equiv i\sigma_1$, $\mathbf{j} \equiv i\sigma_2$, $\mathbf{k} \equiv -i\sigma_3$, $\pm 1 \equiv \pm\mathbb{I}_{2\times 2}$

Problem 8.6.2: Write down all the conjugacy classes of the group.

Problem 8.6.3: Write down all the subgroups of the Quaternionic group. Check that the order of these subgroups are indeed factors of the order of the Quaternionic group.

Problem 8.6.4: Check which of these subgroups are normal subgroups.

Problem 8.6.5: Using the definition of the commutator subgroup show that for the Quaternionic group, the commutator subgroup is $\{1, -1\}$.

Problem 8.6.6: *Abelianization of the Quotient Group*: Since the commutator subgroup is a Normal subgroup, the coset formed out of it $G/[G, G]$ is a Quotient group. We will now check that it is abelian and identify which abelian group it is.

- Find the cosets gH where H is the commutator subgroup. Show that they are $(1, -1)$, $(\mathbf{i}, -\mathbf{i})$, $(\mathbf{j}, -\mathbf{j})$, $(\mathbf{k}, -\mathbf{k})$
- Denoting each of these classes as $[1], [i], [j], [k]$ respectively, show that they form an abelian group. Show explicitly that the composition rule of the Quotient group is $[i][i] = [j][j] = [k][k] = [1]$, $[i][j] = [j][i] = [k]$, $[j][k] = [k][j] = [i]$

  and $[j][i] = [i][j] = [k]$
- Using the notation for abelian groups, where we use $+$ for composition, 0 for identity, so that $[1] = 0$, write the above rules as: $2[i] = 2[j] = 2[k] = 0$ and $[i] + [j] = [k]$, $[j] + [k] = [i]$, $[i] + [k] = [j]$.
- Simplify the above and show that not all are independent relations. Show that the independent relations are:
- $2[i] = 2[j] = 0$ and $[i] + [j] = [k]$.
- Hence show that the Quotient group obtained by abelianization of the Quaternionic group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

8.6.2 Rotations of a rigid body

Now we will consider examples of continuous groups. The first example we consider is that of continuous rotations of a rigid body. These are characterized by 3×3 marices which satisfy

$$RR^\tau = R^\tau R = I_{3\times 3}, \quad \text{with } \det[\mathrm{R}] = +1 \tag{8.1}$$

Here R^τ is the transpose matrix. Its easy to check, the set of all such matrices form a group. This is the group $SO(3)$. Here R is a three dimensional real matrix. The above equation reduces the number of independent elements of the matrix to just three. Under rotations, the length of a vector **V**remains invariant. ie $\mathbf{RV} \cdot \mathbf{RV} = \mathbf{V} \cdot \mathbf{V}$. Let W be an eigenvector of R with eigenvalue λ. Physically we expect that under rotation the vector remains real, so that we take λ to be real. Then,

$$\mathbf{RW} \cdot \mathbf{RW} = \lambda^2 \mathbf{W} \cdot \mathbf{W} = \mathbf{W} \cdot \mathbf{W}$$

Also the determinant condition means that $\lambda_1 \lambda_2 \lambda_3 = 1$. Taken together, this means that there exists atleast one eigenvector with eigenvalue $= +1$. Therefore this eigenvector does not change under rotations. This eigen-direction is called the *axis of rotation*.

Problem 8.6.7: Choose a basis, where the $z - axis$ is the *axis of rotation*, and the x and y axes are any two orthonormal directions. In this basis show, using equation(1), that R can be written as follows:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

The angle θ is called the *angle of rotation*. So that any Rotation matrix is fully characterized by an *angle and an axis of rotation*. It follows from this form of R that $cos(\theta) = \frac{\mathrm{tr}R-1}{2}$. So the angle of rotation is given in terms of the trace of R.

So any rotation is characterized by $(\hat{\mathbf{n}},\ \theta)$. We can now parametrize this space as follows.

- Let $\hat{\mathbf{n}}$ be any direction on the sphere. This means that we distinguish between the axis of rotations $\hat{\mathbf{n}}$ and -$\hat{\mathbf{n}}$.

