Edgeworth Equilibria of Economies and Cores in Multi-choice NTU Games

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Abstract. In this paper, we derive an extension of the payoff-dependent balanced core existence theorem by Bonnisseau and Iehlé [Games Econ. Behav. 61 (2007) 1–26] to multi-choice NTU games which implies a multi-choice extension of Scarf's core existence theorem.

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1 Introduction

In 1881, Edgeworth proved that, in the case of two agents and two commodities, the core of an exchange economy shrinks to the set of Walrasian (competitive) equilibrium allocations. He then claimed that his result applies for an arbitrary number of commodities and agents. Many years later, Debreu and Scarf [6] proved Edgeworth's conjecture by showing that when the economy is replicated, the intersection of the cores of the sequence of the replications coincides with the set of Walrasian equilibrium allocations. Recently, Liu and Liu [13] extended Debreu-Scarf Theorem to coalition production economies.

In 1987, Aliprantis et al. [1] defined Edgeworth equilibrium as any feasible allocation such that the r-fold repetition of it belongs to the core of r-fold replica of the economy for every $r \geq 1$ and proved the existence of Edgeworth equilibrium for pure exchange economies with infinite-dimensional commodity spaces for ordered case. Later, Florenzano [8] proved the existence of Edgeworth equilibrium for exchange economies without ordered preferences. Clearly, the classical result by Debreu and Scarf [6] shows that Edgeworth equilibrium is equivalent to competitive equilibrium for pure exchange economies.

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Edgeworth equilibria of coalition production economies are closely related to cores in multi-choice NTU games. For more on multi-choice games, please see [5,9–11] and [14]. In this paper, we derive an extension of the payoff-dependent balanced core existence theorem by Bonnisseau and Iehlé [4] to multi-choice NTU games which implies a multi-choice extension of Scarf's core existence theorem.

2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be the set of all players. Any non-empty subset of N is called a *(crisp) coalition*. Throughout this paper, we denote the collection of all coalitions (non-empty subsets) of N by $\mathcal{N}$ and for any $a, b \in \mathbb{R}^n$, $a \leq b$ means $a_i \leq b_i$ for each $1 \leq i \leq n$, and $a \gg b$ means each coordinate $a_i > b_i$ for $1 \leq i \leq n$. For each $S \in \mathcal{N}$, denote e^S to be the vector in $\mathbb{R}^n$ with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. We use e^i for $e^{\{i\}}$ for each $i \in N$.

The concept of multi-choice games first introduced by Hsiao and Raghavan [9] (and [10]). Suppose each player $i \in N$ has $m_i + 1$ ($m_i \geq 1$) activity levels from $M_i = \{0, 1, \ldots, m_i\}$ and let $M = (\prod_{i \in N} M_i) \setminus \{0\}$. For each $\mu \in M$, let $A(\mu) = \{i \in N | \mu_i > 0\}$. The following concept of a multi-choice NTU game and subsequent concepts are natural extensions to the corresponding concepts for NTU games (see [16,18]).

Definition 2.1. A multi-choice NTU game in coalition form is (M, V), where V is a mapping that maps each $\mu \in M$ to a subset $V(\mu)$ of $\mathbb{R}^n$ and satisfies the following conditions:

(1) For each $\mu \in M$, $V(\mu)$ is nonempty, closed, comprehensive (i.e., if $x, y \in \mathbb{R}^n$ are such that $y \in V(\mu)$ and $x \leq y$, then $x \in V(\mu)$), bounded from above by $D > 0$ (in the sense that if $x \in V(\mu)$, then $x_i \leq D$ for all $i \in A(\mu)$);
(2) For each $\mu \in M$, $V(\mu)$ is cylindrical in the sense that if $x \in V(\mu)$ and $y \in \mathbb{R}^n$ such that $y_i = x_i$ for each $i \in A(\mu)$, then $y \in V(\mu)$;
(3) For every i, there is a $b_i > 0$ such that $V(m_i e^i) = \{x \in \mathbb{R}^n | x_i \leq b_i\}$.

Denote $m = (m_i)_{i \in N}$. Note that in a multi-choice NTU game (M, V), m plays the same role as the grand coalition N in an NTU game. Clearly, an NTU game V is a special multi-choice NTU game (M, V) with $m_i = 1$ for all $i \in N$.

