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Game Theory and Applications

3rd Joint China-Dutch Workshop
and 7th China Meeting, GTA 2016
Fuzhou, China, November 20–23, 2016
Revised Selected Papers

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Preface

Recently, non-cooperative and cooperative games — particularly cooperative games with coalitional structures, fuzzy non-cooperative and cooperative games, dynamic games, evolutionary games, mechanism design, bargaining games, and auctions — are attracting significant coverage from researchers in many subjects or disciplines such as game theory, operations research, mathematics, decision science, management science, and control theory. Moreover, non-cooperative and cooperative games are successfully applied to various fields such as economics, management, industrial organization, operations and supply chain management, human resources, energy and resource management, biology, and others. In this context, to strengthen the ongoing scientific interaction between the two game theory societies of The Netherlands and China and to promote academic research, exchange, and collaboration among researchers from The Netherlands and China as well as other countries, Fuzhou University of China, Northwestern Polytechnical University of China, University of Twente of The Netherlands, and the Game Theory Subcommittee of Operations Research Society of China hosted the Third Joint China–Dutch Workshop on Game Theory and Applications and the 7th China Meeting on Game Theory and Applications (GTA 2016), which was held during November 20–23, 2016, at Fuzhou University, Fujian, China.

The GTA 2016 received 162 abstract submissions and there were about 180 participants from The Netherlands, USA, UK, Japan, Canada, Russia, and China.

After GTA 2016 in Fuzhou, we prepared the proceedings of GTA 2016 for publication in *Communications in Computer and Information Science* (CCIS) by Springer. Thus, we contacted the experts and scholars attending GTA 2016 and invited them to extend their conference papers for consideration in this publication. Finally, we received and accepted 25 full papers after two rounds of peer review. The 25 papers cover non-cooperative and cooperative games as well as non-cooperative and cooperative games under uncertainty and their applications.

The paper "Repeated Games and Price Wars," written by Ronald Peeters, Hans Peters, Erik Pot, and Dries Vermeulen, discusses collusive equilibria under private and public information and collusive equilibria when market shares form a martingale. The authors show that firms can collude using dynamic price adjustment strategies under the two conditions of public observability and limited volatility of market shares. Particularly, the authors show that collusion can no longer be sustained when the condition of limited volatility of market shares is violated.

The paper "A Game Theory Approach for Deploying Medical Resources in the Emergency Department," by Cheng-Kuang Wu, Yi-Ming Chen, and Dachrahn Wu, proposes a framework for emergency response services that incorporates two game theory models designed to deploy response medical resources when raising three threat advisory levels. The experimental results show that the developed model is feasible, which may provide a method for improving efficiency in emergency department.

The paper “Non-cooperative Monomino Games,” authored by Judith Timmer, Harry Aarts, Peter van Dorenvanck, and Jasper Klomp, investigates monomino games, which are two-player games played on a rectangular board with R rows and C columns. The game pieces are monominoes, which cover exactly one cell of the board. One by one each player selects a column of the board, and places a monomino in the lowest uncovered cell. This generates a payoff for the player. The game ends if all cells are covered by monominoes. The goal of each player is to place his/her monominoes in such a way that his/her total payoff is maximized. The aim of this paper is to derive the equilibrium play and corresponding payoffs for the players.

The paper “Bargaining Model of Mutual Deterrence Among Three Players with Incomplete Information,” by Yan Xiao and Deng-Feng Li, studies the tripartite bargaining problem of mutual deterrence from the perspective of Rubinstein indefinite bargaining and cooperative game theory. The authors mainly establish a tripartite mutual deterrence bargaining model with unilateral and bilateral incomplete information by introducing incomplete information and defining discount factors. Specifically, the analytical formula is obtained to calculate the Nash equilibrium distribution for each player under incomplete information. The developed model and method may provide a new way for solving multiple mutual deterrence or conflict problems with incomplete information.

The paper “Stakeholders’ Behavior Analysis and Enterprise Management Strategy Selection in Chinese Ancient Village Tourism Development,” by Wei Fei, investigates how to exploit and protect ancient villages in tourism development, since Chinese ancient villages are an important type of non-renewable tourism resource. The author firstly identifies the stakeholders (i.e., players) who have an interest and play important roles in Chinese ancient village tourism development and protection. Then, the author systematically analyzes the stakeholders’ relations, interaction, and importance in the exploitation and protection of Chinese ancient village tourism. Finally, the author elaborates on stakeholders’ behaviors and hereby proposes enterprise management strategies for Chinese ancient village tourism.

The paper “Two Bargain Game Models of the Second-Hand Housing Commence,” written by Rui Wang and Deng-Feng Li, discusses the problem of bargaining about final prices of houses for sale on the second-hand house market. Two bargaining models for indefinite and finite periods are established for sellers and buyers. For the indefinite period, the authors derive the complete equilibrium solution of the bargaining game model between the buyers and sellers. Hereby, the game equilibrium solution in the second stage is obtained through imposing some constraints on time. The results show that the game between sellers and buyers depends on the ratio of the discount factor of each seller or buyer.

The paper “Some Relaxed Solutions of Minimax Inequalities for Discontinuous Games,” by Xiaoling Qiu and Dingtao Peng, proves the existence of minimax inequalities under some relaxed assumptions by using the KKMF principle or Fan–Browder fixed point theorem and propose the pseu-solution of minimax inequalities. As applications, the authors introduce pseu-Nash equilibriums for n-person non-cooperative games and obtain some relaxed existence conclusions.

The paper “Dynamic Games of Firm Social Media Disclosure,” written by Bing Wang, Wei Zheng, and Yan Pan, discusses the game problem of firm social media disclosure. The authors propose a three-stage dynamic game model to analyze the process of social media information disclosure. In the first-stage model, firms disclose social media because of low costs and high incomes so that they get more attention in competition. By introducing investors in the second-stage model, firms disclose exaggeratedly in order to get more benefits from investors in the complete information-static game. In the third-stage model, by introducing the external regulators, the authors propose a repeated game model with incomplete information, which has an equilibrium when the repeated time is sufficient.

In the paper “On Stochastic Fishery Games with Endogenous Stage–Payoffs and Transition Probabilities,” Reinoud Joosten and Llea Samuel engineer a stochastic fishery game in which overfishing has a twofold effect: It gradually damages the fish stock inducing lower catches in states high and low, and it gradually causes the system to spend more time in the latter state with lower landings. To analyze the effects of this “double whammy” technically, the authors examine how to determine the set of jointly-convergent pure-strategy rewards supported by the equilibrium involving threats, under the limiting average reward criterion.

The paper “N-Person Credibilistic Non-cooperative Game with Fuzzy Payoffs,” written by Chunqiao Tan and Zhongwei Feng, presents n-person non-cooperative games with fuzzy payoffs. Three credibilistic criteria are introduced to define behavior preferences of players in different game situations based on credibility theory. Hereby the authors propose three solution concepts of credibilistic equilibria and prove their existence theorems. Furthermore, the authors propose three sufficient and necessary conditions for computing credibilistic equilibrium strategies.

The paper “Pareto Optimal Strategies for Matrix Games with Payoffs of Intuitionistic Fuzzy Sets,” written by Jiang-Xia Nan, Cheng-Lin Wei, and Deng-Feng Li, focuses on developing an effective methodology for solving matrix games with payoffs of intuitionistic fuzzy sets. The authors first propose a new ranking method of intuitionistic fuzzy sets and the concept of Pareto Nash equilibrium solutions of matrix games with payoffs of intuitionistic fuzzy sets. Hence it is proven that Pareto Nash equilibrium solutions of matrix games with payoffs of intuitionistic fuzzy sets are equivalent to the Pareto optimal solutions of a pair of bi-objective programming models, which can be easily solved by using existing multi-objective programming methods.

The paper “Marginal Games and Characterizations of the Shapley Value in TU Games,” written by Takumi Kongo and Yukihiko Funaki, discusses axiomatizations and recursive representations of the Shapley value on the class of all cooperative games with transferable utilities (i.e., TU games). Marginal games that are closely related to dual games play central roles in this study. The axiomatizations are based on axioms that are marginal game variations of the well-known balanced contributions property, so that they are interpreted as fair treatment between two players in TU games as the balanced contributions property is. Moreover, the authors propose a general recursive representation that can be used to represent the Shapley value for n-person TU games by those for r-person and (n–r)-person TU games with fixed r being smaller than n.

The paper “Computing the Shapley Value of Threshold Cardinality Matching Games,” written by Lei Zhao, Xin Chen, and Qizhi Fang, discusses the computational and complexity issues on the Shapley value in a particular multi-agent domain, which is called a threshold cardinality matching game. The authors show that the Shapley value can be computed in polynomial time when graphs are restricted to some special graphs, such as linear graphs and the graphs having clique or coclique module decomposition. However, it is proven that computing the Shapley value is P-complete when the threshold is a constant.

The paper “Matrix Analysis for the Shapley Value and Its Inverse Problem,” by Jun Su and Genjiu Xu, deals with algebraic representation and matrix analysis techniques for computing linear values of cooperative games. The authors propose a matrix approach for characterizing linear values with certain essential properties. Some properties are also described for the Shapley standard matrix, which is the representation matrix of the Shapley value. In addition, the authors examine the inverse problem of the Shapley value in terms of the null space of the Shapley standard matrix.

In the paper “The General Nucleolus of n-Person Cooperative Games,” Qianqian Kong, Hao Sun, and Genjiu Xu investigate how to compute and characterize the general nucleolus of n-person cooperative games. To reflect the profit distribution more intuitively on the space of n-person cooperative games, the authors first define the concept of the general nucleolus whose objective function is limited to the players’ complaints. Hereby, the authors propose an algorithm for calculating the general nucleolus under the case of linear complaint functions so that an accurate allocation can be obtained to pay for all players. The authors also propose a system of axioms and the Kohlberg criterion to axiomatically characterize the general nucleolus in terms of balanced collections of coalitions. Furthermore, to normalize the different assignment criteria, the authors prove the equivalence relationships among the general nucleolus, the least square general nucleolus, and the p-kernel.

The paper “A Cooperative Game Approach to Author Ranking in Coauthorship Networks,” authored by Guangmin Wang, Genjiu Xu, and Wenzhong Li, discusses the problem of author ranking in coauthorship networks from the viewpoint of cooperative games. Three weighted coauthorship networks are constructed from different perspectives and thereby three cooperative games are defined. The core and the Shapley value are chosen as allocation rules for the defined cooperative games. Furthermore, the weighted Shapley value and a new value are proposed as the allocation rules to take into consideration the contribution level of the authors to their papers.

The paper “A Reduced Harsanyi Power Solution for Cooperative Games with a Weight Vector,” written by Xianghui Li and Hao Sun, discusses the Harsanyi power solution for cooperative games in which different players may be asymmetric and contribute to different efforts, bargaining powers, or stability in the process of cooperation. The authors use a weight vector to reflect players’ asymmetry and hereby define and characterize a reduced Harsanyi power solution for cooperative games with a weight vector, which is relevant to a loss function of dividends. It is proven that the reduced Harsanyi power solution has a linear relationship with the Harsanyi power solution if the loss function takes particular forms.

The college enrollment plan allocation plays an important role in implementing the reform of higher education and adjusting the structure of qualified personnel in China. In the paper "An Allocation Method of Provincial College Enrollment Plan Based on the Bankruptcy Model," Zhen Wei and Deng-Feng Li regard the provincial college enrollment plan allocation as the bankruptcy problem. Hereby a bankruptcy model and an operable bankruptcy rule are proposed to determine the college enrollment plan allocation according to the eight university educational indexes. This study may provide references for Chinese provincial education administrative departments in the college enrollment plan allocation process.

The paper "Edgeworth Equilibria of Economies and Cores in Multi-choice NTU Games," by Jiuqiang Liu, Xiaodong Liu, Yan Huang, and Wenbo Yang, extends the payoff-dependent balanced core existence theorem to multi-choice cooperative games with non-transferable utilities (i.e., NTU games), which implies a multi-choice extension of Scarf's core existence theorem. The study establishes the connection between Edgeworth equilibria of economies and cores of multi-choice NTU games.

The paper "Two-Phase Nonlinear Programming Models and Method for Interval-Valued Multiobjective Cooperative Games," written by Fang-Xuan Hong and Deng-Feng Li, defines the concepts of interval-valued cores of interval-valued multiobjective n-person cooperative games and a satisfactory degree (or ranking indexes) of comparing intervals with inclusion and/or overlap relations. Hereby the interval-valued cores can be computed by developing a new two-phase method based on the auxiliary nonlinear programming models. The proposed method can provide cooperative chances under the situations of interval inclusion and/or overlap relations in which the traditional interval ranking method may not always assure.

In the paper "Models and Algorithms for Least Square Interval-Valued Nucleoli of Cooperative Games with Interval-Valued Payoffs," Wei-Long Li focuses on developing an effective method for computing least square interval-valued nucleoli of cooperative games with interval-valued payoffs, which are usually called interval-valued cooperative games for short. Based on the square excess that can be intuitionally interpreted as a measure of the dissatisfaction of the coalitions, the author constructs a quadratic programming model for least square interval-valued prenucleolus of any interval-valued cooperative game and obtains its analytical solution, which is used to determine the players' interval-valued imputations via the designed algorithms that ensure the nucleoli always satisfy the individual rationality of players. Hereby the least square interval-valued nucleoli of interval-valued cooperative games are determined in the sense of minimizing the difference of the square excesses of the coalitions. Moreover, the author discusses some useful and important properties of the least square interval-valued nucleolus such as its existence and uniqueness, efficiency, individual rationality, additivity, symmetry, and anonymity.

The paper "Interval-Valued Least Square Prenucleolus of Interval-Valued Cooperative Games with Fuzzy Coalitions," written by Yin-Fang Ye and Deng-Feng Li, describes how to compute interval-valued least square prenucleoli of interval-valued cooperative games with fuzzy coalitions. The authors first determine the fuzzy coalitions' values by using Choquet integral and thereby obtain the interval-valued cooperative games with fuzzy coalitions in Choquet integral forms. Then, the authors develop a simplified method to compute the interval-valued least square prenucleoli of a special subclass of interval-valued cooperative games with fuzzy coalitions in Choquet integral forms. The developed method can always ensure that the lower and upper bounds of the interval-valued least square prenucleolus are directly obtained via utilizing the lower and upper bounds of the interval-valued coalitions' payoffs under some weaker coalition size monotonicity-like conditions.

The paper "Quadratic Programming Models and Method for Interval-Valued Cooperative Games with Fuzzy Coalitions," authored by Deng-Feng Li and Jia-Cai Liu, focuses on developing a quadratic programming method for solving interval-valued cooperative games with fuzzy coalitions. By using the Choquet integral, the interval-valued cooperative games with fuzzy coalitions are converted into the interval-valued cooperative games in which two auxiliary quadratic programming models are constructed to generate their optimal solutions on the basis of the least square method and distance between intervals.

In the paper "Cooperative Games with the Intuitionistic Fuzzy Coalitions and Intuitionistic Fuzzy Characteristic Functions," Jiang-Xia Nan, Hong Bo, and Cheng-Lin Wei present the definition of the Shapley function for intuitionistic fuzzy cooperative games by extending that of the fuzzy cooperative games. Based on the extended Hukuhara difference, the authors derive the specific expression of the Shapley function for intuitionistic fuzzy cooperative games with multilinear extension form and discuss the existence and uniqueness as well as other useful properties.

The paper "A Profit Allocation Model of Employee Coalitions Based on Triangular Fuzzy Numbers in Tacit Knowledge Sharing," written by Shu-Xia Li and Deng-Feng Li, deals with a profit allocation of employee coalitions in tacit knowledge sharing. Due to the existence of uncertain factors, the allocation of profits cannot be accurately estimated and hereby triangular fuzzy numbers are used to express payoffs of coalitions. Taking into consideration the importance of coalitions, a quadratic programming model is constructed to obtain a suitable solution as the profit allocation of employee coalitions. Furthermore, some constraints are imposed on the proposed model so that its optimal solution can always satisfy the efficiency, which implies the pre-imputation of cooperative games with coalition payoffs represented by triangular fuzzy numbers.

We would like to thank the hard work of the academic Program Committee and the Organizing Committee of GTA 2016 as well as all contributors and reviewers, who really understand the meaning of cooperative games. At the same time, we very much appreciate the National Natural Science Foundation of China (NSFC) and the Dutch Organization for Scientific Research (NWO) for their support (No. 71681330662). Particularly, one of the four editors, Prof. Deng-Feng Li, would like to thank his PhD student, Ms. Yin-Fang Ye for her all effort, input, and excellent work for GTA 2016 and for editing the publication.

August 2017

Deng-Feng Li
Xiao-Guang Yang
Marc Uetz
Gen-Jiu Xu

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Repeated Games and Price Wars

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Abstract. This paper is an attempt to reconcile the – at first sight different – views on the determinants of collusion and price wars expressed in Rotemberg and Saloner (1986), Green and Porter (1984), and Stigler (1964). We first argue that the logic of Rotemberg and Saloner (1986) presupposes two determinants for collusion, namely (1) market shares are publicly observable, and (2) volatility of market shares due to exogenous factors is limited. We make our arguments in a model in which firms repeatedly play a Bertrand type price competition game, while market shares are determined by a stochastic process, conditional on current market shares and prices. Following Rotemberg and Saloner (1986), we show under the two conditions of public observability and limited volatility of market shares that firms can collude using dynamic price adjustment strategies. We show that when the first condition (public observability) is violated, we revert to the logic of Green and Porter (1984). When the second condition (limited volatility of market shares) is violated, for example when consumer loyalty has decreased, we also observe that collusion can no longer be sustained, in line with the arguments in Stigler (1964).

1 Introduction

This paper is a contribution to the ongoing discussion on the stability of collusion and the conditions under which pricing agreements among oligopolists can be sustained in equilibrium.

It is well known in the industrial organization literature that fluctuations in levels of both individual and market demand play an important role in the stability of collusion and the occurrence of price wars. An early contribution in this context is Stigler (1964). In the model of Stigler (1964) firms face uncertainty regarding their individual demand, and they cannot directly observe their opponents' behavior. An unexpectedly large drop in a firm's own individual demand may therefore be attributed to an (unobserved) deviation from collusion by one of the opponents, and a price war is seen as the reversion to competitive behavior to punish such deviations. Stigler (1964) argued that, in a market with high consumer loyalty, deviations from a collusive agreement are relatively easy to detect. Therefore collusion is easier to sustain in markets with high consumer loyalty than in markets with low consumer loyalty.

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The first equilibrium based paper showing how price wars can occur on the equilibrium path is Green and Porter (1984). In their model, a firm can experience a period of unexpectedly bad performance both as a result of deviating behavior by one of the firms and as a result of (unobservable) low aggregate demand. Since, because of unobservability, firms are unable to distinguish between these two scenarios, they have to revert to retaliatory behavior in either case in order to discourage deviating behavior. Price wars will thus occur with certainty in periods of low individual demand, even when deviations did not occur.

Another milestone in the discussion is the paper by Rotemberg and Saloner (1986). In a model with volatile aggregate demand and fixed market shares they show that partial collusion can be sustained in equilibrium using countercyclical pricing strategies. The logic of their argument is that, during a boom, the temptation for firms to deviate from the collusive agreement to attract consumers starts to outweigh the decrease in profits resulting from the ensuing retaliatory price war. To counterbalance this threat to collusion, in equilibrium, firms consequently employ a gradual and coordinated downward adjustment of the price levels in response to the increased level of demand in periods where the market is booming. Since the decision to decrease prices during a boom is taken jointly by all competitors as part of the collusive agreement such an orchestrated and voluntary decrease in prices is in fact not a price war, but can better be viewed as a form of dynamic collusion where prices are deliberately adjusted to the circumstances, precisely with the intention to stabilize collusion. A full blown price war in the sense of full reversion to marginal cost pricing does not occur on the equilibrium path. This was concisely put by Ellison (1994): "Rotemberg and Saloner (1986) is commonly associated with the statement that price wars are more likely to occur during booms, and therefore viewed as somehow in opposition to the Green and Porter (1984) theory. The actual Rotemberg and Saloner (1986) model, however, is really about countercyclical pricing – firms have perfect information and adjust prices smoothly in response to demand conditions."

In this paper we present a model where firms interact repeatedly in a Bertrand model for a market with a single homogeneous good. Firms simultaneously choose prices. Given the prices chosen by the firms, the market price is established as the lowest price charged by the firms. Given the market price, aggregate demand is then determined, and aggregate demand is distributed among the firms that charge market prices. Thus, division of the market is primarily decided by price, which implies homogeneity of the good. Only in case several firms charge the market price, the division of market shares among firms that charge market price is governed by other external factors, and market shares may fluctuate over time. Such fluctuations of market shares capture exogenous factors such as perhaps consumer loyalty (or better: lack thereof) or location effects, but also simply random choice by indifferent customers may be a source of demand uncertainty for firms. All that matters in our analysis is that, typically, such factors are outside the control of the firms, but that they are nevertheless known to affect collusive opportunities (Stigler, 1964; Green and Porter, 1984).

Within this model, both with private information on individual market shares and with public information, we derive the conditions under which strategy profiles in trigger strategies, where each firm chooses to collude unless a deviation has been detected in the past (in which case firms revert to marginal cost pricing), can be sustained as a perfect Bayesian Nash equilibrium.

In both the case of private and the case of public information we find that collusive behavior in trigger strategies is harder to sustain when market shares have high volatility over time, and periods with low individual demand are possible. Moreover, in the public information case, opportunities for partial collusion enhance collusion. The logic driving these results is fairly intuitive. Collusive behavior can be sustained in equilibrium by trigger strategies precisely when expected profits for firms adhering to the collusive agreement are higher than the single period gains from deviation. This condition is particularly stringent for periods where individual demand is low, since in such periods the expected profits when a firm follows the agreement are minimal, while the immediate gains from deviation (undercutting) are high. In addition, when a firm also expects its individual demand to be low in the future, which is more likely when current individual demand is low, the punishment ensuing the breaking of the agreement is relatively small. Hence, high volatility of individual market shares hampers collusion. On the other hand, when partial collusion is possible, and market shares are observable, a countercyclical pricing policy with partial collusion as in Rotemberg and Saloner (1986) can be applied to sustain collusive behaviour in equilibrium.

These results can be seen as an attempt to reconcile the views expressed in Stigler (1964) and Green and Porter (1984) with the arguments in Rotemberg and Saloner (1986). In our model with private information, increased volatility of market shares prevents collusion. This is in line with the view of Green and Porter (1984), who argue that collusion is most likely to break down in periods of low demand, and the view of Stigler (1964), who argued that high consumer loyalty, which is directly related to low volatility of market shares, is one of the stabilizing factors for collusion.

On the other hand, in our model with public observability of market shares, the basic logic of the arguments in Rotemberg and Saloner (1986) is still valid. Firms easily collude when differences in individual demand remain relatively small.[1] However, in the presence of fluctuations of market shares collusion becomes more difficult to sustain. The cartel is then stabilized in periods of high

[1] Despite the differences between the R&S model and our model, the intuition is the same in both models. The central issue concerns changes in potential gains from deviation. In the model of R&S these changes are the result of changes in aggregate demand under fixed market shares. In our model these changes are conversely the result of changes in *individual* demand (market shares) under deterministic aggregate demand. Nevertheless the effect is the same: both in the case of high aggregate demand with fixed market shares and in the case of constant aggregate demand with low market shares gains from deviation are increased.

fluctuations in market shares by using a coordinated price adjustment scheme. When market shares are out of balance, a policy of lower collusive price setting discourages deviations by firms with lower market shares.

Thus, our argument here is that the driving force behind the results of Rotemberg and Saloner (1986) is not the presence of shocks on total demand per se, but more in general the public observability of market shares in conjunction with low volatility of these market shares. Public observability of market shares is essential to the implementation of dynamic price strategies and hence partial collusion, while low volatility of market shares guarantees that dynamic adjustment of prices via a collusive agreement can sufficiently decrease gains from deviation.

Note that both conditions, public observability and low volatility, are fulfilled in Rotemberg and Saloner (1986). They assume that total demand is publicly observed, and moreover that total demand is always equally divided over firms, so that implicitly the division of market shares over firms is common knowledge. Thus, firms are assumed to have full information on market shares. And indeed, in an environment where firms can make binding agreements on market shares, the full force of Rotemberg and Saloner (1986)'s arguments applies, and collusion, at least partial collusion, can be sustained in equilibrium using countercyclical pricing strategies.

However, we show that in addition to this basic observation, the logic of Rotemberg and Saloner (1986) breaks down as soon as one of the two conditions is violated. When in our model market shares are no longer publicly observable, collusion via dynamic price schemes is no longer possible, and we effectively revert to a model where the logic of Green and Porter (1984) applies. Hence, the conclusions of Green and Porter (1984) versus Rotemberg and Saloner (1986) are not contradictory, but rather complement each other, and can be observed under different conditions within a single dynamic model of Bertrand competition.

Also, when market shares are still observable but the volatility of market shares is sufficiently high, dynamic pricing schemes can no longer be sustained in equilibrium. This is due to the fact that large changes in market shares can no longer be compensated for by a countercyclical collusive price adjustment scheme. The gains from deviation for firms with low individual demand simply can no longer be counterbalanced in that case. Then, in line with the findings of Stigler (1964), collusion breaks down.

In conclusion, the logic of Rotemberg and Saloner (1986) not only presupposes observability of market shares; also low volatility of individual demand, for example via a sufficient amount of control over individual demand due to forward contracting, is essential to their argument. So, mere observability is not sufficient. When market shares cannot be enforced collectively and are subject to large exogenous fluctuations that are outside the control of the firms, the conclusions of Rotemberg and Saloner (1986) are mitigated by the volatility of market shares, and collusion will be harder, or even impossible, to sustain.

2 Basic Model

In this section we present the basic model we use in our analysis. In our model firms interact repeatedly in a Bertrand-type pricing game. We first define the one-shot game played between firms in each period of time. In each period firms simultaneously and independently choose a pricing strategy. Based on the choice of strategies of the firms each firm receives a payoff. This basic game is then repeated over an infinite time horizon. Firms use time discounting to evaluate the resulting payoff streams.

2.1 The One-Shot Game

The one shot game is a Bertrand model in which n firms compete on price in a market for a homogeneous good. We present the game step by step. First, each firm i chooses a price $p_i \geq 0$. The market price $P \geq 0$ is then determined by

$$P = \min\{p_i \mid i = 1, \ldots, n\}.$$

Given a profile $p = (p_1, \ldots, p_n)$ of prices, $L = \{i \mid p_i = P\}$ is the set of firms that charge the market price. Given the resulting market price P, the market demand is given by $Q = A - P$, with $A > 0$. Marginal costs of production are assumed to be zero.

The second ingredient of the model is a vector of market shares. Market shares only determine division of the total market Q among those firms that charge the market price P. All other firms do not attract customers. Thus, we do have a setting with homogeneous goods, price is the primary decisive factor for demand, and only firms that charge the market price receive strictly positive demand. How the total demand is divided over firms in L is decided by other factors than price, which is modeled by a vector of market shares.

To be precise, market shares are modeled by a vector $\varphi = (\varphi_1, \ldots, \varphi_n)$, where $\varphi_i > 0$ represents the market share of firm i. Market shares divide the total market Q among firms in L that charge the market price. The resulting share of total demand of firm $i \in L$ is $\kappa_i = \frac{\varphi_i}{\sum_{j \in L} \varphi_j}$, so that $\sum_{i \in L} \kappa_i = 1$.

Given these two ingredients of the model, the resulting profit $\Pi_i(p)$ of firm i is computed via

$$\Pi_i(p) = \begin{cases} \kappa_i \cdot P \cdot (A - P) & \text{if } i \in L \\ 0 & \text{otherwise} \end{cases}$$

where $\kappa_i = \frac{\varphi_i}{\sum_{j \in L} \varphi_j}$. Thus, when all firms choose to collude, each firm sets its price p_i equal to $p^m = \frac{A}{2}$, and firms divide monopoly profits $\Pi = \left(\frac{A}{2}\right)^2$ according to their market shares. When at least one of the firms chooses to use marginal cost pricing, and sets $p_i = 0$, all profits are zero. The firm that uses this action attracts the market, but does not make any profit, while the other firms do not have any customers.

As is well known, the only symmetric Nash equilibrium in this Bertrand model is where all firms choose $p_i = 0$. The central question then is whether, in a repeated game setting, collusion of firms, where each firm sets $p_i = \frac{A}{2}$ at each time period, can be sustained, so that strictly positive payoffs can be achieved in equilibrium.

2.2 The Repeated Game

Following, among others, Friedman (1971), we use the repeated game setup to model repeated strategic interaction between firms. In the repeated game the one shot game is repeated over an infinite time horizon. At the start of each period $t = 0, 1, 2, \ldots$ the history h_t is determined. Typically h_t is a record of all actions taken by firms in earlier periods, and the realized market shares of all firms in earlier periods.

Next, the vector $\varphi_t(h_t) = (\varphi_{1t}(h_t), \ldots, \varphi_{nt}(h_t))$ of market shares is determined. This vector is stochastic, and depends in general on both the actions taken previously by the firms and on the realizations of market shares in earlier periods.

Next, when $\varphi_t(h_t)$ is realized, each firm receives information report r_{it}. Typically r_{it} is a record of all actions taken by firms in earlier periods and the realized market shares of all firms in earlier periods (h_t), together with either a firm's own market share in the current period ($\varphi_{it}(h_t)$, *private information*), or all realized market shares in the current period ($\varphi_t(h_t)$, *public information*).

A strategy for firm i in the repeated game is a function s_i that prescribes for each information report r_{it} the price $s_i(r_{it}) \in \mathbb{R}_+$ chosen by firm i. We require $s_i(r_{it})$ to be specified for any conceivable report r_{it}, not just for those that are actually realized by previous price choices of the firms. This is standard practice in game theoretic models and enables us to define the concept of (subgame perfect) Nash equilibrium. In particular, a firm should not only specify what it will do when all firms act according to agreement, but also how it will react to conceivable deviations from the agreement (and by extension also to deviations from deviations from the agreement, etc.).

The initial division of market shares is given by $\varphi_0 = (\varphi_{10}, \ldots, \varphi_{n0})$. Further, the associated information to the firms is denoted by $r_0 = (r_{10}, \ldots, r_{n0})$.

The density function $f_{it+1}(\varphi_{it+1} \mid \varphi_t, p^t)$ denotes the density of the probability distribution of φ_{it+1} conditional on the event that at time t the market shares of firms are divided according to $\varphi_t = (\varphi_{1t}, \ldots, \varphi_{nt})$, and firms charge prices in the profile $p^t = (p_1^t, \ldots, p_n^t)$. Thus, market shares tomorrow only depend on current market shares plus current prices chosen by the firms. We assume that past market shares and past prices do not influence future market shares (via other ways than their influence on current market shares of course).

We write $s(r_t) = (s_1(r_{1t}), \ldots, s_n(r_{nt}))$ for the profile of prices that is played at time t given the report $r_t = (r_{1t}, \ldots, r_{nt})$. Given the profile $s = (s_1, \ldots, s_n)$ of strategies, let $\mathbb{E}\Pi_i^{t+k}(s \mid r_{it})$ denote the present value of the profit of firm i at time $t + k$, given the strategy profile s and information r_{it} to firm i at time t. Firm i evaluates the stream

$$\mathbb{E}\Pi_i^t(s \mid r_{it}),\ \mathbb{E}\Pi_i^{t+1}(s \mid r_{it}),\ \ldots$$

of expected profits via the discounted criterion defined by

$$\Pi_i(s \mid r_{it}) = \sum_{k=0}^{\infty} \delta^k \cdot \mathbb{E}\Pi_i^{t+k}(s \mid r_{it}).$$

We solve the resulting repeated game using the solution concept of perfect Bayesian equilibrium (see for example Fudenberg and Tirole, 1991, or Bonanno, 2013). Given a strategy profile s and a strategy s'_i for firm i, let (s, s'_i) denote the strategy profile where all firms $j \neq i$ play according to the strategy s_j, while firm i plays according to strategy s'_i. Strategy profile s is a perfect Bayesian Nash equilibrium when, at every information set r_{it},

$$\Pi_i(s \mid r_{it}) \geq \Pi_i((s, s'_i) \mid r_{it})$$

holds for every strategy s'_i of firm i.

3 Collusive Equilibria Under Private Information

In this section we assume private information. Thus, for every i and t, r_{it} consists of $\varphi_{it}(h_t)$ together with all realizations of all market shares and all prices chosen by the firms in all previous rounds. We also assume that all realizations of market shares φ_{it} are within an interval $[\underline{\varphi}, \overline{\varphi}]$ with $0 \leq \underline{\varphi} < \overline{\varphi} \leq 1$. We first analyze under what conditions collusion can be sustained as a perfect Bayesian Nash equilibrium via trigger strategies.

Trigger strategies. The trigger strategy T_i of firm i is defined by $T_i(r_{it}) = p^m$ if all firms chose collusive price p^m in all previous rounds according to r_{it}, and $T_i(r_{it}) = 0$ otherwise.
We write $T = (T_i)_{i \in N}$ for the profile of trigger strategies. Further, let $\mathbb{E}(\varphi_{it+k} \mid \varphi_{it})$ denote the expected value of the market share φ_{it+k} of firm i at time $t + k$ given that the current market share at time t is φ_{it}, under the assumption that all firms choose the collusive price p^m at each period[2]. We derive the following necessary and sufficient condition for T to be a perfect Bayesian Nash equilibrium.

Theorem 1. *The strategy profile T is a perfect Bayesian Nash equilibrium if and only if*

$$\sum_{k=0}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \geq 1$$

holds for any possible market share $\varphi_{it} \in [\underline{\varphi}, \overline{\varphi}]$ at time t, for all firms i and at every time t.

[2] Note that, despite the fact that in our model the actions of the firms may influence market share, here market shares are computed given that firms act collusively. In effect then future market shares, given collusive behavior of firms, only depend on current market shares.

Proof. Due to the one deviation property (see e.g. Hendon et al., 1996), the trigger strategy profile is a perfect Bayesian Nash equilibrium exactly when for every firm i, at every time t, and at every information set r_{it} the trigger strategy renders at least the same expected profit as an instantaneous deviation. Thus, consider firm i, at time t, having a market share φ_{it}. Given that in the punishment phase firms make zero profit, the expected loss in this phase equals the discounted sum of expected market shares times Π

$$\delta \cdot \mathbb{E}(\varphi_{it+1} \mid \varphi_{it}) \cdot \Pi + \delta^2 \cdot \mathbb{E}(\varphi_{it+2} \mid \varphi_{it}) \cdot \Pi + \ldots = \sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \cdot \Pi.$$

The gain from optimal deviation is equal to $(1-\varphi_{it})\cdot\Pi$. So the collusive strategy renders at least the same profit when

$$\sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \geq 1 - \varphi_{it}.$$

This concludes the proof. □

As a consequence of this result we derive analogues in our context of the results of Green and Porter (1984) and Stigler (1964). In the remainder of this section we show that, when firms experience low market shares, collusion becomes more difficult to sustain and reversion to non-collusive behavior is more likely to occur. In particular we find that the smaller a firm's market share can get, the higher the discount factor needs to be to guarantee that the trigger strategy profile is an equilibrium for all possible market share realizations. Also, since consumer loyalty reduces the volatility of market shares over time, we find in agreement with Stigler (1964) that collusion becomes easier to sustain when consumer loyalty is high.

For the next corollary we need the following mild assumption.

$$\text{For any } k \geq 1,\ \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \text{ is increasing in } \varphi_{it}. \tag{1}$$

The intuition supporting this assumption is clear. It states that, given the assumption that firms act collusively, the expectation of a firm's future market share is an increasing function of today's market share. In other words, when all firms agree to collude, higher market shares today increase the expected value of tomorrow's market share.

Corollary 1 *Assume (1). The strategy profile T is a perfect Bayesian Nash equilibrium if and only if*

$$\sum_{k=0}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it} = \underline{\varphi}) \geq 1$$

holds for every firm i at every time t.

Proof. By Theorem 1, T is a perfect Bayesian Nash equilibrium if and only if

$$\sum_{k=0}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \geq 1 \tag{2}$$

holds for all firms i, at every time t, for any possible market share $\varphi_{it} \in [\underline{\varphi}, \overline{\varphi}]$ at time t. However, by (1), the left-hand side of (2) is increasing in φ_{it}. Hence, (2) is satisfied for all $\varphi_{it} \in [\underline{\varphi}, \overline{\varphi}]$ if and only if it is satisfied for $\varphi_{it} = \underline{\varphi}$. □

This is a direct analogue of the result of Green and Porter (1984) that the possibility of low market shares hampers collusion in our context. This possibility is reflected in a low value of $\underline{\varphi}$. Then, by (1), the left-hand side of (2) is low. Thus, it will become harder to satisfy the condition for T to be a perfect Bayesian Nash equilibrium, and hence collusion is harder to sustain. It is also in agreement with Stigler (1964), since high consumer loyalty reduces the volatility of market shares. Hence high consumer loyalty increases $\underline{\varphi}$, and collusion becomes easier to sustain.

4 Collusive Equilibria Under Public Information

We now focus on the setting of Rotemberg and Saloner (1986). We show that when firms have public information on realized market shares, the incentives to deviate for a firm that has a low market share can be reduced by jointly choosing a lower collusive price. Thus, public availability of information enables firms to sustain (partial) collusion even in situations where full collusion would break down. The price to pay for this enhancement of collusion is, as also argued in Rotemberg and Saloner (1986), a lower collusive price, and hence lower profits.

We model the phenomenon of partial collusion via collusion at a reduced level of profits in the one shot game. Since under public information the colluding firms observe the vector $\varphi_t = (\varphi_{1t}, \ldots, \varphi_{nt})$ of market shares at the start of each period t, the profit function in the one shot game can now be made contingent on the specific realization of φ_t. Before the start of the game, the colluding firms agree on a threshold level φ^* and a collusive joint profit $\Pi^* < \Pi$. The joint profit Π^* is achieved by letting all firms choose a collective predetermined price level $p_i = p^*$, where $p^* < p^m$.

The agreement is that, as long as all realized market shares φ_{it} are above the threshold φ^*, firms collude at price level p^m, generating monopoly profits Π, while as soon as one or more firms have a realized market share below the threshold, firms collude at price level p^*, generating joint profits Π^* (partial collusion). Note that firms need public information on market shares in order to implement this form of collusion. Thus, strategies are now defined contingent on the realization of the vector $\varphi_t = (\varphi_{1t}, \ldots, \varphi_{nt})$.

Trigger strategies. The trigger strategy T_i^* of firm i is defined by

$$T_i^*(r_{it}) = \begin{cases} p^m & \text{if according to } r_{it} \text{ all firms colluded in all previous rounds,} \\ & \text{and } \varphi_{jt} \geq \varphi^* \text{ for all } j \\ p^* & \text{if according to } r_{it} \text{ all firms colluded in all previous rounds,} \\ & \text{and } \varphi_{jt} < \varphi^* \text{ for at least one } j \\ 0 & \text{otherwise.} \end{cases}$$

We write $T^* = (T_i^*)_{i \in N}$ for the profile of trigger strategies.[3] Given r_t, let $q_{t+k}(r_t)$ denote the probability that $\varphi_{jt+k} \geq \varphi^*$ for all j.

Theorem 2. *The strategy profile T^* is a perfect Bayesian Nash equilibrium if and only if for every firm i and for every information set r_t at every time t, the condition*

$$\sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \cdot \Big(q_{t+k}(r_t)\Pi + (1 - q_{t+k}(r_t))\Pi^*\Big) \geq (1 - \varphi_{it})\Pi \quad (3)$$

holds when $\varphi_{jt} \geq \varphi^*$ for all j, and the condition

$$\sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \cdot \Big(q_{t+k}(r_t)\Pi + (1 - q_{t+k}(r_t))\Pi^*\Big) \geq (1 - \varphi_{it})\Pi^* \quad (4)$$

holds when $\varphi_{jt} < \varphi^*$ for some j.

Proof. The proof generally follows the same steps as the proofs of Theorem 1 and Corollary 1. Consider a firm i at time t with market share φ_{it}. If firm i would deviate from the collusive agreement, the expected loss from the punishment period would be

$$\sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \cdot \Big(q_{t+k}(r_t)\Pi + (1 - q_{t+k}(r_t))\Pi^*\Big).$$

The expected gain when $\varphi_{jt} \geq \varphi^*$ for all j is $(1 - \varphi_{it})\Pi$. So, in this case the equilibrium condition becomes

$$\sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \cdot \Big(q_{t+k}(r_t)\Pi + (1 - q_{t+k}(r_t))\Pi^*\Big) \geq (1 - \varphi_{it})\Pi.$$

When $\varphi_{jt} < \varphi^*$ for some j the expected gain is $(1 - \varphi_{it})\Pi^*$. So, in this case the equilibrium condition becomes

$$\sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \cdot \Big(q_{t+k}(r_t)\Pi + (1 - q_{t+k}(r_t))\Pi^*\Big) \geq (1 - \varphi_{it})\Pi^*. \quad \square$$

[3] Although the definition of T^* looks similar to the definition of the profile of trigger strategies T, due to the different information structures the set of histories on which T^* is defined differs from the set of histories on which T is defined.

The above Theorem shows that in the setting with public information we can, similarly to Rotemberg and Saloner (1986), implement partial collusion as a perfect Bayesian Nash equilibrium in trigger strategies. Also, as in R&S, strategies are based on a collusive dynamic price adjustment strategy.

This form of partial collusion under public information allows for collusion in more environments than the full collusion under private information studied in the previous section. The argument, here as well as in R&S, is straightforward: when we choose $\Pi^* = \Pi$ and $\varphi^* = \underline{\varphi}$ the above conditions are exactly equivalent to the condition in Theorem 1.

Thus, when full observability of market shares is not possible, this type of partial collusion cannot be implemented, and the results from Rotemberg and Saloner (1986) reduce to the results in Green and Porter (1984). Hence, full observability enhances collusion in our model. Partial collusion incorporates the opportunities that full collusion offers, and extends to environments where full collusion is no longer sustainable.

Consumer loyalty guarantees to firms a certain fixed minimum number of consumers. Thus, when the threshold φ^* is sufficiently low, an increase in consumer loyalty tends to increase the probability $q_{t+k}(r_t)$ that $\varphi_{jt+k} \geq \varphi^*$ for all j given the history r_t. All other things being equal, this shows that an increase in consumer loyalty makes it easier to satisfy the conditions in the above Theorem, which is in line with the observations in Stigler (1964).

5 Collusive Equilibria When Market Shares Form a Martingale

An interesting special case in which we can take the above analysis a step further, and give explicit formulas for the breakdown of collusion under both private and public information, is when the stochastic process that governs the market shares forms a martingale.[4]

In order to derive the analogues of Stigler (1964) and Green and Porter (1984) we need a bit more notation together with a mild assumption. Specifically, we assume for the conditional density function $f_{it+1}(\varphi_{it+1} \mid \varphi_{it}, p)$ that it is constant when all firms charge the market price. Thus, the probability density function does not depend on the (collective) choice of market price. This density function then is denoted by $f_{it+1}(\varphi_{it+1} \mid \varphi_{it})$.

We assume that $f_{it+1}(\varphi_{it+1} \mid \varphi_{it}) = 0$ outside the interval $[\underline{\varphi}, \overline{\varphi}]$ and that $f_{it+1}(\varphi_{it+1} \mid \varphi_{it}) > 0$ on the interior of the interval $[\underline{\varphi}, \overline{\varphi}]$.

Further, in accordance with the intuition that a higher market share today increases one's chances to have a higher market share in the future, we assume for the collection of cumulative probability distributions

$$F_{\varphi_{it}}(\varphi) = \int_0^{\varphi} f_{it+1}(\varphi_{it+1} \mid \varphi_{it}) \, d\varphi_{it+1}$$

[4] For firm i, the stochastic process $(\varphi_{it})_{t=0}^{\infty}$ is a martingale when $\mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) = \varphi_{it}$ for every t and k. Formally, a martingale does not satisfy the condition that $f_{it+1}(\varphi_{it+1} \mid \varphi_{it}) > 0$. However, Theorem 1 also holds for martingales.

that $\varphi_{it} \leq \widetilde{\varphi}_{it}$ implies $F_{\varphi_{it}}(\varphi) \geq F_{\widetilde{\varphi}_{it}}(\varphi)$ for every φ. Put slightly differently, when $\varphi_{it} \leq \widetilde{\varphi}_{it}$, the probability distribution $F_{\widetilde{\varphi}_{it}}$ of φ_{it+1} given $\widetilde{\varphi}_{it}$ stochastically dominates the probability distribution $F_{\varphi_{it}}$ of φ_{it+1} given φ_{it}. Under this assumption, we first show that the assumption we imposed earlier is in fact valid in this setting.

Lemma 1. *For any $k \geq 1$, $\mathbb{E}(\varphi_{it+k} \mid \varphi_{it})$ is increasing in φ_{it}.*

A sketch of its proof can be found in the Appendix. Thus, all results we derived in previous sections apply to the current case. We will review our earlier results, and specify the bounds that were only stated qualitatively in the earlier propositions.

We start with the case of private information. In this setting, as a consequence of Theorem 1, we find that the trigger strategy profile is a perfect Bayesian Nash equilibrium precisely when the discount factor exceeds 1 minus the minimum market share. Thus, when the minimum market share is relatively high, and hence uncertainty is relatively low, it is easy for the firms to sustain collusion.

Corollary 2. *Assume private information. When the stochastic variables φ_i form a martingale, the trigger strategy profile T is a perfect Bayesian Nash equilibrium if and only if $\delta \geq 1 - \underline{\varphi}$.*

Proof. When the stochastic variables φ_i form a martingale, we have $\mathbb{E}(\varphi_{it+k} \mid \varphi_{it} = \underline{\varphi}) = \underline{\varphi}$ for all t and k.[5] Thus, the equilibrium condition in Corollary 1 reduces to $\sum_{k=0}^{\infty} \delta^k \cdot \underline{\varphi} \geq 1$, which can be rewritten to $\delta \geq 1 - \underline{\varphi}$. □

We now turn to the case of public information. It turns out there is an appropriate choice of φ^* such that, given δ, the adaptive trigger strategies T^* form an equilibrium whenever the profile of trigger strategies T is an equilibrium.

Corollary 3. *Assume public information. Suppose that the stochastic variables φ_i form a martingale. Let δ be given. Suppose further that*

$$\varphi^* \geq \frac{(1-\delta)\Pi}{\delta \Pi^* + (1-\delta)\Pi}. \tag{5}$$

Then the trigger strategy profile T^* is a perfect Bayesian Nash equilibrium.

Proof. Rewriting of (5) yields

$$\sum_{k=1}^{\infty} \delta^k \cdot \varphi^* \cdot \Pi^* \geq (1-\varphi^*)\Pi. \tag{6}$$

Since the left-hand side of (6) is increasing in φ^* and the right-hand side is decreasing, we obtain

$$\sum_{k=1}^{\infty} \delta^k \cdot \varphi_{it} \cdot \Pi^* \geq (1-\varphi_{it})\Pi$$

[5] For a martingale it even holds that $\varphi_{it+1} = \underline{\varphi}$ with probability one when $\varphi_{it} = \underline{\varphi}$.

for all $\varphi_{it} \geq \varphi^*$. Thus, since the stochastic variables φ_i form a martingale, we find that

$$\sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \cdot \Pi^* \geq (1 - \varphi_{it})\Pi$$

for all $\varphi_{it} \geq \varphi^*$. Hence, since $\Pi^* < \Pi$, also

$$\sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \cdot \Big(q_{t+k}(r_t)\Pi + (1 - q_{t+k}(r_t))\Pi^*\Big) \geq (1 - \varphi_{it})\Pi$$

for all $\varphi_{it} \geq \varphi^*$, which shows that (3) in Theorem 2 is satisfied. In order to obtain (4), notice that the strategy profile T is a perfect Bayesian Nash equilibrium by assumption. So, by Theorem 1

$$\sum_{k=0}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \geq 1$$

for all t and all market shares φ_{it}. Therefore also

$$\sum_{k=1}^{\infty} \delta^k \cdot \mathbb{E}(\varphi_{it+k} \mid \varphi_{it}) \cdot \Pi^* \geq (1 - \varphi_{it})\Pi^*$$

for all t and all market shares φ_{it}. The second set of conditions now follows from the observation that $\Pi^* < \Pi$. □

Finally note that the condition $\varphi^* \geq \frac{(1-\delta)\Pi}{\delta\Pi^* + (1-\delta)\Pi}$ can be satisfied for any given $\delta < 1$ by appropriate choices of $\varphi^* < 1$ and $\Pi^* < \Pi$.

6 Conclusion

We presented a model in which firms repeatedly engage in a Bertrand type competition model. Depending on the strategies chosen, per period profits are distributed among those firms that charge market price according to market shares. Market shares are allowed to fluctuate over time.

Within this model with public information on market shares we derived the conditions under which partial collusion can be implemented via trigger strategies with a dynamic price adjustment policy. Implementability of partial collusion in equilibrium is in line with the logic of Rotemberg and Saloner (1986).

Using this model we argue that both public observability of market shares and low volatility of market shares are essential for the implementation of partial collusion. Absence of public observability prevents firms from using dynamic pricing strategies, and we revert to the logic of Green and Porter (1984). On the other hand, in line with Stigler (1964), we see that in our basic model with full observability, low consumer loyalty also prevents firms from using dynamic pricing strategies. Such strategies can in principle still be executed, but under low consumer loyalty, and hence high volatility of market shares, this form of partial collusion fails to satisfy the equilibrium conditions, and collusion breaks down.

Thus, our model can be seen as reconciliation of the three classical models of Stigler (1964), Rotemberg and Saloner (1986), and Green and Porter (1984). We conclude that these models do not necessarily represent opposing views, but rather complement each other, and each view has its own consistent logic that indeed applies under different, mutually exclusive, conditions within our model.

Appendix

In this appendix we provide a sketch of the proof of Lemma 1. It is well known that stochastic dominance implies the following statement for monotone transformations of φ_{it+1}.

Lemma 2. *Let $g(\varphi_{it+1}) \geq 0$ be (strictly) increasing in φ_{it+1}. Then*

$$\mathbb{E}(g \mid \varphi_{it}) = \int g(\varphi_{it+1}) \cdot f_{it+1}(\varphi_{it+1} \mid \varphi_{it})\ d\varphi_{it+1}$$

is (strictly) increasing in φ_{it}.

Proof of Lemma 1. We write

$$\mathbb{E}(\varphi_{it+1} \mid \varphi_{it}) = \int \varphi_{it+1} \cdot f_{it+1}(\varphi_{it+1} \mid \varphi_{it})\ d\varphi_{it+1}.$$

By Lemma 2, $\mathbb{E}(\varphi_{it+1} \mid \varphi_{it})$ is a strictly increasing function of φ_{it}. Thus, iterating the same argument, also

$$\mathbb{E}(\varphi_{it+2} \mid \varphi_{it}) = \mathbb{E}(\mathbb{E}(\varphi_{it+2} \mid \varphi_{it+1}) \mid \varphi_{it})$$

is a strictly increasing function of φ_{it}. In general we find that $\mathbb{E}(\varphi_{it+k} \mid \varphi_{it})$ is increasing in φ_{it} for any $k \leq 1$. This completes the proof of Lemma 1. □

In order to see that the equation

$$\mathbb{E}(\varphi_{it+2} \mid \varphi_{it}) = \mathbb{E}(\mathbb{E}(\varphi_{it+2} \mid \varphi_{it+1}) \mid \varphi_{it}) \tag{7}$$

in the proof of Lemma 1 indeed holds, it is instructive to derive it for a discrete process. Let M_1, M_2 and M_3 be three finite sets. Suppose we have transition probabilities $P(m_2 \mid m_1)$ and $P(m_3 \mid m_2)$ for all $m_1 \in M_1$, $m_2 \in M_2$, and $m_3 \in M_3$. Then

$$P(m_3 \mid m_1) = \sum_{m_2} P(m_2 \mid m_1) \cdot P(m_3 \mid m_2).$$

So,

$$\begin{aligned}
\mathbb{E}(m_3 \mid m_1) &= \sum_{m_3} P(m_3 \mid m_1) \cdot m_3 \\
&= \sum_{m_3} \sum_{m_2} P(m_2 \mid m_1) \cdot P(m_3 \mid m_2) \cdot m_3 \\
&= \sum_{m_2} \left[\sum_{m_3} m_3 \cdot P(m_3 \mid m_2) \right] \cdot P(m_2 \mid m_1) \\
&= \sum_{m_2} \mathbb{E}(m_3 \mid m_2) \cdot P(m_2 \mid m_1) \\
&= \mathbb{E}\left(\mathbb{E}(m_3 \mid m_2) \mid m_1\right).
\end{aligned}$$

Equation (7) is the continuous variant of the same result. The formula for the continuous case can be shown using the Theorem of Radon-Nikodym and Tonelli's Theorem. For further information we refer to Davidson (1994).

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A Game Theory Approach for Deploying Medical Resources in Emergency Department

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Abstract. Emergency department need a decision support tool to advise the critical medical resources (such as doctors, nurses, or beds) to urgent patients in the different service time requirements. This study proposes a framework for emergency response service that incorporates two game theory models designed to deploy response medical resources when raising three threat advisory levels. First, the interactions between a group of weekly patients and a response agent of emergency department are modeled as a non-cooperative game, after which a threat, vulnerability, and consequence value (i.e., TVC) of each type of patients for emergency event is derived from Nash equilibrium of game. Second, four TVC values of emergency events are utilized for computation of the Shapely value for each type of patient. The deployment of emergency medical resources is carried out based on their expected marginal contribution. Then, the model scheduled daily physicians, nurses, and beds in emergency department. The experimental results show that proposal model is feasible as a method to improve efficiency in emergency department.

Keywords: Emergency response · Nash equilibrium · Shapley value · Medical resources scheduling

1 Introduction

Emergency Department (ED) overcrowding gradually becomes a serious problem which reduces quality of emergency care, increased costs of patients, and decreased physician job satisfaction [6, 18]. However, the more emergency medical resources (such as doctors, nurses, and beds) deploy, the more patients need to care and hence the price to pay is higher. Mass deployments are probably to waste resources. On the contrary, the fewer medical resources deployed, the easier patients to be delayed, and the more difficult it is to respond. The robustness of a completely emergency decision process depends upon a balance between the medical resources requirements of and the urgent of patients. Therefore, there exists in ED a dilemma between overcrowding of visiting patients and the capability of medical resources. The current systems lack

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specific measures for rational decision-making, as well as the capacity to apply mathematical models to capture the interactions between patients and medical resources. The ED administrator should have the tools to measure the strength of overcrowding patients and the resistance capability of the response medical resources. A rating system can be built to assist with decision making by considering the utilities of the moves that the overcrowding patient and the medical resource can take [11].

Game theory tools provide analytical techniques that are already applied in many other research areas, where multiple agents compete and interact with each other within a specific system. In most multiple agent interactions, the overall outcome depends on the choices made by all self-interested agents. The goal is to make choices that optimize the outcome [2]. Game theory also focuses on these adversarial risk analyses may lead to far more effective allocations of limited response resources than current multiplying threat, vulnerability, and consequence (TVC) analyses. Risk analysis and game theory are also support each other and deeply complementary so that they can improve risk assessors to avoid producing potentially irrelevant or misleading TVC risks scores [1].

The proposed model is applied to deploy medical resources for emergency responses in ED. Two game-theoretic models are constructed, representing the two stages needed for economical deployment of the available resources. In the first step, the interactive movements between a group of weekly patients and a response agent (i.e., administrator) of emergency department are modeled and analyzed as a non-cooperative game, after which the TVC value of each type of patients for emergency event is derived from Nash equilibrium of game. This value quantified threat, vulnerability, and consequence of an emergency for visiting patient. In the second step, the interactions of four types of emergency event of patient in a whole administration region are likened to the playing of a cooperative game. Four TVCs of patients for emergency events are utilized to compute each type of emergency's Shapley value. Then, the Shapley value assists in setting priorities for medical resources allocation. Finally, the administrator deploys three types of medical resources for the three shifts working scheduling per day to improve efficiency and enhance patient care in emergency department.

2 Literature Review

The efficiency and cost of emergency systems depend on the performance of deployment and working scheduling of available medical resources. However, constructing optimized work timetables for those resources are NP-hard problem [19, 20]. Luscombe and Kozan [12] proposed a resource allocation model which is heuristic approach to find near-optimal solutions. Their ED model provided the patient appropriate bed and scheduled medical resources with a tabu search. Some simulation-based models analysis the patient requirements and minimize the patient average length of stay so as to produce the appropriate medical resources allocations, such as doctors, nurses, and beds in ED [3, 8]. Although they utilized simulated data to search for the optimal parameters to match the real data, but these optimal approaches could not satisfy the sudden overcrowding patients, particularly when available medical resources are limited in ED. Gupta and Ranganathan [17] surveyed several optimization approaches such as genetic algorithms (GA), the Bayesian search method (BS), as well as random search (RS) and greedy allocation (GS) methodologies. These algorithms cannot be applied to the multi-crisis management problem because they do not provide individual rationality, since some of the members of the population become dominant as the algorithm progresses. However, using game theory, scenarios that optimize multiple competing objectives can be modeled.

Jacobson et al. [7] indicated there is a trade-off between prioritizing urgency jobs and prioritizing higher expected payoff for emergency patient in mass-casualty incidents. An efficient emergency response need to assign the appropriate resources to the urgency of a single patient or a group of patient's requirements. Game theory can model interactions between self-interested agents (e.g., job or resource) and analysis as to which strategies can be designed that will maximize the benefits of an agent in a multiple agent system. Many of the applications of game theory have been to analyze the negotiation and coordination of multiple agents. Game theory can be a useful tool for building future generations of mixed game theory and decision theory [16]. In a non-cooperative game, each player (or self-interested agent) tries to utilize resources at minimum cost and the coordination is not enforced externally but is self-enforcing. All players optimize their decisions which maximize their payoffs in a non-cooperative game. N.E. is a solution concept for a non-cooperative game which identifies a prediction of the game outcome such that every player in the game is satisfied with respect to every other player.

In a cooperative game, the self-interested agents implement their joint actions to form a specification of the coalition. No players can separate from the coalition and take a joint action that makes all of them better off [14]. The Shapley value is a solution concept for cooperative games which computes the power index of an individual for cost allocation [13]. The cooperative game provides a suitable model for the design and analysis of response agent deployment, and it has been shown that the famous Shapley value rule satisfies many nice fairness properties [21]. The Shapely value also identifies a socially fair, good quality allocation for all agents in a multi-agent system (MAS). Here, the individual fairness for each player is optimal and the average fairness of the MAS is high. The social optimality property ensures that each player in the game receives the best utility for himself and for the complete MAS. A power index in the form of the Shapley value is applied to calculate the marginal contribution among agents and achieve mutually agreeable division of cost for MAS deployment.

3 The Proposed Model

This paper employs two solution concepts of game theory to set up a threat level framework for deploying three types of emergency medical resources (e.g., doctors, nurses, and beds) in an ED for working scheduling and dispatch. Two games are constructed, which represent the two stages needed for the economical deployment of emergency medical resources. One is the TVC game, the other is the deployment medical resource game.

A simplified workflow chart describing the principles of optimal medical resource deployment is shown in Fig. 2. First step plays four non-cooperative games for medical, surgical, pediatric, and independent emergency event. In each game, interactions between a group of weekly patients and a response agent (or administrator) of emergency department (ED) are modelled as a two-person, zero-sum game, for each type of emergency event. After considering some of the information needed to measure the efficiency of emergency medical services, such as available resources (e.g., doctors, nurses, and beds), resources requirements, emergency priorities (e.g., five levels of triage categories), mean length of stay per patient (LOS) [4], and the number of patients. This stage simultaneously calculates four payoff matrices which combines the payoffs of each player's strategies. Then, the four payoff matrices respectively obtain expected payoff of a group of weekly patients and four TVCs of emergency events derived from N.E (shown in Fig. 1).

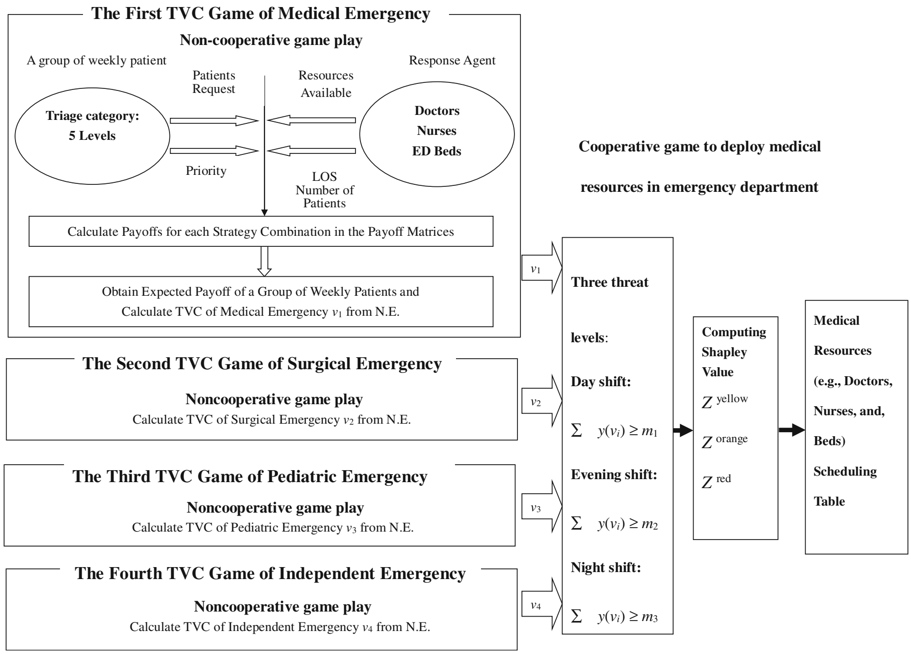

Fig. 1. Optimal resources scheduling workflow of two game models for hospital emergency department

In the second step, four TVCs of emergency events are applied to compute three thresholds so as to confirm three threat levels (i.e., day shift, evening shift, and night shift) in one day. A cooperative game model is applied to establish a rational emergency medical resources allocation by using the Shapley values as the indices of power. Given the three threat levels (i.e., day shift, evening shift, and night shift) the proposed model produces the specific Shapley value vector. Finally, a daily shift schedule of three type of medical resources (i.e., doctors, nurses, and beds) in ED is derived from the total number of each type of medical resources multiple by the Shapley value vector.

3.1 Model Assumptions

Certain assumptions must be made when treating various patients and emergency medical resources response management scenario as a game theory framework. The assumptions are listed below.

1. When patients arriving at the hospital emergency department, a triage nurse will assess and categorized them into 5 levels of urgency — Category 1 (critical), 2 (emergency), 3 (urgent), 4 (standard) and 5 (non-urgent) [8]. According to their categories, patients follow specific procedures and are directed to different treatment areas. ED nurse gives patient directing initial medical care and services required. And then doctors contact with the patient, diagnosis, and examination should be followed by decisions, often required to be made as soon as possible in serious cases. Four types of emergencies (i.e., medical, surgical, pediatric, and independent emergency) may occur simultaneously in different patients in a time-overlapped manner, and resources requests and emergency priorities will be received by the response agent of ED (shown in Table 1).

Table 1. Emergency priorities, types, and the daily resource requests

Triaged category	Priority (1–5)	Doctors				Nurses	Beds
		Internist	Surgeon	Pediatricians	Others		
Level 1	5	13	2	3	1	19	19
Level 2	4	52	11	17	8	87	87
Level 3	3	32	7	10	7	55	55
Level 4	2	2	0	1	1	4	4
Level 5	1	1	0	0	0	1	1
Total		100	20	30	16	166	166

2. This study assumes a group of weekly patients is player 1 and a response agent of ED is player 2. Hence, two game players can utilize a static single step. The first model is a normal form game for modelling the interactions between two players. Player 2 has a different number of resources available, such as doctors, nurses, and beds for four types of patients (shown in Table 2). These four types of patients include medical, surgical, pediatric, and other independent emergencies, which are provided a good standard of treatment and care by three types of medical resources. In TVC game there is complete information regarding the number of patient visits per week.

Table 2. Doctors, nurses and beds available for 4 types of patients

Type of patients	Medical resources		
	Doctors	Nurses	Beds
Medical emergency	20	60	42
Surgical emergency	6	18	42
Pediatric emergency	4	12	42
Independent emergency	28	18	42
Total	58	108	168

3. The weekly mean length of stay (LOS) [4] per triage category patient indicates the time from each type of patient arrival to time of discharge from the ED, calculated as a weekly average (shown in Table 3). Longer length of stay may reflect inefficient use of emergency resources, and length of stay is a key factor of patient satisfaction.

3.2 TVC Game

Four TVC games are applied in the first step of the proposed model, which are designed to obtain the TVC of the i^{th} type of emergency event (e.g., medical, surgical, pediatric, or other independent emergency), $i = 1, 2, 3, 4$. In each non-cooperative game, this study assumes that a group of weekly patients is player 1 and a response agent of ED is player 2. The two players' behaviors are captured with a two-person game theory model. Player 1 generates five levels of urgency patients. Player 2 is a stewardship [9] who is responsible for providing excellent care for these patients within the constraints of medical resources available. This study assumes a group of weekly patients and a response agent of ED to be rational players, a set of noncooperative players $I = \{I_1, I_2\}$, where I_1 is a group of weekly patients and I_2 is a response agent of ED. The parameters for determining the threat, vulnerability, and consequence (TVC) measures are defined in the following paragraphs.

A group of weekly patients is player 1. Five categories of urgency patients could happen simultaneously as a result of the player 1's actions. S_1 denotes the set of strategies available to player 1: $S_1 = \{u_1, u_2, u_3, u_4, u_5\}$ = {category 1 (critical), 2 (emergency), 3 (urgent), 4 (standard) and 5 (non-urgent)}. ED medical resources are available for categories of urgency patients. W denotes the set of resource requirements for five categories of urgency patients in an ED. $W = \{w_1, w_2, \ldots, w_n\}$. The variable $w_{j,k}$ denotes the number of resources required by a u_j triaged category from resource d_k. Moreover, the priority for each category of urgency patient is related to the resource requirements. Typically, in a priority-based system, a high priority patient needs more resources, and a lower priority patient needs fewer resources. Each patient is assigned a priority P on a scale of 1 to 5 indicating the severity of the urgency patients, which is used as a weight in the payoff function to facilitate the calculation of gains. p_j denotes the priority of the j^{th} patient $p_j = 1, 2, 3, 4, 5$ where 1 is lowest and 5 is highest level (shown in Table 1).

Response agent is player 2, which in turn have varied resources to prepare for five triage category patients who will generate four types of emergency events. In this game, player 2 is responsible for providing ED patients with three types of medical resources (i.e., three strategies). S_2 denotes the sets of player 2 strategies: $S_2 = \{d_1, d_2, d_3\}$ = {doctors, nurses, and beds}. O denotes the set of medical resources available in ED. $O = \{o_1, o_2, o_3\}$. o_k denotes the number of resources available at resource d_k; $k \in 1, 2, 3$. Length of emergency department stay (LOS) is a measure of medical care efficiency, and it is often a mainly source of patient complaint and frequently the target of interventions. $t_{i,j}$ denotes the weekly mean length of stay of j^{th} triage category patients in the i^{th} type of emergency events in the ED. Response agent should offer more medical resources to serve the patients when the number of patients in need increases and the average LOS of patient takes long time.

Table 3. Weekly mean length of stay per triage category patient with four types of emergencies

(Minutes) Triage categories	Medical emergency	Surgical emergency	Pediatric emergency	Independent emergency (the others)
Level 1	185	188	165	105
Level 2	163	145	115	99
Level 3	162	123	105	89
Level 4	125	117	120	82
Level 5	104	99	115	75
Total	739	672	620	450

Two players will simultaneously make their strategic decisions. A 5 × 3 payoff matrix for the each TVC game is created based on two players' strategies and interactions as seen in Table 4. The payoff to player 1 for choosing a particular strategy when player 2 makes his selection can be represented as a gain by player 1 (+) or a loss for player 2 (–). In this model, a summation of the losses of player 2 is depicted, and player 1 tries to maximize this loss. Player 2 tries to minimize the losses. Player 1 gains a profit from player 2's effort of resource responses. Player 2 pays as a result of player 1's multi-emergency events and various triage patients. This game assumes that each player aims to achieve as high a payoff for him or her as possible.

Table 4. Payoff matrix for the TVC game in i^{th} type of emergency event (e.g., medical emergency)

Player 1 (patients)	Player 2 (response agent)		
	d_1 (doctor)	d_2 (nurse)	d_3 (bed)
u_1 (level 1)	$\left(\frac{w_{i,1}}{o_{all,1}-w_{1,1}}\right)t_{i,1}p_1$	$\left(\frac{w_{i,2}}{o_{all,2}-w_{1,2}}\right)t_{i,1}p_1$	$\left(\frac{w_{i,3}}{o_{all,3}-w_{1,3}}\right)t_{i,1}p_1$
u_2 (level 2)	$\left(\frac{w_{i,1}}{o_{all,1}-w_{2,1}}\right)t_{i,2}p_2$	$\left(\frac{w_{i,2}}{o_{all,2}-w_{2,2}}\right)t_{i,2}p_2$	$\left(\frac{w_{i,3}}{o_{all,3}-w_{2,3}}\right)t_{i,2}p_2$
u_3 (level 3)	$\left(\frac{w_{i,1}}{o_{all,1}-w_{3,1}}\right)t_{i,3}p_3$	$\left(\frac{w_{i,2}}{o_{all,2}-w_{3,2}}\right)t_{i,3}p_3$	$\left(\frac{w_{i,3}}{o_{all,3}-w_{3,3}}\right)t_{i,3}p_3$
u_4 (level 4)	$\left(\frac{w_{i,1}}{o_{all,1}-w_{4,1}}\right)t_{i,4}p_4$	$\left(\frac{w_{i,2}}{o_{all,2}-w_{4,2}}\right)t_{i,4}p_4$	$\left(\frac{w_{i,3}}{o_{all,3}-w_{4,3}}\right)t_{i,4}p_4$
u_5 (level 5)	$\left(\frac{w_{i,1}}{o_{all,1}-w_{5,1}}\right)t_{i,5}p_5$	$\left(\frac{w_{i,2}}{o_{all,2}-w_{5,2}}\right)t_{i,5}p_5$	$\left(\frac{w_{i,3}}{o_{all,3}-w_{5,3}}\right)t_{i,5}p_5$

The payoff for the j^{th} strategy for player 1 when player 2 chooses the k^{th} strategy to the response can be formulated as

$$\pi_1 = \sum\nolimits_{j=1}^{5} \sum\nolimits_{k=1}^{3} \left(\frac{w_{i,\mathrm{k}}}{o_{all,k} - w_{j,\mathrm{k}}}\right) t_{i,\mathrm{j}} \times p_j, \quad i \in 1, 2, 3, 4 \tag{1}$$

where $o_{all,k}$ is the total number of medical resources available of type k in the whole ED (such as total number of nurses who can care for the patient). The proposed model assumes that the non-cooperative game is a zero-sum game; therefore, the payoff function of player 2 is given by

$$\pi_2 = -\sum_{j=1}^{5}\sum_{k=1}^{3}\left(\frac{w_{i,k}}{o_{all,k} - w_{j,k}}\right)t_{i,j} \times p_j, \quad i \in 1,2,3,4 \tag{2}$$

Player 1 and 2's expected payoff is computed when they use mixed strategies r and q, respectively. If the game has no pure strategy Nash Equilibrium (N.E.), a mixed N.E. pair (r^*, q^*) exist in the game, which also is an optimal strategy [5, 10]. The mixed N.E. for the probability vector is $r^* = \{r^*(u_1), r^*(u_2), r^*(u_3), r^*(u_4), r^*(u_5)\}$ with actions $\{u_1, u_2, u_3, u_4, u_5\}$ by player 1 and the vector $q^* = \{q^*(d_1), q^*(d_2), q^*(d_3)\}$ with actions $\{d_1, d_2, d_3\}$ by player 2. The player 1's expected payoff for a N.E. (pure or mixed strategy) is defined as its TVC of the i^{th} type of emergency event. This study defines the v_i as a TVC of the i^{th} type of emergency event in a week, given by

$$v_i = \sum_{j=1}^{5}\sum_{k=1}^{3} r^*\left(u_j\right)q^*(d_k)\pi_1\left(u_j^*, d_k^*\right), \quad u_j^*, d_k^* \in N.E. \tag{3}$$

Therefore, v_i is derived from the expected payoff of two players' optimal strategies which represents the TVC of the i^{th} type of emergency event in the first game model. The next proposed model applies the value v_i to compute the Shapley value of each type of emergency event within the cooperative game.

3.3 The Deployment Game

This study likens the interaction of four types of emergency events (e.g., medical, surgical, pediatric, or other independent emergency) in a whole hospital ED administration to the playing of a cooperative game. A fair and efficient method is needed to decide the number and priority of medical resources to be deployed when the emergency threat level is raised. The Shapley value is a power index for cost allocation. The cooperative game provides a suitable model for the design and analysis of resources deployment, and it has been shown that the famous Shapley value rule satisfies many fairness properties. Thus, in the proposed model, the Shapley value is applied to create an optimal cost allocation for the deployment of ED resources for working scheduling.

This research defines y: $V \rightarrow R+$ as a one-to-one function by assigning a positive real number to each element of v and $y(0) = 0$, $V = \{v_1, v_2, v_3\}$. The four emergency events for medical resources deployment is based on the concept of the threat level. Three emergency threat levels in one day (e.g., day shift, evening shift, and night shift) are L= $\{l_1, l_2, l_3\}$, where $0 < m_1 < m_2 < m_3$ represents the corresponding threshold values. Given the output vector of four TVCs of the emergency events, the threat level L of the HSAS response region is equal to l_f if the sum of the TVCs of emergencies is greater than or equal to m_f, f = 1, 2, or 3.

$$L = \begin{cases} l_1 & if \sum_{i=1}^{N} y(v_i) \geq m_1 \\ l_2 & if \sum_{i=1}^{N} y(v_i) \geq m_2 \\ l_3 & if \sum_{i=1}^{N} y(v_i) \geq m_3 \end{cases} \tag{4}$$

where $m_1 = v_{Mini} + m_{in}, m_2 = m_1 + m_{in}, m_3 = m_2 + m_{in}, m_{in} = \left(\frac{v_{Max} - v_{Mini}}{4}\right)$

Four TVCs of emergency events can be grouped into three threat levels according to the average value of the interval m_{in} of the threshold. This is divided by three threat levels from the maximum TVC v_{Max} to the minimum v_{Min} value.

Four types of emergency events (e.g., medical, surgical, pediatric, or other independent emergency) can be modeled as a 4-person game with $X = \{1, 2, 3, 4\}$, which includes the set of players and each subset $V \subset N$ and where $v_i \neq 0, \forall_i \in V$ is called a coalition. The coalition of X emergency event groups in the m^{th} threshold of the threat level and each subset of the X (coalition) represents the observed threat pattern for different threat levels of L. The aggregate value of the coalition is defined as the sum of the TVCs of the emergency events,$y(C) = \sum_{i \in C} y(v_i)$ and is called a coalition function. Each emergency event coincides with one or another given m thresholds of the threat level. Therefore, the different priorities for four types of medical resources deployment are derived from three threshold values. Based on the emergency threat for each type of resource with respect to others and the effect of the threshold values on three threat levels, the Shapley value represents the relative importance of each emergency. Now let $y(C) = \sum_{i \in C} y(v_i), v_i \in V, C \subset X$ be the value of the coalition C with cardinality c. The Shapley value of the i^{th} element of the emergency vector is defined by:

$$\omega(i) = \sum_{\substack{C \subset X \\ i \in c}}^{n} \frac{(c-1)!(n-c)!}{n!} [y(C) - y(C - \{i\})] \tag{5}$$

$$\Rightarrow \omega(i) = \sum_{\substack{C' \subset X \\ i \in c'}}^{n} \frac{(c-1)!(n-c)!}{n!} \tag{6}$$

Equation (5) can be simplified to Eq. (6) because the term $y(C) - y(C - \{i\})$ will always have a value of 0 or 1, taking the value 1 whenever C' is a winning coalition. If C' is not a winning coalition, the terms $C - \{i\}$ and $y(C)$ are 0 [15]. Hence, the Shapley value is $\omega(i)$, where C' denotes the winning coalitions with $\sum v_i \geq m_f, i \in C'$. The Shapley value of the i^{th} type of emergency event output indicates the relative TVC for the thresholds l_f (i.e., threat levels). Therefore, a Shapley value represents the strength of emergency event which ED resources should give patient medical care and services required in one day.

We applied Shapley values of i^{th} type of emergency event to compute the number of k^{th} type of resources allocation in the threat level l_f. The numbers of resources of type k allocated to the i^{th} emergency event are defined by

$$e_k(i) = \omega(i) \times o_{all,k} \tag{7}$$

where $o_{all,k}$ is the total number of medical resources available of type k in the whole ED (such as total number of nurses who can care for the patient). The allocated numbers of the i^{th} medical resources $e_k(i)$ are derived from the Shapley value of the i^{th} emergency event $\omega(i)$ multiplied by $o_{all,k}$ the total number of resources available of type k. Finally, the response agent of ED can deploy three types of medical resources to improve efficiency and enhance patient care in emergency department.

4 Simulation Experiments

This study hypothesizes that a response agent of ED possesses a different number of resources available for four types of patients. Given the three threat levels (e.g., day shift, evening shift, and night shift), response agent provides medical resources scheduling and rostering per day in ED. The simulation employs Tables 1, 2 and 3, which show information regarding the resource types and the availability, emergency requests, emergency priorities, and length of stay (LOS) per triage category patient.

This experiment first modeled the noncooperative game and generated simulation sets of TVC measures for the four emergency events in a week. The payoff matrix of the zero-sum TVC game for each emergency event is modeled by Eqs. (1)–(2) and Table 4. Then, four TVC values of emergency events are calculated by Eq. (3) (shown in Fig. 2). Second, four TVCs of the emergency events are utilized to compute the Shapley value of each emergency event (shown in Fig. 3). This study assumes three threat levels for the emergency resources response: night shift, evening shift, and day shift. We adopted the Matlab tool to compute the Shapley value of each emergency event based on the three threat levels. Then, a simulation creates three minimum sets of medical resources (i.e., doctors, nurses, and beds) for working shifts per day in ED by Eqs. (4)–(6) (shown in Figs. 4, 5 and 6).

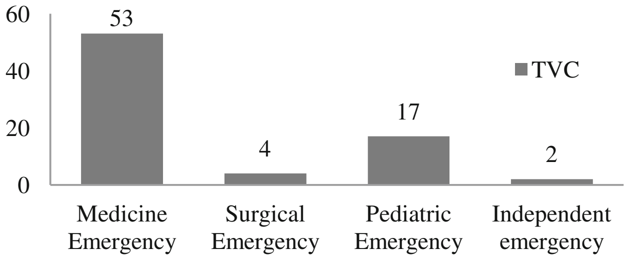

Fig. 2. Four TVCs of emergency events

As can see in Figs. 4, 5 and 6, the medicine emergency need more ED resources (e.g., internist, nurse, and beds) than surgical, pediatric, and independent emergency in day and evening shift. The proposed working schedules or rostering suggest that ED deploys most of physicians, nurses, and beds to care the patients of medicine emergencies. This provides the administrator of ED with a way to prioritize medical resources when allocating limited resources in case of multiple emergencies. It also provides more quantitative values for the various situations than human decision making can provide, and it more fairly deploys the three resources (i.e., doctors, nurses, beds) for the three threat levels. In addition, the administrator can use this model to quantitatively evaluate any emergency threat to response resources and easily discover where response resources are most at risk within the three threat levels (i.e., day shift, evening shift, and night shift). This function also can increase the response agent's vigilance and can secure the critical medical resources to "prevail" against urgency patients.

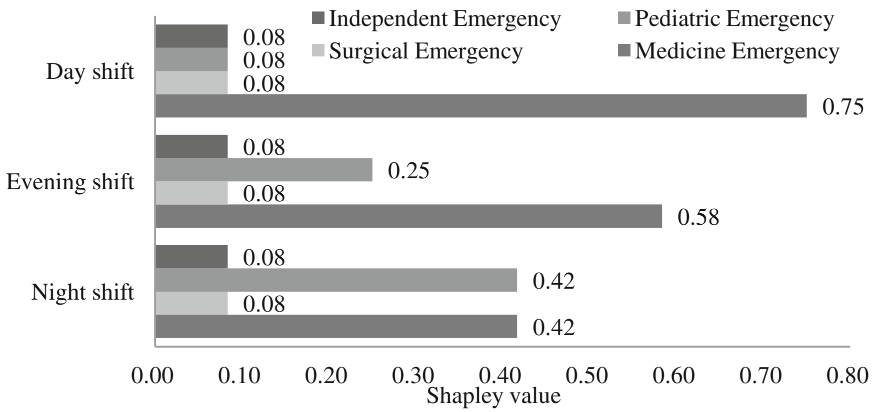

Fig. 3. Four shapley values of emergency events considering the three levels

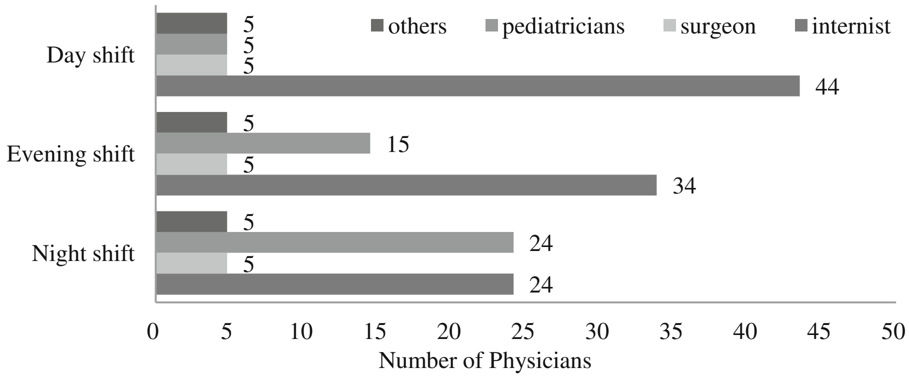

Fig. 4. Three types of ED doctors shift scheduling for the three threat levels

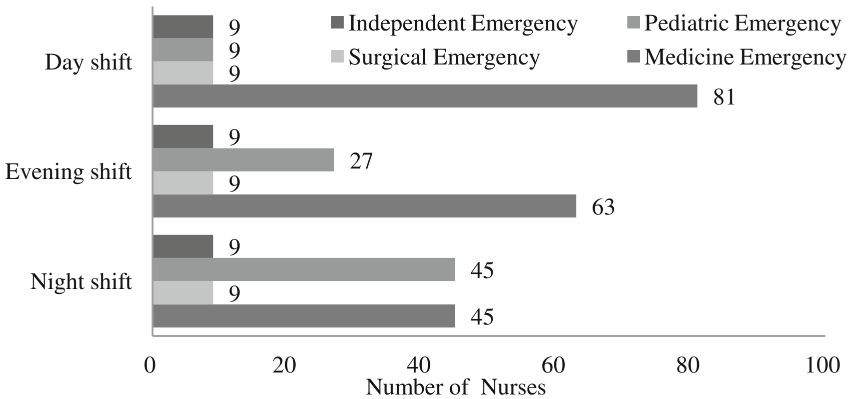

Fig. 5. Three types of ED nurses shift scheduling for the three threat levels

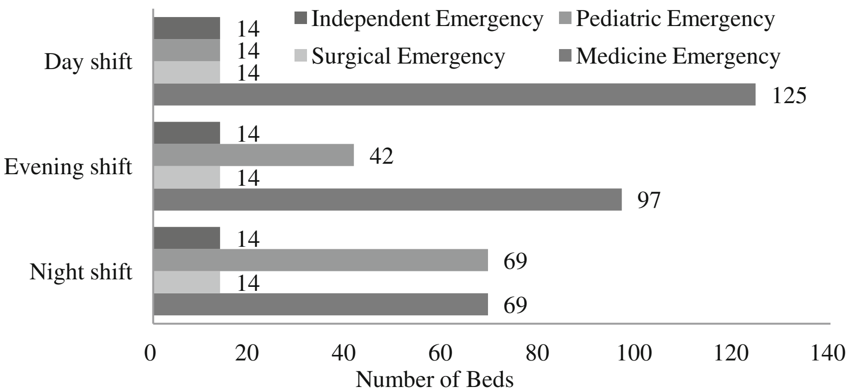

Fig. 6. Three types of ED beds shift scheduling for the three threat levels

5 Conclusions

In the proposed framework, two game theoretic models are applied to develop the optimal deployment of emergency medical resources and to provide three work shifts scheduling per day in emergency department. The first model creates a game strategy using the Nash equilibrium to find the TVC of each emergency event from a non-cooperative game. The second model uses all TVC values of the emergency events to compute the Shapley value for each emergency event given three emergencies levels (i.e., day shift, evening shift, and night shift). The experiments help us to identify the framework connecting the Nash equilibrium and Shapley values, which will enable administrator to prioritize medical resource deployment for the three working shifts in hospital emergency department.

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Non-cooperative Monomino Games

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Abstract. In this paper we study monomino games. These are two player games played on a rectangular board with R rows and C columns. The game pieces are monominoes, which cover exactly one cell of the board. One by one each player selects a column of the board, and places a monomino in the lowest uncovered cell. This generates a payoff for the player. The game ends if all cells are covered by monominoes. The goal of each player is to place his monominoes in such a way that his total payoff is maximized. We derive the equilibrium play and corresponding payoffs for the players.

Keywords: Monomino games · Non-cooperative games · Nash equilibrium · Pure strategies

2010 AMS Subject classification: 91A10 · 91A05

1 Introduction

In this paper we introduce the two-player game of monomino. This is a parlor game like dice games, card games and so on. Instead of determining the player that makes the last move (like in chess, checkers, or the game of Nim), the players are interested in optimizing their payoffs (like in dice games and card games). This game is played on a rectangular board or grid, say it has size 3×3. The cells on the bottom (first) row have a value of 1 unit each, on the middle (second) row the values are 2, and on the top (third) row the cells have a value of 3 units each. The players alternately play a monomino, which is a piece that covers a single cell of the board.

This game has the following rules. The players select one by one a column of the board, and place a monomino in the lowest uncovered cell. A monomino in row i on the board generates a payoff of i units to the player. The game ends if all cells are covered by monominoes. In contrast to games like chess and checkers, the goal of each player is to place his monominoes in such a way that it maximizes his total payoff.

In this paper we analyze non-cooperative monomino games for general rectangular boards. Notice that the game looks a bit like the game of Tetris but it is

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played with monominoes. More general, it resembles a combinatorial game; both have two players, complete information, and no chance involved. The main difference is that we are interested in optimizing the payoffs of the players, instead of determining who makes the last move. The latter question is not interesting in this game since the winner may be determined beforehand. For instance, if the player who plays the last monomino wins, then the winner is determined as follows. If the number of cells is odd, then the player who makes the first move wins, and if the number of cells is even, then the player who makes the second move wins.

In the literature on combinatorial games the focus is on how to win games with dominoes or other pieces like pentominoes. [8] studies winning moves for the game of pentominoes. [6] describes a two-player game played on a square board. One by one the players mark a cell on the board. The first player to form a domino loses; hence the game is named dominono. The author provides winning strategies. Tilings with polyominoes are studied in [10]. Excellent surveys on combinatorial games are [1,5]. In cooperative game theory, attention is also paid to combinatorial games; see [2] for a survey.

The literature on non-cooperative game theory pays among others attention to parlor games like dice games [3], matching pennies, rock-paper-scissors, and two-finger Morra (see e.g. [9]). These are zero-sum games, that is, the gain of one player is the loss of the other player. Also, nearly always these games have no equilibrium in pure strategies.

In this paper we introduce monomino games and study them using non-cooperative game theory. Our results describe the equilibrium play and payoffs for the players. The monomino game is a constant sum game and thus has a Nash equilibrium in pure strategies. An initial study on monomino games is reported in the thesis [4].

The outline of this paper is as follows. In Sect. 2 we introduce monomino games. In Sect. 3 the equilibrium play and payoffs are considered. Section 4 concludes.

2 Monomino Games

A monomino game is played by two players on a rectangular board with R rows and C columns. We denote such a monomino game by $M(R, C)$. Each of the RC cells is square. The game is played with pieces of 1×1 cell; these pieces are named monominoes.

The players are named player 1 and player 2. Player 1 starts. One by one the players put a monomino on the board according to the following rules. A monomino is placed in a cell of the board. If the piece is placed in column i of the board, then this monomino covers the lowest uncovered cell. The game ends if all cells are covered by monominoes; this happens after RC moves.

Each played monomino generates a value for its player. If a player places a monomino in row j then this increases the payoff of this player by j units. Each player wants to maximize his payoff. Hence, this game is a non-cooperative game. The Nash equilibrium [7] is a solution concept for non-cooperative games that describes optimal play of the game. Namely, a pair of strategies (s_1, s_2) for the players is a Nash equilibrium if s_1 optimizes player 1's payoff in case player 2 plays strategy s_2, and conversely, s_2 optimizes player 2's payoff in case player 1 plays strategy s_1. For a game in extensive form we consider subgame perfect equilibria, a refinement of Nash equilibria. A subgame perfect equilibrium is a pair of strategies that induces a Nash equilibrium in every subgame. These notions are illustrated in the example below.

Example 1. Consider the game $M(3, 2)$. This game is played on a board with three rows and two columns. After six moves all six cells on the board are covered and the game is over.

This game already has very many possible plays. To be able to represent the game graphically, assume just for the remainder of this example that if a player can choose among both cells in the same row, then the player selects the cell in column 1.

Figure 1 shows a graphical representation of this game as a game in extensive form, or tree game. At each node of this tree we mention the player that makes a move as well as the game situation $[x_1, x_2]$ with x_i the number of covered cells in column i. The actions are mentioned besides the edges, with V_i the action that column i is selected. At the bottom nodes the payoffs (π_1, π_2), with payoff π_j to player j, are mentioned.

At the first node, player 1 makes the move. According to the extra assumption in this example she has only one action, namely V_1 (select column 1). In the new game situation $[1, 0]$ player 2 can choose between V_1 and V_2. And so on. We find the optimal payoffs and actions by using backward induction. The optimal actions are indicated by thick colored lines; red corresponds to player 1 and blue to player 2. There is a unique subgame perfect equilibrium, that is presented in Table 1. Notice that any subgame corresponds to a game situation. The equilibrium payoff is $(\pi_1, \pi_2) = (5, 7)$. The equilibrium play is as follows.

Table 1. The subgame perfect equilibrium in Fig. 1.

Subgame	[3, 2]	[3, 1]	[2, 2]	[3, 0]	[2, 1]	[2, 0]	[1, 1]	[1, 0]	[0, 0]
Player at move	2	1	1	2	2	1	1	2	1
Optimal action	V_2	V_2	V_1	V_2	V_1	V_1	V_1	V_2	V_1

Player 1 starts in game situation [0, 0] with action V_1, then the new game situation is [1, 0] and player 2 plays V_2. Subsequently, player 1 plays V_1, and so on. See Table 1 or the thick lines in Fig. 1.