- Then we measure θ in counter-clockwise direction of $\hat{\mathbf{n}}$. This way, we can restrict the range of θ to values $0 \leq \theta < \pi$. This is because, in this way of parametrizing, rotation by angle $\pi + \theta$ in counter clockwise direction to $\hat{\mathbf{n}}$, is same as counter clockwise rotation by $\pi - \theta$ around -$\hat{\mathbf{n}}$.

- In particular, rotation by π around $\hat{\mathbf{n}}$ is same as rotation by π around -$\hat{\mathbf{n}}$.

- We can now geometrically represent the space of all $SO(3)$ matrices as points inside a ball of size π, where the radial direction is the $\hat{\mathbf{n}}$, and the length along this direction is θ and because of the identification mentioned in the previous bullet, the surface of this ball has its diametrically opposite points identified. Therefore the configuration space of $SO(3)$ Matrices is in one to one correspondence with the points of a $Ball/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action identifies the opposite points on the surface of the ball.

There is another way in which we can represent these $SO(3)$ matrices. Since these represent continuous rotations, we can build a finite rotation as a series of several smaller rotations, in the limit where each rotation is taken to be infinitesimal and the number of such rotations is taken to be infinity. ie:

$$R = \Pi_{i=1}^{N} R(\epsilon), \;\; lt : (N \to \infty, \; \epsilon \to 0, \;\; N\epsilon = 1). \tag{8.2}$$

Here $R(\epsilon)$ is a infinitesimal rotation. An infinitesimal rotation can be parametrized as

$$R(\epsilon) = \mathbb{I}_{3\times 3} + \epsilon \mathbf{T} \tag{8.3}$$

Using the equation $RR^{\tau} = \mathbb{I}_{3\times 3}$, we can show that $\mathbf{T}$ is an antisymmetric matrix. Any antisymmetric matrix in three dimensions may be parametrized in terms of three *generators* $\{T_i\}$, $\mathbf{T} = \sum_{i=1}^{3} t_i T_i$, where t_i are independent parameters, and

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

These generators satisfy the following algebra: $[T_i, T_j] = \epsilon_{ijk} T_k$, where$[A, B] = AB - BA$ and ϵ_{ijk} is the completely antisymmetric tensor with entries ± 1. $\epsilon_{123} = 1$. We can now build a finite rotation, as a product of such infinitesimal rotations.

$$R(t_1, t_2, t_3) = \lim_{N \to \infty} \left(\mathbb{I}_{3\times 3} + \frac{1}{N} \sum_{i=1}^{3} t_i T_i \right)^N = \exp\left(\sum_{i=1}^{3} t_i T_i \right) \tag{8.4}$$

It is more conventional to express

$$R(t_i) = \exp\left(i \sum_{i=1}^{3} \theta_i L_i \right) = \exp(i\theta \hat{\mathbf{n}} \cdot \mathbf{L}) \tag{8.5}$$

Where $L_i = iT_i$, $\theta_i = -t_i$, $(\theta = \sqrt{t_1^2 + t_2^2 + t_3^2})$, $\theta \; \hat{\mathbf{n}} = (\theta_1, \theta_2, \theta_3)$ and $\mathbf{L} = (L_1, L_2, L_3)$.

The L_i's are Hermitian matrices and satisfy the following algebra: $[L_i, L_j] = i\epsilon_{ijk} L_k$. This algebra is identical to that satisfied by the two dimensional σ matrices. As we will see in the next section, the σ matrices are the generators of the $SU(2)$ matrices in two dimensions. This isomorphism between the two algebras is because of the fact that the $SU(2)$ and $SO(3)$ groups are related by a homomorphism, which we will discuss in the next section.

8.7 $SU(2)$ and $SO(3)$

In this final section, we will discuss the group of two dimensional $SU(2)$ matrices and write down a homomorphism from this group to the group of $SO(3)$ matrices.