A payoff vector to a multi-choice game (M, V) is a vector $(x_{ij})_{1 \leq i \leq n, 0 \leq j \leq m_i}$, where x_{ij} denotes the increase in payoff for player i corresponding to a change of activity from level $j - 1$ to level j and $x_{i0} = 0$ for all $i \in N$. Note that for a multi-choice NTU game (M, V) defined by Definition 2.1, a payoff vector x in each $V(\mu)$ means $x = (x_i)_{i \in N}$ with $x_i = \sum_{0 \leq j \leq m_i} x_{ij}$. Also note that a multi-choice TU game (M, v) with the characteristic function v is a special multi-choice NTU game (M, V) such that for each $\mu \in M$,

$$V(\mu) = \{x \in \mathbb{R}^n | \sum_{i \in A(\mu)} x_i \leq v(\mu)\}. \tag{2.1}$$

Given a multi-choice game (M, V), a payoff vector $x \in V(m)$, and a member $\mu \in M$, we say that μ has an *objection* against x if there exists some $y \in V(\mu)$ such that $y_i > x_i$ for all $i \in A(\mu)$.

Definition 2.2. The core of a multi-choice game (M, V), denoted by $C(M, V)$, consists of all payoff vectors in $V(m)$ that have no objections against them, that is,

$$C(M, V) = V(m) \setminus [\cup_{\mu \in M} \; int(V(\mu))]. \tag{2.2}$$

Let Δ^N be the standard simplex:

$$\Delta^N = \{x \in \mathbb{R}^n | x_i \geq 0 \text{ for each } i \in N \text{ and } \sum_{i=1}^{n} x_i = 1\}.$$

For each $\emptyset \neq S \subseteq N$, denote

$$\Delta^S = \{x \in \Delta^N | x_i = 0 \text{ for each } i \notin S\} = \{x \in \Delta^N | \sum_{i \in S} x_i = 1\}$$

and for each $S \in \mathcal{N}$, define $m^S \in \Delta^N$ by

$$m^S = \frac{e^S}{|S|}.$$

Denote Δ to be the Cartesian product of $\Delta^{A(\mu)}$ over all $\mu \in M$, i.e.,

$$\Delta = (\Delta^{A(\mu)})_{\mu \in M} = \{(\pi_\mu)_{\mu \in M} | \pi_\mu \in \Delta^{A(\mu)} \text{ for each } \mu \in M\}.$$

Definition 2.3. A collection $\mathcal{B} \subseteq M$ is balanced if there exist positive numbers λ_μ for $\mu \in \mathcal{B}$ such that

$$\sum_{\mu \in \mathcal{B}} \lambda_\mu e^{A(\mu)} = e^N. \tag{2.3}$$

The numbers λ_μ are called *balancing coefficients.*

Clearly, (2.3) is equivalent to the following:

$$\sum_{\mu \in \mathcal{B}} \lambda'_\mu m^{A(\mu)} = m^N, \tag{2.4}$$

where each $\lambda'_\mu = \frac{|A(\mu)|}{n} \lambda_\mu$.

The next concept is an extension of the concept of π-balanced collection by Billera [2].

Definition 2.4. Given $\pi \in \Delta$ with $\pi_m \gg 0$, a collection $\mathcal{B} \subseteq M$ is π-balanced if there exist positive numbers λ_μ for $\mu \in \mathcal{B}$ such that

$$\sum_{\mu \in \mathcal{B}} \lambda_\mu \pi_\mu = \pi_m. \tag{2.5}$$

It is clear from (2.4) and (2.5) that a balanced collection $\mathcal{B}$ is π-balanced for the special $\pi \in \Delta$ with $\pi_\mu = m^{A(\mu)}$ for each $\mu \in M$.

Definition 2.5.

(1) A multi-choice NTU game (M, V) is *balanced* if $\cap_{\mu \in \mathcal{B}} V(\mu) \subseteq V(m)$ for every balanced collection $\mathcal{B} \subseteq M$.
(2) Given $\pi \in \Delta$ with $\pi_m \gg 0$, a multi-choice NTU game (M, V) is π-*balanced* if $\cap_{\mu \in \mathcal{B}} V(\mu) \subseteq V(m)$ for every π-balanced collection $\mathcal{B} \subseteq M$.

Clearly, a balanced multi-choice NTU game (M, V) is π-balanced for the special $\pi \in \Delta$ with $\pi_\mu = m^{A(\mu)}$ for each $\mu \in M$.