The bimatrix game that corresponds to this game in extensive form is presented below. The actions of player 1, represented by the rows of the bimatrix, correspond to the nontrivial choices in the game situation $[2, 0]$. The actions of player 2, mentioned in the columns of the bimatrix, are denoted by $a|bc$ with a

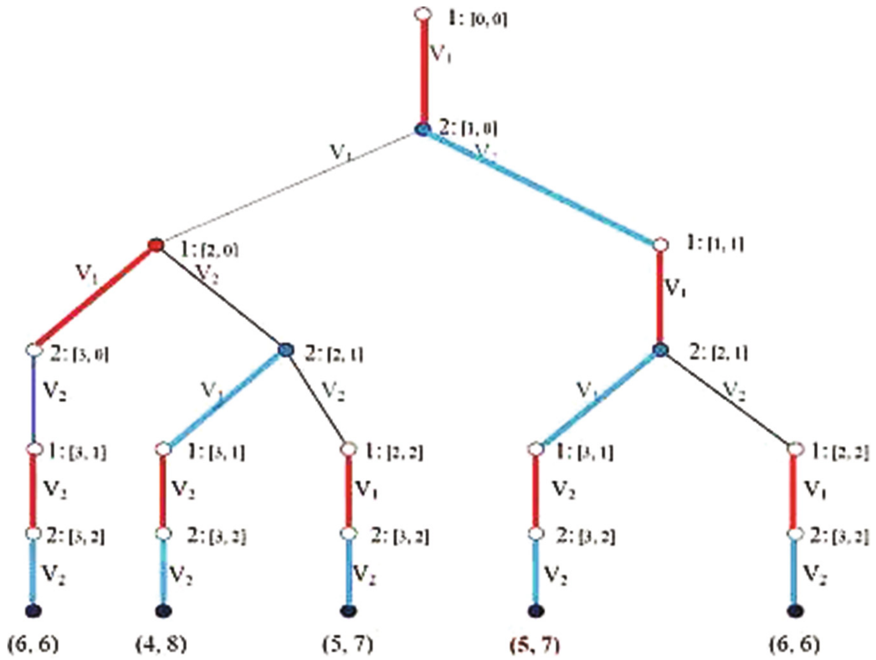

Fig. 1. The game $M(2,3)$ in extensive form with the extra restriction that if a player can choose among both cells in the same row, then the player selects the cell in column 1. (Color figure online)

the choice in situation $[1,0]$, b the choice in situation $[2,1]$ if the previous situation was $[2,0]$, and c the choice in situation $[2,1]$ if the previous situation was $[1,1]$.

	$V_1\|V_1V_1$	$V_1\|V_1V_2$	$V_1\|V_2V_1$	$V_1\|V_2V_2$	$V_2\|V_1V_1$	$V_2\|V_1V_2$	$V_2\|V_2V_1$	$V_2\|V_2V_2$
V_1	$(6,6)$	$(6,6)$	$(6,6)$	$(6,6)$	$(5,7)^*$	$(6,6)$	$(5,7)^*$	$(6,6)$
V_2	$(4,8)$	$(4,8)$	$(5,7)$	$(5,7)$	$(5,7)$	$(6,6)$	$(5,7)$	$(6,6)$

The two Nash equilibria are $(V_1, V_2|V_1V_1)$ and $(V_1, V_2|V_2V_1)$; they are indicated by stars in the bimatrix. The first one is the subgame perfect equilibrium. □

In the example above we temporarily added the restriction that a player should select a cell in column 1 when both cells are available in the same row. Even with this assumption, the game tree is not small. In this paper we consider games without this assumption, leading to even larger game trees.

Given a monomino game $M(R,C)$, the total payoff to the players is fixed, namely $C(1+2+\ldots+R) = CR(R+1)/2$. Any payoff (π_1, π_2) satisfies $\pi_1 + \pi_2 = CR(R+1)/2$. Thus, monomino games are constant sum games. This implies in particular that there exists a Nash equilibrium in pure strategies [9]. Further, if there are multiple Nash equilibria, then the payoff to a player is the same in all equilibria. This was illustrated in Example 1.

3 Game Play and Payoffs in Equilibrium

In this section we analyse the monomino games. We derive the optimal game play and the payoffs in equilibrium.

The following notation is used. The cells of the grid are indicated by pairs (i, j), where $1 \leq i \leq R$ is the row number and $1 \leq j \leq C$ the column number. Let $\mathcal{R} = \{0, 1, 2, \ldots, R\}$ be the set of the number of possible unoccupied cells per column. We consider vectors $\mathbf{p} = (p_1, p_2, \ldots, p_C) \in \mathcal{R}^C$ and denote $P = \sum_{j=1}^{C} p_j$. The vector $\mathbf{p}$ describes for any column j, $1 \leq j \leq C$, that the cells 1 up to $R - p_j$ are occupied with monominoes and the top p_j cells are free. In other words, the vector $\mathbf{p}$ represents a *position* on the playing board when $RC - P$ monominoes have been played.

The following Theorem states the equilibrium, or optimal, actions for the players. The equilibrium payoffs are mentioned in Corollary 3.

Theorem 2. *In the monomino game $M(R, C)$, the game play in equilibrium is as follows.*

(a) If R is even:
In each move, player 2 plays the same column as player 1.

(b) If R is odd and C is even:
Player 2 plays row 1 if player 1 does so, otherwise he plays the same column as player 1.

(c) If both R and C are odd:
Player 1 plays row 1 in his first move. Thereafter he plays row 1 if player 2 does so, otherwise he plays the same column as player 2.

Proof. The theorem is proved by induction to the number of remaining moves if started from a certain position $\mathbf{p}$. That is, we focus on the payoffs for the players for occupying the *remaining* free cells if started at position $\mathbf{p}$ (independent of who placed the monominoes where in the starting position $\mathbf{p}$).

First, consider case (a). We prove that this play is optimal with respect to the remaining payoffs for *each* starting position $\mathbf{p} \in \mathcal{R}^C$, where p_j is even for all $1 \leq j \leq C$. Hence it is also optimal for an empty grid in case R is even.

We use induction to $n = P/2$, the number of moves remaining for player 2 until all cells of the grid are occupied. The induction basis for $n = 1$ is trivial: in the starting position only the two top cells of one column are free and so both players must play this column.

Now let $n \geq 1$ and suppose that the play is optimal for *all* starting positions $\mathbf{p}$ satisfying p_j even for all $1 \leq j \leq C$ and $P/2 = n$. Consider a starting position $\mathbf{q} \in \mathcal{R}^C$ with q_j even for all $1 \leq j \leq C$ and $Q/2 = n + 1$. Since an even number of cells is occupied, it is player 1's turn. Suppose player 1 plays column k, i.e., cell $(R - q_k + 1, k)$ and player 2 plays column $\ell \neq k$; i.e. cell $(R - q_\ell + 1, \ell)$. Now player 1 can play cell $(R - q_\ell + 2, \ell)$ in his next move (because q_ℓ is even). This results in a position that is equivalent to the situation where player 1 plays column k in starting position $\mathbf{p} = (q_1, \ldots, q_\ell - 2, \ldots, q_C)$.

Note that $\mathbf{p}$ satisfies the conditions of the induction hypothesis, so player 2 will now play the same column as player 1 until the end of the game in order to maximize his remaining payoff (and will therefore play column k in his next move).

Now we compare this with the situation where player 2 had played column k after player 1 played this column in position $\mathbf{q}$. In this case the position $(q_1, \ldots, q_k - 2, \ldots, q_C)$ arises, which also satisfies the conditions in the induction hypothesis. So player 2 would then continue with the optimal play, i.e., playing the same column as player 1 until the end of the game.

Consequently, the final position of the pieces on the board at the end of the game only differs in the cells $(R - q_\ell + 1, \ell)$, which is now occupied by a monomino of player 2 and $(R - q_\ell + 2, \ell)$, occupied by a monomino of player 1. Therefore the remaining payoff for player 2's moves from position $\mathbf{q}$ until the end of the game, will be one unit less than the payoff he would have gotten if he had played the same column as player 1 in position $\mathbf{q}$, as the play prescribes. Hence, the play is optimal for starting position $\mathbf{q}$ for player 2 regardless of what player 1 does. Now the proof follows by induction.

Second, consider case (b). Let $\mathbf{p} \in \mathcal{R}^C$ be a starting position on the grid for which the number of nonempty columns is even and each of these columns has an even number of free cells (also fully occupied columns are considered to have an even number of free cells). Obviously, the position on an empty grid satisfies this property: for an empty grid the number of nonempty columns equals zero.

Clearly, P is even for these positions. Again we use induction to $n = P/2$. The induction basis for $n = 1$ is easily verified because in that case there are essentially two possible starting positions: the two top cells of one column are free and so both players must play this column, or $R = 1$, which is also trivial.

Now let $n \geq 1$ and suppose that the play is optimal for *all* starting positions $\mathbf{p}$ satisfying $P/2 = n$ and the number of nonempty columns is even and each of these columns has an even number of free cells. Consider a starting position $\mathbf{q} \in \mathcal{R}^C$ with $Q/2 = n + 1$ satisfying this property. Note that the number of occupied cells in $\mathbf{q}$, $RC - Q$, is even. Hence, it is player 1's turn. We will distinguish three cases.

Case b1: Player 1 plays row 1, say cell $(1, k)$, and player 2 does not, say he plays the nonempty column j, i.e., cell $(R - q_j + 1, j)$. Then, in his next move, player 1 can play cell $(R - q_j + 2, j)$ (because q_j is even). This yields a position that also arises from position $\mathbf{p} = (q_1, \ldots, q_j - 2, \ldots, q_C)$ if player 1 plays cell $(1, k)$. Note that $\mathbf{p}$ satisfies the conditions in the induction hypothesis, so player 2 can optimize his remaining payoff in position $\mathbf{p}$ by using the play described above (starting by playing row 1). Now we compare this with the situation where player 2 had played row 1 after player 1 played this row in position $\mathbf{q}$. Then the position that arises again satisfies the conditions in the induction hypothesis. So player 2 would continue by playing the optimal play until the end of the game.

We find that the remaining payoff for player 2 in position $\mathbf{q}$ will be one unit less than the payoff he would have gotten if he had played row 1, as the play prescribes (the ownership of the cells $(R - q_j + 1, j)$ and $(R - q_j + 2, j)$ is interchanged).

Case b2: Player 1 plays nonempty column j (cell $(R - q_j + 1, j)$) and player 2 plays nonempty column $k \neq j$ (cell $(R - q_k + 1, k)$). Now player 1 can play cell $(R - q_k + 2, k)$ in his next move and we arrive in a position that is equivalent to the situation where player 1 plays column j in starting position $\mathbf{p} = (q_1, \ldots, q_k - 2, \ldots, q_C)$. By the induction hypothesis, player 2 can optimize his remaining payoff in this position by using the play described above (starting by playing column j). Again we derive that player 2's remaining payoff in position $\mathbf{q}$ will be one unit less than if he had started by playing column j, as the play prescribes.

Case b3: Player 1 plays nonempty column j (cell $(R - q_j + 1, j)$) and player 2 plays row 1, say cell $(1, k)$. Then, in his next move, player 1 can play column j again (cell $(R - q_j + 2, j)$). If we now interchange the ownership of the monominoes in cells $(R - q_j + 2, j)$ and $(1, k)$, we arrive in a position that is equivalent to the situation where player 1 plays cell $(1, k)$ in starting position $\mathbf{p} = (q_1, \ldots, q_j - 2, \ldots, q_C)$ (here we use the fact that the remaining payoffs in a starting position are independent of how this position has arisen). By the induction hypothesis, player 2 can optimize his remaining payoff in position $\mathbf{p}$ by using the play described above (starting by playing row 1). If we compare this with the situation where player 2 had played column j in position $\mathbf{q}$ and thereafter followed the (by the induction hypothesis optimal) play in position $(q_1, \ldots, q_j - 2, \ldots, q_C)$, we find that the remaining payoff for player 2 in position $\mathbf{q}$ will be $R - q_j + 1$ units less than if he had started by playing column j, as the play prescribes.

In all three cases the play turns out to be optimal for position $\mathbf{q}$ no matter what player 1 does. Also here, the proof follows by induction.

Finally, consider case (c). Let $\mathbf{p} \in \mathcal{R}^C$ be a starting position on the grid for which the number of nonempty columns is odd and each of these columns has an even number of free cells (also fully occupied columns are considered to have an even number of free cells). Clearly, such a position arises as soon as player 1 played his first move. The proof now follows the same lines as the proof of part (b) with reversed roles for players 1 and 2, and an induction hypothesis for positions with an odd number of nonempty columns, all containing an even number of free cells. Finally, since player 1 must play row 1 in his first move, we conclude that the play described in part (c) of the theorem is the equilibrium play for grids with an odd number of rows and columns. □

Figure 2 shows the optimal game play and payoffs of the monomino games $M(6, 4)$, $M(5, 4)$ and $M(5, 5)$, which correspond to the cases (a), (b) and (c) respectively in Theorem 2. Row 1 is the bottom row, and row R the top row. The number in a cell indicates which player puts a monomino there. The optimal payoffs (π_1, π_2) are $(36, 48)$, $(26, 34)$ and $(43, 32)$ from left to right.

2	2	2	2
1	1	1	1
2	2	2	2
1	1	1	1
2	2	2	2
1	1	1	1

2	2	2	2
1	1	1	1
2	2	2	2
1	1	1	1
1	2	1	2

1	1	1	1	1
2	2	2	2	2
1	1	1	1	1
2	2	2	2	2
1	2	1	2	1

Fig. 2. Optimal game play in monomino games $M(6,4)$, $M(5,4)$ and $M(5,5)$ from left to right with optimal payoffs $(36,48)$, $(26,34)$ and $(43,32)$, respectively. The number in a cell indicates which player puts a monomino there.

The equilibrium payoffs to the players are easy to derive following the game play described in Theorem 2, and illustrated in Fig. 2. In case (a), player 2 forces his monominoes to occupy all cells in the even rows of the grid and player 1's monominoes occupy the odd rows. In case (b) the monominoes of player 2 will occupy half of the cells in row 1 and all cells in the other odd rows, whereas player 1's monominoes will occupy the other cells in the first row and all cells in the even rows. In case (c), player 1 forces his monominoes to occupy $\frac{C+1}{2}$ cells in row 1 and all cells in the other odd rows, and player 2's monominoes will occupy the other cells in the first row and all cells in the even rows. From this, the derivation of the payoffs to the players is straightforward, using $\sum_{k=1}^{R} k = \frac{1}{2}R(R+1)$.

Corollary 3. *The optimal payoffs in the monomino game $M(R,C)$ are as follows.*

$$(\pi_1, \pi_2) = \begin{cases} \left(\frac{CR^2}{4}, \frac{CR(R+2)}{4}\right), & \textit{if } R \textit{ is even}, \\ \left(\frac{C(R^2+1)}{4}, \frac{C(R^2+2R-1)}{4}\right), & \textit{if } R \textit{ is odd and } C \textit{ even}, \\ \left(\frac{CR(R+2)-C+2}{4}, \frac{C(R^2+1)-2}{4}\right), & \textit{if } R \textit{ and } C \textit{ odd}. \end{cases}$$

4 Conclusions

In this paper we introduced a new class of non-cooperative games: the monomino games $M(R,C)$. These are parlor games like dice games, card games and so on. Instead of determining the player that makes the last move (like in chess, checkers, or the game of Nim), the players are interested in optimizing their individual payoffs (like in dice games and card games). These are constant sum games, so Nash equilibria in pure strategies exist. We derived the equilibrium game play and the corresponding payoffs for any size of the board of the game.

Note that the results for game play and corresponding payoffs in Theorem 2 and Corollary 3 can easily be generalized to games with pieces or playing blocks of size $k \times 1$, for any positive integer k, and playing boards of size $kR \times C$ where the blocks must be played vertically. That is, any block is placed in a single column.

For future research, we intend to study non-cooperative 'domino' games. These games are also played on a rectangular board where players one by one put pieces of size $1 \times d$ on the board either in horizontal or vertical direction. (For $d = 2$ the pieces are the well-known domino pieces.) Some initial analysis of these games is done in [4]. There it turned out that these games are much more complex than monomino games. One of the reasons is that in these domino games some cells of the board may remain uncovered.

Finally, in this paper we consider monomino games with two players. It might also be interesting to examine equilibrium game play in case more than two players are involved.

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Bargaining Model of Mutual Deterrence Among Three Players with Incomplete Information

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Abstract. The tripartite bargaining problem of mutual deterrence has been investigated from the perspective of Rubinstein indefinite bargaining and cooperative game theory. Considering the situation of incomplete information in reality, this paper established a tripartite mutual deterrence bargaining model with unilateral and bilateral incomplete information by introducing incomplete information into the model and defining a discount factor. And particularly, the formula is furnished for calculating the Nash equilibrium distribution of every player under the incomplete information. Finally, an illustrative example is presented to show that the established model is feasible and effective and can provide a new way and method to analyze and solve multi mutual deterrence or conflict problems with incomplete information.

Keywords: Deterrence · Bargaining · Incomplete information · Game theory

1 Introduction

Deterrence is an effective way to influence the other player's decision-making, and expect to influence his expected judgment for his own behavioral patterns. Mutual deterrence is a situation of bilateral deterrence that is widespread in real international politics. With the growing complexity of the international situation, the issue of tripartite deterrence is gradually increased. Therefore, the study of tripartite deterrence is of great significance in solving these international conflicts.

In the mid-1960s, Schelling [1] viewed the issue of mutual deterrence as a bargaining problem and defined the ability of mutual deterrence as an ability to harm each other (or the enemy), when the two sides bargain model is relatively simple. From the perspective of cooperative game, Nash [2] proposed the famous Nash bargaining solution in 1950. But from the perspective of non-cooperative game, Rubinstein [3] proposed a limited and indefinite bargaining model. And in the tripartite mutual deterrence bargaining, each player need to consider the deterrence influence of the other two player. Kalandrakis [4] discussed the tripartite bargaining model with majority agreed rules and proposed a Markov refined Nash equilibrium. Calvó-Armengol [5] established an asymmetric tripartite bargaining model. In recent years, Fontenay et al. [15] proposed an analysis of the non-cooperative bilateral bargaining model between network agents, proving that there is a balance which generates an

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alliance bargaining by the surplus arising from externality among agents. Bayati et al. [16] argued that there is a market for a unilateral distribution or utility transferable network and dynamic bargaining model and established a balanced result of a generalized network pair Nash bargaining solution. Aghadadashli et al. [17] studied the bargaining between alternative and complementary trade unions and companies. Collard-Wexler et al. [18] extended the Rubinstein alternative bid bargaining model to a number of upstream and downstream firms on the basis of non-cooperative bargaining, proposed a model of maximizing their bilateral Nash product condition based on agreement for all other negotiating conditions, and proved that there is a perfect Bayesian equilibrium and the result is unique. An et al. [19] analyzed the strategic behavior of the negotiators in the one-to-many and many-to-one negotiations when the agent followed the alternate bid bargaining agreement, and explored how the uncertainty reserve price and duration affect the equilibrium strategy. Abreu et al. [20] found that the refined Nash equilibrium could still be reached when a player with complete information delays to make the decision of his initial demand, and the predicted result depends on the priori probability for patience type of the player with complete information.

Xiang and Wang [6] carried on the analysis of the bargaining model of the mutual deterrence between the two sides, but this model only applied to the situation of two players, and did not involve the tripartite mutual deterrence bargaining problem between three players. Gong et al. [7] studied the tripartite mutual deterrence bargaining model under complete information, but did not introduce incomplete information into the model, so the model was not general. Zhou and Wang [13] investigated the bargaining between a strong supplier and a weak wholesaler in terms of optimal production and optimal order quantity, and established bargaining model of wholesalers and suppliers under complete and asymmetric information.

According to the Rubinstein model, an acceptable agreement can be reached between the rational actors (or subjects) under complete information, so there is no sense of mutual threat and revenge, and the conflict does not occur. Unlike the above-mentioned model, this paper established a tripartite mutual deterrence model based on Rubinstein classic bargaining model in the perspective of cooperative game. And introduced incomplete information into the tripartite mutual deterrence bargaining model to explore the three players' deterrence credibility and conflict possibility under unilateral and bilateral incomplete information conditions, so that the model is more general. Finally, we compared the more general model of this paper with the original complete information tripartite mutual deterrence bargaining model, and found that it can reach a consistent distribution scheme when one player owned unilateral incomplete information, so the conflict would not appear. Meanwhile, in any alliance relationships, the share of the interest of the player with incomplete information was higher than that of player with complete information, and the player with incomplete information had the advantage of unilateral incomplete information. When two players owned incomplete information in the tripartite mutual deterrence bargaining model, each player can not propose a mutually acceptable solution in some cases, where the conflict may occur.

2 Incomplete Information Mutual Deterrence Bargaining Model

2.1 Indefinitely Rubinstein Bargaining Model

Assume that two players distribute the interest of a unit by rotating bidding.

The first stage, the player 1 proposes an allocation program $(v_1, 1 - v_1)$. If the player 2 accept the offer of the player 1, then the distribution plan is generated as $(v_1, 1 - v_1)$; if the player 2 reject the proposal of the player 1, then enter the next stage.

In the second stage, the player 2 puts forward the distribution program $(v_2, 1 - v_2)$. If the player 1 accept the proposal of the player 2, then the allocation of the program is $(\delta_1 v_2, \delta_2(1 - v_2))$, where $\delta_i \in [0, 1](i = 1, 2)$ is the discount factor for player i. If the result has not yet reached Nash equilibrium, then continue to bargain as above.

In the bargaining model of the rotation bidding, if the player can not determine the time limit in advance, they can deal with it through the indefinite bargaining problem. The Nash equilibrium given by the indefinite Rubinstein bargain model is bound to be accepted by the other player.

In the first, third, fifth, seventh, … stage, the player 1 chooses the share of his ossessed interest; the player 2 chooses the share of his possessive interest in the second, fourth, sixth, eighth, … stage. As the game is indefinite, so there is no different between the beginning sub-game in the first stage and the sub-game of the third stage. Rubinstein [3] proposed the following conclusions on the indefinite bargaining model in 1982.

Theorem 1.1 ([3]). In the indefinite rotation bid bargaining game, the only sub-game refinement Nash equilibrium is $v_1^* = \frac{1-\delta_2}{1-\delta_1\delta_2}$ when the player 1 bids in the first, third, fifth, seventh, … stage; the result of the player 2 bids in the second, fourth, sixth, eighth, … stage is $v_1^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}$. The Nash equilibrium interest share obtained by the player 1 is v_1^*, v_2^* is the share of Nash equilibrium interest earned by the player 2.

2.2 Complete Information Indefinitely Tripartite Bargaining Model of Mutual Deterrence

Assume that all three players are rational people. The player 1 first bids, player 2 then bids, and then player 3 bids, and then turn to the player 1 bids again, so making rotation bidding until the Nash equilibrium solution is reached.

The deterrent relationship between the various players is shown in Fig. 1, the discount factor of player $i(i = 1, 2, 3)$ against player $j(j = 1, 2, 3)$ is δ_{ij} if there is no agreement in the game. The smaller the δ_{ij}, the greater the loss of the player j in the game when player i implement deterrence against player j, and vice versa.

2.3 Discount Factor of Deterrence

In the process of bargaining, each of the player in order to get higher interests often spontaneously choose a temporary "alliance", and then the players in the alliance start the bargaining again to allocate the obtained share of the alliance. We use $\delta_{i,jk}$ to

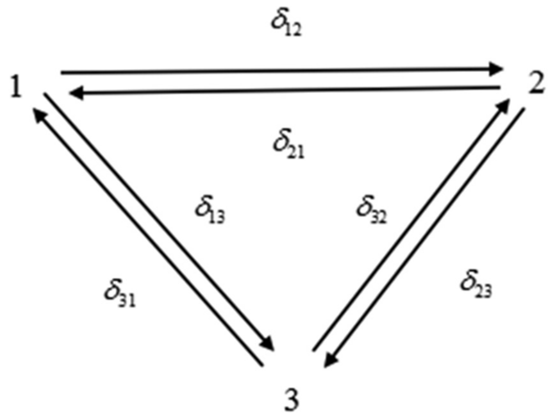

Fig. 1. Tripartite mutual deterrence relationship

express the discount factor of player i against the players in the jk alliance (player j and k form an alliance), $\delta_{ij,k}$ is the discount factor of the players in the ij alliance against player k, and others are similar explanations. Deterrence is the ability to hurt other players in the game, and the ability to accept deterrence is the ability to resist other players' deterrence. The discount factor is the coefficient of loss allocation share caused by the deterrence of the other players. The greater the deterrent ability, the stronger the damage to the other players, that is, the smaller the discount factors of other players are. The ability to resist other players' deterrence is stronger, that is, the greater the discount factor is. The magnitude of the discount factor can represent the relative magnitude of the mutual deterrent capability of two players, and each player has their deterrent ability to deter others and their ability to resist other players' deterrence, which is determined by the attributes of the players. Assuming that the deterrent ability of the player 1, 2 and 3 are represented by x_1, x_2 and x_3, respectively; the ability to resist other players' deterrence are represented by x_1', x_2' and x_3', respectively. In practical problems, the degree of deterrence of a player to the other player is not necessarily 100%, and the degree of deterrence is influenced by the environment, psychology and many other factors, so that we can use a deterrent coefficient on behalf of the degree for deterrence of player i against player j. Similarly, $\sigma_{i,jk}$ is the degree of deterrence for player i against the jk alliance, and $\sigma_{jk,i}$ is the degree of deterrence for the jk alliance against player i. Obviously, the discount factor is between 0 and 1.

Definition 1. In the mutual deterrence bargaining model, the discount factor of the player can be expressed as:

$$discount\,factor = \frac{\text{deterred capability}}{\text{deterred capability} + \text{deterrent capability} \times \text{deterrent coefficient}}$$

Thus, through the quantification calculation of Definition 1, the deterrence in conflict can be transformed into the harm ability to each other, and then use the game theory to establish the corresponding bargaining model. In this way, the discount factor of player i against the player j can be expressed as following:

$$\delta_{ij} = \frac{x'_j}{x'_j + \sigma_{ij} x_i} (i = 1, 2, 3; j = 1, 2, 3 \text{ and } i \neq j).$$

Further, the discount factor of player i against the players in the jk alliance is $\delta_{i,jk}$, the discount factor of the players in the jk alliance against player i is $\delta_{i,jk}$, the discount factors $\delta_{i,jk}$ and $\delta_{i,jk}$ are shown as following respectively:

$$\delta_{i,jk} = \frac{x'_j + x'_k}{x'_j + x'_k + \sigma_{i,jk} x_i}, \delta_{jk,i} = \frac{x'_i}{x'_i + \sigma_{jk,i}(x_j + x_k)}.$$

Obviously, $\delta_{i,jk}$ and $\delta_{jk,i}$ are values on the interval [0, 1], when $\delta_{i,jk} = \delta_{i,kj}$ and $\delta_{jk,i} = \delta_{kj,i}$. The discount factor of the alliance is less than that of the individual in the alliance, and the discount factor is affected by coefficient for degree of deterrence.

Theorem 1.2 ([7]). In the above complete information of the indefinite rotation bid tripartite bargaining model, if the player 2 and 3 choose an alliance, the resulting Nash equilibrium allocation are shown as following respectively:

$$v^*_{11} = \frac{1 - \delta_{1,23}}{1 - \delta_{1,23}\delta_{23,1}} \tag{1}$$

$$v^*_{12} = \frac{(1 - \delta_{23})(\delta_{1,23} - \delta_{1,23}\delta_{23,1})}{(1 - \delta_{23}\delta_{32})(1 - \delta_{1,23}\delta_{23,1})} \tag{2}$$

$$v^*_{13} = \frac{(\delta_{23} - \delta_{23}\delta_{32})(\delta_{1,23} - \delta_{1,23}\delta_{23,1})}{(1 - \delta_{23}\delta_{32})(1 - \delta_{1,23}\delta_{23,1})} \tag{3}$$

Inference 1.1. In the above tripartite bargaining model, if the player 1 and 3 build up an alliance, the resulting Nash equilibrium allocation are displayed as following respectively:

$$v^*_{21} = \frac{(1 - \delta_{13,2})(1 - \delta_{13})}{(1 - \delta_{2,13}\delta_{13,2})(1 - \delta_{13}\delta_{31})} \tag{4}$$

$$v^*_{22} = \frac{\delta_{13,2} - \delta_{13,2}\delta_{2,13}}{1 - \delta_{2,13}\delta_{13,2}} \tag{5}$$

$$v^*_{23} = \frac{(1 - \delta_{13,2})(\delta_{13} - \delta_{13}\delta_{31})}{(1 - \delta_{2,13}\delta_{13,2})(1 - \delta_{13}\delta_{31})} \tag{6}$$

Inference 1.2. In the above tripartite bargaining model, if the player 1 and 2 are in the alliance, the resulting Nash equilibrium allocation are listed as following respectively:

$$v_{31}^{*}=\frac{(1-\delta_{12,3})(1-\delta_{12})}{(1-\delta_{12,3}\delta_{3,12})(1-\delta_{12}\delta_{21})} \tag{7}$$

$$v_{32}^{*}=\frac{(\delta_{12}-\delta_{12}\delta_{21})(1-\delta_{12,3})}{(1-\delta_{12,3}\delta_{3,12})(1-\delta_{12}\delta_{21})} \tag{8}$$

$$v_{33}^{*}=\frac{\delta_{12,3}-\delta_{12,3}\delta_{3,12}}{1-\delta_{3,12}\delta_{12,3}} \tag{9}$$

3 Unilateral Incomplete Information Mutual Deterrence Bargaining Method

If the player 2 and 3 choose to form a alliance, they can be seen as a bargaining game between the player 1 and players 2 and 3 in 23 (or 32) alliance. Assuming that player 1 puts forward a program $(x,\ 1-x)$, the game side 23 alliance can choose to accept and refuse. If he accepts, which means the end of the game, the two sides benefit are x and $1-x$, respectively; if the game side 23 alliance refuses, then the player 1 will implement a retaliation, the two sides benefit are $\delta_{1,23}x$ and $\delta_{23,1}(1-x)$, respectively. Since then, the game side 23 alliance proposes a new program $(y,\ 1-y)$. If the player 1 accepts, the game is over; if the player 1 refuses, then the game side 23 alliance will implement a retaliation, then the player 1 puts forward the new distribution plan in the next step. And so on, until the Nash equilibrium is reached.

Assuming that the player 1 has unilateral incomplete information, the discount factors of the player 2 and 3 against the player 1 are δ_{12} and δ_{13}, the discount factors δ_{12} and δ_{13} are private information, but the discount factors of the player 2 and 3 against the player 1 (δ_{21} and δ_{31}) and the discount factors between the player 2 and player 3 (δ_{23} and δ_{32}) are public knowledge. According to Harsanyi convert ideas, and the player 1 has a high bargaining power of H type (prior probability is p) and a low bargaining power of L type (prior probability is $1-p$), other players can know the real type of player 1 from the degree of damage by retaliation. H type player 1 has a strong ability to hurt, and the discount factors against the player 2 and 3 are δ_{12}^{L} and δ_{13}^{L}, respectively. L type player 1 has a weak ability to hurt, and the discount factors against the player 2 and 3 are δ_{12}^{H} and δ_{13}^{H}, respectively. The damage factor of the player 2 is that the risk factor of the player 3 is. Because the deterrence of H type player 1 is greater than that of L type player 1, so we can get $\delta_{12}^{L}\leq\delta_{12}\leq\delta_{12}^{H}$ and $\delta_{13}^{L}\leq\delta_{13}\leq\delta_{13}^{H}$. δ_{12} and δ_{13} are on behalf of the discount factors of player 1 against the player 2 and 3, respectively. The coefficient σ_{ij} for degree of deterrence is constant $(i=1,2,3; j=1,2,3 \text{ and } i\neq j)$.

Assuming that both the player 1 proposes two programs such as $(\alpha_1, 1-\alpha_1)$ and $(\beta_1, 1-\beta_1)$, which represent the only perfect Nash equilibrium solution for the Rubinstein complete information bargaining of player 1 of L-type and H-type against the players in the 23 alliance, respectively. We can see that the discount factor of L-type player 1 against the players in the 23 alliance is $\delta_{1,23}^{H}$, the discount factor of

H-type player 1 against the players in the 23 alliance is $\delta^L_{1,23}(\delta^L_{1,23} \le \delta_{1,23} \le \delta^H_{1,23})$, the discount factor of the players in the 23 alliance against player 1 is unchanged $\delta_{23,1}$. According to the Rubinstein indefinite bargaining model [3], we can get $\alpha_1 = (1-\delta^H_{1,23})/\ (1-\delta_{23,1}\delta^H_{1,23})$ and $\beta_1 = (1-\delta^L_{1,23})/(1-\delta_{23,1}\delta^H_{1,23})$ by Theorem 1.1. Obviously, the program $(\beta_1, 1-\beta_1)$ is a favorite program of player 1. Thus, when the player 1 proposes a distribution plan $(\beta_1, 1-\beta_1)$, the benefit for the players in the 23 alliance of choosing to accept is $1-\beta_1$, the benefit for the players in the 23 alliance of choosing to refuse is $\delta^L_{1,23}(1-\beta_1)p + \delta^H_{1,23}(1-\alpha_1)(1-p)$. As a result of $p[\delta^L_{1,23}(1-\beta_1) - \delta^H_{1,23}(1-\alpha_1)] < 1-\beta_1 - \delta^H_{1,23}(1-\alpha_1)$, we can get $\delta^L_{1,23}(1-\beta_1)p + \delta^H_{1,23}(1-\alpha_1)(1-p) < 1-\beta_1$. the benefit for the players in the 23 alliance of choosing to accept is are bigger than that of the choice of rejection, so the players in the 23 alliance will accept the distribution plan proposed by the player 1. In this case, the game between the player 1 and the players in the 23 alliance can reach the Nash equilibrium, so the conflict does not occur. At this point, according to (1)–(3) can be calculated, the final Nash equilibrium allocation share of the player 1, 2 and 3 are shown as following respectively:

$$v'_{11} = \beta_1 = \frac{1-\delta^L_{1,23}}{1-\delta_{23,1}\delta^L_{1,23}} \tag{10}$$

$$v'_{12} = \frac{(1-\delta_{23})(\delta^L_{1,23} - \delta^L_{1,23}\delta_{23,1})}{(1-\delta_{23}\delta_{32})(\delta^L_{1,23}\delta_{23,1})} \tag{11}$$

$$v'_{13} = \frac{(\delta_{23} - \delta_{23}\delta_{32})(\delta^L_{1,23} - \delta^L_{1,23}\delta_{23,1})}{(1-\delta_{23}\delta_{32})(1-\delta^L_{1,23}\delta_{23,1})} \tag{12}$$

For $\delta^L_{1,23} \le \delta_{1,23} \le \delta^H_{1,23}$, the Nash equilibrium allocation share of the player 1, 2 and 3 can be obtained as $v'_{11} \ge v^*_{11}$, $v'_{12} \le v^*_{12}$, $v'_{13} \le v^*_{13}$(Equal sign is established only when the player 1, 2 and 3 have full information), respectively. Therefore, when player 2 and 3 choose the alliance, if the player 1 has unilateral incomplete information at this time, the Nash equilibrium allocation share obtained by the player 1 will be increased against that player 1 has the complete information. The Nash equilibrium allocation share of both player 2 and 3 will be reduced. Then the player 1 has the increased incomes because of the advantage of unilateral incomplete information. In this case there is a unique the Nash equilibrium solution for tripartite bargain, so the negotiation is successful and the conflict does not occur.

A similar analysis, When the player 1 has unilateral incomplete information, the player 1 has a high bargaining power of H type and a low bargaining power of L type, when player 1 and 3 choose the alliance, then the bargain of players in the 13 alliance and player 2 can reach Nash balanced, so the conflict does not happen. According to the Eqs. (4)–(6), the final Nash equilibrium allocation share of the player 1, 2 and 3 are shown as following respectively:

$$v_{21}^{*} = \frac{(1 - \delta_{13,2}^{L})(1 - \delta_{13}^{L})}{(1 - \delta_{2,13}\delta_{13,2}^{L})(1 - \delta_{13}^{L}\delta_{31})} \tag{13}$$

$$v_{22}^{*} = \frac{\delta_{13,2}^{L} - \delta_{13,2}^{L}\delta_{2,13}}{1 - \delta_{2,13}\delta_{13,2}^{L}} \tag{14}$$

$$v_{23}^{'} = \frac{(1 - \delta_{13,2}^{L})(\delta_{13}^{L} - \delta_{13}^{L}\delta_{31})}{(1 - \delta_{2,13}\delta_{13,2}^{L})(1 - \delta_{13}^{L}\delta_{31})} \tag{15}$$

When the player 1 has unilateral incomplete information, when 1 and 2 choose the alliance, we can use Heisani conversion, according to the Eqs. (7)–(9), the Nash equilibrium allocation share of the player 1, 2 and 3 are shown as following respectively:

$$v_{31}^{\prime} = \frac{(1 - \delta_{12,3}^{L})(1 - \delta_{12}^{L})}{(1 - \delta_{12,3}^{L}\delta_{3,12})(1 - \delta_{12}^{L}\delta_{21})} \tag{16}$$

$$v_{32}^{\prime} = \frac{(\delta_{12}^{L} - \delta_{12}^{L}\delta_{21})(1 - \delta_{12,3}^{L})}{(1 - \delta_{12,3}^{L}\delta_{3,12})(1 - \delta_{12}^{L}\delta_{21})} \tag{17}$$

$$v_{33}^{\prime} = \frac{\delta_{12,3}^{L} - \delta_{12,3}^{L}\delta_{3,12}}{1 - \delta_{3,12}\delta_{12,3}^{L}} \tag{18}$$

In the case of complete information, the preferred cooperation object of the players is related to the discount factor among the players. The players tend to cooperate with the other players with the relative enhancement of deterrent ability, and the weak will tend to cooperate with the stronger [7]. In this paper, when the player 1 has unilateral incomplete information, we can know that the preferred cooperation object among the players is related to the discount factor (δ_{12}^{L}, δ_{32}, δ_{21}, δ_{13}^{L}, δ_{31} and δ_{23}) from the results of (10)–(18), but it has nothing to do with δ_{12}^{H} and δ_{13}^{H}. Moreover, compared to the complete information, the Nash equilibrium allocation share of the player 1 is increased in any form of alliance, but the Nash equilibrium allocation share of the player 2 and 3 is reduced, so the player 1 has incomplete information advantage. Therefore, in the tripartite mutual deterrence bargaining model with unilateral incomplete information, due to information asymmetry, the player of unilateral incomplete information will plunder some interests of the player with unilateral complete information, so the player of unilateral incomplete information has information advantages.

Similarly, when the player 2 or the player 3 has unilateral incomplete information, since the status of the players has a symmetry, a similar analysis can be performed using the above method, we not repeat them here.

4 The Tripartite Mutual Deterrence Bargaining Method of Bilateral Incomplete Information

Assuming that both the player 1 and 2 have incomplete information, the player 1 has a high bargaining power of 1H type and a low bargaining power of 1L type, with the prior probability is p and $1-p$, respectively. The discount factor of the player 1 against the player 2 and 3 is private information, the discount factor of the 1H type player 1 against the player 2 and 3 is δ_{12}^{L} and δ_{13}^{L} ($\delta_{12}^{L} \le \delta_{12}$, $\delta_{13}^{L} \le \delta_{13}$), respectively; the discount factor of the 1L type player 1 against the player 2 and 3 is δ_{12}^{H} and δ_{13}^{H} ($\delta_{12}^{H} \ge \delta_{12}$, $\delta_{13}^{H} \ge \delta_{13}$), respectively. The player 1 has a high bargaining power of 2H type and a low bargaining power of 2L type, with the prior probability is q and $1-q$, respectively. The discount factor of the player 2 against the player 1 and 3 is private information, the discount factor of the 2H type player 2 against the player 1 and 3 is δ_{21}^{L} and δ_{23}^{L} ($\delta_{21}^{L} \le \delta_{21}$, $\delta_{23}^{L} \le \delta_{23}$), respectively; the discount factor of the 2L type player 2 against the player 1 and 3 is δ_{21}^{H} and δ_{23}^{H} ($\delta_{21}^{H} \ge \delta_{21}$, $\delta_{23}^{H} \ge \delta_{23}$), respectively. While the player 3 has complete information, and the discount factor of the player 3 against the player 1 and 2 is public knowledge, the discount factor of the player 3 against the player 1 and 2 is δ_{31} and δ_{32}, respectively.

Assuming that the player 2 and 3 build up an alliance, the game can be seen as mutual bargaining game of player 1 and players 2 in 23 alliance. The programs proposed by the player 1 are four kinds of $(\alpha, 1-\alpha)$, $(\beta, 1-\beta)$, $(\eta, 1-\eta)$ and $(\zeta, 1-\zeta)$, where $\alpha = (1-\delta_{23,1}^{L})/(1-\delta_{1,23}^{L}\delta_{23,1}^{L})$, $\beta = (1-\delta_{23,1}^{L})/(1-\delta_{1,23}^{H}\delta_{23,1}^{L})$, $\eta = (1-\delta_{23,1}^{H})/\ (1-\delta_{1,23}^{L}\delta_{23,1}^{H})$, $\zeta = (1-\delta_{23,1}^{H})/(1-\delta_{1,23}^{H}\delta_{23,1}^{H})$. If the player 1 proposes each of the four options described above, the posterior probability of 1H type is p_1^B, p_2^B, p_3^B and p_4^B, respectively.

4.1 Strategy Analysis of Player 2 Is 2H Type

(1) If the player 1 puts forward the program $(\beta, 1-\beta)$, the expectation benefit that the player 2 and player 3 of 2H type in alliance accept the program is $(1-\beta)$, and the expectation benefit is $\delta_{23}^{L}(1-\alpha)p_2^B + \delta_{23}^{H}(1-\eta)(1-p_2^B)$ when they reject it. We make $\delta_{23}^{L}(1-\alpha)p_{2H} + \delta_{23}^{H}(1-\eta)(1-p_{2H}) = (1-\beta)$, then we can get $p_{2H} = [\delta_{23}^{H}(1-\eta) - (1-\beta)]/[\delta_{23}^{H}(1-\eta) - \delta_{23}^{L}(1-\alpha)]$. So when $p_2^B \ge p_{2H}$, the player 2 and 3 of 2H type in alliance will accept the program $(\beta, 1-\beta)$, otherwise they will reject it.

(2) If the player 1 puts forward the program $(\alpha, 1-\alpha)$, the expectation benefit that the player 2 and player 3 of 2H type in alliance accept the program is $(1-\alpha)$, and the expectation benefit is $\delta_{23}^{L}(1-\alpha)p_1^B + \delta_{23}^{H}(1-\eta)(1-p_1^B)$ when they reject it. Because of $p_1^B[\delta_{23}^{L}(1-\alpha) - \delta_{23}^{H}(1-\eta)] < (1-\alpha) - \delta_{23}^{H}(1-\eta)$, we can know that $\delta_{23}^{L}(1-\alpha)p_1^B + \delta_{23}^{H}(1-\eta)(1-p_1^B) < (1-\alpha)$. So the expected benefit of rejecting the program is less than the expected benefit of accepting it for the player 2 of 2H type, and he will accept the program $(\alpha, 1-\alpha)$.

(3) If the player 1 proposes the program $(\eta, 1-\eta)$, it is the preferred program of the players in 23 alliance, so they would accept it.
(4) If the player 1 puts forward the program $(\zeta, 1-\zeta)$, the expectation benefit that the players of 2H type in 23 alliance accept the program is $(1-\zeta)$, and the expectation benefit is $\delta_{23}^{L}(1-\alpha)p_{4}^{B}+\delta_{23}^{H}(1-\eta)(1-p_{4}^{B})$ when they reject it. Because of $p_{4H}=[\delta_{23}^{H}(1-\eta)-(1-\zeta)]/[\delta_{23}^{H}(1-\eta)-\delta_{23}^{L}(1-\alpha)]$, so when $p_{4}^{B}\geq p_{4H}$, the players of 2H type in 23 alliance will accept the program $(\zeta, 1-\zeta)$, otherwise they will reject it.

4.2 Strategy Analysis of Player 2 Is 2L Type

(1) If the player 1 puts forward the program $(\beta, 1-\beta)$, the expectation benefit of the players of 2L type in 23 alliance is $(1-\beta)$ when they accept it, the expectation benefit of the players of 2L type in 23 alliance is $\delta_{23}^{L}(1-\beta)p_{2}^{B}+\delta_{23}^{H}(1-\zeta)(1-p_{2}^{B})$ when they reject it. Because of $p_{2}^{B}[\delta_{23}^{L}(1-\beta)-\delta_{23}^{H}(1-\zeta)]<(1-\beta)-\delta_{23}^{H}(1-\zeta)$, we can know that $\delta_{23}^{L}(1-\beta)p_{2}^{B}+\delta_{23}^{H}(1-\zeta)(1-p_{2}^{B})<(1-\beta)$. So the expected benefit of rejecting the program is less than the expected benefit of accepting it for the players of 2L type in 23 alliance, and they will accept the program $(\beta, 1-\beta)$.
(2) If the player 1 puts forward the program $(\alpha, 1-\alpha)$, the expectation benefit of the players of 2L type in 23 alliance is $(1-\alpha)$ when they accept it, the expectation benefit of the players of 2L type in 23 alliance is $\delta_{23}^{L}(1-\alpha)p_{1}^{B}+\delta_{23}^{H}(1-\zeta)(1-p_{1}^{B})$ when they reject it. Because of $p_{1}^{B}[\delta_{23}^{L}(1-\beta)-\delta_{23}^{H}(1-\zeta)]<(1-\beta)-\delta_{23}^{H}(1-\zeta)$, we can obtain the result of $\delta_{23}^{L}(1-\alpha)p_{1}^{B}+\delta_{23}^{H}(1-\zeta)(1-p_{1}^{B})<(1-\alpha)$. So the expected benefit of rejecting the program is less than the expected benefit of accepting it for the players of 2L type in 23 alliance, and they will accept the program $(\alpha, 1-\alpha)$.
(3) If the player 1 proposes the program $(\eta, 1-\eta)$, it is the preferred program of the players in 23 alliance, so they would accept it.
(4) If the player 1 puts forward the program $(\zeta, 1-\zeta)$, the expectation benefit that the players of 2L type in 23 alliance accept the program is $(1-\zeta)$, and the expectation benefit is $\delta_{23}^{L}(1-\beta)p_{4}^{B}+\delta_{23}^{H}(1-\zeta)(1-p_{4}^{B})$ when they reject it. Because of $p_{4}^{B}[\delta_{23}^{L}(1-\beta)-\delta_{23}^{H}(1-\zeta)]<[(1-\beta)-\delta_{23}^{H}(1-\zeta)]$, we can get the result of $\delta_{23}^{L}(1-\beta)p_{4}^{B}+\delta_{23}^{H}(1-\zeta)(1-p_{4}^{B})<(1-\zeta)$. At this point, the expected benefit of rejecting the program is less than the expected benefit of accepting it for the players of 2L type in 23 alliance, and they will accept the program.

When the player 1 is 1H and 1L type, by a similar analysis of the above method, we can see: (1) for the players 1 of 1H type, he proposes the program $(\beta, 1-\beta)$ when $p_{2}^{B}\geq p_{2H}$, two types of the players of 23 alliance agree with it; when $p_{2}^{B}<p_{2H}$ and $q\geq q_{H}$, he would propose a program $(\alpha, 1-\alpha)$, two types of the players of 23 alliance also agree with it; when $p_{2}^{B}<p_{2H}$ and $q<q_{H}$, he proposes a program $(\beta, 1-\beta)$, the players of 2L Type in 23 alliance agree with it, while the players of 2H type in 23 alliance reject it, the conflict may occur. (2) for the players 1 of 1L type, when

$p_2^B \geq p_{2H}$, he would propose a program $(\beta, 1-\beta)$, both types of the players of 23 alliance accept it; when $p_2^B < p_{2H}$ and $q \geq q_H$, he would propose a program $(\alpha, 1-\alpha)$, both types of the players of 23 alliance accept it; when $p_2^B < p_{2H}$ and $q \leq q_{\mathrm{L}}$, he would propose a program $(\beta, 1-\beta)$, the players of 2L Type in 23 alliance agree with it, while the players of 2H type in 23 alliance reject it, the conflict may occur; when $p_2^B < p_{2H}$ and $q_L < q < q_H$, if the player 1 put forward the program $(\eta, 1-\eta)$ or $(\zeta, 1-\zeta)$, and there is no conflict; but if he proposes a program $(\beta, 1-\beta)$, there may be a conflict.

5 Numerical Calculation Case

The retailer (buyer) 1 and the supplier 2, the supplier 3 constitute the supply chain, in which the retailer 1, the supplier 2 and 3 allocate the total profit through the bargaining game. In the cooperating process of buyer 1, supplier 2 and supplier 3, how to formulate a scientific distribution plan to ensure that the residual income of the cooperation can be reasonably allocated among the three players will directly determine the stability and reliability of the cooperation. In other words, the rationality of the distribution of income is the protection for the deep cooperation of the three parties.

The multilateral negotiations of single buyer multi-vendor on supply chain profit distribution, where buyers and suppliers may have different bargaining power, then the negotiation process for retailers 1, suppliers 2 and suppliers 3 is the rotation bargaining. This case mainly considers the bargaining model under the high logistics capabilities, the three players pursue the long-term strategic cooperation. For the higher the cost of replacing the cooperation object, the three players will bargain the co-operative surplus when signing the contract to obtain greater profits, which can guarantee long-term stable and win-win cooperation relations.

Table 1 gives the discount factor of the retailer 1, supplier 2 and supplier 3 in the tripartite mutual deterrence bargaining model under the complete information, and also gives the discount factor of retailer 1 with high bargaining power and low bargaining power under the unilateral incomplete information. Then, according to the discount factor in Table 1, by the Eqs. (10)–(18), we can calculate the Nash equilibrium allocation share of all players in the different cooperation schemes under complete information and the retailer 1 with unilateral incomplete information, then make comparative analysis, as shown in Table 2.

Table 1. The discount factor

	$\delta_{12}/\delta_{12}^L$	δ_{32}	δ_{21}	$\delta_{13}/\delta_{13}^L$	δ_{31}	δ_{23}
Complete information	0.8	0.88	0.75	0.3	0.9	0.35
Unilateral incomplete information	0.5	0.88	0.75	0.2	0.9	0.35

From Table 2, in the complete information of tripartite deterrence bargaining, the supplier 2 will choose to cooperate with the supplier 3. But if the retailer 1 has unilateral incomplete information, the retailer 1 will choose to cooperate with the supplier 3. The interest of the retailer 1 is increased while the interest of the supplier 2 and the supplier 3 is declined as can be seen from Table 2, which also reflects the advantages of the player with incomplete information. When both retailer 1 and supplier 2 have incomplete information, Nash equilibrium can be achieved in some cases, and the Nash equilibrium allocation share is increased. But in other cases, the bargaining can not reach the Nash equilibrium solution, the conflict may occur. According to the above analysis, whether the conflict depends on the specific value of the parameters, such as p, q and p_1^B and so on.

Table 2. Comparison of Nash equilibrium assignment schemes

	2 and 3 alliance	1 and 3 alliance	1 and 2 alliance
Complete information	(0.4233, 0.5417, 0.0350)	(0.6107, 0.3632, 0.0262)	(0.4932, 0.4932, 0.0136)
Unilateral incomplete information	(0.7222, 0.2609, 0.0169)	(0.8172, 0.1624, 0.0204)	(0.7922, 0.0198, 0.0097)

6 Conclusion

In this paper, the bargaining model has been established for the tripartite deterrence bargaining problem of incomplete information, and the factors such as deterrence ability, accepted deterrent ability and deterrent degree are introduced into the discount factor. The expression of the discount factor and influence of the discount factor on the choice of alliance relations in the three bargaining model were given. On the basis of the tripartite mutual deterrence bargaining for complete information, and the Nash equilibrium allocation share model of three players in the unilateral and bilateral incomplete information has been established for the condition of unilateral and bilateral incomplete information, which makes the model more general. The only Nash equilibrium solution of Rubinstein's classic bargaining has been used to analyze the credibility of deterrence and the possibility of conflict. Finally, we compared the more general model of this paper with the tripartite mutual deterrence bargaining model under the original complete information and found that the three players can tend to choose the alliance way of their own maximizing interests and reach an agreed distribution plan when a game side has unilateral incomplete information, so the conflict does not appear. The share of the interests of the game player with incomplete information is higher than that of the game player with the full information in any kind of alliance relationship, and the game player with incomplete information has the advantage of unilateral incomplete information. However, when there are two players of incomplete information in the tripartite mutual deterrence bargaining model, each player can not propose a mutually acceptable solution in some cases, and the conflict may occur.

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Stakeholders' Behavior Analysis and Enterprise Management Strategy Selection in Chinese Ancient Village Tourism Development

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Abstract. Chinese ancient villages are an important type of non-renewable tourism resources. How to exploit and protect ancient villages in the tourism development becomes an urgent problem to be solved. The aim of this paper is to propose enterprise management strategies of Chinese ancient village tourism. In this paper, firstly we identify the stakeholders (i.e., players) who have interest relationships and play important roles in Chinese ancient village tourism development and protection. Secondly, we systematically analyze the stakeholders' relations, interaction, and importance in the exploitation and protection of Chinese ancient village tourism. Thirdly, we elaborately investigate stakeholders' behaviors and hereby propose enterprise management strategies of Chinese ancient village tourism. Finally, our conclusions obtained in this paper are validated and illustrated with Xidi, an ancient village located in Anhui province of China, which was listed in the world heritage list by UNESCO in 2000.

Keywords: Management strategy · Behavior analysis · Ancient village tourism development and protection · Stakeholder · Game theory

1 Introduction

In March 3–13, 2017, Feng Jicai, who is the CPPCC National Committee member, Vice president of China Federation of Literary and Art Circles (CFLAC), and China ancient village protection expert, made a proposal for the protection of Chinese ancient villages at the Fifth Session meeting of the 12th CPPCC National Committee (Annual Democratic National Convention). He suggested that the traditional villages (i.e., Chinese ancient villages) should not be allowed to develop tourism without permit. Currently, many traditional village heritages have not been well sorted out, the villagers have not have a stronger protection sense, and protection measures have not been popularized yet. Consequently, once the tourism projects are firstly launched in these villages, then not only the control power of the projects would fall into the investors, the national protection policies and measures would not be fully implemented, but also these traditional villages will be seriously damaged. The long-term modeled tourism

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development resulted in the traditional villages to be assimilated, homogenized, and commercialized. The intrinsical cultures of these traditional villages have been dismembered, alienated, and distorted [1].

There is other news. Anhui Daily published an article entitled "*Tourism encountered Kan whose villagers complain more*" at the Country page in November 1, 2016. The article reported the story as follows. "Kan" means not only Chengkan village but also that Chengkan village tourism encounters a difficult obstacle to be overcome. Chengkan is an ancient village of the national 5A level scenic spot with a history of one thousand eight hundred years. The ancient buildings and street spaces in Chengkan village are basically in good condition. Earlier than March of 2001, a local person signed a 25-year contract of tourism operation and management with the local government and set up a private travel company, which has invested millions to build roads and renovate buildings. It may be speculated that the private company received enormous grants from the government and profits. However, there is a serious contradiction between the villagers and the private travel company. The company is getting richer. But, the villagers are given a paltry bonus, which is gradually increased from 2.5 Yuan RMB per person to 10, 20, and 50 Yuan RMB. Until recently, the villagers are given a bonus of 150 Yuan RMB per person annually. And then, since March of 2016, the villagers whose relatives and friends come to visit them and/or the outsiders without tour spend in their own houses have to pay the private travel company 30 Yuan RMB for the admission ticket, where the whole admission ticket is 107 Yuan RMB. Correspondingly, in order to get revenge on the travel company, the villagers took some excessive measures to destroy the local tourism industry. For example, they piled livestock manure up and scattered rubbish around to make the village stink everywhere. Even more, the villagers closed their doors to forbidden visitors to visit the houses and disputed with security guards and tourists. For more than 10 years, the villagers' anger accumulated to the extent to burst out. They gathered to request the local government to show the travel contract. Consequently, the villagers found the local government and the travel company forged their signature and signed the contract without their approval [2, 3].

In China, there are series of news about the serious conflicts and disputes between the villagers and tourism companies caused by ancient village tourism development just like the Chengkan. The local governments and tourism companies ignored the village collective interests [4], only when the contradiction is very sharp could they give the villagers a little bonus. This phenomenon has aroused widespread concern in academic [5]. It has extensively studied by many scholars from different angles and disciplines. However, for the problem how to manage tourism under the premise of protecting ancient villages, there has not convincing theoretical and qualitative research results to guide Chinese ancient village development and protection in practice [6, 7]. The aim of this paper is to use the qualitative and quantitative methods to identify stakeholders and study their relations and behavior and hereby propose the innovation theory of income distribution of the stakeholders in Chinese ancient village tourism development, which may further enrich and develop the theoretical point published in 2012 [8].

2 Connotation of the Ancient Village Protection

The ancient village is a term used in this paper to uniformly represent similar various terms or names appearing in different media for different periods [5]. The ancient village protection is an important part of the world heritage protection.

2.1 Interpretations from the International Cultural Heritage Protection Documents

Influenced by industrialization, urbanization, and modernism architecture trends in the early twentieth century, the forms and spaces of human habitation (including urban and countryside) have mutated, and the traditional cultural heritage has suffered a devastating impact. The international community has realized this serious problem. International experts, scholars, and researchers have established the relevant international institutions and organizations to commence the protection movement and draw up a series of international cultural heritage documents to define the system of heritage protection and make protective advice [2].

In 1931, the first International Congress of Historical Monument Architects and Technician Association (the predecessor of the International Council of Monuments and Sites, ICOMOS) was held in Athens. This conference established an international organization for international protection and consultation of rehabilitation, and firstly reached an international consensus about the system of historical site protection and rehabilitation, which is the document "*Athens Charter for the Restoration of Historic Monuments*" (1931). After that, three major international institutions on heritage protection were successively established. Specifically, they are the United Nations Educational Scientific and Cultural Organization (UNESCO), ICOMOS, and International Centre for Conservation and Rehabilitation of Cultural Relics (ICCROM). UNESCO is the special institution of UN for protecting cultural heritage systematically. ICOMOS is the only international non-governmental organization in the field of protection and restoration of monuments. ICCROM is the intergovernmental international cooperation organization for global cultural heritage protection. These three international organizations are the authorities of international cultural heritage protection and successively promulgated corresponding international conventions and charters, which are called *International Cultural Heritage Protection Document* [2]. The theoretical basis of this paper is the authoritative definitions recognized by international protection organizations and experts from existing international conventions and charters.

There are 16 international cultural heritage protection documents cited by this article [2].

(1) The *Athens Charter for the Restoration of Historic monuments*, adopted at the first International Congress of Historical Monument Architects and Technician Association, held in Athens, 1931.

(2) *The Recommendation Concerning the Safeguarding of the Beauty and Character of Landscapes and Sites*, approved at the 12th General Conference of the UNESCO, held in Paris, 1962.

(3) *The International Charter for the Conservation and Restoration of Monuments and Sites (The Venice Charter),* approved at the 2nd International Congress of Historical Monument Architects and Technicians, held in Venice, 1964.
(4) *The Recommendation Concerning the Preservation of the Cultural Property Endangered by the Public or Private Engineering*, approved at the 15th General Conference of the UNESCO, held in Paris, 1968.
(5) *The Recommendation Concerning the Protection of the Cultural and Natural Heritage at the National Level,* approved at the 17th General Conference of the UNESCO, held in Paris, 1972.
(6) *The Recommendation Concerning the Conservation and Contemporary role of Historic Areas (The Nairobi Recommendation),* approved at the 19th General Conference of the UNESCO, held in Nairobi, 1976.
(7) *The Constitution of the ICOMOS,* adopted at the 5th General Conference of the ICOMOS, held in Moscow, 1978.
(8) *The Nara Document on Authenticity,* adopted at the International Conference jointly organized by the UNESCO, ICOMOS, ICCROM, and Japanese Government Cultural Organization Department, held in Nara, 1994.
(9) *The Burra Charter,* adopted *at the* Australian National Council of the ICOMOS, held in Australia, 1999.
(10) *The Charter on the Built Vernacular Heritage,* ratified by the 12th General Conference of the ICOMOS, held in Mexico, 1999.
(11) *The International Cultural Tourism Charter (Principles and Guidelines of Management for Important Cultural Relics and Sites),* ratified by the 12th General Conference of the ICOMOS, held in Mexico, 1999.
(12) *The Principles for the Conservation of Chinese Heritage Sites,* ratified by *China National Council of the ICO*MOS, held in Chengde, 2000.
(13) *The Declaration on Global Cultural Diversity*, adopted by the 31st General Conference of the UNESCO, held in Paris, 2001.
(14) *The Convention for the Safeguarding of the Intangible Cultural Heritage,* adopted by the 32nd General Conference of the UNESCO, held in Paris, 2003.
(15) *The Principles for the Analysis, Conservation and Structural Restoration of Architectural Heritage,* ratified by the 14th General Conference of ICOMOS, held in Victoria Falls, 2003.
(16) *The Hoi An Protocols–Best Conservation Examples in Asia,* ratified by the Seminar of the UNESCO, held in Hoi An of Vietnam, 2005.

The above documents have systematically and detailed described the systems of knowledge, management, and inheritance for the ancient village protection. This paper will focus on the ancient village tourism. In the following sections, we discuss relevant concepts and relations about ancient village heritage landscapes, local people, local governments, protection experts, and tourism management.

2.2 Significance of Ancient Village Heritage Landscape Protection

Ancient villages are rural landscapes of typical natural environments which have site landscapes and characteristics as well as historical, artificial, cultural, and artistic values, as stated in the *Recommendation Concerning the Safeguarding of the Beauty and Character of Landscapes and Sites* (1962) [2].

The value of ancient villages lies not only in the ancient residential buildings, but also in the rural environments where they can find a unique civilization, a meaningful development or a historical event witness, as mentioned in the *International Charter for the Conservation and Restoration of Monuments and Sites (The Venice Charter)* (1964) [2].

The ancient villages being as historical regions also belong to traditional human settlements. From point of view of anthropology, history, and sociology, they are an irreplaceable part of the world heritage and play a role in its diversification. The deterioration and change of historical region environments and architectural styles may result in danger of the simplification of the world environment, as pointed out by the *Recommendation Concerning the Conservation and Contemporary Role of Historic Areas (The Nairobi Recommendation)* (1976) [2].

The architectural complex of ancient villages is all buildings and environments connected by the countryside. They gather from each other because of their architectural styles, types, and regional characteristics with the values of history, art, science, aesthetics and so on. These are described in *the Constitution of the ICOMOS* (1978) [2].

The aboriginal sites such as ancient villages, which have cultural importance, establish a deep and inspired connection between society and landscapes, and between past and real experience. This is stated as in the *Burra Charter* (1999) [2].

The local architectural heritage of ancient villages is a social product, which has usage value and inflexible form. It is the primary point of focus in that era life and the basic manifestation of society and cultures. It is a traditional and natural way for aboriginal people to build their own houses to adapt the continuous process under the social and natural constraints, changes and developments. It is identifiable, environmental suitable, and regional. It is the basic manifestation of social culture and the basic expression of the relationship between the community and its region. These are stated as in the *Charter on the Built Vernacular Heritage* (1999) [2].

2.3 Dominant Position of Aboriginal People in Ancient Villages

The local (or aboriginal) people are the main part of the heritage landscapes.

Ancient villages, especially the sites with scenic features, should be owned by all native communities. The owner of the sites shall not sign agreements with others without permission according to *the Recommendation Concerning the Safeguarding of the Beauty and Character of Landscapes and Sites* (1962) [2].

Historical areas (including ancient villages) and their environments are irreplaceable parts of the world heritage. National governments and citizens have the obligation to integrate them into modern life. Every historical region and its surrounding environment should be regarded as a whole with each other, and their coordination depends on the characteristics of its components. These components include human activities (aboriginal activities), buildings, space structures and the surrounding environments. Therefore, all effective components, no matter how insignificant, are of great significance to the whole, as mentioned in *the Recommendation Concerning the Conservation and Contemporary Role of Historic Areas (The Nairobi Recommendation)* (1976) [2].

The *Nara Document on Authenticity (1994)* made the following statement [2]. All cultures and societies are special forms and methods rooted in tangible and intangible means, which constitute their heritages and should be respected. The responsibility and management of cultural heritages should first be attributed to the cultural communities.

The vernacular architecture heritages (i.e., ancient village heritage landscapes) occupy an important place in human emotion and embody the traditional harmony that form the core of human life. The correct evaluation and successful protection of local architectural heritages depend on the participation and support of the community (all indigenous peoples), and the continuous use and maintenance. The government and the competent authorities should ensure that all communities (villages) have got their traditional living rights through all available legal, administrative, and economic means to protect the traditional aboriginal life and pass it to the offspring. These are well discussed in *the Charter on the Built Vernacular Heritage* (1999) [2].

All activities to protect cultural relics and historical sites (all activities of ancient village aborigines) are aimed at preserving and extending historical information and full values. This viewpoint is gradually acknowledged by t*he Principles for the Conservation of Chinese Heritage Sites* (2000) [2].

The Declaration on Global Cultural Diversity (2001) believes that protecting the diversity of cultural and ethnic groups in the world is one of the best safeguards of international peace and security. Policies that advocate participation of all citizens are a solid guarantee for enhancing social cohesion and the vitality of civil society and maintaining peace [2].

Intangible cultural heritages and tangible cultural heritages depend on each other. Intangible cultural heritages refer to the various practices, performances, and forms of culture heritages which are considered by various groups or individuals, or related tools, objects, handicrafts, and cultural sites of the cultural heritages. The intangible cultural heritages which are handed down from generation to generation can give them a sense of identity and history, as stated in *the Principles for the Analysis, Conservation and Structural Restoration of Architectural Heritage* (2003). Obviously, the aboriginal people in the cultural heritage sites are the inheritors of the intangible cultural heritages. They are an important part of the cultural heritages [2].

Heritage sites such as ancient villages should continue to be managed by traditional managers, who should be empowered and assisted to achieve the protection of authenticity [2].

The responsibility for cultural heritages and management should be shouldered first by the cultural community (the ancient village), and then by the local government. This is discussed by *the HoiAn Protocols–Best Conservation Examples in Asia* (2005) [2].

2.4 Duties of Local Governments in Ancient Villages

All international cultural heritage protection documents require governments to legislative protection. Without governments' guidance and supervision, heritage protection cannot be achieved.

The conference of the ICOROM firstly pointed out that all countries should solve the problem of heritage protection through national legislation. Archaeological sites will be strictly protected by surveillance according to the *Athens Charter for the Restoration of Historic monuments* (1931).

*The Recommendation Concerning the Safeguarding of the Beauty and Character of Landscapes and Sites (*1962*)* stipulates that the member states, by national laws or otherwise, shall formulate measures to affect the standards and principles embodied in this recommendation in the territory under their jurisdiction.

In *the Recommendation Concerning the Preservation of the Cultural Property Endangered by the Public or Private Engineering*, the definition of the cultural property is very broad, including all types of heritages (confirmed and unconfirmed). It recommends that the measures to protect the cultural property should be used extensively in all the territory of a member state rather than confined to certain monuments or sites. The protection and rescue measures required by the member states include legislation, finance, administrative measures, procedures for the protection and rescue of cultural property, penalties, repairs, awards, and educational programs. These legislations require that member states shall formulate or maintain national and local legislative measures necessary for the protection or rescue of the cultural property threatened by public or private projects, according to the aforementioned recommendations and principles. And administrative measures include the following four aspects: (1) Setting up the national coordination organizations composed by the representatives, who are responsible for protecting the cultural property and public and private projects and in charge of urban and rural planning as well as researching and educational institutions to coordinate conflicts of interest and make recommendations; (2) Setting up local government coordination agencies; (3) Providing enough experts and technicians from various disciplines in the administrative organizations for protecting the cultural property; (4) Taking administrative measures.

In order to protect the local architectural heritages (the ancient village landscape heritages), the local government and the competent authorities should ensure that all communities (all aboriginals) can maintain traditional life rights through all available legal, administrative, and economic means to protect the traditional aboriginal life and pass the offspring, as stated in *the Charter on the Built Vernacular Heritage* (1999).

2.5 Authoritative Supervisory Roles of Protection Experts

All heritage protection campaigns are initiated by protection experts. In fact, all international cultural heritage protection documents are drafted and approved by these experts. Without the protection of experts through their guidance, the so-called protective measures may not work, even may be destructive. Experts who have certain professional knowledge play an indispensable role in the process of heritage protection and restoration and management.

The International Charter for the Conservation and Restoration of Monuments and Sites (The Venice Charter) (1964) is a systematic charter devoted to the restoration of monuments. It points out that the purpose of protection and restoration is to regard historical sites as not only historical testimony but also artistic works. Also it believes that protection and restoration must resort to all the science and technology, which are beneficial to the study and protection of heritage sites. The restoration is a highly professional work. Its purpose is to preserve and display the aesthetic and historical value of the monuments and to respect the original materials and authentic documents. If there is speculation, it should immediately stop.

The Recommendation Concerning the Protection of the Cultural and Natural Heritage at the National Level (1972) is the recommendation about the member states' heritage protection and repair technology management. Heritage protection is an extremely complex and professional job, which requires a wide interdisciplinary collaboration. To ensure that cultural heritage can be effectively preserved and displayed, all countries should formulate coordination policies in accordance with their judicial and legislative documents. All science, technology, culture, and other resources can be used by heritage protection.

In the inevitability of changing and development of traditional architectures, the cultural characteristics established by local communities should be preserved. In this case, protection and restoration must be carried out by multi-disciplinary expertise according to *the Charter on the Built Vernacular Heritage* (1999).

Due to the large number of architectural heritages in the entire heritage types and its professionalism, *the Principles for the Analysis, Conservation and Structural Restoration of Architectural Heritage* is the conservation and rehabilitation principle to specifically guide the work of all architectural heritage protection specialists.

2.6 International Principles of Heritage Tourism Management

The diversity and living cultures of natural and cultural heritages (including ancient villages) are the main tourism attractions. Excessive or poorly managed tourism development may threaten the tangible essence, authenticity, and important characteristics of the heritages.

Tourism should bring economic benefits to the village community and provide an important way and impetus for community residents to pay attention to protecting the cultural heritages and lifestyle created by their ancestors. According to the *International Cultural Tourism Charter (Principles and Guidelines of Management for Important Cultural Relics and Sites)*, we briefly summarize some important international principles of heritage tourism management as follows.

(1) Dynamic management. The exchange between the resources and values of cultural heritages and tourism is dynamic and fast changing. Tourism activities and development should achieve positive results and minimize the adverse effects on the heritage and aboriginal lifestyle.
(2) Valuable experiences to visitors. Heritage protection and tourism programs should provide visitors with high-quality information to ensure most visitors aware important features of the heritage and their protection, and allow visitors to enjoy their heritage tours in the right way.
(3) Aboriginal community participation. Communities and indigenous peoples should participate in heritage protection and tourism planning. Heritage tourism management should respect local rights and interests of tourism areas and the owners of the sites and the indigenous people with the rights and obligations of the lands and important sites. In tourism situations, they should participate in setting goals,

policies, strategies and treaties for heritage resources, cultural activities, and contemporary cultural expression.

(4) Benefits for the community and the aborigines. Tourism and protection activities should benefit the community and aborigine people. Policy makers should promote equitable distribution of tourism profits in the countries or regions, improve the level of socio-economic development, and devote to poverty reduction. Through education, training, and the creation of full-time employment opportunities, heritage protection management and tourism activities should provide a fair economic, social, and cultural benefits to indigenous people at all levels of the community. A large portion of the tourism revenues should be used for heritage protection, renovation, and display. Travel plans should encourage training and hiring community aborigines as guides and heritage interpreters to enhance the ability and skills of local aborigines to exhibit and interpret cultural heritage values so that the aborigines have a direct interest in the heritage preservation. The management of heritage protection and tourism programs should provide policy makers, planners, architects, researchers, designers, interpreters, maintainers, and tour operators with educational and employment opportunities.

3 General Situations of Chinese Ancient Village Protection

China is an agricultural country with long history and traditional cultures. It has a huge population and many nationalities. In modern times, it began to develop industries in several important cities. In addition to the Second World War and the civil war, the human heritages were not destroyed by extinction. Although there were some heritages which have been viciously destroyed in 1960s, the level of industrialization was not high, the economy was backward, and Chinese urbanization rate was very slow. Since late 1970s, the reform and opening policy was implemented, China has attached great importance to international exchanges and cooperation in various fields. The international advanced concept is affecting the domestic all walks of life. Industrialization and urbanization are accelerating. The real estate industry has been designated as a national pillar industry, demolition and construction has gradually become the trend. Lots of ancient towns, traditional architectures, and ancient villages quickly disappear. At the same time, China is a country advocating politics and a top-down management model. In 1950s, the concept of heritage protection was highly recommended in the world. Under the influence of the international trend and the appeal of the domestic scholars, Chinese government began to realize the need for protection in the early 1980s. But it acts in the late 1980. The protection of ancient villages is later, basically after 2000.

In 2000, due to the fact that Xidi and Hongcun were listed as the world heritage list, the Ministry of Urban and Rural Construction and the State Bureau of Cultural relics jointly launched the project of "*China Historical Famous Towns and Villages*". Since 2003, 276 villages have been listed as China H*istorical Famous Towns and Villages*. In 2008, the regulation on *China Historical Famous Towns and Villages* was published, but there has been no protection of technical specifications for planning. Because the protection action cannot keep up, some heritage landscapes in the list have been badly damaged and almost lost the significance of protection. Some disordered development of tourism projects become vulgar, and lost the authenticity of cultural heritages.

As mentioned in Introduction, Due to the personal academic prestige of Feng Jicai and his participating heritage protection activities, the Ministry of Housing and Urban-Rural Development, the State Administration of Cultural Heritage, the Ministry of Culture, and the Ministry of Finance jointly launched the project of protecting Chinese traditional villages. Since 2012, 2555 villages have been approved by Chinese government. However, the evaluation criteria is almost the same as that of *China Historical Famous Towns and Villages*. But, the quantity of the latter is far more than the former. Some ancient villages are not only historical and cultural villages but also traditional villages.

The Chinese Traditional Village Protection and Development Research Center, an academic research institution affiliated to Tianjin University, aims at exploring, collecting and sorting out some pictures and intangible cultural heritages of the traditional villages. It does not have the technical guidance ability such as protection, maintenance, and repair. The laws, relevant management regulation and technical manual about the protection of traditional villages have not come out yet. The term ancient village mentioned in this article includes *historical famous towns and villages* and *traditional village*.

4 Chinese Ancient Villages and Tourism Development

Most Chinese ancient villages are located in the poor areas where the traffic is not convenient. They almost have a long history and backward economic development, basically dominated by human and livestock farming. The ancient traditional cultures have not been completely impacted by modern civilization. Globalization and information technology have made people to be interesting in cultural heritage tour, which has a distinctive ethnic identity. For the ancient villages, tourism is the most convenient way and the fastest way to get rich and develop the local economy [8, 9]. In the sequent, we will discuss the status and rights of indigenous in Xidi, which was listed in the world heritage list by UNESCO in 2000. The ancient village is located in the eastern part of Huangshan city in Anhui Province of China. It is a typical tourism pattern in china.

(1) The basic situation of Xidi

Xidi was built in the Northern Song Dynasty (AD 1049–1054). It has 960 years of history. There are 16.4 ha, 370 households and 1060 people in 2008.

(2) The local governments of Xidi

The local governments at all levels have promulgated a number of relevant regulations for the ancient village protection. In 1997, Anhui provincial government promulgated the regulations for the protection of ancient dwellings in southern Anhui. The Yixian county government has promulgated the *Ancient Folk Residence Protection Regulation* in 1998 and *the Protection Regulation of World Cultural Heritage* in 2001.

The village government has promulgated the regulation of Xidi scenic spot management and the Xidi village rules [3].

Under the situation that the village government strives to the autonomous development right for the ancient village community, the county government and relevant departments of tourism management cannot participate in Xidi tourism development.

(3) The protection experts of Xidi

The protection experts are only involved in the designing of village buildings and environmental protection plans, which have been well implemented under the supervision of the local government. However, they do not participate in tourism development and management.

(4) The tourism development of Xidi

In October 1986, the county government took the typical ancient buildings as tourist attractions and the initial tourism development was carried out from it. In September 1994, approved by the county government, the village government and local people jointly established *Xidi Travel Service Company*. The main body of the company is the village government and all the local people. Tickets are the company's major profits. Income distribution is that the company management costs 1.5% and the world heritage protection costs 20%. The rest of the income is allocated to the village government and all the local people in proportion 1:1. And then, all local people redistribute their income. 20% of the income is reserved for the community welfare, and the remaining 80% of the income is allocated based on population and housing area among the local people.

(5) The tourism benefits of Xidi' local people

There are two sources of the tourism income for the local people. One is to obtain a certain percentage of the tickets. The other is to participate in tourism related activities such as catering, selling goods, and operating entertainment facilities, or use their houses to participate in tourism activities to obtain incomes. From the proportion of the ticket income, in addition to the reserve funds and the necessary management cost, the income of the village government is higher than that of all local peoples.

Before the tourism development, Xidi was a poor ancient village. Whereas, after the tourism development, the income of the local peoples has increased a lot, and their living conditions have been improved.

(6) Validity evaluation of Xidi' authenticity protection

As a tangible cultural heritage, Xidi' basic architecture and spatial form are integral. But, its intangible cultural heritage is seriously distorted. The majority of the local people are engaged in tourism commodity management activities to sell local specialties to tourists. The ancient houses selling local products are covered with their own products. Few people engaged in farming. And as a result of the construction of new areas, many local people moved to the new areas so that the ancient village has lost its residential functions. Traditional folk customs almost disappeared.

Although there are some performances and folk customs, which are regularly shown to tourists, the authenticity of the ancient local life scenes is almost replaced by tourist activities.

(7) Cultural heritage research, exhibition and inheritance

There is no protection expert for collection, excavation and archaeological research. There are no local heritage museums. We do not find there is any scientific and professional guide commentary. Naturally, there is no special guide commentary for the local people and there are no training institutions or schools for the heritage inheritance.

5 Tourism Resources and Tourism Enterprise Positioning

5.1 Localization of Tourism Resources

Tourism resources are the basis of tourism development. Usually, tourism resources may be firstly discovered by tourists. Or the local government may create a unique local image to attract visitors. Tourist behaviors are driven by tourists. For tourists, the attractive environments are the tourism resources. Therefore, tourism resources are attractive environments in which a set of elements are specifically associated with each other. Ancient villages have been recognized by the world as very attractive and distinctive tourist resources. Their distinctive features are reflected in the historical continuity, nationality and uniqueness of local social life. Ancient villages are behavior landscapes and social landscapes. If there are not local people in the ancient villages, the ancient villages may become static site landscapes. The distinctive characters of tourist destinations are determined not only by the obvious attractions but also by the unique characteristics that cannot be touched but make the destination different. Undoubtedly, the local residents are the most important tourism resources. The emigration of local residents will change the nature tourism resources. The living security of the local people is the only choice for the sustainable development of the tourism economy.

5.2 Tourism Enterprise Positioning

Stated as earlier, we are concern on what kind of organizational structures the ancient village tourism enterprises should be in China.

In the above discussion, ancient villages managed by non-professional village governments and local people jointly (e.g., Xidi), or just by an outside professional travel company (e.g., Hongcun, which is other ancient village in Anhui province). Tourism development may cause the destruction of authenticity in which all are driven by profits. They are not protected by expert supervision without heritage protection function. They are not sustainable tourism development. In China, in addition that the local people's residence lands are private, all other lands belong to the village collective, which are managed by the local governments, and the incomes are also owned by the local governments. The local peoples are vulnerable groups without any power, and their cultural qualities are rather low. If there is no local government support and guidance, most indigenous people will live in poor. Consequently, to earn a living, young adults choose to work outside while most of the elderly and left behind children have to stay in ancient villages. Based on Chinese national conditions, the protection and utilization of the ancient villages (i.e., tourism development) should be jointly organized by an organization, which consists of the local governments, operators, aborigines, and protection experts. The organization should take the protection of the ancient villages as its mission and carry out the task of the protection plan in detail. Its organizational relations and management operation should be an enterprise management system [8]. The production and life of the local peoples in ancient villages are an important part of tourism resources. Their own lives are at work which may produce values and hereby they should be paid. Thus, in the organizational structure of ancient village tourism enterprises, we need to discuss what kind of roles the local people play in tourism development and management. According to the international principles of heritage tourism management, the following questions need to be identified and analyzed.

First of all, we need to analyze what the goal of ancient village tourism enterprises is. Tourism activities and development should achieve positive results and minimize the adverse effects on heritages and lifestyles of the local people [10, 11]. One of the goals is to have the heritage protection function. That is to say, the aim of the tourism development and management is to maximize the protection of ancient ruins, landscapes, and aboriginal lifestyles. In addition, through education, training, and the creating full time employment opportunities, the tourism development should provide a fair economic, social, and cultural benefit to all indigenous people at all levels of the community. The second goal is to increase economic incomes, cultural quality and social status for the local people.

Secondly, we need to analyze who the biggest beneficiary of the ancient village tourism is. Tourism and protection activities should benefit the community. A large part of the tourism revenues should be used for heritage conservation, renovation and display. The biggest beneficiaries should be the local people except the ancient village heritage landscapes.

Lastly, we need to analyze who are in the making decision levels of the ancient village tourism enterprises. Stated as earlier, it should respect the local rights and interests of tourist areas, the owner of monuments, and all local peoples who have the rights and obligations of lands and important sites. Therefore, the local peoples are the most immediate and important decision-makers and followed by the local governments.

Based on the above analysis, we may conclude that the local peoples, the owners and heirs of the cultural heritages are the biggest beneficiary of tourism business profits and the most direct and important decision makers. According to the economic concept that the enterprise is the producer in the market economy, the goal of an enterprise is to maximize profits for the owner of the enterprise. The biggest beneficiary of the profit is the owner of the business, i.e., the shareholder of the enterprise. Thus, the position of the local people in the tourism enterprise should be the owner, i.e., the trustee of the tourism enterprise. As for the investor, it belongs to the lender, which is equivalent to the bank. Corporate fiduciary and developers belong to the employment agent, which is equivalent to the chairman of the board or the general manager. For example, let us consider what the confused relations are when these new relationships are used to Xidi. Obviously, although the Xidi' village government and aborigines are both holders of the tourism company. But, the local people receive less ticket revenues than the village government. Thus, the role and position of the holders of the local people are not much remarkable. As a result, conflicts between the local people and the village government are bound to occur due to such a serious irrational distribution of tourism interests.

6 Behavior Analysis of Stakeholders in Ancient Village Development

As stated above, there are four stakeholders in ancient village tourism development and management. In the sequent, we discuss relations of some stakeholders and analyze their behaviors through game theory [12, 13].

6.1 Game Model Between Tourism Enterprise and the Local Government and Behavior Analysis

Let us consider the game problem between a tourism enterprise and the local government. Assume that the tourism enterprise has two options (or strategies). One option is to protect the ancient village, denoted by β_1. The other option is to destroy the ancient village, denoted by β_2. Similarly, the local government has two options/strategies. One option is to supervise the tourism development of the ancient village, denoted by α_1. The other option is to un-supervise the tourism development of the ancient village, denoted by α_2. In the tourism development of the ancient village, if the tourism enterprise chooses the option β_2, i.e., destroying the ancient village, then it earns $R + r$; if the tourism enterprise chooses the option β_1, i.e., protecting the ancient village, then it earns R, where $r > 0$ represents the increased invest of the tourism enterprise due to the protection. The supervising cost is c. The government award is h whereas the penalty is a. Then, we have the payoff bimatrix as follows:

$$(\boldsymbol{A}, \boldsymbol{B}) = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \qquad \beta_1 & \qquad\qquad \beta_2 \end{array} \\ \begin{pmatrix} (-c-h, h+R) & (a-c, R+r-c) \\ (0, R) & (0, R+r) \end{pmatrix} \end{array}$$

Assume that the local government chooses α_1 with a probability x and hereby chooses α_2 with the probability $1 - x$. Likewise, the tourism enterprise uses β_1 with a probability y. Thus, the tourism enterprise chooses β_2 with the probability $1 - y$. When y is given, then we can obtain the local government's expected incomes if it chooses α_1 and α_2:

$$E(\alpha_1) = (-c - h)y + (a - c)(1 - y)$$

and

$$E(\alpha_2) = 0y + 0(1 - y) = 0,$$

respectively. Thus, we can obtain the income expectation of the local government as follows:

$$E(x) = xE(\alpha_1) + (1 - x)E(\alpha_2)$$

In the same way, we can obtain the income expectation of the tourism enterprise as follows:

$$E(y) = yE(\beta_1) + (1 - y)E(\beta_2)$$

To take into consideration behavior and strategy choice of the local government and the tourism enterprise, we introduce the time factor and hereby construct the system of duplicate differential dynamic equations as follows:

$$\begin{cases} \dfrac{dx}{dt} = x(E(\alpha_1) - E(x)) \\ \dfrac{dy}{dt} = y(E(\beta_1) - E(y)) \end{cases}$$

which specifically infers as follows:

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x)[-(a + h)y + a - c] \\ \frac{dy}{dt} = y(1 - y)[(a + h)x - r] \end{cases}$$

Let

$$\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = 0 \end{cases}$$

Then, we have four pure equilibrium points: (0, 0), (0, 1), (1, 0), (1, 1) and one equilibrium point $(r/(h + a), (a - c)/(h + a))$ in the sense of mixed strategies if $r/(h + a) \in (0, 1)$ and $(a - c)/(h + a) \in (0, 1)$. But, only two of the above five equilibrium points are stable after specific analysis.

In fact, if $a < c$, i.e., the penalty of the tourism enterprise is smaller than the supervising cost of the local government, then the pure equilibrium point (0, 0) is asymptotic and stable. This may be interpreted as follows: no matter a long term or a short term, the local government tends to choose to un-supervise the tourism development of the ancient village and the tourism enterprise chooses to destroy the ancient village. Therefore, to protect the ancient village, the local government should enlarge the penalty or decrease the supervising cost.

Similarly, if $a > c$ and $h + a < r$, i.e., the penalty is larger than the supervising cost and the sum of the award and the penalty is smaller than the additional income, then the pure equilibrium (1, 0) is asymptotic and stable. Namely, from a viewpoint of a long term, the local government may choose to supervise the tourism development of the ancient village while the tourism enterprise may take a chance to destroy the ancient village due to the fact that the sum of the penalty and the award is smaller than the additional income resulted from destroying the ancient village. Therefore, the local government enlarges the penalty or the award to urge or induce the tourism enterprise to protect the ancient village.

6.2 Game Model Between Tourism Enterprise and the Local People and Behavior Analysis

In this section, we consider the game problem between a tourism enterprise and the local people. Still assume that the tourism enterprise has two options/strategies. One option is to protect the ancient village, denoted by β_1. The other option is to destroy the ancient village, denoted by β_2. Similarly, the local people have two options. One option is to complain the tourism enterprise destroying the ancient village, denoted by σ_1. The other option is to un-complain the tourism enterprise destroying the ancient village, denoted by σ_2. In the tourism development of the ancient village, if the tourism enterprise chooses the option β_1, i.e., protecting the ancient village, then it earns R; if the tourism enterprise chooses the option β_2, i.e., destroying the ancient village, then it earns $R+r$, where $r>0$ represents the increased cost or invest of the tourism enterprise due to the protection. The complaining cost is b and the obtained income is w/n due to the improved environment of the ancient village, where w is the obtained total income and n is the number of the local people. Here, assume that $w/n<b$. The penalty is a if the complaint is confirmed. Then, we have the payoff bimatrix as follows:

$$
\begin{array}{cc} & \begin{array}{cc} \beta_1 & \qquad\qquad \beta_2 \end{array} \\ (\boldsymbol{C},\boldsymbol{D}) = \begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} & \begin{pmatrix} (-b,R) & (w/n-b,R+r-a) \\ (0,R) & (0,R+r) \end{pmatrix} \end{array}
$$

Assume that the proportion of the number of the tourism enterprises choosing to protect the ancient village is y while the proportion of the number of the tourism enterprises choosing to destroy the ancient village is $1-y$. Likewise, the proportion of the number of the local people choosing to complain the tourism enterprise destroying the ancient village is z whereas the proportion of the number of the local people choosing to un-complain the tourism enterprise destroying the ancient village is $1-z$, where $z\in[0,1]$ and $y\in[0,1]$.

Thus, we can the income expectation of the local people as follows:

$$
\begin{aligned}
E(z) &= zE(\sigma_1) + (1-z)E(\sigma_2) \\
&= \frac{w}{n}z - \frac{w}{n}zy - bz
\end{aligned}
$$

Similarly, we can the income expectation of the tourism enterprise as follows:

$$
\begin{aligned}
E(y) &= yE(\beta_1) + (1-y)E(\beta_2) \\
&= r - az - yr + azy + R
\end{aligned}
$$

By introducing the time factor, we can construct the system of duplicate differential dynamic equations as follows:

$$
\begin{cases}
\frac{\mathrm{d}z}{\mathrm{d}t} = z(1-z)(-\frac{w}{n}y - b + \frac{w}{n}) \\
\frac{\mathrm{d}y}{\mathrm{d}t} = y(1-y)(az - r)
\end{cases}
$$

In a parallel way to Subsect. 6.1, let

$$
\begin{cases}
\frac{\mathrm{d}z}{\mathrm{d}t} = 0 \\
\frac{\mathrm{d}y}{\mathrm{d}t} = 0
\end{cases}
$$

Then, we have four pure equilibrium points: (0, 0), (0, 1), (1, 0) and (1, 1). But, only the pure equilibrium point (0, 0) is stable after specific analysis. This may be interpreted as follows: the local people choose to un-complain the tourism enterprise destroying the ancient village due to the fact that they need to spend all the complaining cost whereas they cannot earn the total income resulted from the improved environment. The tourism enterprise chooses to destroy the ancient village due to the psychology and behavior of the local people. Therefore, the local government should decrease the complaining cost. On the other hand, the local people should be given some economic compensation if their complaints are confirmed. At the same time, the local government should enlarge the penalty for destroying the ancient village.

7 The Decision Making Analysis on the Stakeholders of Ancient Village Tourism Companies

In conclusion, Chinese ancient village tourism enterprise organizations should include five elements: protecting ancient villages, all local peoples, protection experts, local governments at all level, investors or management companies. In the following, we make some decision analysis to deal with the above qualitative issues.

The most rational stakeholder relationship is that the reasonable benefit of landscape protection can reflect the perfect protection functions. The reasonable benefit of the local people can reflect the reasonable goals of the tourism enterprises/companies. The local government revenues can reflect the management costs. The government incomes can reflect the improvement of public facilities. Travel and management enterprise can only be paid for meeting their corporate goals. Protection experts belong to non-profit groups/organizations. They participate in the whole process and confirm that tourism enterprise management is consistent with the protection mission.

Through modeling and analysis, the reasonable and right solution to the tourism stakeholders is that ancient village heritage landscapes can obtain sustainable protection, maintenance, excavation, exhibition, research, training, and other funds. The local people can obtain personal incomes which are in line with their own values. These incomes make the local people voluntarily and actively put actions to protective activities, which can reflect their dominant roles. The local governments receive revenues which enable them to make a difference to the public services. Tourism management enterprises can reflect their professional values and social values when they gain reasonable incomes. The number of benefits obtained by tourism enterprises can reflect their professional values. Protection experts belong to social groups/organizations, which reflect their scientific and impartial services to the society.

8 Conclusions

The ancient villages are un-renewable heritage resources, which are of extremely important tourism values. Based on the ecotourism exploitation and management practice and the international cultural heritage protection documents, we systematically discuss some issues of ecotourism environment protection and breakage from a viewpoint of stakeholders such as the local people, tourism enterprises, local governments, and protection experts. Particularly, we discuss the relations among the stakeholders and their roles and hereby analyze their behaviors and strategy choices. These conclusions may provide theory guide and reference for harmonizing relations among the stakeholders in ecotourism exploitation and management of the ancient villages, resolving conflicts between ecotourism exploitation and protection and accelerating the sustainable development of ecological tourism economy.

The tourism development and management of the ancient villages are involved in various factors, including both qualitative and quantitative. In near future, we will investigate on how to evaluate tourism plans of the ancient villages and design the income mechanism of stakeholders by using fuzzy multi-attribute decision making and game theory [14, 15].