8.7.1 $SU(2)$ matrices

$SU(2)$ matrices in two dimensions are complex matrices with determinant +1 and satisfying

$$UU^\dagger = \mathbb{I} \tag{8.6}$$

Its easy to show that any such matrix can be parametrized as

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = \begin{pmatrix} a_0 + ia_3 & a_2 + ia_1 \\ -a_2 + ia_1 & a_0 - ia_3 \end{pmatrix} = a_0\mathbb{I} + i\mathbf{a}\cdot\boldsymbol{\sigma}$$

where, $\alpha = a_0 + ia_3$, $\beta = a_2 + ia_1$ are complex numbers and $(a_0^2 + \mathbf{a}\cdot\mathbf{a} = 1)$ $\mathbf{a} = (a_1, a_2, a_3)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

Problem 8.7.1: Show it.

Therefore it follows that the configuration space of all $SU(2)$ matrices are in one-one correspondence with the points on a S^3.

Parametrizing $a_0 = \cos(\theta)$ and $|\mathbf{a}| = \sin(\theta)$, we can write: $U = e^{i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$, where $\hat{\mathbf{n}} = \frac{\mathbf{a}}{|\mathbf{a}|}$. So the σ matrices are the generators of the $SU(2)$ group in the same sense that $\mathbf{L}_i$'s were of the $SO(3)$ group. Also, the $SU(2)$ and $SO(3)$ generators satisfy the same algebra. We will now see that the two groups are homomorphic.

Homomorphism from $SU(2)$ to $SO(3)$

We start with the observation that for every 3 dimensional vector one can write a corresponding two dimensional traceless, hermitian matrix as follows:

$$\mathbf{V} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \Rightarrow \upsilon = \begin{pmatrix} V_3 & V_- \\ V_+ & -V_3 \end{pmatrix} = \mathbf{V}\cdot\boldsymbol{\sigma}, \quad V_\pm = V_1 \pm iV_2$$

The length of the vector is related to the determinant of the matrix as: $|V|^2 = -\det(\upsilon)$.

After rotation vector $\mathbf{V}$ goes over to vector $\mathbf{V}' = \mathbf{RV}$ with the same length. $|V'| = |V|$. This means that the corresponding two dimensional matrices, υ and $\upsilon' = \mathbf{V}'\cdot\boldsymbol{\sigma}$ must have the same determinant. Since both these matrices are Hermitian, have the same trace and same determinant, it must be that they are related by a unitary transformation as given below:

$$\upsilon' = U_R \upsilon U_R^\dagger \tag{8.7}$$

In fact we can take the U_R to be $SU(2)$ matrices, because the $det(U)$ and $det(U^\dagger)$ cancel each other in the RHS of the above equation. So, we can conclude that the rotation of a vector $\mathbf{V}$ in three dimensions, corresponds, in two dimensions, to acting on the matrix υ by a $SU(2)$ matrix and its inverse from the left and right respectively.

We can use this equation to find an explicit map from $SU(2)$ to the R matrices. We first note that:

$$v' = \sum_i V_i'\sigma_i = \sum_i R_{ij}V_j\sigma_i = (a_0\mathbb{I} + i\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{V}\cdot\boldsymbol{\sigma})(a_0\mathbb{I} - i\mathbf{a}\cdot\boldsymbol{\sigma}), \tag{8.8}$$

where R_{ij}'s are the components of the rotation matrix.

One can then simplify the LHS further by using the following identities:

$$(\mathbf{V}\cdot\boldsymbol{\sigma})(\mathbf{W}\cdot\boldsymbol{\sigma}) = (\mathbf{V}\cdot\mathbf{W})\mathbb{I} + i(\mathbf{V}\times\mathbf{W})\cdot\boldsymbol{\sigma}, \tag{8.9a}$$

$$\mathbf{V}' = a_0^2\mathbf{V} + \mathbf{a}(\mathbf{V}\cdot\mathbf{a}) + (\mathbf{V}\times\mathbf{a})\times\mathbf{a} + 2a_0(\mathbf{V}\times\mathbf{a}). \tag{8.9b}$$