Since NTU games are special multi-choice NTU games with $m_i = 1$ for all $i \in N$, the above concepts yield the corresponding concepts for NTU games when $m_i = 1$ for all $i \in N$. The following are well-known existence theorems for cores in NTU games.

Theorem 2.6 (Scarf, 1967)**.** Any balanced NTU game V has a non-empty core.

Theorem 2.7 (Billera, 1970)**.** Any π-balanced NTU game V has a non-empty core.

Theorem 2.8 (Bondareva, 1963 and Shapley, 1967)**.** A TU game V has a non-empty core if and only if it is balanced.

We will derive extensions of these theorems to multi-choice games in Sect. 4. But, we first provide a close connection between Edgeworth equilibria of coalition production economies and cores of multi-choice NTU games in the next section to show the needs for studying multi-choice NTU games.

3 Connection Between Edgeworth Equilibria of Economies and Cores of Multi-choice NTU Games

In this section, we will give a close connection between Edgeworth equilibria of economies and cores of multi-choice NTU games. First, let us recall the concept of a coalition production economy given in [12] and some necessary preliminaries from [13].

A *coalition production economy* $\mathcal{E} = (\mathbb{R}^L, (X^i, u^i, w^i)_{i \in N}, (Y^S)_{S \in \mathcal{N}})$ with n agents is a collection of the commodity space $\mathbb{R}^L$, where L is the set of commodities, agents' characteristics $(X^i, u^i, w^i)_{i \in N}$, and coalitions' production sets $(Y^S)_{S \in \mathcal{N}}$. The triple (X^i, u^i, w^i) is agent i's characteristics as a consumer: $X^i \subseteq \mathbb{R}^L$ is his consumption set, $u^i : X^i \to \mathbb{R}$ is his utility function, and $w^i \in \mathbb{R}^L$ is his endowment vector. The set $Y^S \subseteq \mathbb{R}^L$ is the production set of the firm (coalition) S for which every agent $i \in S$ works and Y^S consists of all production plans that can be achieved through a joint action by the members of S. We use $Y = Y^N$ for the total production possibility set of the economy.

An *exchange economy* is a coalition production economy with $Y^S = \{0\}$ for every $S \in \mathcal{N}$.

When dealing with replica of an economy $\mathcal{E}$, one usually needs some special conditions on the production possibility sets $(Y^S)_{S \in \mathcal{N}}$. The key assumption is that when $y \in Y^S$, $cy \in Y^S$ for any nonnegative constant c. Here are some common assumptions:

(P.1) $Y^S = \{0\}$ for all $S \in \mathcal{N}$ (exchange economies, see [6]);
(P.2) Y is a convex cone with vertex at the origin and $Y^S = Y$ for all $S \in \mathcal{N}$ (see [6]);
(P.3) Y^S is a convex cone containing 0 for each $S \in \mathcal{N}$ (see [13]).

Clearly, (P.3) contains (P.1) and (P.2). The following assumptions on consumption sets, utility functions, and the sets of attainable allocations are standard:

(A.1) For every agent $i \in N$, $X^i \subseteq \mathbb{R}^L$ is non-empty, closed, and convex,
(A.2) For each $i \in N$, $u^i : X^i \to \mathbb{R}$ is continuous and quasi-concave;
(A.3) for each $S \in \mathcal{N}$, $Y^S \subseteq \mathbb{R}^L$ is non-empty and closed, and the set $F_{\mathcal{E}}(S)$ of feasible
(attainable) S-allocations is nonempty and compact, where

$$F_{\mathcal{E}}(S) = \{(x^i)_{i \in S} | x^i \in X^i \text{ for each } i \in S \text{ and } \sum_{i \in S}(x^i - w^i) \in Y^S\}. \tag{3.1}$$

The set of all attainable allocations of the economy $\mathcal{E}$ is

$$F(\mathcal{E}) = F_{\mathcal{E}}(N) = \{(x^i)_{i \in N} | x^i \in X^i \text{ for each } i \in N \text{ and } \sum_{i \in N}(x^i - w^i) \in Y^N = Y\}$$

which is non-empty and compact.