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Two Bargain Game Models of the Second-Hand Housing Commence

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Abstract. The aim of this paper is to discuss a bargaining problem between sellers and buyers in the case of the final price of the house for sale in the second-hand house market. Two bargaining models are established for the sellers and buyers in indefinite period and finite period. For the indefinite period, the complete equilibrium solution of the bargaining game between the buyers and sellers is obtained. Hereby, imposing some constraints on the time, the game equilibrium solution on the second stage is obtained. At the same time, a multiple game model is constructed and the commence point is discussed. The result shows that the game between sellers and buyers depends on the ratio of each one's discount factor. The time and commitment between sellers and buyers increase the sellers' cost, hence the final price can only be implemented within a certain range, which is related with the proposal cost of sellers, the number of games, and the punishment cost in each round.

Keywords: The second-hand housing commence · Equilibrium · Bargaining · Game model

1 Introduction

With the high-speed development of China's urbanization process after more than ten years, urban construction land supply is increasingly scarce, land acquisition cycle is becoming longer and the cost of construction is increasing. As a result, second-hand housing supply gradually exceed the new one over the past few years in many big cities, and become the main source of urban housing supply. Under the circumstances, the price in second-hand housing market becomes the focus of attention, and the formation process of the transaction price is not only influenced by building its own characteristics, but also affected by both parties decision-making process.

With the further adjustments in the real estate market, sales of second-hand house will become a "major driving force" in the real estate market. That second-hand housing transaction is smooth or not is not only directly affect the operation of the real estate market, and also has close ties with the idea of sticking to a certain place to live and work in peace and contentment. How can they gain maximum benefits between a huge number of sellers and the buyers in the game in such an active market? What

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factors will the bargaining between the two sides be affected by? What kind of impact will it produce for the business of real estate brokerage agency (agency) to bargain during the game process between both the buyer and seller? What message does this give the government departments in the supervision of second-hand house market? For it has not carried out to research in the trading process and its related influencing factors at home, This paper select the bargaining game behavior of second-hand housing transaction between both sides as the research object from the perspective of game theory. By setting up perfect information games models which are time limitation and no time limitation, assuming the utility functions of both the buyer and seller in the second-hand house market is common knowledge for both sides and the provisions of the conditions before negotiations are regarded as exogenous variables of the game, relevant research results will be figured out in this process.

The structure of this thesis is as follows: Section 2 introduces the relevant research on second-hand house market conducted by domestic and international scholar; Section 3 builds timeless and limited game model of bargaining during the second-hand housing transaction. And sub game perfect Nash Equilibrium of both the buyer and seller are gained based on the timeless and limited game model. There was something in the elimination of time of the game based on this. Then equilibriums of game of both sides are worked out in the second stage based on limited game model. In the meantime, several game models are built and game process is discussed at a particular point; Section 4 analyzes and demonstrates using the example; Section 5 is to make conclusion of the paper.

2 Literature Review

In recent decades, the domestic and foreign scholars conducted research in many aspects. As for the research of the sellers' behavior in second-hand housing, for example, Stein [1] first proposed that "constraint of property rights" is used to research the seller's behavior. Bokhari and Geltner [7] first applied theory of loss aversion to commercial real estate pricing, they think the theory of loss aversion significantly influence the behavior of the seller. Hua and Seow [8] examined sellers' price strategy where the reference point depend on the preferences and market is homogeneous, found that the recent transaction price has two distinct signal effects for potential buyers' willingness to pay, and the seller's asking price tends to increase with the increment of the buyer's visit. As for the psychological aspects of expected final price of both sides in the second - hand housing transaction, Black and Diaz [2] simulated the bargaining process through a series of experiments, which show that the artificially fabricated "quotation" is not only affected the buyer "opening offer", but also affect the final transaction price. They concluded that the quotation of seller or agent can be used as an instrument to induce the buyer decision making errors; Clauretie and Thistle [6] used 2828 Las Vegas real trading sample to test the influence of the search cost and anchoring effect on the housing transaction price. In the area of affecting factors of final transaction price in second-hand house market, the opportunity cost model proposed by Krasner [3] state that the seller's psychological price rise amplitude of the price that seller can accept also lower than the buyer's when the housing market change from recession to recovery. Based on the two-stage regression model, Fisher et al. [4, 5] use probity model, Millers model as well as introducing hedonic price model to calculate buyers and sellers' psychological price index, and real estate research center at the Massachusetts institute of technology has put the method mentioned above into practice.

Compared with foreign detailed research on real estate transaction process, bargain model used by domestic scholars mostly concentrate on the game analysis of the trading subject, Such as Yang and Sun [9] who used game analysis based on the government, developers, consumers tripartite game analysis, which clarify clearly the relationship between policy factors and China's real estate market developing path; Huai and Liu [10] analyze non-cooperative bargaining process between the real estate developers and local governments with the application of Rubinstein turn model and Selten non-cooperative game method under the condition of local perfect information; What's more, Wang and Gong [12] built the second-hand housing investors and developers in the real estate market sales bargaining model of profit distribution, discusses the price formation mechanism and benefit distribution mechanism under the interest distribution model. It turned out that the feasibility of these interests distribution is relevant to both real estate product differentiation rate of substitution and repeated game strategy of the discount rate. Lai and Chen [13] found that a government takes control of the land price determines the nature of the regional real estate oligopoly dynamic game equilibrium and equilibrium path under the condition of production technique level and management level for property developers by developing bounded rational real estate dynamic Cournot oligopoly model. Taking China's main body second - hand housing transaction as the research object, Zhao [11] established game model based on the condition of asymmetric information between real estate intermediary and buyers, and puts forward related suggestions to eliminate asymmetric information.

3 Indefinite Period Bargaining Model

3.1 Problem Assumption

To simplify the analysis,this article assumes that there are two participates who are a buyer A and a seller B in the game, and they are numbered respectively 1 and 2. Both of the decision-makers are risk neutral. Both sides of the utility function is their common knowledge without thinking the mediation and other third party for the influence of the utility. In this case, the game between two sides exist the risk of rupture. So in the Nash bargaining solution, the risk of rupture of negotiations can be marked as no agreement point (d_1, d_2). When the negotiation broke down, d_1 represents housing values of the seller A, and d_2 represents monetary value of the buyer B, which represents the lowest housing value the seller A, the highest bidder of the buyer B.

Assuming that before the deal, seller A has the original utility for $\bar{x}_1$, and buyer B has the original utility for $\bar{x}_2$. Add up the original utility and they total x_0. After the deal, seller A will gain the original utility for x_1, and buyer B will gain the original utility for x_2. The total amount of remaining transactions is x_0 because of successful trading between them.

During the real process of Second-hand house transaction, a party will first provides them with a quotation, and the other party from its own utility maximization, decides whether to accept this price or not. If agreed, the two parties will clinch a deal, otherwise both parties will move to the next bargaining game. Assuming the seller will bid the price in accordance with the following process, the behavior of bid is conducted in discontinuous point, i.e. $t\Delta$ $(t = 0, 1, 2, 3, \ldots, N)$.

There is no time and times limit to agreement bargaining for both sides. In the meantime, assuming that at the time of an odd number of times a quotation will be provided by the buyer, the two parties will clinch a deal if the seller accepts this price. Otherwise a price will be offered by the seller, and there will be a deal if the buyer accepts this price. That works like this: bid order of buyer and seller eventually has no effect on game equilibrium solution. Based on the above analysis, when the t is an even number, seller A makes an offers. When the t is an odd number, buyer B makes an offer.

In the case of no time limit, If the buyer refused to offer in order to delay the time at the time $t\Delta$, probability of breakdown in the negotiation is ρ. And the probability is $1 - \rho$ as the game continues at the time $(t + 1)\Delta$.

If the two parties clinch a deal at the time $t\Delta$, the seller A will get monetary value x_1 and the buyer B will own a house worth x_2. The utility function of the two parties is set to $U_i : [0, x] \to R$ which is a strong increasing concave function. And it conforms to the von Neumann - Morgan Stan utility function. That is $U_i(x_i) = u_i(x_i)e^{(-r_i\Delta)}$, and assuming that $\delta_i = \mathrm{e}^{(-r_i\Delta)}$, $\delta_i \in (0, 1)$, δ_i is the discount factor of participant i.

If the disagreement between the two sides leads to a breakdown in the negotiation, the utility received by participant i is $d_i(d_a = \bar{\pi}_a, d_b = \bar{\pi}_b, d_a + d_b = \pi_0)$ in which $U_i(0) < d_i < U_i(\pi)$. At this point, both the buyer and seller can earn as most value as the original value owned by individuals, and they even earn negative value to some extent. For breaking down in the negotiation means both sides benefit no value from the former bargaining, both sides need to continue to look for a new transaction object.

If both parties have been unable to reach a unity, then the utility at the point there is a stalemate for the agreement is:

$$\rho d_i \sum_{k=0}^{\infty} (1 - \rho)^k \delta_i^k \Rightarrow \frac{\rho d_i}{1 - (1 - \rho)\delta_i}$$

Assumption: $\beta_i = \frac{\rho d_i}{1-(1-\rho)\delta_i}$, $\delta_i \in (0, 1)$; because of $0 \leq \frac{\rho}{1-(1-\rho)\delta_i} \leq 1$, then $\beta_i \in [0, d_i]$ $(i = 1, 2)$. There must be a number $x_i^d \in [0, d_i]$ which is satisfied to constraints $U_i(x_i^d) = \beta_i$, $x_i^d = U_i^{-1}(\beta_i)$. After several rounds of bargaining, the negotiations came to a deadlock or even break down, for both the buyer and seller are not satisfied with the terms quoted by themselves. At this time, value gained by both sides is more driven by discount factor. In the meantime, this will further reduce obtained value of both sides compared with breaking down in the negotiation directly.

3.2 Perfect Equilibrium Solution of Subgame

Due to decision-makers are risk neutral, then $U_i(x_i) = x_i$ and the perfect equilibrium solution of both parties is x_1^*, x_2^*. When it comes to the risk of breakdown in the negotiation and the effect of discounting, the only Perfect equilibrium solution of subgame meets the following conditions. When the seller always offer a price x_1^*, when and only when $x_2 \leq x_2^*$, seller A always receives x_2^*; When the buyer B always offer a price x_2^*, when and only when $x_1^* \leq x_1$, the buyer always receives x_1^*. That is:

$$U_i(x - x_i^*) = \rho d_i + (1-\rho)\delta_i U_i(x_i^*) \tag{1}$$

$\delta_i = \mathrm{e}^{(-r_i \Delta)}$, Among them $r_i > 0$ is the discount rate of participant i, it meets the following conditions:

$$x_1^* \geq x_1^d; \quad x_2^* \geq x_2^d; \quad x - x_1^* \geq x_1^d; \quad x - x_2^* \geq x_1^d$$

According to simultaneous Eq. (1), following results can be obtained:

$$x_1^* = \frac{[1-\delta_2(1-\rho)](x-\rho d_1)}{1-(1-\rho)^2\delta_1\delta_2} \tag{2}$$

$$x_2^* = \frac{[1-\delta_1(1-\rho)](x-\rho d_2)}{1-(1-\rho)^2\delta_1\delta_2} \tag{3}$$

$$\frac{x_1^*}{x_2^*} = \frac{(1-\rho)\delta_2 - 1}{(1-\rho)\delta_1 - 1}. \tag{4}$$

Theorem 1. $\frac{x_1^*}{x_2^*}$ is positively correlated to $\frac{\delta_2}{\delta_1}$ through the above results. And the greater the discount factor o the ratio of the buyer and the seller is, the greater surplus value the seller will achieve. Perfect equilibrium solution depends on the ratio of the discount factors. The greater the discount rate (seller to buyer) is, the smaller the proportion of utility will be. Also it infinitely closes to the point of long stalemate. When $\Delta \to 0$, it is assumed that the limit of $\frac{\rho}{\Delta}$ exists, set $\lim\limits_{\Delta \to 0} \frac{\rho}{\Delta} = \lambda$, and $\lambda > 0$. When Δ is quite small, $\delta_i = \mathrm{e}^{-r_i\Delta}$ which can be approximately expressed as:

$$\delta_i = 1 - r_i\Delta. \tag{5}$$

$$\beta_i = \frac{\rho d_i}{1-(1-\rho)\delta_i} \tag{6}$$

Take algebraic expression (5) into algebraic expression (6), then $\lim\limits_{\Delta \to 0} \beta_i = \frac{\lambda d_i}{r_i + \lambda}$. It means the stalemate point in the game of both sides is $(I_1, I_2) = (\frac{\lambda d_1}{r_1+\lambda}, \frac{\lambda d_2}{r_2+\lambda})$ when the game between the buyer and the seller remains deadlocked for a long time.

Due to $U_1(x_1) = x_1, x_1 \in [0, x]$, 且 $U_2(x_2) = x_2$, to get the only Perfect equilibrium solution of subgame is to solve the maximization problem of $(x_1 - I_1)(x_2 - I_2)$. That is:

$$Max(x_1 - I_1)(x_2 - I_2) \tag{7}$$

According to simultaneous Eq. (7), the only perfect equilibrium solution of subgame will be work out

$$R_1 = I_1 + \frac{r_2 + \lambda}{2\lambda + r_1 + r_2}(x - I_1 - I_2), \tag{8}$$

$$R_2 = I_2 + \frac{r_2 + \lambda}{2\lambda + r_1 + r_2}(x - I_1 - I_2). \tag{9}$$

Theorem 2. $I_1 = \frac{\lambda d_1}{r_1 + \lambda}$ shows that if r_1 increases, I_1 strictly decreases but there is no effect on I_2. It will make the buyer fight for more utility through bargaining. It is observed that discount factor has significant impact on the value of both sides. When the discount rates of both sides is determined, once the seller benefits more than the buyer and the buyer is reluctant to give up the deal, the buyer has the tendency to let the negotiation reach an impasse.

Theorem 3. Due to $\lambda = \lim_{\Delta \to 0} \frac{\rho}{\Delta}$, as λ increases, it is going to gets larger the probability of break down in the negotiation, or the time interval of breakdown in the negotiation decreases. If $r_1 = r_2 = r; d_1 = d_2 = d$, Both Conform to the principle of compromise, and both the utilities are $\frac{x}{2}$. If $r_1 = r_2 = r; d_1 \neq d_2$, then utility allocation of the perfect equilibrium solution of subgame is $\frac{x}{2} + \left(\frac{1}{r/\lambda + 1}\right)\left[\frac{d_1 - d_2}{2}\right]$;

When $d_1 > d_2$, the utility of the seller A strictly decreases as r/λ decreases. When $d_1 < d_2$, the utility of the seller A strictly increases as r/λ increases. Therefore, the seller A would prefer r/λ smaller In the case of $d_1 > d_2$. Because $\lambda = \lim_{\Delta \to 0} \frac{\rho}{\Delta}$, with r unchanged, ρ will increase as λ increases and $\Delta \to 0$, which leads to a higher possibility of breakdown in the negotiation. But for seller A, who wants a collapse of negotiation in order to have the opportunity to bargain with the buyer with purchasing desire or conclude a transaction with smaller compensation. When the discount rate is the same for both sides, comparing the value obtained at breakdown point will have great influence on the game result and value distribution of both sides. Both sides hope that they can gain value no fewer than the other side through bargaining. Otherwise it increases likelihood of breakdown in the negotiation. Negotiations between the two sides will break down because the cost is consumed during period of the bargaining without getting favorable expected returns.

All the analysis justifies an obvious view that both the buyer and seller set prices of unsold homes based on the value they earned, and discount factor is an important factor to be considered. The notable one is as follows: both the buyer and seller will not keep going the bargaining forever though there is no time limitation. When value earned by one party is significantly higher than the other party, talks between the buyer and seller will break down because the party who gets less value will stop bidding. The efficiency of second - hand housing transaction is low in this state, and it does not have the nature of operation in practice. But, the situation of bargaining indefinitely is set to study what kind of factors will influence the final results of bargaining in second-hand housing transaction when the process is not disturbed by time. All this provided the reliable theory base for the following research.

4 Finite Period Bargaining Model

Based on the research results of the infinite period bargaining model, The main factors affecting the complete equilibrium solution are the discount factor $\left(\frac{\delta_1}{\delta_2}\right)$ of the seller and the buyer and their attitude to risk. The buyer's optimal strategy is to increase the number of games and delay the transaction time.

In the reality of the second-hand housing market, no buyers and sellers can have an infinite period of the game and waiting. After a certain number of bargaining, through the maximization of the effectiveness of their own measure, they determine whether to collaborate. So the bargaining patience and the number of times are limited. Based on the reality, this paper will improve the original model, adding the relevant parameters, to make it more in line with the reality. In this paper, it is assumed that the interval time between the seller and the buyer is Δ, and the total game time is less than $k\Delta(k \geq 1)$. The seller of each proposed cost including time, experience, communication etc. is c_0. No proposed by the other party within the time required by the party under this misconduct may be wrong with closing conditions for the transaction object, until the next transaction object appears. The resulting loss C' is, in this paper, the penalty cost. In order to simplify the proof and without loss of generality, we assume that the cost of punishment C' in the last time did not offer immediate requirements, and in each stage $i\Delta$of the game in accordance with the possibility of sharing, namely the cost penalty rejected in time for plan is $\frac{j}{k}C'$ $(0 < j \leq k)$. The follows often happen.

In the second-hand housing transactions: because of the seller's responsibilities including emergency in cash, settling abroad, or marketing, they eager to trade as soon as possible and the buyer think various aspects of the property are very satisfactory. If the buyers agreed, price of the transaction may be the high. This is an unwilling result for buyers; if not, the buyers may miss the favorite real estate, but continuous search may not lead to a satisfactory result. In the case of high price transactions, the buyer wants the seller to give some compensation in other ways, such as tax relief, free parking, free transfer etc. In view of the above situation, we assume that when buying in a high price, the cost of capital occupied is c_B, which will be regarded as a component of the buyer discount factor δ_b.

With the advance of the game time, the cost of capital is $j\Delta c_2$ $(1 \leq j \leq k)$. To a certain extent, the seller also has the cost of capital occupation, but because it is fast and the seller in the price does not give a lot of concessions, so its cost is ignored.

4.1 Game Process Description

When buyers and sellers have a clear intention to deal with the sale of real estate in order to facilitate the final transaction, for the first time, the seller try to make appropriate compensation for the buyer including: high compensation scheme and low compensation scheme. Assuming before the compensation, the utility share that the seller could obtain in trading is q_1. The original utility share that the buyer can obtain is q_2. We also assume that the total transaction bargaining formed the remaining share of Q remained unchanged.

In the high compensation scheme, the scheme proposed the following contents: after compensation, the buyer gets q'_{2H} (covered spaces of the house), while the seller shall obtain the share of monetary value q'_{1H}; setting the sale of real estate sale when the unit price is p, the current real estate market unit price is p. After the compensation, the price is p'.

Under a high compensation plan, seller A and buyer B will get the following utilities.

If B agree, A: $q'_{1H}p' - q_1p - c_0$; B: $q'_{1H}p' - q_2p_t$;

If B disagrees, A: $-c_0 - q_1 \times \frac{1}{k}C'$; B: $-q_2 \times \frac{1}{k}C' - \Delta c_2$;

Under a low compensation package, seller A and buyer B will get the following utilities.

If B agrees, A: $q'_{1L}p' - q_1p - c_0$; B: $q'_{2L}p' - q_2p_t$;

If B disagrees, A: $-c_0 - q_1 \times \frac{1}{k}C'$; B: $-q_2 \times \frac{1}{k}C' - \Delta c_2$;

Due to $q'_{2H} > q'_{2L}$, the result of the game is displayed in Fig. 1.

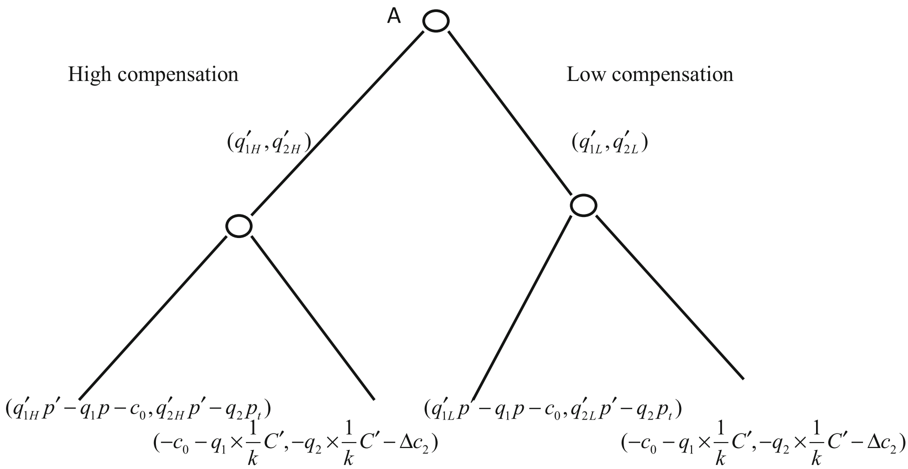

Fig. 1. Dynamic game theory of finite period trading under complete information

4.2 Equilibrium Solution of Game in the Second Round

(a) When $q'_{2L}p' - q_2p_t > -q_2 \times \frac{1}{k}C' - \Delta c_2$

At this point, the strategy of buyer B is (agree, agree). That is to say the most valuable action is to agree in the second stage no matter the compensation given to buyer B by seller A is high or low in the first stage. The optimal choice for seller A in the first stage is to provide buyer B with low compensation, because the action of buyer B in the second stage can be predicted by seller A in the first stage. Perfect equilibrium solved by backward induction is {low compensation (disagree, disagree)}. That means seller A provides buyer B with low compensation in the first stage and buyer B will vote "for" in the second stage.

(b) When $q'_{2L}p' - q_2p_t < -q_2 \times \frac{1}{k}C' - \Delta c_2$

The strategy of buyer B is (disagree, disagree). That is to say the most valuable action is to disagree in the second stage no matter the compensation given to buyer B by seller A is high or low in the first stage. The optimal choice for seller A in the first stage is to provide buyer B with low compensation, because the action of buyer B in the second stage can be predicted by seller A in the first stage. Perfect equilibrium solved by backward induction is {low compensation, (disagree, disagree)}, and the game will move to the next turn.

(c) When $q'_{2H}p' - q_2p_t > -q_2 \times \frac{1}{k}C' - \Delta c_2 > q'_{2L}p' - q_2p_t$

In this case, the decision made by buyer B will be (agree, disagree) in the second stage, then there will be two Nash equilibrium outcomes (high compensation, agree) and (low compensation, disagree). The cost for seller A to the next round is $c_0 + \frac{2}{k}q_1C'$, because of the difference between high compensation and low compensation provided for buyer B by seller A. Compensation for the difference is $\left(q'_{1L} - q'_{1H}\right)(p' - p)$.

When $\left(q'_{1L} - q'_{1H}\right)(p' - p) > c_0 + \frac{2}{k}q_1C'$, seller A will provide high compensation plan for buyer B.

When $\left(q'_{1L} - q'_{1H}\right)(p' - p) < c_0 + \frac{2}{k}q_1C'$, seller A will provide low compensation plan for buyer B.

4.3 Repeated Game Model Establishment

The possible Nash equilibriums in the first game are (Low compensation, agree) and (Low compensation, disagree). In the case of being refuse, game will be repeated for many times. Assuming that the same low compensation plan is put forward by seller A every time, buyer B will classifies the plan and votes on it, and the process of the game is displayed in Fig. 2.

If buyer B agrees with the bidding project offered by seller A in the second round of the game, the utility functions for both parties are these:

Seller A: $U_1 = q'_{1L}p' - q_1p - 2c_0$;

Buyer B: $U_2 = q'_{2L}p' - q_2p_t - \Delta c_2$.

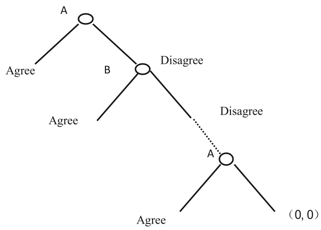

Fig. 2. The finite period dynamic game under the low compensation

If buyer B agrees with the bidding project offered by seller A, the utility functions for both parties are these:

Seller A: $U_1 = -2c_0$; Buyer B: $U_2 = -2\Delta c_2$.

If buyer B and seller A agree with the bidding project offered by each other in the $j(1 \leq j \leq k)$ round of the game, the utility functions for both parties are these:

Seller A: $U_1 = q'_{1L}p' - q_1 p - ic_0$;

Buyer B: $U_2 = q'_{2L}p' - q_2 p_t - i\Delta c_2$.

Assuming that when $j(1 \leq j \leq k)$ the effect of the buyer is zero, that is, $U_2 = q'_{2L}p' - q_2 p_t - i\Delta c_2 = 0$ then $j = \frac{q'_{2L}p' - q_2 p_t}{\Delta c_2}$, it shows that as the seller A of rational man will exercise the freedom before round game, after the turn can make its utility is negative, the exercise of the refusal it do more harm than good. At the same time, when the cost of the buyer is larger, the less number of the participation.

Assuming that the buyer B for utility is not zero in the round $j(1 \leq j \leq k)$, that is, $U_2 = q'_{2L}p' - q_2 p_t - i\Delta c_2 \neq 0$, without the interference of external factors, the j round of j of the game of condition is not the optimal choice of the buyer, the optimal action is his vote to the next, i.e. the $j - 1$ round and the buyer B of the optimal selection is a common knowledge of the buyer and the seller. The subgame perfect Nash equilibrium solution of game process, therefore, is the first round of {low compensation (agree, agree)}.

The analyses above show that the value earned by the seller A is associated with k. When $k \to \infty$, the seller A will gain all the remaining value in the negotiation. The conclusion is the same as that of timelessly repeated game. There is often such a case where the seller has a high quality housing (such as school district housing) with scarce resources or supporting facilities in the second-hand housing transaction, and the number of buyers who take a fancy to the housing and come to bargain is quite large. Under the circumstance, the seller plays a dominant role in the negotiation and the seller will insist on continuous negotiation until gaining its' desired price. The conclusion of this study proves this kind of realistic situation.

Besides, If the data of buyer B can't guaranteed to meet the conditions $q'_{bL}p' - q_b p_t - \Delta c_B \leq q'_{bM}p' - q_b p_t - (k-1)\Delta c_B$ in the $k - 1$ round, voting for agree will be the optimal strategy for buyer B in the first round. Then the subgame Nash Equilibrium is {low compensation, (agree, disagree)}. If seller A predicts the buyer B's decision, there is no need for seller A to improve the compensation plan in the next round of game. Under the circumstance, the optimal strategy of both sides is as follows: seller A put forward the low compensation in the first round and buyer B agreed to the plan immediately. Generally, the buyer only needs to consider the price in the second-hand housing transaction, on the other words the aim of the buyer is to gain expected price set by himself.

However, things are quite different for the buyer. It is usually a group decision making of a family because buying and selling of existing properties is a major decision for a family. Many factors should be taken into account comprehensively. And housing has such natures such as complexity, removability which makes it extremely easy to have a weak negotiating position for the buyer. Therefore, the buyer can not necessarily obtain an ideal trading scheme after several rounds of bargaining. The conclusion of this study also proves this kind of realistic situation in second-hand housing transaction. And plenty of situations where both sides clinch a deal using the initial plan in the second-hand housing transaction are numerically explained after a period of bargaining.

4.4 The Discussion of the Game Process in the Time of $k\Delta$

At the point, utility functions of the buyer and the seller:

The seller: $U_1 = q'_{1L}p' - q_1p - kc_0 - q_1C'$.

The buyer: $U_2 = q'_{2L}p' - q_2p_t - k\Delta c_2 - q_2C'$

Because of consideration of C', the result of $U_1 = q'_{1L}p' - q_1p - kc_0 - q_1C'$ may be zero.

(a) Assuming that $U_1 = q'_{1L}p' - q_1p - kc_0 - q_1C' = 0$, then $k_0 = \frac{q_1p + q_1C' - q'_{1L}p'}{c_0}$, to ensure that value gained by seller A is positive, then $\left[\frac{q_ap + (k-1)c_0}{p'}, \frac{q'_{aL}p' - (k-2)\Delta c_B}{p'}\right]$, the utility will guarantee, $\frac{q'_{aL}p' - (k-2)\Delta c_B}{p'} \geq \frac{q_ap + (k-1)c_0}{p'}$ the number of the game cannot be more than, any rational clinch a deal with the buyer the seller must ensure that at the moment. In order to avoid the point their utility to zero, then the seller will improve when the quotation scheme (compromise), in this scenario can be allocated between the seller and the buyer utility for the scope of generally has, at the moment, clinch a deal the buyer and the seller, at this time for their utility: $U(A) = q'_{aM}p' - q_ap - (k-1)c_0$ $U(B) = q'_{bM}p' - q_bp_t - (k-1)\Delta c_B$

(b) Assuming $U_1 = q'_{1L}p' - q_1p - kc_0 - q_1C' \geq 0$, then $q'_{1M} \geq \frac{q_1p + (k-1)c_0}{p'}$, and, $q'_{2L}p' - q_2p_t - \Delta c_2 \leq q'_{2M}p' - q_2p_t - (k-1)\Delta c_2, q'_{1M} + q'_{2M} = Q, q'_{1L} + q'_{2L} = Q$, then $q'_{aM} \leq \frac{q'_{aL}p' - (k-2)\Delta c_B}{p'}$. Accordingly, scope of q'_{1M} proposed in the compromise by the seller is $\left[\frac{q_1p + (k-1)c_0}{p'}, \frac{q'_{aL}p' - (k-2)\Delta c_2}{p'}\right]$, which secretly means $\frac{q'_{1L}p' - (k-2)\Delta c_2}{p'} \geq \frac{q_1p + (k-1)c_0}{p'}$, it was analyzed to obtain this:

$$q'_{1L}p' - q_1p \geq (k-2)\Delta c_2 + (k-1)c_0 = (c_0 + \Delta c_2)k - 2\Delta c_2 - c_0. \tag{10}$$

Theorem 4. The seller of the revenues from the bargaining is about function, at the time, the seller have all remaining of negotiations, the results and the results of the repeated game indefinitely.

Theorem 5. For the buyer, if can't guarantee a $q'_{bL}p' - q_b p_t - \Delta c_B \leq q'_{bM}p' - q_b p_t - (k-1)\Delta c_B$ t the moment, then he's optimal strategy is in the first round vote agree, at this point the sub-game Nash equilibrium is {low compensation (agree, agree)} If the seller knows the choice of the buyer, he is impossible to have the possible improvements. Therefore, if buyers and sellers are rational, the seller should put forward the minimum compensation plan, and the buyer agrees with the plan in the first round.

Theorem 6. About $\left[\frac{q_a p + (k-1)c_0}{p'}, \frac{q'_{aL}p' - (k-2)\Delta c_B}{p'}\right]$ for the seller, the lower limit value and function, with the increase of, the seller's concessions range is smaller, the requirements in the greater the share of compromise; With the relevant instructions clinch a deal finally the price is higher, seller A can concessions space is, the greater the total utility in bargaining surplus for the smaller. The bigger space for seller A to concede the less fight for total remaining utility caused by bargaining will have. The analyses above show that the seller in order to save the effort and time cost,will settle for second best to choose compromise in order to complete the transaction as soon as possible during the second-hand housing actual transaction so that the seller can save the effort and time cost. With the increase of the times of bargaining, the seller will not make more concessions unless the price of for-sale houses is quite high. Therefore, the situation where the seller discussed in Sect. 4.3 is in a strong position in the negotiation is part of the phenomenon of trading during second-hand housing transaction.

5 A Real Example of the Bargaining Game Model

In some part of our country second-hand house market has two types of players which include seller A and buyer B. Seller A and buyer B determine the final sale price of unsold homes through bargaining. Both sides are risk-neutral, whose utility function is based on their common knowledge without thinking the influence of third parties such as an intermediary. Table 1 shows the specific parameters related to game of bargaining during second-hand housing transactions in this paper. The game models of bargaining during second-hand housing transactions are built respectively based on indefinite period and finite period. Then indefinite sub game perfect equilibrium of both sides could be figured out using Eq. (1)–(9). Based on this, finite game equilibrium of both sides could be figured out in the second stage by putting the brakes on the time of game. Repeated game models are developed and the game process of transaction at a specific point is discussed as listed in Tables 2 and 3.

Table 1. List of basic parameters (Unit: ten thousand yuan)

	$\bar{x}_i$	x'_i	ρ	q_i	r_i	p	p_t	p'	c_0	C'
Seller	100	135	0.35	135	0.35	1.15	1.25	1.11	0.05	0.1
Buyer	120	115	0.65	120	0.17					

Table 2. Game of bargaining during second-hand housing transactions analysis (Indefinite period and finite period) (Unit: ten thousand yuan)

	Indefinite period		Finite period			
	PES	PES Payoffs	The first round	The second round		The jth round j=4
Seller	128	70.41	[−1.61, 2.84]	−1.60	−0.10	−1.70
Buyer	40.21	72.50	[−0.83, 18.48]	−0.93	−0.20	−1.63

Table 3. Value gained by both sides during second-hand housing transactions at the point of $k\Delta$ (Unit: ten thousand yuan)

	The time of $k\Delta$	The penalty cost C'	
	k=3	$U_1 \neq 0$	$U_1 \geq 0$
Seller	−1.78	−1.18	[−0.14, 1.49]
Buyer	−1.25	3.75	

The result shows that the game between sellers and buyers depends on the ratio of each one's discount factor. The time and commitment between sellers and buyers increase the sellers' cost, hence the final price can only be implemented within a certain range, which is related with the proposal cost of sellers, the number of games, and the punishment cost in each round.

6 Concluding Remarks

In this article a model is developed to discuss problem of bargaining game about second-hand house within indefinite duration or a time limit. The premise of the research in this paper is both the seller and buyer as participants who are risk neutral. And utility functions of both parties are based on their common knowledge without thinking the influence of a third party, such as brokers or other counterparties. The model reveals that the result of bargaining game is related to discount factors of both parties. The difference between utility functions of both parties will leave the division of residual made by both parties in an inconsistent state when there is a risk of breakdown in the negotiation. If the seller with strong communication skills negotiates in the initial stage, the smaller the relative discount factor is in the meanwhile, the less risk aversion the buyer wants. Then the seller gains more trade surplus.

On the other hand, if game of the buyer and seller goes on indefinitely and discount factor of the buyer is bigger than seller,that the negotiation is delayed indefinitely or breaks down is the optimal strategy for seller. In this way, the seller can gain all trade surplus values. The trading platform which the buyer and seller trade is reluctant to see the result of the game, because real-estate brokerage firms can't finish the business any more. If it happens, firms couldn't cover their costs for payment, even worse, the previous efforts will change into a sunk cost. There for some restricted conditions to game process must be made in order to change the result of the game. There is no doubt that these external factors have impacts on the whole results of the game.

Therefore, some restrictions for the game process needs to be set in order to change the outcome of the game, and this will affect and change the final game result. In this paper, the vision of the past research in the field is to apply into two categories main trading body of second-hand house market: the buyer and seller. The paper detailed depicts the real trading game process of two kinds of market players. The research of paper is not only provides practical reference value for third-sector organizations such as second-hand housing transaction brokerage agencies and government regulators, but also offer certain reference and implications for research on the second-hand house market players.

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Some Relaxed Solutions of Minimax Inequality for Discontinuous Game

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Abstract. In this paper, we first prove the existence results of minimax inequality under some relaxed assumptions by virtue of KKMF principle or Fan-Browder fixed point theorem and propose the pseu-solution of minimax inequality. Mild continuity named pseudocontinuity and mild convexity are introduced for the existence results and generalize the present results in the literature. Some other cases of minimax inequality with pseudocontinuity are given in various ways. As applications, we introduce some pseu-Nash equilibrium for n-person noncooperative game and obtain some relaxed existence theorems.

Keywords: Minimax inequality · KKMF lemma · Pseudocontinuity · Diagonal-quasi-concavity · Pseu-solution · pseu-Nash equilibrium

Mathematics Subject Classification (MSC2010): 47H10 · 91A10

1 Introduction

Ky Fan [4] introduced an important inequality named the minimax inequality or Ky Fan inequality which plays a very important role in many fields such as variational inequalities, game theory, mathematical economics, control theory and fixed point theory, etc. The solutions of minimax inequality are also called Ky Fan's points first by Tan, Yu and Yuan in 1995, to see [9]. Because of its wide applications, the existence and stability of Ky Fan's points have been generalized in various ways. A great deal of fruitful results have been achieved on how to improve and apply the important inequality such as [7–9,11,12] and references therein. Peng [8] proved the existence result of weak Ky Fan's point for the functions with no continuity on a non-compact set and Park [7] showed various forms of the minimax inequality by virtue of the KKM principle for a convex space. In [11], Yu obtained the existence of Ky Fan's points for reflexive Banach spaces and its applications to Nash equilibrium points of noncooperative games. From the point of view of the stability, Yu and Xiang [12] proposed the essential components of Ky Fan's points and proved that, there exist at least one essential components of the set of Nash equilibrium points for n-persons noncooperative game.

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In this paper, we introduce the pseudocontinuity and diagonal-quasi-convexity for the existence of Ky Fan's points, and propose the pseu-solutions of minimax inequality. Also we show the new other cases of minimax inequality and generalized minimax inequality to establish some pseu-solution existence results. As applications, we introduce some pseu-Nash equilibrium for n-person noncooperative game and obtain some relaxed existence theorems.

2 Preliminaries

Now let us begin with some definitions and lemmas which we will use.

Definition 2.1 [5]**.** *Let Y be a Hausdorff topological space and let f be a function defined on Y. The function f is said to be lower semicontinuous at $y_0 \in Y$ if and only if*

$$f(y_0) \leq \lim_{y' \to y_0} inf f(y');$$

The function f is said to be upper semicontinuous at $y_0 \in Y$ if and only if

$$\lim_{y' \to y_0} sup f(y') \leq f(y_0).$$

Definition 2.2 [6]**.** *Let Y be a Hausdorff topological space and $f : Y \to R$ be a function.*

(i) f is said to be upper pseudocontinuous at $y_0 \in Y$ if for all $y \in Y$ such that $f(y_0) < f(y)$, we have

$$\limsup_{y \to y_0} f(y) < f(y);$$

f is said to be upper pseudocontinuous on Y if it is upper pseudocontinuous at each y of Y;

(ii) f is said to be lower pseudocontinuous at $y_0 \in Y$ if for all $y \in Y$ such that $f(y) < f(y_0)$, we have

$$f(y) < \liminf_{y \to y_0} f(y);$$

f is said to be lower pseudocontinuous on Y if it is lower pseudocontinuous at each y of Y;

(iii) f is said to be pseudocontinuous at $y \in Y$ if f is both upper pseudocontinuous and lower pseudocontinuous at y; f is said to be pseudocontinuous on Y if f is pseudocontinuous at each y of Y.

Remark 2.1. If f is upper pseudocontinuous on Y, then $-f$ is lower pseudocontinuous on Y.

Remark 2.2. Each upper (resp. lower) semicontinuous function is also upper (resp. lower) pseudocontinuous. But the converse is not true. For example: Let $Y = [0, 2]$, $f_i : Y \to R, i = 1, 2$ be defined as follows:

$$f_1(y) = \begin{cases} 1 - y, & 0 \leq y < 1, \\ -1, & 1 \leq y \leq 2. \end{cases}; \quad f_2(y) = \begin{cases} y, & 0 \leq y < 1, \\ 2, & 1 \leq y \leq 2. \end{cases}$$

We can easily check that f_1 is upper pseudocontinuous but not upper semicontinuous at $y = 1$ and that f_2 is not lower semicontinuous but lower pseudocontinuous at $y = 1$.

Lemma 2.1 [6]. *Let Y be a Hausdorff topological space and $f : Y \to R$ be lower pseudocontinuous, then $\forall b \in f(Y)$, the set $\{y \in Y : f(y) \leq b\}$ is closed.*

Definition 2.3 [5]. *Let X be a nonempty convex subset of Hausdorff topological space E, $f : X \to R$ be a function. $\forall x_1, x_2 \in X, \forall \lambda \in (0, 1)$,*

(i) f is said to be convex function on X if there holds

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2);$$

(ii) f is said to be concave function on X if $-f$ is convex function on X. That means

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda f(x_1) + (1 - \lambda)f(x_2);$$

(iii) f is said to be quasi-convex function on X if there holds

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \max\{f(x_1), f(x_2)\};$$

(iv) f is said to be quasi-concave function on X if $-f$ is quasi-convex function on X. That means

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{f(x_1), f(x_2)\}.$$

Lemma 2.2 [5]. *Let X be a nonempty convex subset of Hausdorff topological space E and $f : X \to R$ be a function, the following hold*

(i) f is a quasi-concave function on X if and only if $\forall r \in R, \{x \in X : f(x) > r\}$ is convex;
(ii) f is a quasi-convex function on X if and only if $\forall r \in R, \{x \in X : f(x) < r\}$ is convex.

Definition 2.4 [13]. *Let X be a nonempty convex subset of a Hausdorff topological space E and $\varphi : X \times X \to R$ be a function.*

(i) For any $x \in X, y \mapsto \varphi(x, y)$ is said to be diagonal-quasi-concave on X if for any finite subset $\{y_1, \cdots, y_n\} \subset X$ and any $y_0 \in Co\{y_1, \cdots, y_n\}$, we have

$$\varphi(y_0, y_0) \geq \min_{1 \leq i \leq n} \{\varphi(y_0, y_i)\}.$$

(ii) For any $x \in X, y \mapsto \varphi(x, y)$ is said to be diagonal-quasi-convex on X if $-\varphi(x, y)$ is diagonal-quasi-concave in the second variable.

Remark 2.3. (i) If $y \mapsto \varphi(x, y)$ is quasi-concave on X for every given x, then $y \mapsto \varphi(x, y)$ is diagonally quasi-concave on X. But the converse does not hold. See ([13]) for a counterexample.

(ii) Since the sum of two quasi-concave functions does not remain quasi-concave in general speaking, the same holds for the property of the diagonally quasi-concave functions.

The following well-known KKMF Lemma is an important generalization of KKM theorem to the infinite dimensional space by Ky Fan.

Lemma 2.3 (KKMF Lemma) [3]**.** *Let X be a nonempty convex subset of Hausdorff topological vector space E, let $F : X \rightrightarrows X$ be a set-valued mapping. For each $x \in X, F(x)$ is closed, and there exists some $x_0 \in X$ such that $F(x_0)$ is compact. If $Co\{x_1, x_2, \cdots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$, where $Co\{x_1, x_2, \cdots, x_n\}$ is the convex hull of $\{x_1, x_2, \cdots, x_n\}$, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

The following fixed theorem is Fan-Browder fixed point theorem.

Lemma 2.4 (Fan-Browder fixed point theorem) [2]**.** *Let X be a nonempty convex and compact subset of a Hausdorff topological vector space E. Suppose a set-valued mapping $F : X \rightrightarrows X$ has the following properties:*

(i) for each $x \in X, F(x)$ is nonempty and convex;
(ii) for each $y \in X$, the inverse valued $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X.

Then F has at least one fixed point.

3 Some Relaxed Solutions of Minimax Inequality

In this section, we first consider the existence of Ky Fan's points for minimax inequality with pseudocontinuous functions on a compact set.

Theorem 3.1. *Let X be a nonempty convex and compact subset of Hausdorff topological space E. The function $\phi : X \times X \to R$ is satisfying:*

(i) for each $y \in X, x \mapsto \phi(x, y)$ is lower pseudocontinuous on X;
(ii) for each $x \in X, y \mapsto \phi(x, y)$ is diagonal-quasi-concave on X;
(iii) for each $x \in X, \phi(x, x) = 0$.

Then there exists $x^* \in X$ such that $\phi(x^*, y) \leq 0$ for any $y \in X$.

Proof. $\forall y \in X$, We define the set-value function $F : X \rightrightarrows X$ as follows

$$F(y) = \{x \in X : \phi(x, y) \leq 0\}.$$

By (iii), we obtain $y \in F(y)$, so $F(y) \neq \emptyset$ and $\phi(y, y) = 0 \in \phi(X, y)$. According to Lemma 2.1, $F(y)$ must be a closed set. Since X is compact set, $F(y)$ is compact.

Next we will prove that F is a KKM mapping, i.e., for any finite subset $\{y_1, y_2, \cdots, y_n\}$ of X, we have

$$Co\{y_1, y_2, \cdots, y_n\} \subset \bigcup_{i=1}^{n} F(y_i).$$

Assume by contradiction that there exists $y_0 \in Co\{y_1, y_2, \cdots, y_n\} \subset X$ and $y_0 = \sum_{i=1}^{n} \alpha_i y_i$ with $\alpha_i \geq 0, i = 1, 2, \cdots, n, \sum_{i=1}^{n} \alpha_i = 1$ but $y_0 \notin \bigcup_{i=1}^n F(y_i)$. Then for any $i = 1, \cdots, n, y_0 \notin F(y_i)$, that is $\phi(y_0, y_i) > 0$. By (ii), we obtain

$$\phi(y_0, y_0) \geq \min_{1 \leq i \leq n} \phi(y_0, y_i) > 0,$$

which is a contradiction with the condition (iii). Then F must be a KKM mapping.

Applying now KKMF Lemma, we have $\cap_{y \in X} F(y) \neq \emptyset$. Take $x^* \in \cap_{y \in X} F(y)$, then $x^* \in F(y)$ for all $y \in X$. Thus we have $\phi(x^*, y) \leq 0$ for any $y \in X$. ■

Remark 3.1. We can also derive Theorem 3.1 by Fan-Browder fixed point theorem. We argue by contradiction. For $\forall y \in X, F(y) = \{x \in X : \phi(x, y) > 0\} \neq \emptyset$, by (ii) $F(y)$ is convex. For $\forall x \in X, F^{-1}(x) = \{y \in X : x \in F(y)\} = \{y \in X : \phi(x, y) > 0\}$, by Lemma 2.1, $F^{-1}(x)$ is open in X. Applying Fan-Browder fixed point Theorem, there exists $y^* \in X$ such that $y^* \in F(y^*)$, i.e., $\phi(y^*, y^*) > 0$. That is a contradiction with the condition (iii).

Remark 3.2. Note that the solution set $S = \{x \in X : \phi(x, y) \leq 0, \forall y \in X\}$. It is easy to see that S is compact. In fact, $S = \cap_{y \in X} \{x \in X : \phi(x, y) \leq 0\} = \cap_{y \in X} F(y)$. Since $F(y)$ is compact for each $y \in X$, S is also compact.

Remark 3.3. We say x^* is a pseu-solution of the function ϕ if x^* is a solution of minimax inequality and ϕ satisfies the condition (i) of Theorem 3.1

From Remarks 2.2 and 2.3 (i), we can obtain the following the corollaries.

Corollary 3.1. *Let X be a nonempty convex and compact subset of Hausdorff topological space E. The function $\phi : X \times X \to R$ is satisfying:*

(i) for each $y \in X, x \mapsto \phi(x, y)$ is lower semicontinuous on X;
(ii) for each $x \in X, y \mapsto \phi(x, y)$ is diagonal-quasi-concave on X;
(iii) for each $x \in X, \phi(x, x) = 0$.

Then there exists $x^* \in X$ such that $\phi(x^*, y) \leq 0$ for any $y \in X$.

Corollary 3.2. *Let X be a nonempty convex and compact subset of Hausdorff topological space E. The function $\phi : X \times X \to R$ is satisfying:*

(i) for each $y \in X, x \mapsto \phi(x, y)$ is lower pseudocontinuous on X;
(ii) for each $x \in X, y \mapsto \phi(x, y)$ is quasi-concave on X;
(iii) for each $x \in X, \phi(x, x) = 0$.

Then there exists at least a pseu-solution x^*, that is $\phi(x^*, y) \leq 0$ for any $y \in X$.

Corollary 3.3. *Let X be a nonempty convex and compact subset of Hausdorff topological space E. The function $\phi : X \times X \to R$ is satisfying:*

(i) for each $y \in X, x \mapsto \phi(x, y)$ is lower semicontinuous on X;
(ii) for each $x \in X, y \mapsto \phi(x, y)$ is quasi-concave on X;
(iii) for each $x \in X, \phi(x, x) \leq 0$.

Then there exists $x^* \in X$ *such that* $\phi(x^*, y) \leq 0$ *for any* $y \in X$*.*

Let X be a nonempty convex and compact subset of Hausdorff topological space E. The function $\psi : X \times X \to R$, if there exists $x^* \in X$ such that $\psi(x^*, y) \geq 0$ for all $y \in X$, x^* is called the solution of equilibrium problem introduced in [1].

From the above, we get the sufficient conditions for the solution of equilibrium problem which is parallel to Theorem 3.1.

Theorem 3.2. *Let* X *be a nonempty convex and compact subset of Hausdorff topological space* E*. The function* $\psi : X \times X \to R$ *is satisfying:*

(i) for each $y \in X, x \mapsto \psi(x, y)$ *is upper pseudocontinuous;*
(ii) for each $x \in X, y \mapsto \psi(x, y)$ *is diagonal-quasi-convex;*
(iii) for each $x \in X, \psi(x, x) = 0$*.*

Then there exists $x^* \in X$ *such that* $\psi(x^*, y) \geq 0$ *for any* $y \in X$*.*

Proof. $\forall x \in X, \forall y \in X$, Set $\phi(x, y) = -\psi(x, y)$. It is easy to check that

(i) for each $y \in X, x \mapsto \phi(x, y)$ is lower pseudocontinuous on X;
(ii) for each $x \in X, y \mapsto \phi(x, y)$ is diagonal-quasi-concave on X;
(iii) for each $x \in X, \phi(x, x) = 0$.

By Theorem 3.1, there exist $x^* \in X$ such that $\phi(x^*, y) \leq 0$ for any $y \in X$. That implies $\psi(x^*, y) \geq 0$ for any $y \in X$. ■

Similarly, we have the following Corollaries.

Corollary 3.4. *Let* X *be a nonempty convex and compact subset of Hausdorff topological space* E*. The function* $\psi : X \times X \to R$ *is satisfying:*

(i) for each $y \in X, x \mapsto \psi(x, y)$ *is upper semicontinuous on* X*;*
(ii) for each $x \in X, y \mapsto \psi(x, y)$ *is diagonal-quasi-convex on* X*;*
(iii) for each $x \in X, \psi(x, x) = 0$*.*

Then there exists $x^* \in X$ *such that* $\psi(x^*, y) \leq 0$ *for any* $y \in X$*.*

Corollary 3.5. *Let* X *be a nonempty convex and compact subset of Hausdorff topological space* E*. The function* $\psi : X \times X \to R$ *is satisfying:*

(i) for each $y \in X, x \mapsto \psi(x, y)$ *is upper pseudocontinuous on* X*;*
(ii) for each $x \in X, y \mapsto \psi(x, y)$ *is quasi-convex on* X*;*
(iii) for each $x \in X, \psi(x, x) = 0$*.*

Then there exists $x^* \in X$ *such that* $\psi(x^*, y) \leq 0$ *for any* $y \in X$*.*

Corollary 3.6. *Let* X *be a nonempty convex and compact subset of Hausdorff topological space* E*. The function* $\psi : X \times X \to R$ *is satisfying:*

(i) for each $y \in X, x \mapsto \psi(x, y)$ *is upper semicontinuous on* X*;*
(ii) for each $x \in X, y \mapsto \psi(x, y)$ *is quasi-convex on* X*;*
(iii) for each $x \in X, \psi(x, x) \leq 0$*.*

Then there exists $x^* \in X$ *such that* $\psi(x^*, y) \leq 0$ *for any* $y \in X$*.*

4 The Other Cases with Pseudocontinuity

In the above section, we discussed the existence results of Ky Fan's points of minimax inequality in the case of nonempty convex compact set. In this section, we will transfer our interests to the existence results of Ky Fan's points of minimax inequality with pseudocontinuity in a noncompact or nonconvex set.

Theorem 4.1. *Let* $X = \bigcup_{n=1}^{\infty} C_n$, *where* C_n *is a sequence nonempty convex compact subset of Hausdorff linear topological space* E, $n = 1, 2, \cdots$, *and* $C_1 \subset C_2 \subset \cdots$. *The function* $f : X \times X \to R$ *satisfies the following conditions:*

(i) for each $y \in X, x \mapsto f(x, y)$ *is lower pseudocontinuous on* X;
(ii) for each $x \in X, y \mapsto f(x, y)$ *is diagonal-quasi-concave on* X;
(iii) for each $x \in X, f(x, x) = 0$;
(iv) for any sequence $\{x_n\} \subset X$ *with* $x_n \in C_n, n = 1, 2, 3, \cdots$, *and for any* $n, \exists x_m \notin C_n$, *there exists a positive integer* n_0 *and a point* $y_{n_0} \in C_{n_0}$ *such that* $f(x_{n_0}, y) > 0$.

Then there exists $x^* \in X$ *such that* $f(x^*, y) \leq 0$ *for any* $y \in X$.

Proof. $\forall n = 1, 2, \cdots$, Since C_n is nonempty convex and compact in X, by Theorem 3.1, there exists $x_n \in C_n$ such that $f(x_n, y) \leq 0$ for any $y \in C_n$.

For the sequence $\{x_n\}_{n=1}^{\infty}$ in X, we will show that there exists a positive integer N_1 such that $\{x_n\}_{n=1}^{\infty} \subset C_{N_1}$. Otherwise, for each n, there exists $x_m \notin C_n$. By (iv) there exists a positive integer n_0 and $y_{n_0} \in C_{n_0}$ such that $f(x_{n_0}, y_{n_0}) > 0$, which contradicts that $f(x_{n_0}, y) \leq 0$ for all $y \in C_{n_0}$.

For any $y \in X$, since $X = \bigcup_{n=1}^{\infty} C_n$, there exists a positive integer N_2 such that $y \in C_{N_2}$. When $n_k \geq N_2$, there holds $f(x_{n_k}, y) \leq 0$. By (i), Lemma 2.1 and $\{x_{n_k}\} \to x^*$, It follows $f(x^*, y) \leq 0$. The proof is thus complete. ■

Remark 4.1. Theorem 4.1 shows existence of a solution of minimax inequality with pseudocontinuity while the set X is not compact but $X = \bigcup_{n=1}^{\infty} C_n$ with compact C_n for $\forall n = 1, 2, \cdots$.

Theorem 4.2. *Let* X *be a nonempty convex closed subset of Hausdorff linear topological space* E. *Suppose the function* $f : X \times X \to R$ *satisfies the following conditions:*

(i) for each $y \in X, x \mapsto f(x, y)$ *is lower pseudocontinuous on* X;

(ii) for each $x \in X, y \mapsto f(x, y)$ *is diagonal-quasi-concave on* X;

(iii) for each $x \in X, f(x, x) = 0$;

(iv) there exists a nonempty compact set $K \subset X$ and $y_0 \in X$ such that $f(x, y_0) > 0$ for any $x \in X \setminus K$.

Then there exists $x^* \in X$ such that $f(x^*, y) \leq 0$ for any $y \in X$.

Proof. For any $y \in X$, Set $F(y) = \{x \in X : f(x, y) \leq 0\}$, by (i) (iii) and Lemma 2.1, $F(y)$ is closed. For any finite subset $\{y_1, \cdots, y_n\}$ of X, we will prove that

$$Co\{y_1, \cdots, y_n\} \subset \bigcup_{i=1}^{n} F(y_i).$$

Assume by contradiction that there exists $y_0 \in Co\{y_1, y_2, \cdots, y_n\}$ and $y_0 = \sum_{i=1}^{n} \alpha_i y_i$ with $\alpha_i \geq 0, i = 1, \cdots, n, \sum_{i=1}^{n} \alpha_i = 1$ but $y_0 \notin \bigcup_{i=1}^{n} F(y_i)$. Then for $i = 1, \cdots, n, y_0 \notin F(y_i)$, i.e., $f(y_0, y_i) > 0$. By (ii), we obtain

$$f(y_0, y_0) \geq \min_{1 \leq i \leq n} f(y_0, y_i) > 0,$$

which is a contradiction with (iii).

By (iv), we know $F(y_0) \cap (X \setminus K) = \emptyset$. Thus $F(y_0) \subset K$ and $F(y_0)$ is compact since K is compact. By KKMF Lemma, we obtain $\cap_{y \in X} F(y) \neq \emptyset$. We take $x^* \in \cap_{y \in X} F(y)$, then

$$f(x^*, y) \leq 0 \quad \text{for} \quad \text{any} \quad y \in X.$$

The proof is thus finished. ■

Theorem 4.3. *Let X be a nonempty convex compact subset of Hausdorff linear topological space E. The two function $f, g : X \times X \to R$ satisfy the following conditions:*

(i) for each $y \in X, \forall x \in X, f(x, y) \leq g(x, y)$;
(ii) for each $y \in X, x \mapsto f(x, y)$ is lower pseudocontinuous on X;
(iii) for each $x \in X, y \mapsto g(x, y)$ is quasi-concave on X;
(iv) for each $x \in X, f(x, x) = g(x, x) = 0$;

Then there exists $x^* \in X$ such that $f(x^*, y) \leq 0$ for any $y \in X$.

Proof. $\forall y \in X$, Set $F(y) = \{x \in X : f(x, y) \leq 0\}$, by (ii) (iv) and Lemma 2.1, $F(y)$ is closed and compact.

$\forall \{y_1, \cdots, y_n\} \subset X$, we will prove that

$$Co\{y_1, \cdots, y_n\} \subset \bigcup_{i=1}^{n} F(y_i).$$

We argue by contradiction that there exists $y_0 \in Co\{y_1, y_2, \cdots, y_n\} \subset X$ but $y_0 \notin \bigcup_{i=1}^{n} F(y_i)$. $y_0 = \sum_{i=1}^{n} \lambda_i y_i$, with $\lambda_i \geq 0, i = 1, \cdots, n, \sum_{i=1}^{n} \lambda_i = 1$. Then for

$i = 1, \cdots, n, y_0 \notin F(y_i), f(y_0, y_i) > 0$. Since $g(\sum_{i=1}^{n} \lambda_i y_i, y_i) \geq f(\sum_{i=1}^{n} \lambda_i y_i, y_i) > 0$ and the condition of (iii), it follows

$$g(y_0, y_0) \geq \min_{1 \leq i \leq n} g(y_0, y_i) > 0,$$

which is a contradiction with (iv).

Thus, by KKMF Lemma, we know $\bigcap_{y \in X} F(y) \neq \emptyset$. We take $x^* \in \bigcap_{y \in X} F(y)$. That implies

$$f(x^*, y) \leq 0 \quad \text{for} \quad \text{any} \quad y \in X.$$

The proof is finished. ■

Remark 4.2. Theorems 4.1, 4.2 and 4.3 generalize Theorem 3.3.2, Theorem 3.3.3, Theorem 3.3.5 of [10], where $f(x, y)$ is lower semicontinuous in the first variable and $f(y, y) \leq 0$ for any $y \in X$.

Theorem 4.4. *Let X be a nonempty convex compact subset of Hausdorff linear topological space E. Suppose the function $f : X \times X \mapsto R$ satisfies the following conditions:*

(i) for each $y \in X, x \mapsto f(x, y)$ is lower pseudocontinuous on X;

(ii) for any $\{y_1, \cdots, y_n\} \subset X, Co\{y_1, \cdots, y_n\} \subset \bigcup_{i=1}^{n} \{x \in X : f(x, y) \leq 0\}$,;

(iii) for each $x \in X, f(x, x) = 0$.

Then there exists $x^* \in X$ such that $f(x^*, y) \leq 0$ for any $y \in X$.

Proof. For any $y \in X$, set $F(y) = \{x \in X : f(x, y) \leq 0\}$, by (i) and Lemma 2.1, $F(y)$ is closed and compact. By (ii) (iii), $\forall \{y_1, \cdots, y_n\} \subset X$, there holds

$$Co\{y_1, \cdots, y_n\} \subset \bigcup_{i=1}^{n} F(y_i).$$

Thus, by KKMF Lemma, we know $\bigcap_{y \in X} F(y) \neq \emptyset$. We take $x^* \in \bigcap_{y \in X} F(y)$. That implies

$$f(x^*, y) \leq 0 \quad \text{for} \quad \text{any} \quad y \in X.$$

The proof is completed. ■

Theorem 4.5. *Let X be a nonempty unbounded closed convex subset of a reflexive Banach space B. If $f : X \times X \to R$ satisfies the following conditions:*

(i) for each $y \in X, x \mapsto f(x, y)$ is pseudocontinuous on X;
(ii) for each $x \in X, y \mapsto f(x, y)$ is diagonal-quasi-concave on X;
(iii) for each $x \in X, f(x, x) = 0$;
(iv) for any sequence $\{x_m\}_{m=1}^{\infty}$ with $\|x_m\| \to \infty$, there exists a positive integer m_0 and a point $y_{m_0} \in X$ such that $\|y_{m_0}\| \leq \|x_{m_0}\|$ and $\phi(x_{m_0}, y_{m_0}) > 0$.

Then there exists $x^* \in X$ such that $f(x^*, y) \leq 0$ for any $y \in X$.

Proof. For each $m = 1, 2, \cdots$, set $C_m = \{x \in X : \|x\| \leq m\}$. We may assume that $C_m \neq \emptyset$. C_m is a bounded closed convex subset in X since X is closed convex subset of a reflexive Banach space B. By Theorem 3.1, there exists $x_m \in C_m$ such that $f(x_m, y) \leq 0$ for all $y \in C_m$.

If the sequence $\{x_m\}_{m=1}^{\infty}$ is unbounded in X, we can suppose that $\|x_m\| \to \infty$ (otherwise subsequence). By (iv), there exists a positive integer m_0 and $y_{m_0} \in X$ such that $\|y_{m_0}\| \leq \|x_{m_0}$ and $f(x_{m_0}, y_{m_0}) > 0$ which is a contradiction with $\|y\| \leq \|x_{m_0}\| \leq m_0, y \in C_{m_0}, f(x_{m_0}, y) \leq 0$. Thus $\{x_m\}_{m=1}^{\infty}$ is bounded in X and there is a positive integer M such that $\|x_m\| \leq M$. Since C_M is bounded and we may assume $x_m \to x^* \in C_M \subset X$.

For any $y \in X$, There is a positive integer K such that $y \in C_k$ and $C_k \subset C_M, y \in C_M, f(x_m, y) \leq 0$ when $m \geq k$. Set $F(y) = \{x \in X : f(x, y) \leq 0\}$, by (i) and Lemma 2.1, then $F(y)$ is closed. Since $x_m \to x^*$, then $x^* \in F(y)$, which implies $f(x^*, y) \leq 0$. The proof is thus completed. ■

5 Some Relaxed Nash Equilibrium

As we all know, minimax inequality plays a very important role in game theory. According to theorems above, we can obtain some existence results of Nash equilibrium for n-persons non-cooperative game.

Let $N = \{1, 2, \cdots, n\}$ be the set of players. For each $i \in N$, let X_i be the strategy set for player i, $X = \prod_{i=1}^{n} X_i$, $f_i : X \to R$ be a payoff function of player i. Every $x \in X$ is denoted by $x = (x_i, x_{\hat{i}})$, where $x_i \in X_i$ and $x_{\hat{i}} \in \prod\limits_{j \neq i} X_j$. This normal form game is denoted by $\Gamma = \{X_i, f_i\}_{i \in N}$. A strategy profile $x^* = (x_1^*, \cdots, x_n^*) \in X$ is called a Nash equilibrium of Γ if for each $i \in N$,

$$f_i(y_i, x_{\hat{i}}^*) < f_i(x_i^*, x_{\hat{i}}^*), \forall y_i \in X_i,$$

and x^* is a pseu-Nash equilibrium of Γ if x^* is a pseu-solution to the minimax inequality corresponding to the function ϕ_Γ, where

$$\phi_\Gamma(x, y) = \sum_{i=1}^{n} (f_i(y_i, x_{\hat{i}}) - f_i(x_i, x_{\hat{i}})), \quad \forall (x, y) \in X \times X.$$

Theorem 5.1. *Let $\Gamma = \{X_i, f_i\}_{i \in N}$ be a game, for any $i = 1, \cdots, n$, let X_i be a nonempty convex and compact subset of Hausdorff topological space E_i, $f_i : X \to R$ satisfy the following properties:*

(i) for any $y = (y_1, \cdots, y_n) \in X, x \mapsto \sum_{i=1}^{n} [f_i(y_i, x_{\hat{i}}) - f_i(x_i, x_{\hat{i}})]$ is lower pseudocontinuous on X.
(ii) for any $x = (x_1, \cdots, x_n) \in X, y \mapsto \sum_{i=1}^{n} [f_i(y_i, x_{\hat{i}}) - f_i(x_i, x_{\hat{i}})]$ is diagonal-quasi-concave on X.

Then there exists at least a pseu-Nash equilibrium of Γ.

Proof. For any $x = (x_1, \cdots, x_n) \in X, \forall y = (y_1, \cdots, y_n) \in X$, denote by

$$\phi(x, y) = \sum_{i=1}^{n} [f_i(y_i, x_{\hat{i}}) - f_i(x_i, x_{\hat{i}})].$$

It is easy to see that $\phi(y, y) = 0$ for any $y \in X$. By the conditions (i) (ii) and Theorem 3.1, there exists $x^* \in X$ such that $\phi(x^*, y) \le 0$ for any $y \in X$. $\forall i \in N, \forall y_i \in X_i$, Set $y = (y_i, x^*_{\hat{i}})$ and $y \in X$.
Then

$$\phi(x^*, y) = f_i(y_i, x^*_{\hat{i}}) - f_i(x^*_i, x^*_{\hat{i}}) \le 0$$

That is

$$f_i(y_i, x^*_{\hat{i}}) \le f_i(x^*_i, x^*_{\hat{i}}) \qquad \forall y_i \in X_i.$$

Therefore x^* is a pseu-Nash equilibrium of Γ. ■

Corollary 5.1. *Let $\Gamma = \{X_i, f_i\}_{i \in N}$ be a game, for any $i = 1, \cdots, n$, let X_i be a nonempty convex and compact subset of Hausdorff topological space E_i, $f_i : X \to R$ satisfy the following properties:*

(i) for any $y = (y_1, \cdots, y_n) \in X, x \mapsto \sum_{i=1}^{n} [f_i(y_i, x_{\hat{i}}) - f_i(x_i, x_{\hat{i}})]$ is lower pseudocontinuous on X;
(ii) for any $x = (x_1, \cdots, x_n) \in X, y \mapsto \sum_{i=1}^{n} [f_i(y_i, x_{\hat{i}}) - f_i(x_i, x_{\hat{i}})]$ is quasi-concave on X.

Then there exists at least a pseu-Nash equilibrium.

Theorem 5.2. *For each $i \in N$, let X_i be a nonempty closed convex subset of a reflexive Banach space B. $f^i : X \to R$ satisfies the following conditions:*

(i) for any $y = (y_1, \cdots, y_n) \in X, x \mapsto \sum\limits_{i=1}^{n} [f^i_j(y_i, x_{\hat{i}}) - f^i_j(x_i, x_{\hat{i}})]$ is lower pseudo-continuous;
(ii) for any $x = (x_1, \cdots, x_n) \in X, y \mapsto \sum\limits_{i=1}^{n} [f^i_j(y_i, x_{\hat{i}}) - f^i_j(x_i, x_{\hat{i}})]$ is diagonal-quasi-concave;
(iii) for any sequence $\{x^m = (x^m_1, \cdots, x^m_n)\}$ with $\|x^m\| = \sum\limits_{i=1}^{n} \|x^m_i\|_i \to \infty$, where $\|x^m_i\|_i$ means the norm of x^m_i in X_i, there exists some $i \in N$, a positive integer m_0 and $y \in X$ such that $\|y_i\| \le \|x^m_i\|_i$ and $f^i(y_i, x^{m_0}_{\hat{i}}) - f^i(x^{m_0}_i, x^{m_0}_{\hat{i}}) > 0$.

Then there exists a pseu-Nash equilibrium of n-persons noncooperative game.

Proof. For any $x = (x_1, \cdots, x_n) \in X, y = (y_1, \cdots, y_n) \in X$, denote by

$$\phi(x, y) = \sum_{i=1}^{n} [f^i(y_i, x_{\hat{i}}) - f^i(x_i, x_{\hat{i}})].$$

It is easy to check

(i) $\forall y \in X, x \mapsto \phi(x, y)$ is lower pseudocontinuous on X;
(ii) $\forall x \in X, y \mapsto \phi(x, y)$ is diagonal-quasi-concave on X;
(iii) $\forall x \in X, \phi(x, x) = 0$;

By (iv), for any sequence $\{x^m = (x_1^m, \cdots, x_n^m)\}$ with $\|x^m\| = \sum_{i=1}^{n} \|x_i^m\|_i \to \infty$, there exist some $i_0 \in N$, a positive integer m_0 and $y_{m_0} \in X_0$ such that $\|y_{i_0}\| \leq \|x_i^{m_0}\|_{i_0}$ and $f^i(y_i, x_{\hat{i}_0}^{m_0}) - f^i(x_{i_0}^{m_0}, x_{\hat{i}_0}^{m_0}) > 0$. Set $y = (y_i, x_{\hat{i}_0}^{m_0})$, then $y \in X, \|y\| \leq \|x^{m_0}\|$, but

$$\phi(x^{m_0}, y) = f^{i_0}(y_{i_0}, x_{\hat{i}_0}^{m_0}) - f^i(x_{i_0}^{m_0}, x_{\hat{i}_0}^{m_0}) > 0.$$

Thus, by Theorem 4.5, there exists $x^* \in X$ such that $\phi(x^*, y) \leq 0$ for any $y \in X$.
That means $x^* \in X$ is a pseu-Nash equilibrium of n-persons noncooperative game. ■

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Dynamic Games of Firm Social Media Disclosure

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Abstract. Firm social media disclosure is a complex game. This paper proposes three stages dynamic games model to analysis the process of social media information disclose. In the first stage model, firms disclose on social media because of low cost and high income, and this can make firms obtain more attention in competition. We introduced investors in the second stage model. Firms disclose exaggeratedly in order to get more benefits from investors in the complete information static game. And investors would not believe social media disclosures and not invest. When reputation model of KMRW is introduced in this stage, the model becomes repeated game with incomplete information. If the game is repeated enough times, the cooperative equilibrium can be achieved. But investors always act in the short run and the model of KMRW does not work. So, the external regulators are introduced in the third stage model. If the benefits which firms get from exaggerated disclosure can be given to the investors through punishment mechanism firms finally disclose truly on social media.

Keywords: Disclosure · Social media · Dynamic games

1 Introduction

With China government's national big data strategy, internet becomes the key industry of economic growth. According to the statistics released by China Internet Network Information Centre, until December 2015, China had a population of cyber citizen for 688 million, instant messenger user for 624 million and the usage rate was 90.7%. And the online news audience number was 562 million, and the usage rate was 82%.

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Micro-blog, We-Chat and other social media are very popular. They modify daily life and working style and dramatically change the way people receive and use message.

Social media also alters the approach of firm disclosure. Firms use social media to spread their news, products and corporate cultures. In the United States, the acceptance rate of social media disclosure is about 70% (Zhang 2015). Firms take advantages of social media to enhance disclosures' frequency, timeliness, coverage radio and to reduce information cost and information asymmetry (Blankespoor et al. 2014). Firms start using Micro-blog and We-Chat in China. Shenzhen Stock Exchange launches "Hu Dong Yi" mobile app of firm user and common user.

However, what determines social media disclosure? How the information disclosed? And what is the impact of relevant parties on the social media disclosure? These problems need to be solved. This paper studies the game situation of interest related parties which disclose their information on social media. We investigate the choice of firm disclosure and analyze the game with the investors and external supervise under perfect condition. The contributions of this paper are as following: first, we use progressive three phase method to analyze multi-party game of firm disclosure and enrich the studies on game theory. Second, we argue that firms will consider their own costs and benefits to determine social media disclosure. Third, the KMRW reputation model is used to analyze the dynamic repeated game between firms and investors. And we argue that dynamic repeated game will make firms disclose real information on social media. However, the short-term nature of investors may lead to reputation model invalid. Fourth, we argue that in the strong regulatory environment firm self-interested behavior of exaggerated disclosure can be curbed. Firms will disclose true information and investors will trust social media disclosure. So under strong supervise social media disclosure can reach Pareto Optimality. Next section analyzes firm social media disclosure by game models.

2 Related Research

Scholars study the motivations (Healy and Palepu 2001), levels (Hooghiemstra et al. 2015), quality (Ecker et al. 2006), impact factors (Welker 1995), market effects (Bushee et al. 2010) of firms disclosure. They believe managers voluntarily disclose for their own interests (Lang and Lundholm 2000). The levels and quality of disclosures depend on the internal and external factors (Muiño and Núñez-Nickel 2016). Most of them believe that disclosures can affect the market and can reduce information asymmetry (Heinle and Verrecchia 2016). Some scholars analyze the decision equilibrium of disclosures from the perspective of game theory. They mainly consider the game equilibrium between the investors and the managements (Verrecchia 1983). And some use game-theory to study the coopetition of insurance agents and insurance firms (Mahito 2012). Most of the existing analyses are based on disclosure from traditional ways. And the game-theory studies are always from the players of the traditional firm disclosure. Fewer people analyze game-theory from the aspect of social media channels.

Social media has been widespread used. Some Scholars believe that people use Twitter to interact, make announcements, provide information, express opinions and share experience (Blankespoor et al. 2014). Some investigate the situation of city police department to release information through Twitter (Heverin and Zach 2010). In the securities market, Scholars have studied XBRL technology. Their research focuses on security issues and their impact on information transparency (Boritz and No 2005; Hodge et al. 2004). Scholars studied the market impact of firms' use of Twitter (Blankespoor et al. 2014). They find social media disclosures can enhance market liquidity and reduce information asymmetry. But voluntary disclosures are difficult to assurance (Koonce et al. 2016). Managers may disclose for their own interests. Sometimes managers exaggerate disclosing and even deceive investors. So, there are a lot of researches on social media, but the analysis from the perspective of game theory is still less.

Thus, there are many researches on firm disclosure and game theory of disclosure. The balance of disclosure lies in the cost and benefits. Scholars also analyze the social media disclosure. But there are still less game theory researches on social media disclosure. We introduce investors, firm managers and regulators in the game model. We analyze the game equilibrium from the perspective of the interests of all parties in social media disclosure. This research has strong practical and theoretical significance.

3 The First Stage Game Model of Social Media Disclosure

We assume that managers face the choice of disclosing on social media or not. If they use traditional channels they have poor timeliness and narrow coverage of information disclosure. The firm gets smaller interests (R1). If managers want to disclose on social media, the gains are greater (R2). The social media channels need costs C_S, but the costs are much smaller than gains. The first stage game model is shown in Fig. 1.

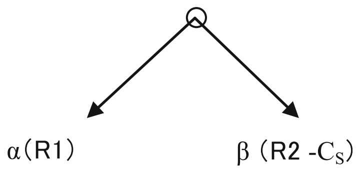

Fig. 1. The first stage game model of social media disclosure

As shown in Fig. 1, the circle represents the choice of the managers. If they do not disclose on social media, the earning is R1. If they disclose on social media, the gain is $R2 - C_S$, because R2 is much larger than R1, and C_S is so small that can be neglected. Therefore we can believe that the optimal choice of managers is to use social media (β strategy) to disclose in order to obtain the maximum benefits.

Firms' decision of social media disclosure also depends on the choice of competitors. We assume that disclosures are true and investors are rational. They can accurately identify firms' disclosures and make right response. Disclosures on social media can get more attention from investors. We assume there are two firms (marked firm1 and firm2) with a competitive relationship (The competition is not only for investors but also in product markets). If the firm discloses on social media while the competitor does not disclose, then the firm can get larger income 4 and competitor only can get 1. If both sides do not disclose on social media they all can get only 2. If they both disclose on social media they both get 3 (Although social media disclosure needs maintenance cost, but it is very low). The payoff matrix of both static games is shown in Table 1. Nash equilibrium for both parties is to disclose on social media and both get 3.

Table 1. Social media disclosure payment matrix

Firm1	Firm2	
	Disclose	Not disclose
Disclose	3,3	4,1
Not disclose	1,4	2,2

So, no matter from the perspective of their own game analysis or from game analysis of considering the competitors firms would disclose on social media in order to obtain more investor attention and trust, and then get more income.

However, social media is not official channel in China. Most of firm disclosures on social media are non-financial or non-significant information. If investors are not mature or the social media coverage is less, investors do not reflect the information rightly. Then the firm would get little gain and lead to $(R2 - C_S) < R1$. The gain deducting cost $(R2 - C_S)$ is less than the benefits of traditional disclosure (R1). Then firms have no incentive to disclose on social media.

4 The Second Stage Game Model of Social Media Disclosure

4.1 The Second Stage Static Game

We introduce investors into the model. Managers can be honest (γ) or exaggerated (δ) on social media disclosure. The second stage game model is shown in Fig. 2.

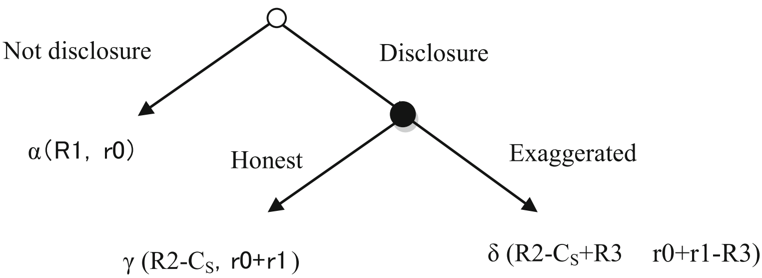

Fig. 2. The second stage game model of social media disclosure

If traditional disclosure, managers obtain income R1, and the investors get return r0. If true disclosure on social media, firms get the gain $R2 - C_S$, investors get the gain r0 + r1. The managers and the investors all get better because of social media disclosure. But if the managers disclose exaggeratedly, they obtain income $R2 - C_S + R3$. The gain R3 comes from investors. So the gain of investors is r0 + r1 − R3. If R3 greatly exceeds the sum of r0 and r1, social media disclosure is harmful to investors. This eventually leads investors not to trust social media disclosure.

4.2 The Second Stage Complete Information Game

If investors can choose to invest or not to invest, the game of second stage becomes more complex and realistic. Assuming that firms face two choices: true or exaggerated disclosures. And investors face two choices: collecting information and investing or not collecting information through social media and not investing.

The choices and payments are shown in Table 2. Supposing firms disclose and investors collect information on social media. Firms and investors both get 5. But if firms truly disclose and investors do not collect through social media, firms get lost of −2 because of the high cost of real disclosure. And investors get −2 because they do not invest so produce regret costs.

Table 2. Social media disclosure payoff matrix for companies and investors

Firms	Investors	
	Collect information through social media and invest	Not collect information through social media and not invest
True	(5, 5)	(−2, −2)
Exaggerated	(8, −3)	(−1, 0)

If firms disclose exaggeratedly and investors collect information through social media, firms can get a bigger payment of 8. The revenue is 3 more than the actual disclosure. This comes from investors' losses. Investors get −3 because they gather exaggerated information and have been cheated. If investors do not collect information through social media and do not invest then the exaggerated disclosure could not play a role and investors get 0. And firms get lower cost of −1 because they do not need to disclose diligently like truly disclosing.

In this payment matrix, the dominant strategy of firms is exaggerated disclosure. Investors have no dominant strategy and they will choose a mixed strategy. If firms disclose exaggeratedly investors do not collect information on social media and not invest. If firms disclose truly the best choice of investors is to collect information on social media and invest. When firms and investors carry one-shot game, the game becomes complete information static game, both sides know each other's choices and payments, the result of the game is that firms disclose exaggeratedly and investors know the dominant choice of firms they do not collect on social media and not invest. Ultimately, firms do not disclose on social media and social media disclosures have not beneficial effect on information disclosure.

4.3 The Second Stage Incomplete Information Repeated Game

If the information is incomplete, investors do not know whether firms are true or exaggerated and they only know the probability. And we assume that the game is repeated. So we can introduce KMRW reputation model. This model is established by Kreps, Milgrom, Roberts and Wilson in 1982, and it is an incomplete information repeated game model. They believe that one party's information about the other party's payoff function and policy space are incomplete. This incomplete information has an important influence on the equilibrium result of the game. As long as the game is repeated enough cooperative behavior will appear in a limited number of games.

Firms disclose exaggeratedly in a single game if they are rational (F). They disclose truly if they are irrational (T). The probabilities of rational and irrational disclosure are (1 − p) and p. And investors are rational they will choose the tit for tat strategy. That is, if firms disclose truly, investors believe the social media and invest, if firms disclose exaggeratedly, investors do not believe social media and not invest.

The order of the game is as follows:

(1) The natural Select the type of firms: The firms know their type but investors only know the probabilities of rational and irrational disclosure.
(2) Firms and investors carry the first game and after observing the results of the first game, they begin second game. And after observing the results of the second game, they begin the third game and so on.
(3) The game payoff of the firms and the investor is the discounted sum of all the games (Assume the discount factor is 1).

When the game is repeated twice (t = 2), as with the complete information game, in the last stage (t = 2), rational firms disclose exaggeratedly (F), investors do not invest (N). In the first stage, assume that irrational firms choose T. The choice (X2) of the second stage depends on the investor's choice(X1) of the first stage. If investors collect information through social media and invest (Y), firms choose T and else choose F (As show in Table 3).

Table 3. The game repeated two times

Players	t = 1	t = 2
Irrational company	T	X2
Rational company	F	F
Investor	X1	N

If investors invest in the first stage (X1 = Y) and the expected return of investors is:

$$[p \times 5 + (1 - p) \times (-3)] + [p \times (-2) + (1 - p) \times 0] = 6p - 3$$

Among them, the first item on the left is the expected income of the first stage, the second item on the left is the expected income of the second stage.

If investors do not invest in the first stage (X1 = N) and the expected return of investors is:

$$[p \times (-2) + (1 - p) \times 0] + 0 = -2p$$

If 6p − 3 > −2p, i.e., p > 3/8, investors choose X1 = Y. In other words, when the probability of firm belongs to irrational (true disclosure) is not less than 3/8, investors would choose to invest.

If the game is repeated three times (T = 3), suppose p > 3/8. Firms and investors in the first stage would disclose truly and invest. The equilibrium path of the second stage and the third stage are the same as Table 3. The total path is shown in Table 4.

Table 4. The game repeated three times

Players	t = 1	t = 2	t = 3
Irrational company	T	T	T
Rational company	T	F	F
Investor	Y	Y	N

In the first stage, when both sides of the game know they will play three times, F is not the firm's optimal strategy because this exposes the type of firm is rational although the choice F in the first stage will give the firm gain of 8 (if investors choose Y). And investors will choose N in the second stage. Then the biggest benefit of a rational firm in the second stage is −1. But if firms choose Y and do not expose themselves they may get 5 in the first stage and get 8 in the second stage.

Supposing investors choose Y in the first stage. If the firm chooses T the posterior probability of investors will stay invariant. In the second and the third stage investors select Y and N. The expected earning of rational firm is: 5 + 8 − 1 = 12.

If rational firm chooses F in the first stage, it exposes itself. Investors will choose N in the second and third stages. The expected earning of rational firm is: 8 − 1 − 1 = 6.

Because 12 > 6, so the best choice of rational firm in the first stage is T.

There are three strategies of investors: (Y, Y, N), (N, N, N), (N, Y, N). Assuming that rational firms choose T in the first stage (second and third stage select F) and investor's choice is (Y, Y, N). The expected return of investor is:

$$5 + [p \times 5 + (1 - p)(-3)] + [p(-2) + (1 - p) \times 0] = 6p + 2$$

If investors choose (N, N, N), the expected return is:

$$-2 - 0 - 0 = -2$$

So if 6p + 2 > −2 (p > −2/3), then (Y, Y, N) is better than (N, N, N).

If investors choose (N, Y, N), the expected return is:

$$-2 - 3 + [p(-2) + (1 - p) \times 0] = -5 - 2p$$

And If 6p + 2 > −5 − 2p, i.e. p > −7/8, then (Y, Y, N) is better than (N, Y, N).

So, as long as the probability of firm's irrational is greater than 3/8 and the strategy combination of Table 4 is a refined Bias equilibrium. Rational firms choose T in the first stage, and then select F in the second and third stages. Investors choose Y in the first and second stages, and choose N in the third stage. As long as T > 3 and investors choose Y, then the cooperation situation which firms choose T will appear. The non cooperative stages are 2, which are not related to the number of T.

Of course, if p < 3/8, cooperative equilibrium may not appear. Although the strategy of investors' willingness and firm's disclosure are private information, as long as the probability of irrational is more than zero (p > 0) and the number of repeated game enough, the cooperative equilibrium will eventually appear. That is, as long as there are enough games between firms and investors, no matter how low their irrational probability is firms will eventually disclose truly on social media. And Investors will also gather information and invest. Finally they can achieve cooperative equilibrium.

So, if the game is repeated firms in order not to expose their type early and obtain a higher income in the long run they will disclose truly in the previous stage. Investors will trust and invest. Ultimately they can reach a cooperative equilibrium and social media disclosure is effective.

But there are many individual investors in China. They always make short-term investments. So the investors which firms facing are not always the same. Managers may assume that investors are different in different stages. Firms would disclose exaggeratedly on social media to maximize their gains. From the overall and long-term perspective, investors will not believe social media disclosure after being cheated many times. And ultimately this leads social media disclosure invalid.

5 The Third Stage Game Model of Social Media Disclosure

5.1 Add Regulators in the Game

Under the premise of no other parties interfere, firms disclose on social media based on their own interests, and their nature of pursuit interests leads to exaggerated way to deceive investors and get the gain of investors. To make social media disclosure more feasible for all parties, it is necessary to introduce regulation. Regulators have two choices: punish (ε) or not punish (η). The third stage game model is shown in Fig. 3.

If supervision is loose, the regulators would regardless of the exaggerated behavior on social media. Then the managers get R2 − CS + R3 and investors only get r0 + r1 − R3. Social media disclosure does not benefit the investors. If regulators strict managers and severely punish those exaggerated behavior and give the gain of punishment to the investors the revenue of managers is R2 − CS + R3 − R4 and the investors get r0 + r1 − R3 + R4. Generally speaking, R4 is more than R3 which makes the managers do not want to exaggerate. Because the return is R2 − CS when they disclose truly on social media, which is more than the return of being punished. When the probability of punishment is high managers dare not to disclose

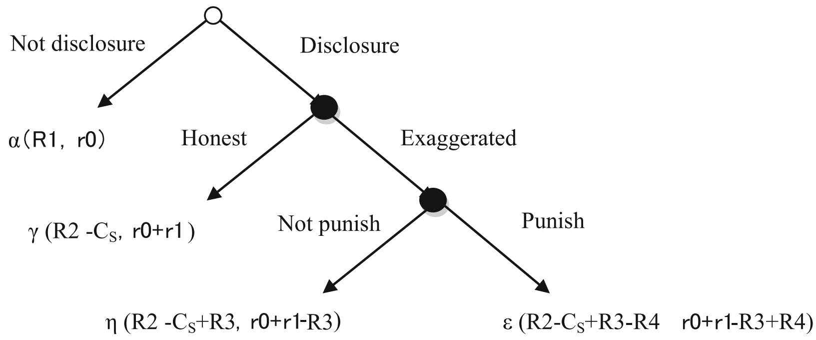

Fig. 3. The third stage game model of social media disclosure

exaggeratedly. And the result of the game is in the state of γ which is true disclosure. The state has become the best choice. Social media disclosure can expand the interests of managers and investors. This obtains the Pareto optimal and improves the market efficiency.

However, if R4 is less than R3 the condition is different. The illegal cost of firms' exaggerated disclosure is too low or the penalty amount is too small. When investors want to take legal action, their cost of collecting evidence is too large or the amount of compensation is too small. This leads managers continue to disclose exaggeratedly and investors do not believe social media disclosure. Social media could not play a role in improving the market efficiency of information disclosure.

5.2 Game Analysis of Regulators and Firm Social Media Disclosure

The analysis above does not consider the gains and losses of regulators. If considering the restriction of the regulators we get social media disclosure game between regulators and firms (Table 5). Assuming that the earnings of firms' exaggerated disclosure are a ($a > 0$). And true disclosure need to pay a greater cost, the earnings are −a. If exaggerated disclosure has been found the penalty is D ($D > 0$) and litigation losses are R. Regulator's regulatory costs are c ($c > 0$). If firms disclose truly and regulators do not regulate, regulatory costs c have been saved. It is similar to opportunity costs. If firms disclose exaggeratedly regulators have supervised and found no problems. Firms gain and regulators cost. When firms disclose exaggeratedly and regulators do not regulate they would be reported. Regulators lose L which contains X and Y. X is the image loss in the public and Y is the punishment of their neglect of duty. Thus the payment which regulators do not supervise is $-L(X, Y)$, and L is an increasing function of X and Y ($L > 0$). If there is no report regulators have no cost and loss. There will be no sense of cost savings. That is to say there is no opportunity cost.

When firms disclose exaggeratedly and have been found by regulators they would be fined and compensation, the paid is $-D - R$. Regulators obtain the fine and pay the

Table 5. Payment matrix regulatory authorities and listed companies

Firms	Regulatory authorities			
	Regulate		Not regulate	
	Found	Not found	Reported	Not reported
Exaggerated	(–D–R, D–c)	(a, –c)	(a, –L(X,Y))	(a, 0)
True	(–a, –c)	(–a, –c)	(–a, c)	(–a, c)

regulatory cost. Their gain is D − c. If regulators have regulated but did not find exaggerated disclosure firms get gain "a" and regulators paid regulatory cost. When firms disclose exaggeratedly and regulators do not regulate, firms earning is a. And if regulators do not regulate and have been reported the payment is −L(X, Y) and if have not been reported the payment is 0. When firms disclose truly the payment is –a whether the regulators monitored or not. If regulators regulate the regulatory cost is –c. If regulators do not regulate they save regulatory costs and the payment is c.

We assume that the game is a static game with incomplete information and the probability of exaggerated and true disclosure are p and 1 − p. The probability of regulation and non regulation are q and 1 − q. The probability of finding and not finding the problem are r and 1 − r when regulation. The probability of being and not being reported are w and 1 − w when regulators do not regulate.

The expected return of the regulator is:

$$\begin{aligned} \Pi 1 = {} & q \cdot \{r \cdot [p \cdot (D - c) + (1 - p) \cdot (-c)] + (1 - r) \cdot [p \cdot (-c) + (1 - p) \cdot (-c)]\} \\ & + (1 - q) \cdot \{w \cdot [p \cdot (-L)] + (1 - p) \cdot c] + (1 - w) \cdot [p \cdot 0 + (1 - p) \cdot c]\} \end{aligned} \tag{1}$$

The partial derivatives of the formula (1) for q, and then let it equals to zero. We can get the first order condition of regulator's maximizing expected return:

$$\partial \Pi 1 / \partial q = 0;$$

Then the equilibrium condition of the regulator is:

$$p* = 2c/(rD + wL + c) \tag{2}$$

Because 0 $\leq$ p $\leq$ 1, that is: 0 $\leq$ c $\leq$ rD + wL

Thus, When the probability of exaggerated disclosure on social media is p < 2c/(r D + w L + c) and the optimal strategy for regulators is "non regulation". When the probability is p > 2c/ (r D + w L + c) and the optimal strategy for regulators is "regulation". When the probability is p = 2c/ (r D + w L + c) and the optimal strategy of regulators is to carry out "regulation" and "non regulation" randomly.

The expected return of firm is:

$$\begin{aligned} \Pi 2 = p \cdot \{q \cdot [r \cdot (-D - R) + (1 - r) \cdot a] + (1 - q) \cdot [w \cdot a + (1 - w) \cdot a]\} \\ + (1 - p)\{q \cdot [r \cdot (-a) + (1 - r) \cdot (-a)] + (1 - q) \cdot [w \cdot (-a) + (1 - w) \cdot (-a)]\} \end{aligned} \tag{3}$$

The partial derivatives of the formula (3) for p, and then let it equals to zero. We can get the first order condition of firm's maximizing expected return:

$$\partial \Pi 2 / \partial p = 0;$$

Then the equilibrium condition of firm is:

$$q* = 2a/[r(D + R + a)] \tag{4}$$

Because: $0 \leq q \leq 1$ and R, D, a, r are greater than zero.
That is:

$$0 \leq 2a/[r(D + R + a)] \leq 1, \quad 0 \leq a \leq (rD + rR)/(2 - r)$$

When the probability of regulators' regulating in social media is $q < 2a/[r(D + R + a)]$ the optimal strategy of firm is "exaggerated disclosure". When the probability is $q > 2a/[r(D + R + a)]$ and the optimal strategy of firm is "real disclosure". When the probability is $q = 2a/[r(D + R + a)]$ and the optimal strategy of firm is to carry out "exaggerated disclosure" and "real disclosure" randomly.