We then get

$$R_{ij}V_j = \Big((a_0^2 - \mathbf{a}\cdot\mathbf{a})\delta_{ij} + 2a_ia_j + 2a_0\epsilon_{ijk}a_k\Big)V_j. \tag{8.10}$$

Then comparing the coefficient of V_j on both sides we get the desired explicit form of R_{ij}

$$R = \begin{pmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & 2(a_1a_2 + a_0a_3) & 2(a_1a_3 - a_0a_2) \\ 2(a_1a_2 - a_0a_3) & a_0^2 + a_2^2 - a_1^2 - a_3^2 & 2(a_2a_3 + a_0a_1) \\ 2(a_1a_3 + a_0a_2) & 2(a_2a_3 - a_0a_1) & a_0^2 + a_3^2 - a_1^2 - a_2^2 \end{pmatrix},$$

where as before, $a_0^2 + \mathbf{a}\cdot\mathbf{a} = 1$. $R(a_0, \mathbf{a})$ is therefore the desired map which takes an element of $SU(2)$ labelled by $(a_0, \mathbf{a})$ to an element of $SU(3)$. We can easily show that the kernel of this map is $a_0^2 = 1$, $\mathbf{a} = \mathbf{0}$. ie: $R(\pm 1, \mathbf{0}) = \mathbb{I}$. Therefore the kernel is simply $\pm\mathbb{I}_{2\times 2}$. This is the subgroup $\mathbb{Z}_2$. $\mathbb{Z}_2$ is called the *centrer* of $SU(2)$. In general, the *centre* of a group is the subset of all elements which commutes with every other element. For $SU(N)$ group the centre is $\mathbb{Z}_N$. So we get the relation that $SU(2)/\mathbb{Z}_2 \cong SO(3)$. Thus there is two-one relation between elements of the $SU(2)$ and $SO(3)$.

We can further identify the angle and axis of rotation with the φ and $\hat{\mathbf{n}}$ appearing in the parameterization of the $SU(2)$ matrix $U_R = e^{i\varphi\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$ as follows: $\theta = 2\varphi$ and angle of rotation is same as $\hat{\mathbf{n}}$. This is left as an exercise below. This relation between the angles θ and φ is related to the existence of half integral spin particles in quantum mechanics— a direct consequence of the $SU(2)$ representation of rotations.

Problem 8.7.2: Prove the identity given by Eq. (8.9a).

Problem 8.7.3: Using the identity of Eq. (8.9a), show Eq. (8.9b).

Problem 8.7.4: By choosing $\mathbf{V} \parallel \mathbf{a}$, show that $\mathbf{V}' = \mathbf{V}$, and thus prove that $\hat{\mathbf{n}}$ is indeed the direction of the axis of rotation.

Problem 8.7.5: By choosing $\mathbf{V} \perp \mathbf{a}$, and using the fact that the angle of rotation $\cos(\theta) = \mathbf{V}'\cdot\mathbf{V}/|V|^2$, show that $\theta = 2\varphi$. So that U_R can be parametrized directly in terms of the axis and angle of rotation as $U_R = e^{i(\theta/2)\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}$.

8.8 Conformal transformations

Conformal transformations are transformations which preserve the angle between curves. The angle between two curves at a point of intersection p is the angle that the tangents to the two curves make at p. It is given by

$$\cos(\theta) = \frac{d\mathbf{s}_1.d\mathbf{s}_2}{|d\mathbf{s}_1||d\mathbf{s}_2|},$$

where $d\mathbf{s}_1$ and $d\mathbf{s}_2$ are infinitesimal vectors along the tangent directions at the point p. It is clear, from the expression of the angle that it is invariant under:

- global translations $(x \to x + a)$,
- global rotations(or Lorentz transformations, depending on whether the space is Euclidean or Minkowskian)$(x_i \to R_{ij}x_j)$
- global scaling of coordinates $(x \to \lambda x)$. In this case, Both $d\mathbf{s}_1.d\mathbf{s}_2$ and $|d\mathbf{s}_1||d\mathbf{s}_2|$ scale by a factor of λ^2, but these factors cancel in the denominator and numerator terms.
- In fact, the expression would be invariant if the numerator and denominator both scaled locally by the same function $f(x)$, which could then cancel among each other. This would mean demanding that $ds_1.ds_2 \to f(x)ds_1.ds_2$ and $|ds_1||ds_2| \to f(x)|ds_1||ds_2|$. We can now ask, what kind of coordinate transformations $x \to x'(x)$ can achieve such a local scaling.