In an effort to connect the two concepts of core and competitive equilibrium in exchange economies (more generally, coalition production economies satisfying (P.2)), Debreu and Scarf [6] considered r-fold replica of an economy. For each positive integer r, the r-fold replica of the economy $\mathcal{E}$, denoted by $\mathcal{E}_r$, is defined to be the economy composed of r subeconomies identical to $\mathcal{E}$ with a set of consumers

$$N_r = \{(i, q) | i = 1, \ldots, n \text{ and } q = 1, \ldots, r\}.$$

The first index of consumer (i, q) refers to the type of the individual and the second index distinguishes different individuals of the same type. It is assumed that all consumers of type i are identical in terms of their consumption sets, endowments, and utility functions. Let S be a non-empty subset of N_r. An allocation $(x^{(i,q)})_{(i,q) \in S}$ is S-attainable in the economy $\mathcal{E}_r$ if

$$\sum_{(i,q) \in S} (x^{(i,q)} - w^{(i,q)}) \in Y^{S'} \tag{3.2}$$

where $S' = \{i \in N | (i, q) \in S\}$, $x^{(i,q)} \in X^i$ and $w^{(i,q)} = w^i$ for every q. Thus, (3.2) can be written as

$$\sum_{i \in S'} \sum_{q \in S(i)} x^{(i,q)} - \sum_{i \in S'} |S(i)| w^i \in Y^{S'}. \tag{3.3}$$

where $S(i) = \{q \in \{1, 2, \ldots, r\} | (i, q) \in S\}$ and $|S(i)|$ denotes the number of elements in $S(i)$.

Let $\mathcal{E}$ be a coalition production economy. From an r-fold replica $\mathcal{E}_r$ of $\mathcal{E}$, we form a multi-choice NTU game (M^r, V) as follows: Let $M_i^r = \{0, 1, \ldots, r\}$ each $i \in N$ and let $M^r = (\prod_{i \in N} M_i^r) \setminus \{0\}$. For each $\mu \in M^r$, define $V(\mu) = \{v \in \mathbb{R}^n |$ there exists $(x^i)_{i \in N} \in F_{\mathcal{E}}(A(\mu))$ such that $\sum_{i \in A(\mu)} \mu_i (x^i - w^i) \in Y^{A(\mu)}$ and

$$v_i \leq u^i(x^i) \text{ for every } i \in A(\mu)\}, \tag{3.4}$$

where $A(\mu) = \{i \in N | \mu_i > 0\}$. Note that for $m^r = (r, r, \ldots, r)$, $A(m^r) = N$ for all $r \geq 1$. Under assumption (P.3), we have $V(m^r) = V(e^N)$ for all $r \geq 1$. By (2.2), $C((M^{r_2}, V)) \subseteq C((M^{r_1}, V))$ whenever $r_1 < r_2$. It follows that

$$\lim_{r \to \infty} C((M^r, V)) = \cap_{r \geq 1} C((M^r, V)). \tag{3.5}$$

Recall that for an economy $\mathcal{E}$, an allocation $x = (x^1, \ldots, x^n)$ is blocked by a coalition S if there is an S-attainable partial allocation $(\overline{x}^i)_{i \in S}$ such that $u^i(\overline{x}^i) > u^i(x^i)$ for each $i \in S$. The *core* $C(\mathcal{E})$ of an economy $\mathcal{E}$ is the set of all attainable allocations which can not be blocked by any coalition. The following concept of Edgeworth equilibrium is given in [1] (see also [8]), where the r-fold repetition of an allocation $x = (x^1, \ldots, x^n)$ is $r \circ x = (x^{(i,q)})_{(i,q) \in N_r}$ with $x^{(i,q)} = x^i$ for all $q \leq r$ and for every $i \in N$.

Definition 3.1. An *Edgeworth equilibrium* of an economy $\mathcal{E}$ is an attainable allocation $x \in F(\mathcal{E})$ such that for any positive integer r, the r-fold repetition $r \circ x$ of x belongs to the core of the r-fold replica $\mathcal{E}_r$ of the economy $\mathcal{E}$. We will denote by $C^E(\mathcal{E})$ the set of all Edgeworth equilibria of $\mathcal{E}$.

Debreu and Scarf [6] proved that in an exchange economy or a coalition production economy satifying (A.1)–(A.3) and (P.2), when the set of economic agents is replicated, the set of core allocations of the replica economy shrinks to the set of competitive equilibria. This result has been extended to coalition production economies satisfying (A.1)–(A.3) and (P.3) by Liu and Liu [13]. The following theorem shows that the core of the multi-choice NTU game (M^r, V) arising from the r-fold replica economy $\mathcal{E}_r$ shrinks to a subset of the set of Edgeworth equilibria of $\mathcal{E}$ by (3.5).