Simultaneous Eqs. (2) and (4) we can calculate the mixed Nash equilibrium of the game:

$$S^* = (S1^*, S2^*) = \{2c/(rD + wL + c), 2a/[r(D + R + a)]\} \tag{5}$$

The equilibrium shows: When the probability of regulation is $q* = 2a/[r(D + R + a)$ and the probability of firms exaggerated disclosure is $p* = 2c/(rD + wL + c)$ the expected utilities of both sides are maximization. Of course, this also shows that there are $2a/(r(D + R + a))$ ratio of firms exaggerated disclosure. And if regulators carry out random regulation there are $2c/(rD + wL + c)$ ratio of firms real disclosure on social media.

At the same time, from the Eq. (2) we can get the following conclusions. The equilibrium probability of firms' exaggerated disclosure is p* which is inversely proportional to D, r, w and L. This shows that if increasing the punishment of exaggerated disclosure, the level of supervision, the punishment and the loss of regulators' image, the probability of regulators being reported we can reduce the probability of exaggerated disclosure. The grassroots and interactive of social media would greatly enhance the probability of regulator's being reported and this reduce the probability of exaggerated disclosure. And p* is proportional to c. This means the higher the supervision cost, the greater the probability of firm's exaggerated disclosure on social media.

Thus, reducing regulatory costs c, increasing the cost of firm's exaggerated disclosure D and enhancing the regulatory costs of dereliction of duty are conducive to reduce the probability of exaggerated disclosure. We can improve the efficiency of supervision to reduce the cost of supervision. And we also can prohibit disclosure on social media to reduce the cost of supervision, but this is an extreme measure which like give up eating for fear of choking.

From the Eq. (4) we can get the following conclusions. The equilibrium probability of regulation is q* which is inversely proportional to D, R and "r". This shows that if increasing the punishment of exaggerated disclosure D and the amount of compensation the probability of supervision will reduce. Improve the level of supervision and the probability of finding the problem "r", we can reduce the probability of supervision. q* is proportional to a. This means the greater the gain of exaggerated disclosure the greater the probability of supervision. It can be seen that the reduction of firm's exaggerated disclosure and firm's operating cost can reduce the probability of supervision. And they can reduce the workload of regulators.

6 Conclusions and Implications

6.1 Research Conclusion

Social media makes the cost lower, the speed faster and the range broader of information disclosure. In the first stage game model the players of the game are only firms. Considering their own interests and the industry competition firms take the advantage of social media to obtain higher market attention and gain.

We add investors in the second stage game model. Under the premise of one-off game, managers disclose exaggeratedly and can get more returns. Investors do not believe social media disclosure after being cheated. This eventually leads social media disclosure become lemon market. If firms consider long-term interests the games are repeated games of continuous disclosure and acceptance of information. After introducing the reputation model we find that if firms maximize the income of the first game at the beginning, and after investors have damaged, they would no longer believe social media disclosure. And this would make firms' total revenue become lower. But if firms do not expose their types at the beginning and choose true disclosure, investors would believe social media disclosure and buy the stock. So in the long run, firms make greater profits.

Therefore, the dynamic repeated game with the consideration of firms' reputation and investor's response can achieve the equilibrium of the game. Firms disclose truly on social media and investors believe and invest. But if investors appear short-term behavior and firms will not consider their reputation and they will disclose exaggeratedly at the first time which results the investors suffering loss. Eventually the entire social media disclosure becomes lemon market and investors do not believe it and firms' disclosure could not reach the desired effect.

Thus, due to the possibility of short-term behavior of investors and firms the introduction of regulation in the model is necessary. In the third stage game model we think that as long as the regulation can let the interests occupied by the firms return to the investors no matter what form of regulation is. If exaggerated firms lose more to compensate investors, social media disclosure is feasible and can achieve the Pareto optimal and make firms and investors all benefit.

We further consider the cost and return of regulators. We find that the regulators' supervision depends on the firms' probability of exaggerated disclosure on social media. If the probability of exaggerated disclosure is high the regulators increase supervision. Exaggerated disclosure depends on the regulations. If the regulation is greater the probability of exaggerated disclosure is small. In addition, improving the efficiency of regulators, reducing the cost of regulation, strengthening regulatory penalties will also urge firms to disclose truly. So, in a strong regulatory environment firms use social media to disclose information will promote them to disclose true information, which will benefit both firms and investors.

6.2 Inspirations and Suggestions

Social media improves the efficiency of disclosure, but this needs firms, investors and regulators work together. The inspirations and suggestions are as follows:

Firstly, firms should actively embrace social media. Social media allows firms get rid of the shackles of traditional media and face investors directly. This is a revolution of information disclosure. Social media has the characteristics of low cost, timely, interactive and extensive coverage. So, firms should actively embrace social media and get the attention and trust of investors which can reduce the cost of capital and improve the market value of firms.

Secondly, we should improve the quality of investors. Because of the low entry barriers all kinds of information (bad and good) mixed together on social media. Investors should have the ability to identify information and are careful to the information disclosed by firms on social media. Investors should select the firms which disclose truly so that there is no market for exaggerated disclosure on social media.

Thirdly, we must vigorously develop social media disclosure and strengthen the supervision. Social media disclosure is a new channel. Managers disclose all kinds of information through social media which have significant impact on stock market. The characteristics of social media can make information disclosure more effective. Therefore, we should encourage the healthy development of it. The contents of social media disclosure are in huge volume including voluntary and compulsory information. The regulators could not have the powerful energy to supervise all aspects. So this requires the perfection of the legal system. Only playing the initiative of investors and exerting the authority of law we can truly kill the violation of laws and regulations of the disclosure. And this can promote the healthy and steady development of social media disclosure and can improve the efficiency of the securities market.

The parties to the game of social media disclosure may be more complex. We have done a detailed analysis of the situation, but we could not take all the parties and all kinds of circumstances into consideration. These needs to be further improved and revised in the future work.

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Non-cooperative Games Under Uncertainty

On Stochastic Fishery Games with Endogenous Stage-Payoffs and Transition Probabilities

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Abstract. We engineered a stochastic fishery game in which overfishing has a twofold effect: it gradually damages the fish stock inducing lower catches in states *High* and *Low*, and it gradually causes the system to spend more time in the latter state with lower landings.

To analyze the effects of this 'double whammy' technically, we demonstrate how to determine the set of jointly-convergent pure-strategy rewards supported by equilibria involving threats, under the limiting average reward criterion.

Keywords: Stochastic games · Limiting average rewards · Endogenous transition probabilities · Endogenous stage payoffs

1 Introduction

We introduce a new *Small Fish War*[1] featuring endogenous stage-payoffs and endogenous transition probabilities, thus combining two earlier distinct research lines, e.g., [31,32,34,36]. Our present analysis benefits from auxiliary research on improving efficiency of algorithms enabling an analysis of such stochastic games [37].

In a(ny) *Small Fish War*, several (here two) agents possess the fishing rights on a body of water. The agents have several (here two) options, to fish with or without restraint. Fishing with restraint by the agents is (assumed to be) sustainable in the long run, as the resource is (assumed to be) able to recover; unrestrained fishing yields higher immediate catches to the agent(s) overfishing than restrained fishing, but harms the resource.

The game is played at discrete points in time called stages and at each stage the resource is in one of two states, *High* or *Low*. The system is subject to random transitions between these states induced by phenomena with well established distributions, e.g., weather, influencing the stocks. In *High*, the fish are more abundant and catches are larger than in *Low*.

The damage to the resource by overexploitation becomes apparent in the sizes of the landings, as these (may) decrease (significantly). Furthermore, overfishing

[1] A word play on [41] who show that strategic interaction in a fishery may induce a 'tragedy of the commons' [21].

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may affect the stochastic dynamics of the resource system resulting in a higher proportion of time being spent in *Low*.

The first manifestation of damage by overfishing has been studied in [31,32,34]. The common denominator of results was in line with results from similar models of the tragedy of the commons: sufficiently patient agents avoid depletion of the resource in equilibrium by diligently managing it. Even stronger, Pareto efficient equilibrium outcomes rely on maintaining rather high fish stocks for high proportions of time.

However, this 'good news' result depends on conditions not necessarily met by real-world systems, and further research relaxed several of these. Joosten [32,33] examined the influence of lack and asymmetry of patience, of information on own time-preferences and those of the opponent(s), on the state and the dynamics of the system, and on equilibrium behavior and the distribution of long-term incomes.

Joosten [34] added rarity value, a complex price-scarcity mechanism suggested by biologists [9], to a *Small Fish War*. Under strong rarity value Pareto-efficient equilibrium behavior of patient agents induces a 'tragedy of the herd' accompanied by a 'manna for the shepherd.' Increases in prices under increased scarcity of the resource overcompensate the effects of overfishing on the sizes of the landings as well as of the increased search costs involved. A real-world example of such a phenomenon may be the blue-fin tuna, which saw prices in the Far East soar over decades.[2]

The second sign of damage to the resource has been studied in [36] to analyze hysteresis (e.g., [8,9,18]), i.e., recovery programs do not show effects in replenishing stocks decimated by overfishing for long periods of time. Mature cod spawn a considerably higher number of eggs than younger ones, cf., e.g., [40]. Estimations suggest linear fecundity-weight relations [2,48], or even exponential ones [52]. Modern catching methods target mature cod, cohorts most productive in providing offspring. To regain full reproductive capacity, younger cohorts must reach ages well beyond adulthood.

We engineered a stochastic game[3] in which overfishing has a detrimental effect on the underlying payoff structure and on the associated transition probabilities between states. To start with the latter, Nature moves the play from one state to the other dependent on the current action choices of the agents, but also on their past catching behavior. We designed *endogenously changing* stochastic

[2] On 01-31-2017, fresh blue fin tuna from Japan registered prices on the Tsukiji wholesale market 8–10 times those for fresh herring, 6–8 times the price for salmon and roughly 16–20 times the price for pollock. Yellowfin, bigeye and southern bluefin tuna did between a quarter and half of the price of the top bluefin tuna (http://www.st.nmfs.noaa.gov/st1/market_news/japan-wholesale.txt. on 02-08-2017). On special New Year auctions prices of more than $1 M have been recorded for a 222 kg blue fin tuna in 2013. Public outcry caused prices to drop after then, but in 2017, a 210 kg fish fetched between $600,000 and $866,000 (internet does not agree on prices).

[3] 'Engineered' as in [3]. Stochastic games were introduced in [54], see also [1] for links to difference and differential games to which much work on fisheries belongs, cf., e.g., [23,42] for overviews.

variation.[4] The probability of moving to *High* decreases in time in each state and for each action combination if the agents show prolonged lack of restraint, i.e., overfish frequently. We designed an *endogenously changing* payoff structure as well, i.e., if the agents overfish more frequently, then the stage, i.e., immediate, payoffs decrease over time.

The agents are assumed to wish to maximize their long term average catches and we formalized this by using the limiting average reward criterion to evaluate the infinite stream of stage payoffs. We adopt a Folk Theorem type analysis and validate relevant procedures. First, we show how to establish the rewards for any pair of jointly-convergent pure strategies. Then, we determine the set of jointly-convergent pure-strategy rewards. Next, we establish for each player the threat point reward, i.e., the highest amount this player can guarantee himself if his opponent tries to minimize his rewards. Finally, we obtain a large set of rewards which can be supported by equilibria using threats, namely all jointly-convergent pure-strategy rewards giving each player at least his threat point reward.

Mahohoma [43] independently studied a model which is a special case of ours. This work differs from ours in that a more inductive approach is used by generating long sequences of play resulting from various types of strategies conditioned on several levels of memory sizes to record the history of play. For these sequences, the average rewards are determined and compared to threat point rewards from strategies exhibiting the exact same type having exactly the same levels of memory size. However, Mahohoma [43] remains silent on the theme of equilibria.

As one important objective of Game Engineering is to draw lessons from analyzing the games designed, we analyze one example of this new type of *Small Fish War* to obtain insights relevant to the management of such a renewable resource system. Our findings reveal that ecological and economic goals may be served simultaneously. Here, full restraint by the agents yields Pareto-optimal rewards, i.e., total rewards are maximized globally, giving 5.15 times the total never-restraint rewards. Meanwhile, the lowest equilibrium fish stock is about 53% of the maximum (i.e., full restraint) fish stock whereas permanent overfishing (never restraint) reduces the resource to about 35% of the same.[5] So, a plausible ecological maximalistic goal coincides perfectly with a plausible economic maximalistic one.

Next, we introduce our model with endogenous stage payoffs and endogenous transition probabilities. In Sect. 3, we focus on strategies and restrictions desirable or resulting from the model. Section 4 treats rewards in a very general sense, and equilibrium rewards more specifically. Also some attention is paid to the complexity of computing threat point rewards. Section 5 concludes.

2 Endogenizing Payoffs and Transitions

A *Small Fish War* is played by row player A and column player B at discrete moments in time called stages. Each player has two actions and at each stage

[4] So, the Markov property of standard stochastic games ([54]) is lost.

[5] We refer to Examples 2 and 3 for a motivation of these numbers.

$t \in \mathbb{N}$ the players independently and simultaneously choose an action. Action 1 for either player denotes the action for which some restriction exists allowing the resource to recover, e.g., catching with wide-mazed nets or catching a low quantity. Action 2 denotes the action with little restraint. Logic dictates that in every state, fishing with restraint always yields lower immediate catches to an agent than fishing without.

We assume catches to vary due to random shocks, which we model by means of a stochastic game with two states at every stage of the play. As we have both the stage payoffs and transition probabilities being determined endogenously, i.e., by past play, let us first capture this formally. The past play until stage $t, t > 1$, is captured by the following two matrices

$$QH^t = \begin{bmatrix} q_1^t \ q_2^t \\ q_3^t \ q_4^t \end{bmatrix} \text{ and } QL^t = \begin{bmatrix} q_5^t \ q_6^t \\ q_7^t \ q_8^t \end{bmatrix}.$$

Here, e.g., q_1^t is the relative frequency with which action pair top-left in *High* has occurred until stage t, and q_7^t is the relative frequency of action pair bottom-left in *Low* having occurred during past play. So, we must have $q^t = (q_1^t, ..., q_8^t) \in \Delta^7 = \{x \in \mathbb{R}^8 | x_i \geq 0 \text{ for all } i = 1, ..., 8 \text{ and } \sum_{j=1}^8 x_j = 1\}$.

A formal way of writing down these eight relative frequencies recording past play is the following where $j = 1, 2,\ t > 1$:

$$\begin{aligned} q_j^t &\equiv \tfrac{\#\left\{\left(j_u^{A,H}, j_u^{B,H}\right) \mid\ j_u^{A,H}=1,\ j_u^{B,H}=j,\ 1 \leq u < t\right\}}{t-1}, \\ q_{j+2}^t &\equiv \tfrac{\#\left\{\left(j_u^{A,H}, j_u^{B,H}\right) \mid\ j_u^{A,H}=2,\ j_u^{B,H}=j,\ 1 \leq u < t\right\}}{t-1}, \\ q_{j+4}^t &\equiv \tfrac{\#\left\{\left(j_u^{A,L}, j_u^{B,L}\right) \mid\ j_u^{A,H}=1,\ j_u^{B,H}=j,\ 1 \leq u < t\right\}}{t-1}, \\ q_{j+6}^t &\equiv \tfrac{\#\left\{\left(j_u^{A,L}, j_u^{B,L}\right) \mid\ j_u^{A,H}=2,\ j_u^{B,H}=j,\ 1 \leq u < t\right\}}{t-1}. \end{aligned}$$

Here, $j_u^{A,X}$ $(j_u^{B,X})$ denotes the action taken by player A (B) while being in state $X = H, L$ at stage u. So, for instance q_4^t is the relative frequency of action pair $(2, 2)$ in state H being chosen until stage t. We refer to such a vector q^t as the **relative frequency vector**.

Let the interaction at stage t of the play be represented by the following two matrices:

$$H^t = \begin{bmatrix} \theta_1 (q^t), p_1 (q^t) \ \theta_2 (q^t), p_2 (q^t) \\ \theta_3 (q^t), p_3 (q^t) \ \theta_4 (q^t), p_4 (q^t) \end{bmatrix},$$

$$L^t = \begin{bmatrix} \theta_5 (q^t), p_5 (q^t) \ \theta_6 (q^t), p_6 (q^t) \\ \theta_7 (q^t), p_7 (q^t) \ \theta_8 (q^t), p_8 (q^t) \end{bmatrix}.$$

Here H^t (L^t) indicates state *High* (*Low*) at stage t of the play. Each entry of the two matrices contains an ordered pair denoting the pair of immediate payoffs to the players $\theta_i (q^t) = \left(\theta_i^A (q^t), \theta_i^B (q^t)\right)$ if the corresponding action pair is chosen and the probability $p_i (q^t)$ that the system moves to *High* at stage $t + 1$. All functions $p_i : \Delta^7 \to [0, 1]$, $\theta_i : \Delta^7 \to \mathbb{R}_+ \cup \{0\}$ are assumed continuous.

To make sense in the framework studied, we have to impose several restrictions on the numbers and functions above. The first one relates to the catches in *Low* being universally lower than in *High* for comparable action combinations of the agents

$$\theta_i\left(q^t\right) \geq \theta_{i+4}\left(q^t\right) \text{ for } i = 1,2,3,4 \text{ and all } q^t \in \Delta^7.$$

With respect to restraint yielding lower landings than fishing without restraint, logic dictates that we must have for all $q^t \in \Delta^7$

$$\theta_1^A\left(q^t\right) \leq \theta_3^A\left(q^t\right), \theta_2^A\left(q^t\right) \leq \theta_4^A\left(q^t\right), \theta_5^A\left(q^t\right) \leq \theta_7^A\left(q^t\right), \theta_6^A\left(q^t\right) \leq \theta_8^A\left(q^t\right),$$

$$\theta_1^B\left(q^t\right) \leq \theta_2^B\left(q^t\right), \theta_3^B\left(q^t\right) \leq \theta_4^B\left(q^t\right), \theta_5^B\left(q^t\right) \leq \theta_6^B\left(q^t\right), \theta_7^B\left(q^t\right) \leq \theta_8^B\left(q^t\right).$$

If overfishing is to yield decreasing catches over time, the functions θ_i^A, θ_i^B, $i = 1, ..., 8$ should be non-increasing in $q_2, q_3, q_4, q_6, q_7, q_8$ which means that if these functions are continuously differentiable that

$$\frac{\partial\theta_i^A}{\partial q_j}, \frac{\partial\theta_i^B}{\partial q_j} \leq 0 \text{ for all } i = 1, ..., 8,\ j = 2,3,4,6,7,8.$$

Finally, overfishing decreases the transition probabilities to *High*, hence the functions p_i $i = 1, ..., 8$ should be decreasing over time in $q_2, q_3, q_4, q_6, q_7, q_8$ as well. If these functions are continuously differentiable

$$\frac{\partial p_i}{\partial q_j} \leq 0 \text{ for all } i = 1, ..., 8,\ j = 2,3,4,6,7,8.$$

We now give a numeric example.

Example 1. We assume that in both states Action 1, i.e., catching with restraint, is dominated by the alternative. Let the following be given

$$H^t = \begin{bmatrix} \lambda(q^t)\cdot(16,16), p_1\left(q^t\right) & \lambda(q^t)\cdot(14,28), p_2\left(q^t\right) \\ \lambda(q^t)\cdot(28,14), p_3\left(q^t\right) & \lambda(q^t)\cdot(24,24), p_4\left(q^t\right) \end{bmatrix},$$

$$L^t = \begin{bmatrix} \lambda(q^t)\cdot(4,4), p_1\left(q^t\right) & \lambda(q^t)\cdot\left(\frac{7}{2},7\right), p_2\left(q^t\right) \\ \lambda(q^t)\cdot\left(7,\frac{7}{2}\right), p_3\left(q^t\right) & \lambda(q^t)\cdot(6,6), p_4\left(q^t\right) \end{bmatrix},$$

where normalization factor $\lambda : \Delta^7 \to [0,1]$ is given by

$$\lambda(q^t) = 1 - \frac{q_2^t + q_3^t}{4} - \frac{q_4^t}{3} - \frac{q_6^t + q_7^t}{2} - \frac{2q_8^t}{3},$$

and the functions governing the transition probabilities $p_i : \Delta^7 \to [0,1], i = 1, ..., 8$, are given by

$$\begin{aligned}
&p_1(q^t) = 0.8 - 0.35\left(q_2^t + q_3^t + q_6^t + q_7^t\right) - 0.7(q_4^t + q_8^t)\\
&p_2(q^t) = p_3(q^t) = 0.7 - 0.3\left(q_2^t + q_3^t + q_6^t + q_7^t\right) - 0.6(q_4^t + q_8^t)\\
&p_4(q^t) = 0.6 - 0.25\left(q_2^t + q_3^t + q_6^t + q_7^t\right) - 0.5(q_4^t + q_8^t)\\
&p_5(q^t) = 0.5 - 0.2\left(q_2^t + q_3^t + q_6^t + q_7^t\right) - 0.4(q_4^t + q_8^t)\\
&p_6(q^t) = p_7(q^t) = 0.4 - 0.15\left(q_2^t + q_3^t + q_6^t + q_7^t\right) - 0.3(q_4^t + q_8^t)\\
&p_8(q^t) = 0.15 - 0.05\left(q_2^t + q_3^t + q_6^t + q_7^t\right) - 0.1(q_4^t + q_8^t).
\end{aligned}$$

So, if at given stage t, with history q^t, the play happens to be in *High*, and the players choose action pair Top-Right, then their immediate payoffs are

$$\begin{aligned}\theta_2\left(q^t\right) &= \left(\theta_2^A\left(q^t\right), \theta_2^B\left(q^t\right)\right) = \lambda(q^t)\cdot(14,28)\\ &= \left[1-\frac{q_2^t+q_3^t}{4}-\frac{q_4^t}{3}-\frac{q_6^t+q_7^t}{2}-\frac{2q_8^t}{3}\right]\cdot(14,28)\end{aligned}$$

and with probability

$$p_2(q^t) = 0.7 - 0.3\left(q_2^t+q_3^t+q_6^t+q_7^t\right) - 0.6(q_4^t+q_8^t)$$

the system will stay in *High*, and with the complementarity probability the play will occur in *Low* at the next stage.

Note that for every fixed q^t the catches in *Low* are a mere quarter of those in *High*, for every comparable action pair, this proportion does not change.

The number

$$\lambda(q^t) = 1-\frac{q_2^t+q_3^t}{4}-\frac{q_4^t}{3}-\frac{q_6^t+q_7^t}{2}-\frac{2q_8^t}{3},$$

may be interpreted as a normalization of the current level of the fish stock to the unit interval. If $q_1^t+q_5^t=1$, both agents have never overexploited the resource, then $\lambda(q^t)=1$, which implies that the fish stock is at its maximum. If both agents always overfish in both states then it can be shown that the fish stock will reach $\lambda_{\min}=\frac{20}{57}=0.350\,88$.

Similarly, two-sided full restraint is sustainable and in this case, a good period (the system was in *High*) is followed by another one with probability $\frac{8}{10}$; *Low* is followed by *Low* with probability at most $\frac{1}{2}$. So, there is some switching between states, but *High* occurs nearly 70% of the time and landings may be disappointing in less than 30% of the time.

Overfishing alters these probabilities, as for moderate rates of overfishing, it appears that *High* is predominantly followed by *High* as $p_1^t, p_2^t, p_3^t, p_4^t > \frac{1}{2}$, and *Low* by *Low* as $1-p_5^t, 1-p_6^t, 1-p_7^t, 1-p_8^t > \frac{1}{2}$. If overexploitation is more severe, the system being in *High* switches to *Low* with an increasing probability, and when in *Low*, stays in that state with a higher probability. Hence, the system spends less and less time in state *High*. ■

3 Strategies and Restrictions

Since a strategy is a game plan for the entire infinite time horizon, allowing it to depend on any condition makes a comprehensive analysis of infinitely repeated games quite impossible. Most restrictions in the literature put requirements on what aspects the strategies are conditional upon. For instance, a *history-dependent* strategy prescribes a possibly mixed action to be played at each stage conditional on the current stage and state, as well as on the full history until then, i.e., all states visited and all action combinations realized before. Less general strategies are for instance, *action independent* ones which condition on all states visited before, but not on the action combinations chosen [25]. *Markov* strategies condition on the current state and the current stage, and *stationary* strategies only condition on the current state (cf., e.g., [15]). Let $\mathcal{X}^k$ denote the set of history-dependent strategies of player $k = 1, 2$.

A strategy is **pure**, if at *each* stage a **pure action** is chosen, i.e., an action is chosen with probability 1. The set of pure strategies for player k is $\mathcal{P}^k$, and $\mathcal{P} \equiv \mathcal{P}^A \times \mathcal{P}^B$.

The strategy pair $(\pi, \sigma) \in \mathcal{X}^A \times \mathcal{X}^B$ is **jointly convergent** if and only if $q \in \Delta^7$ exists such that for all $\varepsilon > 0$, $i \in \{1, 2, ..., 8\}$:

$$\limsup_{t\to\infty} \Pr_{\pi,\sigma} \left[|q_i^t - q_i| \geq \varepsilon \right] = 0. \tag{1}$$

$\Pr_{\pi,\sigma}$ denotes the probability under strategy-pair (π, σ). $\mathcal{JC}$ denotes the set of jointly-convergent strategy pairs. Under such a pair of strategies, the relative frequency of each action pair in both states as play goes to infinity converges to a fixed number with probability 1 in the terminology of Billingsley [6, p. 274]. The **set of jointly-convergent pure-strategy rewards** $P^{\mathcal{JC}}$ is then the set of pairs of rewards obtained by using a pair of jointly-convergent pure strategies.

For a pair of jointly-convergent pure strategies, let $p_i \equiv \lim_{t\to\infty} p_i \left(q^t\right) = p_i(q)$ for $i = 1, ..., 8$. These notions are well defined as the relevant functions are continuous (cf., e.g., [6]). The following equation implicitly holds:

$$\sum_{i=1}^{4} q_i \left(1 - p_i\right) = \sum_{i=5}^{8} q_i p_i. \tag{2}$$

Equation (2) is a conservation of flow equation dictated by the logic of Markov chains: play takes place on both states infinitely often, therefore, due to the law of large numbers the actual instances of leaving *High* (*Low*) must be proportional to the long run probability of leaving that state and the latter must be equal to the probability of returning.

4 On Rewards and Equilibrium Rewards

The players receive an infinite stream of stage payoffs, they are assumed to wish to maximize their average rewards. For a given pair of strategies (π, σ), $R_t^k (\pi, \sigma)$ is the expected payoff to player k at stage t under strategy combination (π, σ), then player k's **average reward,** $k = A, B$, is $\gamma^k (\pi, \sigma) = \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} R_t^k (\pi, \sigma)$, and $\gamma (\pi, \sigma) \equiv \left(\gamma^A (\pi, \sigma), \gamma^B (\pi, \sigma)\right)$. Moreover, for vector $q \in \Delta^7$, the ***q*-averaged payoffs** $(x, y)_q$ are given by

$$(x, y)_q = \textstyle\sum_{i=1}^{8} q_i \theta_i(q).$$

In the analysis of repeated games, another helpful measure to reduce complexity is to focus on rewards instead of strategies. It is more rule than exception that the same reward combination can be achieved by several distinct strategy combinations. Here, we focus on rewards to be obtained by jointly-convergent pure strategies.

4.1 Jointly Convergent Pure Strategy Rewards

The next result connects notions introduced in the previous sections.

Proposition 1. *Let strategy-pair* $(\pi, \sigma) \in \mathcal{JC}$ *and let* $q \in \Delta^7$ *for which* (1) *is satisfied, then the average payoffs are given by* $\gamma(\pi, \sigma) = (x, y)_q$.

Proof: Let $(\pi, \sigma) \in \mathcal{JC}$ and $E\{\theta_u^{\pi,\sigma}\} \equiv \left(R_u^1(\pi, \sigma), R_u^2(\pi, \sigma)\right)$, then

$$\gamma(\pi, \sigma) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} E\{\theta_t^{\pi,\sigma}(q^t)\} = \lim_{T\to\infty} E\left\{\frac{1}{T}\sum_{t=1}^{T} \theta_t^{\pi,\sigma}(q^t)\right\}.$$

The second equation is standard. Consider an infinite sequence of realizations of play under (π, σ)

$$\left(\theta_1^{\pi,\sigma}(q^1), \theta_2^{\pi,\sigma}(q^2), \theta_3^{\pi,\sigma}(q^3), ...\right).$$

Let $\theta_i(q^t) = \theta_t^{\pi,\sigma}(q^t)$, i.e., the realization $\theta_t^{\pi,\sigma}(q^t)$ under (π, σ) at stage t was payoff entry $\theta_i(q^t)$, where $i = 1, ..., 2$. Let $\theta_i(q) = \lim_{t\to\infty} \theta_i(q^t)$, and this notion is meaningful due to [6, p. 274].

Then, focus without loss of generality only on those instances $t_1, t_2, t_3, ...$ in which $\theta_{t_j}^{\pi,\sigma}(q^{t_j}) = \theta_1(q^{t_j}) \neq 0$, $j = 1, 2, 3, ...$ and consider

$$\left(..., \frac{\theta_1(q^{t_1})}{\theta_1(q)}, ..., \frac{\theta_1(q^{t_2})}{\theta_1(q)}, ..., \frac{\theta_1(q^{t_3})}{\theta_1(q)}, ...\right)$$

then for fixed very large j :

$$\left(..., \frac{\theta_1(q^{t_{j+1}})}{\theta_1(q)}, ..., \frac{\theta_1(q^{t_{j+2}})}{\theta_1(q)}, ..., \frac{\theta_1(q^{t_{j+3}})}{\theta_1(q)}, ...\right) \to (..., 1, ..., 1, ..., 1, ...).$$

So, this implies that $\sum_{k=1}^{T} \frac{\theta_1(q^{t_{j+k}})}{\theta_1(q)}$ simply records how often entry 1 occurs after stage t_j, and as T becomes very large it must hold that $\sum_{k=1}^{T} \frac{\theta_1(q^{t_{j+k}})}{\theta_1(q)} \to Tq_1$, hence

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} \frac{\theta_1(q^{t_{j+k}})}{\theta_1(q)} = q_1.$$

Therefore, as $\lim_{T\to\infty} \frac{1}{T+j} \sum_{t=1}^{j} \frac{\theta_1(q^t)}{\theta_1(q)} = 0$,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \frac{\theta_i(q^t)}{\theta_i(q)} = \lim_{T\to\infty} \frac{1}{T+j} \sum_{t=1}^{j} \frac{\theta_1(q^t)}{\theta_1(q)} + \lim_{T\to\infty} \frac{1}{T+j} \sum_{t=1}^{T} \frac{\theta_1(q^{t_{j+k}})}{\theta_1(q)}$$

$$= \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \frac{\theta_1(q^{j+t})}{\theta_1(q)} = q_i.$$

Hence,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\theta_t^{\pi,\sigma}(q^t)=\lim_{T\to\infty}\sum_{i:\theta_1(q)\neq 0}\theta_i(q)\left(\frac{1}{T}\sum_{t=1}^{T}\frac{\theta_i(q^t)}{\theta_i(q)}\right)$$
$$=\sum_{i=1}^{8}\theta_i(q)\left(\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\frac{\theta_i(q^t)}{\theta_i(q)}\right)=\sum_{i=1}^{8}\theta_i(q)q_i.$$

So, we may conclude that $\gamma(\pi,\sigma)=(x,y)_q$. ■

The following example visualizes the set of jointly-convergent rewards.

Example 2. We proceed with the numbers given in Example 1 in order to illustrate the above. We have computed a large number of pairs of rewards to be obtained by pairs of jointly convergent strategies with Matlab (code available from the authors) and have made a visualization of this set in Fig. 1. The Pareto-efficient boundary of this set consists of the rewards represented at the most right hand upper elements of this set.

The bigger dots in Fig. 1 indicate the levels of six stage payoffs of the safely exploited resource, i.e., $\lambda=1$. Observe that the set of jointly convergent strategy rewards does not intersect with the convex hull of the four points associated with state *High*. On the other hand the convex hull of the four points connected to *Low* in a safely exploited resource is covered by the set of jointly convergent strategy rewards.

The upper right hand extreme point of the shape representing the set of jointly convergent strategy rewards corresponds to the situation in which the agents never overfish, i.e., $q_1+q_5=1$. The analysis of the Markov chain implied yields $\lambda(q)=1$, $q_1=\frac{5}{7}$ and $q_5=\frac{2}{7}$. So, the safely exploited system spends about 71% of the time in *High* and the complementary percentage in *Low*. The corresponding rewards are therefore given by

$$1\left(\frac{5}{7}(16,16)+\frac{2}{7}(4,4),\frac{5}{7}(16,16)+\frac{2}{7}(4,4)\right)$$
$$=\left(\frac{88}{7},\frac{88}{7}\right)\approx(12.571,12.571).$$

The lower left hand extreme point of the shape corresponds to the situation in which both agents always overfish, i.e., $q_4+q_8=1$. The analysis of the Markov chain implied yields $\lambda(q)=\frac{20}{57}$, $q_1=\frac{5}{95}$ and $q_5=\frac{90}{95}$. So, the overexploited system spends less than 6% of the time in *High* and more than the complementary percentage in *Low*. The corresponding rewards are therefore given by

$$\frac{20}{57}\left(\frac{5}{95}(24,24)+\frac{90}{95}(6,6),\frac{5}{95}(24,24)+\frac{90}{95}(6,6)\right)$$
$$=\left(\frac{880}{361},\frac{880}{361}\right)\approx(2.437\,7,2.437\,7).$$

The safely exploited fish stock is more than 85% $(=1/\frac{20}{57}-1)$ higher than the maximally overexploited one, and the system spends more than 13.571 $(=\frac{5}{7}/\frac{5}{95})$ times the amount of time in *High* than in the overexploited system.

The maximal total rewards from jointly-convergent pure strategies yield 5.15 $(=\frac{12.571}{2.4377})$ times the never-restraint total rewards. Permanent overfishing reduces the resource to about 35% $(=\frac{20}{57})$ of the full restraint stock. ■

4.2 Equilibrium Rewards

We work towards establishing pairs of rewards from equilibria involving threats. Our approach is similar to a well-established one in the repeated games literature (cf., e.g., [16,22]), linked to the Folk Theorem (see e.g., [60]) and applied to stochastic games as well (cf., e.g., [35,53,57]).

We call $v = \left(v^A, v^B\right)$ the **threat point**, where $v^A = \min_{\sigma\in\mathcal{X}^B}\max_{\pi\in\mathcal{X}^A} \gamma^A(\pi,\sigma)$, and $v^B = \min_{\pi\in\mathcal{X}^A}\max_{\sigma\in\mathcal{X}^B}\gamma^B(\pi,\sigma)$. So, v^A is the highest amount A can get if B tries to minimize A's average payoffs. Under a pair of **individually rational** (feasible) rewards each player receives at least the threat-point reward. We can now present the following formal result.

Theorem 2. *Let E be the set of all individually-rational jointly-convergent pure-strategy rewards. Then, each pair of rewards in E can be supported by an equilibrium.*

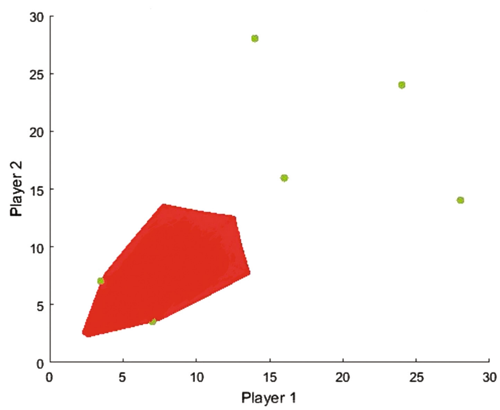

Fig. 1. The dense area depicts rewards feasible by jointly convergent strategies. The six dots concur with entries of the matrices associated with an unexploited resource. Pareto-efficient rewards are located at the upper, right hand boundary of this set.

Proof: Let π^p (σ^p) be a punishment-strategy of A (B), i.e., a strategy holding his opponent to at most v^B (v^A) . Let $(x, y) \in E$, then a pure-strategy combination $(\pi, \sigma) \in \mathcal{JC}$ exists such that $\gamma(\pi, \sigma) = (x, y) \geq \left(v^A, v^B\right)$. Let

$$\pi_t^* \equiv \begin{cases} \pi_t \text{ if } j_k = \sigma_k^* \text{ for all } k < t, \\ \pi_t^p \text{ otherwise.} \end{cases}$$

$$\sigma_t^* \equiv \begin{cases} \sigma_t \text{ if } i_k = \pi_k^* \text{ for all } k < t, \\ \sigma_t^p \text{ otherwise.} \end{cases}$$

Here, i_t (j_t) denotes the action taken by player A (B) at stage t of the play. Clearly, $\gamma(\pi^*, \sigma^*) = \gamma(\pi, \sigma) = (x, y)$. Suppose player A were to play π' such that $\pi'_k \neq \pi_k^*$ for some k, then player B would play according to σ^p from then on. Since, $\gamma^A(\pi', \sigma^p) \leq v^A \leq x$, it follows immediately that player A can not improve against σ^*. A similar statement holds in case player B deviates unilaterally. Hence, (π^*, σ^*) is an equilibrium. ■

Such a pair of strategies (π^*, σ^*) in the proof above is commonly referred to as an equilibrium involving threats. Joosten *et al.* [35] prove by construction that rewards in the convex hull of E correspond to equilibria.

Folk Theorem type results are standard in the analysis of repeated and stochastic games, e.g., [22,57,60]. They hinge on the possibility of punishing unilateral deviations, as in e.g., [19]. So, we need history-dependent strategies. To prevent misconception: there is no contradiction between an equilibrium being jointly-convergent and subgame-perfect: if the equilibrium path in the terminology of [22] induces *convergence with probability 1*, the off-equilibrium part may be of arbitrary sophistication. Neither is there a contradiction between strategy pairs being both jointly-convergent and history-dependent, or for that matter cooperative, e.g., [39,58,59], or incentive strategies, or combinations, e.g., [11–14].

4.3 On Computing Threat Points

We illustrate Theorem 2 and the notions introduced. Moreover, we use the examples to show the scope of the problem of computing threat points. Hordijk *et al.* [28] show that a pure stationary strategy suffices as a best reply against a fixed stationary strategy. Any pair of stationary strategies here is clearly jointly-convergent. The next example shows that linear programs may not suffice.

Example 3. Assume that player B uses a strategy consisting of playing his second, i.e., right hand, action in state *High* and his first, i.e., left hand action in state *Low* **at all stages of the play**. Then, consider the following (nonlinear) program for Player A to minimize player B's rewards

$$\begin{array}{l}
\min_{q_2,q_4,q_5,q_7} \lambda(q)[28q_2 + 24q_4 + 4q_5 + 3.5q_7] \\
\text{s.t. } 1 = q_2 + q_4 + q_5 + q_7 \\
\lambda(q) = 1 - \frac{q_2+2q_7}{4} - \frac{q_4}{3} \\
0 = (1-p_2)q_2 + (1-p_4)q_4 - p_7q_7 - p_5q_5 \\
p_2 = 0.7 - 0.3\,(q_2+q_7) - 0.6q_4 \\
p_4 = 0.6 - 0.25\,(q_2+q_7) - 0.5q_4 \\
p_5 = 0.5 - 0.2\,(q_2+q_7) - 0.4q_4 \\
p_7 = 0.4 - 0.15\,(q_2+q_7) - 0.3q_4 \\
0 \le q_2, q_4, q_6, q_8.
\end{array}$$

Clearly, putting as much weight on q_4 and q_7 as possible minimizes the part between brackets in the objective function, while decreasing the part before the brackets and decreasing the four probabilities most, is a good intuition to look for a candidate solving the minimization problem. In fact in the Appendix we show that this indeed yields a minimum. The associated reward to player B is 4.4588, so B has a strategy guaranteeing him at least this amount. This implies $v^B \geq 4.4588$.

Suppose player A uses his second action at all stages of the play to punish his opponent. Consider the (nonlinear) program

$$\begin{array}{l}
\max_{q_3,q_4,q_7,q_8} \lambda(q)\,[14q_3 + 24q_4 + 3.5q_7 + 6q_8] \\
\text{s.t. } 1 = q_3 + q_4 + q_7 + q_8 \\
\lambda(q) = 1 - \frac{q_3+2q_7}{4} - \frac{q_4+2q_8}{3} \\
0 = (1-p_3)q_3 + (1-p_4)q_4 - p_7q_7 - p_8q_8 \\
p_3 = 0.7 - 0.3\,(q_3+q_7) - 0.6(q_4+q_8) \\
p_4 = 0.6 - 0.25\,(q_3+q_7) - 0.5(q_4+q_8) \\
p_7 = 0.4 - 0.15\,(q_3+q_7) - 0.3(q_4+q_8) \\
p_8 = 0.15 - 0.05\,(q_3+q_7) - 0.1(q_4+q_8) \\
0 \le q_3, q_4, q_7, q_8.
\end{array}$$

In the Appendix we argue that B can get at most 4.4588, so A has a strategy keeping B at this amount. This implies $v^B \leq 4.4588$. This together with the previous finding implies $v^B = 4.4588$.

Figure 2 visualizes all Nash equilibrium rewards, and they are located to the North-East of the threat point. Observe that the threat point minimizes the total rewards to the players within the set of equilibrium rewards. In the Appendix we found that the solution $x = y = 0$ implies that

$$q = \frac{-1.05 + \sqrt{1.05^2 - 4 \cdot 0.1 \cdot (-0.25)}}{2 \cdot 0.1} = 0.232\,93$$

$$\lambda(q) = \frac{2}{12}q + \frac{1}{2} = \frac{1}{6} \cdot 0.232\,93 + \frac{1}{2} = 0.538\,82.$$

Hence, the fish stock associated to the lowest total equilibrium rewards, is about 53% of the maximum fish stock obtained by full restraint. ■

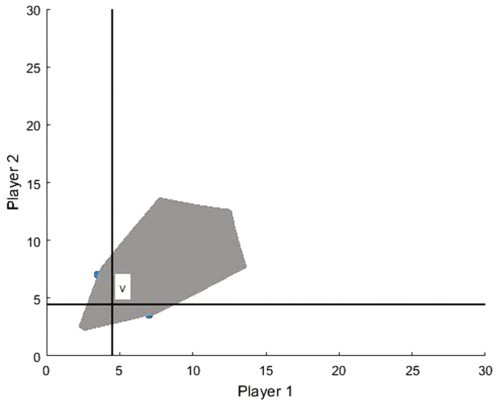

Fig. 2. Jointly-convergent pure-strategy rewards to the North-East of $v = (4.4855, 4.4855)$ can be obtained by Nash equilibria involving threats.

Example 3 and our endeavors in the appendix illustrate that finding threat points may be cumbersome. Moreover, for games with more states than two, the present approach gives a hunch where to look for the solution to the minimization and the maximization problems, can be followed as well, but clearly a visualization will not be possible. Hence, not helpful in reaching conclusions. What remains then, is the determination of the partial derivatives and this may add another cumbersome step to the search.

The *general* problem is equivalent to finding the value of a zero-sum stochastic game which can be done by taking the limit point for stationary discounted optimal strategies, cf., [4,5]. By [47], we have the right solution, but only ε-optimal strategies are known to exist. The Big Match [17], solved in [7], shows that neither ε-optimal stationary strategies, nor ε-optimal Markov strategies need to exist. However, for the game of Example 3 both states always communicate, i.e., the system may move from one state to the other at any stage, for all strategy pairs. So, every pair of stationary strategies generates a Markov chain with a unique ergodic set (being simply the two states). That implies that the game is unichain and stationary optimal strategies exist, cf., e.g., [27,56].

5 Conclusions

We added an innovation to the framework of *Small Fish Wars* (e.g., [31,32,34]) by combining variants quite close to the original approach with the one introduced in [36]. In the latter contribution, transition probabilities between the various states in a stochastic game change as a result of the actions taken by the agents. So, this new approach presented here entails that we allow endogenous stage payoffs as in the original setup as well as endogenous transition probabilities as in [36]. There is a slight but notable difference of the present approach with [36]. Here, states will not become absorbing *temporarily*.

The combined model had not been tackled partly due to computational problems in generating visualizations of jointly-convergent pure-strategy rewards in [36], deemed necessary for an analysis. The algorithm developed there proved a tremendous improvement of earlier ones for older models, but turned out to be extremely slow precisely in the complex setting studied for which it had been developed. Recent research in improving algorithms, [37], took care of this bottleneck and the present state of our computing prowess is such that we can use it to facilitate the analysis in the present set up.

Our approach generalizes standard stochastic games,[6] too. We propose methods of analysis originally introduced in [35] inspired by Folk Theorems for stochastic games e.g., [29,30,53,57] and developed further in for instance [31,32,34]. Crucial is the notion of jointly-convergent strategies to justify the steps in creating analogies to the Folk Theorem.[7]

Our analysis of a special example shows that a ‘tragedy of the commons’ may be averted by sufficiently patient[8] rational agents maximizing their utilities non-cooperatively. All equilibrium rewards yield more than the amounts associated to the permanent ruthless exploitation of the resource. Pareto optimal equilibrium rewards correspond to strategy pairs involving a considerable amount of restraint on the part of the agents, and are up to more than 5 times higher than no-restraint rewards. Case-by-case analysis is however required, hence no general claims are implied.

To present a tractable model and to economize on notations, we imposed symmetry and used the three ‘twos’: two states, two players and two actions. Two distinct states allow to model the kind of transitions we had in mind; two agents are minimally required to model strategic interaction; two stage-game actions leave something to choose. The three twos can be interpreted as a lower bound to have interesting dynamic strategic interaction.

The three twos also serve as an upper bound as the main purpose of our paper is to introduce an innovation in modeling. Notational challenges may then distract from the main message. Just take three threes instead of three twos. Then in each state 27 action combinations will be relevant, each inducing both a stage payoff vector to the three players and a transition probability vector to the three states. So, one needs to formulate 162 vector functions[9] instead of the

[6] Although our games fall into the class of stochastic games with infinitely many states, we prefer our presentation because we were able to obtain results due to it. We have no idea about which results from the analysis of stochastic games with infinitely many states help to obtain results, too.

[7] We want our models to resemble repeated games for reasons of ease of communication for instance with politicians. Many have learned about the repeated prisoners' dilemma in education, so offering our model in a simple fashion may offer windows of opportunity for communication.

[8] Agents are not individuals, but rather countries, regions, or cooperatives. Individual fisherman's preferences seem too myopic (cf., e.g., [26]). It is well known that other factors influence the outcome of a tragedy of the commons, too (cf., e.g., [38,49,50, 55]).

[9] The general three Ns case would require $2 \cdot N^{N+1}$ vector functions of dimension N.

modest 16 for the case with the three twos. Furthermore, by keeping the model and its analysis relatively simple, we invite comparisons with contributions in the social dilemma literature, cf., e.g., [24,38,44]. Our resource game is to be associated primarily with a social trap, see e.g., [10,20,51] of which the 'tragedy of the commons' cf., e.g., [21,45,46], is a notorious example.

In order to capture additional real-life phenomena such as seasonalities or other types of correlations, a larger number of states may be required. Furthermore, more levels or dimensions of restraining measures may be necessary. Adding states, (asymmetric) players or actions changes nothing to our approach conceptually in *obtaining* large sets of rewards. Obviously, analysis of models with more than three players rules out visualization.

A Appendix

Validation of the claim in Example 3. A quick and dirty preliminary analysis yielded a first candidate, namely the strategy pair where Player B always plays his second action in *High* and the first action in *Low*; Player A always plays his second action. So, A is the player punishing his opponent, and all calculations will induce a threat point reward for B and symmetry then implies that the same reward is the threat point reward for A, as well.

We start with the **minimization program**. We reduce the four dimensional system to a two dimensional one as follows. We set $q_2 = qx, q_4 = q(1-x), q_5 = (1-q)y, q_7 = (1-q)(1-y)$ where $0 \leq x, y, q \leq 1$. The interpretation is that to minimize his opponents rewards even further, A may be allowed to shift an arbitrary weight from the original $q_4 = 1$ to q_2, hence after the shift we have $q_2 = qx, q_4 = q(1-x)$. Similarly, A may shift weight from $q_7 = 1$ to q_5, such that after the shift we have $q_5 = (1-q)y, q_7 = (1-q)(1-y)$.

So, the variables to minimize over are x and y, q will result from the analysis of the balance Eq. (1) for given x, y using this transformation of the four original variables. The transformed transition probabilities are

$$\begin{aligned} p_2 &= 0.3y + 0.4 - 0.3q\,(1 - x + y) \\ p_4 &= 0.25y + 0.35 - 0.25q\,(1 - x + y) \\ p_5 &= 0.2y + 0.3 - 0.2q\,(1 + x - y) \\ p_7 &= 0.15y + 0.25 - 0.15q\,(1 - x + y) \end{aligned}$$

The transformed balance equation for this setting is

$$\begin{aligned} 0 = {} & q^2\left(-0.05x^2 - 0.3xy - 0.05x + 0.35y^2 + 0.05y + 0.1\right) + \\ & q\left(0.3xy - 0.2x - 0.3y^2 + 0.15y + 1.05\right) + \left(-0.05y^2 - 0.2y - 0.25\right) \end{aligned}$$

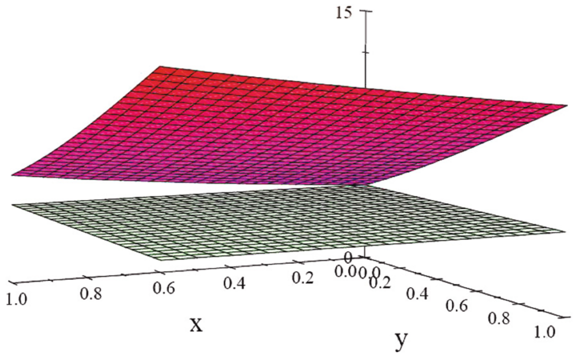

Fig. 3. The upper manifold represents all rewards to B possible if B plays the fixed strategy: Right (2) in *High*, left (1) in *Low*. The horizontal hyperplane indicates level $v_{4,7} = 4.4588$. This justifies the conclusion that B's rewards are minimized at $x = y = 0$, i.e., Player A should play bottom (2) in both states forever. Partial derivatives are positive on $[0, 1] \times [0, 1]$.

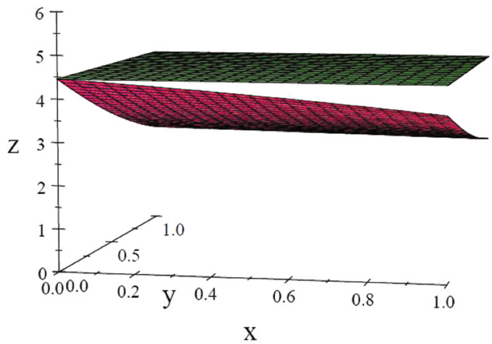

Fig. 4. The lower manifold represents all rewards to B on the interval $[0, 1]\times$ $[0, 1]$, if Player A plays a fixed strategy of bottom (2) in both states. The horizontal hyperplane indicates level $v_{4,7}$. Clearly the highest rewards are to be found at $x = y = 0$ which means that B should use his right action (2) in *High* and his left action (1) in *Low*. It is also easy to see that the partial derivatives are negative on the same interval.

Hence, the minimization problem reduces and simplifies to the following.

$$
\begin{aligned}
&\min_{x,y} \lambda(q)[20.5q + 0.5y + 4.0qx - 0.5qy + 3.5]\\
&\lambda(q) = \tfrac{1}{12}q\,(2 + x - 6y) + \tfrac{1}{2}y + \tfrac{1}{2}\\
&q = \tfrac{-B+\sqrt{B^2-4AC}}{2A}\\
&A = \left(-0.05x^2 - 0.3xy - 0.05x + 0.35y^2 + 0.05y + 0.1\right)\\
&B = \left(0.3xy - 0.2x - 0.3y^2 + 0.15y + 1.05\right)\\
&C = \left(-0.05y^2 - 0.2y - 0.25\right)\\
&0 \le x, y, q \le 1.
\end{aligned}
$$

The **other root of the balance equation** is real too, but yields nonsense.

There are alternative options to proceed now, but we show a plot of all rewards to B as a function of $x, y \in [0,1] \times [0,1]$ under the conditions of the minimization problem. Figure 3 shows immediately that the rewards of B are minimized for $x = y = 0$.

Proceeding in the same fashion for the **maximization problem**, we generate Fig. 4 showing that maximizes B's rewards for $x = y = 0$.

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n-Person Credibilistic Non-cooperative Game with Fuzzy Payoffs

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Abstract. In real game situations, the players are often lack of the information about their opponents' or even their own payoffs. The existing literature on non-cooperative games with uncertain payoffs just focused on two-person zero-sum games or bimatrix games. In this paper, we consider a n-person non-cooperative game with fuzzy payoffs. First, based on credibility theory, three credibilistic criteria are introduced to define the behavior preferences of players in different game situations. Then, three solution concepts of credibilistic equilibria and their existence theorems are proposed. Finally, three sufficient and necessary conditions are presented for finding the credibilistic equilibrium strategies to illustrate the usefulness of the theory developed in this paper.

Keywords: n-person credibilistic game · Credibility theory · Fuzzy payoff · Credibility measure

1 Introduction

As a collection of mathematical models that is used to model and analyze interaction among a group of rational players, game theory was formulated as an independent theory by von Neumann and Morgenstern (1944). Non-cooperative game theory was proposed by Nash (1950, 1951) and developed by Kuhn (1950, 1953), Harsanyi (1966), Aumann (1974) and so on based on the assumption that the payoffs of each player are crisp variables over his strategy space, which has been applied extensively in many fields. In a real game, however, this assumption restricts its applications to real problems. For instance, it is hard to exactly estimate players' payoffs. To deal with such situations, Bayesian game and fuzzy game are introduced.

The Bayesian game model was proposed by Harsanyi (1967), where private information is considered as players' behavior types characterized by a random variable with some probability distribution. Blau (1974), Cassidy et al. (1972) and Charnes et al. (1968) investigated a two-person zero-sum game with random payoffs, and Berg (2000), Ein-Dor and Kanter (2001) and Roberts (2006) considered a bimatrix game with random payoffs, respectively. However, many situations of interest have not enough historical records for probabilistic reasoning, that is, the payoffs in a game cannot be modeled by random variables. Fuzzy set theory that is proposed by Zadeh (1965) offers an approach to such situations (Roy 2010). Campos (1989), Maeda (2003), Mula et al. (2015), Roy (2010), Roy and Mula (2016), Bhaumik et al. (2017) studied a two-person zero-sum matrix game with fuzzy payoffs, respectively. Maeda

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(2000), Roy and Mula (2013) and Das and Roy (2010, 2013) discussed a bimatrix game with fuzzy payoffs. A multi-objective matrix game with fuzzy payoffs is studied by Nishizaki and Sakawa (2001). Vijay et al. (2005) and Cevikel and Ahlatçıoğlu (2010) studied two-person zero-sum matrix games based on fuzzy goals and fuzzy payoffs. Li (2011), Nan et al. (2014) and Chandra and Aggarwal (2015) provided a solution to solve matrix games based on the payoff of interval values, triangular intuitionistic fuzzy numbers and triangular fuzzy numbers, respectively. Furthermore, Gao and his co-workers developed a spectrum of credibilistic games within the framework of credibility theory, including credibilistic strategic game (Gao and Liu 2005; Gao 2007; Gao et al. 2009; Liang et al. 2010; Roy et al. 2011; Gao and Yang 2013), credibilistic coalitional game (Shen and Gao 2010) and credibilistic extensive game (Gao and Yu 2013).

The existing literature on non-cooperative games with fuzzy payoffs just focused on two-person zero-sum games or bimatrix games. In real games, however, the multi-person non-cooperative games with fuzzy payoffs are in line with many decision situations. Thus, how to solve the n-person non-cooperative games with fuzzy payoffs is a valuable research problem, which is just the topic of this paper.

In this paper, we focus on the n-person credibilistic non-cooperative games. Firstly, three credibilistic criteria from credibility theory, which are the expected value criterion, the optimistic value criterion and the credibility criterion, are used to characterize the behaviors of the players in the different decision environment. Furthermore, the concepts of three credibilistic equilibria are defined and their existence theorems are presented. Finally, the sufficient and necessary conditions of the three credibilistic equilibria are developed.

The rest of this paper is arranged as follows: In Sect. 2, the classical n-person non-cooperative game and some basic results of credibility theory are briefly reviewed. In Sect. 3, we define the concepts of expected equilibrium, α_i-optimistic equilibrium and credibility equilibrium and prove their existence theorems. In Sect. 4, the sufficient and necessary conditions of the three credibilistic equilibria are given. In Sect. 5, a numerical example is given. We show the conclusion in Sect. 6.

2 Preliminaries

2.1 Classical *n*-Person Non-cooperative Games

Consider a game $G = \langle N, (S_i)_{i \in N}, (v_i)_{i \in N} \rangle$ with a finite player set $N = \{1, 2, \ldots, n\}$ and, each player $i \in N$, with a finite pure strategy set S_i including m_i pure strategies and a payoff function $v_i(s_i, s_{-i})$ depending on the pure strategy combination (s_i, s_{-i}) played, where $s_{-i:} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ indicates a pure strategy combination for all players except player i. Let $S{:} = S_1 \times S_2 \times \ldots \times S_n$ be the set of pure strategy combinations for all players, and $S_{-i} := S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$ be the set of pure strategy combinations which players other than i could choose. The set of probability measures over S_i is denoted by Λ_i. An element $\sigma_i \in \Lambda_i$ is defined as a mixed strategy for player $i \in N$, where σ_i is a function $\sigma_i : S_i \to [0, 1]$. Therefore, if player $i \in N$ chooses σ_i, then she/he chooses pure strategy s_i with the probability $\sigma_i(s_i)$. A mixed strategy combination is denoted by (σ_i, σ_{-i}), where σ_{-i} indicates a mixed strategy combination for all players except player i. Let $\Lambda := \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$ be the set of mixed strategy combinations for all players and $\Lambda_{-i} := \Lambda_1 \times \cdots \times \Lambda_{i-1} \times \Lambda_{i+1} \times \cdots \times \Lambda_n$ be the set of mixed strategy combinations which players other than i could choose. A mixed strategy game is denoted by $G_1 = \langle N, (\Lambda_i)_{i \in N}, (v_i)_{i \in N} \rangle$.

The expected payoff u_i that player i obtains for some mixed strategy combination (σ_i, σ_{-i}) is given by

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i, s_{-i} \in S_{-i}} \sigma_1(s_1) \cdots \sigma_i(s_i) \cdots \sigma_n(s_n) v_i(s_i, s_{-i}) \tag{1}$$

Definition 2.1. A mixed strategy combination $\sigma^* = (\sigma_i^*, \sigma_{-i}^*)$ is called a Nash equilibrium in the game G_1, if it satisfies

$$u_i(\sigma^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \quad \forall\, i \in N, \sigma_i \in \Lambda_i, \tag{2}$$

where $\sigma_{-i}^* = (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_n^*) \in \Lambda_{-i}$.

Lemma 2.1 (Nash 1950). There exists at least one mixed strategy Nash Equilibrium in a finite n-person non-cooperative game.

2.2 Credibility Theory

As a branch of mathematics, Credibility theory is often used to model the behavior of fuzzy phenomena. Since proposed by Liu (2004), credibility theory has been used extensively in many fields, such as portfolio selection (Chen et al. 2012; Liu et al. 2012) and transportation planning (Li et al. 2013; Yang et al. 2012).

Definition 2.1 (Liu 2007). Let Θ be a nonempty set and $\Xi(\Theta)$ be the power set of Θ, a set function $Cr\{\cdot\}$ is called a credibility measure if it satisfies the following four axioms.

Axiom 1. (Normality) $Cr\{\Theta\} = 1$.

Axiom 2. (Monotonicity) $Cr\{A\} \leq Cr\{B\}$, where $A \subset B$.

Axiom 3. (Self-Duality) $Cr\{A\} + Cr\{A^c\} = 1$ for any $A \in \Xi(\Theta)$.

Axiom 4. (Maximality) $Cr\{\cup_i A_i\} = sup_i Cr\{A_i\}$ for any $\{A_i\}$ with $sup_i Cr\{A_i\} < 0.5$.

Definition 2.2 (Liu 2007). Let Θ be a nonempty set, $\Xi(\Theta)$ be the power set of Θ and Cr be a credibility measure, then the triplet $(\Theta, \Xi(\Theta), Cr)$ is called a credibility space.

Definition 2.3 (Liu 2007). A fuzzy variable $\tilde{v}$ is a measurable function from a credibility space $(\Theta, \Xi(\Theta), Cr)$ to the set of real numbers.

Lemma 2.1 (Liu 2007). Let $\tilde{v}$ be a fuzzy variable with membership function μ, then for any set B of real numbers, we have

$$Cr\{\tilde{v} \in B\} = \frac{1}{2}(\sup_{x \in B} \mu(x) + 1 - \sup_{x \notin B^c} \mu(x)) \quad (3)$$

Definition 2.4 (Liu 2007). The fuzzy variables $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m$ are said to be independent if

$$Cr\{\bigcap_{i=1}^{m}(\tilde{v}_i \in B_i)\} = \min_{1 \le i \le m} Cr\{\tilde{v}_i \in B_i\} \quad (4)$$

for any sets $B_1, B_2, \ldots, B_m$ of R.

Definition 2.5 (Liu and Liu 2002). Let $\tilde{v}$ be a fuzzy variable, then the expected value of $\tilde{v}$ is defined by

$$E[\tilde{v}] = \int_0^{+\infty} Cr\{\tilde{v} \ge r\}dr - \int_{-\infty}^{0} Cr\{\tilde{v} \le r\}dr \quad (5)$$

provided that at least one of the two integrals is finite.

Lemma 2.3 (Liu and Liu 2003). Let $\tilde{v}$ and $\tilde{\eta}$ be independent fuzzy variables with finite expected values, then for any numbers a and b, we have

$$E[a\tilde{v} + b\tilde{\eta}] = aE[\tilde{v}] + bE[\tilde{\eta}] \quad (6)$$

Definition 2.6 (Liu and Liu 2003). Let $\tilde{v}$ be a fuzzy variable and $\alpha \in (0, 1]$ be a confidence level, then, for a real number r,

$$\tilde{v}_{\sup}(\alpha) = \sup\{r | Cr\{\tilde{v} \ge r\} \ge \alpha\} \quad (7)$$

is called the α-optimistic value to $\tilde{v}$.

This means that the fuzzy variable $\tilde{v}$ will reach upwards of the α-optimistic value $\tilde{v}_{\sup}(\alpha)$ with credibilityα. In other words, the α-optimistic value $\tilde{v}_{\sup}(\alpha)$ is the supremum value that $\tilde{v}$ achieves with credibilityα.

Lemma 2.4 (Liu and Liu 2003). Let $\tilde{v}$ and $\tilde{\eta}$ be independent fuzzy variables, then for any $\alpha \in (0, 1]$ and any nonnegative numbers a and b, we have

$$(a\tilde{v} + b\tilde{\eta})_{\sup}(\alpha) = a\tilde{v}_{\sup}(\alpha) + b\tilde{\eta}_{\sup}(\alpha) \quad (8)$$

Lemma 2.5 (Liu and Liu 2007). Let $\tilde{v}$ be a fuzzy variable, then for any $\alpha \in (0, 1]$, we have

$$\text{if } a \ge 0, \text{ then } (a\tilde{v})_{\sup}(\alpha) = a\tilde{v}_{\sup}(\alpha) \quad (9)$$

In order to rank fuzzy variables, three credibilistic approaches are often used.

Definition 2.7 (Liu 2002). Let $\tilde{\nu}$ and $\tilde{\eta}$ be independent fuzzy variables, then we have

(1) Expected value criterion: $\tilde{\nu} < \tilde{\eta}$ if and only if $E[\tilde{\nu}] < E[\tilde{\eta}]$;
(2) Optimistic value criterion: $\tilde{\nu} < \tilde{\eta}$ if and only if $\tilde{\nu}_{\sup}(\alpha) < \tilde{\eta}_{\sup}(\alpha)$ for some predetermined confidence level $\alpha \in (0, 1]$;
(3) Credibility Criterion: $\tilde{\nu} < \tilde{\eta}$ if and only if $Cr\{\tilde{\nu} \geq r\} < Cr\{\tilde{\eta} \geq r\}$ for some predetermined level r.

The expected value criterion is used to deal with the situation where a player wants to optimize the expected value of her/his payoff. The optimistic value criterion is applied to coping with the situation in which a player strives to optimize the optimistic value of her/his payoff at given a confidence level α. While the credibility criterion applies to the situation that a player would set a payoff level and wants to maximize the chance of his achieving the given payoff level.

3 *n*-Person Credibilistic Non-cooperative Game

Since the decision environment is often characterized by a great many possible strategies, intricate relations between strategic choices and their influences to players' payoffs, it is impossible that a player makes accurate or probabilistic estimation of her/his own payoffs. For such situations, we consider a n-person game with fuzzy payoffs. Specifically, for each player $i \in N$, the payoff $\tilde{\nu}_i(s_i, s_{-i})$ is modeled as a fuzzy variable for all $(s_i, s_{-i}) \in S$, then for any mixed strategy combination (σ_i, σ_{-i}), expected payoff of player $i \in N$ is also fuzzy variable and given by

$$\tilde{u}_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i, s_{-i} \in S_{-i}} \sigma_1(s_1)\sigma_2(s_2) \cdots \sigma_n(s_n)\tilde{\nu}_i(s_i, s_{-i}).$$

This game is denoted by $G_2 = \langle N, (\Lambda_i)_{i \in N}, (\tilde{\nu}_i)_{i \in N} \rangle$.

3.1 Three Credibilistic Equilibrium Strategies

Firstly, assuming that player i for $i \in N$ wants to optimize the expected value of her expected fuzzy payoff. Then, the best responses of player $i \in N$ to her/his opponents' strategy combination $\sigma^*_{-i} \in \Lambda_{-i}$ are the optimal solutions of the fuzzy expected value model

$$\max_{\sigma_i \in \Lambda_i} E[\tilde{u}_i(\sigma_i, \sigma^*_{-i})] \tag{10}$$

Definition 3.1. A mixed strategy combination $\sigma^* = (\sigma^*_i, \sigma^*_{-i})$ is called an expected Nash equilibrium in the game G_2, if it satisfies the following inequality

$$u^*_i = E[\tilde{u}_i(\sigma^*_i, \sigma^*_{-i})] \geq E[\tilde{u}_i(\sigma_i, \sigma^*_{-i})], \ \forall \sigma_i \in \Lambda_i,$$

where $\sigma^*_{-i} = (\sigma^*_1, \ldots, \sigma^*_{i-1}, \sigma^*_{i+1}, \ldots, \sigma^*_n) \in \Lambda_{-i}$ and u^*_i is the optimistic value of player i's expected payoff $\tilde{u}_i(\sigma^*_i, \sigma^*_{-i})$.

Secondly, if player i, for $i \in N$, is risk averse, by optimistic value criterion, he wants to optimize the optimistic value of her expected fuzzy payoff at confidence level α_i so that he can avoid potential considerable losses. This game is denoted by $G_3 = \langle N, (\Lambda_i)_{i\in N}, (\tilde{v}_i)_{i\in N}, (\alpha_i)_{i\in N}\rangle$. Then the best responses of player $i \in N$ to her/his opponents' strategy combination $\sigma^*_{-i} \in \Lambda_{-i}$ are the optimal solutions of the fuzzy chance-constrained programming model:

$$\max_{\sigma_i\in\Lambda_i} \max_{u_i'} \ Cr\{\tilde{u}_i(\sigma_i, \sigma^*_{-i}) \geq u_i'\} \geq \alpha_i \tag{11}$$

where $\tilde{u}_i(\sigma_i, \sigma^*_{-i}) = \sum_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma^*_{i-1}\sigma_i\sigma^*_{i+1} \cdots \sigma^*_n \tilde{v}_i(s_i, s_{-i})$ and u_i' is a real number.

A n-person credibilistic game is denoted by $G_3 = \langle N, (\Lambda_i)_{i\in N}, (\tilde{v}_i)_{i\in N}, (\alpha_i)_{i\in N}\rangle$, where α_i is the confidence level of player i.

Definition 3.2. A mixed strategy combination $\sigma^* = (\sigma_i^*, \sigma^*_{-i})$ is called a α_i-optimistic Nash equilibrium in the game G_3, if it satisfies

$$u_i'^* = \max\{u_i' | Cr\{\tilde{u}_i(\sigma_i^*, \sigma^*_{-i}) \geq u_i'\} \geq \alpha_i\} \geq \max\{u_i' | Cr\{\tilde{u}_i(\sigma_i, \sigma^*_{-i}) \geq u_i'\} \geq \alpha_i\} \text{ for } \forall i \in N, \sigma_i \in \Lambda_i,$$

where $\sigma^*_{-i} = (\sigma_1^*, \ldots, \sigma^*_{i-1}, \sigma^*_{i+1}, \ldots, \sigma_n^*) \in \Lambda_{-i}$ and $u_i'^*$ is the optimistic value of player i's expected payoff $\tilde{u}_i(\sigma_i^*, \sigma^*_{-i})$ at the confidence level α_i.

Thirdly, in many situations, a player may be concerned with an event, such as objective function's being greater than prospective value, and want to maximize the chance of the event. That is, player i has specified a predetermined payoff level u_i and wants to maximize the credibility measure of the event $\tilde{u}_i(\sigma_i, \sigma_{-i}) \geq u_i, \ \forall(\sigma_i, \sigma_{-i}) \in \Lambda$. Then the best responses of player $i \in N$ to her/his opponents' strategy combination $\sigma^*_{-i} \in \Lambda_{-i}$ are the optimal solutions of the fuzzy chance- constrained programming model

$$\max_{x_i\in\Lambda_i} \ Cr\{\tilde{u}_i(\sigma_i, \sigma^*_{-i}) \geq u_i''\} \tag{12}$$

Then, in this case, a n-person credibilistic game is denoted by $G_4 = \langle N, (\Lambda_i)_{i\in N}, (\tilde{v}_i)_{i\in N}, (u_i'')_{i\in N}\rangle$, where u_i'' is the predetermined payoff levels of player i.

Definition 3.3. A mixed strategy combination $\sigma^* = (\sigma_i^*, \sigma^*_{-i})$ is called a most credibility Nash equilibrium in the game G_4, if it satisfies the following inequality

$$\alpha_i^* = Cr\{\tilde{u}_i(\sigma_i^*, \sigma^*_{-i}) \geq u_i''\} \geq Cr\{\tilde{u}_i(\sigma_i, \sigma^*_{-i}) \geq u_i''\}, \ \forall\sigma_i \in \Lambda_i.$$

3.2 Existence Theorem of Three Credibilistic Equilibria

Theorem 3.1. Let payoffs $\tilde{v}_1(s_1, s_{-1}), \tilde{v}_2(s_2, s_{-2}), \ldots, \tilde{v}_n(s_n, s_{-n})$ be independent fuzzy variables in the game G_2. Then there exists at least one expected Nash equilibrium strategy.

Theorem 3.2. Let payoffs $\tilde{v}_1(s_1, s_{-1}), \tilde{v}_2(s_2, s_{-2}), \ldots, \tilde{v}_n(s_n, s_{-n})$ be independent fuzzy variables in the game G_3. Then there exists at least one α_i-optimistic Nash equilibrium strategy.

Theorem 3.3. Let payoffs $\tilde{v}_1(s_1, s_{-1}), \tilde{v}_2(s_2, s_{-2}), \ldots, \tilde{v}_n(s_n, s_{-n})$ be independent fuzzy variables in the game G_4. Then there exists at least one most credibility Nash equilibrium strategy.

Proof: Since the proofs of Theorems 3.1 and 3.3 are similar to that of Theorem 3.2, for simplicity, we show the proof of Theorem 3.2.
For any mixed strategy combination $(\sigma_i, \sigma_{-i}) \in \Lambda$ in the game G_3, it follows from Definition 2.6 that

$$\max\{u_i' | Cr\{\{\tilde{u}_i(\sigma_i, \sigma_{-i}) \geq u_i'\} \geq \alpha_i\} = (\tilde{u}_i(\sigma_i, \sigma_{-i}))_{\sup}(\alpha_i). \tag{13}$$

Since $\tilde{u}_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i, s_{-i} \in S_{-i}} \sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n)\tilde{v}_i(s_i, s_{-i})$, we have that

$$(\tilde{u}_i(\sigma_i, \sigma_{-i}))_{\sup}(\alpha_i) = (\sum_{s_i \in S_i, s_{-i} \in S_{-i}} \sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n)\tilde{v}_i(s_i, s_{-i}))_{\sup}(\alpha_i).$$

It follows from Lemma 2.4 that

$$\begin{aligned}(\tilde{u}_i(\sigma_i, \sigma_{-i}))_{\sup}(\alpha_i) &= (\sum_{s_i \in S_i, s_{-i} \in S_{-i}} \sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n)\tilde{v}_i(s_i, s_{-i}))_{\sup}(\alpha_i) \\ &= \sum_{s_i \in S_i, s_{-i} \in S_{-i}} \sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n)(\tilde{v}_i(s_i, s_{-i}))_{\sup}(\alpha_i)\end{aligned} \tag{14}$$

It follows from Eq. (9) that

$$\max\{u_i' | Cr\{\{\tilde{u}_i(\sigma_i, \sigma_{-i}) \geq u_i'\} \geq \alpha_i\} = (\tilde{u}_i(\sigma_i, \sigma_{-i}))_{\sup}(\alpha_i) = \sum_{s_i \in S_i, s_{-i} \in S_{-i}} \sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n)(\tilde{v}_i(s_i, s_{-i}))_{\sup}(\alpha_i) \tag{15}$$

A game is denoted by $G_3' = \langle N, (\Lambda_i)_{i \in N}, ((\tilde{v}_i(s_i, s_{-i}))_{\sup}(\alpha_i))_{i \in N} \rangle$, where $(s_i, s_{-i}) \in S$.

It follows from Definition 3.1 and Eq. (11) that the existence of a α_i-optimistic Nash equilibrium in the game G_3 is equivalent to the existence of a Nash equilibrium in the game G_3'.

It follows from Lemma 2.1 that there exists at least one mixed strategy Nash equilibrium in the game G_3'. Thus, there exists at least one α_i-optimistic Nash equilibrium in the game G_3.

These complete the proof of Theorem 3.2. □

4 Sufficient and Necessary Condition of Three Nash Equilibria

Three sufficient and necessary conditions are presented to find credibilistic equilibrium strategies in the n-person credibilistic non-cooperative game, respectively.

Theorem 4.1. Let payoffs $\tilde{v}_1(s_1, s_{-1}), \tilde{v}_2(s_2, s_{-2}), \ldots, \tilde{v}_n(s_n, s_{-n})$ be independent fuzzy variables, Then a strategy profile $(\sigma_i^*, \sigma_{-i}^*) \in \Lambda$ is an expected Nash equilibrium in the game G_2 if and only if the point $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*, u_1^*, u_2^*, \ldots, u_n^*)$ is an optimal solution to the following non-linear programming model

$$\max_{\sigma_1,\ldots,\sigma_n,u_1,\ldots,u_n} \sum_{i=1}^{n} \Big(\sum_{s_i \in S_i, s_{-i} \in S_{-i}} \sigma_1\sigma_2\cdots\sigma_n E[\tilde{v}_i(s_i, s_{-i})]\Big) - \sum_{i=1}^{n} u_i$$
$$s.t. \begin{cases} \sum\limits_{s_{-1}\in S_{-1}} \sigma_2\cdots\sigma_n E[\tilde{v}_1(s_1^1, s_{-1})] \le u_1 \\ \vdots \\ \sum\limits_{s_{-1}\in S_{-1}} \sigma_2\cdots\sigma_n E[\tilde{v}_1(s_1^{m_1}, s_{-1})] \le u_1 \\ \vdots \\ \sum\limits_{s_{-i}\in S_{-i}} \sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_n E[\tilde{v}_i(s_i^1, s_{-i})] \le u_i \\ \vdots \\ \sum\limits_{s_{-i}\in S_{-i}} \sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_n E[\tilde{v}_i(s_i^{m_i}, s_{-i})] \le u_i \\ \vdots \\ \sum\limits_{s_{-n}\in S_{-n}} \sigma_1\sigma_2\cdots\sigma_{n-1} E[\tilde{v}_n(s_n^1, s_{-n})] \le u_n \\ \vdots \\ \sum\limits_{s_{-n}\in S_{-n}} \sigma_1\sigma_2\cdots\sigma_{n-1} E[\tilde{v}_n(s_n^{m_n}, s_{-n})] \le u_n \\ \forall i \in N, \sigma_i \in \Lambda_i, u_i \in R \end{cases} \tag{16}$$

where s_i^j is the pure strategy that player $i \in N$ chooses jth $j \in \{1, 2, \ldots, m_i\}$ pure strategy from the pure strategy set S_i.

Theorem 4.2. Let payoffs $\tilde{v}_1(s_1, s_{-1}), \tilde{v}_2(s_2, s_{-2}), \ldots, \tilde{v}_n(s_n, s_{-n})$ be independent fuzzy variables. Then a strategy profile $(\sigma_i^*, \sigma_{-i}^*) \in \Lambda$ is an α_i-optimistic Nash equilibrium in the game G_3 if and only if the point $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*, u_1^{'*}, u_2^{'*}, \ldots, u_n^{'*})$ is an optimal solution to the following non-linear programming model

$$
\max_{\sigma_1,\ldots,\sigma_n,u'_1,\ldots,u'_n} \sum_{i=1}^{n} \Big(\sum_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1\sigma_2\cdots\sigma_n(\tilde{v}_i(s_i,s_{-i}))_{\sup}^{\alpha_i} \Big) - \sum_{i=1}^{n} u'_i
$$
$$
s.t. \begin{cases}
\sum\limits_{s_{-1}\in S_{-1}} \sigma_2\cdots\sigma_n(\tilde{v}_1(s_1^1,s_{-1}))_{\sup}^{\alpha_1} \le u'_1 \\
\vdots \\
\sum\limits_{s_{-1}\in S_{-1}} \sigma_2\cdots\sigma_n(\tilde{v}_1(s_1^{m_1},s_{-1}))_{\sup}^{\alpha_1} \le u'_1 \\
\vdots \\
\sum\limits_{s_{-i}\in S_{-i}} \sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_n(\tilde{v}_i(s_i^1,s_{-i}))_{\sup}^{\alpha_i} \le u'_i \\
\vdots \\
\sum\limits_{s_{-i}\in S_{-i}} \sigma_1\cdots\sigma_{i-1}\sigma_{i+1}\cdots\sigma_n(\tilde{v}_i(s_i^{m_i},s_{-i}))_{\sup}^{\alpha_i} \le u'_i \\
\vdots \\
\sum\limits_{s_{-n}\in S_{-n}} \sigma_1\sigma_2\cdots\sigma_{n-1}(\tilde{v}_n(s_n^1,s_{-n}))_{\sup}^{\alpha_n} \le u'_n \\
\vdots \\
\sum\limits_{s_{-n}\in S_{-n}} \sigma_1\sigma_2\cdots\sigma_{n-1}(\tilde{v}_n(s_n^{m_n},s_{-n}))_{\sup}^{\alpha_n} \le u'_n \\
\forall i\in N, \sigma_i\in \Lambda_i, u'_i\in R
\end{cases} \tag{17}
$$

where s_i^j is the pure strategy that player $i \in N$ chooses jth $j \in \{1,2,\ldots,m_i\}$ pure strategy from the pure strategy set S_i.

Theorem 4.3. Let payoffs $\tilde{v}_1(s_1,s_{-1}), \tilde{v}_2(s_2,s_{-2}), \ldots, \tilde{v}_n(s_n,s_{-n})$ be independent fuzzy variables. Then a strategy profile $(\sigma_i^*, \sigma_{-i}^*) \in \Lambda$ is a most credibility Nash equilibrium in the game G_4 if and only if the point $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*, \alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$ is an optimal solution to the following non-linear programming model

$$
\max_{\sigma_1,\ldots,\sigma_n,\alpha_1,\ldots,\alpha_n} \sum_{i=1}^{n} Cr\{ \sum_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1\sigma_2\cdots\sigma_n\tilde{v}_i(s_i,s_{-i}) \ge u''_i \} - \sum_{i=1}^{n} \alpha_i
$$
$$
s.t. \begin{cases}
Cr\{ \sum\limits_{s_1\in S_1, s_{-1}\in S_{-1}} \sigma_1\sigma_2\cdots\sigma_n\tilde{v}_1(s_1,s_{-1}) \ge u''_1 \} \ge \alpha_1 \\
\vdots \\
Cr\{ \sum\limits_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1\sigma_2\cdots\sigma_n\tilde{v}_i(s_i,s_{-i}) \ge u''_i \} \ge \alpha_i \\
\vdots \\
Cr\{ \sum\limits_{s_n\in S_n, s_{-n}\in S_{-n}} \sigma_2\cdots\sigma_{n-1}\tilde{v}_n(s_n,s_{-n}) \ge u''_n \} \ge \alpha_n \\
\forall \sigma_i\in \Lambda_i,\ \sigma_{-i}\in \Lambda_{-i},\ \alpha_i, u''_i\in R,\ i\in N
\end{cases} \tag{18}
$$

Proof: Since the proofs of the three theorems are similar, for simplicity, Theorem 4.2 is only proved.

First, assuming that $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*, u_1'^*, u_2'^*, \ldots, u_n'^*)$ is an optimal solution to the following non-linear programming model (17), then we have

$$
\begin{aligned}
&\sum_{s_{-1}\in S_{-1}} \sigma_2^* \cdots \sigma_n^* (\tilde{v}_1(s_1^1, s_{-1}))_{\sup}^{\alpha_1} \le u_1'^* \\
&\vdots \\
&\sum_{s_{-1}\in S_{-1}} \sigma_2^* \cdots \sigma_n^* (\tilde{v}_1(s_1^{m_1}, s_{-1}))_{\sup}^{\alpha_1} \le u_1'^* \\
&\vdots \\
&\sum_{s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma_{i-1}^* \sigma_{i+1}^* \cdots \sigma_n^* (\tilde{v}_i(s_i^1, s_{-i}))_{\sup}^{\alpha_i} \le u_i'^* \\
&\vdots \\
&\sum_{s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma_{i-1}^* \sigma_{i+1}^* \cdots \sigma_n \sigma_n^* (\tilde{v}_i(s_i^{m_i}, s_{-i}))_{\sup}^{\alpha_i} \le u_i'^* \\
&\vdots \\
&\sum_{s_{-n}\in S_{-n}} \sigma_1^* \sigma_1^* \cdots \sigma_{n-1}^* (\tilde{v}_n(s_n^1, s_{-n}))_{\sup}^{\alpha_n} \le u_n'^* \\
&\vdots \\
&\sum_{s_{-n}\in S_{-n}} \sigma_1^* \sigma_1^* \cdots \sigma_{n-1}^* (\tilde{v}_n(s_n^{m_n}, s_{-n}))_{\sup}^{\alpha_n} \le u_n'^*
\end{aligned}
$$

Thus, we have

$$u_i'(\sigma_i, \sigma_{-i}^*) = \sum_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma_{i-1}^* \sigma_i \sigma_{i+1}^* \cdots \sigma_n^* (\tilde{v}_i(s_i, s_{-i}))_{\sup}^{\alpha_i} \le \sum_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma_{i-1}^* \sigma_i^* \sigma_{i+1}^* \cdots \sigma_n^* (\tilde{v}_i(s_i, s_{-i}))_{\sup}^{\alpha_i} = u_i'^*$$

which implies $(\sigma_i^*, \sigma_{-i}^*) \in \Lambda, \ \forall\, i \in N$ is an α_i-optimistic Nash equilibrium, and $u_i'^*$ is the α_i-optimistic value of player $i \in N$ in the game G_3. The sufficient condition is proved.

Second, assuming that $(\sigma_i^*, \sigma_{-i}^*) \in \Lambda, \ \forall\, i \in N$ is an α_i-optimistic Nash equilibrium, then we have

$$u_i'(\sigma_i, \sigma_{-i}^*) = \sum_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma_{i-1}^* \sigma_i \sigma_{i+1}^* \cdots \sigma_n^* (\tilde{v}_i(s_i, s_{-i}))_{\sup}^{\alpha_i} \le \sum_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma_{i-1}^* \sigma_i^* \sigma_{i+1}^* \cdots \sigma_n^* (\tilde{v}_i(s_i, s_{-i}))_{\sup}^{\alpha_i} = u_i'^*,$$

that is

$$\sum_{s_{-1}\in S_{-1}} \sigma_2^* \cdots \sigma_n^* (\tilde{v}_1(s_1^1, s_{-1}))_{\sup}^{\alpha_1} \le u_1'^*$$
$$\vdots$$
$$\sum_{s_{-1}\in S_{-1}} \sigma_2^* \cdots \sigma_n^* (\tilde{v}_1(s_1^{m_1}, s_{-1}))_{\sup}^{\alpha_1} \le u_1'^*$$
$$\vdots$$
$$\sum_{s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma_{i-1}^* \sigma_{i+1}^* \cdots \sigma_n^* (\tilde{v}_i(s_i^1, s_{-i}))_{\sup}^{\alpha_i} \le u_i'^*$$
$$\vdots$$
$$\sum_{s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma_{i-1}^* \sigma_{i+1}^* \cdots \sigma_n \sigma_n^* (\tilde{v}_i(s_i^{m_i}, s_{-i}))_{\sup}^{\alpha_i} \le u_i'^*$$
$$\vdots$$
$$\sum_{s_{-n}\in S_{-n}} \sigma_1^* \sigma_1^* \cdots \sigma_{n-1}^* (\tilde{v}_n(s_n^1, s_{-n}))_{\sup}^{\alpha_n} \le u_n'^*$$
$$\vdots$$
$$\sum_{s_{-n}\in S_{-n}} \sigma_1^* \sigma_1^* \cdots \sigma_{n-1}^* (\tilde{v}_n(s_n^{m_n}, s_{-n}))_{\sup}^{\alpha_n} \le u_n'^*$$

which means that $(u_1'^*, u_2'^*, \ldots, u_n'^*)$ is a feasible solution of the non-linear programming model (17). In fact, obviously,

$$\sum_{i=1}^{n} \Big(\sum_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1 \cdots \sigma_n (\tilde{v}_i(s_i, s_{-i}))_{\sup}^{\alpha_i} \Big) - \sum_{i=1}^{n} u_i' \le 0.$$

It follows from

$$\sum_{i=1}^{n} \Big(\sum_{s_i\in S_i, s_{-i}\in S_{-i}} \sigma_1^* \cdots \sigma_n^* (\tilde{v}_i(s_i, s_{-i}))_{\sup}^{\alpha_i} \Big) - \sum_{i=1}^{n} u_i'^* = 0$$

that $(u_1'^*, u_2'^*, \ldots, u_n'^*)$ is an optimal solution to the following non-linear programming model (17). The necessary condition is proved. □

5 Numerical Example

Assume that there are three players in some market, which are denoted by players 1, 2, respectively. The payoffs of them are characterized as triangular fuzzy numbers and are shown in Table 1.

Table 1. 2-player game with triangular fuzzy payoffs

	s_2^1	s_2^2
s_1^1	(90,100,110), (80,105,110)	(110,160,170), (110,150,190)
s_1^2	(130,140, 190), (120,140,200)	(60,110,120), (80,90,140)

s_3^1

First, assume players adopt the expected value criterion, then the payoff matrix are shown as follows:

	s_2^1	s_2^2
s_1^1	100, 100	150, 150
s_1^2	150, 150	100, 100

s_3^1

According to Theorem 4.1, the solution to the model (16) is (0.5, 0.5, 0.5, 0.5, 125, 125). It means that (0.5, 0.5, 0.5, 0.5) is an expected Nash equilibrium. It leads to an expected payoff 125.

Second, if the two players are risk averse, then the optimistic value criterion is adopted, assume that the confidence levels of players 1 and 2 are 0.85 and 0.7, respectively. It leads to

	s_2^1	s_2^2
s_1^1	93, 95	125, 134
s_1^2	133, 132	75, 86

s_3^1

According to Theorem 4.2, by solving the model (17), the solution (0.5412, 0.4588, 0.5556, 0.4444, 107.2, 112.0) is obtained, which means that (0.5412, 0.4588, 0.5556, 0.4444) is a (0.85, 0.7)-optimistic Nash equilibrium. The optimal strategy of player 1 is (0.5412, 0.4588) that leads to a payoff 107.2 with uncertain measure 85%, and the optimal strategy of player 2 is (0.5556, 0.4444) that yields a payoff level 112.0 with uncertain measure 70%.

$*$Third, if 110 and 107 is the pursuing payoffs of players 1 and 2, respectively. According to Theorem 4.3, we can obtain the optimal solution to the model (18), i.e., (0.5600, 0.4400, 0.5556, 0.4444, 0.8, 0.9). It implies that player 1's strategy (0.5600, 0.4400) leads to a payoff 110 with uncertain measure 80% and that player 1's strategy (0.5556, 0.4444) leads to a payoff 107 with uncertain measure 90%.

6 Conclusion

In this paper, we consider a n-person credibilistic non-cooperative game with fuzzy payoffs. According to different credibilistic criteria from credibility theorem, the concepts of three credibilistic equilibria are given to deal with different behavior types of players and their existence theorems are shown. Finally, the sufficient and necessary conditions of the three credibilistic equilibria are presented for finding the credibilistic equilibrium strategies.

This n-person credibilistic non-cooperative game is presented. Topics for further studies include that we can extend the proposed game model to the situation with asymmetric information.

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Pareto Optimal Strategies for Matrix Games with Payoffs of Intuitionistic Fuzzy Sets

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Abstract. The aim of this paper is to develop an effective methodology for solving matrix games with payoffs of intuitionistic fuzzy sets (IFSs). In this methodology, a new ranking order relation of IFSs is proposed and the concept of Pareto Nash equilibrium solutions of matrix games with IFS payoffs is firstly defined. It is proven that the solutions of matrix games with IFS payoffs are equivalent to those of a pair of bi-objective programming models. The models and method proposed in this paper are illustrated with a numerical example and compared with other methods to show the validity, applicability and superiority.