To find such coordinate transformations, it is useful to consider the invariant distance $dS^2 = g_{ij}dx^i dx^j$. The distance dS^2 is defined to be invariant under any general coordinate transformations, which means that under any coordinate transformations, when

$$dx^i \to \frac{\partial x'^i}{\partial x^j}dx^j, \quad \text{the metric} \quad g_{ij} \to \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}g_{kl},$$

so that the statement that $ds_1.ds_2 \to f(x)|_p ds_1.ds_2$ is the same as demanding that the metric $g_{ij} \to \frac{1}{f(x)}g_{ij}$.

- So a conformal transformation is a coordinate transformation $x'(x)$ under which $g_{ij}(x) \to g'_{ij}(x') = h(x)g_{ij}(x)$ where $h(x)$ is any local function of the coordinates.
- It is easy to see that these transformations form a group. Under two such transformations, $x \to x'(x) \to x''(x'(x))$, the metric will transform as

  $$g_{ij}(x) \to g'_{ij}(x'(x)) = h'(x)g_{ij}(x),$$

  $$g'_{ij}(x') \to g''_{ij}(x'') = h''(x')g'_{ij}(x') = h''(x'(x))h'(x)g_{ij}(x) = \hat{h}(x)g_{ij}(x),$$

  where $\hat{h}(x) = h''(x'(x))h'(x)$. The inverse coordinate transformation, assuming it exists, is the inverse element while the identity is $x'(x) = x$.

We study these transformations by first looking at the infinitesimal forms of coordinate transformations. So, we look at $x'(x) = x + \epsilon(x)$ and $h(x) = 1 + g(x)$. Under these transformations, the metric change is as follows:

$$\frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} g_{ab}(x) = h(x) g_{ij}(x), \tag{8.11}$$

which for infinitesimal transformations becomes

$$(\delta_i^a - \partial_i \epsilon^a)(\delta_j^b - \partial_j \epsilon^b) g_{ab} = (1 + g(x)) g_{ij}(x) \tag{8.12}$$

We will be working mostly on flat space, so that we can take $g_{ij} = \delta_{ij}$. Then the above equation simplifies to:

$$\partial_i \epsilon_j + \partial_j \epsilon_i = -g(x)\delta_{ij} \tag{8.13}$$

We will first analyze these equations in two dimensions, where it is simple. In two dimensions, i, j take values $1, 2$. Then the above two equations are:

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \quad \text{and} \quad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \tag{8.14}$$

These are just the Cauchy-Riemann conditions, and it means that in two dimensions, under a conformal transformations, $(z = x + iy) \to f(z)$ and similarly for $\bar{z} = x - iy \to \bar{f}(\bar{z})$. So any holomorphic transformation is a conformal transformation in two dimensions.

We now try to construct the generators of these conformal transformations. To find the generators, let us first recall how it is done for the case of translations and rotations.

- The generator of translations is simply the derivative operator $\frac{\partial}{\partial x^i}$. Any function $f(x + a) = e^{a\partial_x} f(x)$. For a infinitesimal,

  $$f(x + a) = f(x) + a\partial_x f(x) = (1 + a\partial_x) f(x).$$

  In particular for $f(x) = x$, $x' = x + a = (1 + a\partial_x)x$.

- Similarly for the case of two dimensional infinitesimal rotations $x' = x + y\epsilon$ and $y' = y - \epsilon x$, we know that the generator is simply $(x\partial_y - y\partial_x)$, as for any function

  $$f(x', y') = f(x + \epsilon y, y - x\epsilon) = f(x, y) + \epsilon(y\partial_x - x\partial_y) f(x, y).$$

  In particular, for $f(x) = x$ we have $x' = (1 + \epsilon(y\partial_x - x\partial_y))x$ and similarly $y' = (1 + \epsilon(y\partial_x - x\partial_y))y$.