Theorem 3.2. Let $\mathcal{E}$ be a coalition production economies satisfying (A.1)–(A.3) and (P.3). Then $v \in \cap_{r \geq 1} C((M^r, V))$ implies that x is an Edgeworth equilibrium, that is, $x \in C^E(\mathcal{E})$, where $x = (x^i)_{i \in N} \in X$ is an attainable allocation satisfying $v_i = u^i(x^i)$ for every $i \in N$.

Proof. Let $v = (v^i)_{i \in N} \in C((M^r, V))$ for all $r \geq 1$. We show that the r-fold repetition of x is in $C(\mathcal{E}_r)$ for all $r \geq 1$, where $x = (x^i)_{i \in N} \in X$ is an attainable allocation satisfying $v_i = u^i(x^i)$ for every $i \in N$. By (3.4), $v \in V(m^r)$, where $m^r = (r, r, \ldots, r)$, implies that there exists $x = (x^i)_{i \in N} \in X$ such that

$$\sum_{i \in N} r(x^i - w^i) \in Y^N = Y \text{ and } v_i \leq u^i(x^i) \text{ for every } i \in N. \tag{3.6}$$

By (2.2) and (3.4), $v \in C((M^r, V))$ implies that $v_i = u^i(x^i)$ for every $i \in N$. We claim that for any $r \geq 1$, $r \circ x = (x^{(i,q)})_{(i,q)\in N_r} \in C(\mathcal{E}_r)$, where $x^{(i,q)} = x^i$ for all $q \leq r$ and every $i \in N$. Suppose that $(x^{(i,q)})_{(i,q)\in N_r} \notin C(\mathcal{E}_r)$. Then there exists $S \subseteq N_r$ such that $(x^{(i,q)})_{(i,q)\in N_r}$ is blocked by S through a partial S-attainable vector $(\overline{x}^{(i,q)})_{(i,q)\in S}$. Let $S' = \{i \in N | (i,q) \in S\}$ and $S(i) = \{q \in \{1, 2, \ldots, r\} | (i,q) \in S\}$ for each $i \in N$. Then for each $i \in S'$ and all $q \in S(i)$, $\overline{x}^{(i,q)} \in X^i$ and

$$u^i(\overline{x}^{(i,q)}) > u^i(x^{(i,q)}) = u^i(x^i). \tag{3.7}$$

Let $\mu \in M^r$ be defined by $\mu_i = |S(i)|$ for each $i \in N$. Then $A(\mu) = S'$. By (3.2) and (3.3), $(\overline{x}^{(i,q)})_{(i,q)\in S}$ is S-attainable implies

$$\sum_{i\in S'} \mu_i [\frac{1}{\mu_i} \sum_{q\in S(i)} \overline{x}^{(i,q)}] - \sum_{i\in S'} \mu_i w^i \in Y^{S'}. \tag{3.8}$$

For each $i \in S'$, since $\overline{x}^{(i,q)} \in X^i$ for each $1 \leq q \leq r$ and X^i is convex by (A.1),

$$x^i_\mu = \frac{1}{\mu_i} \sum_{q\in S(i)} \overline{x}^{(i,q)} = \frac{1}{|S(i)|} \sum_{q\in S(i)} \overline{x}^{(i,q)} \in X^i.$$

It follows from (3.8) that

$$\sum_{i\in A(\mu)} \mu_i(x^i_\mu - w^i) \in Y^{A(\mu)}. \tag{3.9}$$

For each $i \in S' = A(\mu)$, since $u^i(\overline{x}^{(i,q)}) > u^i(x^i)$ for every $q \in S(i)$ by (3.7) and u^i is quasi-concave by (A.2),

$$u^i(x^i) < min_{q\in S(i)}\{u^i(\overline{x}^{(i,q)})\} \leq u^i(\frac{1}{|S(i)|} \sum_{q\in S(i)} \overline{x}^{(i,q)}) = u^i(x^i_\mu).$$