Keywords: Intuitionistic fuzzy set · Game theory · Multiobjective programming · Pareto nash equilibrium strategy

1 Introduction

There exist some competitive or antagonistic situations in many parts of real life such as economics, business, management and e-commerce. Game theory gives a mathematical tool for dealing with such conflicting events and has achieved a success. In the real world, due to the lack of adequate information and/or imprecision of the available information on the environment, players are not able to estimate exactly payoffs of outcomes in the games. In order to develop an effective methodology, fuzzy, interval and stochastic approaches are frequently used to describe the imprecise and uncertain factors appearing in real game problems. Hence, the fuzzy games, interval-valued games and the stochastic games have been studied. Lots of papers and books (Bector et al. 2004; Collins and Hu 2008; Li 2011; Larbani 2009; Nishizaki and Sakawa 2001; Nayak and Pal 2009; Vijay et al. 2005, 2007) have been published on these topics in which several types of games have been investigated. In some sense, it seems to be more natural for players to describe their negative feelings than positive attitudes. But simultaneously there may be hesitation degrees for players' judgment. However, the fuzzy, interval and stochastic approaches are no means to represent the negative feelings and the hesitation degrees of players. The intuitionistic fuzzy set (IFS) seems to be very useful for modeling situations like this. Atanassov (1986, 1999) introduced

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the concept of an IFS, which is characterized by two functions expressing the membership degree and the non-membership degree, respectively. The hesitation degree is equal to 1 minus the membership degree and the non-membership degree. The three elements of IFSs can express simultaneously the satisfaction degrees, the non-satisfaction degrees and the hesitation degrees of players for the payoff values in the games. As far as we know, there exists little investigation on games using the IFSs. Atanassov (1999) firstly defined and described a game problem using the IFSs. Nayak and Pal (2007, 2010) studied the bi-matrix games with intuitionistic fuzzy goals and intuitionistic fuzzy payoffs, respectively. Aggarwal et al. (2012) studied the matrix games with IFS goals. Li (2010) proposed matrix games with payoffs represented by Atanassov's interval-valued IFSs which are an extension of IFSs. Li and Liu (2014) studied a parameterized non-linear programming approach to solve matrix games with payoffs of I-fuzzy numbers. Li and Yang (2013) given a bilinear programming approach to solve bi-matrix games with payoffs of trapezoidal intuitionistic fuzzy numbers. Rahaman et al. (2015) studied bi-matrix games with pay-offs of triangular intuitionistic fuzzy numbers. Nan et al. (2010) studied matrix games with payoffs of triangular intuitionistic fuzzy numbers which are a special case of IFSs. Li and Nan (2009) studied the matrix games with IFS payoffs and transformed solving matrix games with IFS payoffs into solving the nonlinear programming models in terms of the inclusion relation of IFSs and their operations. However, from the viewpoint of logic and the concept of matrix games with IFS payoffs, Player I's gain-floor and Player II's loss-ceiling of matrix games with IFS payoffs should be IFSs, while they can not be explicitly obtained by Li and Nan's model (2009) even though these are very much desirable. Obviously, this case is not rational and effective. On the other hand, in Li and Nan's method (2009) the bi-objective nonlinear programming models are solved by using the weighted average method to aggregate constraints and objective functions, which may loss some information. Thus, this paper focuses on developing an effective method to determine the Pareto Nash equilibrium strategies and the IFS-type values for Player I's gain-floor and Player II's loss-ceiling of matrix games with IFS payoffs.

A new ranking order relation of IFSs is proposed and the concept of Pareto Nash equilibrium strategies of matrix games with IFS payoffs is firstly defined. It has been proven that the Pareto Nash equilibrium strategies, Player I's gain-floor and Player II's loss-ceiling of matrix games with IFS payoffs are equivalent to solving a pair of bi-objective linear programming models.

The rest of this paper is organized as follows. In Sect. 2, the concept of the IFSs and their operations are briefly reviewed. In addition, a new ranking order relation of IFSs is proposed. In Sect. 3, matrix games with IFS payoffs are formulated and the concept of solutions of matrix games with IFS payoffs is defined. Pareto Nash equilibrium strategies for two players, Player I's gain-floor and Player II's loss-ceiling of matrix games with IFS payoffs are obtained by solving a pair of bi-objective programming models. In Sect. 4, the method proposed in this paper is illustrated with a numerical example and compared with other methods to show the validity, applicability and superiority. Conclusion is made in Sect. 5.

2 Definitions and Notations

2.1 IFSs and Operations

The concept of an IFS was firstly introduced by Atanassov (1986, 1999).

Definition 1 (Atanassov 1986, 1999). Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universal set. An IFS $\tilde{A}$ in X may be mathematically expressed as $\tilde{A} = \{ < x_l, \mu_{\tilde{A}}(x_l), \upsilon_{\tilde{A}}(x_l) > |x_l \in X\}$, where $\mu_{\tilde{A}} : X \mapsto [0, 1]$ and $v_{\tilde{A}} : X \mapsto [0, 1]$ are the membership degree and the non-membership degree of an element $x_l \in X$ to the set $\tilde{A} \subseteq X$, respectively, such that they satisfy the following condition: $0 \le \mu_{\tilde{A}}(x_l) + \upsilon_{\tilde{A}}(x_l) \le 1$ for every $x_l \in X$.

Let $\pi_{\tilde{A}}(x_l) = 1 - \mu_{\tilde{A}}(x_l) - \upsilon_{\tilde{A}}(x_l)$, which is called the intuitionistic index (or hesitancy degree) of an element x_l to the set $\tilde{A}$. It is the degree of indeterminacy membership of the element x_l to the set $\tilde{A}$. Obviously, $0 \le \pi_{\tilde{A}}(x_l) \le 1$.

If an IFS $\tilde{C}$ in X is a singleton set, i.e., $\tilde{C} = \{ < x_k, \mu_{\tilde{C}}(x_k), \upsilon_{\tilde{C}}(x_k) > \}$, then it is usually denoted by $\tilde{C} = < \mu_{\tilde{C}}(x_k), \upsilon_{\tilde{C}}(x_k) >$ for short.

Definition 2 (Atanassov 1986, 1999). Let $\tilde{A}$ and $\tilde{B}$ be two IFSs in the set X, and $\lambda > 0$ be a real number. Designate:

(1) $\tilde{A} + \tilde{B} = \{ < x_l, \mu_{\tilde{A}}(x_l) + \mu_{\tilde{B}}(x_l) - \mu_{\tilde{A}}(x_l)\mu_{\tilde{B}}(x_l), \upsilon_{\tilde{A}}(x_l)\upsilon_{\tilde{B}}(x_l) > |x_l \in X\}$;
(2) $\tilde{A}\tilde{B} = \{ < x_l, \mu_{\tilde{A}}(x_l)\mu_{\tilde{B}}(x_l), \upsilon_{\tilde{A}}(x_l) + \upsilon_{\tilde{B}}(x_l) - \upsilon_{\tilde{A}}(x_l)\upsilon_{\tilde{B}}(x_l) > |x_l \in X\}$;
(3) $\lambda\tilde{A} = \{ < x_l,\ 1 - (1 - \mu_{\tilde{A}}(x_l))^{\lambda},\ (\upsilon_{\tilde{A}}(x_l))^{\lambda} > |x_l \in X\}$.
(4) $A \subseteq B$ if and only if for any $x_l \in X$, $\mu_A(x_l) \le \mu_B(x_l)$ and $\upsilon_A(x_l) \ge \upsilon_B(x_l)$.

When the IFSs are used to model game problems, the comparison or ranking order of IFSs is important. In fact, the IFS $\tilde{C} = < \mu_{\tilde{C}}(x_l), \upsilon_{\tilde{C}}(x_l) >$ is mathematically equivalent to the interval-valued fuzzy set, denoted by $[\mu_{\tilde{C}}(x_l), \mu_{\tilde{C}}(x_l) + \pi_{\tilde{C}}(x_l)]$ or $[\mu_{\tilde{C}}(x_l), 1 - \upsilon_{\tilde{C}}(x_l)]$. In order to facilitate the sequent discussions, inspired by the ranking order relation of intervals, we propose a new ranking method of IFSs, which is equivalent to the inclusion relation of IFSs, i.e., Eq. (4) in Definition 2.

Definition 3. Let $\tilde{A} = < \mu_{\tilde{A}}(x_l), \upsilon_{\tilde{A}}(x_l) >$ and $\tilde{B} = < \mu_{\tilde{B}}(x_l), \upsilon_{\tilde{B}}(x_l) >$ be two IFSs in the set X. Then, the IFS order relation "$\le_{IF}$" is defined as $\tilde{A} \le_{IF} \tilde{B}$ if and only if $\mu_{\tilde{A}}(x_l) \le \mu_{\tilde{B}}(x_l)$ and $1 - \upsilon_{\tilde{A}}(x_l) \le 1 - \upsilon_{\tilde{B}}(x_l)$.

The symbol "$\le_{IF}$" is an intuitionistic fuzzy version of the order relation "$\le$" in the real line and has the linguistic interpretation "essentially smaller than or equal to".

3 The Models and Method for Matrix Games with IFS Payoffs

Let us consider any matrix games with IFS payoffs. Assume that $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ and $S_2 = \{\beta_1, \beta_2, \cdots, \beta_n\}$ are sets of pure strategies for Players I and II, respectively. Denote $I = \{1, 2, \cdots, m\}$ and $J = \{1, 2, \cdots, n\}$. The vectors $\boldsymbol{x} = (x_1, x_2, \cdots, x_m)^{\mathrm{T}}$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)^{\mathrm{T}}$ are mixed strategies for Players I and II, respectively, where

$x_i (i \in I)$ and $y_j (j \in J)$ are probabilities which Players I and II choose their pure strategies $\alpha_i \in S_1 (i \in I)$ and $\beta_j \in S_2 (j \in J)$, respectively. Sets of mixed strategies for Players I and II are denoted by X and Y, respectively, i.e., $X = \{\boldsymbol{x} | \sum_{i=1}^{m} x_i = 1, x_i \geq 0 \ (i \in I)\}$ and $Y = \{\boldsymbol{y} | \sum_{j=1}^{n} y_j = 1, y_j \geq 0 \ (j \in J)\}$. If Player I chooses pure strategy $\alpha_i \in S_1 (i \in I)$ and Player II chooses pure strategy $\beta_j \in S_2 (j \in J)$, then at the outcome (α_i, β_j) Player I gains a payoff a_{ij} expressed with an IFS $a_{ij} = <\mu_{ij}, \upsilon_{ij}>$ $(i \in I; j \in J)$, while Player II loses the payoff a_{ij}, i.e., the IFS $<\mu_{ij}, \upsilon_{ij}>$. Thus, a payoff matrix for Player I is concisely expressed in the matrix format as follows:

$$\tilde{\boldsymbol{A}} = (<\mu_{ij}, \upsilon_{ij}>)_{m \times n} = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{array} \begin{array}{c} \begin{array}{cccc} \beta_1 & \beta_2 & \cdots & \beta_n \end{array} \\ \begin{pmatrix} <\mu_{11}, \upsilon_{11}> & <\mu_{12}, \upsilon_{12}> & \cdots & <\mu_{1n}, \upsilon_{1n}> \\ <\mu_{21}, \upsilon_{21}> & <\mu_{22}, \upsilon_{22}> & \cdots & <\mu_{2n}, \upsilon_{2n}> \\ \vdots & \vdots & \vdots & \vdots \\ <\mu_{m1}, \upsilon_{m1}> & <\mu_{m2}, \upsilon_{m2}> & \cdots & <\mu_{mn}, \upsilon_{mn}> \end{pmatrix} \end{array}$$

In the following, the matrix games $\tilde{\boldsymbol{A}}$ with IFS payoffs are usually called the IFS matrix games $\tilde{\boldsymbol{A}}$ for short. Often they are used interchangeably.

If Player I chooses a mixed strategy $\boldsymbol{x} \in X$ and Player II chooses a mixed strategy $\boldsymbol{y} \in Y$, then an expected payoff for Player I is obtained as follows:

$$\tilde{E}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y} \tag{1}$$

According to Definition 2, the expected payoff $\tilde{E}(\boldsymbol{x}, \boldsymbol{y})$ in Eq. (1) is an IFS and can be calculated as follows:

$$\begin{aligned} \boldsymbol{x}^{\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y} &= (x_1, x_2, \cdots, x_m) \begin{pmatrix} <\mu_{11}, \upsilon_{11}> & <\mu_{12}, \upsilon_{12}> & \cdots & <\mu_{1n}, \upsilon_{1n}> \\ <\mu_{21}, \upsilon_{21}> & <\mu_{22}, \upsilon_{22}> & \cdots & <\mu_{2n}, \upsilon_{2n}> \\ \vdots & \vdots & \vdots & \vdots \\ <\mu_{m1}, \upsilon_{m1}> & <\mu_{m2}, \upsilon_{m2}> & \cdots & <\mu_{mn}, \upsilon_{mn}> \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\ &= <1 - \prod_{j=1}^{n} \prod_{i=1}^{m} (1 - \mu_{ij})^{x_i y_j}, \quad \prod_{j=1}^{n} \prod_{i=1}^{m} \upsilon_{ij}^{x_i y_j}> \end{aligned} \tag{2}$$

It is customary to assume that Player I is a maximizing player and Player II is a minimizing player. That is to say, Player II is interesting in finding a mixed strategy $\boldsymbol{y} \in Y$ so as to minimize $\tilde{E}(\boldsymbol{x}, \boldsymbol{y})$, denoted by

$$\upsilon(\boldsymbol{x}) = \min_{\boldsymbol{y} \in Y} \{\boldsymbol{x}^{\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}\}$$

Hence, Player I should choose a mixed strategy $\boldsymbol{x} \in X$ that maximizes the minimum expected gain, i.e.,

$$\upsilon^* = \max_{\boldsymbol{x} \in X} \min_{\boldsymbol{y} \in Y} \{\boldsymbol{x}^{\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}\} \tag{3}$$

Such υ^* is called Player I's gain-floor.

Similarly, Player I is interesting in finding a mixed strategy $\boldsymbol{x} \in X$ so as to maximize $\tilde{E}(\boldsymbol{x}, \boldsymbol{y})$, denoted by

$$\omega(\boldsymbol{y}) = \max_{\boldsymbol{x} \in X} \{\boldsymbol{x}^{\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}\}$$

Hence, Player II should choose a mixed strategy $\boldsymbol{y} \in Y$ that minimizes the maximum expected loss, i.e.,

$$\omega^* = \min_{\boldsymbol{y} \in Y} \max_{\boldsymbol{x} \in X} \{\boldsymbol{x}^{\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}\} \tag{4}$$

Such ω^* is called Player II's loss-ceiling.

It has been proved that Player I's gain-floor υ^* and the Player II's loss-ceiling ω^* are IFSs and such that $\upsilon^* \leq_{IF} \omega^*$ (Li and Nan 2009).

According to maximin and minimax principles for Players I and II, the Nash equilibrium strategy for the IFS matrix games $\tilde{\boldsymbol{A}}$ is defined as follows:

Definition 4. A pair $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in X \times Y$ is a Nash equilibrium strategy for the IFS matrix games $\tilde{\boldsymbol{A}}$ if

(1) $\boldsymbol{x}^{\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}^* \leq_{IF} \boldsymbol{x}^{*\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}^*$ for $\boldsymbol{x} \in X$;
(2) $\boldsymbol{x}^{*\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}^* \leq_{IF} \boldsymbol{x}^{*\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}$ for $\boldsymbol{y} \in Y$.

The expected payoff $\tilde{E}(\boldsymbol{x}, \boldsymbol{y})$ is an IFS with the membership degree and the non-membership degree which usually are conflict. Thus usually there do not exist above Nash equilibrium strategy defined in Definition 4 for the IFS matrix games $\tilde{\boldsymbol{A}}$. Then, the concepts of solutions of the IFS matrix games $\tilde{\boldsymbol{A}}$ may be given in a similar way to that of the Pareto optimal solutions as follows.

Definition 5. A pair $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in X \times Y$ is a Pareto Nash equilibrium strategy for the IFS matrix games $\tilde{\boldsymbol{A}}$ if

(1) There exists no $\mathbf{x} \in X$ such that $\boldsymbol{x}^{*\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}^* \leq_{IF} \boldsymbol{x}^{\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}^*$;
(2) There exists no $\boldsymbol{y} \in Y$ such that $\boldsymbol{x}^{*\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y} \leq_{IF} \boldsymbol{x}^{*\mathrm{T}} \tilde{\boldsymbol{A}} \boldsymbol{y}^*$.

Based on the ranking method of IFSs defined in Definition 3, the Theorem 1 is obtained as follows.

Theorem 1. For the IFS matrix games $\tilde{\boldsymbol{A}}$, $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in X \times Y$ is a Pareto Nash equilibrium strategy for the IFS matrix games $\tilde{\boldsymbol{A}}$ if and only if

(1) There exists no $\boldsymbol{x} \in X$ such that $\boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}^* \leq \boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}^*$ and $\boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}^* \leq \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}^*$;
(2) There exists no $\mathbf{y} \in Y$ such that $\mathbf{x}^{*\mathrm{T}}\boldsymbol{\mu}\mathbf{y} \leq \mathbf{x}^{*\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}^*$ and $\boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y} \leq \boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}^*$,

where $\boldsymbol{\mu} = (\mu_{\tilde{\boldsymbol{A}}ij})_{m\times n}$ $\boldsymbol{e} = (1)_{m\times n}$ and $\boldsymbol{\upsilon} = (\upsilon_{\tilde{\boldsymbol{A}}ij})_{m\times n}$.

Proof: Let $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ be a Nash equilibrium strategy for the IFS matrix games $\tilde{\boldsymbol{A}}$. First, we assume that there exists a strategy $\bar{\boldsymbol{x}} \in X$ such that

$$\boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}^* \leq \bar{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}^* \text{ and } \boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}^* \leq \bar{\boldsymbol{x}}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}^*,$$

i.e.,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\mu_{ij}y_j^* \leq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i\mu_{ij}y_j^* \text{ and } \sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*(1-\upsilon_{ij})y_j^* \leq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i(1-\upsilon_{ij})y_j^*,$$

which are equivalent to the inequalities as follows:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*(1-\mu_{ij})y_j^* \geq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i(1-\mu_{ij})y_j^* \tag{5}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^*\upsilon_{ij}y_j^* \geq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i\upsilon_{ij}y_j^* \tag{6}$$

respectively.

As $\ln x$ is a monotonically increasing function, $x_i^* \geq 0$, $y_j^* \geq 0$, $1-\mu_{ij} \geq 0$ and $1 \geq \upsilon_{ij} \geq 0$. Then it is easily seen that Eqs. (5) and (6) are equivalent to

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* \ln(1-\mu_{ij})y_j^* \leq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i \ln(1-\mu_{ij})y_j^* \tag{7}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i^* \ln \upsilon_{ij}y_j^* \leq \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_i \ln \upsilon_{ij}y_j^* \tag{8}$$

respectively.

Equations (7) and (8) imply that

$$\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\mu_{ij})^{x_i^*y_j^*} \leq \prod_{j=1}^{n}\prod_{i=1}^{m}(1-\mu_{ij})^{\bar{x}_iy_j^*} \tag{9}$$

and

$$\prod_{j=1}^{n}\prod_{i=1}^{m}\upsilon_{ij}^{x_i^* y_j^*} \leq \prod_{j=1}^{n}\prod_{i=1}^{m}\upsilon_{ij}^{\bar{x}_i y_j^*} \tag{10}$$

respectively. Hence, we obtain

$$1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\mu_{ij})^{x_i^* y_j^*} \geq 1-\prod_{j=1}^{n}\prod_{i=1}^{m}(1-\mu_{ij})^{\bar{x}_i y_j^*} \tag{11}$$

and

$$1-\prod_{j=1}^{n}\prod_{i=1}^{m}\upsilon_{ij}^{x_i^* y_j^*} \geq 1-\prod_{j=1}^{n}\prod_{i=1}^{m}\upsilon_{ij}^{\bar{x}_i y_j^*} \tag{12}$$

respectively.

It is derived from Definition 3 and Eqs. (2), (11) and (12) that

$$\boldsymbol{x}^{*\mathrm{T}}\tilde{\boldsymbol{A}}\boldsymbol{y}^* \leq_{IF} \bar{\boldsymbol{x}}^{\mathrm{T}}\tilde{\boldsymbol{A}}\boldsymbol{y}^* \tag{13}$$

Combining with Eq. (13) and according to Definition 5, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is not a Nash equilibrium strategy for the IFS matrix games $\tilde{\boldsymbol{A}}$. Hence, there exists a contradiction with the assumption.

Analogously, we can prove that there exists no $\boldsymbol{y} \in Y$ such that $\boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y} \leq \boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}^*$ and $\boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y} \leq \boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}^*$.

Theorem 2. The strategy $\boldsymbol{x}^*$ is a Pareto Nash equilibrium strategy and $<\mu^*, \upsilon^*>$ is the Player I's gain-floor for the IFS matrix games $\tilde{\boldsymbol{A}}$ if and only if $(\boldsymbol{x}^*, \mu^*, \upsilon^*)$ is an efficient solution of the bi-objective programming model as follows:

$$\begin{aligned} &\max\{\mu, 1-\upsilon\} \\ &s.t.\begin{cases} \sum_{i=1}^{m}\mu_{ij}x_i \geq \mu \quad (j=1,2,\cdots,n) \\ \sum_{i=1}^{m}(1-\upsilon_{ij})x_i \geq 1-\upsilon \quad (j=1,2,\cdots,n) \\ 0 \leq \mu+\upsilon \leq 1 \\ \sum_{i=1}^{m}x_i = 1 \\ \mu \geq 0,\ \upsilon \geq 0,\ x_i \geq 0 \quad (i=1,2,\cdots,m) \end{cases} \end{aligned} \tag{14}$$

Proof: Let $\boldsymbol{x}^*$ be a Pareto Nash equilibrium strategy for Player I. Then, there is no $\boldsymbol{x} \in \boldsymbol{X}$ such that

$$\boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}^* \leq \boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}^* \text{ and } \boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}^* \leq \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}^*$$

namely,

$$\min_{\boldsymbol{y}\in Y}\{\boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}\} \leq \min_{\boldsymbol{y}\in Y}\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}\} \text{ and } \min_{\boldsymbol{y}\in Y}\{\boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}\} \leq \min_{\boldsymbol{y}\in Y}\{\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}\} \quad (15)$$

which can be rewritten as follows:

$$\min_{\boldsymbol{y}\in Y}\{\boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}, \boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}\} \leq \min_{\boldsymbol{y}\in Y}\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mu}\boldsymbol{y}, \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{y}\}. \quad (16)$$

It makes sense to consider only the extreme points of set $\boldsymbol{Y}$ in Eq. (22). Then, we have

$$\min_{j\in J}\{\boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\mu}\boldsymbol{e}_1, \boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{e}_1\} \leq \min_{j\in J}\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mu}\boldsymbol{e}_1, \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{e}_1\}, \quad (17)$$

where $\boldsymbol{e}_1 = (1, \cdots, 1)$.

Note that Eq. (23) states that $\boldsymbol{x}^*$ is an efficient solution of the model as follows:

$$\max_{\boldsymbol{x}\in \boldsymbol{X}}\{\min_{j\in J}\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mu}\boldsymbol{e}_1\}, \min_{j\in J}\{\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{e}_1\}\}. \quad (18)$$

Let $\mu = \min_{j\in J}\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mu}\boldsymbol{e}_1\}$ and $1-\upsilon = \min_{j\in J}\{\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\upsilon})\boldsymbol{e}_1\}$. Then, Eq. (18) can be expressed as Eq. (14).

Analogous to Theorem 2, we have the following theorem for Player II.

Theorem 3. The strategy $\boldsymbol{y}^*$ is a Pareto Nash equilibrium strategy and $<\sigma^*, \rho^*>$ is the Player II's loss-ceiling for the IFS matrix games $\tilde{\boldsymbol{A}}$ if and only if $(\boldsymbol{y}^*, \sigma^*, \rho^*)$ is an efficient solution of the bi-objective programming model as follows:

$$\min\{\sigma, 1-\rho\}$$
$$s.t.\begin{cases} \sum_{j=1}^{n} \mu_{ij}y_j \leq \sigma \quad (i = 1, 2, \cdots, m) \\ \sum_{j=1}^{n} (1-\upsilon_{ij})y_j \leq 1-\rho \quad (i = 1, 2, \cdots, m) \\ 0 \leq \sigma + \rho \leq 1 \\ \sum_{j=1}^{n} y_j = 1 \\ \sigma \geq 0,\ \rho \geq 0,\ y_j \geq 0 \quad (j = 1, 2, \cdots, n) \end{cases} \quad (19)$$

Proof: Let $\boldsymbol{y}^*$ be a Pareto Nash equilibrium strategy for Player II. Then, there is no $\boldsymbol{y} \in \boldsymbol{Y}$ such that

$$\boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\sigma y} \leq \boldsymbol{x}^{*\mathrm{T}}\boldsymbol{\sigma y}^* \text{ and } \boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y} \leq \boldsymbol{x}^{*\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y}^*,$$

namely,

$$\max_{\boldsymbol{x}\in\boldsymbol{X}}\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\sigma y}\} \leq \max_{\boldsymbol{x}\in\boldsymbol{X}}\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\sigma y}^*\} \text{ and } \max_{\boldsymbol{x}\in\boldsymbol{X}}\{\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y}\} \leq \max_{\boldsymbol{x}\in\boldsymbol{X}}\{\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y}^*\}, \tag{20}$$

which can be rewritten as follows:

$$\max_{\boldsymbol{x}\in\boldsymbol{X}}\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\sigma y}, \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y}\} \leq \max_{\boldsymbol{x}\in\boldsymbol{X}}\{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\sigma y}^*, \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y}^*\}. \tag{21}$$

It makes sense to consider only the extreme points of set $\boldsymbol{X}$ in Eq. (27). Then, we have

$$\max_{i\in I}\{\boldsymbol{e}_2^{\mathrm{T}}\boldsymbol{\sigma y}, \boldsymbol{e}_2^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y}\} \leq \max_{i\in I}\{\boldsymbol{e}_2^{\mathrm{T}}\boldsymbol{\sigma y}, \boldsymbol{e}_2^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y}^*\}, \tag{22}$$

where $\boldsymbol{e}_2 = (1, \cdots, 1)$.

Note that Eq. (22) states that $\boldsymbol{y}^*$ is an efficient solution of the model as follows:

$$\min_{\boldsymbol{y}\in\boldsymbol{Y}}\{\max_{i\in I}\{\boldsymbol{e}_2^{\mathrm{T}}\boldsymbol{\sigma y}\}, \max_{i\in I}\{\boldsymbol{e}_2^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y}\}\}. \tag{23}$$

Let $\sigma = \max_{i\in I}\{\boldsymbol{e}_2^{\mathrm{T}}\boldsymbol{\sigma y}\}$ and $1-\rho = \max_{i\in I}\{\boldsymbol{e}_2^{\mathrm{T}}(\boldsymbol{e}-\boldsymbol{\rho})\boldsymbol{y}\}$. Then, Eq. (23) can be expressed as Eq. (19).

4 A Numerical Example and Comparative Analysis

4.1 A Numerical Example

A numerical example is adopted from Li and Nan (2009) in order to conduct comparison between Li and Nan's method (2009) and the method proposed in this paper. The computational results are analyzed and compared to show the validity, applicability and superiority of the method proposed in this paper.

Let us consider the specific IFS matrix games $\tilde{\boldsymbol{A}}$, where the IFS payoff matrix is given as follows (Li and Nan 2009):

$$\tilde{\boldsymbol{A}} = \begin{matrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{matrix}\overset{\begin{matrix}\alpha_1 & \qquad\qquad \alpha_2 & \qquad\qquad \alpha_3\end{matrix}}{\begin{pmatrix} <0.95, 0.05> & <0.70, 0.25> & <0.50, 0.40> \\ <0.25, 0.70> & <0.95, 0.05> & <0.70, 0.25> \\ <0.50, 0.40> & <0.05, 0.95> & <0.95, 0.05> \end{pmatrix}} \tag{24}$$

According to Eq. (14), the bi-objective programming model is constructed as follows:

$$\max\{\mu, 1-\upsilon\}$$
$$s.t.\begin{cases} 0.95x_1 + 0.25x_2 + 0.5x_3 \geq \mu \\ 0.7x_1 + 0.95x_2 + 0.05x_3 \geq \mu \\ 0.5x_1 + 0.7x_2 + 0.95x_3 \geq \mu \\ 0.95x_1 + 0.3x_2 + 0.6x_3 \geq 1-\upsilon \\ 0.75x_1 + 0.95x_2 + 0.05x_3 \geq 1-\upsilon \\ 0.6x_1 + 0.75x_2 + 0.95x_3 \geq 1-\upsilon \\ x_1 + x_2 + x_3 = 1 \\ \mu + \upsilon \leq 1 \\ x_i, \mu, \upsilon \geq 0 \quad (i = 1, 2, 3) \end{cases} \tag{25}$$

where x_1, x_2, x_3, μ and υ are decision variables. Using the software package ADBASE (Steuer 1995), the efficient solution $(\boldsymbol{x}^*, \mu^*, \upsilon^*)$ of Eq. (25) can be obtained, where $\boldsymbol{x}^* = (0.543, 0.3, 0.157)$, $\mu^* = 0.649$ and $\upsilon^* = 0.286$. Thus, $\boldsymbol{x}^* = (0.5, 0.304, 0.196)$ is a Pareto Nash equilibrium strategy of Player I and Player I's gain-floor is given by the IFS $<\mu^*, \upsilon^*> = <0.649, 0.286>$ for the IFS matrix games $\tilde{\boldsymbol{A}}$.

Similarly, according to Eq. (19), the bi-objective programming model is constructed as follows:

$$\min\{\sigma, 1-\rho\}$$
$$s.t.\begin{cases} 0.95y_1 + 0.7y_2 + 0.5y_3 \leq \sigma \\ 0.25y_1 + 0.95y_2 + 0.7y_3 \leq \sigma \\ 0.5y_1 + 0.05y_2 + 0.95y_3 \leq \sigma \\ 0.95y_1 + 0.75y_2 + 0.6y_3 \leq 1-\rho \\ 0.3y_1 + 0.95y_2 + 0.75y_3 \leq 1-\rho \\ 0.6y_1 + 0.05y_2 + 0.95y_3 \leq 1-\rho \\ y_1 + y_2 + y_3 = 1 \\ \sigma + \rho \leq 1 \\ y_i \geq 0, \sigma \geq 0, \rho \geq 0 \quad (i = 1, 2, 3) \end{cases} \tag{26}$$

where y_1, y_2, y_3, σ and ρ are decision variables. Using the software package ADBASE (Steuer 1995), the efficient solution $(\boldsymbol{y}^*, \sigma^*, \rho^*)$ of Eq. (26) can be obtained, where $\boldsymbol{y}^* = (0.234, 0.218, 0.548)$, $\sigma^* = 0.646$ and $\rho^* = 0.314$. Thus, $\boldsymbol{y}^* = (0.234, 0.203, 0.563)$ is a Pareto Nash equilibrium strategy of Player II and Player II's loss-ceiling is given by the IFS $<\sigma^*, \rho^*> = <0.646, 0.314>$ for the IFS matrix games $\tilde{\boldsymbol{A}}$.

4.2 Comparative Analysis of the Results Obtained by Li and Nan's Method and the Proposed Method

Li and Nan (2009) studied the matrix games with IFS payoffs. According to the definition of solutions for the IFS matrix games $\tilde{\boldsymbol{A}}$, the inclusion relation and the operations of IFSs, Player I's gain-floor $\bar{\theta} = <\bar{\mu}, \bar{\upsilon}>$ and corresponding optimal strategy $\bar{\boldsymbol{x}}$ for the IFS matrix games $\tilde{\boldsymbol{A}}$ can be generated by solving the bi-objective non-linear programming model constructed as follows:

$$\max\{\mu\},\ \min\{\upsilon\}$$
$$s.t.\begin{cases} 1-\prod_{i=1}^{m}(1-\mu_{ij})^{x_i} \geq \mu \quad (j=1,2,\cdots,n) \\ \prod_{i=1}^{m}\upsilon_{ij}^{x_i} \leq \upsilon \quad (j=1,2,\cdots,n) \\ 0 \leq \mu+\upsilon \leq 1 \\ \sum_{i=1}^{m}x_i = 1 \\ \mu \geq 0,\ \upsilon \geq 0,\ x_i \geq 0 \quad (i=1,2,\cdots,m) \end{cases} \tag{27}$$

where μ, υ and x_i $(i=1,2,\cdots,m)$ are decision variables. By using a variable substitution technique and the weighted average method to aggregate constraints and objective functions, the bi-objective nonlinear programming model (i.e., Eq. (27)) is transformed as follows:

$$\min\{p\}$$
$$s.t.\begin{cases} \prod_{i=1}^{m}[(1-\mu_{ij})^{\lambda}\ \upsilon_{ij}^{1-\lambda}]^{x_i} \leq p \quad (j=1,2,\cdots,n) \\ \sum_{i=1}^{m}x_i = 1 \\ x_i \geq 0 \quad (i=1,2,\cdots,m) \end{cases} \tag{28}$$

where $\lambda \in [0,1]$.

Similarly, Player II's loss-ceiling $\bar{\omega} = <\bar{\sigma}, \bar{\rho}>$ and corresponding optimal strategy $\bar{\boldsymbol{y}}$ for the IFS matrix games $\tilde{\boldsymbol{A}}$ can be generated by solving the bi-objective programming model constructed as follows:

$$\min\{\sigma\},\ \max\{\rho\}$$
$$s.t.\begin{cases} 1-\prod_{j=1}^{n}(1-\mu_{ij})^{y_j} \leq \sigma \quad (i=1,2,\cdots,m) \\ \prod_{j=1}^{n}\upsilon_{ij}^{y_j} \geq \rho \quad (i=1,2,\cdots,m) \\ 0 \leq \sigma+\rho \leq 1 \\ \sum_{j=1}^{n}y_j = 1 \\ \sigma \geq 0,\ \rho \geq 0,\ y_j \geq 0 \quad (j=1,2,\cdots,n) \end{cases} \tag{29}$$

In the same analysis to that of Player I, the bi-objective nonlinear programming model (i.e., Eq. (29)) is transformed as follows:

$$\max\{q\}$$
$$s.t.\begin{cases} \prod_{j=1}^{n} [(1-\mu_{ij})^{\lambda} \upsilon_{ij}^{1-\lambda}]^{y_j} \geq q \quad (i = 1, 2, \cdots, m) \\ \sum_{j=1}^{n} y_j = 1 \\ y_j \geq 0 \quad (j = 1, 2, \cdots, n) \end{cases} \tag{30}$$

For the IFS matrix games $\tilde{\boldsymbol{A}}$, according to Eqs. (24) and (28), the nonlinear programming model is constructed as follows:

$$\min\{p\}$$
$$s.t.\begin{cases} 0.05^{x_1}(0.75^{\lambda}0.7^{1-\lambda})^{x_2}(0.5^{\lambda}0.4^{1-\lambda})^{x_3} \leq p \\ (0.3^{\lambda}0.25^{1-\lambda})^{x_1}0.05^{x_2}0.95^{x_3} \leq p \\ (0.5^{\lambda}0.4^{1-\lambda})^{x_1}(0.3^{\lambda}0.25^{1-\lambda})^{x_2}0.05^{x_3} \leq p \\ x_1 + x_2 + x_3 = 1 \\ x_i \geq 0 \quad (i = 1, 2, 3) \end{cases} \tag{31}$$

Similarly, according to Eqs. (24) and (30), the nonlinear programming model is constructed as follows:

$$\max\{q\}$$
$$s.t.\begin{cases} 0.05^{y_1}(0.3^{\lambda}0.25^{1-\lambda})^{y_2}(0.5^{\lambda}0.4^{1-\lambda})^{y_3} \geq q \\ (0.75^{\lambda}0.7^{1-\lambda})^{y_1}0.05^{y_2}(0.3^{\lambda}0.25^{1-\lambda})^{y_3} \geq q \\ (0.5^{\lambda}0.4^{1-\lambda})^{y_1}0.95^{y_2}0.05^{y_3} \geq q \\ y_1 + y_2 + y_3 = 1 \\ y_j \geq 0 \quad (j = 1, 2, 3) \end{cases} \tag{32}$$

For some given values $\lambda \in (0, 1)$, using the nonlinear programming methods, optimal solutions of Eqs. (31) and (32) are obtained as in Table 1, respectively.

Table 1. Optimal solutions of Eqs. (31) and (32)

λ	$\bar{\boldsymbol{x}}$	$\bar{p}$	$\bar{\boldsymbol{y}}$	$\bar{q}$	$\bar{\boldsymbol{x}}^{\mathrm{T}}\tilde{\boldsymbol{A}}\bar{\boldsymbol{y}}$
0.1	$(0.414,0.335,0.251)^{\mathrm{T}}$	0.206	$(0.261,0.294,0.445)^{\mathrm{T}}$	0.206	<0.773, 0.203>
0.3	$(0.411,0.333,0.256)^{\mathrm{T}}$	0.210	$(0.265,0.295,0.440)^{\mathrm{T}}$	0.210	<0.759, 0.217>
0.5	$(0.408,0.332,0.260)^{\mathrm{T}}$	0.215	$(0.268,0.296,0.436)^{\mathrm{T}}$	0.215	<0.773, 0.204>
0.8	$(0.403,0.331,0.266)^{\mathrm{T}}$	0.222	$(0.275,0.297,0.428)^{\mathrm{T}}$	0.222	<0.773, 0.204>
0.9	$(0.402,0.330,0.268)^{\mathrm{T}}$	0.225	$(0.275,0.297,0.428)^{\mathrm{T}}$	0.225	<0.773, 0.204>

From Table 1, for given $\lambda \in (0, 1)$, Player I's optimal strategy $\bar{\boldsymbol{x}}$ and Player II's optimal strategy $\bar{\boldsymbol{y}}$ are obtained, respectively, whereas Player I's gain-floor and Player II's loss-ceiling of the IFS matrix games $\tilde{\boldsymbol{A}}$ can not be obtained. Furthermore, the expected payoff value obtained Li and Nan's method (2009) is the value <0.773, 0.204>, which is approximate to the expected payoff value <0.7603, 0.2103> obtained the proposed method in this paper.

Comparing Li and Nan's method (2009) and the method proposed in this paper, it is not difficult to draw the following conclusions.

(1) The proposed method not only provides the Nash equilibrium strategies but also the IFS-type values for Player I's gain-floor and Player II's loss-ceiling of the IFS matrix games $\tilde{\boldsymbol{A}}$, which cannot be obtained by Li and Nan's method (2009). The latter method only obtains the optimal strategies for two players, respectively. The method proposed in this paper obtains the IFS-type Player I's gain-floor $<\mu^*, \upsilon^*> \; = \; <0.649, 0.286>$ and Player II's loss-ceiling $<\sigma^*, \rho^*> \; = \; <0.646, 0.314>$ as well as the Pareto Nash equilibrium strategies for Player I and Player II.

(2) The amount of computation and the complexity of solving process of the proposed method are less than that using Li and Nan's method (2009). Player I's gain-floor, Player II's loss-ceiling and corresponding Pareto Nash equilibrium strategies can be obtained through solving the bi-objective linear programming models with the membership degrees and the non-membership degrees of the IFS-type payoffs in the payoff matrix, respectively. However, in Li and Nan's method (2009) the optimal strategies for two players are obtained by the bi-objective non-linear programming models, which are not easily solved.

(3) The method proposed in this paper does not involve in any subjective factor. However, in Li and Nan's method (2009), the obtained solutions closely depend on the parameter $\lambda \in [0, 1]$ and more or less involve in subjective factors such as attitude and preference. Thus, the computational results obtained by the method proposed in this paper is more rational, reliable and convincing than that obtained by Li and Nan's method (2009).

5 Conclusion

This paper develops a simple and an effective method for solving IFS matrix games. A new ranking order relation of IFSs is proposed and the concept of Pareto Nash equilibrium strategies of matrix games with IFS payoffs is firstly defined. The Pareto Nash equilibrium strategies and the IFS-type values for Player I's gain-floor and Player II's loss-ceiling of matrix games with IFS payoffs are obtained through solving the derived a pair of bi-objective linear programming models.

It is easily seen from the aforementioned discussion and comparison that our study is significantly different from Li and Nan's work (2009). Moreover, the method proposed in this paper has some remarkably advantages over Li and Nan's method (2009) from the aspects of the scale, solving process, validity and computation amount of the derived auxiliary programming models.

It is not difficult to see that the idea of this paper may be applicable to other games with payoff of IFSs such as two-person nonzero-sum games (i.e., bi-matrix games) and n-person non-cooperative games, which will be investigated in the near future.

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Cooperative Games

Marginal Games and Characterizations of the Shapley Value in TU Games

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Abstract. Axiomatizations and recursive representations of the Shapley value on the class of all cooperative games with transferable utilities are given. Marginal games, which are closely related to dual games, play central roles in our results. Our axiomatizations are based on axioms that are marginal game variations of the well-known balanced contributions property, so that they are interpreted as fair treatment between two players in games as the balanced contributions property is. Our general recursive representation enables us to represent the Shapley value for n-person games by those for r-person and $(n-r)$-person games with fixed $r < n$. The particular case of $r = 1$ has a clear contrasting interpretation to the existing recursive formula.

Keywords: Shapley value · Marginal game · Dual game · Balanced contribution · Recursive representation

1 Introduction

Duality plays important roles in cooperative games with transferable utilities (henceforth, TU games). Particularly, duality between games (i.e., duality between characteristic functions in games) has been extensively studied in the literature on TU games. Let N be a set of players and v a characteristic function on N. Given a TU game (N, v), its dual game is a game in which every coalition $T \subseteq N$ obtains the worth that is lost when the coalition leave from the grand coalition in the original game, that is, $v(N) - v(N \setminus T)$. Dual games are, for example, studied by [18] in voting games. In addition, [8] studies a dual game representation of bankruptcy games. [1] show that dual airport games are line-graph peer group games. [3] clarify that there is a duality relation between auction games and the ring games. [6] investigates the Shapley value of weighted cost games and their duals. [4] axiomatize classes of values by focusing on duality between so-called the CIS value and the ENSC value. Recently, based on duality between games, [13] define duality between solutions, axioms, and axiomatizations, and investigate duality between axiomatizations of solutions of TU games.

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Marginal games also play substantial role in TU games. Given a TU game (N, v) and a coalition $S \subseteq N$, the marginal game with respect to S is a game on the complement coalition of the coalition, that is, $N \setminus S$. In the game, any coalition $T \subseteq N \setminus S$ wins the cooperation of S by paying the worth of S in the original games, that is, T obtains $v(S \cup T) - v(S)$. The marginal games are used for construction an appropriate algorithm for generating the constrained egalitarian solution for convex game in [9]. Also, they characterize convex games and total clan games (see [2]). Marginal games are obtained by dual games. More precisely, a marginal game with respect to S is obtained by the following procedure: (i) consider a dual game of the game, (ii) consider a restriction of the dual game on $N \setminus S$, and (iii) again consider a dual game of the restricted game. Consequently, marginal games can be seen as a kind of dual games. In addition, this mechanism gives that properties of dual games can be translated to marginal games.

The Shapley value ([15]), is one of the central solution of TU games, and it has been subjected to extensive research. Among many researches on the Shapley value, a characterization by [12] using the balanced contributions property and efficiency not only sheds lights on reasonableness of the Shapley value but also gives us new viewpoints of it. By these axioms, the Shapley value is represented in a recursive manner with respect to numbers of players in games. In other words, the Shapley value of an n-person game is represented by the Shapley values of $(n-1)$-person games (subgames) which are restrictions of the original n-person game (see [10,11], [16]).

In this paper, we give new characterizations of the Shapley value, by focusing on marginal games. Our new characterizations are classified into two categories: axiomatizations and recursive representations.

For axiomatizations, we consider two variations of the balanced contributions property of [12] by using marginal games instead of subgames. The balanced contributions property of [12] requires that fair treatment of two players with respect to departure of each of the players from games. Our two variations are obtained by using (extended) marginal games instead of subgames, and thus, they can be interpreted as fair requirement between two players, as the original Myerson's balanced contributions property is. Although each of our new balanced contributions properties is different to the original balanced contributions property of [12], it characterizes the Shapley value in conjunction with efficiency (and the dummy player out property of [17]) in a similar manner as the original one.

For recursive representations, we revert the interpretation of the Shapley value as an expectation of each player's marginal contributions among all possible orders on the set of all players. This interpretation and marginal games fit together well, and we obtain a general recursive representation of the Shapley value for n-person games by that for r-person and $(n-r)$-person games with fixed $r < n$. The particular case of our general recursive representation when $r = 1$ is in contrast to an existing recursive formula of the Shapley value of [10,11], [16].

The paper is organized as follows. Basic notation and definitions are presented in Sect. 2. Marginal games and extended marginal games are defined and their properties are discussed in Sect. 3. Axiomatic characterizations of the Shapley value are given in Sect. 4. Recursive formulas of the Shapley value are provided in Sect. 5. A remark on non-cooperative foundation of the Shapley value is included in Sect. 6.

2 Preliminaries

A *TU game* is a pair (N, v) where $N \subseteq \mathbb{N}$ is a finite set of *players* and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is a *characteristic function*. Let $\mathcal{G}$ be a set of all TU games. Let $|N| = n$ where $|\cdot|$ represents the cardinality of a set. A subset S of N is called a *coalition*. For any $S \subseteq N$, $v(S)$ represents the worth of the coalition. For simplicity, each singleton is represented as i instead of $\{i\}$ when there exists no fear of confusion. For any two games $(N, v), (N, w) \in \mathcal{G}$, a game $(N, v + w)$ is defined as $(v + w)(S) = v(S) + w(S)$ for any $S \subseteq N$. For any $S \subseteq N$, the *subgame of* (N, v) *on* S is a pair $(S, v|_S)$ where $v|_S(T) = v(T)$ for any $T \subseteq S$. We write the subgame on S as (S, v), for simplicity. Given a game $(N, v) \in \mathcal{G}$, its *dual game* (N, v^*) is the game that assigns to each coalition $S \subseteq N$ the worth that is lost by the grand coalition N if S leaves N, that is, for each $S \subseteq N$, $v^*(S) = v(N) - v(N \setminus S)$. Given a game $(N, v) \in \mathcal{G}$, a player $i \in N$ is *dummy* in (N, v) if for any $S \subseteq N$ with $S \ni i$, it holds that $v(S) = v(S \setminus i) + v(i)$, and a player $i \in N$ is *null* in (N, v) if i is dummy and if $v(i) = 0$.

A *value* φ is a mapping from $\mathcal{G}$ to $\mathbb{R}^N$. A value φ

- is *additive* (ADD) if $\varphi(N, v+w) = \varphi(N, v)+\varphi(N, w)$ for any $(N, v), (N, w) \in \mathcal{G}$,
- is *self-dual* (SD) if $\varphi(N, v) = \varphi(N, v^*)$ for any $(N, v) \in \mathcal{G}$,
- satisfies the *dummy player out property* (DPO, [17]) if $\varphi_j(N, v) = \varphi_j(N \setminus i, v)$ for any dummy player i in (N, v), any $j \neq i$, and any $(N, v) \in \mathcal{G}$,
- satisfies the *null player property* (NP) if $\varphi_i(N, v) = 0$ for any null player i in (N, v) and any $(N, v) \in \mathcal{G}$,
- satisfies the *null player out property* (NPO, [7]) if $\varphi_j(N, v) = \varphi_j(N \setminus i, v)$ for any null player i in (N, v), any $j \neq i$, and any $(N, v) \in \mathcal{G}$,
- satisfies the *balanced contributions property* (BC, [12]) if $\varphi_i(N, v) - \varphi_i(N \setminus j, v) = \varphi_j(N, v) - \varphi_j(N \setminus i, v)$ for any $i, j \in N$ with $i \neq j$, and any $(N, v) \in \mathcal{G}$, and
- is *efficient* (EFF) if $\sum_{i \in N} \varphi_i(N, v) = v(N)$ for any $(N, v) \in \mathcal{G}$.

One of the well-known values in TU games is the Shapley value ([15]). Let π be an order on N and Π be the set of all orders on N. Given $(N, v) \in \mathcal{G}$, the *Shapley value* $\phi(N, v)$ is defined as follows: For each $i \in N$,

$$\phi_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi} \Big(v(\{j | \pi(j) \leq \pi(i)\}) - v(\{j | \pi(j) < \pi(i)\}) \Big).$$

The term $v(\{j|\pi(j) \leq \pi(i)\}) - v(\{j|\pi(j) < \pi(i)\})$ is called i's *marginal contribution* in order π. Player i's Shapley value can be interpreted as the expected value of i's marginal contributions with respect to the discrete uniform distribution on the set of all possible orders of all players. It is well-known that the Shapley value satisfies all of the above seven properties.

3 Marginal Games and Extended Marginal Games

Given a game $(N, v) \in \mathcal{G}$ and a coalition $S \subseteq N$, the *S-marginal game* $(N \setminus S, v^S)$ is the game that assigns to each coalition $T \subseteq N \setminus S$ the worth of $S \cup T$ minus the worth of S, that is, for each $T \subseteq N \setminus S$, $v^S(T) = v(S \cup T) - v(S)$. In the S-marginal game, any subset of $N \setminus S$ can win the cooperation of S by paying the value $v(S)$ to S. Between the dual games and the marginal games, the following holds.

Proposition 1. *For any $(N, v) \in \mathcal{G}$ and any $S \subseteq N$, it holds $v^S = (v^*|_{N \setminus S})^*$.*

Proof. For any $S \subseteq N$ and any $T \subseteq N \setminus S$, $(v^*|_{N \setminus S})^*(T) = v^*(N \setminus S) - v^*((N \setminus S) \setminus T) = v(N) - v(S) - v(N) + v(S \cup T) = v^S(T)$. □

By comparing dual games and marginal games, the set of players are different. By extending marginal games on the same set of players as the original games and dual games, we introduce an alternative approach.

Given a game $(N, v) \in \mathcal{G}$ and a coalition $S \subseteq N$, the *extended S-marginal game* $(N, \bar{v}^S)$ is the game that assigns to each coalition $T \subseteq N$ the worth of $T \cup S$ minus the worth of $S \setminus T$, that is, for each $T \subseteq N$, $\bar{v}^S(T) = v(S \cup T) - v(S \setminus T)$. By this definition, for any $T \subseteq N \setminus S$, $\bar{v}^S|_{N \setminus S}(T) = v(S \cup T) - v(S) = v^S(T)$, that is, the difference between the extended S-marginal game and the S-marginal game is whether or not the players of coalition S are included in the player set. In addition, let $T \subseteq N \setminus S$ and let $R \subseteq S$. Then by definition, $\bar{v}^S(T \cup R) = v(S \cup T) - v(S \setminus R)$, $\bar{v}^S(T) = v(S \cup T) - v(S)$, and $\bar{v}^S(R) = v(S) - v(S \setminus R)$. Thus, $\bar{v}^S(T \cup R) = \bar{v}^S(T) + \bar{v}^S(R)$. This means that any $i \in N$ is a dummy player in $(N, \bar{v}^i)$, and further, this additive feature of extended marginal games essentially reduces the computational complexity of the Shapley value in the following way.

Proposition 2. *Given $(N, v) \in \mathcal{G}$, for any $S \subseteq N$ and any $i \in N$, $\phi_i(N, \bar{v}^S) = \phi_i(S, v)$ if $i \in S$, and $\phi_i(N, \bar{v}^S) = \phi_i(N \setminus S, v^S)$ if $i \notin S$.*

Proof. Given $(N, v) \in \mathcal{G}$ and $S \subseteq N$, let $w^S(T) = v(S) - v(S \setminus T)$, and $u^S(T) = v(S \cup T) - v(S)$, for any $T \subseteq N$. Then, $w^S + u^S = \bar{v}^S$. By ADD of the Shapley value, $\phi(N, \bar{v}^S) = \phi(N, w^S) + \phi(N, u^S)$. By definition, any $i \notin S$ is a null player in (N, w^S) and any player $i \in S$ is a null player in (N, u^S), respectively. By NP and NPO of the Shapley value, $\phi_i(N, \bar{v}^S) = \phi_i(S, w^S)$ if $i \in S$, and $\phi_i(N, \bar{v}^S) = \phi_i(N \setminus S, u^S)$ if $i \notin S$.

By definition, $(w^S|_S)^*(T) = w^S(S) - w^S(S \setminus T) = v(S) - v(S \setminus S) - v(S) + v(S \setminus (S \setminus T)) = v(T)$ for any $T \subseteq S$, and $u^S(T) = v^S(T)$ for any $T \subseteq N \setminus S$. Thus, for any $i \in S$, SD of the Shapley value implies that $\phi_i(S, w^S) = \phi_i(S, (w^S)^*) = \phi_i(S, v)$, and for any $i \in N \setminus S$, $\phi_i(N \setminus S, u^S) = \phi_i(N \setminus S, v^S)$. □

4 Marginal Games and Axiomatic Characterizations of the Shapley Value

[13] introduce duality between axioms, and review axiomatizations of the Shapley value in [10] and [5]. Similarly, by focusing on marginal games, we consider axiomatizations of the Shapley value for that in [12] that axiomatized the Shapley value by BC and EFF. The following is a marginal game variation of BC.

Balanced M(marginal)-contributions property (BMC): For each $(N, v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N, v) - \varphi_i(N \setminus j, v^j) = \varphi_j(N, v) - \varphi_j(N \setminus i, v^i).$$

BC and BMC are different. To see this, let us consider two values φ^1 and φ^2 such that $\varphi^1_i(N, v) = \sum_{S \ni i, S \subseteq N} v(S)$, and $\varphi^2_i(N, v) = \varphi^1_i(N, v^*) = 2^{n-1}v(N) - \sum_{S \not\ni i, S \subseteq N} v(S)$, for any $i \in N$ and any $(N, v) \in \mathcal{G}$. Note that φ^2 is a dual of φ^1 because it applies the same value to dual games (see [13]). φ^1 satisfies BC, however, not BMC. Meanwhile, φ^2 satisfies BMC, however, not BC. Therefore, BC and BMC are independent with each other. However, the two properties are the same under SD.

Proposition 3. *Under SD, BC and BMC are equivalent.*

Proof. Given $(N, v) \in \mathcal{G}$, by applying BC to its dual game (N, v^*), we obtain $\varphi_i(N, v^*) - \varphi_i(N \setminus j, v^*) = \varphi_j(N, v^*) - \varphi_j(N \setminus i, v^*)$. By Proposition 1, SD of φ implies that $\varphi_i(N, v^*) = \varphi_i(N, v)$, $\varphi_i(N \setminus j, v^*) = \varphi_i(N \setminus j, v^j)$, $\varphi_j(N, v^*) = \varphi_j(N, v)$ and $\varphi_j(N \setminus i, v^*) = \varphi_j(N \setminus i, v^i)$. □

Despite the difference, BMC characterizes the Shapley value in conjunction with EFF as BC does.

Theorem 1. *The Shapley value is the unique value which satisfies BMC and EFF.*

Proof. SD of the Shapley value and Proposition 3 together imply that the Shapley value satisfies BMC.

For uniqueness, we use induction with respect to the number of players. Let φ be a value on $\mathcal{G}$ that satisfies BMC and EFF. In the case of $|N| = 1$, $\varphi_i(N, v) = v(i) = \phi_i(N, v)$ for $i \in N$. Let $|N| = n \geq 2$ and suppose that $\varphi = \phi$ in case of there are less than n players. Consider the case of n players. Fix $i \in N$; by BMC and the supposition above, for any $j \in N \setminus i$, $\varphi_i(N, v) - \varphi_j(N, v) = \varphi_i(N \setminus j, v^j) - \varphi_j(N \setminus i, v^i) = \phi_i(N \setminus j, v^j) - \phi_j(N \setminus i, v^i) = \phi_i(N, v) - \phi_j(N, v)$. Summing up the above equalities over $j \in N \setminus i$ (and making simple calculations), we obtain $\varphi_i(N, v) = \phi_i(N, v)$. For any $j \neq i$, $\varphi_j(N, v) = \phi_j(N, v)$ is shown in the same manner. Hence, $\varphi = \phi$ in the case of n players. □

For the independence of axioms, $\varphi^3(N,v) = (0,0,\ldots,0)$, for any $(N,v) \in \mathcal{G}$ satisfies BMC but not EFF, and $\varphi^4(N,v) = (\frac{v(N)}{|N|}, \frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|})$, for any $(N,v) \in \mathcal{G}$ satisfies EFF but not BMC.

The following corresponding axiom using the extended marginal games instead of the marginal games is equivalent to BMC under DPO, and hence, it also characterizes the Shapley value in conjunction with EFF and DPO.

Balanced EM(extended-marginal)-contributions property (BEMC): For each $(N,v) \in \mathcal{G}$ and any $i, j \in N$ with $i \neq j$,

$$\varphi_i(N,v) - \varphi_i(N,\bar{v}^j) = \varphi_j(N,v) - \varphi_j(N,\bar{v}^i).$$

Proposition 4. *Under DPO, BMC and BEMC are equivalent.*

Proof. Since j is a dummy player in $(N,\bar{v}^j)$, DPO implies that $\varphi_i(N,\bar{v}^j) = \varphi_i(N \setminus j, v^j)$ for any $i \neq j$. □

Theorem 1 and Proposition 4 together imply the following,

Corollary 1. *The Shapley value is the unique value which satisfies BEMC, EFF and DPO.*

For the independence of axioms, a value φ^5 defined by for any $(N,v) \in \mathcal{G}$ and any $i \in N$, $\varphi^5_i(N,v) = v(N)$ if $i = \min_{j \in N} j$, and $\varphi^5_i(N,v) = 0$ if $i \neq \min_{j \in N} j$, satisfies BEMC and EFF but not DPO. A value φ^3 satisfies BEMC and DPO but not EFF. Let $D(N,v) \subseteq N$ is a set of all dummy players in $(N,v) \in \mathcal{G}$. A value φ^6 defined by for any $(N,v) \in \mathcal{G}$ and any $i \in N$, $\varphi^6_i(N,v) = v(i)$ if $i \in D(N,v)$, and $\frac{v(N \setminus D(N,v))}{|N \setminus D(N,v)|}$ if $i \notin D(N,v)$, satisfies EFF and DPO but not BEMC.

5 Marginal Games and Recursive Formulas of the Shapley Value

[10,11], [16] show that the Shapley value for n-person games is represented by using the Shapley value for $(n-1)$-person games in the following manner.

$$\phi_i(N,v) = \frac{1}{n}(v(N) - v(N \setminus i)) + \frac{1}{n}\sum_{j \neq i} \phi_i(N \setminus j, v). \quad (1)$$

Along with the expected value interpretation of the Shapley value, the above recursive formula can be interpreted as follows. The first term $\frac{1}{n}(v(N) - v(N \setminus i))$ is the expected value of i's marginal contribution conditional on i appears *last* among all players, and the second term $\frac{1}{n}\sum_{j \neq i} \phi_i(N \setminus j, v)$ is that conditional on other player j appears *last*.

By using the marginal games, we give a new recursive representation of the Shapley value.

Proposition 5. *Take any integer r such that $1 \leq r \leq n$. Consider any $R \subseteq N$ with $|R| = r$. Then for each $(N, v) \in \mathcal{G}$ and each $i \in N$,*

$$\phi_i(N,v) = \frac{r!(n-r)!}{n!} \sum_{R \subseteq N, |R|=r, R \ni i} \phi_i(R,v) + \frac{r!(n-r)!}{n!} \sum_{R \subseteq N, |R|=r, R \not\ni i} \phi_i(N \setminus R, v^R). \tag{2}$$

Proof. Let $\Pi^R = \{\pi \in \Pi | \pi(j) \leq r \text{ for each } j \in R\}$. Then for any $R \neq R'$ which satisfy $|R| = |R'| = r$, we have $\Pi^R \cap \Pi^{R'} = \emptyset$ and $\bigcup_{R \subseteq N, |R|=r} \Pi^R = \Pi$. Thus,

$$\begin{aligned}
\phi_i(N,v) &= \frac{1}{n!} \sum_{R \subseteq N, |R|=r} \sum_{\pi \in \Pi^R} (v(\{j | \pi(j) \leq \pi(i)\}) - v(\{j | \pi(j) < \pi(i)\})) \\
&= \frac{1}{n!} \sum_{R \subseteq N, |R|=r, R \ni i} \sum_{\pi \in \Pi^R} (v|_R(\{j | \pi(j) \leq \pi(i)\}) - v|_R(\{j | \pi(j) < \pi(i)\})) \\
&+ \frac{1}{n!} \sum_{R \subseteq N, |R|=r, R \not\ni i} \sum_{\pi \in \Pi^R} (v^R(\{j | r < \pi(j) \leq \pi(i)\}) - v^R(\{j | r < \pi(j) < \pi(i)\})) \\
&= \frac{1}{n!} \sum_{R \subseteq N, |R|=r, R \ni i} r!(n-r)! \phi_i(R,v) + \frac{1}{n!} \sum_{R \subseteq N, |R|=r, R \not\ni i} r!(n-r)! \phi_i(N \setminus R, v^R) \\
&= \frac{r!(n-r)!}{n!} \sum_{R \subseteq N, |R|=r, R \ni i} \phi_i(R,v) + \frac{r!(n-r)!}{n!} \sum_{R \subseteq N, |R|=r, R \not\ni i} \phi_i(N \setminus R, v^R).
\end{aligned}$$

□

The particular case of the above Proposition 5 when $r = 1$ has a contrasting interpretation to Eq. (1). If $r = 1$, Eq. (2) is simplified to the following.

$$\phi_i(N,v) = \frac{1}{n} v(i) + \frac{1}{n} \sum_{j \neq i} \phi_i(N \setminus j, v^j). \tag{3}$$

In Eq. (3), the first term $\frac{1}{n} v(i)$ is the expected value of i's marginal contribution conditional on i appears *first* among all players, and the second term $\frac{1}{n} \sum_{j \neq i} \phi_i(N \setminus j, v^j)$ is that conditional on other player j appears *first*. In this sense, Eq. (3) can be contrast to the existing recursive formula of Eq. (1). Further, by using the extended marginal games instead of the marginal games, the following is obtained as a corollary of Propositions 2 and 5.

Corollary 2. *Take any integer r such that $1 \leq r \leq n$. Consider any $R \subseteq N$ with $|R| = r$. For each $(N, v) \in \mathcal{G}$ and each $i \in N$,*

$$\phi_i(N,v) = \frac{r!(n-r)!}{n!} \sum_{R \subseteq N, |R|=r} \phi_i(N, \bar{v}^R).$$

6 Final Remark

Based on a recursive formula of the Shapley value in Eq. (1), [14] construct a set of non-cooperative games in which subgame perfect equilibrium payoffs coincide with the Shapley value (if games are zero-monotonic). In their games, (i) players bid with each other to choose a proposer, (ii) the chosen proposer offers payoffs to all of the others, and (iii) each of the offered players chooses either accept the offer or not. Unanimous acceptance of the offer determines the payoffs and the game is over. Otherwise, players other than the proposer go to the same game defined on subgames on them.

Our characterization of the Shapley value based on marginal games also applicable to this non-cooperative foundation of the Shapley value. That is, the Shapley value is obtained by subgame perfect equilibrium payoffs of the set of non-cooperative games obtained by replacing subgames with marginal games with respect to the singleton of the proposer in the case of rejection at (iii). The proof goes along the same line as that of [14], and hence, we just mention the fact as a remark.

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Computing the Shapley Value of Threshold Cardinality Matching Games

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Abstract. The Shapley value is one of the most important solutions on the scheme of distributing the profits among agents in cooperative games. In this paper, we discuss the computational and complexity issues on the Shapley value in a particular multi-agent domain, a threshold cardinality matching game (TCMG). We show that the Shapley value can be calculated in polynomial time when graphs are restricted to some special graphs, such as linear graphs and the graphs having clique or coclique modules decomposition. For general graphs, we prove that calculating the Shapley value is *#P-complete* when the threshold is a constant.

Keywords: Shapley value · Threshold matching game · *#P-complete* · Efficient algorithm

1 Introduction

Cooperative games provide a framework for profit or cost distribution in multi-agent systems, such as network flow game [10], weighted voting games [6]. In cooperative game, the *Shapley* value is an important distribution scheme aiming to capture the notion of fairness of the distribution, based on the intuition that the payment that each agent receives should be proportional to his contribution [16]. Algorithmic issues on computing the Shapley value have been the topic of detailed studies, varieties of complexity results are presented. In Deng and Papadimitriou's work [6], it was shown that computing the Shapley value can be done in polynomial time in weighted subgraph games, while it is *#P-complete* in weighted majority games. Matsui and Matsui [12,13] showed that in weighted voting games, although computing the Shapley value is *NP*-hard, it can be done by a pseudo-polynomial time algorithm.

Matching game is one of the most important cooperative game models established on optimal matching problems, that has attracted much attention from researchers [7]. Shapley and Shubik [17] introduced assignment games, a special case of matching games defined on bipartite graph, to formulate the interaction between buyers and sellers in exchange markets. The solutions of matching

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games related to stability, such as the core, least-core and the nucleolus, have been extensively discussed in [1,4,7,9,11,17,18]. Shapley *et al.* [17] and Deng *et al.* [7] showed that the core was characterized efficiently by the dual theorem of linear programming. In [4,11,18] it was shown that computing the nucleolus for both assignment games and matching games in the "unweighted case" can be done in polynomial time. Aziz *et al.* [1] and Fang et al. [9] introduced a natural variation of matching games, called threshold matching games, and investigated the algorithmic aspect on the solutions, the least-core and the nucleolus.

However, less attention has been paid to the Shapley value in matching games. Recently, Aziz and Keijzer [2] studied the algorithmic problems on the Shapley value in cardinality matching games. Although the Shapley value is hard to compute (#*P-complete*), it can be computed efficiently when restricted on two special graphs (paths and graphs with a constant number of clique or coclique modueles). Bousquet [5] extended Aziz and Keijzer's results by showing that the Shapley value of trees can be computed in polynomial time.

We note that the computational difficulty on game solutions may be quite different in matching games and its threshold versions. In this paper, we investigate the computational complexity of computing the Shapley value for threshold cardinality matching games (TCMGs). We give positive answers on computing the Shapley value when graphs restricted to some classes of graphs: linear graphs, graph consists of clique or coclique modules and complete k-partite graphs. While in general case, computation of the Shapley value is shown to be #*P-complete*.

The organization of the paper is as follows. In Sect. 2, we introduce the definition of threshold cardinality matching games (TCMGs) and the Shapley value. In Sect. 3, we discuss the algorithms on the computation of the Shapley value of TCMG defined on some special graphs. Section 4 is dedicated to the intractability of the Shapley value in general case. Further discussion is given in Sect. 5.

2 Preliminary and Definition

2.1 Cooperative Game and Shapley Value

A *cooperative game* $\Gamma = (N, \nu)$ (transferable utility) consists of a set of players $N = \{1, 2, ..., n\}$ and a characteristic function $\nu : 2^N \rightarrow R$. For each $S \subseteq N$ (named a coalition), $\nu(S)$ is called the value of S, representing the benefits achieved by the players in S collectively; $\nu(N)$ is the total benefits that the whole group N achieves. One of the central problems in cooperative game is to seek a fair or stable distribution of the total benefits $\nu(N)$ between the players in N. A distribution of the total benefits $\nu(N)$ is given by a vector $x = (x_1, x_2, ..., x_n)$ with $\sum_{i=1}^{n} x_i = \nu(N)$, where each component x_i is the payoff for player i. For convenience, throughout the paper we denote $x(S) = \sum_{i \in S} x_i, \forall S \subseteq N$. Different criteria of fairness and stability on distributions give rise to different solution concepts. The Shapley value [16], which we focus on in this work, is an important solution concept.

Let $\Gamma = (N, \nu)$ $(|N| = n)$ be a cooperative game. For coalition $S \subseteq N$ and player $i \notin S$, the value $\nu(S \cup \{i\}) - \nu(S)$ is referred to as the marginal contribution of i *w.r.t.* S. The Shapley value is intended to reflect the average marginal contribution of each player over all coalitions S the player may join. Formally, the Shapley value $\varphi(\Gamma) = (\varphi_1, \varphi_2, ..., \varphi_n)$ is defined as follows:

$$\varphi_i(\Gamma) = \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} |S|!(n - |S| - 1)! \left[\nu(S \cup \{i\}) - \nu(S)\right] \forall i \in N.$$

The importance of the Shapley value lies in the fact that it is the unique solution satisfying the following properties [20]:

1. **Efficiency:** $\sum_{i \in N} \varphi_i(\Gamma) = \nu(N)$;
2. **Null player:** If $\nu(S \cup \{i\}) - \nu(S) = 0$ for all $S \subseteq N \setminus \{i\}$ (player i is called a null player), then $\varphi_i(\Gamma) = 0$;
3. **Symmetry:** If $\nu(S \cup \{i\}) - \nu(S) = \nu(S \cup \{j\}) - \nu(S)$ for all $S \subseteq N \setminus \{i, j\}$ (players i and j are called symmetric), then $\varphi_i(\Gamma) = \varphi_j(\Gamma)$;
4. **Additivity:** For any two games $\Gamma^1 = (N, \nu)$ and $\Gamma^2 = (N, w)$ and their combined game $\Gamma^1 + \Gamma^2 = (N, \nu + w)$, $\varphi_i(\Gamma^1 + \Gamma^2) = \varphi_i(\Gamma^1) + \varphi_i(\Gamma^2)$.

2.2 Threshold Cardinality Matching Game (TCMG)

Now we introduce the definition of threshold cardinality matching games. For more detailed introduction, please refer to [1,8,11]. All the graphs we consider in this paper are simple undirected graphs.

Let $G = (V, E)$ be a graph, V be the vertex set, E be the edge set. A matching M of G is an edge subset in which no edges have a common endpoint, and the size (cardinality) of M is denoted by $|M|$. A matching is maximum if its size is maximum over all the matchings in G, and the size of a maximum matching of G is denoted by $\gamma^*(G)$.

Given a graph $G = (V, E)$ and a threshold value $T \in Z^+$, the corresponding *threshold cardinality matching game* (TCMG), denoted by $\Gamma(G) = (V, \mu; T)$, is defined as:

- The player set is the vertex set V;
- $\forall S \subseteq V$, $\mu(S) = \begin{cases} 1 & \text{if } \gamma^*(G[S]) \geq T \\ 0 & \text{otherwise.} \end{cases}$,
  where, $G[S]$ is the subgraph of G induced by S.

Note that, TCMG is a threshold version of (cardinality) matching game. In a matching game on graph $G = (V, E)$, the value of each coalition $S \subseteq V$ is defined as $\gamma^*(G[S])$. Some basic ideas in Aziz and Keijzer's work [2] on matching games are used for reference in our work.

2.3 Observations on the Shapley Value of TCMG

In this subsection, we give some general observations about the Shapley value of TCMG.

Let $\Gamma(G) = (V, \mu; T)$ be the TCMG defined on graph $G = (V, E)$. Given player $i \in V$ and coalition $S \subseteq N \setminus \{i\}$, if $\mu(S \cup \{i\}) - \mu(S) = 1$, then the player i is called *pivotal* for coalition S. That is, if player i is pivotal for coalition S, then $\gamma^*(G[S]) = T - 1$ and $\gamma^*(G[S \cup \{i\}]) = T$, respectively.

Denote by $\mathcal{P}_i$ the set of coalitions for which player i is pivotal. Then the Shapley value φ_i of TCMG $\Gamma(G)$ can be rewritten via the size of $\mathcal{P}_i$:

$$\varphi_i(\Gamma) = \sum_{s=1}^{n-1} \frac{s!(n-s-1)!}{n!} \big|\{S \in \mathcal{P}_i : |S| = s\}\big|.$$

Lemma 1. *Let $G = (V, E)$ be a graph with $|V| = n$ and $i \in V$. In the corresponding TCMG $\Gamma(G)$, for each $s = 1, 2, ..., n-1$, if the number of subsets in $\{S \in \mathcal{P}_i : |S| = s\}$ can be determined in time $f(n)$, then the Shapley value of i in $\Gamma(G)$ can be computed in time $nf(n)$.*

3 Efficient Algorithms in Special Cases

Based on the properties of Efficiency and Symmetry, the Shapley value can be obtained easily for TCMGs on symmetric graphs, such as, the complete graph K_n, complete bipartite graph $K_{n\times n}$ and cycles. In the following, we shall discuss the algorithms on the Shapley value for TCMGs defined on two special kinds of graphs: linear graphs and graphs consisting of clique or coclique modules.

3.1 Linear Graphs

A linear graph (or a path) is a graph containing two end-vertices of degree 1 and the remaining vertices of degree 2. Throughout this subsection, a linear graph is denoted as $G_L = (V, E)$, where

- the vertex set is $V = \{1, 2, ..., n\}$, 1 and n are end-vertices (the degree is 1);
- the edge set is $E = \{(j, j+1) : j = 1, 2, ..., n-1\}$.

To compute the Shapley value, we first give the following result on linear graphs. For each $s = 1, 2, ..., n-1$ and $0 \le k \le \lfloor \frac{n}{2} \rfloor$, denote $\mathcal{H}^s[k]$ the set of subsets $S \subseteq V$ such that $|S| = s$ and $\gamma^*(G[S]) = k$.

Lemma 2. *Given a linear graph $G_L = (V, E)$ with $|V| = n$, the size of the set $\mathcal{H}^s[k]$ can be computed in polynomial time ($\forall s = 1, 2, ..., n-1$ and $0 \le k \le \lfloor \frac{n}{2} \rfloor$).*

Proof. We prove the result by induction on the parameter k.

When $k = 0$. It is clear that the set $S \in \mathcal{H}^s 0$ is an independent set of size s. Therefore, the size of $\mathcal{H}^s 0$ equals the number of ways to choose s non-adjacent vertices from n vertices on the line, that is,

$$\big|\mathcal{H}^s[0]\big| = C_{n-s+1}^s.$$

Where

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

When $k = 1$. There is only one matching edge in $G[S]$. We distinguish two cases.

Case 1. $G[S]$ has only a couple of connected vertices, and the other vertices are independent. See Fig. 1(a). Analogous to the analysis for $k = 0$, we have

$$\left|\mathcal{H}^s[1]\right| = C_{s-1}^1 \cdot C_{n-s+1}^{s-1}.$$

Case 2. $G[S]$ has three connected vertices, the other vertices are independent. See Fig. 1(b). We also have

$$\left|\mathcal{H}^s[1]\right| = C_{s-2}^1 \cdot C_{n-s+1}^{s-2}.$$

Hence, the result is true when $k = 1$.

(a) (b)

Fig. 1. The sets with two connected vertices(a) and three connected vertices(b)

We assume that the result is true for $k = p \geq 1$, that is, $\left|\mathcal{H}^s[p]\right|$ can be computed in polynomial time.

Then we prove the result for $k = p + 1$. For this purpose, we use induction for the number of vertices $|V|$ in G_L.

For $|V| = 2(p+1)$, G_L has a unique maximum matching of size $p+1$. Hence, the size of $\mathcal{H}^s[p+1]$ is 1 for $s = 2(p+1)$, and 0 for other values of s.

For $|V| = 2(p+1)+1$, $\gamma^*(G_L) = p+1$. It is easy to verify that the size of $\mathcal{H}^s[p+1]$ is $p+2$ for $s = 2(p+1)+1$, 1 for $2(p+1)$, and 0 for other values of s.

Assume that size of $\mathcal{H}^s[p+1]$ can be determined in polynomial time for $|V| = n$. Consider a linear graph $G_L = (V, E)$ with $|V| = n+1$. Denote

$$V = \{0, 1, 2, ..., n\} \text{ and } E = \{(j, j+1) : j = 0, 1, ..., n-1\}.$$

We divide the set $\mathcal{H}^s[p+1]$ into two subsets:

$$\mathcal{H}_{+0}^s[p+1] = \{S \in \mathcal{H}^s[p+1] : 0 \in S\}$$
$$\mathcal{H}_{-0}^s[p+1] = \{S \in \mathcal{H}^s[p+1] : 0 \notin S\}.$$

(i) The size of $\mathcal{H}_{-0}^s[p+1]$ equals the size of $\mathcal{H}^s[p+1]$ in graph $G_L \backslash \{0\}$ (containing n vertices), which can be counted in polynomial time followed by induction assumption.

(ii) For $\mathcal{H}_{+0}^s[p+1]$, $\forall t = 0, 1, ..., n$, we denote

$$\mathcal{H}_{+0}^s[p+1](t) = \{S \in \mathcal{H}_{+0}^s[p+1] : \{0, 1, 2, ..., t\} \subseteq S \text{ and } t+1 \notin S\}.$$

When $t = 0$, S does not contain vertex 1. Hence, we just need to count the number of the subset S in graph $G'_L = G_L \backslash \{0, 1\}$, such that the $\gamma^*(G'_L[S]) = p+1$ and $|S| = s - 1$. That is, the size of $\mathcal{H}^s_{+0}[p+1](0)$ equals the size of $\mathcal{H}^{s-1}[p+1]$ in graph $G'_L = G_L \setminus \{0, 1\}$, where G'_L has only $n - 1$ vertices. By induction assumption, the size of $\mathcal{H}^s_{+0}[p+1](0)$ can be determined in polynomial time.

When $t = 1$, S contains vertices 0,1 and does not contain vertex 2. Obviously, the size of $\mathcal{H}^s_{+0}[p+1](1)$ equals the size of $\mathcal{H}^{s-2}[p]$ in graph $G"_L = G_L \setminus \{0, 1, 2\}$, where $G"_L$ contains only $n - 2$ vertices. Also by induction assumption, the size of $\mathcal{H}^s_{+0}[p+1](1)$ can also be determined in polynomial time.

With similar analysis, we claim that the size of $\mathcal{H}^s_{+0}[p+1](t)$ can be determined in polynomial time for $t = 2, 3, ..., n$. Since

$$|\mathcal{H}^s_{+0}[p+1]| = \sum_{t=0}^{min\{s-1,2(p+1)\}} |\mathcal{H}^s_{+0}[p+1](t)|,$$

it is followed directly that $\left|\mathcal{H}^s[p+1]\right|$ can be determined in polynomial time.

The proof is done. □

Lemma 3. *If graph $G = (V, E)$ has K connected components (K is a fixed number independent of $|V|$) and each component is a linear graph, then the size of the set $\mathcal{H}^s[k]$ can be determined in polynomial time.*

In the TCMG defined on $G_L = (V, E)$ with $V = \{1, 2, ..., n\}$, for each player i and a coalition S for which i is pivotal, we give some notations for convenience of discussion. For $i \in V$ and $1 \leq s \leq n - 1$, denote:

$$\mathcal{P}^s_i = \{S \subseteq V \setminus \{i\} : \ i \text{ is pivotal for } S, \ |S| = s\}.$$

And

$$\mathcal{P}^s_{i,R} = \{S \subseteq V \setminus \{i-1, i\} : i+1 \in S, S \in \mathcal{P}^s_i\};$$
$$\mathcal{P}^s_{i,L} = \{S \subseteq V \setminus \{i, i+1\} : i-1 \in S, S \in \mathcal{P}^s_i\};$$
$$\mathcal{P}^s_{i,C} = \{S \subseteq V \setminus \{i\} : i+1, i-1 \in S, S \in \mathcal{P}^s_i\}.$$

It is easy to see that $\mathcal{P}^s_{i,R}$, $\mathcal{P}^s_{i,L}$ and $\mathcal{P}^s_{i,C}$ are disjoint, and

$$|\mathcal{P}^s_i| = |\mathcal{P}^s_{i,R}| + |\mathcal{P}^s_{i,L}| + |\mathcal{P}^s_{i,C}|.$$

Theorem 1. *The Shapley value of the TCMG defined on linear graph $G_L = (V, E)$ ($|V| = n$) can be computed in polynomial time for any threshold $T \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. Since the Shapley value φ_i ($i = 1, 2, ..., n$) in TCMG $\Gamma(G_L)$ can be rewritten as

$$\varphi_i = \sum_{s=1}^{n-1} \frac{s!(n-s-1)!}{n!} \left|\mathcal{P}^s_i\right|,$$

we need only to show that the size of $\mathcal{P}^s_i$ (that is, the sizes of $\mathcal{P}^s_{i,R}$, $\mathcal{P}^s_{i,L}$ and $\mathcal{P}^s_{i,C}$) can be determined in polynomial time.

(1) When $T = 1$. It is clear that a coalition $S \in \mathcal{P}^s_{i,R}$ is exactly an independent set with size $s-1$ in G_L not containing the vertices $i-1, i, i+1$ and $i+2$. That is, S must be the union of two independent sets: one is in $G[\{1, 2, ..., i-2\}]$ of size s_1 and the other is in $G[\{i+3, i+4, ..., n\}]$ of size s_2, where $s_1 + s_2 = s-1$. Hence, following from Lemma 3, the size of $\mathcal{P}^s_{i,R}$ can be determined in polynomial time. Similarly, the size of $|\mathcal{P}^s_{i,L}|$ and $|\mathcal{P}^s_{i,C}|$ can also be computed efficiently.

(2) We prove the result for $T = k \geq 2$. We first discuss the size of $\mathcal{P}^s_{i,R}$. Denote by $\mathcal{P}^s_{i,R}(t)$ the set of coalitions $S \in \mathcal{P}^s_{i,R}$, such that $i+1, i+2, ..., i+t+1 \in S$ and $i+t+2 \notin S$. It is easy to see that the size of $\mathcal{P}^s_{i,R}(t)$ is 0, if t is odd.

When $t = 0$, the size of $\mathcal{P}^s_{i,R}(0)$ equals the size of $\mathcal{H}^{s-1}[k]$ in $G' = G[V \setminus \{i-1, i, i+1, i+2\}]$ (recall the notation $\mathcal{H}^{s-1}[k]$ in Lemma 2), yielding that the size of $\mathcal{P}^s_{i,R}(0)$ can be determined in polynomial time by Lemma 2. Similar analysis can be given for t is even. Also since

$$|\mathcal{P}^s_{i,R}| = \sum_{t=0}^{min\{s-1,2(k-1)\}} |\mathcal{P}^s_{i,R}(t)|,$$

$\mathcal{P}^s_{i,R}$ can be counted in polynomial time.

The size of $\mathcal{P}^s_{i,L}$ and $\mathcal{P}^s_{i,C}$ can be obtained in a similar way, meaning that the size of $\mathcal{P}^s_i$ can be determined in polynomial time.

The proof is done. □

3.2 Graphs with a Constant Number of Clique or Coclique Modules

Given a graph $G = (V, E)$, a subset $S \subseteq V$ is a module if all the vertices in S have the same neighbors in $N \setminus S$. A subset $S \subseteq V$ is a clique (resp. coclique) module means that S is a clique (resp. coclique) and a module, *i.e.*, all vertices in the module S are pairwise connected (resp. disconnected). Obviously, the partition of vertex set V into singletons is a trivial modular decomposition. In [1], Aziz and Keijzer showed that for graph G, a minimum cardinality module decomposition into cocliques or cliques can be found in polynomial time.

In a cooperative game, a set of players S is said to be of the same player type if all players in S are pairwise symmetric. Ueda *et al.* [19] showed that for a cooperative game $\Gamma = (N, \nu)$ in which $\nu(S)$ ($S \subseteq N$) can be computed in polynomial time, and there is a fixed size k partition of the players into the same player type, then the Shapley value can be computed in polynomial time (in n) via dynamic programming. For a TCMG defined on graph G, which can be decomposed into k coclique modules or clique modules, all the players in the same coclique module or clique module of G are of the same player type. Based on these analysis, we have the following theorem.

Theorem 2. *Let $G = (V, E)$ ($|V| = n$) be a graph in which there exists a modular decomposition into k cocliques or k cliques, where k is independent of n. Then the Shapley value of the TCMG defined on G can be computed in polynomial time for any threshold $T \leq \lfloor \frac{n}{2} \rfloor$.*

Corollary 1. *Let $G = (V, E)$ be a complete k-partite graph (k is independent of n). Then the Shapley value of the TCMG defined on G can be computed in polynomial time.*

4 Computational Complexity in General Case

In this section, we discuss the computational complexity of the problem of computing the Shapley value for TCMGs in general case.

For this purpose, we introduce a well known *#P-complete* problem, *the Cardinality Vertex Cover problem* [14]:

- Input: A graph $G = (V, E)$, integer k.
- Output: The size (cardinality) of the set $\{S \subseteq V : S$ is a vertex cover for G and $|S| = k\}$.

Given a graph $G = (V, E)$, we know that a vertex subset $S \subseteq V$ is a vertex cover if and only if $V \setminus S$ is an independent set in G. Then, we define the problem of *Cardinality Independent Set.* Denote by $\alpha_k(G)$ the number of independent sets $S \subseteq V$ with $|S| = k$. For $k = 0$, we define $\alpha_0(G) = 0$. And for $k = 1$, $\alpha_0(G) = |V|$. The *Cardinality Independent Set* problem will be:

- Input: A graph $G = (V, E)$, integer k.
- Output: The size (cardinality) of the set $\{S \subseteq V : S$ is an independent set of G and $|S| = k\}$.

Lemma 4 [14]**.** *The problem of Cardinality Independent Set is #P-complete.*

In the next theorem, we discuss the computational complexity of computing the Shapley value in the special case of TCMG where the threshold is $T = 1$.

Theorem 3. *Given a graph $G = (V, E)$ ($|V| = n$), computing the Shapley value of the TCMG defined on graph G for threshold $T = 1$ is #P-complete.*

Proof. We prove the intractability of computing the Shapley value by making use of a polynomial-time Turing reduction from the problem of Cardinality Independent Set.

We first construct a series of $n + 1$ new graphs based on G.

For $i = 1, 2, ..., n, n + 1$, we construct graph G_i as follows (In Fig. 2):

(1) Add a star graph T_i with center vertex y and the other vertices $x_1, x_2, ..., x_i$;
(2) The graph G_i is composed of two components: the original graph G and the star graph T_i.

For $i = 1, 2, ..., n, n + 1$, we denote the TCMG defined on graph G_i by Γ_i. Then we focus on the Shapley value of player y in each Γ_i:

$$\varphi_y(\Gamma_i) = \sum_{s=1}^{n+i} \frac{s!(n + i - s)!}{(n + i + 1)!} (\mu(S \cup \{y\}) - \mu(S)),$$

where $S \in V \cup \{x_1, x_2, ..., x_i\}$ is the subset of players in Γ_i (i.e., the subset of vertices in G_i with size of $|S| = s$). To simplify the proof, we consider the "raw Shapley value":

$$\kappa_y(\Gamma_i) = \sum_{s=1}^{n+i} s!(n+i-s)!(\mu(S \cup \{y\}) - \mu(S)),$$

which has the same computational complexity with the Shapley value.

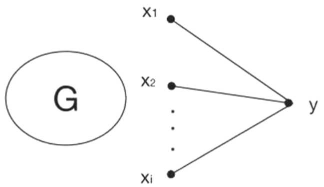

Fig. 2. G_i

Obviously, if $\mu(S \cup \{y\}) - \mu(S) = 1$, then there will be at least one vertex of $x_1, x_2, ..., x_i$ in S. By carefully calculating, we have

$$\begin{aligned}\kappa_y(\Gamma_i) &= \sum_{s=1}^{n+i} s!(n+i-s)![C_i^1 \alpha_{s-1}(G) + C_i^2 \alpha_{s-2}(G) + ... + C_i^i \alpha_{s-i}(G)] \\ &= \sum_{s=1}^{n+1} \Big[C_i^1 s!(n+i-s)! + C_i^2 (s+1)!(n+i-s-1)! \\ &\quad + ... + C_i^i (s+i-1)!(n-s+1)!\Big] \alpha_{s-1}(G).\end{aligned} \tag{4.1}$$

Denote the coefficient of $\alpha_{s-1}(G)$ in the formula (4.1) by b_{is} for $s = 1, 2, ..., n+1$, that is,

$$\begin{aligned}b_{is} = C_i^1 s!(n+i-s)! + C_i^2 (s+1)!(n+i-s-1)! \\ + ... + C_i^i (s+i-1)!(n-s+1)!.\end{aligned}$$

Then $\kappa_y(\Gamma_i)$ can be written as:

$$\kappa_y(\Gamma_i) = \sum_{s=1}^{n+1} b_{is} \alpha_{s-1}(G). \tag{4.2}$$

Denote the coefficient of $\alpha_{s-1}(G)$ in the (4.2) by b_{is} for $s = 1, 2, ..., n+1$, and denote by $\Theta = \left(b_{is}\right)_{(n+1)\times(n+1)}$ the matrix. Putting all the formulas (4.2) for $i = 1, 2, ..., n+1$ together, we have

$$\begin{pmatrix} \kappa_y(\Gamma_1) \\ \kappa_y(\Gamma_2) \\ \vdots \\ \kappa_y(\Gamma_{n+1}) \end{pmatrix} = \Theta \cdot \begin{pmatrix} \alpha_0(G) \\ \alpha_1(G) \\ \vdots \\ \alpha_n(G) \end{pmatrix}. \tag{4.3}$$

We then prove that $\alpha_s(G)$ can be computed by solving $k_y(\Gamma_i)$ in polynomial time.

Lemma 5. *The determinant of matrix A defined by $A_{ij} = (i+j)!$ is equal to $\Pi_{i=0}^{n} i!^2 \neq 0$.*

A is a matrix that is related to Pascal triangle [3], and we will show Θ also is related to Pascal triangle and is nonsingular.

Note that $C_n^m = C_{n-1}^m + C_{n-1}^{m-1}$. We can rewrite the formulation of $\kappa_y(\Gamma_i)$ in (4.1), for $i \geq 2$:

$$\begin{aligned}\kappa_y(\Gamma_i) = &\sum_{s=1}^{n+1} \Big[(C_{i-1}^0 + C_{i-1}^1)s!(n+i-s)! \\ &+(C_{i-1}^1 + C_{i-1}^2)(s+1)!(n+i-s-1)! \\ &+ ... + (C_{i-1}^{i-1} + C_{i-1}^i)(s+i-1)!(n-s+1)!\Big]\alpha_{s-1}(G) \\ = &\sum_{s=1}^{n+1} s!(n+i-s)!\alpha_{s-1}(G) + (n+i+1)\kappa_y(\Gamma_{i-1}) \\ = &\sum_{s=1}^{n=1} \Big[s!(n+i-s)! + (n+i+1)b_{i-1s}\Big]\alpha_{s-1}(G).\end{aligned} \tag{4.4}$$

The last "equation" in (4.4) holds based on the formulation of $\kappa_y(\Gamma_{i-1})$ (4.2). From Eq. (4.4), the matrix Θ in (4.3) can be transformed into

$$\Theta' = \begin{pmatrix} 1!n! & \dots & (n+1)!0! \\ 1!(n+1)! + (n+2)b_{11} & \dots & (n+1)!1! + (n+2)b_{1n+1} \\ \vdots & \ddots & \vdots \\ 1!(2n)! + (2n+1)b_{n1} & \dots & (n+1)!n! + (2n+1)b_{n2n+1} \end{pmatrix} \tag{4.5}$$

Based on the relationship of the coefficient of $\alpha_{s-1}(G)$ in (4.2) and (4.4):

$$b_{is} = s!(n+i-s)! + (n+i+1)b_{i-1s},$$

we use the matrix elementary operations on the matrix Θ' to transform it into the following form:

$$\Theta'' = \begin{pmatrix} 1!n! & 2!(n-1)! & \dots & (n+1)!0! \\ 1!(n+1)! & 2!n! & \dots & (n+1)!1! \\ \vdots & \vdots & \ddots & \vdots \\ 1!(2n)! & 2!(2n-1)! & \dots & (n+1)!n! \end{pmatrix} \tag{4.6}$$

From Lemma 5, we conclude that the matrice Θ, Θ' and Θ'' are all nonsingular, it follows from (4.3) that we can solve $\alpha_s(G)$ by solving $k_y(\Gamma_i)$ in polynomial time, and vice versa. □

Note that, the case of threshold $T = 1$ is a special case for $TCMGs$, so we have the general complexity result.

Theorem 4. *Computing the Shapley value of a TCMG is* #P-complete*.*

Based on Deng and Papadimitriou's work [6], Aziz and Brandt [1] also conclude that Computing the Shapley value of threshold matching games is *#P-complete*. But in their proof, the threshold was set to be related to the size of the graph, rather than a fixed number. Therefore, Theorem 4 generalizes Aziz and Brandt's result.

5 Conclusion and Further Discussion

In this paper, we focus on the computation of the Shapley value for TCMGs. We show that the Shapley value can be computed in polynomial time on special graphs: linear graphs and graphs consist of clique or coclique modules. For general graphs, we prove that computing the Shapley value is *#P-complete*. However, there are still quite a few problems for further discussion.

Firstly, given a graph G with k connected components $G_1, G_2, ..., G_k$, how to obtain the Shapley value on G through the Shapley values of each components. The difficulty is that the TCMG defined G can not be viewed as the sum of the TCMGs defined on $G_1, G_2, ..., G_k$, that is, the property of Additivity does not holds. For example, for both linear graphs and cycles, the Shapley value can be computed efficiently, but till now we have no evidence to show the same result for non-connected graphs with vertex degree at most two.