So, now we can use the same method to find the generators for the conformal transformations in two dimensions. We first write

$$z' = z + \epsilon(z) = \left[1 + \left(\sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}\right)\partial_z\right] z, \tag{8.15}$$

where in the RHS, we have done a Laurent expansion of $\epsilon(z)$ and ϵ_n are its infinitesimal coefficients. So the generators are $L_n = z^{n+1}\partial_z$. We have similar generators from the anti holomorphic transformations on $\bar{z}$, viz., $\tilde{L}_n = \bar{z}^{n+1}\partial_{\bar{z}}$. These generators, satisfy the following algebra:

$$[L_n, L_m] = (m-n)L_{m+n}, \quad [\tilde{L}_n, \tilde{L}_m] = (m-n)\tilde{L}_{m+n} \tag{8.16}$$

These are the generators of the conformal transformations in two dimensions. In $d > 2$ however, the analysis of Eq. (8.12) will lead to a finite set of symmetries. The analog of these in two dimensions are the ones generated by the following subset, which form a subalgebra— the set generated by (L_{-1}, L_0, L_{+1}) and $(\tilde{L}_{-1}, \tilde{L}_0, \tilde{L}_{+1})$. We will try to see what finite transformations these correspond to.

1. $L_{-1} + \tilde{L}_{-1} = \partial_z + \partial_{\bar{z}} = \partial_x$,
2. $i(L_{-1} - \tilde{L}_{-1}) = \partial_y$,
3. $L_0 + \tilde{L}_0 = z\partial_z + \bar{z}\partial_{\bar{z}} = x\partial_x + y\partial_y = r\partial_r = \partial_{\ln(r)}$, (r is the usual radial coordinate)
4. $L_0 - \bar{L}_0 = x\partial_y - y\partial_x$,
5. $L_1 = z^2\partial_z = -\partial_{z^{-1}}$.

From the above, we can conclude that:

1. $L_{-1} \pm \tilde{L}_{-1}$ generate translations,
2. $L_0 + \tilde{L}_0 = \partial_{\ln(r)}$, from analogy with translations, generates translations along $\ln r$. But translations along $\ln r$ corresponds to constant scaling of r and therefore of x, y. So this is the generator for constant scaling.
3. $L_0 - \tilde{L}_0$ is the generator of rotation.
4. $L_1 = -\partial_{z^{-1}}$, again generates translations in $\frac{1}{z}$. So it takes
$$\frac{1}{z} \to \frac{1}{z} + a \Rightarrow z \to \frac{z}{1+az},$$
where a is complex. Written in terms of $\mathbf{X} = (x, y)$ the transformation is
$$\mathbf{X}' = \frac{\mathbf{X} + \mathbf{a}(|X|^2)}{1 + 2\mathbf{a}\cdot\mathbf{X} + |a|^2|X|^2}, \quad \mathbf{a} = (a_1, a_2). \tag{8.17}$$
This is known as the *Special Conformal transformation(SCT)*
5. Taken together the translations, rotations, scaling and special conformal transformations together form a group — the global conformal group. For any $d \geq 2$ this group is isomorphic to $SO(d+1, 1)$, which is the Lorentz group in $d+2$ dimensions. For $d > 2$, this is the full conformal group, while in $d = 2$ this group of transformations is a part of the infinite dimensional transformations, that we discussed.

References

[1] For the discussion on abelian groups, I have mostly followed the book by M. Nakahara, *Geometry, Topology and Physics* (Taylor & Francis, Boca Raton, FL, USA, 2003). *Chapter-3*, *section- 3.1*

[2] The discussion on $SO(3)$ and the homomorphism with $SU(2)$ is discussed in the classical mechanics textbook by J. V. José and E. J. Saletan, *Classical Dynamics: A contemporary approach*, (Cambridge U. Press, 1998). *chapter-8*, *section-8.2.1, 8.4*.

[3] The discussion on conformal transformations is available in many places. One place is the text book by P. Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 2012).