It follows from (3.6) that $v_i \leq u^i(x^i) < u^i(x^i_\mu)$ for each $i \in A(\mu)$. By (3.4) and (3.9), we conclude that $v \in int(V(\mu))$, contradicting $v \in C((M^r, V))$ by (2.2). Therefore, we have $r \circ x = (x^{(i,q)})_{(i,q)\in N_r} \in C(\mathcal{E}_r)$ and the theorem follows. □

4 Existence of Cores in Multi-choice NTU Games

Throughout this section, we use ∂D to denote the boundary of a subset D of $\mathbb{R}^n$ and $co\{X\}$ for the convex hull of the set X. Give an NTU game V, set $W = \cup_{S\in\mathcal{N}} V(S)$ and $\mathcal{S}(x) = \{S \in \mathcal{N} | x \in \partial V(S)\}$. The following concept is Definition 2.2 from Bonnisseau and Iehlé [4].

Definition 4.1. Let V be an NTU game.

(i) A *transfer rate rule* is a collection of set-valued mappings $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that: for each $S \in \mathcal{N}$, $\varphi_S : \partial V(S) \mapsto \Delta^S$ is an upper semi-continuous correspondence with non-empty compact and convex values; $\psi : \partial V(N) \mapsto \Delta^N$ is an upper semi-continuous correspondence with non-empty compact and convex values.

(ii) The game V is *payoff-dependent balanced* if there exists a transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that, for each $x \in \partial W$,

$$\text{if } co\{\varphi_S(x)|S \in \mathcal{S}(x)\} \cap \psi(P_N(x)) \neq \emptyset, \text{ then } x \in V(N),$$

where P_N is a projection of $\mathbb{R}^n$ to $\partial V(N)$ defined by $P_N(x) = proj(x) - \lambda_N(proj(x))e^N$ which is continuous.

Bonnisseau and Iehlé [4] proved the following payoff-dependent core existence theorem.

Theorem 4.2 (Bonnisseau and Iehlé, 2007). If an NTU game V is payoff-dependent balanced with respect to some transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$, then there exists a core payoff vector x satisfying:

$$co\{\varphi_S(x)|S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset.$$

Next, we extend the concept of payoff-dependent balancedness to multi-choice NTU games. For a multi-choice NTU game(M, V), let $W' = \cup_{\mu\in M} V(\mu)$ and $\mathcal{S}'(x) = \{\mu \in M|x \in \partial V(\mu)\}$. Recall that $M = (\prod_{i\in N} M_i) \setminus \{0\}$, $m = (m_i)_{i\in N}$, and $A(\mu) = \{i \in N|\mu_i > 0\}$ for each $\mu \in M$.

Definition 4.3. Let (M, V) be a multi-choice NTU game.

(i) A *transfer rate rule* is a collection of set-valued mappings $((\varphi_\mu)_{\mu\in M}, \psi)$ such that: for each $\mu \in M$, $\varphi_\mu : \partial V(\mu) \mapsto \Delta^{A(\mu)}$ is an upper semi-continuous correspondence with non-empty compact and convex values; $\psi : \partial V(m) \mapsto \Delta^N$ is an upper semi-continuous correspondence with non-empty compact and convex values.

(ii) The multi-choice game (M, V) is *payoff-dependent balanced* if there exists a transfer rate rule $((\varphi_\mu)_{\mu\in M}, \psi)$ such that, for each $x \in \partial W'$,

$$\text{if } co\{\varphi_\mu(x)|\mu \in \mathcal{S}'(x)\} \cap \psi(P_N(x)) \neq \emptyset, \text{ then } x \in V(m),$$

where P_N is a projection of $\mathbb{R}^n$ to $\partial V(m)$ defined by $P_N(x) = proj(x) - \lambda_N(proj(x))e^N$.

Theorem 4.2 can be extended to multi-choice NTU games as follows.

Theorem 4.4. If a multi-choice NTU game (M, V) is payoff-dependent balanced with respect to some transfer rate rule $((\varphi_\mu)_{\mu\in M}, \psi)$, then there exists a core payoff vector x satisfying:

$$co\{\varphi_\mu(x)|\mu \in \mathcal{S}'(x)\} \cap \psi(x) \neq \emptyset,$$

where $\mathcal{S}'(x) = \{\mu \in M|x \in \partial V(\mu)\}$.