Secondly, Bousquet [5] recently proved that the Shapley value on trees can be computed in polynomial time. We conjecture that the ideas in [5] can be used to compute the Shapley value for TCMGs. Another algorithmic problem is that when the computation of the Shapley value is hard, how to design the approximation algorithms.

Thirdly, as a similar solution concept as the Shapley value, the Banzhaf index of TCMG has not been discussed. Like the Shapley value, the Banzhaf index measures agents marginal contributions over all coalitions. Given a characteristic function game $\Gamma = (N, v)$ with $|N| = n$, the Banzhaf index of a player $i \in N$ is

$$\beta_i(G) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} \big[v(S \cup \{i\} - v(S)\big].$$

In our opinion, the efficiency on computation of the Shapley value would yield the same result on the Banzhaf index. However, the computational complexity on the computation of the Banzhaf index for TCMGs in general case is still open.

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Matrix Analysis for the Shapley Value and Its Inverse Problem

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Abstract. In the framework of cooperative game theory, any linear value of games is a linear operator on game space, implying that algebraic representations and matrix analysis are possibly justifiable techniques for studying linear values. For any linear value, the payoff vector of any game is represented algebraically by the product of a column-coalitional representation matrix and the worth vector. The analysis of the structure of these representation matrices covers the study of the class of linear values. We achieve a matrix approach for characterizing linear values with some essential properties. Also, some properties are described for the Shapley standard matrix, which is the representation matrix of the Shapley value. Furthermore, the inverse problem of the Shapley value is studied in terms of the null space of the Shapley standard matrix.

Keywords: Matrix analysis · The Shapley value · The Shapley standard matrix · Inverse problem

1 Introduction

In economic situations, players may cooperate to obtain more profits with assuming that they are rational. It is a common and important issue to distribute the surplus of cooperation among the players. Cooperative game theory provides general mathematical techniques for analyzing such cooperation and distribution issues. The solution part of cooperative game theory deals with the allocation problem of how to divide the overall earnings (worth) among the players in the game. As is well-known, every cooperative transferable utility game (TU-game) can be identified with a column vector, where the components of such a vector represent the worths of nonempty coalitions. The set of all TU-games with a certain player set spans a vector space when the operations of addition and multiplication are involved in the set. Stimulated from the fact, linear values on the game space were well-studied and became the most important class of solution concepts in cooperative game theory, of which the Shapley value is the most important representative.

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From the review of linearity in cooperative game theory, the algebraic representation and the matrix analysis to cooperative game theory come forward as natural and powerful. Some initial ideas related to the algebraic approach appeared in the literature (see Weber [11], Dragan [2]). Kleinberg and Weiss [8] constructed a direct-sum decomposition of the null space of the Shapley value into invariant subspaces by using the representation theory of symmetric groups, and derived a characterization of a very general type of values, of which the Shapley value is one particular example. The matrix approach was applied to study associated game consistency for the Shapley value by Xu et al. [12,13] and Hamiache [7].

However, the algebraic representation and the matrix analysis have not been used systematically. It is still a neglected technique in cooperative game theory. For any linear value, the payoff vector of any game is represented algebraically by the product of a column-coalitional representation matrix and the worth vector. This paper mainly provides new and intuitive proofs of some known results for characterizing linear values within the matrix approach to cooperative games. Mostly, the approach yields insight into structure of representation matrix, named the Shapley standard matrix, of the Shapley value. Furthermore, the inverse problem of the Shapley value is also studied in terms of the null space of the Shapley standard matrix.

The paper is organized as follows. Section 2 introduces the matrix representation for linear values, and provides new and intuitive proofs of some results for characterizing linear values by applying the matrix approach to cooperative games. Section 3 studies mainly the properties of the Shapley standard matrix as well as the Shapley value. In Sect. 4 we develop a matrix approach to analyze the inverse problem of the Shapley value. Section 5 concludes the paper.

2 Matrix Approach to Linear Values

A *cooperative game* with transferable utility (TU) is a pair $\langle N, v\rangle$, where N is a nonempty, finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. An element of N (notation: $i \in N$) and a subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) are called a *player* and *coalition* respectively, and the associated real number $v(S)$ is called the *worth* of coalition S. The size of coalition S is denoted by s. We denote by $\mathcal{G}^N$ the set of all these TU-games with player set N and by $\Omega = 2^N \setminus \emptyset$ the set of all coalitions.

Concerning the solution theory for cooperative TU-games, a *value* is a mapping $\Phi : \mathcal{G}^N \to \mathbb{R}^n$ that associates a vector $\Phi(v) \in \mathbb{R}^n$ with every game $\langle N, v\rangle \in \mathcal{G}^N$, where the real number $\Phi_i(v)$ represents the *payoff* to player i in the game. The well-known *Shapley value* $Sh(v)$ is defined by Shapley [10] as follows.

$$Sh_i(v) := \sum_{S\subseteq N, S\ni i} \frac{(s-1)!(n-s)!}{n!}\Big[v(S) - v(S\setminus\{i\})\Big], \quad \text{for all } i \in N.$$

It is characterized [10] as the unique value satisfying linearity, efficiency, symmetry, and the dummy player property. A value Φ on the universal game space $\mathcal{G}^N$ is said to be *linear*, if $\Phi(\alpha \cdot v + \beta \cdot w) = \alpha \cdot \Phi(v) + \beta \cdot \Phi(w)$ for all games $\langle N, v\rangle$, $\langle N, w\rangle$, and all $\alpha, \beta \in \mathbb{R}$; *efficient*, if $\sum_{i\in N} \Phi_i(v) = v(N)$ for all games $\langle N, v\rangle$; *symmetric*, if $\Phi_{\pi(i)}(\pi v) = \Phi_i(v)$ for all games $\langle N, v\rangle$, all $i \in N$, and every permutation π on N. Here the game $\langle N, \pi v\rangle$ is given by $(\pi v)(S) = v(\pi^{-1}(S))$ for all $S \in \Omega$. Φ satisfies *dummy* (respectively, *null*) *player property*, if $\Phi_i(v) = v(\{i\})$ for any game $\langle N, v\rangle$ and any dummy (null) player $i \in N$. A player i is a *dummy player* (respectively, *null player*) in $\langle N, v\rangle$ if $v(S \cup \{i\}) - v(S) = v(\{i\})$ (respectively, $= 0$) for all $S \subseteq N \setminus \{i\}$. Obviously, any linear value is also a linear operator on $\mathcal{G}^N$.

The *lexicographic order* is introduced for the set Ω of coalitions as follows.[1] For two coalitions $S = \{i_1, i_2, \cdots, i_s\}$ and $T = \{j_1, j_2, \cdots, j_t\}$ with $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_t$, S *precedes* T in this order, if and only if the sizes of these two coalitions verify either $s < t$, or $s = t$ and for some k, $1 \leq k < s$, it holds that $i_l = j_l$, for all $1 \leq l \leq k-1$ and $i_k < j_k$.

In this paper, a game $\langle N, v\rangle$ is always presented as the column vector $\mathbf{v}$ of worths $v(S)$ of all lexicographically ordered coalitions $S \in \Omega$, *i.e.*, $\mathbf{v} = (v(S))_{S\in\Omega}$. For example, a 3-person game $\langle N, v\rangle$ will be presented as

$$\mathbf{v} = \big(v(\{1\}), v(\{2\}), v(\{3\}), v(\{1,2\}), v(\{1,3\}), v(\{2,3\}), v(\{1,2,3\})\big)^T.$$

If no confusion arises, we write v instead of $\mathbf{v}$. In a sense, the set $\mathcal{G}^N$ of all n-person games with player set N is isomorphic to the vector space $\mathbb{R}^{2^n-1}$, for always $v(\emptyset) = 0$.

Stimulated by the fact that the game space $\mathcal{G}^N$ spans a vector space, linear values on $\mathcal{G}^N$ are well-studied and become the most important class of solution concepts in cooperative game theory. The algebraic representation and matrix approach come forward as a natural and powerful technique to study linear values. Xu et al. [12] introduced a new type of matrix, named coalitional matrix, in order to apply algebraic representation and matrix analysis to cooperative game theory.

Definition 1 (Xu et al. [12]). *A matrix M is called row (respectively, column)-coalitional if its rows (respectively, columns) are indexed by all lexicographically ordered coalitions $S \in \Omega$. M is called square-coalitional if it is both row-coalitional and column-coalitional.*

In this section, we represent algebraically linear values in terminology of coalitional matrices. Next we apply matrix analysis to investigate characterizations of the class of linear values. First of all, by linear algebra we know that any linear value on game space can be represented uniquely by a corresponding column-coalitional matrix.

[1] To line up the coalitions, any order is suitable for the set Ω.

Theorem 1 (Weber [11]). *For any value Φ on $\mathcal{G}^N$, it is linear if and only if there exists a unique column-coalitional matrix $M^\Phi = [M^\Phi]_{i\in N, S\in\Omega}$ such that $\Phi(v) = M^\Phi \cdot v$, for all games $\langle N, v\rangle$.*

For the determination of the matrix M^Φ, recall that any game $\langle N, v\rangle$ can be represented as $v = \sum_{S\in\Omega} v(S)\cdot e_S$, where e_S denotes the worth vector of *unity game* $\langle N, e_S\rangle$, $S\in\Omega$, defined by
$e_S(T) = \begin{cases} 1, \text{ if } S = T; \text{By the linearity of } \Phi, \text{we have} \\ 0, \text{ otherwise.} \end{cases}$

$$\Phi(v) = \sum_{S\in\Omega} v(S)\cdot \Phi(e_S) = M^\Phi \cdot v,$$

where the entries of the column-coalitional matrix M^Φ are given by $[M^\Phi]_{i,S} = \Phi_i(e_S)$, for all $i\in N, S\in\Omega$.[2]

Therefore, for any linear value, the payoff vector of any game is represented algebraically by the product of a column-coalitional matrix and the worth vector. We call this associated matrix the *representation matrix* of the linear value. In order to study the linear value, we may analyze the structure of this representation matrix. We start with linear values which possess some other essential properties. Denote by $\mathbf{1}_N \in \mathbb{R}^N$ the n-dimensional column vector with all entries equal to one.

Proposition 2. *Let Φ be a linear value on $\mathcal{G}^N$. Then Φ is efficient if and only if each column sum of the representation matrix M^Φ equals 0 except for the unitary sum of the last column indexed by N,* i.e.,

$$\mathbf{1}_N^{'} \cdot M^\Phi = (0, 0, \cdots, 0, 1).$$

Proof. Let Φ be a linear value on $\mathcal{G}^N$ and M^Φ be its representation matrix. For any game $\langle N, v\rangle$ and any player $i\in N$, by Theorem 1, it follows that

$$\begin{aligned}\sum_{i\in N}\Phi_i(v) &= \sum_{i\in N}\sum_{S\in\Omega}[M^\Phi]_{i,S}v(S) = \sum_{S\in\Omega}\sum_{i\in N}[M^\Phi]_{i,S}v(S)\\ &= \sum_{\substack{S\in\Omega\\ S\neq N}} v(S)\sum_{i\in N}[M^\Phi]_{i,S} + v(N)\sum_{i\in N}[M^\Phi]_{i,N}.\end{aligned}$$

[2] Actually, any basis can be chosen here to determine the matrix of M^Φ. The advantage of the chosen basis of unity games is the fact the argumentation in the proofs of the reported results can be in line with the standard approach in cooperative game theory.

Note that the game $\langle N, v\rangle$ is arbitrary and so, the worths $v(S)$, $S \in \Omega$, can be chosen arbitrarily. Thus, Φ is efficient, *i.e.*, $\sum_{i\in N} \Phi_i(v) = v(N)$ for all games $\langle N, v\rangle$, if and only if

$$\sum_{i\in N}[M^{\Phi}]_{i,S} = 0, \text{ for all } S \in \Omega, S \neq N, \text{ and } \sum_{i\in N}[M^{\Phi}]_{i,N} = 1.$$

That is, $\mathbf{1}_N^{'} \cdot M^{\Phi} = (0, 0, \cdots, 0, 1)$. □

Proposition 3. *Let Φ be a linear value on $\mathcal{G}^N$. Then Φ possesses the null player property if and only if the representation matrix M^{Φ} satisfies the condition*

$$[M^{\Phi}]_{i,S} = -[M^{\Phi}]_{i,S\setminus\{i\}}, \quad \textit{for all } i \in N, \textit{ and all } S \in \Omega, S \ni i, S \neq \{i\}.$$

Proof. Let Φ be a linear value on $\mathcal{G}^N$ and M^{Φ} be its representation matrix.

"$\Leftarrow$": Suppose that M^{Φ} satisfies

$$[M^{\Phi}]_{i,S} = -[M^{\Phi}]_{i,S\setminus\{i\}}, \quad \text{for all } i \in N, \text{ and all } S \in \Omega, S \ni i, S \neq \{i\}.$$

For any game $\langle N, v\rangle$, by Theorem 1, we have

$$\begin{aligned}
\Phi_i(v) &= \sum_{S\in\Omega}[M^{\Phi}]_{i,S}v(S) = \sum_{S\ni i}[M^{\Phi}]_{i,S}v(S) + \sum_{\substack{S\not\ni i\\ S\neq\emptyset}}[M^{\Phi}]_{i,S}v(S)\\
&= \sum_{\substack{S\ni i\\ S\neq\{i\}}}[M^{\Phi}]_{i,S}v(S) + [M^{\Phi}]_{i,\{i\}}v(\{i\}) + \sum_{\substack{S\ni i\\ S\neq\{i\}}}[M^{\Phi}]_{i,S\setminus\{i\}}v(S\setminus\{i\})\\
&= \sum_{\substack{S\ni i\\ S\neq\{i\}}}[M^{\Phi}]_{i,S}\big[v(S) - v(S\setminus\{i\})\big] + [M^{\Phi}]_{i,\{i\}}v(\{i\}).
\end{aligned}$$

For any null player i in the game $\langle N, v\rangle$, it holds that $v(\{i\}) = 0$ as well as $v(S) = v(S \setminus \{i\})$ for all $S \in \Omega$, $S \ni i, S \neq \{i\}$. Hence, $\Phi_i(v) = 0$, and so, Φ satisfies the null player property.

"$\Rightarrow$": For any $i \in N$, any $S \in \Omega, S \ni i$, $S \neq \{i\}$, consider the game $\langle N, v_{S_i}\rangle$ given by

$$v_{S_i}(S) = v_{S_i}(S \setminus \{i\}) = 1, \text{ and for any other coalition } T \in \Omega,\ v_{S_i}(T) = 0.$$

By Theorem 1, we obtain that

$$\Phi_i(v_{S_i}) = \sum_{R\in\Omega}[M^{\Phi}]_{i,R}v_{S_i}(R) = [M^{\Phi}]_{i,S} + [M^{\Phi}]_{i,S\setminus\{i\}}.$$

Obviously, i is a null player in the game $\langle N, v_{S_i}\rangle$. By the null player property, $\Phi_i(v_{S_i}) = 0$. Therefore, $[M^{\Phi}]_{i,S} = -[M^{\Phi}]_{i,S\setminus\{i\}}$. □

Consider a weight system $m = (m_{i,S})_{i\in S}^{S\in \Omega}$ such that $\sum_{S\ni i} m_{i,S} = 1,$ for all $i \in N$. Weber introduced in [11] a value Φ on $\mathcal{G}^N$ as follows:

$$\Phi_i(v) = \sum_{S\ni i} m_{i,S}\big[v(S) - v(S\setminus\{i\})\big], \quad \text{for all games } \langle N, v\rangle, \text{ and all } i \in N.$$

It is called a *Weber value* by Derks [1]. And Weber's characterization [11] of the class of Weber values is derived directly from Proposition 3.

Corollary 4 (Weber [11]). *A value Φ on $\mathcal{G}^N$ possesses linearity and the dummy player property if and only if it is a Weber value.*

Proof by the matrix approach. It is easy to check that every Weber value satisfies linearity and the dummy player property.

Let Φ be a linear value on $\mathcal{G}^N$ possessing the dummy player property and M^Φ be its representation matrix. Since the dummy player property for the value Φ implies the null player property, by Proposition 3, for all $i \in N$, we have $[M^\Phi]_{i,S} = -[M^\Phi]_{i,S\setminus\{i\}}$ for all $S \in \Omega$, $S \ni i, S \neq \{i\}$. So, for any game $\langle N, v\rangle$, since $v(\emptyset) = 0$, we conclude that

$$\begin{aligned}\Phi_i(v) &= \sum_{S\in\Omega}[M^\Phi]_{i,S}v(S) = \sum_{S\ni i}[M^\Phi]_{i,S}v(S) + \sum_{\substack{S\not\ni i\\ S\neq\emptyset}}[M^\Phi]_{i,S}v(S)\\ &= \sum_{\substack{S\ni i\\ S\neq\{i\}}}[M^\Phi]_{i,S}v(S) + [M^\Phi]_{i,\{i\}}v(\{i\}) + \sum_{\substack{S\ni i\\ S\neq\{i\}}}[M^\Phi]_{i,S\setminus\{i\}}v(S\setminus\{i\})\\ &= \sum_{S\ni i}[M^\Phi]_{i,S}\big[v(S) - v(S\setminus\{i\})\big]\end{aligned}$$

Let $\langle N, v\rangle$ be a game with the dummy player i. Then $v(S) - v(S\setminus\{i\}) = v(\{i\})$, for all $S \in \Omega, S \ni i$. By the dummy player property, it holds that $\Phi_i(v) = v(\{i\})$. So $\sum_{S\ni i}[M^\Phi]_{i,S} = 1$. Therefore, Φ is the Weber value with the weights $m_{i,S} = [M^\Phi]_{i,S}$, for all $S \in \Omega$, $S \ni i$. □

Proposition 5. *Let Φ be a linear value on $\mathcal{G}^N$. Then Φ is symmetric if and only if the representation matrix M^Φ satisfies, for all players $i, j \in N$, and all coalitions $S, T \in \Omega$ with equal sizes $s = t$,*

$$[M^\Phi]_{i,S} = [M^\Phi]_{j,T}, \quad \textit{when}\, i \in S, j \in T \,\textit{or}\, i \notin S, j \notin T. \tag{1}$$

Proof. Let Φ be a linear value on $\mathcal{G}^N$ and M^Φ be its representation matrix.

"⇐": Let $\pi \in \Pi^N$ be a permutation on N. Then for any $i \in N$ and any $T \in \Omega$, $\pi(i) \in \pi(T)$ if and only if $i \in T$, as well as $\pi(i) \notin \pi(T)$ if and only if $i \notin T$. So by (1), we have

$$[M^\Phi]_{\pi(i),\pi(T)} = [M^\Phi]_{i,T}.$$

Therefore, for all $i \in N$,

$$\begin{aligned}
\Phi_{\pi(i)}(\pi v) &= \sum_{S\in\Omega}[M^{\Phi}]_{\pi(i),S}(\pi v)(S) \\
&= \sum_{S\ni\pi(i)}[M^{\Phi}]_{\pi(i),S}(\pi v)(S) + \sum_{\substack{S\not\ni\pi(i)\\ S\neq\emptyset}}[M^{\Phi}]_{\pi(i),S}(\pi v)(S) \\
&= \sum_{\pi^{-1}(S)\ni i}[M^{\Phi}]_{\pi(i),S}v(\pi^{-1}(S)) + \sum_{\substack{\pi^{-1}(S)\not\ni i\\ \pi^{-1}(S)\neq\emptyset}}[M^{\Phi}]_{\pi(i),S}v(\pi^{-1}(S)) \\
&= \sum_{T\ni i}[M^{\Phi}]_{\pi(i),\pi(T)}v(T) + \sum_{\substack{T\not\ni i\\ T\neq\emptyset}}[M^{\Phi}]_{\pi(i),\pi(T)}v(T) \\
&= \sum_{T\ni i}[M^{\Phi}]_{i,T}v(T) + \sum_{\substack{T\not\ni i\\ T\neq\emptyset}}[M^{\Phi}]_{i,T}v(T) = \Phi_i(v).
\end{aligned}$$

The symmetry property of Φ holds.

"$\Rightarrow$": From the fact that the columns $[M^{\Phi}]_S$ of M^{Φ} are the values $\Phi(e_S)$ for all $S \in \Omega$, symmetry simply implies that $\Phi_i(e_S) = \Phi_j(e_T)$ for each S, T with $s = t$, and $i \in S, j \in T$ or $i \notin S, j \notin T$. That is $[M^{\Phi}]_{i,S} = [M^{\Phi}]_{j,T}$. □

From this, for a linear, symmetric value Φ, of which the representation matrix is M^{Φ}, each entry $[M^{\Phi}]_{i,S}$ of M^{Φ} is only related to the size s of the coalition S and the membership or nonmembership between the player i and the coalition S. Therefore, we denote the entry $[M^{\Phi}]_{i,S}$ as m_s^{Φ} for $i \in S$, otherwise as $m_{s^-}^{\Phi}$, for all $S \in \Omega$.

Corollary 6. *Any linear and symmetric value Φ on $\mathcal{G}^N$ can be expressed as*

$$\Phi_i(v) = \sum_{S\ni i} m_s^{\Phi} v(S) + \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} m_{s^-}^{\Phi} v(S), \quad \textit{for all games } \langle N, v\rangle, \textit{ and all } i \in N.$$

Furthermore, we have the following theorem.

Theorem 7. *Let Φ be a linear, symmetric value on $\mathcal{G}^N$. Then Φ is efficient if and only if the representation matrix M^{Φ} satisfies*

$$m_{s^-}^{\Phi} = -\frac{s}{n-s} m_s^{\Phi}, \quad \textit{for all } 1 \le s < n, \textit{ and } \quad m_n^{\Phi} = \frac{1}{n}. \tag{2}$$

Proof. Let Φ be a linear, symmetric value on $\mathcal{G}^N$ with the representation matrix M^{Φ}. By Corollary 6, for any game $\langle N, v\rangle$, we have

$$\begin{aligned}
\sum_{i\in N}\Phi_i(v) &= \sum_{i\in N}\sum_{S\in\Omega}[M^{\Phi}]_{i,S}v(S) = \sum_{S\in\Omega}\Big(\sum_{i\in N}[M^{\Phi}]_{i,S}\Big)v(S) \\
&= \sum_{S\in\Omega}\Big[\sum_{i\in S} m_s^{\Phi} + \sum_{i\notin S} m_{s^-}^{\Phi}\Big]v(S) = \sum_{S\in\Omega}\big[s m_s^{\Phi} + (n-s)m_{s^-}^{\Phi}\big]v(S).
\end{aligned}$$

Together with efficiency, we have $\sum_{S\in\Omega}\left[sm_s^{\Phi}+(n-s)m_{s^-}^{\Phi}\right]v(S)=v(N)$. Note that the game $\langle N,v\rangle$ is arbitrary and so, the worths $v(S)$, $S\in\Omega$, can be chosen arbitrarily. Thus, $sm_s^{\Phi}+(n-s)m_{s^-}^{\Phi}=0$, for all $S\in\Omega$, $S\neq N$, and $nm_n^{\Phi}=1$ for $S=N$. That is,

$$m_{s^-}^{\Phi}=-\frac{s}{n-s}m_s^{\Phi},\ \text{for all } 1\leq s<n, \text{ and } m_n^{\Phi}=\frac{1}{n}.$$

□

By this theorem and Corollary 6, we can get the formula given by Ruiz *et al.* [9] for the class of linear, symmetric and efficient values.

Corollary 8 (Ruiz et al. [9]). *A value Φ on $\mathcal{G}^N$ possesses linearity, symmetry and efficiency if and only if there exists $m_s^{\Phi}, s=1,2,\cdots,n-1$, such that for any game $\langle N,v\rangle$,*

$$\Phi(v)=M^{\Phi}\cdot v,\quad where\quad [M^{\Phi}]_{i,S}=\begin{cases}\frac{1}{n}, & if\ S=N;\\ m_s^{\Phi}, & if\ i\in S, S\neq N;\\ -\frac{s}{n-s}m_s^{\Phi}, & if\ i\notin S,\end{cases}$$

i.e.,

$$\Phi_i(v)=\frac{v(N)}{n}+\sum_{\substack{S\subsetneq N\\ S\ni i}}m_s^{\Phi}v(S)-\sum_{\substack{S\not\ni i\\ S\neq\emptyset}}\frac{s}{n-s}m_s^{\Phi}v(S),\quad for\ all\ i\in N.$$

3 The Shapley Standard Matrix

In this section, we characterize the representation matrix of the Shapley value named Shapley standard matrix, by using the classic axioms: linearity, efficiency, symmetry and the null player property. Moreover, some other properties of the Shapley standard matrix are described.

Theorem 9. *Let $M^{\Phi}=\left[M^{\Phi}\right]_{i\in N,S\in\Omega}$ be the representation matrix of a linear value Φ on $\mathcal{G}^N$ possessing symmetry, efficiency and the dummy player property (i.e., the Shapley value). Then*

$$[M^{\Phi}]_{i,S}=\begin{cases}\frac{(s-1)!(n-s)!}{n!}, & if\ i\in S;\\ -\frac{s!(n-s-1)!}{n!}, & if\ i\notin S.\end{cases}\tag{3}$$

Proof. Let Φ be a linear value on $\mathcal{G}^N$ possessing symmetry, efficiency, the dummy player property and M^{Φ} be its representation matrix. For any player $i\in N$, we consider a game $\langle N,v\rangle$ as follows. For all $S\in\Omega, S\not\ni i$, the worths $v(S)$ and

$v(\{i\})$ are chosen arbitrarily, then let $v(S\cup\{i\}) = v(S) + v(\{i\})$, to ensure that i is a dummy player in $\langle N, v\rangle$. By Corollary 6, we have

$$\begin{aligned}
\Phi_i(v) &= \sum_{S\ni i} m_s^{\Phi} v(S) + \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} m_{s^-}^{\Phi} v(S)\\
&= \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} m_{s+1}^{\Phi} v(S\cup\{i\}) + m_1^{\Phi} v(\{i\}) + \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} m_{s^-}^{\Phi} v(S)\\
&= \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} m_{s+1}^{\Phi} v(S) + \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} m_{s+1}^{\Phi} v(\{i\}) + m_1^{\Phi} v(\{i\}) + \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} m_{s^-}^{\Phi} v(S)\\
&= \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} \left[m_{s+1}^{\Phi} + m_{s^-}^{\Phi}\right] v(S) + \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} m_{s+1}^{\Phi} v(\{i\}) + m_1^{\Phi} v(\{i\}).
\end{aligned}$$

By the dummy player property, $\Phi_i(v) = v(\{i\})$. Note that the worths $v(S)$, $S\in\Omega, S\not\ni i$ can be arbitrary values. Particularly, consider the unanimity games $\langle N, u_S\rangle$, for all $S\in\Omega, S\not\ni i$. We conclude that

$$m_{s+1}^{\Phi} + m_{s^-}^{\Phi} = 0, \quad \text{for all } s = 1, 2, \cdots, n-1.$$

By (2) in Theorem 7, we have $m_n^{\Phi} = \frac{1}{n}$ as well as

$$m_{s+1}^{\Phi} = -m_{s^-}^{\Phi} = \frac{s}{n-s} m_s^{\Phi}, \quad \text{for all } s = 1, 2, \cdots, n-1. \tag{4}$$

We obtain $m_{s+1}^{\Phi}, m_{s^-}^{\Phi}$ for all $s = n-1, n-2, \cdots, 1$ recursively as follows.

$$\begin{cases} m_s^{\Phi} = \frac{(s-1)!(n-s)!}{n!}, \\ m_{s^-}^{\Phi} = -\frac{s!(n-s-1)!}{n!}; \end{cases} \quad i.e., \ [M^{\Phi}]_{i,S} = \begin{cases} \frac{(s-1)!(n-s)!}{n!}, & \text{if } i\in S;\\ -\frac{s!(n-s-1)!}{n!}, & \text{if } i\notin S. \end{cases}$$

What is more,

$$\begin{aligned}
\Phi_i(v) &= \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} m_{s+1}^{\Phi} v(\{i\}) + m_1^{\Phi} v(\{i\}) = v(\{i\}) \sum_{s=1}^{n-1} \binom{n-1}{s} m_{s+1}^{\Phi} + \frac{1}{n} v(\{i\})\\
&= v(\{i\}) \sum_{s=1}^{n-1} \frac{1}{n} + \frac{1}{n} v(\{i\}) = v(\{i\}).
\end{aligned}$$

□

We denote by M^{Sh} the representation matrix of the Shapley value on $\mathcal{G}^N$, and we call it the *Shapley standard matrix* (Xu et al. [12]). We restate the Shapley value in terminology of the Shapley standard matrix as follows.

Definition 2. *Given any game $\langle N, v\rangle$, the Shapley value $Sh(v)$ is represented by the Shapley standard matrix M^{Sh} as:*

$$Sh(v) = M^{Sh} \cdot v,$$

where the Shapley standard matrix $M^{Sh} = \left[M^{Sh}\right]_{i\in N, S\in\Omega}$ is given by (3).

Now we study some properties of the Shapley value in terms of the Shapley standard matrix.

Proposition 10. *Let $\left[M^{Sh}\right]_T$ be the column of M^{Sh} indexed by coalition $T \in \Omega$. Then it holds that $\left[M^{Sh}\right]_S = -\left[M^{Sh}\right]_{N\setminus S}$, for all $S \in \Omega \setminus \{N\}$.*

Proof. For any coalition $S \in \Omega, S \neq N$, it is sufficient to show that

$$\left[M^{Sh}\right]_{i,S} + \left[M^{Sh}\right]_{i,N\setminus S} = 0, \quad \text{for all } i \in N.$$

Without loss of generality, we suppose that $i \in S$. By the definition of the Shapley standard matrix M^{Sh}, we conclude that

$$\left[M^{Sh}\right]_{i,S} + \left[M^{Sh}\right]_{i,N\setminus S} = \frac{(s-1)!(n-s)!}{n!} - \frac{(n-s)!(s-1)!}{n!} = 0.$$

□

The above *anti-complementarity property* of the Shapley standard matrix implies an alternative formula for the Shapley value, due to Driessen [5].

Corollary 11 (Driessen[5]). *The Shapley value is of the following form:*

$$Sh_i(v) = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!}\Big[v(S) - v(N \setminus S)\Big], \qquad for\ any\ \langle N, v\rangle\ and\ i \in N.$$

Proof by the matrix approach. For any game $\langle N, v\rangle$, since $v(\emptyset) = 0$, for all $i \in N$, by Proposition 10, we have

$$\begin{aligned}
Sh_i(v) &= \sum_{S\in\Omega} [M^{Sh}]_{i,S} v(S) = \sum_{S\ni i} [M^{Sh}]_{i,S} v(S) + \sum_{\substack{S\not\ni i\\ S\neq\emptyset}} [M^{Sh}]_{i,S} v(S)\\
&= \sum_{S\ni i} [M^{Sh}]_{i,S} v(S) + \sum_{\substack{T\ni i\\ T\neq N}} [M^{Sh}]_{i,N\setminus T} v(N\setminus T)\\
&= \sum_{S\ni i} [M^{Sh}]_{i,S} v(S) - \sum_{T\ni i} [M^{Sh}]_{i,T} v(N\setminus T)\\
&= \sum_{S\ni i} \Big\{[M^{Sh}]_{i,S} v(S) - [M^{Sh}]_{i,S} v(N\setminus S)\Big\}\\
&= \sum_{S\ni i} [M^{Sh}]_{i,S} \big[v(S) - v(N\setminus S)\big]\\
&= \sum_{S\ni i} \frac{(s-1)!(n-s)!}{n!}\big[v(S) - v(N\setminus S)\big].
\end{aligned}$$

□

For a given game $\langle N, v\rangle$, its *dual game* $\langle N, v^*\rangle$ is defined as $v^*(S) := v(N) - v(N \setminus S)$, for all $S \subseteq N$. The dual game is a linear operator on $\mathcal{G}^N$. By the dual matrix Q, the dual game is represented as $v^* = Q \cdot v$, where $Q = \big[Q\big]_{S,T\in\Omega}$ is square-coalitional given by

$$\big[Q\big]_{S,T} = \begin{cases} -1, & \text{if } T = N \setminus S \text{ and } S \neq N; \\ 1, & \text{if } T = N; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

The self-duality of the Shapley value can be translated into matrix interpretation as follows.

Proposition 12. *The Shapley standard matrix M^{Sh} satisfies $M^{Sh}Q = M^{Sh}$. That is to say, the Shapley value satisfies the self-duality property in that the Shapley values of the initial game and its dual game are equal.*

Proof by the matrix approach. It is sufficient to check the column equalities $\big[M^{Sh}Q\big]_T = \big[M^{Sh}\big]_T$ for all coalitions $T \in \Omega$. Due to the algebraic representation of a column of a matrix product, it holds that

$$\big[M^{Sh}Q\big]_T = \sum_{S\in\Omega} [Q]_{S,T}[M^{Sh}]_S$$

By (5) and Proposition 10, we obtain the following. If $T \neq N$, then

$$\big[M^{Sh}Q\big]_T = [Q]_{N\setminus T,T}[M^{Sh}]_{N\setminus T} = -[M^{Sh}]_{N\setminus T} = [M^{Sh}]_T.$$

If $T = N$, then

$$\big[M^{Sh}Q\big]_N = \sum_{S\in\Omega} \big[Q\big]_{S,N}\big[M^{Sh}\big]_S = \sum_{S\in\Omega} \big[M^{Sh}\big]_S = \big[M^{Sh}\big]_N.$$

That is to say, for a game $\langle N, v\rangle$ and its dual game $\langle N, v^*\rangle$, we have

$$Sh(v^*) = M^{Sh} \cdot v^* = M^{Sh} \cdot Q \cdot v = M^{Sh} \cdot v = Sh(v).$$

In terms of the similarity of matrices, by combining the dual operator Q and the linear mapping M_λ with respect to Hamiache's associated game on the game space, the Shapley value for TU-games is also axiomatized by Xu et al. [13] as the unique value verifying dual similar associated consistency, continuity, and the inessential game property.

4 The Inverse Problem

For a value Φ on $\mathcal{G}^N$, the following problem is often considered. How to find the set of games $\langle N, v\rangle$ such that $\Phi(v) = b$ for any given vector $b \in \mathbb{R}^N$? We call it the *inverse problem* of the value Φ. The null space $\mathcal{N}_\Phi$ of Φ is defined as the subspace of these games $v \in \mathbb{R}^{2^n-1}$ such that $\Phi(v) = 0$. And two games $\langle N, v\rangle, \langle N, w\rangle$ are called equivalent if $\Phi(v) = \Phi(w)$. Clearly, the inverse problem of a linear value Φ is to solve the null space $\mathcal{N}_\Phi$. To demonstrate the equivalence of two games $v, w \in \mathbb{R}^{2^n-1}$, we need to show that the difference game $v - w$ is in the null space $\mathcal{N}_\Phi$ of the linear value Φ. On the other hand, every game can be decomposed in a unique manner as the sum of its value game (a game with the same value) and an game of $\mathcal{N}_\Phi$. We thus see that, quite apart from what one might think, inessential games, as one type of value games, play a significant role in the characterization of the Shapley value.

Using the representation theory of symmetric groups, Kleinberg and Weiss [8] constructed a direct-sum decomposition of the null space of the Shapley value into invariant subspaces. Then they used the same theory to derive a characterization of a very general type of value, of which the Shapley value is one particular example. The inverse problem of the Shapley value was also studied by Dragan [2]. He presented a potential basis for the null space and an explicit representation of all games with an apriori given Shapley value. The same potential approach was used to analyze the null space of the Banzhaf value (Dragan [3]), as well the family of semivalues (Dragan [4]).

In terms of the Shapley standard matrix M^{Sh}, we have the matrix representation $Sh(v) = M^{Sh} \cdot v$ for all $v \in \mathbb{R}^{2^n-1}$. Therefore, the null space $\mathcal{N}_{Sh}$ of the Shapley value agrees with the null space of the matrix M^{Sh}, *i.e.*,

$$\mathcal{N}_{Sh} = \mathrm{Null}(M^{Sh}) = \{x \in \mathbb{R}^{2^n-1} \mid M^{Sh}x = 0\}.$$

First of all, we show that the Shapley standard matrix is full row rank.

Proposition 13. *The rank of the Shapley standard matrix M^{Sh} with respect to the game space $\mathcal{G}^N$ satisfies rank$(M^{Sh}) = n$.*

Proof. Obviously, $\mathrm{rank}(M^{Sh}) \leq n$ because of n rows. Let us consider the columns of M^{Sh} indexed by single player coalitions and the grand coalition:

$$M^{Sh} = \begin{pmatrix} \frac{1}{n} & -\frac{1}{n(n-1)} & \cdots & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} \\ -\frac{1}{n(n-1)} & \frac{1}{n} & \cdots & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ -\frac{1}{n(n-1)} & -\frac{1}{n(n-1)} & \cdots & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

Adding the multiplication of last column by $\frac{1}{n-1}$ to all of columns indexed by single player coalitions, we get

$$\begin{pmatrix} \frac{1}{n-1} & 0 & \cdots & 0 & \cdots & \frac{1}{n} \\ 0 & \frac{1}{n-1} & \cdots & 0 & \cdots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{n-1} & \cdots & \frac{1}{n} \end{pmatrix}.$$

There are n linear independent columns, $\mathrm{rank}(M^{Sh}) \geq n$. Therefore, $\mathrm{rank}(M^{Sh}) = n$. □

Without proof, we emphasize that for every permutation $\pi \in \Pi^N$, the set of columns $[M^{Sh}]_{\pi(\{1\})}, [M^{Sh}]_{\pi(\{1,2\})}, \cdots, [M^{Sh}]_{\pi(N)}$ of M^{Sh} are maximal linearly independent.

For a matrix A, the dimension of its null space is denoted by $\text{dimNull}(A)$. The well-known Rank Theorem in algebra theory is as follows.

Theorem 14 (The Rank Theorem). *If A is an $n \times m$ matrix, then*

$$rank(A) + dimNull(A) = m.$$

Hamiache [6] defined the *associated game* $\langle N, v_\lambda^{Sh} \rangle$ for any game $\langle N, v \rangle$ and $\lambda \in \mathbb{R}$ as

$$v_\lambda^{Sh}(S) := v(S) + \lambda \sum_{j \in N \setminus S} \big[v(S \cup \{j\}) - v(S) - v(\{j\}) \big], \quad \text{for all } S \subseteq N.$$

The associated game is a linear operator on the game space. Hamiache [7] and Xu *et al.* [12] employed the matrix representation for this operator as

$$v_\lambda^{Sh} = M_\lambda \cdot v,$$

and developed a matrix approach for Hamiache's axiomatization [6] of the Shapley value by means of associated consistency, continuity, and the inessential game property.

The axiom of associated consistency implies that the Shapley value behaves invariant under the adaptation of a game into its associated game. In terminology of the matrix representation, it turns out that the Shapley standard matrix M^{Sh} is invariant under multiplication with the associated transformation matrix M_λ, i.e.,

$$M^{Sh} M_\lambda = M^{Sh}. \tag{6}$$

Inspired by the associated consistency, the inverse problem of the Shapley value is studied in terms of the associated transformation matrix M_λ.

Theorem 15. *The null space of M^{Sh} is the column space of $M_\lambda - I$,* i.e.*,*

$$Null(M^{Sh}) = Col(M_\lambda - I).$$

Proof. By the associated consistency of the Shapley value, we have $M^{Sh} M_\lambda = M^{Sh}$, or equivalently, $M^{Sh}(M_\lambda - I) = \mathbf{0}$. Hence,

$$\text{Col}(M_\lambda - I) \subseteq \text{Null}(M^{Sh}).$$

It is sufficient to show that these two spaces have the same dimension. By Theorem 2.4 in [12], we have 1 is an eigenvalue of M_λ and $\text{rank}(M_\lambda - I) = 2^n - 1 - n$. Therefore, the dimension of the column space $\text{Col}(M_\lambda - I)$ is

$$\text{dimCol}(M_\lambda - I) = \text{rank}(M_\lambda - I) = 2^n - 1 - n.$$

By the Rank Theorem and Proposition 13,

$$\text{dimNull}(M^{Sh}) = 2^n - 1 - \text{rank}(M^{Sh}) = 2^n - 1 - n.$$

Therefore, $\text{Null}(M^{Sh}) = \text{Col}(M_\lambda - I)$. □

Remark 1. Since $M_\lambda = PD_\lambda P^{-1}$ (*see* Lemma 2.6 in [12]), *it follows that*

$$M_\lambda - I = P(D_\lambda - I)P^{-1},$$

and the columns in the diagonal matrix $D_\lambda - I$ *corresponding to the eigenvalue 1 of* M_λ*, which are indexed by all single player coalitions, is a zero-vector. The other columns span the column space* $\text{Col}(M_\lambda - I)$*, equivalently, they form a basis for the null space* $\mathcal{N}_{Sh}$*.*

5 Conclusion

In this paper, the algebraic representation and the matrix approach are applied to study linear operators on the game space, more precisely, the property of linear values. In terms of the basic notion of a coalitional matrix, linear values are represented algebraically by the products of corresponding coalitional matrix and worth vector. We preform a matrix analysis in the setting of cooperative game theory, to study axiomatizations of linear values, by investigating appropriate properties of these representation matrices. Particularly, the Shapley value is the most important representative. This paper is an initialization for using systematically the algebraic representation and the matrix approach in the research field of cooperative game theory. There are still many more open problems, such as how to apply the matrix approach to derive new axioms of proposed linear values, or to define new linear values for TU-games? To generalize the matrix approach to game-theoretic models with coalitional structures is another challenging issue in cooperative game theory.

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The General Nucleolus of n-Person Cooperative Games

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Abstract. In this paper, we define the concept of the general nucleolus whose objective function is limited to the player complaint, to reflect the profit distribution more intuitively on the space of n-person cooperative games. An algorithm for calculating the general nucleolus under the case of linear complaint functions is given so that we can get an accurate allocation to pay for all players. A system of axioms are proposed to characterize the general nucleolus axiomatically and the Kohlberg Criterion is also given to characterize it in terms of balanced collections of coalitions. Finally, we prove the equivalence relationship of the general nucleolus, the least square general nucleolus and the p-kernel to normalize the different assignment criteria.

Keywords: Player complaint · General nucleolus · Kohlberg Criterion · Least square · p-kernel

1 Introduction

In cooperative game theory, Schmeidler (1969) uses the lexicographic order to compare the coalition excess vector, from which he gets the solution concept of the nucleolus over the imputation set. Further, Kohlberg (1971) proposes the Kohlberg Criterion that the nucleolus of n-person games is characterized in terms of balanced collections of coalitions and then a generalization of the Kohlberg Criterion is raised to extend the sets of payoff vectors for various coalition structures by Owen (1977). However, the idea of the definition of the nucleolus is not applied only to TU games under some applications, Maschler et al. (1992) propose the general nucleolus, which is a generalization of the nucleolus to an arbitrary pair (Π, F), where Π is a topological space and F is a finite vector whose components are real and continuous functions defined on Π.

Moreover, Davis and Maschler (1965) introduce the solution concept of kernel based also on the idea of excess and maximal surplus. Later, a suitable reformation of the surplus concept is introduced by Ruiz et al. (1996), with which they define the average kernel. And instead of minimizing the maximum complaint function vector according to the lexicographical order, they (Ruiz et al. 1996) select the payoff vector which minimizes the variance of the resulting excess of

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the coalitions, which is named with the least square nucleolus over the imputation set by means of an assessment of its relative fairness. Furthermore, they obtain that the least square nucleolus is the unique point of the average kernel and Molina and Tejada (2000) prove that the least square nucleolus and the lexicographical solution (Sakawa and Nishizaki 1994) choose the same imputation for any game with nonempty imputation set.

As the solutions mentioned above are all based on the coalition excess $e\,(S, x)$ only regarded as the dissatisfaction or complaint of any coalition $S \subseteq N$, it can not reflex complaints of the players themselves intuitively. Moreover, Sakawa and Nishizaki (1994) firstly propose the player excess to evaluate everyone's payoff by summing up all of the excesses of coalitions to which he belongs. Further, Vanam and Hemachandra (2013) define the excess sum wherein a player views entire coalitions excess and per-capita excess-sum of a player as sum of normalized excesses of coalitions involving this player to view them as two measures of player's dissatisfaction or complaint towards a payoff vector. Sun et al. (2015) present another criterion to measure the payment of each player by summing up all of the differences between the marginal contribution of a coalition to which he belongs and the corresponding payment. Thus, we can define the general nucleolus whose complaint function can be applied to the case of a class of player complaints.

In fact, we restrict the objective function F to be the player complaint of a more general form over the imputation set and define the concept of the general nucleolus on the space of n-person cooperative games. Similarly, the least square value and p-kernel are defined to be as two different methods to distribute the overall earnings. The conclusion that three allocation schemes are the same assignment criteria, is proved. In order to obtain the allocation of the general nucleolus under the case of the linear complaint functions accurately, we explore an algorithm to calculate it. We also give a series of axioms to characterize the general nucleolus besides with the Kohlberg Criterion.

The paper is organized as follows: In Sect. 2, we get some preliminary knowledge served for the later contexts. Section 3 introduces an algorithm of the general nucleolus. Section 4 proposes a system of axioms to characterize the general nucleolus. A similar result of the Kohlberg Criterion is proved in Sect. 4. Section 5 concludes with a brief summary.

2 Preliminaries

A cooperative game with transferable utility (TU) game is a pair (N, v), where $N = \{1, 2, \cdots, n\}$ is a finite player set and $v : 2^N \to R$ is a real-valued function satisfying $v(\emptyset) = 0$ with 2^N representing the subset of N named with coalitions. For each coalition $S \subseteq N$, $v\,(S)$ stands for the worth that coalition S achieves with its members cooperative altogether and $|S|$ denotes the cardinality of S. Noting that $n = |N|$ for symbolic convenience. If there is no ambiguity, we identify the game (N, v) with its characteristic function v. The set of all cooperative games over N is denoted by G^N.

The solution part of cooperative game theory is to deal with the allocation problem of how to divide the overall earnings the amount of $v(N)$ among the players in the TU game and it may be thought of as a vector whose coordinates are indexed by the players. There is associated a single allocation called the value of the TU game. A payoff vector $x \in R^n$ is feasible for a game v if $x(N) \le v(N)$, where $x(S) = \sum_{i \in S} x_i$ for any $S \subseteq N$. For convenience, we simply write $x(i)$ instead of $x(\{i\})$ for each $i \in N$. A payoff vector x is called *efficient* or a *preimputation* if $x(N) = v(N)$, and an *imputation* if, besides, it holds that $x_i \ge v(i)$ for any $i \in N$. $I^*(N,v)$ and $I(N,v)$ denote the preimputation and imputation set respectively.

A well-established one-point solution concept is the nucleolus, proposed by Schmeidler (1969). Let $v \in G^N$, the excess of the coalition $S \in 2^n$ with respect to the payoff vector x is defined as

$$e(S,x) = v(S) - x(S).$$

Thus the excess $e(S,x)$ can be regarded as the loss or complaint for coalition S facing the final payoff vector x. Moreover, we denote $\theta(x) \in R^{2^n}$ with the vector whose components are the excesses $e(S,x)$ arranged in non-increasing order. The nucleolus for any $(N,v) \in G^N$ denoted by $n(N,v)$, is defined by

$$\mathcal{N}(N,v) = \{x \in I(N,v) \,|\theta(x) \le_L \theta(y) \text{ for all } y \in I(N,v)\}.$$

Also, Schmeidler (1969) proves that the nucleolus of a game with a nonempty imputation set is a unique point.

However, the idea of lexicographically minimizing (maximizing) a vector of objective functions need not be applied only to TU games. Indeed, it is applied in several other conflict situations. Thus Maschler et al. (1992) introduce the *general nucleolus*, which is a generalization of the nucleolus to an arbitrary pair $(\Pi, F) \in \Omega$, where Π is a topological space and $F = \{F_j\}_{j \in M}$ is a finite set of real continuous function whose domain is Π. They identify the general nucleolus with

$$\mathcal{GN}(\Pi,F) = \{x \in \Pi \,|\theta \circ F(x) \le_L \theta \circ F(y), \text{ for all } y \in \Pi\},$$

where $\theta : R^M \to R^m$ is the coordinate ordering map ($\theta \circ F(x)$ is an m-vector, with the same components as in $F(x)$, but ordered in non-increasing order) and $\le_L$ is the lexicographic order on R^m $(m = |M|)$. And they got that the general nucleolus $\mathcal{GN}(\Pi,F) \ne \emptyset$ if Π is a nonempty compact set and includes at least a point.

3 The Calculation of the General Nucleolus

Given a cooperative game (N,v) and the objective function F is restricted to the player complaint vector, then we can get the concept of the general nucleolus of n-person cooperative games.

Definition 1. The general nucleolus for any $v \in G^N$ is defined to be

$$\mathcal{GN}(N,v) = \{x \in I(N,v) \,|\theta \circ F(x) \le_L \theta \circ F(y) \text{ for all } y \in I(N,v)\},$$

where $\theta : R^n \to R^n$ is the coordinate ordering map, $F = \{F_i\}_{i \in N}$ is a finite set of real continuous functions with the variable F_i seen as the complaint for player i or the player i's excess function and $\theta \circ F(x)$ being an n-tuple vector whose components are ordered in nonincreasing order and $\le_L$ is the lexicographic order.

It is reasonable to think that the more the player i gets, the less complaint player i has when facing the final payoff vector x. Assume that player $j \in N \backslash i$ has nothing to do with the payoff of player i. Inspired by the proposed player excesses (Sakawa and Nishizaki 1994; Sun et al. 2015, 2017; Vanam and Hemachandra 2013; Kong et al. 2017), we consider the case that the complaint function F_i is affine in allocation x_i in our paper, i.e., $F_i(x) = a_i - bx_i, i \in N$ with a_i being a constant of player i and $b > 0$. We firstly explore an algorithm to calculate the accurate allocation of the general nucleolus under the case of the linear complaint functions. Thus, we need to prove the general prenucleolus consist of a unique point before the algorithm.

Lemma 1. *The general nucleolus of n-person cooperative games* $\mathcal{GN}(N,v)$ *is nonempty if the imputation set is nonempty and consists of at most one point.*

Proof. The non-emptiness of $\mathcal{GN}(N,v)$ is easy to be obtained, next we only need to prove its uniqueness. Supposed that $x, y \in \mathcal{GN}(N,v)$ and $x \ne y$, we have that

$$\theta \circ F(x) = \theta \circ F(y) \le_L \theta \circ F(z) \text{ for any } z \in I(N,v).$$

Especially, let $z = \lambda x + (1-\lambda) y$ for any $0 < \lambda < 1$, it yields that $\theta \circ F(x) = \theta \circ F(y) \le_L \theta \circ F(\lambda x + (1-\lambda) y)$. Without loss of generality, consider $1, 2, \cdots, n$ to be an order satisfying that

$$\begin{aligned} \theta \circ F(\lambda x + (1-\lambda) y) &= (F_1(\lambda x + (1-\lambda) y), \cdots, F_n(\lambda x + (1-\lambda) y)) \\ &= \lambda (F_1(x), \cdots, F_n(x)) + (1-\lambda)(F_1(y), \cdots, F_n(y)) \\ &\le_L \lambda \theta \circ F(x) + (1-\lambda) \theta \circ F(y) = \theta \circ F(x) = \theta \circ F(y). \end{aligned}$$

Thus, we get that $\theta \circ F(x) = \theta \circ F(y) = \theta \circ F(\lambda x + (1-\lambda) y)$.

Hence, x and y have all player complaints equal, which means that $x = y$. □

Now, we can state the algorithm to calculate the general nucleolus under the case of the linear complaint functions for any n-person cooperative games based on the general prenucleolus (Kong et al. 2017) of n-person cooperative games.

Algorithm. Construct a sequence of pairs $\left(x^i, D_i\right)$ $(i = 1, 2, \cdots)$, where x^i is a payoff vector and D_i a subset of N, inductively defined by

Step 1. $x_i{}^1 = \frac{v(N)}{n} + \frac{na_i - \sum_{j \in N} a_j}{nb} (i = 1, 2, \cdots)$ and $D_1 = \{i \in N \,|x_i < v(i)\}$;

Step 2. $x_i{}^{l+1} = \begin{cases} \frac{v(N)}{n-\sum\limits_{i=1}^{l}|D_i|} + \frac{\left(n-\sum\limits_{i=1}^{l}|D_i|\right)a_i - \sum\limits_{j\in N\setminus(D_1\cup\cdots\cup D_l)} a_j}{\left(n-\sum\limits_{i=1}^{l}|D_i|\right)b}, & i \notin D_1\cup\cdots\cup D_l \\ v(i), & i \in D_1\cup\cdots\cup D_l \end{cases}$ and

$D_{l+1} = \{i \in N\setminus(D_1\cup\cdots\cup D_l)\,|x_i < v(i)\}$(with $D_0 = \emptyset$);

Step 3. The sequence stops when $D_{l+1} = \emptyset$ and you will get the general nucleolus x^{l+1}. Otherwise, let $l = l+1$ and go to Step 2.

From the process of the algorithm, the solution of each step is to give $v(i)$ for the player i whose payment is smaller than the amount $v(i)$ by going alone and average the summation of the remaining players' complaints so as to pay for the rest of players. It should be noted that when $D_1 = \emptyset$, the closing payoff vector is x^1 and x^1 is obtained by $F_i = F_j$ for any $i, j \in N$ and $i \neq j$. That is to say that x^1 is the optimal solution by taking the same complaint for all of players. This process must end at most n steps and the closing payoff vector $x^{l+1}(0 \leq l \leq n-1)$ is the general nucleolus.

Theorem 1. *The closing payoff vector* $x^{l+1}(0 \leq l \leq n-1)$ *is the general nucleolus.*

Proof. Assume that the algorithm stops at l steps, it is obvious that the closing payoff vector is x^{l+1} with

$$x_i{}^{l+1} = \begin{cases} \frac{v(N)}{n-\sum\limits_{i=1}^{l}|D_i|} + \frac{\left(n-\sum\limits_{i=1}^{l}|D_i|\right)a_i - \sum\limits_{j\in N\setminus(D_1\cup\cdots\cup D_l)} a_j}{\left(n-\sum\limits_{i=1}^{l}|D_i|\right)b}, & i \notin D_1\cup\cdots\cup D_l \\ v(i), & i \in D_1\cup\cdots\cup D_l \end{cases}$$

and $M^{l+1} = \emptyset$. As x^{l+1} is obvious to be an imputation, we only prove that x^{l+1} is the optimal solution in the lexicographic order. From the process of the algorithm, we naturally have that

$$F_i\left(x^{l+1}\right) = \begin{cases} \frac{\sum\limits_{i\in N} F_i\left(x^{l+1}\right) - \sum\limits_{i\in D_1\cup\cdots\cup D_l}(a_i - bv(i))}{n-\sum\limits_{i=1}^{l}|D_i|}, & i \notin D_1\cup\cdots\cup D_l \\ a_i - bv(i), & i \in D_1\cup\cdots\cup D_l \end{cases}$$

and $F_i\left(x^{l+1}\right) \geq \frac{1}{n}\sum\limits_{i=1}^{n} F_i(x) \geq F_j\left(x^{l+1}\right)$ for any $i \in N\setminus(D_1\cup\cdots\cup D_l)$ and $j \in D_1\cup\cdots\cup D_l$. Then there are two cases,

(1) if $F_i\left(x^{l+1}\right) = F_j\left(x^{l+1}\right)$ for any $i \in N\setminus(D_1\cup\cdots\cup D_l)$, $j \in D_1\cup\cdots\cup D_l$, then x^{l+1} is the general prenucleolus, which is of course the optimal solution in the lexicographic order.

(2) if $F_i\left(x^{l+1}\right) > F_j\left(x^{l+1}\right)$ for any $i \in N \backslash (D_1 \cup \cdots \cup D_l)$, $j \in D_1 \cup \cdots \cup D_l$, then for any $y \in I(N, v)$,

$$F_j\left(x^{l+1}\right) = a_j - bv(j) \geq a_j - by_j = F_j(y), j \in D_1 \cup \cdots \cup D_l.$$

Thus,

$$\sum_{i \in N \backslash (D_1 \cup \cdots \cup D_l)} F_i\left(x^{l+1}\right) \leq \sum_{i \in N \backslash (D_1 \cup \cdots \cup D_l)} F_i(y).$$

Therefore, there must exist an integer $1 \leq h \leq n$ such that $\left(\theta \circ F\left(x^{l+1}\right)\right)_g = (\theta \circ F(y))_g$ for $1 \leq g < h$, whereas $\left(\theta \circ F\left(x^{l+1}\right)\right)_h = (\theta \circ F(y))_h$, that is, x^{l+1} is the optimal solution in the lexicographic order. □

Next, we will take two examples to illustrate the algorithm of which one of the objective functions is the special case proposed by Sakawa and Nishizaki Sakawa and Nishizaki (1994) and the other by Sun et al. (2015).

Example 1. Consider a game $v \in G^N$ with $N = \{1, 2, 3, 4\}$ defined by and the objective function is that $F_i(x) = \sum_{\substack{S \subseteq N \\ i \in S}} e(S, x)$ for all $i \in N$ (Table 1).

Table 1. Worths of the given 4-person game

S	{1}	{2}	{3}	{4}	{1,2}	{1,3}	{1,4}	{2,3}
$v(S)$	59	80	60	54	150	108	86	132
S	{2,4}	{3,4}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}	{1,2,3,4}	
$v(S)$	110	88	210	182	145	170	255	

According the the mentioned algorithm, we firstly get the payoff vector x^1,

$$x^1 = (66.125, 89.625, 59.375, 39.875).$$

Next give 60 to the player 3 and 54 to player 4 and then divide the summation of the remaining complaints of player 1 and 2 equally, so that we obtain

$$x^2 = (58.75, 82.25, 60, 54).$$

Now we give 59 to player 1 and the rest to player 2, we finally get the general nucleolus in this way,

$$x^3 = (59, 82, 60, 54) = \mathcal{GN}(N, v).$$

Example 2. Let $v \in G^N$ with $N = \{1, 2, 3, 4\}$ defined to be and $F_i(x) = \sum_{\substack{S \subseteq N \\ i \in S}} (v(S) - v(S \backslash i) - x_i)$ for all $i \in N$ (Table 2).

Similarly, we firstly get the payoff vector x^1,

$$x^1 = (6, 4, 6, 4) .$$

And then let the payoffs of player 1 and 3 both be 8 and equally distribute the remaining complaints to play 2 and 4, thus we get the general nucleolus

$$\mathcal{GN}\,(N, v) = (8, 2, 8, 2) .$$

Table 2. Worths of the given 4-person game

S	{1}	{3}	{1,2}	{3,4}	{1,2,3,4}	otherwise
$v(S)$	8	8	8	8	20	0

4 The Properties of the General Nucleolus of N-Person Cooperative Games

As is stated in above section, the general nucleolus is a single-valued solution concept. In this part, we will characterize it based on the mentioned algorithm for the general nucleolus. Firstly, we introduce a system of axioms.

For any $(N, v) \in G^N$, we firstly give several properties that a solution $\varphi : G^N \to R^n$ might have.

(i) Efficiency: For any game $(N, v) \in G^N$, $\sum\limits_{i\in N} \varphi_i(N, v) = v(N)$.

(ii) Individual rationality: For any game $(N, v) \in G^N$, $\varphi_i(N, v) \geq v(i)$ for any $i \in N$.

(iii) Inessential game property: For any inessential game (N, v), the value verifies $\varphi_i\,(N, v) = v\,(i)$ for all $i \in N$.

(iv) Continuity: For every convergent sequence of games $\{(N, v_k)\}_{k=0}^{\infty}$ the limit of which is game $(N, \tilde{v})$, we have $\lim\limits_{k\to\infty} \varphi(N, v_k) = \varphi(N, \tilde{v})$.

(v) Player complaint property: Let $Y = (v\,(1)\,, v\,(2)\,, \cdots, v\,(n)) \in R^n$ and $H = \left\{ i \in N \,\middle|\, F_i\,(Y) < \frac{1}{n} \sum\limits_{i\in N} F_i\,(x),\ \text{for any } x \in I\,(N, v) \right\}$, then for any $i \in H$, $\varphi_i\,(N, v) = v\,(i)$ and for any $j, k \in N\backslash H$, $F_j\,(\varphi) = F_k\,(\varphi)$.

(vi) Anonymity: Let v be a game and $\sigma : N \to N$ be a permutation of player set N. Then if $\sigma v\,(S) = v\,(\sigma\,(S))$, we have $\varphi_{\sigma(i)}\,(N, \sigma v) = \varphi_i\,(N, v)$ for all $i \in N$.

(vii) Covariance: φ is covariance if v is a game, λ is a positive number and $a \in R^n$, then $\varphi\,(\lambda v + \bar{a}) = \lambda\varphi\,(v) + a$, where $\bar{a}$ is the additive game generated by v.

Proposition 1. *The general nucleolus under the case of the linear complaint functions verifies the above properties with a nonempty imputation set.*

The readers can easily check that the mentioned propertied can be deduced by the definition of the general nucleolus whenever $F_i(x) = a_i - bx_i$ for all $i \in N$.

Theorem 2. *The general nucleolus under the case of the linear complaint functions is the unique value verifying the Efficiency and Player complaint property.*

Proof. Suppose that there exist $y^1, y^2 \in R^n$ verifying the two properties, then it should be $y^1 = y^2$.

Indeed, for any $i \in H$,

$$y_i^1 = v(i) = y_i^2$$

and for any $k, j \in N \backslash H$,

$$F_j\left(y^1\right) = F_k\left(y^1\right),\ F_j\left(y^2\right) = F_k\left(y^2\right).$$

Then we can get that $F_j\left(y^1\right) = F_j\left(y^2\right)$, otherwise, without loss of generality, let $F_j\left(y^1\right) = a_j - by_j{}^1 > a_j - by_j{}^2 = F_j\left(y^2\right)$. Thus, we get that $y_j^1 < y_j^2$, $j \in N \backslash H$.

Therefore,

$$y^1(N) = \sum_{i \in H} y_i^1 + \sum_{i \in N \backslash H} y_i^1 < \sum_{i \in H} y_i^2 + \sum_{i \in N \backslash H} y_i^2 = y^2(N),$$

which is contradicted with the efficiency.

Eventually, we get that $y_i^1 = y_i^2$ for any $i \in N$. □

Remark 1. Logical independence of axioms in Theorem 2 is shown by the following alternative values.

(1) The value $\varphi^1 \in R^n$ defined as

$$\varphi_i^1 = \begin{cases} v(i), & \text{if } x_i{}^1 < v(i) \\ x_i{}^1, & \text{otherwise} \end{cases},$$

satisfies the Player complaint property except Efficiency.

(2) Let $\varphi^2 \in R^n$ be defined to be

$$\varphi_i^2(N, v) = \frac{v(N)}{n} \text{ for any } i \in N,$$

which is obviously verifies the Efficiency but not the Player complaint property.

It is obviously obtained that the Player complaint property is a great reflection of the algorithm. And the reason why the axiomatization doesn't include the individual rationality is that the individual rationality can be deduced by the Player complaint property.

4.1 The Least Square General Nucleolus Is the General Nucleolus

When the summation of all players' complaint functions is a constant, some players' complaints will be increased, which is caused by the decrease of partial players' complaints. By minimizing the variance of the resulting complaints of players different from that given by the lexicographic order, we define the least square general nucleolus on the space of n-person cooperative games. Formally, we consider the following Problem 1 for any game $v \in G^N$,

Problem 1.

$$\begin{aligned} \min \quad & \sum_{i \in N} \left(F_i(x) - \bar{F}(x)\right)^2 \\ s.t. \quad & x(N) = v(N) \ \ and\, x_i \geq v(i) \ \ for\ all\, i \in N, \end{aligned}$$

where $\bar{F}(x)$ is the average player complaint at x, given by

$$\bar{F}(x) = \frac{1}{n} \sum_{i \in N} F_i(x).$$

Definition 2. The least square general nucleolus of n-person cooperative games is the solution of Problem 1 for a game $v \in G^N$ with respect with the payoff vector x.

Remark 2.(1) The average player complaint at x under the case of the linear complaint functions is the same for any $x \in I(N, v)$ since

$$\bar{F}(x) = \frac{1}{n} \sum_{i \in N} F_i(x) = \frac{1}{n} \sum_{i \in N} (a_i - bx_i) = \frac{1}{n} \left(\sum_{i \in N} a_i - bv(N) \right),$$

where the last equation is due to efficiency. Thus, we can substitute $\bar{F}(x)$ for $\bar{F}$ even for any constant k as the optimal solution of Problem 1 remains unchanged.

(2) Moreover, for any $k \in R$,

$$\sum_{i \in N} (F_i(x) - k)^2 = \sum_{i \in N} F_i^{\,2}(x) + k^2 - 2k \sum_{i \in N} F_i(x),$$

the last summation is the same for all efficient payoff vector under the case of the linear complaint functions so that the resulting objective function differs from that of Problem 1 on a constant on their feasible sets. In particular, for $k = 0$ we conclude that the optimal solution of Problem 1 is that of the following Problem 2.

Problem 2.

$$\begin{array}{ll} \min & \sum_{i \in N} F_i^{\,2}(x) \\ s.t. & x(N) = v(N) \;\; and\, x_i \geq v(i) \;\; for\; all\, i \in N. \end{array}$$

So far, we have raised two methods to divide the overall earnings, i.e. the thoughts of lexicographic order and the least square criterion, to make players be satisfied with the resulting payoff vector as much as possible. However, what makes us interested in is to normalize the different assignment criteria, which is of great practical significance. Now we will prove that the least square general nucleolus is the general nucleolus under the case of the linear complaint functions. Then the following Lemma is required.

Lemma 2. *For any game $v \in G^N$, then an imputation x is the least square general nucleolus of v under the case of the linear complaint functions iff for all $j \in N$,*

$$x_j > v(j) \Rightarrow F_j(x) = \max\{F_i(x) \,|i \in N\}.$$

Proof. As the objective function is strictly convex and the imputation set is compact, there exists a unique solution of Problem 2. Further, it is easy to prove that the Karush-Kuhn-Tucker conditions are necessary and sufficient conditions for the following Problem 3, whose optimal solution is equivalent to that of Problem 2,

Problem 3.

$$\begin{array}{ll} \min & f(x) \\ s.t. & h(x) \leq 0, \;\; i = 1, 2, \cdots, n, \\ & g(x) = 0, \\ & x \in R^N, \end{array}$$

where $f(x) = \sum_{i \in N} F_i^{\,2}(x)$, $h(x) = v(i) - x_i$ and $g(x) = x(N) - v(N)$. Then $x \in R^N$ is the optimal solution of the Problem 3 iff there exists scalars $v \in R$, $u_i \geq 0, i = 1, 2, \cdots, n$, such that

$$\begin{array}{l} \nabla f(x) + \sum_{i=1}^{n} u_i \nabla h_i(x) + v \nabla g(x) = 0 \\ u_i h_i(x) = 0, i = 1, 2, \cdots, n, \end{array}$$

which is equivalent to that

$$\begin{array}{l} -2bF_i(x) - u_i + v = 0, i = 1, 2, \cdots, n, \\ u_j = 0 \; with\, x_j > v(j) \;\; for\; every\, j \in N. \end{array}$$

Thus, it yields that for $u_i \geq 0, i = 1, 2, \cdots, n$

$$F_i(x) = \frac{v - u_i}{2b} \leq \frac{v}{2b} = F_j(x), j = 1, 2, \cdots, n.$$

□

Theorem 3. *For any game $v \in G^N$ with nonempty imputation set, the least square general nucleolus of v is the general nucleolus under the case of the linear complaint functions.*

Proof. Let $x = \mathcal{GN}(N, v)$ and y be the least square general nucleolus, it is obvious that $\theta \circ F(x) \leq_L \theta \circ F(y)$ by the definition of the general nucleolus. Moreover, let us consider the sets $N_1 = \{j \in N \,|y_j = v(j)\}$, $N_2 = \{j \in N \,|y_j > v(j)\}$. Then we get that

(1) $F_j(x) \leq F_j(y)$ with $y_j = v(j) \leq x_j$ for all $j \in N_1$.
(2) $F_j(x) \leq (\theta \circ F(x))_1 \leq (\theta \circ F(y))_1$ for all $j \in N_2$. By Lemma 2, it directly yields that $(\theta \circ F(y))_1 = F_j(y)$. That is to say, $F_j(x) \leq F_j(y)$ for all $j \in N_2$.

Thus, $F_j(x) \leq F_j(y)$ for all $j \in N$. Then we get that $F_j(x) = F_j(y)$ for all $j \in N$ with $\sum\limits_{j \in N} F_j(x) = \sum\limits_{j \in N} F_j(y)$.

Eventually, $x = y$ for $F_j(x) = a_j - bx_j$, $j \in N$. □

4.2 The General Nucleolus Is the Unique Point of p-kernel

The concept of the kernel introduced in Davis and Maschler (1965), is based on the idea of excess and maximum surplus $S^v_{ij}(x)$, which can be regarded as a measure of the power of player i to threaten player j with respect to the imputation x. Then the kernel is defined as the set of all imputations for which no player outweights another player. We now propose a parallel solution concept but based on the player excess. Thus, the p-kernel is described as follows.

Definition 3. The p-kernel $pk(N, v)$ of a game v is the set of all imputations $x \in I(N, v)$ satisfying for all $i, j \in N$, $i \neq j$,

$$(F_i(x) - F_j(x))(x_j - v(j)) \leq 0, \text{ or}$$
$$(F_j(x) - F_i(x))(x_i - v(i)) \leq 0.$$

By the definition, it is interpreted as the player j get the amount $v(j)$ by going alone when the player j poses a threat to i, in other words, the complaint of player i is larger than j. And vice versa.

As it is well-known, the nucleolus is always in the kernel (Schmeidler 1969). A similar but stronger result can be established about the p-kernel.

Proposition 2. *For any $v \in G^N$, the least square general nucleolus under the case of the linear complaint functions is the unique point of the p-kernel.*

Proof. By Lemma 2, a direct conclusion that an imputation x is the least square general nucleolus if and only if $F_j(x) < F_i(x) \Rightarrow x_j = v(j)$ for all $i, j \in N, i \neq j$, is obtained. Moreover, the p-kernel can be equivalently defined as

$$pk(N, v) = \{x \in I(N, v) \,|F_j(x) < F_i(x) \Rightarrow x_j = v(j), i,\; j \in N, i \neq j\}.$$

Thus, the result is established.

Theorem 4. *The general nucleolus under the case of the linear complaint functions is the unique point of the p-kernel for any* $v \in G^N$.

The conclusion is directly obtained by Theorem 3.

5 The Kohlberg Criterion

Kohlberg (1971) proposes that the nucleolus of n-person games is characterized in terms of balanced collections of coalitions. A similar result can be derived for the general nucleolus of n-person cooperative games over the imputation set in terms of balanced sets.

For any game v and any payoff vector x, let

$$a_1 = \max\{F_i(x) \,| i \in N\},\ B_1 = \{i \in N \,| F_i(x) = a_1\};$$

$$a_2 = \max\{F_i(x) \,| i \in N \backslash B_1\},\ B_2 = \{i \in N \,| F_i(x) = a_2\};$$

$$\cdots$$

$$a_p = \max\{F_i(x) \,| i \in N \backslash B_1 \backslash \cdots \backslash B_{p-1}\},\ B_p = \{i \in N \,| F_i(x) = a_p\},$$

where $B_{p+1} = \emptyset$, which means that $T = \{B_1, B_2, \cdots, B_p\}$ is a partition of N, that is, $B_1 \cup B_2 \cup \cdots \cup B_p = N$ and $B_1 \cap B_2 \cap \cdots \cap B_p = \emptyset$. Finally, $B_0 = \{i \in N \,| x_i \geq v(i)\}$. It is obvious that $a_1 < a_2 < \cdots < a_p$. Consider $C_j = B_1 \cup B_2 \cup \cdots \cup B_j,\ j = 1, 2, \cdots, n$.

Proposition 3. *For any game* v*, the following two statements are equivalent.*

(1) An imputation x *is the general nucleolus under the case of the linear complaint functions.*

(2) For all $j = 1, 2, \cdots, p$ *and any* $y \in R^n$,

$$y_i \geq 0, i \in C_j \cup B_0 \text{ and } y(N) = 0$$

imply that $y_i = 0$ *for all* $i \in C_j$.

Proof. Assumed that $\sum_{i \in C_j} y_i \neq 0$, there exists $i \in C_j$ satisfying $y_i > 0$ since $y_i \geq 0, i \in C_j \cup B_0$. It is obvious that $z = x + ty \in I(N, v)\,(t > 0)$ when t is small enough. Then for all $i \in C_j$ and $k \notin C_j$, there exists $l \leq j, m > j$ satisfying

$$F_i(x + ty) = a_l > a_m = F_j(x + ty) \text{ with } i \in B_l,\ k \in B_m$$

and

$$F_i(x + ty) = a_i - b(x_i + ty_i) < a_i - bx_i = F_i(x),$$

the inequality is because of $b > 0,\ t > 0,\ y_i > 0$. Thus it is contradicted with $x = \mathcal{GN}(N, v)$.

On the other hand, let $x \in I(N,v)$, $z = \mathcal{GN}(N,v)$ with $x \neq z$ and $y = z - x$ satisfying $y(N) = 0$, it directly follows that $F_i(x) = a_1 \geq F_i(z)$ for all $i \in B_1$ since $\theta \circ F(z) \leq_L \theta \circ F(x)$. By the condition (2), we get that

$$F_i(x) - F_i(z) = a_i - bx_i - (a_i - bz_i) = by_i = 0, \text{ for all } i \in B_1.$$

Thus, it implies that $F_i(x) = F_i(z)$ and obviously $x_i = z_i,\ i \in B_1$. Similarly, we have that $F_i(x) = a_2 \geq F_i(z)$ for all $i \in B_2$ and $i \notin B_1$ and $F_i(x) - F_i(z) = a_i - bx_i - (a_i - bz_i) = by_i = 0,$ for all $i \in B_1 \cup B_2$. Then, $x_i = z_i,\ i \in B_2$ is obtained since $F_i(x) = F_i(z),\ i \in B_2$ and so on. We finally gain that $x_i = z_i,\ i \in B_p$, that is, $x_i = z_i,\ i \in N$. Thus, $x = z = \mathcal{GN}(N,v)$.

Now Kohlberg Criterion is proved to characterized by the general nucleolus in terms of balanced collections of coalitions.

Proposition 4 (Kohlberg). *For any game v and any imputation x, then x is the general nucleolus if and only if T is a balanced collection.*

Proof. It is obvious that T is a balanced collection as T is a partition of the grand coalition N.

While T is a balanced collection, we have that $\sum\limits_{i \in B_k} \lambda(B_k) e^{B_k} = e^N$. If y is a payoff vector verifying (2) in Proposition 3, it yields that $\sum\limits_{i \in B_k} \lambda(B_k) e^{B_k} \cdot y = y(N) = 0$. Thus, we have that $y(B_k) = 0$ which is equivalent to $y_i = 0,\ i \in B_k$ under the condition of (2) in Proposition 3. Therefore, Proposition 3 implies that x is the general nucleolus.

6 Conclusions

In this paper, we have generalize three concepts of the general nucleolus, the least square general nucleolus and p-kernel for n-person cooperative games over the imputation set. And the allocation to divide the overall earnings is equivalent. An algorithm is proposed to calculate an accurately general nucleolus. A few of axioms are given to characterize the general nucleolus besides with the Kohlberg Criterion.

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A Cooperative Game Approach to Author Ranking in Coauthorship Networks

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Abstract. In this paper, we propose a cooperative game approach to the issue of author ranking in coauthorship networks. We first construct three weighted coauthorship networks from different perspectives and define three cooperative games according to the corresponding coauthorship networks. Then we use the core and the Shapley value as allocation rules for the games. Finally, considering the contribution level of the authors to their papers, we give the weighted Shapley value and a new value as the allocation rules. These allocation rules can be used to rank the authors in coauthorship networks.

Keywords: Cooperative game · Coauthorship network · Author ranking · Core · Shapley value · Weighted shapley value

1 Introduction

Nowadays, scientific collaboration is a universal phenomenon in the scientific field. Research paper is the main form of the author's research achievement. Coauthorship of a paper can be thought of as documenting a collaboration between two or more authors, and these collaborations form a "coauthorship network".

Coauthorship networks which basically formed by papers and authors were widely studied in recent years. Constructing and analysing the suitable coauthorship networks as to some research field can help us to know the research ability or influence of every author or research group. The potential researchers, hot research topics and new research directions also can be spotted by analysing the coauthorship networks. Solving these problems is very important for government's scientific research management. For example, how to allocate their research funds to the researchers and how to set up scientific research projects.

The research of coauthorship networks in the literature is mainly focused on complex network (Newman [8–10], Acedo [1], Xie [14], Ding [4] etc.). As far as I know, there is no research that applies the pure cooperative game theory to the study of coauthorship network. The cooperative game theory is just a tool for them to construct or build the networks in some papers. For example, Narayanam et al. [7] defines the cooperative game to capture the information

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diffusion process in the social network, and use Shapley value of the nodes to discover the influential nodes, coauthorship network is a typical example in their paper. Coauthorship network actually is a kind of hypergraph, there are virous studies that refer to hypergraph cooperative game. The Myerson value (Nouweland et al. [11]) and the position value (Borm et al. [2], Casajus [3], Shan [12]) are two main research topics in hypergraph cooperative game.

In this paper, we apply cooperative game to rank the authors in the coauthorship network. We use the allocation rules of the games that induced from the coauthorship networks to evaluate the authors' scientific research ability and rank them. We first construct three weighted coauthorship networks according to the amount of authors' papers, the citation number and the rank factor of the papers. And we also give three cooperative games though the weighted coauthorship networks. Then we study the core and the Shapley value of the games. Finally, from respective of author's contribution level we define the weight system and study the weighted Shapley value, and we also give a new value which is similar to the position value. These allocation rules are needed for our author ranking.

Formal definitions are provided in Sect. 2. In Sect. 3 we construct three coauthorship networks and define three related cooperative games. The core and the Shapley value for the games are given in Sect. 4. Considering the contribution level of the authors to the related papers, we also give other two value in Sect. 4. Section 5 is the conclusions and remarks.

2 Preliminaries

2.1 Coauthorship Network

Coauthorship network is a complex network that represents the scientific collaboration relationships of the researchers, it's basically formed by authors and their related papers. The studies of such networks turns out to reveal many interesting features of academic communities.

Formally, the nodes $N = \{1, 2, \cdots, n\}$ in the coauthorship network (N, L) are the authors, and two authors are connected by a undirected line $l \in L$ if they have coauthored one or more papers (Newman [8]).

There exists another type or definition of the coauthorship network (Estrada and Rodrŕguez-Velázquez [5]). The authors still represented by the nodes N in the coauthorship network (N, H), every author is connected by the hyperlinks. The hyperlink in the coauthorship network means there exists at least one paper that collaborated by the related authors, and H represents the set of all the hyperlinks. In fact, this kind of coauthorship network is a hypergraph. In this paper, we use the second type of the coauthorship network, we also assume that every author in the coauthorship network at least have one paper, and one author connected by only one hyperlink is permitted.

2.2 Cooperative Game

A cooperative TU game is a pair (N, v) where $N = \{1, 2, \cdots, n\}$ is a player set and $v : 2^n \to \mathbb{R}$ with $v(\emptyset) = 0$ is a characteristic function. The subsets of N is called coalitions and $v(T)$, $T \subseteq N$ is called the worth of the coalition T. We denote the t for the cardinality of the coalition T.

A game (N, v) is additive game if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$. A game is convex if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. A convex game (N, v) always has a nonempty core solution, for a game (N, v) the core is

$$C(v) = \{x \in I(v) | x(S) \geq v(S)\},$$

where imputation set

$$I(v) = \{x \in \mathbb{R}^n | x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

A value is an operator φ that assigns payoff vectors to all games, $\varphi \in \mathbb{R}^N$.

A conference structure for cooperative TU game (N, v) is a hypergraph (N, H) where H is a system of non-singleton subsets of N, $H \subseteq H^N = \{h \subseteq N || h| > 1\}$. The elements h in H is called hyperlink or conference. We allow the hyperlinks in the hypergraph can connect single player in this paper. The set of hyperlinks that contain player i is denoted by $H_i = \{h \in H | i \in h\}$. A game with hypergraph communication situation i.e. conference structure is a triple (N, v, H). We denote $C(N, H)$ the set of connect components split by H.

The Shapley value for a TU game (N, v) is defined as follow

$$Sh_i(N, v) = \sum_{i \in S; S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \backslash \{i\})],$$

where n and s are the cardinalities of N and S respectively.

The Myerson value μ for a game with conference structure (N, v, H) is defined as follow

$$\mu(N, v, H) = Sh(N, v^H), \quad v^H(k) = \sum_{S \in C(k, H|_k)} v(S), \quad K \subseteq N,$$

where $H|_K = \{h \in H | h \subseteq K\}$ denote the restriction of H to $K \subseteq N$. The position value π is defined by

$$\pi_i(N, v, H) = \sum_{h \in H_i} \frac{1}{|h|} Sh_h(H, v^N),$$

where $v^N(F) = \sum_{S \in C(N,F)} v(S)$, $F \subseteq H$ denotes the hyperlink/link game.

3 The Weighted Coauthorship Networks and Related Cooperative Games

In this section, we define the weighted coauthorship networks and related cooperative games.

Using the hyperlink to connect the authors who have coauthored papers, then we can get a basic coauthorship network (N, H) for the specific research field. H is the collection of all the hyperlinks. The hyperlinks in the coauthorship network represent there exists papers that written by the related authors. In this paper, we think every author in the coauthorship network at least have one paper, and every single author also can have his or her own papers. So, there is no any author who has no related hyperlinks and there exists hyperlinks that connect only one author in our coauthorship networks.

Obviously, this simple coauthorship network (N, H) just describes the relationship of the authors's scientific collaboration, and it's not enough for us to construct the corresponding cooperative game models. When we want to connect the coauthorship network with the cooperative game we should acquire the characteristic function from the coauthorship network. Essentially, we should know how to measure the worth of every coalition.

The scientific research ability or academic influence of some author often embodied in the number of papers he has written, the total citation number of his papers and the real quality of his papers. So, we can integrate these factors into coauthorship network as the weights of the hyperlinks and acquire the related cooperative games. We will construct three kinds of weighted coauthorship networks and define the related cooperative games in the rest of this section.

3.1 The Number of the Papers

The number of the papers is a basic standard to show the scientific research ability of the authors. If we give every hyperlink h a positive weight $\alpha(h)$ that represents how many papers have been written by the authors h, then we have a weighted hypergraph or weighted network (N, H^α). The related cooperative game can be easily defined.

So, we can define a cooperative game with hypergraph communication structure (N, v, H^α). The player set N in the cooperative game are the authors in the coauthorship networks. Then the characteristic function is as follows,

$$v(S) = v_\alpha^H(S) = \sum_{h \in C(S)} \alpha(h) \text{ for all } S \in N, \tag{1}$$

where S represents the coalition and $C(S)$ is the collection of hyperlinks whose vertices are completely contained in coalition S, h is the hyperlink that contains in $C(S)$, $\alpha(h)$ is a positive integer that represents the number of the papers that coauthored by the authors h. Then $v_\alpha^H(S)$ is the total amount of the papers that written by authors in S. And we call the (N, v_α^H) the point game of (N, v, H^α). It's easy to verify that the game (N, v_α^H) is a convex game and we omit it.

3.2 The Citation of the Papers

The above simple model just considers the amount of the papers that written by the authors, and this is not enough to evaluate the worth of the coalitions. So we should take the quality of the papers into account. The number of citations not only can identify the research quality but also can reveal the development potential of scientific research directions. For example, we can collect the citation data of papers in recent years, higher citation in some research direction means the higher development potential.

So, if we give a positive integer $\beta(h)$ which represents the total citation of the papers written by authors h to every hyperlink h in the basic coauthorship network (N, H), then we can get a weighted hypergraph or weighted network (N, H^β) respect to the citation of the papers. We also have (N, v, H^β) with characteristic function given by

$$v(S) = v_\beta^H(S) = \sum_{h \in C(S)} \beta(h) \text{ for all } S \in N, \tag{2}$$

where $\beta(h)$ represents the total number of citation of the papers coauthored by the authors h. Then $v_\beta^H(S)$ is the total citation number of the papers written by authors in coalition S. The point game (N, v_β^H) of (N, v, H^β) is also a convex game.

3.3 The Ranking Factor of the Papers

In scientific research field, there are many ways to evaluate the papers. In China, Chinese Academy of Sciences JCR partition table and Journal Ranking of Thomson Reuters JCR are two widely accepted ways to rank the scientific research papers. Both ranking ways divide all the papers into four levels. We can give every level a positive coefficient according to one of these two evaluating ways. Here we call this coefficient the ranking factor. Every papers belongs to one of these levels and has a ranking factor. Naturally, we think the higher ranking correspond to bigger ranking factor. Then we can use the total ranking factor of the papers respect to some authors to evaluate the worth of them.

First, we construct a weighted coauthorship network (N, H^γ) where every hyperlink connect the authors who have some coauthored papers. There is a positive number $\gamma(h)$ for every hyperlink h represents the total ranking factor of the papers that coauthored by the related authors h. Then, we also can define the related cooperative game. The player set N still the authors in the coauthorship network. As mentioned before, we can use the total ranking factor of the papers written by coalition S represents the worth of it. So, we have the following characteristic function of cooperative game(N, v, H^γ)

$$v(S) = v_\gamma^H(S) = \sum_{h \in C(S)} \gamma(h) \text{ for all } S \in N, \tag{3}$$

where $\gamma(h) = w_1t_1(h) + w_2t_2(h) + w_3t_3(h) + w_4t_4(h)$, and w_i $(i = 1, 2, 3, 4)$ are the ranking factors respect to the four levels, $t_i(h)$ $(i = 1, 2, 3, 4)$ are the number of the ith level papers which coauthored by h. Then $v_\gamma^H(S)$ represents the total ranking factor of the coalition S. And we also call (N, v_γ^H) is the point game of the (N, v, H^γ).

Now, we have following theorem showing that the game (N, v_γ^H) is a convex game.

Theorem 1. *The game (N, v_γ^H) induced from weighted coauthorship network (N, H^γ) is a convex game.*

Proof. From the definition of the game (N, v_γ^H), we know that $v_\gamma^H(S)$ is a convex combination of the $t_i(h)$ $(i = 1, 2, 3, 4)$ and h is the hyperlink in coalition S. So, given any two sub-coalitions S and T in N, if $S \cap T = \varnothing$, then $v_\gamma^H(S \cup T) = v_\gamma^H(S) + v_\gamma^H(T)$, if $S \cap T \neq \varnothing$, then $v_\gamma^H(S \cup T) + v_\gamma^H(S \cap T) \geq v_\gamma^H(S) + v_\gamma^H(T)$ because there may exists a hyperlink whose vertices belongs to $S \backslash T$ and $T \backslash S$. So, we have $v_\gamma^H(S \cup T) + v_\gamma^H(S \cap T) \geq v_\gamma^H(S) + v_\gamma^H(T)$ for any sub-coalitions S and T in N, and it means that the game (N, v_γ^H) is convex game. □

For simplicity, we mainly consider game (N, v_γ^H) in the rest of this paper.

Now, we give an simple example to show the third weighted coauthorship network (N, H^γ) and related cooperative game (N, v_γ^H).

Example 1. *Suppose there are 5 authors in a coauthorship network (N, H), and their scientific collaboration relationship is showing as Fig. 1, where $N = \{A, B, C, D, E\}$, $H = \{a, b, c, d, e, f\}$.*

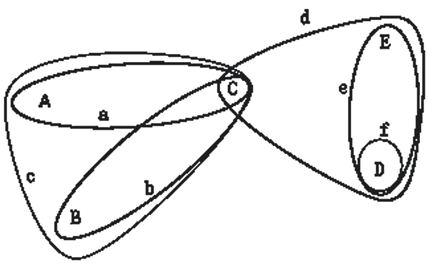

Fig. 1. The coauthorship network (N, H).

We assume that $w = \{1, 0.75, 0.5, 0.25\}$ is the ranking factor vector and $t_1(a) = t_1(b) = t_1(d) = t_4(f) = 1$, $t_3(b) = t_1(c) = t_2(e) = t_2(d) = 2$, $t_i(h) = 0$ otherwise. Then we can calculate the vector

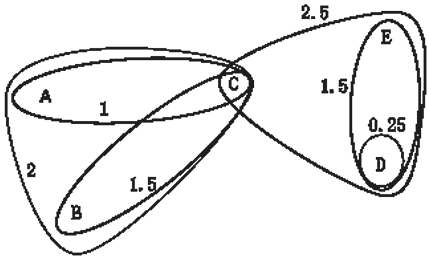

Fig. 2. The weighted coauthorship network (N, H^{γ}).

$$\gamma = \{\gamma(a), \gamma(b), \gamma(c), \gamma(d), \gamma(e)\gamma(f)\} = \{1, 1.5, 2, 2.5, 1.5, 0.25\}.$$

So we have the weighted coauthorship network (N, v^{γ}) *showing in Fig. 2. According to* (N, v^{γ}) *we can acquire the game* (N, v_{γ}^{H})*, for example we can get* $v_{\gamma}^{H}(\{A, C\}) = v_{\gamma}^{H}(\{A, C, E\}) = 1$*,* $v_{\gamma}^{H}(\{A, C, D\}) = 1.25$*, and* $v_{\gamma}^{H}(N) = 8.75$*.*

4 Author Ranking with the Induced Cooperative Games

In the former section, we obtained three different weighted coauthorship networks from different perspectives, and induced three cooperative games respect to these networks. Now, we try to rank the authors with the help of the feasible allocation schemes. We use these allocation schemes to divide the total number, citation, weight of papers written by all the authors to every author in the coauthorship network. Then we use the allocation results to evaluate the scientific research ability of every single author and rank them.

4.1 The Core and the Shapley Value for the Games

The core and the Shapley value have very important positions in the cooperative game theory.

Core. The core is a set value that every element in the core give every coalition at least it own worth, no coalition will not satisfy the allocation. So, the allocation in the core has stable property. For our induced game (N, v_{γ}^{H}) and any coalition $S \subseteq N$, the core $C(N, v_{\gamma}^{H})$ is

$$C(N, v_{\gamma}^{H}) = \{x \in I(N, v_{\gamma}^{H}) | x(S) \geq v_{\gamma}^{H}(S) \text{ for all } S \subseteq N\} \tag{4}$$

where $I(N, v_{\gamma}^{H}) = \{x \in \mathbb{R}^n | x(N) = v_{\gamma}^{H}(N) \text{ and } x_i \geq v_{\gamma}^{H}(\{i\}) \text{ for all } i \in N\}$ is the imputation set of game (N, v_{γ}^{H}).

In cooperative game theory, the convex game has a nonempty core. We have already shown that games (N, v_{α}^{H}), (N, v_{β}^{H}) and (N, v_{γ}^{H}) are convex games so they have nonempty cores. Although the core solutions have stable property and the games (N, v_α^H), (N, v_β^H) and (N, v_γ^H) have nonempty cores, the core is not a single value and we don't know which allocation is the most suitable one.

The Shapley Value. The Shapley value is a widely used single value in cooperative game theory, and it's a expectation of all the players' marginal contribution. For the game (N, v_γ^H) and a player $i \in N$, the Shapley value $Sh_i(N, v_\gamma^H)$ of i is

$$Sh_i(N, v_\gamma^H) = \sum_{i \in S; S \subseteq N} \frac{(s-1)!(n-s)!}{n!}[v_\gamma^H(S) - v_\gamma^H(S \backslash \{i\})]. \tag{5}$$

This value satisfies many good allocation properties, such as efficiency, symmetry, dummy player property and additivity. These four properties also construct a characterization of the Shapley value.

We have already construct three weighted coauthorship networks and induced three cooperative games (N, v_α^H), (N, v_β^H) and (N, v_γ^H), so we can calculate every players' Shapley value and use them to evaluate every player's scientific research ability. From the allocation results, we can calculate every author's weight in the coauthorship network. Then, we use these weights to rank the authors easily. Here, we give an example to show how to use the Shapley value to rank the authors.

Example 2. *Consider the coauthorship network (N, H^γ) and game (N, v_γ^H) in Example 1. We can get their Shapley value $Sh_A(N, v_\gamma^H) = \frac{7}{6}$, $Sh_B(N, v_\gamma^H) = \frac{17}{12}$, $Sh_C(N, v_\gamma^H) = \frac{33}{12}$, $Sh_D(N, v_\gamma^H) = \frac{11}{6}$, $Sh_E(N, v_\gamma^H) = \frac{19}{12}$.*

Assuming that there is a weight 1 for the whole coauthorship network, and we can acquire every author's weight $W_A \approx 0.133$, $W_B \approx 0.162$, $W_C \approx 0.314$, $W_D \approx 0.210$, $W_E \approx 0.181$ by their Shapley value. Then the rank of them can be easily find.

4.2 Contribution Level Based Allocation Rules

In the former subsection, the core and the Shapley value have been used for evaluate the author's scientific research ability. However these allocation rules are not enough to solve the allocation problem here. As we know that every coauthored paper has first author, second author etc., these authors have different contribution to the coauthored paper. When we consider the allocation rule we shouldn't ignore this issue otherwise we will overestimate some author's contribution or scientific research ability. Consider the following simple example.

Example 3. *Author A and author B coauthored a paper q^*, the contribution of themselves to the paper must be different so there are first author and second author among them. When we use the Shapley value shown in the former subsection as their allocation, they will get the same because the Shapley value views this situation as being symmetric. The pure Shapley value overestimates the second author and reduces the first author's payoff. Obviously, the pure Shapley value is not a fair allocation in this situation.*

The Weighted Shapley Value. In above example, suppose there exists a weight $\lambda = (\lambda(A), \lambda(B)) \in \mathbb{R}^2_{++}$ such that $\lambda(A) + \lambda(B) = 1$ to describe their contribution level to the coauthored paper. Then, a generalized version of the Shapley value–the weighted Shapley value using this kind of weight λ would be a better allocation for author A and author B.

The weighted Shapley value attempts to give a fair way of dividing up the worth of the grand coalition by assigning to each player a weighted average marginal contribution he makes to all possible coalitions, with weights vector and the size of the coalition. Actually, we can construct a weight system $\lambda \in \mathbb{R}^n_{++}$ for games that we have defined according to every papers' author order, the author order of the paper just reflect the contribution level. We can use this weight system as the weight of the weighted Shapley value.

So, how to capture the contribution level of every author respect to their papers? Ming et al. [6] use following weight to characterize every author's contribution level. They think every paper has weight 1, the first author and other authors have different weights to the coauthored paper, but the total weight of the authors respect to one coauthored paper must be 1.

Table 1. Every author's contribution level to the coauthored papers.

Author number	First	Second	Third	Forth	Fifth	Sixth
1	1					
2	0.6	0.4				
3	0.6	0.25	0.15			
4	0.6	0.2	0.1	0.1		
5	0.6	0.1	0.1	0.1	0.1	
6	0.6	0.1	0.1	0.1	0.05	0.05

From this table, we know that any author i have weight $\kappa(i, q)$ respect to paper $q \in Q_i$ according to which author he or she is, where Q_i represents the collection of the papers that related to the author i and Q represents the total number of the papers that have been written by all the authors. Following simple example show how to find the weights of the author to the related papers.

Example 4. *Considering the author A and author B in example 3, supposing that A is the first author to the paper q^*. Then we have $\kappa(A, q^*) = 0.6$ and $\kappa(B, q^*) = 0.4$ according to the table.*

We should calculate every author's weight λ_i respect to the grand coalition according to these weights. A natural idea is that summing up all of author i's weights as his or her weight λ_i^* respect to grand coalition N, i.e.

$$\lambda_i^* = \sum_{q \in Q_i} \kappa(i, q). \tag{6}$$

However, the sum of all the authors' weights is $|Q|$ not 1 , where $|Q|$ is the cardinality of the Q. Thus, it needs a normalization process, i.e.

$$\lambda_i = \frac{1}{|Q|}\lambda_i^* = \frac{1}{|Q|}\sum_{q\in Q_i}\kappa(i,q). \tag{7}$$

It's easy to verify that $\lambda \in \mathbb{R}^n_{++}$ and $\sum_{i\in N}\lambda_i = 1$. Now, we have the weight system λ calculated from author's contribution level to the related paper.

Next, we give the definition of the weighted Shapley value that have been introduced by Shapley [13]. Consider a cooperative game (N, v), let $\Sigma(N)$ be the set of all permutations on N. For any permutation $\sigma \in \Sigma(N)$, let $\sigma^i = \{j \in N|\sigma(j) < \sigma(i)\}$ is the set of the players preceding i in the permutation σ. Given $\sigma = (i_1, \cdots, i_n) \in \Sigma(N)$ and $\omega \in \mathbb{R}^n_{++}$, $P_\omega(\sigma) = \prod_{k=1}^n(\omega_{i_k}/\sum_{t=1}^k \omega_{i_t})$.

Definition 1. *For any game (N, v) and any player $i \in N$. Given $\omega \in \mathbb{R}^n_{++}$ such that $\sum_{i\in N}\omega_i = 1$, the weighted Shapley value with weights ω, Sh^ω is defined by*

$$Sh_i^\omega(N, v) = \sum_{\sigma\in\Sigma(N)} P_\omega(\sigma)[v(\sigma^i \cup \{i\}) - v(\sigma^i)] \tag{8}$$

From the definition, we know that the weighted Shapley value assigns to each player a weighted average of the marginal contributions he or she makes to all possible coalitions. We have got the weight system λ, so we can use the weighted Shapley value $Sh_i^\lambda(N, v_\gamma^H)$ of game (N, v_γ^H) to evaluate the author's scientific research ability. And we can use the allocation results to rank the authors.

Example 5. *Consider there are three authors $N = \{A, B, C\}$, They coauthored one paper q_{ABC} whose ranking factor is 0.75 and A is the first author and B is the second author. A and B coauthored one paper q_{AB}, the ranking factors is 1 and first author is B. A and C have one coauthored paper whose ranking factor is 0.5 and C is the first author. So we have weighted coauthorship network (N, H, γ) respect to ranking factor shown in Fig. 3.*

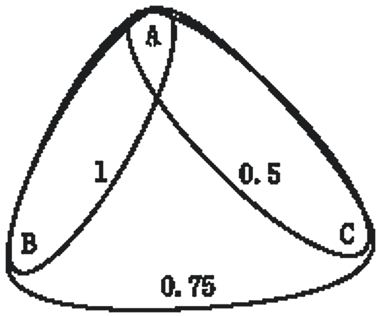

Fig. 3. The coauthorship network (N, H^γ) about A, B, C.

Corresponding cooperative game (N, v_γ^H) *is* $v_\gamma^H(\{A,B\}) = 1$, $v_\gamma^H(\{A,C\}) = 0.5$, $v_\gamma^H(N) = 2.25$, $v_\gamma^H(S) = 0$ *otherwise. And we can get* $\lambda = (\frac{7}{15}, \frac{17}{60}, \frac{1}{4})$. *Finally, we have* $Sh_A^\lambda(N, v_\gamma^H) = 1.299$, $Sh_B^\lambda(N, v_\gamma^H) = 0.589$, $Sh_C^\lambda(N, v_\gamma^H) = 0.362$. *In order to rank the authors, we also can get every author's weight to the whole coauthorship network* $W_A \approx 0.577$, $W_B \approx 0.262$, $W_C \approx 0.161$ *by the weighted Shapley value.*

The Position Value with Contribution Level. From table 1, we know how to measure every authors' contribution level to the related papers, so if we can give an allocation rule that allocates the grand coalition's worth $v_\alpha^H(N)$, $v_\beta^H(N)$, $v_\gamma^H(N)$ to every hyperlink, then we can allocate every hyperlink's payoff to the related authors proportionally according to the contribution levels to the hyperlinks.

This idea is a bit similar to the position value. The position value for games with conference structure first allocates every hyperlink the Shapley value respect to hyperlink game, and then divides these hyperlinks' payoff to their related players equally. The total payoff of the players in the game is the sum of the payoffs that they acquire from the hyperlinks he or she belongs to.

So, we can adopt this idea to our allocation problem. First, we should obtain hyperlink games from (N, v, H^α), (N, v, H^β), (N, v, H^γ).

Definition 2. *For game* (N, v, H^γ) *consider hyperlink game* (H, v_γ^N) *where*

$$\begin{aligned} v_\gamma^N(F) &= \sum_{S \subset A(N,F)} v_\gamma^H(S) \\ &= \sum_{S \subset A(N,F)} \sum_{h \in C(S)} \gamma(h) \\ &= \sum_{h \in F} \gamma(h), \quad \textit{for all } F \subseteq H. \end{aligned} \tag{9}$$

where $A(N, F)$ *represents the set of authors or players that constitute the hyperlinks in* $F \subseteq H$. *It's easy to prove that* $v_\gamma^N(H) = v_\gamma^H(N)$.

Then we can use the Shapley value as the allocation for game (H, v_γ^H) and divide all the hyperlinks' payoff to their related authors according to the contribution levels as to hyperlinks. So we have following allocation rule:

$$P_i(N, v_\gamma^H) = \sum_{h \in H_i} \kappa(i,h) Sh_h(H, v_\gamma^N). \tag{10}$$

H_i is the collection of the hyperlinks that related to author i and $\kappa(i,h)$ represents the contribution level of author i correspond to hyperlink h. We use i's average contribution level to his or her related papers which have same authors as connected by hyperlink h as his or her contribution level $\kappa(i,h)$ to h. Formally, $\kappa(i,h)$ is defined as follow

Definition 3. *For any author i, his contribution level to hyperlink $h \in H_i$ is*

$$\kappa(i, h) = \frac{1}{|Q_h|} \sum_{q \in Q_h} \kappa(i, q),$$

where Q_h represents the papers that written by authors h.

This allocation rule first allocates the worth of grand coalition to every hyperlink by the Shapley value of hyperlink game (H, v_γ^N). Then, every hyperlink divides their payoff to their related authors proportionally by authors' contribution level κ, and this is the different with position value whose allocation in second step is equally divided.

However, one can easily verify that the induced hyperlink game (H, v_γ^H) is an additive game. From the Shapley value of additive game we have

$$\begin{aligned} P_i(N, v_\gamma^H) &= \sum_{h \in H_i} \kappa(i, h) Sh_h(H, v_\gamma^N) \\ &= \sum_{h \in H_i} \kappa(i, h) v_\gamma^N(h) \\ &= \sum_{h \in H_i} \kappa(i, h) \gamma(h). \end{aligned} \tag{11}$$

Actually, this allocation rule is a proportion rule as to authors' participation or contribution level although we adopt the idea of position value.

Example 6. *Consider the situation in example 5. First, we can get the contribution levels $\kappa(A, h_{AB}) = \kappa(A, h_{AC}) = 0.4$, $\kappa(A, h_{ABC}) = \kappa(B, h_{AB}) = \kappa(C, h_{AC}) = 0.6$, $\kappa(B, h_{ABC}) = 0.25$, $\kappa(C, h_{ABC}) = 0.15$. So we have the position value with contribution levels $P_A(N, v_\gamma^H) = 1.05$, $P_B(N, v_\gamma^H) = 0.7875$, $P_C(N, v_\gamma^H) = 0.1125$. We also can get every author's weight to the whole coauthorship network easily and we omit it.*

5 Conclusion

In this paper, we use cooperative game to rank the authors in coauthorship networks. At first, we construct three weighted coauthorship networks from different aspects. We consider the amount of authors' papers, the citation number and the weight of the papers. And we also give three cooperative games though the weighted coauthorship networks. Then we study the core and the Shapley value of the games. Finally, from respective of author's contribution level we define the weight system and study the weighted Shapley value, and we also give a new value which is similar to the position value. These allocation results can be used for ranking the authors in the coauthorship network.

As mentioned before, cooperative game theory is an old and still hot research field and coauthorship network is a interesting topic that has been studying by complex network theory. This is the first paper that applies pure cooperative game theory to the coauthorship network. Although we have constructed three kinds of coauthorship networks, defined the related games and given the allocation rules to rank the authors, our models in this paper still a little simple. So, one can construct more complex models in the future work.

The coauthorship networks are the complex network and there are many authors in them. We use core, Shapley value and weighted Shapley value as the allocation rules from the angle of cooperative game but we ignore the computation complexity. Actually, it's a big problem for application when there exists thousands of authors in the network.

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A Reduced Harsanyi Power Solution for Cooperative Games with a Weight Vector

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Abstract. The Harsanyi power solution for cooperative games allocates dividends generated by coalitions proportionally to each player's power index. Normally, cooperative games tacitly treat all players symmetric. However, the fact is that different players may be asymmetric and contribute to different efforts, bargaining powers, or stability in the process of cooperation. A weight vector is used to reflect players' asymmetry. In view of these weights are possible to be less than 1, that is, not all players are absolutely important, a loss of dividends of coalitions can happen. We define and characterize a reduced Harsanyi power solution for cooperative games with a weight vector, which is relevant to a loss function of dividends. Moreover, when the loss function takes particular forms, the reduced Harsanyi power solution has a linear relationship with the Harsanyi power solution.

Keywords: Harsanyi dividend · Loss function · Reduced Harsanyi power solution

1 Introduction

A cooperative game over a finite set of players is a function assigning to any coalition a profit achieved by cooperation. The main research for cooperative games is to discuss how to allocate these profits among all players. The Shapley value is a famous solution concept, in which the dividend of every coalition is divided equally. One axiom that axiomatizes Shapley value is symmetry. The underlying assumption of this axiom is that all players are symmetric. However, there are many realistic situations where the assumption seems unrealistic. For example, in a cooperative game with two players, player one may need more efforts than player two in order to accomplish a project. Player one may be comparatively more pleased to promote the cooperation. Also, the two players may possess different bargaining powers.

Shapley [5] introduced the weighted Shapley value. Each player is attached to a positive weight and shares the dividend of coalitions proportional to its weight. In the Shapley value, all the weights are equal. Aubin [1] considered the asymmetry about the participation levels of players and introduced the fuzzy

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cooperative game. The Shapley value for fuzzy cooperative games have been studied by Butnariu [2], Tsurumi et al. [7], Meng and Zhang [3]. The study of core for fuzzy games see [6,9].

Vasil'ev [8] proposed the Harsanyi solution for cooperative games. The Harsanyi solution divides the Harsanyi dividends over players in the corresponding coalition according to power indexes of players. However, for cooperative games with a weight vector, the gap between weights and 1 can lead to a part of loss of Harsanyi dividends. As a matter of fact, one player occupies absolutely important status only when the weight of this player is 1. Based on the background, in this paper, we propose the concept of loss function, and characterize a newly defined reduced Harsanyi power solution. If the loss function takes particular forms, the corresponding reduced Harsanyi power solution for cooperative games with a weight vector is a linear combination of Harsanyi power solution of several subgames with respect to the weight vector.

This paper is organized as follows. In Sect. 2, we get some preliminary knowledge served for the latter contexts. In Sect. 3, we propose a loss function and define a solution concept of reduced Harsanyi power solution for cooperative games with a weight vector. Also, the axiomatization of this new solution is given. In Sect. 4, we list several particular kinds of loss functions and illustrate the relationship between the corresponding reduced Harsanyi power solutions and the crisp Harsanyi power solutions. Finally, some conclusions are given in Sect. 5.

2 Preliminaries

2.1 Cooperative Games

A *cooperative game* can be described by a pair (N, v) where $N = \{1, 2, \ldots, n\}$ denotes the set of players and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ the corresponding characteristic function. $|S|$ denotes the number of members in coalition S. $S \subseteq N$ is called a *coalition* and $v(S)$ the *worth* of S. (T, v) for any $T \subseteq N$ is called a *subgame* of $(N, v) \in G^N$. If there is no ambiguity, we identify the game (N, v) with its characteristic function v. The set of all cooperative games over N is denoted by G^N.

The *solution concept* in a cooperative game defines how to allocate the profit $v(N)$ of grand coalition N among the players. Therefore, a solution is a vector $x \in \mathbb{R}^n$ and satisfies that $\sum\limits_{i \in N} x_i = v(N)$. As the important solution concepts for cooperative games, the *Shapley value* of a game v is defined for all player $i \in N$ as

$$Sh_i(v) = \sum_{S \subseteq N; i \in S} \frac{(|N| - |S|)!(|S| - 1)!}{|N|!} \cdot [v(S) - v(S \backslash \{i\})]$$

Denote the Shapley value of subgame (T, v) by $Sh(v)(T)$. For each $T \subseteq N$, the *unanimity game* u_T is defined to be

$u_T(S) = \begin{cases} 1; & \text{if } S \subseteq T, \\ 0; & \text{otherwise.} \end{cases}$ It is well known that the unanimity games $u_T, \emptyset \neq T \subseteq N$ construct a basis for G^N, and then any game $v \in G^N$ be can written as the following form, i.e.,

$$v = \sum_{\emptyset \neq T \subseteq N} \Delta_v(T) u_T,$$

where the coefficients $\Delta_v(T)$ are the Harsanyi dividends generated by players in coalition T. Note that

$$Sh_i(v) = \sum_{i \in T \subseteq N} \Delta_v(T) \frac{1}{|T|},$$

which implies that the Shapley value views all players symmetric and the dividend of every coalition is equally allocated among the players in this coalition. However, under the most situations, the bargaining power of each player is not the same. Shapley [5] assigned a positive weight to each player as a measure of its individual ability and introduced the weighted Shapley value. In the weighted Shapley value, the dividend is shared proportional to each player's weight. Given a positive weight vector $w = (w_1, w_2, \ldots, w_n)$, $w_i \in (0, 1]$ for any $i \in N$, the weighted Shapley value with weights w, Sh^w is defined as

$$Sh_i^w(v) = \sum_{i \in T \subseteq N} \Delta_v(T) \frac{w_i}{\sum_{j \in T} w_j}$$

for any $v \in G^N$, $i \in N$. Afterwards, Vasil'ev [8] propose the Harsanyi solution. A sharing system over N is $p = \big((p^S)_{S \subseteq N}\big)$, where p^S is an $|S|$ dimensional vector and $\sum_{i \in S} p_i^S = 1$. The Harsanyi payoff for a cooperative game $v \in G^N$ with a sharing system p is $H_i^p(v) = \sum_{\emptyset \neq T \subseteq N} \Delta_v(T) p_i^T$ for any $i \in N$. Given a power measure, which is a function σ that assigns to any $S \subseteq N$ a power vector $\sigma(S)$ and $\sigma_i(S)$ reflects the positive power of player i in S. Denote $p_i^S = \frac{\sigma_i(S)}{\sum_{i \in S} \sigma_i(S)}$, the corresponding Harsanyi power solution H^σ is defined to be

$$H_i^\sigma(v) = \sum_{\emptyset \neq T \subseteq N} \Delta_v(T) \frac{\sigma_i(T)}{\sum_{i \in T} \sigma_i(T)}$$

for any $i \in N$.

2.2 Fuzzy Cooperative Games

There are some situations where players do not fully participate in a coalition, but take action according to the level of participation. Aubin [1] studied the problem at first by the proposal of fuzzy coalition.

Let N be a finite set of players. We call $U = (U_1, U_2, \ldots, U_n), U_i \in [0, 1]$ for each $i \in N$, a *fuzzy coalition*. Here U_i denotes the participation level of player i in the coalition U. The set of all fuzzy coalitions is denoted by $\mathcal{F}^N$. We define the *carrier* of fuzzy coalition U by $U^{cr} = \{i \in N | U_i \neq 0\}$ and denote $Q(U) = \{U_i | U_i > 0, i \in N\}$. For $s, t \in \mathcal{F}^N$, we use the notation $t \leq s$ iff $t_i \leq s_i$ for each $i \in N$. Each fuzzy coalition e^S with $e_i^S = 1$ if $i \in S$ and otherwise $e_i^S = 0$ corresponds to the situation where the players within S fully cooperate. We write e^i instead of $e^{\{i\}}$.

A *fuzzy cooperative game* is a function $v^f : \mathcal{F}^N \to \mathbb{R}_+^n$ such that $v^f(e^{\emptyset}) = 0$. $v^f(e^S) = v(S)$ for each $S \subseteq N$. The set of fuzzy cooperative games is denoted by FG^N.

In the following context, we fix $U \in \mathcal{F}^N$, with $|Q(U)| = m$. We write the non-zero elements in $Q(U)$ in a increasing order $h_1 < h_2 < \cdots < h_m$ and $h_0 = 0$. In 1980, Butnariu [2] firstly introduced a limited subclass of fuzzy cooperative games as follows.

Definition 1. A fuzzy cooperative game v^f is said to be *with proportional form* if and only if

$$v^f(U) = \sum_{k=1}^{m} h_k \cdot v([U]^{h_k})$$

for each $U \in \mathcal{F}^N$, here $[U]^{h_k}$ is the set of players in fuzzy coalition U with the same participation level h_k, i.e. $[U]^{h_k} = \{i \in N | U_i = h_k\}$ for each $k \in \{1, 2, \ldots, m\}$.

Denote the set of fuzzy cooperative games with proportional forms by ${FG_p}^N$. Butnariu gave the explicit form of Shapley value with fuzzy coalition U over ${FG_p}^N$

$$\phi^{pro}(v^f)(U) = \sum_{k=1}^{m} h_k \cdot \phi(v)([U]^{h_k}).$$

Following Butnariu's approach, in 2001 Tsurumi et al. [7] proposed a new class of fuzzy cooperative games.

Definition 2. A fuzzy cooperative game v^f is said to be *with Choquet integral form* if and only if

$$v^f(U) = \sum_{k=1}^{m} [h_k - h_{k-1}] \cdot v([U]_{h_k})$$

for each $U \in \mathcal{F}^N$, where $[U]_{h_k}$ is the set of players in fuzzy coalition U with participation level not smaller than h_k, i.e. $[U]_{h_k} = \{i \in N | U_i \geq h_k\}$ for each $k \in \{1, 2, \ldots, m\}$.

Denote the set of fuzzy cooperative games with Choquet integral forms by ${FG_c}^N$. Tsurumi et al. also gave the expression of Shapley value with fuzzy coalition U over FG_c^N

$$\phi^{ch}(v^f)(U) = \sum_{k=1}^{m} [h_k - h_{k-1}] \cdot \phi(v)([U]_{h_k}).$$

Owen [4] introduced the fuzzy cooperative games with multilinear extension form.

Definition 3. A fuzzy cooperative game v^f is said to be with *multilinear extension form* if and only if

$$v^f(U) = \sum_{T \subseteq U^{cr}} \prod_{i \in T} U_i \prod_{i \in U^{cr} \setminus T} (1 - U_i) \cdot v(T)$$

for any $U \in \mathcal{F}^N$.

Denote the set of fuzzy cooperative games with multilinear extension forms by ${FG_m}^N$. The Shapley value with fuzzy coalition U over FG_m^N can be written as

$$\phi^{mul}(v^f)(U) = \sum_{T \subseteq N} \prod_{i \in T} U_i \prod_{i \in N \setminus T} (1 - U_i) Sh(v)(T).$$

3 Reduced Harsanyi Power Solution

The previous proposed Harsanyi solution implies that the dividend $\Delta_v(S)$ of any coalition S can be reached. It seems unrealistic to give this ideal assumption that all players are equally important and contribute their entire efforts to promote the cooperation. It is more natural to suppose that each player makes different contributions in the generation of Harsanyi dividends, from the effort levels, reliability to bargaining powers. Based on this consideration, in this section we use a weight vector to represent different contributions of players and propose a new solution.

Definition 4. *A reduced Harsanyi power solution* for any cooperative game $v \in G^N$ with a weight vector w and power measure σ is defined to be

$$H_i^{(w,\sigma)}(v) = \sum_{i \in T \subseteq N} \Delta_v(T) h^T(w) \frac{\sigma_i^w(T)}{\sum\limits_{i \in T} \sigma_i^w(T)},$$

here $h^T(w) \in [0, 1]$ is a loss function of $\Delta_v(T)$ with respect to w, indicating the loss of dividend $\Delta_v(T)$ caused by the weights less than 1, which is why we say this solution "reduced". We suppose that $h^T(w)$ and $\sigma_i^w(T)$ only rely on the weights of players in T.

Remark 1. Because some players in coalition T only elaborate a part of efforts and do not possess enough bargaining powers, or have a certain possibility to split off this coalition, a loss of Harsanyi dividends will happen in view of different coalition formation possibilities.

In an ideal, $h^T(w) = 1$ for any $T \subseteq N$, i.e., any coalition $T \subseteq N$ can form and achieve the dividend $\Delta_v(T)$, the reduced Harsanyi dividend solution is called the *crisp Harsanyi power solution*, denoted by $H^{cr(w,\sigma)}(v)$. The crisp Harsanyi power solution is exactly the weighted Shapley value [5] Sh^w if $\sigma_i^w(S) = w_i$ for any $S \subseteq N$. Therefore, the weighted Shapley value is a specific case of reduced Harsanyi power solution.

In the following, several axioms for a solution ψ over $v \in G^N$ with the weight vector w and power measure σ are presented. We say a cooperative game $v \in G^N$ with the weight vector w and power measure σ is *player-anonymous* if $v(S) = 0$ for any $S \subseteq N$ and $v(N) \neq 0$.

Axiom 1. Power measure property If $v \in G^N$ with w and σ^w is player-anonymous, there exists a constant α such that $\sum\limits_{i \in N} \psi_i^{(w,\sigma)}(N, v) = \alpha \sigma_i^w(N)$.

This axiom shows that for a player-anonymous cooperative game with a weight vector and a power measure, each player $i \in N$ obtains a constant times of its power $\sigma_i^w(N)$ in grand coalition N.

Axiom 2. Additivity For all $u, v \in G^N$, $\psi^{(w,\sigma)}(u+v) = \psi^{(w,\sigma)}(u) + \psi^{(w,\sigma)}(v)$

Axiom 3. Loss function efficiency If $v \in G^N$ with w and and σ^w is player-anonymous, $\sum\limits_{i \in N} \psi_i^{(w,\sigma)}(v) = v(N) h^N(w)$.

Loss function efficiency says that for a player-anonymous game, the payoff of all players is equal to the reduced $v(N)$ after losing part of Harsanyi dividend of grand coalition N.

Axiom 4. Inessential player property Let T be a non-empty coalition. If $v = cu_T$, $\psi_i^{(w,\sigma)}(N, v) = \psi_i^{(w_T,\sigma)}(T, w)$ for all $i \in T$ and 0 otherwise.

Inessential player property states that when v is a unanimity game on a non-empty coalition T, the final payoffs of players in T do not depend on the players outside T.

Lemma 3.1. The reduced Harsanyi power solution for any cooperative game $v \in G^N$ with a weight vector w and power measure σ satisfies **Axioms** 1–4

Proof. From Definition 4, we know that $H_i^{(w,\sigma)}(v) = \sum\limits_{i \in T \subseteq N} \Delta_v(T) h^T(w) \frac{\sigma_i^w(T)}{\sum\limits_{i \in T} \sigma_i^w(T)}$. When $v = u_T$, T is a non-empty coalition,

$$H_i^{(w,\sigma)}(u_T) = \begin{cases} h^T(w) \frac{\sigma_i^w(T)}{\sum\limits_{i \in T} \sigma_i^w(T)}; & \text{if } i \in T, \\ 0; & \text{otherwise.} \end{cases}$$

Now we begin to successively verify that the reduced Harsanyi power solution satisfies these four axioms.

Axiom 1. If $v \in G^N$ with w and σ is player-anonymous,

$$H_i^{(w,\sigma)}(v) = \Delta_v(N)h^N(w)\frac{\sigma_i^w(N)}{\sum\limits_{i\in N}\sigma_i^w(N)}.$$

Take $\alpha = \frac{\Delta_v(N)h^N(w)}{\sum\limits_{i\in N}\sigma_i^w(N)}$, we obtain that $H_i^{(w,\sigma)}(v) = \alpha\sigma_i^w(N)$

Axiom 2. Because $(u+v)(S) = u(S) + v(S)$ for any $u, v \in G^N$, $S \subseteq N$, it is obvious that

$$\Delta_{u+v}(S) = \Delta_u(S) + \Delta_v(S)$$

for all $S \subseteq N$. Thus,

$$\begin{aligned} H_i^{(w,\sigma)}(u+v) &= \sum_{i\in T\subseteq N}\Delta_{(u+v)}(T)h^T(w)\frac{\sigma_i^w(T)}{\sum\limits_{i\in T}\sigma_i^w(T)} \\ &= \sum_{i\in T\subseteq N}\Delta_u(T)h^T(w)\frac{\sigma_i^w(T)}{\sum\limits_{i\in T}\sigma_i^w(T)} + \sum_{i\in T\subseteq N}\Delta_v(T)h^T(w)\frac{\sigma_i^w(T)}{\sum\limits_{i\in T}\sigma_i^w(T)} \\ &= H_i^{(w,\sigma)}(u) + H_i^{(w,\sigma)}(v). \end{aligned}$$

Axiom 3. Because

$$H_i^{(w,\sigma)}(v) = \Delta_v(N)h^N(w)\frac{\sigma_i^w(N)}{\sum\limits_{i\in N}\sigma_i^w(N)}$$

if $v \in G^N$ with w and σ is player-anonymous, we get that $\sum\limits_{i\in N} H_i^{(w,\sigma)}(v) = \Delta_v(N)h^N(w) = v(N)h^N(w)$.

Axiom 4. When $v = u_T$, $\emptyset \neq T \subseteq N$, obviously $H_i^{(w,\sigma)}(N, u_T) = 0$ for any $i \notin T$ and $H_i^{(w,\sigma)}(N, u_T) = h^T(w)\frac{\sigma_i^w(T)}{\sum\limits_{i\in T}\sigma_i^w(T)}$, $H_i^{(w_T,\sigma)}(T, u_T) = h^T(w_T)\frac{\sigma_i^{w_T}(T)}{\sum\limits_{i\in T}\sigma_i^{w_T}(T)}$ for all $i \in T$. For the reason that $h^T(w_T) = h^T(w)$ and $\sigma_i^w(T) = \sigma_i^{w_T}(T)$, we have $H_i^{(w,\sigma)}(N, u_T) = H_i^{(w_T,\sigma)}(T, u_T)$ for any $i \in T$. □

Lemma 3.2. If any solution satisfies Axioms 1–4, it is the reduced Harsanyi power solution.

Proof. Let ψ be a solution satisfying all four axioms. In view of when $v = u_T$, any subgame (T, v) of game $v \in G^N$ is player-anonymous about a weight vector w_T and power measure σ, we get that $\psi_i^{(w_T,\sigma)}(u_T) = \alpha\sigma_i^{w_T}(T) = \alpha\sigma_i^w(T)$ for any $i \in T$ according to the power measure property. We also have that

$\sum_{i\in T} \psi_i^{(w_T,\sigma)}(u_T) = u_T(T)h^T(w) = h^T(w)$ from the loss function efficiency. Therefore, $\psi_i^{(w_T,\sigma)}(u_T) = h^T(w)\frac{\sigma_i^w(T)}{\sum_{i\in T}\sigma_i^w(T)}$ for any $i \in T$. It follows from the inessential player property that $\psi_i^{(w,\sigma)}(N, u_T) = \psi_i^{(w_T,\sigma^{w_T})}(T, u_T) = h^T(w)\frac{\sigma_i^w(T)}{\sum_{i\in T}\sigma_i^w(T)}$ if $i \in T$ and otherwise $\psi_i^{(w,\sigma)}(N, u_T) = 0$. Using the additivity, we obtain that $\psi_i^{(w,\sigma)}(N, v) = \sum_{i\in T} \psi_i^{(w,\sigma)}(N, u_T) = \sum_{i\in T} h^T(w)\frac{\sigma_i^w(T)}{\sum_{i\in T}\sigma_i^w(T)} = H_i^{(w,\sigma)}(N, v)$ for all $i \in N$.

We complete this proof. □

Theorem 3.3. Reduced Harsanyi power solution is a unique solution for any cooperative game with a weight vector and power measure satisfying **Axioms** 1–4.

Proof. This result can be easily obtained by Lemmas 3.1 and 3.2.

4 Three Particular Loss Functions

In the previous context, we have referred that the weights of players can lead to a loss of Harsanyi dividends to some extent. The weights can have many explanations, such as the player's effort level, reliability and so on. The loss function captures different information if weights have different meanings and thus has different forms. In this section, we focus on the study of deterministic and concrete loss functions. Firstly, we list several particular expressions of loss functions $h^S(w)$ for any $S \subseteq N$.

1. **Minimal weight loss function**: $h_m^S(w) = min\{w_i | i \in S\}$.
2. **Equal weight loss function**: $h_e^S(w) = \begin{cases} w_i; & \text{if } w_i = w_j \text{ for all } i, j \in S, \\ 0; & \text{otherwise.} \end{cases}$
3. **Product weight loss function**: $h_p^S(w) = \prod_{i\in S} w_i$.

Suppose w_i is the effort level player i contributes in a cooperation. Less w_i reflects that player i makes less effort. The minimal weight loss function takes the minimal weight of players in coalition S as an evaluation of strength of S. If a coalition is less strong, i.e., players in it have lower participation levels, the players in it is not so pleased to form this coalition and naturally S can not achieve the ideal Harsanyi dividend $\Delta_v(S)$. When all players in S work together to form this coalition and spend their entire effort on the project, the whole $\Delta_v(S)$ can be obtained, i.e., $h_m^S(w) = 1$.

Because weights of all players are not the same, under the rational considerations, players with higher abilities are not willing to cooperate with players with lower abilities. That is, not all coalitions can form and only those players who have the same weights are possible to gather together. The equal weight loss function considers this fact and takes the same weight of players in a formed coalition as a measurement of its strength. In a stronger coalition, players have stronger abilities. Moreover, not enough strong ability $w_i \le 1$ $(i \in S)$ will certainly influence the obtainment of $\Delta_v(S)$. Specifically, if $w_i = 1$ for all $i \in S$, $h_e^S(w) = 1$.

When w_i is treated as its reliability to support the formation of coalitions including i, the product of all w_i $(i \in S)$ can be regarded as the cohesion level of coalition S. A coalition with a higher cohesion is relatively more stable and thus players in it do not want to break away from this coalition. Product weight loss function embodies the stability of coalitions in the generation of Harsanyi dividends. A more stable coalition achieves larger part of its Harsanyi dividend. When $w_i = 1$ for all $i \in S$, $h_e^S(w) = 1$.

Proposition 4.1. Denote $Q(w) = \{w_i | i \in N\} = \{s_1, s_2, \ldots, s_q\}$, $s_1 < s_2 < \cdots s_q$ and $s_0 = 0$. The following statements hold:

(1) When $h^S(w) = h_m^S(w)$, $H_i^{(w,\sigma)}(v) = \sum_{k=1}^{q} (s_k - s_{k-1}) H_i^{cr(w,\sigma)}(v)([w]_{s_k})$, where $[w]_{s_k} = \{i | w_i \ge s_k\}$, $H_i^{(w,\sigma)}(v)([w]_{s_k})$ is the reduced Harsanyi power solution of subgame $([w]_{s_k}, v)$;

(2) When $h^S(w) = h_e^S(w)$, $H_i^{(w,\sigma)}(v) = \sum_{k=1}^{q} s_k H_i^{cr(w,\sigma)}(v)([w]^{s_k})$, where $[w]^{s_k} = \{i | w_i = s_k\}$, $H_i^{(w,\sigma)}(v)([w]^{s_k})$ is the reduced Harsanyi power solution of subgame $([w]^{s_k}, v)$;

(3) When $h^S(w) = h_p^S(w)$, $H_i^{(w,\sigma)}(v) = \sum_{T \subseteq N} \prod_{i \in T} w_j \prod_{i \in N \setminus T} (1 - w_i)$ $H_i^{cr(w,\sigma)}(v)(T)$.

Proof. (1) When $h^S(w) = h_m^S(w)$, we get that

$$\begin{aligned}
&\sum_{k=1}^{q} (s_k - s_{k-1}) H_i^{cr(w,\sigma)}(v)([w]_{s_k}) \\
&= \sum_{k=1}^{q} (s_k - s_{k-1}) \sum_{i \in S \subseteq [w]_{s_k}} \Delta_v(S) \frac{\sigma_i^w(S)}{\sum_{i \in S} \sigma_i^w(S)} \\
&= \sum_{i \in S \subseteq N} \Delta_v(S) \cdot min\{w_i | i \in S\} \frac{\sigma_i^w(S)}{\sum_{i \in S} \sigma_i^w(S)} = H_i^{(w,\sigma)}(v).
\end{aligned}$$

(2) When $h^S(w) = h_e^S(w)$, we have

$$\begin{aligned}
&\sum_{k=1}^{q} s_k H_i^{cr(w,\sigma)}(v)([w]^{s_k})\\
=&\sum_{k=1}^{q} s_k \cdot \sum_{i\in S\subseteq [w]^{s_k}} \Delta_v(S) \cdot \frac{\sigma_i^w(S)}{\sum\limits_{i\in S}\sigma_i^w(S)}\\
=&\sum_{i\in S\subseteq N} \Delta_v(S)\cdot h_e^S(w)\frac{\sigma_i^w(S)}{\sum\limits_{i\in S}\sigma_i^w(S)}\\
=&H_i^{(w,\sigma)}(v).
\end{aligned}$$

(3) When $h^S(w) = h_p^S(w)$, note that

$$\begin{aligned}
H_i^{(w,\sigma)}(v) &= \sum_{i\in S\subseteq N}\prod_{j\in S} w_j H_i^{cr(w,\sigma)}(v)(S)\\
&= \sum_{i\in S\subseteq N}\prod_{j\in S} w_j \sum_{i\in T\subseteq S}\Delta_v(T)\frac{\sigma_i^w(T)}{\sum\limits_{i\in T}\sigma_i^w(T)}\\
&= \sum_{i\in S\subseteq N}\sum_{T\supseteq S}\prod_{i\in T} w_j \prod_{i\in N\setminus T}(1-w_i)\Delta_v(T)\frac{\sigma_i^w(T)}{\sum\limits_{i\in T}\sigma_i^w(T)}\\
&= \sum_{i\in S\subseteq N}\sum_{T\supseteq S}\Delta_v(S)\prod_{i\in T} w_j \prod_{i\in N\setminus T}(1-w_i)\frac{\sigma_i^w(T)}{\sum\limits_{i\in T}\sigma_i^w(T)}\\
&= \sum_{T\subseteq N}\sum_{i\in S\subseteq T}\Delta_v(S)\prod_{i\in T} w_j \prod_{i\in N\setminus T}(1-w_i)\frac{\sigma_i^w(T)}{\sum\limits_{i\in T}\sigma_i^w(T)}\\
&= \sum_{T\subseteq N}\prod_{i\in T} w_j \prod_{i\in N\setminus T}(1-w_i)\sum_{i\in S\subseteq T}\Delta_v(S)\frac{\sigma_i^w(T)}{\sum\limits_{i\in T}\sigma_i^w(T)}\\
&= \sum_{T\subseteq N}\prod_{i\in T} w_j \prod_{i\in N\setminus T}(1-w_i) H_i^{cr(w,\sigma)}(v)(T).
\end{aligned}$$

This proposition is verified true. □

Proposition 4.2. Let w_i be the participation level of player i, when $\sigma_i^w(S) = 1$ for any $i \in S$ and $w = U$, then $H^{(w,\sigma)}(v) = \begin{cases} Sh^{ch}(v^f)(U); & \text{if } h^S(w) = h_m^S(w),\\ Sh^{pro}(v^f)(U); & \text{if } h^S(w) = h_e^S(w),\\ Sh^{mul}(v^f)(U); & \text{if } h^S(w) = h_p^S(w).\end{cases}$

This proposition says that the reduced Harsanyi solution is also an extension of Shapley value for fuzzy cooperative games and we omit the proof.

5 Conclusion

Under most situations in our life, each player involved in a cooperation is not equally important. For example, one of players may contribute more effort or have a stronger bargaining ability. Thus, to assume that players are symmetric seems unrealistic in the cooperative game theory.

In this paper, we assign to each player a positive weight and extend the Harsanyi solution for cooperative games to reduced Harsanyi power solution. It is worthwhile that in view of weights can influence the possibilities of coalitions formation, here we introduce the concept of loss function, which reflects the loss of Harsanyi dividends generated by coalitions. The reduced Harsanyi power solution is relevant to the loss function. We also give an axiomatization of this new solution. Furthermore, we present some particular forms of loss functions and give the corresponding specific solutions.

In our solution, Harsanyi dividend of any coalition is only distributed among players in this coalition. In the further study, we will continue to discuss such class of solutions where a player can share the dividends of coalitions not including it.

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An Allocation Method of Provincial College Enrollment Plan Based on Bankruptcy Model

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Abstract. The college enrollment plan allocation plays an important role in implementing the reform of higher education and adjusting the structure of qualified personnel in society. In this paper, the provincial college enrollment plan allocation is regarded as the bankruptcy problem. By transforming the eight university educational indexes into membership degrees, which are taken into account in the allocation process. A bankruptcy model and an operable bankruptcy rule are proposed. This study provides references for the provincial education administrative departments in college enrollment plan allocation process.

Keywords: Enrollment plan allocation · Membership degrees · Bankruptcy problem · Cooperative game

1 Introduction

With the deepening of the higher education reform in China, it is necessary for us to accelerate innovations and improve the quality of the higher education in order to promote the scientific development of education programs and improve the level of educational modernization. The enrollment plan has become the baton that conducts the higher education orchestra, which is an important reflection of educational equality, playing a significant role in the education reform and regional talent structure adjustment.

The key question is how to allocate the quotas of new students reasonably in the process of enrollment plan-making. Under the premise of provincial enrollment total scale approved by the education ministry, the provincial education administrations are responsible for the coordination and distribution of the quotas of college enrollment plan in their regions. However, the complexity of the factors leads to the difficulty of the college enrollment scale allocation. The reasons are as follows: (1) The numbers of students have declined, which requires higher requirements to reasonable enrollment plan allocation. (2) The enrollment plan allocation affects directly the sizes of universities and the long-term development, so colleges pursuit blindly the quotas of enrollment plan, which interfere with the progress of enrollment plan allocation.

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(3) Because of the complexity of policy factors and educational indexes, it makes allocation inefficient and affects normal operation of college enrollment.

Some scholars have carried out relevant researches to solve these problems, by establishing mathematical models to distribute the numbers of enrollment plan. Zheng et al. [1] study the provincial enrollment plan allocation model from the perspective of colleges and universities, using the method combining the grey theory and the fuzzy set theory. Dong [2] constructs the index system of the regional allocation in the enrollment plan of local universities and uses AHP to simulate the allocation of the enrollment plan of local colleges. Jing et al. [3] set up the bankruptcy model to distribute enrollment plan in colleges and universities by the APL rule. However, the above researches are carried out only from the perspective of the state or universities instead of resolving the difficulties in the actual operation process from the perspective of the provincial education administrations.

In this paper, we describe the process of enrollment plan distribution from the perspective of the provincial education administrations based on the enrollment total scale approved by the Education Ministry and claims from the local colleges. We will analyze this process and calculate the membership degree to evaluate the college index system by using the minimum membership degree deviation method of fuzzy multi-objective decision making. Then we use the above conclusion to set up the bankruptcy model, which includes membership degree, a factor taken into account in the allocation of the enrollment plan. The example is analyzed to show the rationality and validity of the developed distribution approach.

2 Comprehensive Evaluation of the University Educational Indexes of Enrollment Plan

2.1 Description of the University Educational Indexes

The provincial education administrative departments distribute local university enrollment plan by considering some factors such as the need of the economic and social development, the university development, and the base conditions of universities, the enrollment and claim of universities. By investigating in the provincial education administrative departments, we describe eight university educational indexes, taken into account in the allocation process, which are shown in Fig. 1.

1. The basic educational conditions indexes. The Education Ministry issued the No. 2 document of *the Basic Educational Conditions Index of University* in 2004, which points out that the five basic educational condition indexes are the basis to measure the sizes and qualities of universities and re-evaluate the annual recruitment of students scale, including the student-teacher ratio, the percentage of teachers obtained master degrees in the full-time teachers, the average size of teaching and administration buildings per student, the average numbers of books in the library per student, the average cost of equipment for teaching and scientific researches per student. These five basic education conditions indexes above are represented by β_1, β_2, β_3, β_4, β_5, respectively.

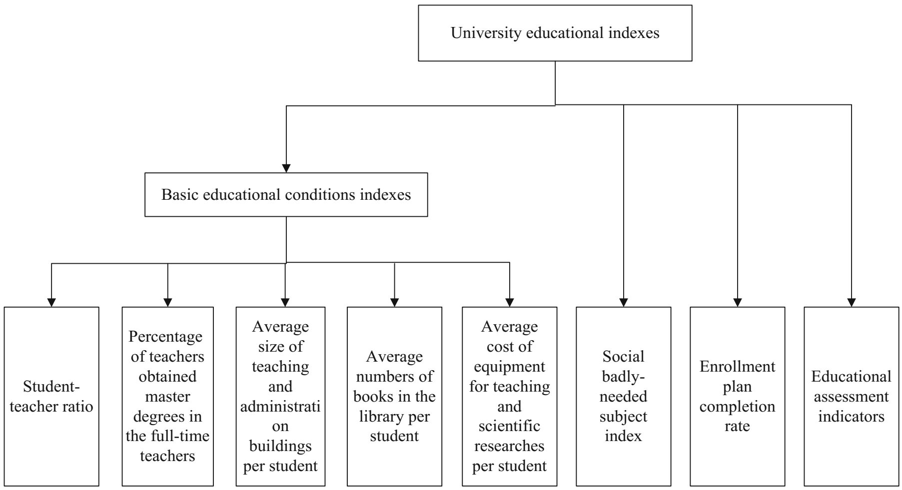

Fig. 1. The system of university educational indexes

2. The social badly-needed subject index. To guarantee the stable development of regional economy, we must focus on regional talents structure and support the development of shortage subject to meet the demand of the regional key projects and talents' need in remote areas under harsh conditions, based on the diversity and rationality of the subject development in universities. According to the demand of province regional economic development, there are the badly-needed subjects in the kinds of universities, including engineering, agriculture, forestry and medical universities. β_6 represents the social shortage subject indicator.
3. The enrollment plan completion rate. Enrollment plan reflects that the chances of obtaining higher education resources for citizens, while unfinished enrollment plan causes the waste of education resources. University enrollment plan completion rate embodies the full use of education resources, also indirectly reflects the social recognition and satisfaction of colleges and universities. The enrollment plan completion rate is defined as β_7.
4. The educational assessment indicators. University education evaluation monitors the national macro management and the teaching work in colleges and universities, used in the evaluation of college teaching work and the level of meeting personnel training requirements. The evaluation results reflect the teaching quality, the teaching efficiency and the reform of the national education. Therefore, the evaluation results, as the educational index, are taken into account in the allocation of the enrollment plan. The educational assessment indicator is denoted by β_8.

2.2 Comprehensive Evaluation Based on the Minimum Membership Degree Deviation Method

The Minimum membership degree deviation method is a kind of fuzzy multi-objective decision-making methods. The main ideas of this method include two points [4].

1. Determining the membership function. Because of the different dimension of the objectives, if the objectives are used to make decision, the results of decision-making will be unreasonable. The objectives are transformed into the non-dimensional numbers by the membership function, which are used to measure the object membership degrees. According to the actual situation, the membership functions are considered to fall into four types: Efficiency, Cost, Fixed-value and Interval.
2. Determining the gathering function. The Minimum membership degree deviation method is used to evaluate deviation degree between the object and the ideal value. If the objectives value is closer to the ideal value, it is more superior. Generally, the weighted Minkowski distance function is considered as gathering function. Then the multi-objective problem is transformed into single-objective evaluation problem, applying the membership degree obtained above to evaluate the schemes.

We denote the set of universities to participate in the provincial enrollment plan distribution by $N = \{1, 2, \ldots, n\}$. $\beta_{ij}(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, 8)$ is the education index j of the university i. $m_j = [a_j, b_j]$ is the interval. $f_{ij}(\beta_{ij}) \in [0, 1]$ represents the membership function as follows.

(a) Cost membership function.

$$f_{ij}\left(\beta_{ij}\right) = \begin{cases} 1, & \beta_{ij} \in \left[0, a_j\right] \\ \frac{b_j - \beta_{ij}}{b_j - a_j}, & \beta_{ij} \in \left(a_j, b_j\right) \\ 0, & \beta_{ij} \in \left[b_j, +\infty\right) \end{cases} \tag{1}$$

(b) Efficiency membership function

$$f_{ij}\left(\beta_{ij}\right) = \begin{cases} 0, & \beta_{ij} \in \left[0, a_j\right] \\ \frac{\beta_{ij} - a_j}{b_j - a_j}, & \beta_{ij} \in \left(a_j, b_j\right) \\ 1, & \beta_{ij} \in \left[b_j, +\infty\right) \end{cases} \tag{2}$$

According to the No. 2 document of the Education Ministry in 2004, the qualified criterion and the unqualified criterion in the different college classes are shown in Tables 1 and 2, respectively. The restricted universities are identified according to their cost type of education indexes, which are higher than the unqualified criterion, or lower than the unqualified criterion in terms of efficiency index. The qualified universities are identified according to their cost type of education indexes, which are lower than the qualified criterion, or higher than the qualified criterion in terms of efficiency indexes. Therefore, the qualified criterion equals to upper bound b_j, and the unqualified criterion equals to lower bound a_j.

Ideal solution of each evaluation index is denoted by $\boldsymbol{g} = (g_1, g_2, \ldots, g_8)$. When the membership degree of all the objective is maximum, that is, when $g_i = 1$ represents the ideal solution. The weighted Minkowski distance function is used to evaluate the distance between the membership degree and the ideal value. The optimal membership μ_i can be defined as

Table 1. Qualifed criterion of eight education indexes

College classes	Qualified criterion of eight education indexes							
	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8
class 1	18	30	14	100	5000	1	1	1
class 2	18	30	16	80	5000	1	1	1
class 3	16	30	16	80	5000	1	1	1
class 4	18	30	9	100	3000	1	1	1
class 5	11	30	22	70	4000	1	1	1
class 6	11	30	18	80	4000	1	1	1

Table 2. Unqualified criterion of eight education indexes

College classes	Unqualified criterion of eight education indexes							
	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8
Class 1	22	10	8	50	3000	0	0	0
Class 2	22	10	9	40	3000	0	0	0
Class 3	22	10	9	40	3000	0	0	0
Class 4	23	10	5	50	2000	0	0	0
Class 5	17	10	13	35	2000	0	0	0
Class 6	17	10	11	40	2000	0	0	0

$$\mu_i = \left\{ \sum_{j=1}^{8} \left[w_j (g_j - f_{ij}) \right]^q \right\}^{\frac{1}{q}} \tag{3}$$

w_j is the weight of objectives, which is obtained through the investigation of the actual allocation process in the provincial education administrative departments. This paper respectively selects $q = 1$, named the weighted hamming distance method and $q = 2$, named the weighted Euclidean distance method, to calculate the membership degree μ_i. In the following paper, we will set up the bankruptcy model to distribute local university enrollment plan combining the membership degree.

3 Bankruptcy Model of Provincial College Enrollment Plan Allocation

A bankruptcy problem aims at looking for a fair and rational distribution rule in the case of insufficient resources. For instance, when a firm went bankrupt, some debts will be left behind. However, the estate available is not sufficient to meet all the claims from the other firms or agents. More generally, the bankruptcy problem may be the situation as follows: a scare resource is distributed in the certain rule the by a set of agents.

O'Neil [5] proposes a corresponding cooperative bankruptcy game method for every bankruptcy problem. Aumann and Maschler [6] discuss the bankruptcy problems from the Talmud by the game theory and prove that the allocation rule of bankruptcy problems in Talmud is the nucleolus of the corresponding bankruptcy games [7, 8].

The classical bankruptcy problem [9–11] is a triplet $(N, \boldsymbol{c}, E)$, satisfying the following conditions:

$$E \leq \sum_{i=1}^{n} c_i \tag{4}$$

where $N = \{1, 2, \ldots, n\}$ is the set of agents involved in the situation, $\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n_+$ represents the claims of agents, while the estate is denoted by $E \in \mathbb{R}_+$, divided among all the claimants.

Given Bankruptcy problem, a solution is vector $\boldsymbol{x} = (x_1, x_2, \ldots x_n)$, assigned to claimants in the certain rule, can be defined as

$$0 \leq x_i \leq c_i (i \in N) \text{ and } \sum_{i=1}^{n} x_i = E. \tag{5}$$

The above function shows that the estate has to be divided completely among the claimants, and each claimant has to obtain a nonnegative quantity not greater than their claim.

From the perspective of the provincial education administrative department, the distribution of enrollment plan can be regarded as a classic bankruptcy problem. The Provincial education administrative departments request local colleges and universities to report enrollment plan scale according to the actual running situation of the universities. However, the pursuing of the long-term development and their interests, colleges and universities applies for more quotas than the ones that would be absorbed by the universities, which leads to the situation that the total demands is greater than the scale of provincial enrollment plan every year, even more than 20% of the total scale some times. The education administrative departments must allocate reasonably the university enrollment plan by evaluating the quality of teaching level and policy factors. Moreover, they have to make sure that the enrollment plan obtained by every university is not only a non-negative integer but also not greater than its claim. In this situation, $N = \{1, 2, \ldots, n\}$ represents the local university participate in enrollment allocation, E is the total scale of enrollment approved by the education ministry. The demand of enrollment for university is denoted by $\boldsymbol{c}$. $\boldsymbol{x}$ is the solution in the certain rule, namely enrollment plan assigned to the university. The optimal membership μ_i reflects the scale and level of colleges and universities.

According to the No. 2 document of the Education Ministry in 2004, universities are restricted to recruit students if one of the five basic education condition indicators of the university reach the unqualified criterion (that is $\sum_{j=1}^{5} \operatorname{sgn} f_{ij} = 4$); the university is suspended to admit students, which means enrollment plan will not be assigned to this university this year under the condition that two, or more than two, of the five basic education condition indicators of this university reach the unqualified criterion (that is $\sum_{j=1}^{5} \operatorname{sgn} f_{ij} \leq 3$). The paper proposes the divided rule of bankruptcy problem in this situation based on obtained membership degree in Sect. 2. In this article, we discuss the situation that $\sum_{k \in N \setminus S_3} \mu_k$ do not equal to zero. Because the $\sum_{k \in N \setminus S_3} \mu_k$ do not equal to zero in the enrollment allocation process. If $\sum_{k \in N \setminus S_3} \mu_k$ equal to zero, it means that all of eight education indexs for each colleges reach the qualified criterions. In fact, the qualified criterions from the Education Ministry are difficult to be reached for colleges. The qualified criterions maybe be reached for several key colleges, which is less than the fifth provincial colleges in China.

1. The algorithm of non-integer solutions $\bar{x}_i$. We denote Parameters as follows: S_4 is the set of restricted-enrollment universities, S_3 represents the set of the pause-enrollment universities. $\lambda_i \in \mathbb{N}_+$ is denoted by the number of graduates of university i. $\sum_{j=1}^{5} \operatorname{sgn} f_{ij}$ is defined as the number of the basic educational condition indexes failing to reach the unqualified criterion in the university i.
   (a) Primary distribution of universities, the solutions x_i' is as follows:

$$x_i' = \begin{cases} c_i - \dfrac{\mu_i}{\sum_{k \in N \setminus S_3} \mu_k} \left(\sum_{k \in N \setminus S_3} c_k - E \right) + \\ \dfrac{(1-\mu_i)}{\sum_{o \in N \setminus (S_3 \cup S_4)} (1-\mu_o)} \sum_{t \in S_4} \max \left\{ c_t - \dfrac{\mu_t}{\sum_{k \in N \setminus S_3} \mu_k} \left(\sum_{k \in N \setminus S_3} c_k - E \right) - \lambda_t, 0 \right\}, & \sum_{j=1}^{5} \operatorname{sgn} f_{ij} = 5 \\ \min \left\{ c_i - \dfrac{\mu_i}{\sum_{k \in N \setminus S_3} \mu_k} \left(\sum_{k \in N \setminus S_3} c_k - E \right), \lambda_i \right\}, & \sum_{j=1}^{5} \operatorname{sgn} f_{ij} = 4 \\ 0, & \sum_{j=1}^{5} \operatorname{sgn} f_{ij} \leq 3 \end{cases} \quad (6)$$

   (b) x_i' shall be transformed in order to meet the requirements, as follows:

$$x_i' = \begin{cases} 0, & x_i' < 0 \\ x_i', & 0 \leq x_i' \leq c_i \\ c_i, & x_i' > c_i \end{cases} \quad (7)$$

   (c) Note that if $x_i' < 0$, negative residual x_i' come into being; if $x_i' > c_i$, positive residual $x_i' - c_i$ appear. Suppose Q is the sum of residuals, S_1 represents the set of universities, satisfying with $0<x'_i<c_i$ and $\sum_{j=1}^{5} \operatorname{sgn} f_{ij} = 5$. The integer solutions $\bar{x}_i$ can be obtained in the following way:

$$\bar{x}_i = \begin{cases} x'_i + \frac{\mu_i}{\sum_{k\in S_1}\mu_k} Q, & i \in S_1 \text{ and } Q<0 \\ x'_i + \frac{(1-\mu_i)}{\sum_{k\in S_1}(1-\mu_k)} Q, & i \in S_1 \text{ and } Q>0 \\ x'_i, & i \notin S_1 \text{ or } Q=0 \end{cases} \tag{8}$$

Steps (b) and (c) are repeated until $(0\le\bar{x}_i\le c_i$ and $Q=0)$ or $(\bar{x}_i = c_i(\bar{x}_i \in N\backslash(S_3\cup S_4))$ and $Q>0)$. If the total scale of enrollment may not run out (that is $Q>0$), the provincial education administrative will reduce the scale of enrollment. Residual quotas will be used for other types of enrollment.

The main ideas of allocation above are as follows:

(i) The enrollment-suspended universities are assigned to 0.

(ii) Except the enrollment-suspended universities, other universities are divided by the method above, namely

$$c_i - \frac{\mu_i}{\sum_{k\in N\backslash S_3}\mu_k}\left(\sum_{k\in N\backslash S_3} c_k - E\right)$$

(iii) The enrollment-restricted universities obtain the smaller value as the enrollment plan, in the comparison of the enrollment plan allocation in step (b) with the number of graduates.

If $c_i - \frac{\mu_i}{\sum_{k\in N\backslash S_3}\mu_k}\left(\sum_{k\in N\backslash S_3} c_k - E\right) > \lambda_i$, the positive residual appeared is denoted by

$$c_i - \frac{\mu_i}{\sum_{k\in N\backslash S_3}\mu_k}\left(\sum_{k\in N\backslash S_3} c_k - E\right) - \lambda_i$$

(iv) Except the two kinds of universities mentioned above, others obtain the sum that the value of the primary allocation in step (b) adds to the compensation value together. Compensation value is obtained according to the proportion of the membership degree to allocate the sum of positive residual.

(v) If the condition does not satisfy with $0\le\bar{x}_i\le c_i$, x'_i shall be transformed into (7).

(vi) The sum of residuals in step (c) is assigned to the universities that satisfy with $0<x'_i<c_i$ and $\sum_{j=1}^{5} \operatorname{sgn} f_{ij} = 5$.

(vii) Steps (b) and (c) are repeated until $(0 \le x_i' \le c_i$ and $Q = 0)$ or $(x_i' = c_i(x_i' \in N \backslash (S_3 \cup S_4))$ and $Q > 0)$.

2. Algorithm of the final integer solution x_k. The enrollment plan has to be the integer, while the non-integer solution $\bar{x}_i$ may have decimal part. Therefore, it is necessary to transform non-integer solutions into the final integer solution.

Suppose $\text{int}(\bar{x}_i)$ represents the largest integer that does not exceed $\bar{x}_i$, namely the integer part of $\bar{x}_i$. The decimal part is denoted by $\bar{x}_i - \text{int}(\bar{x}_i)$, then ∂_i can be defined as

$$\partial_i = \frac{\bar{x}_i - \text{int}(\bar{x}_i)}{\text{int}(\bar{x}_i)} \tag{9}$$

Where ∂_i reflect the degrees that decimal part influences on the integer part. According to the descending order of ∂_i, $\text{int}(\bar{x}_i)$ is ordered in turn. If $\partial_i = \partial_j$, then the universities are ordered according to the objective evaluation indexes. The objective evaluation index can be regarded as the ranking of the university, which is different from each other. The universities are ordered finally, namely $ord(\bar{x}_1) > ord(\bar{x}_2) > \ldots > ord(\bar{x}_k) > \ldots > ord(\bar{x}_n)$.

Suppose $m = E - \sum_{i=1}^{n} \text{int}(\bar{x}_i)$, θ_k is such that

$$\theta_k = \begin{cases} 1, & k \le m \\ 0, & m < k \le n \end{cases} \tag{10}$$

Then, the universities obtain the following enrollment plan.

$$x_k = \text{int}(\bar{x}_k) + \theta_k \tag{11}$$

4 Analysis of the Example

There are 18 provincial public colleges in a province in 2016. The scale of enrollment plan is 72240 approved by the education ministry. The claims and the eight educational indexes for these colleges are shown in Table 3.

We use the proposed approach to distribute enrollment plan, under the process as follows:

1. Determine the membership function. The membership function of β_1 is selected as the cost membership function in (1), β_2, β_3, β_4 and β_5 are selected as the efficiency membership function in (2). For β_6 and β_8, 1 is used for yes and 0 is used for no. The indexes will be calculated according to the membership function selected.
2. Under the condition when $q = 1$ or $q = 2$, the membership degree will be calculated in (3) respectively.
3. Calculate the non-integer solutions x_i in (6), (7), (8).
4. The non-integer solutions are transformed into the final integer solutions in (10), (11).

Table 3. Claims, eight educational indexes for the public colleges in a province

Colleges	College classes	λ_i	c_i	Eight education indexes and their weights							
				β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8
				$w_1 = 0.06$	$w_2 = 0.06$	$w_3 = 0.06$	$w_4 = 0.06$	$w_5 = 0.06$	$w_6 = 0.2$	$w_7 = 0.3$	$w_8 = 0.2$
1	Class 2	5379	5920	17	86	17	87	25569	Yes	1	Yes
2	Class 1	4853	5850	19	84	15	116	14827	No	0.995	Yes
3	Class 2	5193	5700	16	86	23	90	20944	Yes	0.978	Yes
4	Class 3	2257	2830	14	82	15	69	14647	Yes	1	Yes
5	Class 3	2002	2030	10	71	16	81	16705	Yes	1	Yes
6	Class 1	5984	6500	17	78	16	94	15883	No	1	Yes
7	Class 1	4912	5050	17	82	12	107	7464	No	1	Yes
8	Class 2	5639	5820	18	78	14	86	10771	Yes	1	Yes
9	Class 4	1955	3900	18	80	21	95	3901	No	1	Yes
10	Class 4	96	1300	19	71	17	144	11982	No	1	Yes
11	Class 2	3480	4565	17	77	16	81	9848	Yes	0.991	Yes
12	Class 1	2810	3985	19	71	11	63	9211	No	1	Yes
13	Class 1	3524	4420	20	69	12	78	8213	No	1	Yes
14	Class 2	4138	5300	19	82	11	67	16532	Yes	1	Yes
15	Class 1	2319	3000	21	73	13	72	8875	No	1	Yes
16	Class 1	2861	3700	19	71	12	91	6773	No	0.995	Yes
17	Class 1	2909	4550	19	74	17	85	7214	No	0.988	Yes
18	Class 1	946	2080	15	74	26	101	10442	No	0.996	Yes

The distribution results by using the proposed method in this paper are shown in Table 4 and Fig. 2. The comparison show that the results reached by the proposed method of this study is in line with the actual enrollment plan allocation issued by the provincial education administrative department, which embodies the feasibility and operability of this model. Little deviation appeared in Fig. 2 is caused by the subjective factors, or actual allocation policy and the interests from all players.

The proposed method has the following advantages. Firstly, we use the membership degree to evaluate the level of actual operation condition of universities based on the education indexes and establish the mathematical model of the enrollment plan distribution (membership degree included), which makes the enrollment plan distribution work of the provincial education administrative department more scientific, reasonable and normative. Secondly, the allocation process is modeled to avoid subjective factors in the actual allocation process. In addition, this model assists the work of provincial education administrative departments in avoiding tedious distribution work and repeated work in the process of the actual distribution, reducing the workload in the distribution process, which enhances efficiency of the distribution of enrollment plan. Finally, we bring these important indexes into the evaluation system, which could encourage colleges and universities to obtain more enrollment quotas by improving hardware conditions and educational quality.

Table 4. Results of using the proposed method vs. Results of the actual allocation

Colleges	Claim	Actual allocation results	Membership degree μ_i		Results by using the proposed the method x_i	
			$q = 1$	$q = 2$	$q = 1$	$q = 2$
1	5920	5880	0	0	5920	5920
2	5850	5850	0.217	0.201	5511	5481
3	5700	5530	0.007	0.007	5689	5687
4	2830	2830	0.025	0.019	2791	2795
5	2030	2030	0	0	2030	2030
6	6500	6150	0.207	0.2	6177	6133
7	5050	4680	0.22	0.201	4706	4681
8	5820	5750	0.017	0.017	5793	5789
9	3900	3560	0.206	0.2	3579	3533
10	1300	1250	0.212	0.2	970	933
11	4565	4260	0.003	0.003	4560	4560
12	3985	3590	0.289	0.208	3534	3603
13	4420	4100	0.276	0.205	3989	4044
14	5300	5030	0.077	0.049	5180	5210
15	3000	2750	0.289	0.208	2549	2618
16	3700	3300	0.247	0.202	3315	3329
17	4550	3890	0.237	0.201	4180	4181
18	2080	1810	0.201	0.2	1767	1713

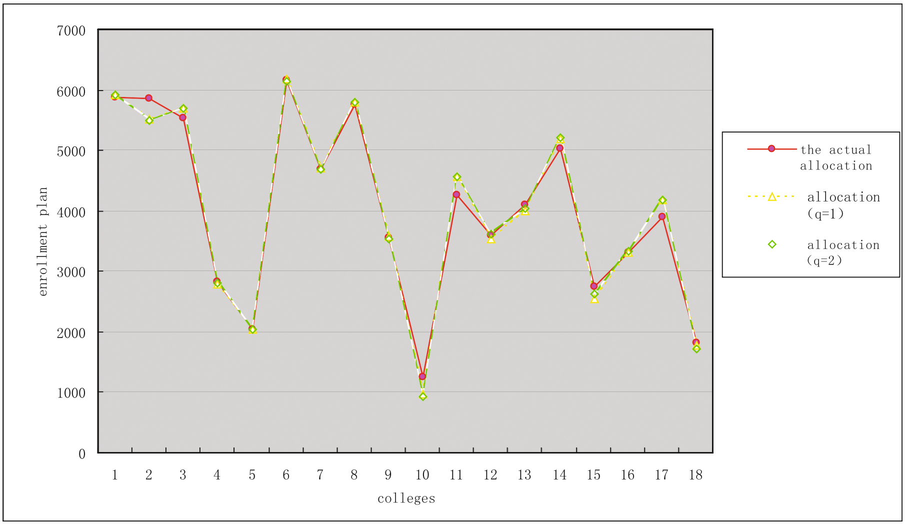

Fig. 2. Results of using the proposed method vs. Results of the actual allocation

5 Conclusion

In this paper, we study the provincial college enrollment plan allocation from the perspective of the provincial education administrative department and find that the actual distribution work is tedious, and some subjective factors may exist to influence the fairness of enrollment plan allocation. Therefore, we describe the eight educational indexes that are taken into account in the process of the actual distribution work by investigating in the provincial education administrative departments. Then the minimum membership degree deviation method is used to evaluate the education condition indexes. The membership degree is obtained by the minimum membership degree deviation method, which actually embodies the hardware conditions and educational quality of the colleges.

Moreover, the provincial enrollment plan assignment problem is regarded as a type of bankruptcy problems. Because the total quotas of enrollment plan which the colleges apply for are greater than the scale of provincial enrollment plan approved by the Education Ministry. In this bankruptcy problem, the quotas applied for by colleges can be regarded as the claims, which the scale of provincial enrollment plan approved by the Education Ministry is the estate. Then we propose the enrollment plan allocation method combined with the membership degree is obtained by the mathematical model and. Finally, an example is analyzed to shows that this proposed method is provided with superiority and feasibility in the process of the enrollment plan distribution.

In the future, we will study the cooperative game nature of the proposed distribution method based on the corresponding cooperative bankruptcy game and extend our approach to apply to another similar situation.

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Cooperative Games Under Uncertainty

Edgeworth Equilibria of Economies and Cores in Multi-choice NTU Games

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Abstract. In this paper, we derive an extension of the payoff-dependent balanced core existence theorem by Bonnisseau and Iehlé [Games Econ. Behav. 61 (2007) 1–26] to multi-choice NTU games which implies a multi-choice extension of Scarf's core existence theorem.

Keywords: Cores · Edgeworth equilibrium · Balanced collections · Balancedness · Payoff-dependent balancedness · NTU games · Multi-choice NTU games

JEL Classification: C60 · C71

1 Introduction

In 1881, Edgeworth proved that, in the case of two agents and two commodities, the core of an exchange economy shrinks to the set of Walrasian (competitive) equilibrium allocations. He then claimed that his result applies for an arbitrary number of commodities and agents. Many years later, Debreu and Scarf [6] proved Edgeworth's conjecture by showing that when the economy is replicated, the intersection of the cores of the sequence of the replications coincides with the set of Walrasian equilibrium allocations. Recently, Liu and Liu [13] extended Debreu-Scarf Theorem to coalition production economies.

In 1987, Aliprantis et al. [1] defined Edgeworth equilibrium as any feasible allocation such that the r-fold repetition of it belongs to the core of r-fold replica of the economy for every $r \geq 1$ and proved the existence of Edgeworth equilibrium for pure exchange economies with infinite-dimensional commodity spaces for ordered case. Later, Florenzano [8] proved the existence of Edgeworth equilibrium for exchange economies without ordered preferences. Clearly, the classical result by Debreu and Scarf [6] shows that Edgeworth equilibrium is equivalent to competitive equilibrium for pure exchange economies.

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Edgeworth equilibria of coalition production economies are closely related to cores in multi-choice NTU games. For more on multi-choice games, please see [5,9–11] and [14]. In this paper, we derive an extension of the payoff-dependent balanced core existence theorem by Bonnisseau and Iehlé [4] to multi-choice NTU games which implies a multi-choice extension of Scarf's core existence theorem.

2 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be the set of all players. Any non-empty subset of N is called a *(crisp) coalition*. Throughout this paper, we denote the collection of all coalitions (non-empty subsets) of N by $\mathcal{N}$ and for any $a, b \in \mathbb{R}^n$, $a \leq b$ means $a_i \leq b_i$ for each $1 \leq i \leq n$, and $a \gg b$ means each coordinate $a_i > b_i$ for $1 \leq i \leq n$. For each $S \in \mathcal{N}$, denote e^S to be the vector in $\mathbb{R}^n$ with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. We use e^i for $e^{\{i\}}$ for each $i \in N$.

The concept of multi-choice games first introduced by Hsiao and Raghavan [9] (and [10]). Suppose each player $i \in N$ has $m_i + 1$ ($m_i \geq 1$) activity levels from $M_i = \{0, 1, \ldots, m_i\}$ and let $M = (\prod_{i \in N} M_i) \setminus \{0\}$. For each $\mu \in M$, let $A(\mu) = \{i \in N | \mu_i > 0\}$. The following concept of a multi-choice NTU game and subsequent concepts are natural extensions to the corresponding concepts for NTU games (see [16,18]).

Definition 2.1. A multi-choice NTU game in coalition form is (M, V), where V is a mapping that maps each $\mu \in M$ to a subset $V(\mu)$ of $\mathbb{R}^n$ and satisfies the following conditions:

(1) For each $\mu \in M$, $V(\mu)$ is nonempty, closed, comprehensive (i.e., if $x, y \in \mathbb{R}^n$ are such that $y \in V(\mu)$ and $x \leq y$, then $x \in V(\mu)$), bounded from above by $D > 0$ (in the sense that if $x \in V(\mu)$, then $x_i \leq D$ for all $i \in A(\mu)$);
(2) For each $\mu \in M$, $V(\mu)$ is cylindrical in the sense that if $x \in V(\mu)$ and $y \in \mathbb{R}^n$ such that $y_i = x_i$ for each $i \in A(\mu)$, then $y \in V(\mu)$;
(3) For every i, there is a $b_i > 0$ such that $V(m_i e^i) = \{x \in \mathbb{R}^n | x_i \leq b_i\}$.

Denote $m = (m_i)_{i \in N}$. Note that in a multi-choice NTU game (M, V), m plays the same role as the grand coalition N in an NTU game. Clearly, an NTU game V is a special multi-choice NTU game (M, V) with $m_i = 1$ for all $i \in N$.

A payoff vector to a multi-choice game (M, V) is a vector $(x_{ij})_{1 \leq i \leq n, 0 \leq j \leq m_i}$, where x_{ij} denotes the increase in payoff for player i corresponding to a change of activity from level $j - 1$ to level j and $x_{i0} = 0$ for all $i \in N$. Note that for a multi-choice NTU game (M, V) defined by Definition 2.1, a payoff vector x in each $V(\mu)$ means $x = (x_i)_{i \in N}$ with $x_i = \sum_{0 \leq j \leq m_i} x_{ij}$. Also note that a multi-choice TU game (M, v) with the characteristic function v is a special multi-choice NTU game (M, V) such that for each $\mu \in M$,

$$V(\mu) = \{x \in \mathbb{R}^n | \sum_{i \in A(\mu)} x_i \leq v(\mu)\}. \tag{2.1}$$

Given a multi-choice game (M, V), a payoff vector $x \in V(m)$, and a member $\mu \in M$, we say that μ has an *objection* against x if there exists some $y \in V(\mu)$ such that $y_i > x_i$ for all $i \in A(\mu)$.

Definition 2.2. The core of a multi-choice game (M, V), denoted by $C(M, V)$, consists of all payoff vectors in $V(m)$ that have no objections against them, that is,

$$C(M, V) = V(m) \setminus [\cup_{\mu \in M}\ int(V(\mu))]. \tag{2.2}$$

Let Δ^N be the standard simplex:

$$\Delta^N = \{x \in \mathbb{R}^n | x_i \geq 0 \text{ for each } i \in N \text{ and } \sum_{i=1}^{n} x_i = 1\}.$$

For each $\emptyset \neq S \subseteq N$, denote

$$\Delta^S = \{x \in \Delta^N | x_i = 0 \text{ for each } i \notin S\} = \{x \in \Delta^N | \sum_{i \in S} x_i = 1\}$$

and for each $S \in \mathcal{N}$, define $m^S \in \Delta^N$ by

$$m^S = \frac{e^S}{|S|}.$$

Denote Δ to be the Cartesian product of $\Delta^{A(\mu)}$ over all $\mu \in M$, i.e.,

$$\Delta = (\Delta^{A(\mu)})_{\mu \in M} = \{(\pi_\mu)_{\mu \in M} | \pi_\mu \in \Delta^{A(\mu)} \text{ for each } \mu \in M\}.$$

Definition 2.3. A collection $\mathcal{B} \subseteq M$ is balanced if there exist positive numbers λ_μ for $\mu \in \mathcal{B}$ such that

$$\sum_{\mu \in \mathcal{B}} \lambda_\mu e^{A(\mu)} = e^N. \tag{2.3}$$

The numbers λ_μ are called *balancing coefficients.*

Clearly, (2.3) is equivalent to the following:

$$\sum_{\mu \in \mathcal{B}} \lambda'_\mu m^{A(\mu)} = m^N, \tag{2.4}$$

where each $\lambda'_\mu = \frac{|A(\mu)|}{n}\lambda_\mu$.

The next concept is an extension of the concept of π-balanced collection by Billera [2].

Definition 2.4. Given $\pi \in \Delta$ with $\pi_m \gg 0$, a collection $\mathcal{B} \subseteq M$ is π-balanced if there exist positive numbers λ_μ for $\mu \in \mathcal{B}$ such that

$$\sum_{\mu \in \mathcal{B}} \lambda_\mu \pi_\mu = \pi_m. \tag{2.5}$$

It is clear from (2.4) and (2.5) that a balanced collection $\mathcal{B}$ is π-balanced for the special $\pi \in \Delta$ with $\pi_\mu = m^{A(\mu)}$ for each $\mu \in M$.

Definition 2.5.

(1) A multi-choice NTU game (M, V) is *balanced* if $\cap_{\mu \in \mathcal{B}} V(\mu) \subseteq V(m)$ for every balanced collection $\mathcal{B} \subseteq M$.
(2) Given $\pi \in \Delta$ with $\pi_m \gg 0$, a multi-choice NTU game (M, V) is π-*balanced* if $\cap_{\mu \in \mathcal{B}} V(\mu) \subseteq V(m)$ for every π-balanced collection $\mathcal{B} \subseteq M$.

Clearly, a balanced multi-choice NTU game (M, V) is π-balanced for the special $\pi \in \Delta$ with $\pi_\mu = m^{A(\mu)}$ for each $\mu \in M$.

Since NTU games are special multi-choice NTU games with $m_i = 1$ for all $i \in N$, the above concepts yield the corresponding concepts for NTU games when $m_i = 1$ for all $i \in N$. The following are well-known existence theorems for cores in NTU games.

Theorem 2.6 (Scarf, 1967)**.** Any balanced NTU game V has a non-empty core.

Theorem 2.7 (Billera, 1970)**.** Any π-balanced NTU game V has a non-empty core.

Theorem 2.8 (Bondareva, 1963 and Shapley, 1967)**.** A TU game V has a non-empty core if and only if it is balanced.

We will derive extensions of these theorems to multi-choice games in Sect. 4. But, we first provide a close connection between Edgeworth equilibria of coalition production economies and cores of multi-choice NTU games in the next section to show the needs for studying multi-choice NTU games.

3 Connection Between Edgeworth Equilibria of Economies and Cores of Multi-choice NTU Games

In this section, we will give a close connection between Edgeworth equilibria of economies and cores of multi-choice NTU games. First, let us recall the concept of a coalition production economy given in [12] and some necessary preliminaries from [13].

A *coalition production economy* $\mathcal{E} = (\mathbb{R}^L, (X^i, u^i, w^i)_{i \in N}, (Y^S)_{S \in \mathcal{N}})$ with n agents is a collection of the commodity space $\mathbb{R}^L$, where L is the set of commodities, agents' characteristics $(X^i, u^i, w^i)_{i \in N}$, and coalitions' production sets $(Y^S)_{S \in \mathcal{N}}$. The triple (X^i, u^i, w^i) is agent i's characteristics as a consumer: $X^i \subseteq \mathbb{R}^L$ is his consumption set, $u^i : X^i \to \mathbb{R}$ is his utility function, and $w^i \in \mathbb{R}^L$ is his endowment vector. The set $Y^S \subseteq \mathbb{R}^L$ is the production set of the firm (coalition) S for which every agent $i \in S$ works and Y^S consists of all production plans that can be achieved through a joint action by the members of S. We use $Y = Y^N$ for the total production possibility set of the economy.

An *exchange economy* is a coalition production economy with $Y^S = \{0\}$ for every $S \in \mathcal{N}$.

When dealing with replica of an economy $\mathcal{E}$, one usually needs some special conditions on the production possibility sets $(Y^S)_{S \in \mathcal{N}}$. The key assumption is that when $y \in Y^S$, $cy \in Y^S$ for any nonnegative constant c. Here are some common assumptions:

(P.1) $Y^S = \{0\}$ for all $S \in \mathcal{N}$ (exchange economies, see [6]);
(P.2) Y is a convex cone with vertex at the origin and $Y^S = Y$ for all $S \in \mathcal{N}$ (see [6]);
(P.3) Y^S is a convex cone containing 0 for each $S \in \mathcal{N}$ (see [13]).

Clearly, (P.3) contains (P.1) and (P.2). The following assumptions on consumption sets, utility functions, and the sets of attainable allocations are standard:

(A.1) For every agent $i \in N$, $X^i \subseteq \mathbb{R}^L$ is non-empty, closed, and convex,
(A.2) For each $i \in N$, $u^i : X^i \to \mathbb{R}$ is continuous and quasi-concave;
(A.3) for each $S \in \mathcal{N}$, $Y^S \subseteq \mathbb{R}^L$ is non-empty and closed, and the set $F_{\mathcal{E}}(S)$ of feasible
(attainable) S-allocations is nonempty and compact, where

$$F_{\mathcal{E}}(S) = \{(x^i)_{i\in S} | x^i \in X^i \text{ for each } i \in S \text{ and } \sum_{i\in S}(x^i - w^i) \in Y^S\}. \tag{3.1}$$

The set of all attainable allocations of the economy $\mathcal{E}$ is

$$F(\mathcal{E}) = F_{\mathcal{E}}(N) = \{(x^i)_{i\in N} | x^i \in X^i \text{ for each } i \in N \text{ and } \sum_{i\in N}(x^i - w^i) \in Y^N = Y\}$$

which is non-empty and compact.

In an effort to connect the two concepts of core and competitive equilibrium in exchange economies (more generally, coalition production economies satisfying (P.2)), Debreu and Scarf [6] considered r-fold replica of an economy. For each positive integer r, the r-fold replica of the economy $\mathcal{E}$, denoted by $\mathcal{E}_r$, is defined to be the economy composed of r subeconomies identical to $\mathcal{E}$ with a set of consumers

$$N_r = \{(i, q) | i = 1, \ldots, n \text{ and } q = 1, \ldots, r\}.$$

The first index of consumer (i, q) refers to the type of the individual and the second index distinguishes different individuals of the same type. It is assumed that all consumers of type i are identical in terms of their consumption sets, endowments, and utility functions. Let S be a non-empty subset of N_r. An allocation $(x^{(i,q)})_{(i,q)\in S}$ is S-attainable in the economy $\mathcal{E}_r$ if

$$\sum_{(i,q)\in S} (x^{(i,q)} - w^{(i,q)}) \in Y^{S'} \tag{3.2}$$

where $S' = \{i \in N | (i, q) \in S\}$, $x^{(i,q)} \in X^i$ and $w^{(i,q)} = w^i$ for every q. Thus, (3.2) can be written as

$$\sum_{i\in S'} \sum_{q\in S(i)} x^{(i,q)} - \sum_{i\in S'} |S(i)| w^i \in Y^{S'}. \tag{3.3}$$

where $S(i) = \{q \in \{1, 2, \dots, r\} | (i, q) \in S\}$ and $|S(i)|$ denotes the number of elements in $S(i)$.

Let $\mathcal{E}$ be a coalition production economy. From an r-fold replica $\mathcal{E}_r$ of $\mathcal{E}$, we form a multi-choice NTU game (M^r, V) as follows: Let $M_i^r = \{0, 1, \dots, r\}$ each $i \in N$ and let $M^r = (\prod_{i \in N} M_i^r) \setminus \{0\}$. For each $\mu \in M^r$, define $V(\mu) = \{v \in \mathbb{R}^n |$ there exists $(x^i)_{i \in N} \in F_{\mathcal{E}}(A(\mu))$ such that $\sum_{i \in A(\mu)} \mu_i (x^i - w^i) \in Y^{A(\mu)}$ and

$$v_i \le u^i(x^i) \text{ for every } i \in A(\mu)\}, \tag{3.4}$$

where $A(\mu) = \{i \in N | \mu_i > 0\}$. Note that for $m^r = (r, r, \dots, r)$, $A(m^r) = N$ for all $r \ge 1$. Under assumption (P.3), we have $V(m^r) = V(e^N)$ for all $r \ge 1$. By (2.2), $C((M^{r_2}, V)) \subseteq C((M^{r_1}, V))$ whenever $r_1 < r_2$. It follows that

$$\lim_{r \to \infty} C((M^r, V)) = \cap_{r \ge 1} C((M^r, V)). \tag{3.5}$$

Recall that for an economy $\mathcal{E}$, an allocation $x = (x^1, \dots, x^n)$ is blocked by a coalition S if there is an S-attainable partial allocation $(\overline{x}^i)_{i \in S}$ such that $u^i(\overline{x}^i) > u^i(x^i)$ for each $i \in S$. The *core* $C(\mathcal{E})$ of an economy $\mathcal{E}$ is the set of all attainable allocations which can not be blocked by any coalition. The following concept of Edgeworth equilibrium is given in [1] (see also [8]), where the r-fold repetition of an allocation $x = (x^1, \dots, x^n)$ is $r \circ x = (x^{(i,q)})_{(i,q) \in N_r}$ with $x^{(i,q)} = x^i$ for all $q \le r$ and for every $i \in N$.

Definition 3.1. An *Edgeworth equilibrium* of an economy $\mathcal{E}$ is an attainable allocation $x \in F(\mathcal{E})$ such that for any positive integer r, the r-fold repetition $r \circ x$ of x belongs to the core of the r-fold replica $\mathcal{E}_r$ of the economy $\mathcal{E}$. We will denote by $C^E(\mathcal{E})$ the set of all Edgeworth equilibria of $\mathcal{E}$.

Debreu and Scarf [6] proved that in an exchange economy or a coalition production economy satifying (A.1)–(A.3) and (P.2), when the set of economic agents is replicated, the set of core allocations of the replica economy shrinks to the set of competitive equilibria. This result has been extended to coalition production economies satisfying (A.1)–(A.3) and (P.3) by Liu and Liu [13]. The following theorem shows that the core of the multi-choice NTU game (M^r, V) arising from the r-fold replica economy $\mathcal{E}_r$ shrinks to a subset of the set of Edgeworth equilibria of $\mathcal{E}$ by (3.5).

Theorem 3.2. Let $\mathcal{E}$ be a coalition production economies satisfying (A.1)–(A.3) and (P.3). Then $v \in \cap_{r \ge 1} C((M^r, V))$ implies that x is an Edgeworth equilibrium, that is, $x \in C^E(\mathcal{E})$, where $x = (x^i)_{i \in N} \in X$ is an attainable allocation satisfying $v_i = u^i(x^i)$ for every $i \in N$.

Proof. Let $v = (v^i)_{i \in N} \in C((M^r, V))$ for all $r \ge 1$. We show that the r-fold repetition of x is in $C(\mathcal{E}_r)$ for all $r \ge 1$, where $x = (x^i)_{i \in N} \in X$ is an attainable allocation satisfying $v_i = u^i(x^i)$ for every $i \in N$. By (3.4), $v \in V(m^r)$, where $m^r = (r, r, \dots, r)$, implies that there exists $x = (x^i)_{i \in N} \in X$ such that

$$\sum_{i \in N} r(x^i - w^i) \in Y^N = Y \text{ and } v_i \le u^i(x^i) \text{ for every } i \in N. \tag{3.6}$$

By (2.2) and (3.4), $v \in C((M^r, V))$ implies that $v_i = u^i(x^i)$ for every $i \in N$. We claim that for any $r \geq 1$, $r \circ x = (x^{(i,q)})_{(i,q)\in N_r} \in C(\mathcal{E}_r)$, where $x^{(i,q)} = x^i$ for all $q \leq r$ and every $i \in N$. Suppose that $(x^{(i,q)})_{(i,q)\in N_r} \notin C(\mathcal{E}_r)$. Then there exists $S \subseteq N_r$ such that $(x^{(i,q)})_{(i,q)\in N_r}$ is blocked by S through a partial S-attainable vector $(\overline{x}^{(i,q)})_{(i,q)\in S}$. Let $S' = \{i \in N | (i,q) \in S\}$ and $S(i) = \{q \in \{1, 2, \ldots, r\} | (i,q) \in S\}$ for each $i \in N$. Then for each $i \in S'$ and all $q \in S(i)$, $\overline{x}^{(i,q)} \in X^i$ and

$$u^i(\overline{x}^{(i,q)}) > u^i(x^{(i,q)}) = u^i(x^i). \tag{3.7}$$

Let $\mu \in M^r$ be defined by $\mu_i = |S(i)|$ for each $i \in N$. Then $A(\mu) = S'$. By (3.2) and (3.3), $(\overline{x}^{(i,q)})_{(i,q)\in S}$ is S-attainable implies

$$\sum_{i\in S'} \mu_i [\frac{1}{\mu_i} \sum_{q\in S(i)} \overline{x}^{(i,q)}] - \sum_{i\in S'} \mu_i w^i \in Y^{S'}. \tag{3.8}$$

For each $i \in S'$, since $\overline{x}^{(i,q)} \in X^i$ for each $1 \leq q \leq r$ and X^i is convex by (A.1),

$$x^i_\mu = \frac{1}{\mu_i} \sum_{q\in S(i)} \overline{x}^{(i,q)} = \frac{1}{|S(i)|} \sum_{q\in S(i)} \overline{x}^{(i,q)} \in X^i.$$

It follows from (3.8) that

$$\sum_{i\in A(\mu)} \mu_i (x^i_\mu - w^i) \in Y^{A(\mu)}. \tag{3.9}$$

For each $i \in S' = A(\mu)$, since $u^i(\overline{x}^{(i,q)}) > u^i(x^i)$ for every $q \in S(i)$ by (3.7) and u^i is quasi-concave by (A.2),

$$u^i(x^i) < min_{q\in S(i)} \{u^i(\overline{x}^{(i,q)})\} \leq u^i(\frac{1}{|S(i)|} \sum_{q\in S(i)} \overline{x}^{(i,q)}) = u^i(x^i_\mu).$$

It follows from (3.6) that $v_i \leq u^i(x^i) < u^i(x^i_\mu)$ for each $i \in A(\mu)$. By (3.4) and (3.9), we conclude that $v \in int(V(\mu))$, contradicting $v \in C((M^r, V))$ by (2.2). Therefore, we have $r \circ x = (x^{(i,q)})_{(i,q)\in N_r} \in C(\mathcal{E}_r)$ and the theorem follows. □

4 Existence of Cores in Multi-choice NTU Games

Throughout this section, we use ∂D to denote the boundary of a subset D of $\mathbb{R}^n$ and $co\{X\}$ for the convex hull of the set X. Give an NTU game V, set $W = \cup_{S\in\mathcal{N}} V(S)$ and $\mathcal{S}(x) = \{S \in \mathcal{N} | x \in \partial V(S)\}$. The following concept is Definition 2.2 from Bonnisseau and Iehlé [4].

Definition 4.1. Let V be an NTU game.

(i) A *transfer rate rule* is a collection of set-valued mappings $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that: for each $S \in \mathcal{N}$, $\varphi_S : \partial V(S) \mapsto \Delta^S$ is an upper semi-continuous correspondence with non-empty compact and convex values; $\psi : \partial V(N) \mapsto \Delta^N$ is an upper semi-continuous correspondence with non-empty compact and convex values.

(ii) The game V is *payoff-dependent balanced* if there exists a transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$ such that, for each $x \in \partial W$,

$$\text{if } co\{\varphi_S(x)|S \in \mathcal{S}(x)\} \cap \psi(P_N(x)) \neq \emptyset, \text{ then } x \in V(N),$$

where P_N is a projection of $\mathbb{R}^n$ to $\partial V(N)$ defined by $P_N(x) = proj(x) - \lambda_N(proj(x))e^N$ which is continuous.

Bonnisseau and Iehlé [4] proved the following payoff-dependent core existence theorem.

Theorem 4.2 (Bonnisseau and Iehlé, 2007). If an NTU game V is payoff-dependent balanced with respect to some transfer rate rule $((\varphi_S)_{S\in\mathcal{N}}, \psi)$, then there exists a core payoff vector x satisfying:

$$co\{\varphi_S(x)|S \in \mathcal{S}(x)\} \cap \psi(x) \neq \emptyset.$$

Next, we extend the concept of payoff-dependent balancedness to multi-choice NTU games. For a multi-choice NTU game(M, V), let $W' = \cup_{\mu\in M} V(\mu)$ and $\mathcal{S}'(x) = \{\mu \in M|x \in \partial V(\mu)\}$. Recall that $M = (\prod_{i\in N} M_i) \setminus \{0\}$, $m = (m_i)_{i\in N}$, and $A(\mu) = \{i \in N|\mu_i > 0\}$ for each $\mu \in M$.