Proof. Let (M,V) be a multi-choice NTU game which is payoff-dependent balanced with respect to some transfer rate rule $((\varphi_\mu)_{\mu\in M},\psi)$. For each $S\in\mathcal{N}$, set $V^*(S)=\cup_{A(\mu)=S}V(\mu)$. Then each $V^*(S)$ is closed as it is a union of finite number of closed sets and V^* is an NTU game. For each $S\in\mathcal{N}$, define $\varphi^*_S=co\{\varphi_\mu|A(\mu)=S\}$. Then φ^*_S is an upper semi-continuous correspondence with non-empty compact and convex values for each $S\in\mathcal{N}$. Define $\psi^*=\psi$. Then V^* is payoff-dependent balanced with respect to the transfer rate rule $((\varphi^*)_{S\in\mathcal{N}},\psi^*)$. Now, Theorem 4.4 follows from Theorem 4.2 easily. □

By (2.4), the following extension of Scarf's Theorem (Theorem 2.6) follows from Theorem 4.4 by setting $\varphi_\mu(x)=\{m^{A(\mu)}\}$ for each $\mu\in M$ and $\psi=\varphi_m=\{m^N\}$.

Theorem 4.5. Any balanced multi-choice NTU game (M,V) has a non-empty core.

By (2.5), the next extension of Billera's Theorem (Theorem 2.7) follows from Theorem 4.4 by setting $\varphi_\mu(x)=\{\pi_\mu\}$ for each $\mu\in M$ and $\psi=\varphi_m=\{\pi_m\}$.

Theorem 4.6. Any π-balanced multi-choice NTU game (M,V) has a non-empty core.

Next, we show that for multi-choice TU games, the converses of Theorems 4.5 and 4.6 hold. The following theorem is an extension of Bondareva - Shapley Theorem (Theorem 2.8) to multi-choice games.

Theorem 4.7. A multi-choice TU game (M,V) has a non-empty core if and only if it is balanced.

Proof. The sufficiency follows from Theorem 4.5. We now prove the necessity. Assume that (M,V) is a multi-choice TU game (M,V) with a nonempty core $C(M,V)$. Let $x^*\in C(M,V)=V(m)\setminus[\cup_{\mu\in M}\ int(V(\mu))]$ (see (2.2)). Then $x^*\in\partial V(m)$ and $x^*\notin V(\mu)$ for all $\mu\in M$. By (2.1), we have that $\sum_{i=1}^n x^*_i=v(m)$ and $x^*\cdot e^{A(\mu)}=\sum_{i\in A(\mu)}x_i\geq v(\mu)$ for every $\mu\in M$.

We now show that V is balanced. Let $\mathcal{B}\subseteq M$ be any balanced collection. Then, by (2.3), we have $\sum_{\mu\in\mathcal{B}}\lambda_\mu e^{A(\mu)}=e^N$ with some positive numbers λ_μ for $\mu\in\mathcal{B}$. We need to show that $\cap_{\mu\in\mathcal{B}}V(\mu)\subseteq V(m)$. Let $x\in\cap_{\mu\in\mathcal{B}}V(\mu)$. Then $x\in V(\mu)$ for each $\mu\in\mathcal{B}$ which implies that $x\cdot e^{A(\mu)}=\sum_{i\in A(\mu)}x_i\leq v(\mu)$ by (2.1). It follows that

$$\sum_{i=1}^n x_i = x\cdot e^N = x\cdot\sum_{\mu\in\mathcal{B}}\lambda_\mu e^{A(\mu)}$$

$$=\sum_{\mu\in\mathcal{B}}\lambda_\mu(x\cdot e^{A(\mu)})\leq\sum_{\mu\in\mathcal{B}}\lambda_\mu v(\mu)\leq\sum_{\mu\in\mathcal{B}}\lambda_\mu(x^*\cdot e^{A(\mu)})$$

$$=x^*\cdot\sum_{\mu\in\mathcal{B}}\lambda_\mu e^{A(\mu)}=x^*\cdot e^N=\sum_{i=1}^n x^*_i=v(m),$$

which implies that $x\in V(m)$ by (2.1). Thus (M,V) is balanced. □

Recall that a balanced multi-choice NTU game (M, V) is π-balanced for the special $\pi \in \Delta$ with $\pi_{A(\mu)} = m^{A(\mu)}$ for each $\mu \in M$. The next characterization follows from Theorems 4.6 and 4.7 immediately.

Theorem 4.8. A multi-choice TU game (M, V) has a non-empty core if and only if it is π-balanced.

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