Definition 4.3. Let (M, V) be a multi-choice NTU game.

(i) A *transfer rate rule* is a collection of set-valued mappings $((\varphi_\mu)_{\mu\in M}, \psi)$ such that: for each $\mu \in M$, $\varphi_\mu : \partial V(\mu) \mapsto \Delta^{A(\mu)}$ is an upper semi-continuous correspondence with non-empty compact and convex values; $\psi : \partial V(m) \mapsto \Delta^N$ is an upper semi-continuous correspondence with non-empty compact and convex values.

(ii) The multi-choice game (M, V) is *payoff-dependent balanced* if there exists a transfer rate rule $((\varphi_\mu)_{\mu\in M}, \psi)$ such that, for each $x \in \partial W'$,

$$\text{if } co\{\varphi_\mu(x)|\mu \in \mathcal{S}'(x)\} \cap \psi(P_N(x)) \neq \emptyset, \text{ then } x \in V(m),$$

where P_N is a projection of $\mathbb{R}^n$ to $\partial V(m)$ defined by $P_N(x) = proj(x) - \lambda_N(proj(x))e^N$.

Theorem 4.2 can be extended to multi-choice NTU games as follows.

Theorem 4.4. If a multi-choice NTU game (M, V) is payoff-dependent balanced with respect to some transfer rate rule $((\varphi_\mu)_{\mu\in M}, \psi)$, then there exists a core payoff vector x satisfying:

$$co\{\varphi_\mu(x)|\mu \in \mathcal{S}'(x)\} \cap \psi(x) \neq \emptyset,$$

where $\mathcal{S}'(x) = \{\mu \in M|x \in \partial V(\mu)\}$.

Proof. Let (M, V) be a multi-choice NTU game which is payoff-dependent balanced with respect to some transfer rate rule $((\varphi_\mu)_{\mu\in M}, \psi)$. For each $S \in \mathcal{N}$, set $V^*(S) = \cup_{A(\mu)=S} V(\mu)$. Then each $V^*(S)$ is closed as it is a union of finite number of closed sets and V^* is an NTU game. For each $S \in \mathcal{N}$, define $\varphi^*_S = co\{\varphi_\mu | A(\mu) = S\}$. Then φ^*_S is an upper semi-continuous correspondence with non-empty compact and convex values for each $S \in \mathcal{N}$. Define $\psi^* = \psi$. Then V^* is payoff-dependent balanced with respect to the transfer rate rule $((\varphi^*)_{S\in\mathcal{N}}, \psi^*)$. Now, Theorem 4.4 follows from Theorem 4.2 easily. □

By (2.4), the following extension of Scarf's Theorem (Theorem 2.6) follows from Theorem 4.4 by setting $\varphi_\mu(x) = \{m^{A(\mu)}\}$ for each $\mu \in M$ and $\psi = \varphi_m = \{m^N\}$.

Theorem 4.5. Any balanced multi-choice NTU game (M, V) has a non-empty core.

By (2.5), the next extension of Billera's Theorem (Theorem 2.7) follows from Theorem 4.4 by setting $\varphi_\mu(x) = \{\pi_\mu\}$ for each $\mu \in M$ and $\psi = \varphi_m = \{\pi_m\}$.

Theorem 4.6. Any π-balanced multi-choice NTU game (M, V) has a non-empty core.

Next, we show that for multi-choice TU games, the converses of Theorems 4.5 and 4.6 hold. The following theorem is an extension of Bondareva - Shapley Theorem (Theorem 2.8) to multi-choice games.

Theorem 4.7. A multi-choice TU game (M, V) has a non-empty core if and only if it is balanced.

Proof. The sufficiency follows from Theorem 4.5. We now prove the necessity. Assume that (M, V) is a multi-choice TU game (M, V) with a nonempty core $C(M, V)$. Let $x^* \in C(M, V) = V(m) \setminus [\cup_{\mu\in M} \; int(V(\mu))]$ (see (2.2)). Then $x^* \in \partial V(m)$ and $x^* \notin V(\mu)$ for all $\mu \in M$. By (2.1), we have that $\sum_{i=1}^n x^*_i = v(m)$ and $x^* \cdot e^{A(\mu)} = \sum_{i\in A(\mu)} x_i \geq v(\mu)$ for every $\mu \in M$.

We now show that V is balanced. Let $\mathcal{B} \subseteq M$ be any balanced collection. Then, by (2.3), we have $\sum_{\mu\in\mathcal{B}} \lambda_\mu e^{A(\mu)} = e^N$ with some positive numbers λ_μ for $\mu \in \mathcal{B}$. We need to show that $\cap_{\mu\in\mathcal{B}} V(\mu) \subseteq V(m)$. Let $x \in \cap_{\mu\in\mathcal{B}} V(\mu)$. Then $x \in V(\mu)$ for each $\mu \in \mathcal{B}$ which implies that $x \cdot e^{A(\mu)} = \sum_{i\in A(\mu)} x_i \leq v(\mu)$ by (2.1). It follows that

$$\sum_{i=1}^n x_i = x \cdot e^N = x \cdot \sum_{\mu\in\mathcal{B}} \lambda_\mu e^{A(\mu)}$$

$$= \sum_{\mu\in\mathcal{B}} \lambda_\mu (x \cdot e^{A(\mu)}) \leq \sum_{\mu\in\mathcal{B}} \lambda_\mu v(\mu) \leq \sum_{\mu\in\mathcal{B}} \lambda_\mu (x^* \cdot e^{A(\mu)})$$

$$= x^* \cdot \sum_{\mu\in\mathcal{B}} \lambda_\mu e^{A(\mu)} = x^* \cdot e^N = \sum_{i=1}^n x^*_i = v(m),$$

which implies that $x \in V(m)$ by (2.1). Thus (M, V) is balanced. □

Recall that a balanced multi-choice NTU game (M, V) is π-balanced for the special $\pi \in \Delta$ with $\pi_{A(\mu)} = m^{A(\mu)}$ for each $\mu \in M$. The next characterization follows from Theorems 4.6 and 4.7 immediately.

Theorem 4.8. A multi-choice TU game (M, V) has a non-empty core if and only if it is π-balanced.

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Two-Phase Nonlinear Programming Models and Method for Interval-Valued Multiobjective Cooperative Games

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Abstract. In this paper, we define the concepts of interval-valued cores of interval-valued multiobjective n-person cooperative games and satisfactory degree (or ranking indexes) of comparing intervals with the features of inclusion and/or overlap relations. Hereby, the interval-valued cores can be computed by developing a new two-phase method based on the auxiliary nonlinear programming models. The proposed method can provide cooperative chances under the situations of inclusion and/or overlap relations between intervals in which the traditional interval ranking method may not always assure. The feasibility and applicability of the models and method proposed in this paper are illustrated with a numerical example.

Keywords: Cooperative games · Core · Interval ranking · Mathematical programming · Satisfactory degree

1 Introduction

More and more researchers have become interested in the multiple objectives decision making considered in the real application problems, which are described as the type of multiobjective n-person cooperative games problem. In this games, the worth of each coalition is measured by multiple criteria, and therefore it is given as a set in a multidimensional real space [1–3].

In real situations, because of incompleteness and uncertainty of decision information and the complexity of players' behavior, payoffs (or values) of players' coalitions in n-person cooperative games may be imprecise and vague. In some cases, we only can estimate the lower and upper bounds of payoffs, and the payoffs vary with these ranges, which can be described as intervals [4–7]. Interval computing and ranking method are a complex problem, which is different from that of real numbers and has attracted lots of attention [5, 6, 8–12]. Thanks to Moore [10], interval computing has been a well-established field and has been successfully applied to some areas. Branzei et al. [13] studied the cooperative games under interval uncertainty and the convexity of the interval-valued undominated cores. Alparslan-Gök et al. [14] investigates interval-type solution concepts of interval-valued cooperative games such as the

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interval-valued core, interval-valued dominance core and stable sets. Wang and Zhang [15, 16] further discuss some properties of fuzzy interval cooperative games and propose the sufficient conditions for the non-emptiness of the interval core. But, all these researches are based on the traditional ranking methods of intervals such as Moore method and LR method. These traditional ranking methods are relatively strict since they only consider the strictly relationships including intersection and being greater while they do not consider inclusion and/or overlap relations between intervals. Additionally, players may accept the inclusion and/or relations between intervals of coalitions' values at some satisfactory degrees in the practical cooperative games.

Hence, the aim of this paper is to study how to solve such a type of interval-valued multiobjective n-person cooperative games with the maximum satisfactory degree of interval inclusion and/or overlap relations between intervals of coalitions' values.

The rest of this paper is organized as follows. Section 2 briefly reviews some notations and definitions such as arithmetic operations over intervals and satisfactory degrees of comparing intervals. In Sect. 3, we formulate the interval-valued core and solution method for interval-valued multiobjective n-person cooperative games with satisfactory degrees of comparing intervals. A new two-phase approach with the auxiliary nonlinear programming models are derived to obtain the interval-valued cores and corresponding maximum satisfactory degrees that the players in coalitions accept the interval-type inclusion and/or overlap relations. In Sect. 4, implementation of the model and method proposed in this paper is conducted with a numerical example. Conclusion is made in Sect. 5.

2 Arithmetic Operations over Intervals and Concept of Satisfactory Degrees of Comparing Intervals

2.1 Arithmetic Operations over Intervals

Let $\Re$ be the set of real numbers. An interval may be expressed as $\hat{a} = [\underline{a}, \bar{a}] = \{a | \underline{a} \leq a \leq \bar{a}, \underline{a} \in \Re, \bar{a} \in \Re\}$, where $\underline{a}$ and $\bar{a}$ are called the lower and upper bounds of the interval $\hat{a}$, respectively. If $\underline{a} = \bar{a}$, then $\hat{a} = [\underline{a}, \bar{a}]$ is reduced to a real number a, where $a = \underline{a} = \bar{a}$.

Alternatively, an interval $\hat{a}$ may be expressed in mean-width or center-radius form as $\hat{a} = <m(\hat{a}), w(\hat{a})>$, where $m(\hat{a}) = (\underline{a} + \bar{a})/2$ and $w(\hat{a}) = (\bar{a} - \underline{a})/2$ are the mid-point and half-width of the interval $\hat{a}$, respectively. The set of intervals in the real number set $\Re$ is denoted by $I(\Re)$.

For any intervals $\hat{a} = [\underline{a}, \bar{a}]$ and $\hat{b} = [\underline{b}, \bar{b}]$, we stipulate their operations as follows:

(1) $\hat{a} + \hat{b} = [\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}]$;

(2) $\gamma \hat{a} = \gamma [\underline{a}, \bar{a}] = \begin{cases} [\gamma \underline{a}, \gamma \bar{a}] & \text{if } \gamma \geq 0 \\ [\gamma \bar{a}, \gamma \underline{a}] & \text{if } \gamma < 0; \end{cases}$.

Using the aforementioned mean-width or center-radius form, we can rewrite the former three operations of the above intervals' operations as follows:

$$(1') \quad \begin{aligned} \hat{a}+\hat{b} &= <m(\hat{a}),w(\hat{a})> + <m(\hat{b}),w(\hat{b})> \\ &= <m(\hat{a})+m(\hat{b}),w(\hat{a})+w(\hat{b})> \end{aligned};$$

$$(2') \quad \begin{aligned} \gamma\hat{a} = \gamma<m(\hat{a}),w(\hat{a})> &= <\gamma m(\hat{a}),|\gamma|w(\hat{a})> \\ &= \begin{cases} <\gamma m(\hat{a}),\gamma w(\hat{a})> & \text{if } \gamma \geq 0 \\ <\gamma m(\hat{a}),-\gamma w(\hat{a})> & \text{if } \gamma < 0. \end{cases} \end{aligned}$$

2.2 Concept of Satisfactory Degrees of Comparing Intervals and Properties

The ranking order of intervals is a difficult problem, which has been discussed by some researchers. And now most of the researches about interval-valued games are based on the opinions of Moore and Ishihuchi, especially the LR method about ranking order of intervals. Moore [10] held that $\hat{a} \leq \hat{b}$ if $\bar{a} \leq \underline{b}$. However, Ishihuchi [9] considered that $\hat{a} \leq \hat{b}$ if $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$ in the LR method. All these traditional ranking methods are relatively strict in that they only considered the strictly relationships including intersection and being greater rather than the inclusion and overlap relations between intervals. In fact, in terms of the fuzzy set, the statement "the interval $\hat{a}$ is not greater than the interval $\hat{b}$" may be regarded as a fuzzy relation between $\hat{a}$ and $\hat{b}$, which is denoted by $\hat{a} \leq_I \hat{b}$. Thus, inspired by Li [5], we define a fuzzy partial order relation for intervals, taking full account of the inclusion relation between intervals, which is current and with proven mathematical rigor.

Definition 1. Let $\hat{a} = [\underline{a}, \bar{a}]$ and $\hat{b} = [\underline{b}, \bar{b}]$ be two intervals. The premise "$\hat{a} \leq_I \hat{b}$" is regarded as a fuzzy set, whose membership function is defined as follows:

$$\varphi(\hat{a} \leq_I \hat{b}) = \begin{cases} 1 & \text{if } \bar{a} < \underline{b} \\ 1^- & \text{if } \underline{a} < \underline{b} \leq \bar{a} < \bar{b} \\ \frac{\bar{b}-\bar{a}}{2(w(\hat{b})-w(\hat{a}))} & \text{if } \underline{b} \leq \underline{a} \leq \bar{a} \leq \bar{b} \text{ and } w(\hat{b}) > w(\hat{a}) \\ 0.5 & \text{if } w(\hat{a}) = w(\hat{b}) \text{ and } \underline{a} = \underline{b}, \end{cases} \quad (1)$$

where "1^-" is a fuzzy number being less than 1, which linguistically indicates the fact that the interval $\hat{a}$ is weakly not greater than the interval $\hat{b}$.

The symbol "$\leq_I$" is an interval-valued version of the order relation "$\leq$" in the real number set $\Re$ and has the linguistic interpretation "essentially not greater than". The symbols "$\geq_I$" and "$=_I$" are similarly explained.

Obviously, the satisfactory index $\varphi(\hat{a} \leq_I \hat{b})$ is $0 \leq \varphi(\hat{a} \leq_I \hat{b}) \leq 1$. Thus, $\varphi(\hat{a} \leq_I \hat{b})$ may be interpreted as the satisfactory degree of the premise (or order relation) $\hat{a} \leq_I \hat{b}$. If $\varphi(\hat{a} \leq_I \hat{b}) = 0$, then the premise $\hat{a} \leq_I \hat{b}$ is not accepted. If $0 < \varphi(\hat{a} \leq_I \hat{b}) < 1$, then players accept the premise $\hat{a} \leq_I \hat{b}$ with different satisfactory degrees between 0 and 1. If $\varphi(\hat{a} \leq_I \hat{b}) = 1$, then players are absolutely satisfied with the premise $\hat{a} \leq_I \hat{b}$. Namely, the players believe that the premise $\hat{a} \leq_I \hat{b}$ is true.

According to Definition 1, we can find that while both the intervals entirely overlap, the satisfactory degree of the premise (or order relation) $\hat{a} \leq_I \hat{b}$ is equal to 0.5. Moreover, while both the intervals degenerate to the identical real number, the satisfactory degree is also equal to 0.5. And it is not difficult to find that the satisfactory degree is 1 where $\bar{a} < \underline{b}$. When the intervals is inclusion relation and $w(\hat{b}) > w(\hat{a})$, we can also easily obtain that the satisfactory degree is between 0 and 1 according to the above formula. With the condition $w(\hat{b}) > w(\hat{a})$, both the intervals are not reduced to the real numbers at the same time, and even $w(\hat{a})$ is equal to 0 or approaches to 0, the satisfactory degree is also between 0 and 1.

Analogously, we can define the following premise $\hat{a} \geq_I \hat{b}$ which indicates the statement "the interval $\hat{a}$ is not less than the interval $\hat{b}$".

Definition 2. Let $\hat{a} = [\underline{a}, \bar{a}]$ and $\hat{b} = [\underline{b}, \bar{b}]$ be two intervals. The premise "$\hat{a} \geq_I \hat{b}$" is regarded as a fuzzy set, whose membership function is defined as $\varphi(\hat{a} \geq_I \hat{b}) = 1 - \varphi(\hat{a} \leq_I \hat{b})$, i.e.,

$$\varphi(\hat{a} \geq_I \hat{b}) = \begin{cases} 0 & \text{if } \bar{a} < \underline{b} \\ 0^+ & \text{if } \underline{a} < \underline{b} \leq \bar{a} < \bar{b} \\ \frac{\underline{a} - \underline{b}}{2(w(\hat{b}) - w(\hat{a}))} & \text{if } \underline{b} \leq \underline{a} \leq \bar{a} \leq \bar{b} \text{ and } w(\hat{b}) > w(\hat{a}) \\ 0.5 & \text{if } w(\hat{a}) = w(\hat{b}) \text{ and } \underline{a} = \underline{b}, \end{cases} \quad (2)$$

where "0^+" is a fuzzy number being greater than 0, which linguistically indicates the fact that the interval $\hat{a}$ is weakly not less than the interval $\hat{b}$.

Thus, the equality relation "$=_I$" can be defined that $\hat{a} =_I \hat{b}$ is equivalent to both $\underline{a} = \underline{b}$ and $\bar{a} = \bar{b}$. Linguistically, "$\hat{a} =_I \hat{b}$" may be interpreted as "the interval $\hat{a}$ is equal to the interval $\hat{b}$" in the sense of Definitions 1 and 2. Moreover, $\hat{a} >_I \hat{b}$ if and only if $\hat{a} \geq_I \hat{b}$ and $\hat{a} \neq_I \hat{b}$. $\hat{a} <_I \hat{b}$ if and only if $\hat{a} \leq_I \hat{b}$ and $\hat{a} \neq_I \hat{b}$.

In the sequent, the above fuzzy ranking index φ is often called the satisfactory degree (or index). It is easy to prove that the satisfactory degree φ is continuous except a single special case, i.e., $\underline{a} = \underline{b}$ and $w(\hat{a}) = w(\hat{b})$. Moreover, for any intervals $\hat{a}$ and $\hat{b}$, we can easily prove that the following properties are valid:

(1) $0 \leq \varphi(\hat{a} \leq_I \hat{b}) \leq 1$;
(2) $\varphi(\hat{a} \leq_I \hat{a}) = 0.5$;
(3) $\varphi(\hat{a} \leq_I \hat{b}) + \varphi(\hat{b} \geq_I \hat{a}) = 1$;
(4) For any interval $\hat{c}$, if $\varphi(\hat{a} \leq_I \hat{b}) \geq 0.5$ and $\varphi(\hat{b} \leq_I \hat{c}) \geq 0.5$, then $\varphi(\hat{a} \leq_I \hat{c}) \geq 0.5$; or if $\varphi(\hat{a} \leq_I \hat{b}) \leq 0.5$ and $\varphi(\hat{b} \leq_I \hat{c}) \leq 0.5$, then $\varphi(\hat{a} \leq_I \hat{c}) \leq 0.5$.

Thus, "$\geq_I$" and "$\leq_I$" have well established fuzzy partial orders for intervals. Definitions 1 and 2 may provide quantitative methods to determine the exact degree of satisfactory for ranking two intervals. In the sequent, the satisfactory degree φ is used to define satisfactory crisp equivalent forms of interval-valued inequality relations.

3 Interval-Valued Cores and Solution Method for Interval-Valued Multiobjective n-Person Cooperative Games

3.1 The Concept of Interval-Valued Multiobjective n-Person Cooperative Games

An interval-valued multiobjective n-person cooperative games in coalitional form is an ordered pair $<N, \hat{\upsilon}>$, where $N=\{1,2,\ldots,n\}$ is the set of players, and $\hat{\upsilon}: 2^n \to (I(\Re))^m$ is the characteristic function vector which assigns to each coalition $S \in 2^n$ a closed interval $\hat{\upsilon}(S) \in (I(\Re))^m$, with given $\hat{\upsilon}(\varnothing) = [0,0]$. For each $S \in 2^n$, the worth set (or worth interval) $\hat{\upsilon}_k(S)$ of the coalition S for the objective O_k $(k = 1,2,\cdots,m)$ in the interval-valued multiobjective n-person cooperative games is a closed interval, which will be denoted by $[\underline{\upsilon}_k(S), \bar{\upsilon}_k(S)]$, where $\underline{\upsilon}_k(S)$ and $\bar{\upsilon}_k(S)$ are the lower and upper bounds of $\hat{\upsilon}_k(S)$, respectively.

The family of all interval-valued multiobjective n-person cooperative games with the player set N is denoted by IG^n. Note that if all the worth intervals are degenerate intervals, i.e., $\underline{\upsilon}_k(S) = \bar{\upsilon}_k(S)$, then the interval-valued multiobjective n-person cooperative games $<N, \hat{\upsilon}>$ is reduced to the classical multiobjective n-person cooperative games $<N, \upsilon>$, where $\hat{\upsilon}_k(S) = \upsilon_k(S)$. This means that traditional multiobjective n-person cooperative games can be embedded in the class of interval-valued multiobjective n-person cooperative games in a natural way. Alparslan-Gök et al. [14] confirmed that if all the worth intervals of an interval-valued n-person cooperative games $<N, \hat{\upsilon}>$ are degenerate intervals then strong balancedness is reduced to balancedness and strong unbalancedness is reduced to unbalancedness for the classical n-person cooperative games $<N, \upsilon>$, respectively.

3.2 Interval-Valued Cores for Interval-Valued Multiobjective n-Person Cooperative Games

For further use, we denote by IG^n the set of all n-dimensional vectors whose components are elements in $I(\Re)$. Let $\hat{x}_{ik}$ be the interval-valued payoff of the objective O_k for player i, and $\hat{\boldsymbol{x}}_k = (\hat{x}_{1k}, \hat{x}_{2k}, \ldots, \hat{x}_{nk})$ be an n-person interval-valued payoff vector of the objective O_k. Then, according to Moore [10], we have $\sum_{i \in S} \hat{x}_{ik} = [\sum_{i \in S} \underline{x}_{ik}, \sum_{i \in S} \bar{x}_{ik}] \in I(\Re)$ for each $S \in 2^n \backslash \varnothing$. Next, we define an interval-valued solution concept for interval-valued multiobjective n-person cooperative games $\hat{\upsilon} \in IG^n$. Instead of $\hat{\upsilon}_k(\{i\})$,$\hat{\upsilon}_k(\{i,j\})$, etc., we often write $\hat{\upsilon}_k(i)$,$\hat{\upsilon}_k(i,j)$, etc. Then an interval-valued imputation set $I(\hat{\upsilon})$ of the interval-valued multiobjective n-person cooperative games $\hat{\upsilon}$, is defined as follows:

$$I(\hat{\upsilon}) = \left\{ (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \ldots, \hat{\boldsymbol{x}}_m) \in (I(\Re))^{nm} \middle| \sum_{i \in N} \hat{x}_{ik} =_I \hat{\upsilon}_k(N), \hat{x}_{ik} \geq_I \hat{\upsilon}_k(i), \text{for all } i \in N \text{ and } k = 1,2,\ldots,m \right\}$$

Noting that $\sum_{i \in N} \hat{x}_{ik} =_I \hat{\upsilon}_k(N)$ is equivalent with $\sum_{i \in N} \bar{x}_{ik} =_I \bar{\upsilon}_k(N)$ and $\sum_{i \in N} \underline{x}_{ik} =_I \underline{\upsilon}_k(N)$, which are described as efficiency condition and shows that the sum of all the players' imputations should be equal to the value of the grand coalition N for the objective O_k. $\hat{x}_{ik} \geq_I \hat{\upsilon}_k(i)$ is described as individual rationality, which shows that each player's imputation should not be smaller than the payoff of the player alone for the objective O_k.

Generally, the interval-valued imputation set $I(\hat{\upsilon})$ is always non empty. More specifically, if an interval-valued multiobjective n-person cooperative games $\hat{\upsilon}$ is inessential, i.e., $\hat{\upsilon}_k(N) =_I \sum_{i \in N} \hat{\upsilon}_k(i)$, then the interval-valued imputation set $I(\hat{\upsilon})$ is no empty and singleton, i.e., $I(\hat{\upsilon}) = \{(\hat{\upsilon}_k(1), \hat{\upsilon}_k(2), \cdots, \hat{\upsilon}_k(n))\}$. Conversely, if an interval-valued multiobjective n-person cooperative games is essential, i.e., $\hat{\upsilon}_k(N) >_I \sum_{i \in N} \hat{\upsilon}_k(i)$, then players benefit from cooperation and the interval-valued imputation set $I(\hat{\upsilon})$ is always non empty which usually has infinite elements. Therefore, our interest focuses on essential interval-valued multiobjective n-person cooperative games.

Moreover, we can easily prove that interval-valued imputation sets of interval-valued multiobjective n-person cooperative games are convex. In fact, for any $\hat{x}' \in I(\hat{\upsilon})$, $\hat{x}'' \in I(\hat{\upsilon})$ and $\lambda \in [0, 1]$, we can easily check that

$$\sum_{i \in N} [\lambda \hat{x}'_{ik} + (1-\lambda)\hat{x}''_{ik}] = \lambda \sum_{i \in N} \hat{x}'_{ik} + (1-\lambda) \sum_{i \in N} \hat{x}''_{ik} =_I \lambda \hat{\upsilon}_k(N) + (1-\lambda)\hat{\upsilon}_k(N)$$
$$=_I \hat{\upsilon}_k(N)$$

and

$$\lambda \hat{x}'_{ik} + (1-\lambda)\hat{x}''_{ik} \geq_I \lambda \hat{\upsilon}_k(i) + (1-\lambda)\hat{\upsilon}_k(i) =_I \hat{\upsilon}_k(i).$$

In other words, $\lambda \hat{x}' + (1-\lambda)\hat{x}''$ satisfies the efficiency and individual rationality. Namely, $\lambda \hat{x}' + (1-\lambda)\hat{x}'' \in I(\hat{\upsilon})$. Therefore, the interval-valued imputation set $I(\hat{\upsilon})$ is convex.

Definition 3. The interval-valued core $C(\hat{\upsilon})$ of an interval-valued multiobjective n-person cooperative games $\hat{\upsilon}$, is defined as follows:

$$C(\hat{\upsilon}) = \left\{ (\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \ldots, \hat{\boldsymbol{x}}_m) \in (I(\Re))^{mn} \middle| \sum_{i \in N} \hat{x}_{ik} =_I \hat{\upsilon}_k(N), \sum_{i \in S} \hat{x}_{ik} \geq_I \hat{\upsilon}_k(S), \text{for all S} \subset N; k = 1, 2, \cdots, m \right\}$$

Here, $\sum_{i \in N} \hat{x}_{ik} =_I \hat{\upsilon}_k(N)$ is the efficiency condition and $\sum_{i \in S, S \neq N} \hat{x}_{ik} \geq_I \hat{\upsilon}_k(S)$ is the stability condition of the interval-valued payoff vectors. Clearly, due to $\{i\} \subseteq N$, then $\hat{x}_{ik} \geq_I \hat{\upsilon}_k(i)$. Therefore, we can deduct that $C(\hat{\upsilon}) \subseteq I(\hat{\upsilon})$ for each $\hat{\upsilon} \in IG^n$.

Alparslan-Gök et al. [14] consider that some basic properties of the interval-valued core are extensions of the corresponding properties of the core of traditional cooperative games. Specifically, the interval-valued core $C(\hat{\upsilon})$ is convex and relatively invariant with respect to strategic equivalence. Also, they claim that the interval-valued core $C(\hat{\upsilon})$ is non empty if the interval-valued multiobjective n-person cooperative games $\hat{\upsilon}$ is balanced.

Theoretically, according to the Definition 3, $C(\hat{\upsilon})$ may be computed by solving the system of interval-valued inequalities as follows:

$$\begin{cases} \sum\limits_{i \in S, S \neq N} \hat{x}_{ik} \geq_I \hat{\upsilon}_k(S), & (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m;\ S \subset N) \\ \sum\limits_{i \in N} \hat{x}_{ik} =_I \hat{\upsilon}_k(N), & (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m) \\ \bar{x}_{ik} \geq \underline{x}_{ik}, & (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m) \end{cases} \tag{3}$$

However, due to the fact that Eq. (3) involves interval comparison or ranking order of intervals and the preference of different objectives, it is very difficult to solve Eq. (3), which will be focused on the next section.

3.3 Solution Method for Interval-Valued Multiobjective n-Person Cooperative Games with Satisfactory Degrees of Comparing Intervals

By using the concept of satisfactory degree φ given above, we can establish the following satisfactory crisp equivalent forms of interval-valued inequality constraints, which will be used to construct auxiliary nonlinear programming models of interval-valued multiobjective *n*-person cooperative games.

Let $\alpha \in [0, 1]$ denote the satisfactory degree of the interval-valued inequality constraint which may be satisfied. For the situation $\bar{a} \leq \bar{b}$, $\underline{a} \geq \underline{b}$ and $w(\hat{b}) > w(\hat{a})$, from Definition 1, a satisfactory crisp equivalent form of an interval-valued inequality constraint $\hat{a} \leq_I \hat{b}$ is defined as follows:

$$\begin{cases} \bar{a} \leq \bar{b} \\ \underline{a} \geq \underline{b} \\ \varphi(\hat{a} \leq_I \hat{b}) \geq \alpha, \end{cases} \tag{4}$$

which can be further written as the following system of inequalities:

$$\begin{cases} \bar{a} \leq \bar{b} \\ \underline{a} \geq \underline{b} \\ (\bar{b} - \bar{a}) / [2(w(\hat{b}) - w(\hat{a}))] \geq \alpha. \end{cases} \tag{5}$$

It is easy to see from Eq. (5) that $w(\hat{b}) > w(\hat{a})$ due to $\bar{a} \leq \bar{b}$ and $\alpha \in [0, 1]$.

Similarly, for the situation $\bar{a} \leq \bar{b}$, $\underline{a} \geq \underline{b}$ and $w(\hat{b}) > w(\hat{a})$ from Definition 2, a satisfactory crisp equivalent form of an interval-valued inequality constraint $\hat{a} \geq_I \hat{b}$ is defined as follows:

$$\begin{cases} \underline{a} \geq \underline{b} \\ \bar{a} \leq \bar{b} \\ \varphi(\hat{a} \geq_I \hat{b}) \geq \alpha, \end{cases} \tag{6}$$

which can be further written as the following system of inequalities:

$$\begin{cases} \underline{a} \geq \underline{b} \\ \bar{a} \leq \bar{b} \\ (\underline{a} - \underline{b})/[2(w(\hat{b}) - w(\hat{a}))] \geq \alpha. \end{cases} \tag{7}$$

It is easy to see from Eq. (7) that $w(\hat{b}) > w(\hat{a})$ due to $\underline{a} \geq \underline{b}$ and $\alpha \in [0, 1]$.

Analogously, for other situations $\bar{a} < \underline{b}$, $\underline{a} < \underline{b} \leq \bar{a} < \bar{b}$ or $w(\hat{a}) = w(\hat{b})$ and $\underline{a} = \underline{b}$, we can respectively obtain the satisfactory crisp equivalent forms of interval-valued inequality constraint $\hat{a} \leq_I \hat{b}$ according to Definition 1 (omitted).

In the sequent, we focus on using Eq. (6) (or Eq. (7)) to establish the auxiliary nonlinear programming model for Eq. (3).

For any coalitions $S \subset N$, let $\alpha_{Sk} = \varphi(\sum_{i \in S} \hat{x}_{ik} \geq_I \hat{\upsilon}_k(S))$ denote the satisfactory degree of the interval-valued inequality $\sum_{i \in S} \hat{x}_{ik} \geq_I \hat{\upsilon}_k(S)$ which may be satisfied.

For the situation $\sum_{i \in S} \underline{x}_{ik} \geq \underline{\upsilon}_k(S)$ and $\sum_{i \in S} \bar{x}_{ik} \leq \bar{\upsilon}_k(S)$ $(S \subset N)$, according to Eq. (6) and the above discussion on the "$=_I$", we propose a two-phase approach for solving the interval-valued core of the interval-valued multiobjective n-person cooperative games.

In the first phase, the satisfactory crisp equivalent mathematical programming model for Eq. (3) can be constructed as follows:

$$\max\{\max_{1 \leq k \leq m} \min_{S \subset N}\{\alpha_{Sk}\}\}$$
$$\text{s.t.} \begin{cases} \sum_{i \in S} \underline{x}_{ik} \geq \underline{\upsilon}_k(S) \quad (S \subset N; k = 1, 2, \cdots, m) \\ \sum_{i \in S} \bar{x}_{ik} \leq \bar{\upsilon}_k(S) \quad (S \subset N; k = 1, 2, \cdots, m) \\ \alpha_{Sk} = \varphi(\sum_{i \in S} \hat{x}_{ik} \geq_I \hat{\upsilon}_k(S)) \quad (S \subset N; k = 1, 2, \cdots, m) \\ \sum_{i \in N} \hat{x}_{ik} =_I \hat{\upsilon}_k(N) \ (k = 1, 2, \cdots, m) \\ \bar{x}_{ik} \geq \underline{x}_{ik} \ (i = 1, 2, \cdots, n; k = 1, 2, \cdots, m) \end{cases} \tag{8}$$

Let $\beta = \max_{1 \le k \le m} \min_{S \subset N}\{\alpha_{Sk}\}$. Then, $0 \le \beta \le 1$. Thereby, according to Definition 2, Eq. (8) can be rewritten as the following mathematical programming model:

$$\max\{\beta\}$$
$$\text{s.t.}\begin{cases} \sum_{i \in S} \underline{x}_{ik} \ge \underline{\upsilon}_k(S) \quad (S \subset N; k = 1, 2, \ldots, m) \\ \sum_{i \in S} \bar{x}_{ik} \le \bar{\upsilon}_k(S) \quad (S \subset N; k = 1, 2, \ldots, m) \\ (1-\beta)\sum_{i \in S} \underline{x}_{ik} + \beta \sum_{i \in S} \bar{x}_{ik} \ge (1-\beta)\underline{\upsilon}_k(S) + \beta\bar{\upsilon}_k(S) \quad (S \subset N; k = 1, 2, \ldots, m) \\ \sum_{i \in N} \bar{x}_{ik} = \bar{\upsilon}_k(N) \quad (k = 1, 2, \ldots, m) \\ \sum_{i \in N} \underline{x}_{ik} = \underline{\upsilon}_k(N) \quad (k = 1, 2, \ldots, m) \\ \bar{x}_{ik} \ge \underline{x}_{ik}, (i = 1, 2, \ldots, n; k = 1, 2, \ldots, m) \\ 0 \le \beta \le 1 \end{cases} \tag{9}$$

where β, $\bar{x}_{ik}$ and $\underline{x}_{ik}$ $(i = 1, 2, \cdots, n; k = 1, 2, \cdots, m)$ are decision variables to be determined.

Solving Eq. (9) by the bisection method and algorithms in the following section, we can obtain its solution of this nonlinear programming model, denoted by $(\beta^*, \hat{\mathbf{x}}^*)$.

In the second phase, we construct the following mathematical programming model:

$$\max\{\sum_{k=1}^{m} \sum_{S \subset N} \omega_k \alpha_{Sk}\}$$
$$\text{s.t.}\begin{cases} \sum_{i \in S} \underline{x}_{ik} \ge \underline{\upsilon}_k(S) \quad (S \subset N; k = 1, 2, \cdots, m) \\ \sum_{i \in S} \bar{x}_{ik} \le \bar{\upsilon}_k(S) \quad (S \subset N; k = 1, 2, \cdots, m) \\ \alpha_{Sk} = (\sum_{i \in S} \underline{x}_{ik} - \underline{\upsilon}_k(S))/[2(w(\hat{\upsilon}_k(S) - w(\sum_{i \in S} \hat{x}_{ik}))] \quad (S \subset N; k = 1, 2, \cdots, m) \\ \alpha_{Sk} \ge \beta^* \quad (S \subset N; k = 1, 2, \cdots, m) \\ \sum_{i \in N} \bar{x}_{ik} = \bar{\upsilon}_k(N) \quad (k = 1, 2, \cdots, m) \\ \sum_{i \in N} \underline{x}_{ik} = \underline{\upsilon}_k(N) \quad (k = 1, 2, \cdots, m) \\ \bar{x}_{ik} \ge \underline{x}_{ik} \ (i = 1, 2, \cdots, n; k = 1, 2, \cdots, m) \end{cases} \tag{10}$$

where ω_k is the weight of the objective O_k, which satisfies the normalized conditions: $\omega_k \ge 0$ $(k = 1, 2, \cdots, m)$ and $\sum_{k=1}^{m} \omega_k = 1$; $\alpha_{Sk} \in [0, 1]$, $\bar{x}_{ik}$ and $\underline{x}_{ik}$ $(S \subset N, i = 1, 2, \cdots, n, k = 1, 2, \cdots, m)$ are decision variables need to be determined. And Eq. (10) may be rewritten as the following nonlinear programming model:

$$\max\{\sum_{k=1}^{m}\sum_{S\subseteq N,S\neq N}\omega_k\alpha_{Sk}\}$$
$$\text{s.t.}\begin{cases}\sum_{i\in S}\underline{x}_{ik}\geq\underline{\upsilon}_k(S)\quad(S\subset N;k=1,2,\ldots,m)\\ \sum_{i\in S}\bar{x}_{ik}\leq\bar{\upsilon}_k(S)\quad(S\subset N;k=1,2,\ldots,m)\\ (1-\alpha_{Sk})\sum_{i\in S}\underline{x}_{ik}+\alpha_{Sk}\sum_{i\in S}\bar{x}_{ik}=(1-\alpha_{Sk})\underline{\upsilon}_k(S)+\alpha_{Sk}\bar{\upsilon}_k(S)\quad(S\subset N;k=1,2,\ldots,m)\\ \sum_{i\in N}\bar{x}_{ik}=\bar{\upsilon}_k(N)\quad(k=1,2,\ldots,m)\\ \sum_{i\in N}\underline{x}_{ik}=\underline{\upsilon}_k(N)\quad(k=1,2,\ldots,m)\\ \alpha_{Sk}\geq\beta^*\quad(S\subset N;k=1,2,\ldots,m)\\ \bar{x}_{ik}\geq\underline{x}_{ik},\ (i=1,2,\ldots,n;k=1,2,\ldots,m)\end{cases}\tag{11}$$

where $\alpha_{Sk}\in[0,1]$, $\bar{x}_{ik}$ and $\underline{x}_{ik}$ $(i=1,2,\cdots,n;k=1,2,\cdots,m)$ are decision variables need to be determined.

Solving Eq. (11) with the given weight of objectives, we can obtain its solution, denoted by $\left(\alpha^*_{Sk},\hat{\mathbf{x}}^{**}\right)$. Thus, $\hat{\mathbf{x}}^{**}$ is an element of the interval-valued core of the interval-valued multiobjective n-person cooperative games with the maximum satisfactory degree α^*_{Sk}.

Obviously, if $\alpha^*_{Sk}=1$, then we can obtain the element of the interval-valued core in which the satisfactory degree is equal to 1 for the coalition S and the objective O_k

Analogously, for the situations $w(\sum_{i\in S}\hat{x}_{ik})=w(\hat{\upsilon}_k(S))$ and $\sum_{i\in S}\underline{x}_{ik}=\underline{\upsilon}_k(S)$, $\sum_{i\in S}\underline{x}_{ik}<\underline{\upsilon}_k(S)\leq\sum_{i\in S}\bar{x}_{ik}<\bar{\upsilon}_k(S),\sum_{i\in S}\bar{x}_{ik}<\underline{\upsilon}_k(S)(S\subset N;k=1,2,\ldots,m)$, according to Definition 2 and the above discussion on the "$=_I$", the satisfactory crisp equivalent form of Eq. (3) can be discussed.

3.4 Algorithms for Solving Interval-Valued *n*-Person Cooperative Games with Satisfactory Degrees

By using the bisection method [17], we can obtain the global optimal solution of Eq. (9), denoted by $(\beta^*,\hat{\mathbf{x}}^*)$ in the first phase.

Obviously, if $\beta^*=1$, then we can obtain the global optimal solution of Eq. (9) in which the satisfactory degree is equal to 1.

The bisection procedures and algorithms which can be used to estimate the global optimal solution of Eq. (9) at a given precision $\varepsilon\in(0,1]$ (hereby the number of the iteration is the positive integer m_0, which is not smaller than $-\ln\varepsilon/\ln 2$) are summarized as follows:

Step 1: Let $t = 0$, and take $\bar{\beta}_t = 1$ and this nonlinear programming problem (i.e., Eq. (9)) can be transformed to the linear programming. Solving Eq. (9) with $\bar{\beta}_t = 1$ by using the LINGO tool (or the simplex method of linear programming), if we can obtain its feasible solution $\hat{\mathbf{x}}_t^*$, then $\beta^* = \bar{\beta}_t = 1$ is the optimal value of the objective function of Eq. (9). The algorithm stops. On the contrary, if there is not any feasible solution, go to Step 2.

Step 2: Take $\underline{\beta}_t = 0$ and solve Eq. (9) by using the LINGO tool (or the simplex method of linear programming), if there is not any feasible solution, which means that this linear programming problem (hereby Eq. (9)) has no solutions, then the algorithm stops. On the contrary, if we can obtain its feasible solution $\hat{\mathbf{x}}_t^*$, then we can determine that the optimal value of the objective function of Eq. (9) is between 0 and 1 (i.e., $\beta^* \in (0, 1)$), go to Step 3.

Step 3: According to the bisection method, and let $m(\hat{\beta}_t)$ be the mean of the lower bound $\underline{\beta}_t$ and the upper bound $\bar{\beta}_t$ of the interval $\hat{\beta}_t = [\underline{\beta}_t, \bar{\beta}_t]$. Namely, $m(\hat{\beta}_t) = (\underline{\beta}_t + \bar{\beta}_t)/2 = (0+1)/2 = 0.5$, then we solve Eq. (9) by using the LINGO tool (or the simplex method of linear programming). If there is not any feasible solution, then the optimal value of the objective function of Eq. (9) falls into the range which is between the lower bound $\underline{\beta}_t$ and the mean $m(\hat{\beta}_t)$ of the interval $\hat{\beta}_t$ (i.e., $\beta^* \in (\underline{\beta}_t, m(\hat{\beta}_t)) = (0, 0.5)$), thereby the interval $\hat{\beta}_t$ is narrowed. Let $\bar{\beta}_{t+1} = m(\hat{\beta}_t) = 0.5$ and $\underline{\beta}_{t+1} = \underline{\beta}_t = 0$, then go to Step 4. On the contrary, if we can obtain the feasible solution $\hat{\mathbf{x}}_t^*$, then the optimal value of the objective function of Eq. (9) falls into the range which is between the mean $m(\hat{\beta}_t)$ and the upper bound $\bar{\beta}_t$ of the interval $\hat{\beta}_t$ (i.e., $\beta^* \in (m(\hat{\beta}_t), \bar{\beta}_t) = (0.5, 1)$), thereby the interval $\hat{\beta}_t$ is narrowed also. Let $\underline{\beta}_{t+1} = m(\hat{\beta}_t) = 0.5$ and $\bar{\beta}_{t+1} = \bar{\beta}_t = 1$, then go to Step 4.

Step 4: Let $t := t + 1$, and repeat Step 3 in the new smaller interval $\hat{\beta}_t = [\underline{\beta}_t, \bar{\beta}_t]$ until the m_0-th iteration. Then, go to Step 5.

Step 5: The length of the narrowed interval $\hat{\beta}_{m_0} = [\underline{\beta}_{m_0}, \bar{\beta}_{m_0}]$ of the m_0-th iteration is not greater than the given precision ε. Let $\beta^* = (\underline{\beta}_{m_0} + \bar{\beta}_{m_0})/2$, which is the mean of the lower and upper bounds of the interval $\hat{\beta}_{m_0}$. Namely, β^* is the optimal value of the objective function of Eq. (9) at a given precision ε.

4 A Numerical Example

Suppose that there are three business companies in the electronic product supply chain aiming to cooperation to develop a new type of electronic production. Each company has different superior resources and can't produce alone. All companies not only care for their short-term profits in the process of profit distribution, but also many other elements, such as the degree of technology spillover, production efficiency, product industrialization time, development risk and so on. For simplicity sake, we only consider two objectives including the short-term profits and degree of technology spillover in this paper. Due to a lack of information or imprecision of the available information, the managers of these three companies usually are not able to exactly forecast the payoffs of the companies' product cooperative innovation. Usually, business companies only can predict the optimistic and the pessimistic payoffs of cooperation. Hence, intervals are suitable to represent the payoffs from three companies' perspectives. This problem may be regarded as an interval-valued bi-objective 3-person cooperative games. Namely, these three business companies may be regarded as players 1, 2 and 3, respectively. Let $<\{1,2,3\},\hat{\upsilon}>$ be denoted by this interval-valued bi-objective 3-person cooperative games with the characteristics function vector-valued just as followed:

$$\begin{cases} \hat{\upsilon}(1,2) = ([22,30],[40,60])^{\mathrm{T}} \\ \hat{\upsilon}(1,3) = ([24,28],[20,30])^{\mathrm{T}} \\ \hat{\upsilon}(2,3) = ([20,32],[16,44])^{\mathrm{T}} \\ \hat{\upsilon}(1,2,3) = ([40,44],[61,66])^{\mathrm{T}} \\ \hat{\upsilon}(1) = \hat{\upsilon}(2) = \hat{\upsilon}(3) = (0,0)^{\mathrm{T}} \end{cases}$$

where the components $\hat{\upsilon}_1(1,2) = [22,30]$ and $\hat{\upsilon}_2(1,2) = [40,60]$ of $\hat{\upsilon}(1,2) = ([22,30],[40,60])^{\mathrm{T}}$ denote characteristics function value for coalition $\{1,2\}$ to obtain the short-term profit and degree of technology spillover objectives, respectively. Other vector-valued can be similarly understood.

4.1 Computational Results Obtained by the Proposed Method

According to Eq. (9), the nonlinear programming model in the first phase can be constructed and solved by the above bisection method, then we can narrow the range of β constantly and infer that $\beta \in (0.875, 0.8750625)$. Therefore, the global optimal solution $(\beta^*, \hat{\mathbf{x}}^*)$ at a given precision in the first phase can be estimated, where $\beta^* = 0.875$, $\hat{x}_1^* = ([9.5,13.5],[19,24])^{\mathrm{T}}$, $\hat{x}_2^* = ([16,16],[36,36])^{\mathrm{T}}$ and $\hat{x}_3^* = ([14.5,14.5],[6,6])^{\mathrm{T}}$.

According to Eq. (11) in the second phase, the nonlinear programming model can be constructed. Suppose that these three business companies agree that the short-term profit is more important than the degree of technology spillover in the process of profit distribution, and let $\omega_1 = 0.8$ and $\omega_2 = 0.2$. We can obtain the optimal solution $(\alpha_{Sk}^*, \hat{\mathbf{x}}^{**})$, where $\alpha_{\{1,2\}1}^* = 0.875$, $\alpha_{\{1,3\}1}^* = 0.875$, $\alpha_{\{2,3\}1}^* = 0.875$, $\alpha_{\{1,2\}2}^* = 1$, $\alpha_{\{1,3\}2}^* = 1$, $\alpha_{\{2,3\}2}^* = 0.929$, $\hat{x}_1^{**} = ([12.290, 13.101],[19,24])^{\mathrm{T}}$, $\hat{x}_2^{**} = ([13.210, 16.399],[36,36])^{\mathrm{T}}$ and $\hat{x}_3^{**} = ([14.5,14.5],[6,6])^{\mathrm{T}}$.

Thus, we obtain an element $\hat{\mathbf{x}}^{**}$ of the interval-valued core $C(\hat{\upsilon})$ of the interval-valued bi-objective 3-person cooperative games with the maximum satisfactory degree α_{S1}^* of the short-term profit objective and α_{S2}^* of the degree of technology spillover objective. In other words, if the satisfactory degree of $\sum\limits_{i\in S, S\neq N} \hat{x}_{ik} \geq_I \hat{\upsilon}_k(S)$ for these three business companies is not greater than α_{Sk}^*, the interval-valued core of the interval-valued bi-objective 3-person cooperative games exists, and hereby these three companies may choose cooperative innovation.

Furthermore, using the nonlinear programming model (i.e., Eq. (11)), we also can obtain the interval-valued core of this interval-valued bi-objective 3-person cooperative games with players' different weight preference, where $\omega_k \in [0, 1]$, $\sum \omega_k = 1$, $k = 1, 2$, depicted as in Table 1.

Table 1. Interval-valued cores of the interval-valued bi-objective 3-person cooperative games

(ω_1, ω_2)	$\alpha^*_{\{1,2\}1}$	$\alpha^*_{\{1,3\}1}$	$\alpha^*_{\{2,3\}1}$	$\alpha^*_{\{1,2\}2}$	$\alpha^*_{\{1,3\}2}$	$\alpha^*_{\{2,3\}2}$	$\hat{x}_1^{**T}$	$\hat{x}_2^{**T}$	$\hat{x}_3^{**T}$
(0,1)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(0.1,0.9)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(0.2,0.8)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(0.3,0.7)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(0.4,0.6)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(0.5,0.5)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(0.6,0.4)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(0.7,0.3)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(0.8,0.2)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(0.9,0.1)	0.875	0.875	0.875	1	1	0.929	([12.290, 13.101], [19, 24])	([13.210, 16.399], [36, 36])	([14.5, 14.5], [6, 6])
(1,0)	0.875	0.875	0.875	0.908	0.875	1	([12.290, 13.101], [17, 22])	([13.210, 16.399], [36.625, 36.625])	([14.5, 14.5], [7.375, 7.375])

Here, we observe that if these three business companies are absolutely concerned about the short-term profit in the process of profit distribution, then they will choose $\omega_1 = 1$, $\omega_2 = 0$ and get the final solution, where $\hat{x}_1^{**} = ([12.290, 13.101], [17, 22])^{\mathrm{T}}$, $\hat{x}_2^{**} = ([13.210, 16.399], [36.625, 36.625])^{\mathrm{T}}$ and $\hat{x}_3^{**} = ([14.5, 14.5], [7.375, 7.375])^{\mathrm{T}}$. Similarly if these three business companies are absolutely concerned about the degree of technology spillover in the process of profit distribution, then they will choose $\omega_1 = 0$, $\omega_2 = 1$ and get the final distribution solution. Further, $\omega_1 = 0.5$ and $\omega_2 = 0.5$ represent the neutral or indifference scenario of these three business companies.

Analogously, for the situations $w(\sum_{i \in S} \hat{x}_{ik}) = w(\hat{v}_k(S))$ and $\sum_{i \in S} \underline{x}_{ik} = \underline{v}_k(S)$, $\sum_{i \in S} \underline{x}_{ik} < \underline{v}_k(S) \leq \sum_{i \in S} \bar{x}_{ik} < \bar{v}_k(S), \sum_{i \in S} \bar{x}_{ik} < \underline{v}_k(S) (S \subset N; k = 1, 2, \ldots, m)$, we find there is no feasible solution of these situations and hereby these three companies may have not any cooperative desire.

These results are consistent with the reality, where companies not only consider the short-term profit, but also care for other factors with different weight preferences in the process of the decision making about cooperative innovation.

4.2 Computational Results with the LR Method

According to Eq. (3), we construct the linear programming model and we find that there is no feasible solution by using the LR method (i.e., ranking relation: if $\underline{a} \leq \underline{b}$ and $\bar{a} \leq \bar{b}$, then $\hat{a} \leq \hat{b}$). Hereby, these three companies may have not any desire for cooperative innovation.

Obviously, it is shown that there is not feasible solution with traditional LR interval ranking method. On the contrary, we can obtain the alternative feasible solution by introducing the satisfactory degrees of comparing intervals, which can give more scientific suggestions for players (or managers).

Therefore, it is shown that the traditional LR ranking method is relatively strict, which may affect the cooperative desire and the decision making in the real situation with interval-valued payoffs. Moreover, we can easily obtain the maximum satisfactory degrees that companies accept and corresponding feasible solution with this new two-phase approach. It can give more scientific suggestions for decision makers. These conclusions agree with the actual situation as expected. On the other hand, it is shown that it is necessary to consider the special conditions and different possibility in real situations.

5 Conclusion

Interval-valued multiobjective n-person cooperative games can provide a basic conceptual framework for formulating and analyzing cooperative decision problems. In this paper, we introduce a satisfactory degree of comparing intervals including the feature of inclusion and/or overlap relations, then hereby propose a new two-phase nonlinear programming models and bisection solving method of interval-valued cores for any interval-valued multiobjective n-person cooperative games. It is shown that the method of interval ranking order is very important, which can give more scientific suggestions for decision makers. Furthermore, this approach takes into full account the internal/external fuzzy decision making environment, including many different objectives and weight preference.

It is obvious that interval-valued multiobjective n-person cooperative games is special case of multiobjective n-person cooperative games. In fact, if all the interval value degenerate to the real number, i.e.,$\upsilon_{ik} = \bar{\upsilon}_{ik} = \underline{\upsilon}_{ik}$, then the interval-valued multiobjective n-person cooperative games are reduced to the classical multiobjective n-person cooperative games.

Obviously, although we propose the new two-phase method for solving the interval-valued core $C(\hat{\upsilon})$ with the maximum satisfactory degree, the $C(\hat{\upsilon})$ of the interval-valued multiobjective n-person cooperative games maybe empty or non-unique which is the same as that of the classical multiobjective n-person cooperative games. Moreover, intervals are just a special case of fuzzy number and core is one of solution of multiobjective n-person cooperative games. In reality, there are various forms of fuzzy numbers such as trapezoidal fuzzy number, triangular fuzzy number. And, there are various forms of solutions such as Shapley value, stable sets and τ value. Also, the coalitions in the interval-valued multiobjective n-person cooperative games may be restricted and complex. These are the further study directions in the future.

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Models and Algorithms for Least Square Interval-Valued Nucleoli of Cooperative Games with Interval-Valued Payoffs

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Abstract. The aim of this paper is to develop a new method for computing least square interval-valued nucleoli of cooperative games with interval-valued payoffs, which usually are called interval-valued cooperative games for short. In this methodology, based on the square excess which can be intuitionally interpreted as a measure of the dissatisfaction of the coalitions, we construct a quadratic programming model for least square interval-valued prenucleolus of any interval-valued cooperative game and obtain its analytical solution, which is used to determine players' interval-valued imputations via the designed algorithms that ensure the nucleoli always satisfy the individual rationality of players. Hereby the least square interval-valued nucleoli of interval-valued cooperative games are determined in the sense of minimizing the difference of the square excesses of the coalitions. Moreover, we discuss some useful and important properties of the least square interval-valued nucleolus such as its existence and uniqueness, efficiency, individual rationality, additivity, symmetry, and anonymity.

Keywords: Game algorithm · Cooperative game · Interval computing · Quadratic programming · Optimization model

1 Introduction

Due to uncertainty and imprecision in real economic management situations, player coalitions' values usually have to be estimated. Recently, intervals seem to be suitable for employing to deal with inherited imprecision or vagueness in coalitions' values and hereby there appears an important type of cooperative games with interval-valued data, which often are called interval-valued cooperative games for short [1, 2]. A good example may be the interval bankruptcy games with interval claims [3]. Specifically, Branzei et al. [3] introduced interval-valued cooperative games which are used to handle bankruptcy situations where the estate is known with certainty while claims belong to known bounded intervals of real numbers and hereby defined two Shapley-like values for solving the interval-valued cooperative games. Obviously, interval-valued cooperative games are remarkably different from classical cooperative games from the point of view of the data type of the player coalitions' values.

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The coalitions' values of interval-valued cooperative games are expressed with intervals but that of classical cooperative games are expressed with real numbers [4, 5].

Lately, interval-valued cooperative games have attracted attention of researchers and their solution concepts have applied to many fields such as business [6], operations research [7], economy, modern finance, climate negotiations and policy, tourism management [2], environmental management, and pollution control. To be more precise, Branzei et al. [3] firstly defined two Shapley-like values, which associate vectors of intervals with interval-valued cooperative games of the interval bankruptcy problems with interval claims, and studied the interrelations among using the arithmetic of intervals [8]. To place the models of interval-valued cooperative games within the cooperative game theory and to motivate continued interest in theory and application development, Branzei et al. [1] gave a good survey that discussed how the models of interval-valued cooperative games extended the cooperative game theory, and reviewed their existing and potential applications in economic management and business situations with interval data. Alparslan Gök et al. [4] studied the properties of the interval-valued Shapley value on the class of size monotonic interval-valued cooperative games and gave an axiomatic characterization of the interval-valued Shapley value on a special subclass of interval-valued cooperative games. Kimms and Drechsel [6] proposed a general mathematical programming algorithm which can be used to find an element in the interval-valued core. Hong and Li [9] constructed an auxiliary nonlinear programming model and hereby proposed a corresponding effective bisection method for computing elements of interval-valued cores of interval-valued n-person cooperative games by introducing the satisfaction degree index (or fuzzy ranking index) of interval comparison. Theoretically, Branzei et al. [10] defined the interval-valued cores of interval-valued cooperative games through discussing the interval-valued square dominance and interval-valued dominance imputations. Alparslan Gök et al. [11] introduced some set-valued solution concepts of interval-valued cooperative games, which include the interval-valued core, the interval-valued dominance core, and the interval-valued stable sets. Alparslan Gök et al. [12] extended the classical two-person cooperative game theory to two-person cooperative games with interval data and studied the interval-valued core, balancedness, superadditivity, and some other properties.

However, it is easy to find that most of the aforementioned works used the Moore's interval operations [8], especially the Moore's interval subtraction, which usually enlarges uncertainty of the resulted interval. This case usually is not accordant with real economic management situations. Therefore, the aim of this paper is to develop simple and effective quadratic programming methods for solving interval-valued cooperative games. More precisely, based on the differences of the square excesses of the player coalitions, we construct two quadratic programming models and obtain their analytical solutions, i.e., least square interval-valued prenucleoli and nucleoli, which are used to determine players' interval-valued imputations through using the designed algorithms which ensure that they satisfy the individual rationality of players. Hereby, the least square interval-valued prenucleoli and nucleoli of interval-valued cooperative games are determined in the sense of minimizing the difference of the square excesses of the player coalitions. The quadratic programming methods proposed in this paper are remarkably different from the aforementioned methods. On the one hand, the developed methods can provide analytical formulae for determining the least square interval-valued prenucleoli and nucleoli of interval-valued cooperative games and hereby obtain the interval-valued imputations of players. On the other hand, the developed methods can effectively avoid the Moore's interval subtraction [8].

The rest of this paper is arranged as follows. In the next section, we briefly review the interval-valued cooperative games and their solution concepts and hereby define the square excesses of players' coalitions on interval-valued payoff vector to measure the dissatisfaction of the coalitions. In Sect. 3, we construct two quadratic programming models based on the square excesses of the player coalitions to compute the least square interval-valued prenucleoli and nucleoli of interval-valued cooperative games. The effective algorithm is designed to determine players' interval-valued imputations through considering the individual rationality of players. Furthermore, we discuss some important and useful properties of the least square interval-valued prenucleoli and nucleoli of interval-valued cooperative games. The quadratic programming models and algorithms are illustrated with a numerical example about the optimal allocation of the cooperative profits of joint production and the computational result is analyzed in Sect. 4. The validity, applicability, and advantages of the methods proposed in this paper are shown and some remarks on further research are discussed in the last section.

2 Notations of Intervals and Interval-Valued Cooperative Games

2.1 Interval Notations and Arithmetic Operations

To facilitate introducing interval-valued cooperative games, we firstly review the concepts of intervals and their distances as well as interval arithmetic operations.

Usually, $\bar{a} = [a_L, a_R] = \{a | a \in \mathrm{R}, a_L \leq a \leq a_R\}$ is used to express an interval, where R is the set of real numbers, $a_L \in \mathrm{R}$ and $a_R \in \mathrm{R}$ are called the lower bound and the upper bound of the interval $\bar{a}$, respectively. Let $\bar{R}$ be the set of intervals on the set R.

Clearly, if $a_L = a_R$, then the interval $\bar{a} = [a_L, a_R]$ degenerates to a real number, denoted by a, where $a = a_L = a_R$. Conversely, a real number a may be written as an interval $\bar{a} = [a, a]$. Therefore, intervals are a generalization of real numbers. That is to say, real numbers are a special case of intervals [2, 8].

If $a_L \geq 0$, then $\bar{a} = [a_L, a_R]$ is called a non-negative interval, denoted by $\bar{a} \geq 0$. Likewise, if $a_R \leq 0$, then $\bar{a}$ is called a non-positive interval, denoted by $\bar{a} \leq 0$. If $a_L > 0$, then $\bar{a}$ is called a positive interval, denoted by $\bar{a} > 0$. If $a_R < 0$, then $\bar{a}$ is called a negative interval, denoted by $\bar{a} < 0$.

To facilitate the sequent discussion, we briefly review arithmetic operations of intervals such as the equality, the addition, and the scalar multiplication [2, 8, 13].

Assume that $\bar{a} = [a_L, a_R]$ and $\bar{b} = [b_L, b_R]$ be two intervals on the set $\bar{R}$. Then, $\bar{a}$ is equal to $\bar{b}$ if and only if $a_L = b_L$ and $a_R = b_R$, denoted by $\bar{a} = \bar{b}$.

$$\bar{a} + \bar{b} = [a_L + b_L, a_R + b_R]. \tag{1}$$

The scalar multiplication of a real number $\gamma \in R$ and an interval $\bar{a}$ is defined as follows:

$$\gamma\bar{a} = \begin{cases} [\gamma a_L, \gamma a_R] & (\gamma \geq 0) \\ [\gamma a_R, \gamma a_L] & (\gamma < 0) \end{cases} \tag{2}$$

Clearly, the above arithmetic operations of intervals are a generalization of those of real numbers.

In most management situations, we usually have to compare or rank intervals. However, ranking intervals or interval comparison is a difficult and an important problem. In a parallel way to comparison of the real numbers, Moore [8] firstly proposed the order relation between intervals as follows:

$$\bar{a} \leq \bar{b} \text{ if and only if } a_L \leq b_L \text{ and } a_{\mathrm{R}} \leq b_R, \tag{3}$$

which is simply called the Moore's order relation between intervals.

To measure differences between intervals, we give the distance concept as follows.

Definition 1. Assume that $\bar{a}$ and $\bar{b}$ be two intervals on the set $\bar{R}$. If a mapping $d: \bar{R} \times \bar{R} \mapsto \mathrm{R}$ satisfies the three properties (1)–(3) as follows:

(1) Non-negativity: $d(\bar{a}, \bar{b}) \geq 0$;
(2) Symmetry: $d(\bar{a}, \bar{b}) = d(\bar{b}, \bar{a})$;
(3) Trigonometrical inequality relation: $d(\bar{a}, \bar{b}) \leq d(\bar{a}, \bar{c}) + d(\bar{c}, \bar{b})$ for any interval $\bar{c}$ on the set $\bar{R}$, then $d(\bar{a}, \bar{b})$ is called the distance between the intervals $\bar{a}$ and $\bar{b}$.

It is easy to see from Definition 1 that the distance between intervals is a natural generalization of that of the set of real numbers.

Obviously, there are various forms of distances between intervals which satisfy Definition 1. For instance, to meet the need of modeling interval-valued cooperative games in the subsequent sections, we define the distance between two intervals $\bar{a} \in \bar{R}$ and $\bar{b} \in \bar{R}$ as follows:

$$D(\bar{a}, \bar{b}) = (a_L - b_L)^2 + (a_R - b_R)^2. \tag{4}$$

It is easy to see that Eq. (4) is very similar to the distance between two points in the two-dimension Euclidean space.

Theorem 1. $D(\bar{a}, \bar{b})$ defined by Eq. (4) is the distance between the intervals $\bar{a} \in \bar{R}$ and $\bar{b} \in \bar{R}$.

Proof. We need to validate that $D(\bar{a}, \bar{b})$ defined by Eq. (4) satisfies the three properties (1)–(3) of Definition 1, respectively.

It is easy to see from Eq. (4) that $D(\bar{a}, \bar{b}) \geq 0$ and $D(\bar{a}, \bar{b}) = D(\bar{b}, \bar{a})$ for any intervals $\bar{a}$ and $\bar{b}$. Namely, $D(\bar{a}, \bar{b})$ satisfies the properties (1) and (2) of Definition 1.

For any interval $\bar{c}$ on the set $\bar{R}$, where $\bar{c} = [c_L, c_R]$, it is easily derived from Eq. (4) that

$$\begin{aligned} D(\bar{a},\bar{b}) =&(a_L-b_L)^2+(a_R-b_R)^2 \\ &\leq [(a_L-c_L)^2+(c_L-b_L)^2]+[(a_R-c_R)^2+(c_R-b_R)^2] \\ &=[(a_L-c_L)^2+(a_R-c_R)^2]+[(c_L-b_L)^2+(c_R-b_R)^2] \\ &=D(\bar{a},\bar{c})+D(\bar{c},\bar{b}), \end{aligned}$$

i.e.,

$$D(\bar{a},\bar{b}) \leq D(\bar{a},\bar{c})+D(\bar{c},\bar{b}).$$

Therefore, $D(\bar{a},\bar{b})$ satisfies the property (3) of Definition 1. Thus, we have proven that $D(\bar{a},\bar{b})$ defined by Eq. (4) is the distance between the intervals $\bar{a}$ and $\bar{b}$.

It is noted that the square appears in Eq. (4). In fact, Eq. (4) is the square of the distance between the intervals. In the sequent, the distance between two intervals is referred to the square of the distance given by Eq. (4) unless otherwise specified.

2.2 Interval-Valued Cooperative Games and Notations

A n-person interval-valued cooperative game in characteristic function form is an ordered-pair $<N,\bar{\upsilon}>$, where $N=\{1,2,\cdots,n\}$ is the set of n players, each subset $S\subseteq N$ is called a coalition of the player set N, and $\bar{\upsilon}:2^N\to\mathbb{R}$ is the interval-valued characteristic function of players' coalitions. 2^N denotes the set of coalitions of the player set N. Obviously, N is the grand coalition. For each coalition $S\subseteq N$, its size is denoted by s, which represents the number of players in the coalition S. The interval $\bar{\upsilon}(S)=[\upsilon_L(S),\upsilon_R(S)]$ represents the range of reward (or profit) that the coalition S can achieve on its own if all the players in it act together, where the lower bound $\upsilon_L(S)$ of the interval $\bar{\upsilon}(S)$ is the minimal reward of the coalition S and the upper bound $\upsilon_R(S)$ of the interval $\bar{\upsilon}(S)$ is the maximal reward of the coalition S. The interpretation of interval-valued cooperative games is that a coalition $S\subseteq N$ can obtain for its members a worth that is somewhere in the interval $\bar{\upsilon}(S)$. Stated as the above Sect. 2.1, we stipulate $\bar{\upsilon}(\varnothing)=[0,0]$, where $\varnothing$ is an empty set. Note that usually $\bar{\upsilon}(\varnothing)$ can be simply written as $\bar{\upsilon}(\varnothing)=0$ according to the notation of intervals in Sect. 2.1. For convenience, $\bar{\upsilon}(S\cup\{i\})$, $\bar{\upsilon}(S\backslash\{i\})$, $\bar{\upsilon}(\{i,j\})$, and $\bar{\upsilon}(\{i\})$ are usually written as $\bar{\upsilon}(S\cup i)$, $\bar{\upsilon}(S\backslash i)$, $\bar{\upsilon}(i,j)$, and $\bar{\upsilon}(i)$, respectively. In the sequent, a n-person interval-valued cooperative game $<N,\bar{\upsilon}>$ is simply called the interval-valued cooperative game $\bar{\upsilon}$. The set of n-person interval-valued cooperative games $\bar{\upsilon}$ is denoted by $\bar{G}^n$.

3 Quadratic Programming Model for Least Square Interval-Valued Prenucleoli of Interval-Valued Cooperative Games

For any interval-valued cooperative game $\bar{\upsilon}\in\bar{G}^n$, it is obvious that each player's payoff obtained from cooperation should be also an interval due to the fact that the payoff (or characteristic value) of each coalition $S\subseteq N$ is an interval. Thus, let $\bar{x}_i=$

$[x_{Li}, x_{Ri}]$ be the interval-valued payoff which is allocated to the player $i \in N$ when the grand coalition N is reached. Denote $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^{\mathrm{T}}$, which is the interval-valued payoff vector of n players in the grand coalition N. For any coalition $S \subseteq N$, denote $\bar{x}(S) = \sum_{i \in S} \bar{x}_i$, which represents the collective (or aggregated) interval-valued payoff of all the players in the coalition S. According to Eq. (1), $\bar{x}(S) = [x_L(S), x_R(S)]$ can be expressed as the following interval:

$$\bar{x}(S) = [\sum_{i \in S} x_{Li}, \sum_{i \in S} x_{Ri}].$$

In a similar way to the definitions of the efficiency and individual rationality of the classical cooperative game [2, 14], if an interval-valued payoff vector $\bar{\boldsymbol{x}}$ satisfies both the efficiency and individual rationality conditions as follows:

$$\sum_{i=1}^{n} \bar{x}_i = \bar{\upsilon}(N) \tag{5}$$

and

$$\bar{x}_i \geq \bar{\upsilon}(i) \ (i = 1, 2, \cdots, n), \tag{6}$$

respectively, then $\bar{\boldsymbol{x}}$ is called an imputation of the interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$. In other word, an interval-valued payoff vector $\bar{\boldsymbol{x}}$ is said to be efficient or a preimputation if the efficiency condition $\bar{x}(N) = \bar{\upsilon}(N)$ is valid. Further, $\bar{\boldsymbol{x}}$ is said to be an imputation if the individual rationality conditions $\bar{x}_i \geq \bar{\upsilon}(i)$ for all players $i \in N$ are also satisfied. $\bar{I}^{\mathrm{Pr}}(\bar{\upsilon})$ and $\bar{I}(\bar{\upsilon})$ denote the sets of interval-valued preimputations and imputations of the interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, respectively.

Using Eqs. (1) and (3), Eqs. (5) and (6) can be rewritten as follows:

$$\begin{cases} \sum_{i=1}^{n} x_{Li} = \upsilon_L(N) \\ \sum_{i=1}^{n} x_{Ri} = \upsilon_R(N) \end{cases} \tag{7}$$

and

$$\begin{cases} x_{Li} \geq \upsilon_L(i) \ (i = 1, 2, \cdots, n) \\ x_{Ri} \geq \upsilon_R(i) \ (i = 1, 2, \cdots, n), \end{cases} \tag{8}$$

respectively.

For any interval-valued payoff vector $\bar{\boldsymbol{x}}$ and any coalition $S \subseteq N$, where $S \neq \emptyset$, according to Eq. (4), denote

$$e(S, \bar{x}) = (\upsilon_L(S) - x_L(S))^2 + (\upsilon_R(S) - x_R(S))^2, \tag{9}$$

which is just the square of the distance between the intervals $\bar{x}(S)$ and $\bar{\upsilon}(S)$. Then, $e(S,\bar{x})$ is called the square excess of the coalition S on the interval-valued payoff vector $\bar{\boldsymbol{x}}$.

Usually, for any coalition $S \subseteq N$, we use $e_L(S,\bar{x})$ to express $\upsilon_L(S) - x_L(S)$, i.e.,

$$e_L(S,\bar{x}) = \upsilon_L(S) - x_L(S), \tag{10}$$

which is called the lower bound of the excess of the coalition S on the interval-valued payoff vector $\bar{\boldsymbol{x}}$. Likewise, we use $e_R(S,\bar{x})$ to represent $\upsilon_R(S) - x_R(S)$, i.e.,

$$e_R(S,\bar{x}) = \upsilon_R(S) - x_R(S), \tag{11}$$

which is called the upper bound of the excess of the coalition S on $\bar{\boldsymbol{x}}$. Therefore, $e(S,\bar{x})$ can be rewritten as follows:

$$e(S,\bar{x}) = (e_L(S,\bar{x}))^2 + (e_R(S,\bar{x}))^2.$$

It is noted that $e(S,\bar{x})$ can be interpreted as a measure of the dissatisfaction of the coalition S if $\bar{\boldsymbol{x}}$ were suggested as a final interval-valued payoff vector for all the players in the grand coalition. Obviously, $e(S,\bar{x}) \geq 0$. Further, the square excess $e(N,\bar{x})$ of the grand coalition N on $\bar{x}$ is equal to 0 if the interval-valued payoff vector $\bar{\boldsymbol{x}}$ satisfies the efficiency. Hence, the greater $e(S,\bar{x})$ the more unfair the coalition S.

Least square interval-valued prenucleoli and nucleoli are an important type of solutions for interval-valued cooperative games. In a paralleled way to the definitions of the prenucleoli and nucleoli [15, 16] of classical cooperative games, we can define the least square interval-valued prenucleoli and nucleoli of interval-valued cooperative games based on the square excesses of coalitions on the interval-valued payoff vectors.

The least square interval-valued prenucleolus of an interval-valued cooperative game would choose an interval-valued payoff vector to minimize the sum of the square excesses from the preimputation set according to the lexicographical order. Whereas, the least square interval-valued nucleolus would choose an interval-valued payoff vector to minimize the sum of the square excesses from the imputation set. In both cases, the key problem to obtain least square interval-valued prenucleoli and nucleoli of interval-valued cooperative games is to minimize the maximal complaint with the square excess of a coalition on an interval-valued payoff vector. This selection is regarded as equitable and reasonable. To attain the minimum of $\sum_{S \subseteq N} e(S,\bar{x})$ and balance the gain of each player $i \in N$, we will choose the interval-valued payoff vector to minimize the sum of the squares of the differences between the excesses of the coalitions and their means (or average excesses). Namely, we try to find the interval-valued payoff vector so that the resulting excesses are the closest to the means under the least square criterion. Thus, combining with Eq. (7), solving a least square interval-valued prenucleolus of any interval-valued cooperative game can be transformed into solving the constructed quadratic programming model as follows:

$$\min\left\{\sum_{S\subseteq N,S\neq\varnothing}\left[(e_L(S,\bar{x})-e_{Lm}(S,\bar{x}))^2+(e_R(S,\bar{x})-e_{Rm}(S,\bar{x}))^2\right]\right\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^{n}x_{Li}=\upsilon_L(N)\\ \sum_{i=1}^{n}x_{Ri}=\upsilon_R(N),\end{cases} \tag{12}$$

where the summation is taken over all nonempty coalitions $S\subseteq N$, and $e_{Lm}(S,\bar{x})$ and $e_{Rm}(S,\bar{x})$ are the means of the excesses of coalitions on $\bar{\boldsymbol{x}}$, i.e.,

$$e_{Lm}(S,\bar{x})=\frac{1}{2^{n-1}}\sum_{S\subseteq N,S\neq\varnothing}e_L(S,\bar{x}) \tag{13}$$

and

$$e_{Rm}(S,\bar{x})=\frac{1}{2^{n-1}}\sum_{S\subseteq N,S\neq\varnothing}e_R(S,\bar{x}). \tag{14}$$

Analogously, combining with Eqs. (7) and (8), solving a least square interval-valued nucleolus of any interval-valued cooperative game can be converted into solving the constructed quadratic programming model as follows:

$$\min\{\sum_{S\subseteq N,S\neq\varnothing}[(e_L(S,\bar{x})-e_{Lm}(S,\bar{x}))^2+(e_R(S,\bar{x})-e_{Rm}(S,\bar{x}))^2]\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^{n}x_{Li}=\upsilon_L(N)\\ \sum_{i=1}^{n}x_{Ri}=\upsilon_R(N)\\ x_{Li}\geq\upsilon_L(i)\ \ (i=1,2,\cdots,n)\\ x_{Ri}\geq\upsilon_R(i)\ (i=1,2,\cdots,n).\end{cases} \tag{15}$$

It is easily observed the following conclusion: for any interval-valued cooperative game $\bar{\upsilon}\in\bar{G}^n$, if an interval-valued payoff vector $\bar{\boldsymbol{x}}$ satisfies the efficiency, then the sum of the lower (or upper) bounds of the excesses of all coalitions $S\subseteq N$ on $\bar{\boldsymbol{x}}$ is the same as that on any other interval-valued payoff vector which also satisfies the efficiency. In fact, due to the assumption that $\bar{\boldsymbol{x}}$ is an interval-valued payoff vector which satisfies the efficiency, then we have

$$
\begin{aligned}
\sum_{S\subseteq N,S\neq\varnothing} e_L(S,\bar{x}) &= \sum_{S\subseteq N,S\neq\varnothing} (\upsilon_L(S) - x_L(S)) \\
&= \sum_{S\subseteq N,S\neq\varnothing} \upsilon_L(S) - \sum_{S\subseteq N,S\neq\varnothing} x_L(S) \\
&= \sum_{S\subseteq N,S\neq\varnothing} \upsilon_L(S) - \frac{1}{2}[\sum_{S\subseteq N,S\neq\varnothing} x_L(S) + \sum_{S\subseteq N,S\neq\varnothing} x_L(N\backslash S) + x_L(N)] \\
&= \sum_{S\subseteq N,S\neq\varnothing} \upsilon_L(S) - \frac{1}{2}\sum_{S\subseteq N,S\neq\varnothing} (x_L(S) + x_L(N\backslash S)) - \frac{1}{2}x_L(N) \\
&= \sum_{S\subseteq N,S\neq\varnothing} \upsilon_L(S) - \frac{1}{2}(2^n - 1)x_L(N) - \frac{1}{2}x_L(N) \\
&= \sum_{S\subseteq N,S\neq\varnothing} \upsilon_L(S) - 2^{n-1}x_L(N) \\
&= \sum_{S\subseteq N,S\neq\varnothing} \upsilon_L(S) - 2^{n-1}\upsilon_L(N),
\end{aligned}
$$

i.e.,

$$
\sum_{S\subseteq N,S\neq\varnothing} e_L(S,\bar{x}) = \sum_{S\subseteq N,S\neq\varnothing} \upsilon_L(S) - 2^{n-1}\upsilon_L(N), \tag{16}
$$

which easily implies that $\sum_{S\subseteq N,S\neq\varnothing} e_L(S,\bar{x})$ is a constant for any interval-valued payoff vector which satisfies the efficiency.

Likewise, we can easily obtain

$$
\sum_{S\subseteq N,S\neq\varnothing} e_R(S,\bar{x}) = \sum_{S\subseteq N,S\neq\varnothing} \upsilon_R(S) - 2^{n-1}\upsilon_R(N). \tag{17}
$$

Thus, Eqs. (16) and (17) show that the sums of the lower (or upper) bounds of the excesses of all coalitions are identical for all interval-valued payoff vectors which satisfy the efficiency.

Further, it is easy to see from Eqs. (16) and (17) that the means of the lower (or upper) bounds of the excesses of all coalitions are identical for all interval-valued payoff vectors which satisfy the efficiency.

Using Eqs. (10)–(14) and Eqs. (16) and (17), then Eq. (15) can be rewritten as follows:

$$
\begin{aligned}
\min\{ &\sum_{S\subseteq N,S\neq\varnothing} [\upsilon_L(S) - x_L(S) - \frac{1}{2^n - 1}(\sum_{S\subseteq N,S\neq\varnothing} \upsilon_L(S) - 2^{n-1}\upsilon_L(N))]^2 \\
&+ \sum_{S\subseteq N,S\neq\varnothing} [\upsilon_R(S) - x_R(S) - \frac{1}{2^n - 1}(\sum_{S\subseteq N,S\neq\varnothing} \upsilon_R(S) - 2^{n-1}\upsilon_R(N))]^2\}
\end{aligned} \tag{18}
$$

$$\text{s.t.}\begin{cases}\sum_{i=1}^{n} x_{Li} = \upsilon_L(N)\\ \sum_{i=1}^{n} x_{Ri} = \upsilon_R(N).\end{cases}$$

4 A Fast Method for Computing Least Square Interval-Valued Prenucleoli of Interval-Valued Cooperative Games

In this section, based on the square excess, we focus on developing an effective and a fast quadratic programming method for solving interval-valued cooperative games as stated in Sect. 2.2. It is easy to see from Eq. (18) that computing the least square interval-valued prenucleolus of an interval-valued cooperative game becomes solving the quadratic programming model.

Using the Lagrange multiplier method, the Lagrange function of Eq. (18) can be constructed as follows:

$$\begin{aligned}L(\bar{x},\lambda,\mu) = &\sum_{S\subseteq N, S\neq\varnothing}[\upsilon_L(S) - x_L(S) - \frac{1}{2^n-1}(\sum_{S\subseteq N, S\neq\varnothing}\upsilon_L(S) - 2^{n-1}\upsilon_L(N))]^2\\ &+ \sum_{S\subseteq N, S\neq\varnothing}[\upsilon_R(S) - x_R(S) - \frac{1}{2^n-1}(\sum_{S\subseteq N, S\neq\varnothing}\upsilon_R(S) - 2^{n-1}\upsilon_R(N))]^2\\ &+ \lambda(\sum_{i=1}^{n} x_{Li} - \upsilon_L(N)) + \mu(\sum_{i=1}^{n} x_{Ri} - \upsilon_R(N)),\end{aligned}$$

where λ and μ are Lagrange multipliers.

The partial derivatives of $L(\bar{x},\lambda,\mu)$ with respect to the variables x_{Lj}, x_{Rj} ($j \in S \subseteq N$), λ, and μ are obtained as follows:

$$\frac{\partial L(\bar{x},\lambda,\mu)}{\partial x_{Lj}} = -2\sum_{S:i\in S}[\upsilon_L(S) - x_L(S) - \frac{1}{2^n-1}(\sum_{S\subseteq N, S\neq\varnothing}\upsilon_L(S) - 2^{n-1}\upsilon_L(N))] + \lambda,$$

$$\frac{\partial L(\bar{x},\lambda,\mu)}{\partial \lambda} = \sum_{i=1}^{n} x_{Li} - \upsilon_L(N),$$

$$\frac{\partial L(\bar{x},\lambda,\mu)}{\partial x_{Rj}} = -2\sum_{S:i\in S}[\upsilon_R(S) - x_R(S) - \frac{1}{2^n-1}(\sum_{S\subseteq N, S\neq\varnothing}\upsilon_R(S) - 2^{n-1}\upsilon_R(N))] + \mu,$$

and

$$\frac{\partial L(\bar{x},\lambda,\mu)}{\partial \mu} = \sum_{i=1}^{n} x_{Ri} - \upsilon_R(N),$$

respectively.

Let the partial derivatives of $L(\bar{x}, \lambda, \mu)$ with respect to the variables x_{Lj}, x_{Rj} $(j \in S \subseteq N)$, λ, and μ be equal to 0, respectively. Consequently, we have

$$-2\sum_{S:i\in S}[\upsilon_L(S) - x_L^{*\mathrm{E}}(S) - \frac{1}{2^n - 1}(\sum_{S\subseteq N, S\neq\varnothing} \upsilon_L(S) - 2^{n-1}\upsilon_L(N))] + \lambda^{*\mathrm{E}} = 0, \tag{19}$$

$$\sum_{i=1}^{n} x_{Li}^{*\mathrm{E}} = \upsilon_L(N), \tag{20}$$

$$-2\sum_{S:i\in S}[\upsilon_R(S) - x_R^{*\mathrm{E}}(S) - \frac{1}{2^n - 1}(\sum_{S\subseteq N, S\neq\varnothing} \upsilon_R(S) - 2^{n-1}\upsilon_R(N))] + \mu^{*\mathrm{E}} = 0, \tag{21}$$

and

$$\sum_{i=1}^{n} x_{Ri}^{*\mathrm{E}} = \upsilon_R(N), \tag{22}$$

respectively.

It is obvious that

$$\sum_{S:i\in S} x_L^{*\mathrm{E}}(S) = 2^{n-1}x_{Li}^{*\mathrm{E}} + \sum_{j\in N\backslash i} 2^{n-2}x_{Lj}^{*\mathrm{E}} \ (i,j \in N) \tag{23}$$

It can be easily derived from Eqs. (19) and (23) that

$$-2\sum_{S:i\in S}\upsilon_L(S) + 2\times 2^{n-1}x_{Li}^{*\mathrm{E}} + 2\sum_{j\in N\backslash i} 2^{n-2}x_{Lj}^{*\mathrm{E}} + \frac{2}{2^n - 1}\sum_{S\subseteq N, S\neq\varnothing}\upsilon_L(S) - \frac{2^n}{2^n - 1}\upsilon_L(N) + \lambda^{*\mathrm{E}} = 0$$

Combining with the equality:

$$x_{Li}^{*\mathrm{E}} + \sum_{j\in N\backslash i} x_{Lj}^{*\mathrm{E}} = \upsilon_L(N) \ (i,j \in N),$$

we can directly obtain

$$-2\sum_{S:i\in S}\upsilon_L(S) + 2^{n-1}x_{Li}^{*\mathrm{E}} + (2^{n-1} - \frac{2^n}{2^n - 1})\upsilon_L(N) + \frac{2}{2^n - 1}\sum_{S\subseteq N, S\neq\varnothing}\upsilon_L(S) + \lambda^{*\mathrm{E}} = 0, \tag{24}$$

which can be rewritten as follows:

$$x_{Li}^{*\mathrm{E}} = \frac{2\sum_{S:i\in S}\upsilon_L(S) - (2^{n-1} - \frac{2^n}{2^n-1})\upsilon_L(N) - \frac{2}{2^n-1}\sum_{S\subseteq N, S\neq\varnothing}\upsilon_L(S) - \lambda^{*\mathrm{E}}}{2^{n-1}} \tag{25}$$

Thus, the key to solve x_{Li}^{*E} $(i = 1, 2, 3, \cdots, n)$ becomes obtaining λ^{*E}. It is easily derived from Eq. (20) that

$$\sum_{i \in N} \frac{2\sum_{S:i \in S} \upsilon_L(S) - (2^{n-1} - \frac{2^n}{2^n - 1})\upsilon_L(N) - \frac{2}{2^n - 1}\sum_{S \subseteq N, S \neq \varnothing} \upsilon_L(S) - \lambda^{*E}}{2^{n-1}} = \upsilon_L(N),$$

i.e.,

$$2\sum_{S \subseteq N, S \neq \varnothing} s\upsilon_L(S) - n(2^{n-1} - \frac{2^n}{2^n - 1})\upsilon_L(N) - \frac{2n}{2^n - 1}\sum_{S \subseteq N, S \neq \varnothing} \upsilon_L(S) - n\lambda^{*E} = 2^{n-1}\upsilon_L(N),$$

where s denotes the cardinality of the coalition $S \subseteq N$, i.e., $s = |S|$. Hence, we can easily obtain

$$\lambda^{*E} = \frac{2\sum_{S \subseteq N, S \neq \varnothing} s\upsilon_L(S)}{n} - (2^{n-1} - \frac{2^n}{2^n - 1})\upsilon_L(N) - \frac{2}{2^n - 1}\sum_{S \subseteq N, S \neq \varnothing} \upsilon_L(S) - \frac{2^{n-1}}{n}\upsilon_L(N), \tag{26}$$

which is substituted into Eq. (25), we directly have

$$\begin{aligned}
x_{Li}^{*E} &= \frac{2\sum_{S:i \in S} \upsilon_L(S) - (2^{n-1} - \frac{2^n}{2^n - 1})\upsilon_L(N) - \frac{2}{2^n - 1}\sum_{S \subseteq N, S \neq \varnothing} \upsilon_L(S)}{2^{n-1}} \\
&\quad - \frac{\frac{2\sum_{S \subseteq N, S \neq \varnothing} s\upsilon_L(S)}{n} - (2^{n-1} - \frac{2^n}{2^n - 1})\upsilon_L(N) - \frac{2}{2^n - 1}\sum_{S \subseteq N, S \neq \varnothing} \upsilon_L(S) - \frac{2^{n-1}}{n}\upsilon_L(N)}{2^{n-1}} \\
&= \frac{2\sum_{S:i \in S} \upsilon_L(S) - \frac{2\sum_{S \subseteq N, S \neq \varnothing} s\upsilon_L(S)}{n} + \frac{2^{n-1}}{n}\upsilon_L(N)}{2^{n-1}} \\
&= \frac{\upsilon_L(N)}{n} + \frac{2\sum_{S:i \in S} \upsilon_L(S) - \frac{2\sum_{S \subseteq N, S \neq \varnothing} s\upsilon_L(S)}{n}}{2^{n-1}} \\
&= \frac{\upsilon_L(N)}{n} + \frac{1}{n2^{n-2}}(n\sum_{S:i \in S} \upsilon_L(S) - \sum_{S \subseteq N, S \neq \varnothing} s\upsilon_L(S)) \\
&= \frac{\upsilon_L(N)}{n} + \frac{1}{n2^{n-2}}(na_{Li}(\upsilon) - \sum_{j \in N}\sum_{S:j \in S} \upsilon_L(S)) \\
&= \frac{\upsilon_L(N)}{n} + \frac{1}{n2^{n-2}}(na_{Li}(\upsilon) - \sum_{j \in N} a_{Lj}(\upsilon)),
\end{aligned}$$

i.e.,

$$x_{Li}^{*E} = \frac{\upsilon_L(N)}{n} + \frac{1}{n2^{n-2}}(na_{Li}(\upsilon) - \sum_{j\in N} a_{Lj}(\upsilon))\ (i \in N), \tag{27}$$

where $a_{Li}(\upsilon) = \sum_{S:i\in S} \upsilon_L(S)$.

Likewise, using Eqs. (21) and (22), the upper bounds of the interval-valued optimal solution $\bar{\boldsymbol{x}}^{*E}$ of Eq. (18) can be obtained as follows:

$$x_{Ri}^{*E} = \frac{\upsilon_R(N)}{n} + \frac{1}{n2^{n-2}}(na_{Ri}(\upsilon) - \sum_{j\in N} a_{Rj}(\upsilon))\ (i \in N), \tag{28}$$

where $a_{Ri}(\upsilon) = \sum_{S:i\in S} \upsilon_R(S)\ (i \in N)$.

Then, we obtain the interval-valued optimal solution $\bar{\boldsymbol{x}}^{*E} = (\bar{x}_1^{*E}, \bar{x}_2^{*E}, \cdots, \bar{x}_n^{*E})^{T}$ of Eq. (18), whose components' lower and upper bounds consist of Eqs. (27) and (28), respectively, where $\bar{x}_i^{*E} = [x_{Li}^{*E}, x_{Ri}^{*E}]\ (i \in N)$. Therefore, the least square interval-valued prenucleolus of the interval-valued cooperative game $\bar{\upsilon}$ is $\bar{\boldsymbol{x}}^{*E}$.

In what follows, we discuss some useful and important properties of the least square interval-valued prenucleoli for interval-valued cooperative games.

Theorem 2. Assume that $\bar{\upsilon} \in \bar{G}^n$ is any interval-valued cooperative game. Then, there always exists a unique least square interval-valued prenucleolus, which is determined by Eqs. (27) and (28).

Proof. It is straightforward to prove Theorem 2 according to Eqs. (27) and (28).

Theorem 3. Assume that $\bar{\upsilon} \in \bar{G}^n$ is any interval-valued cooperative game. Then, its least square interval-valued prenucleolus $\bar{\boldsymbol{x}}^{*E}$ satisfies the efficiency, i.e., $\sum_{i=1}^{n} \bar{x}_i^{*E} = \bar{\upsilon}(N)$.

Proof. According to Eq. (1), it is easily derived from Eqs. (27) and (28) that

$$\begin{aligned}\sum_{i=1}^{n} \bar{x}_i^{*E} &= [\sum_{i=1}^{n} x_{Li}^{*E}, \sum_{i=1}^{n} x_{Ri}^{*E}] \\ &= [\sum_{i=1}^{n} \frac{\upsilon_L(N)}{n} + \frac{1}{2^{n-2}}\sum_{i=1}^{n} a_{Li}(\upsilon) - \frac{1}{2^{n-2}}\sum_{j\in N} a_{Lj}(\upsilon), \sum_{i=1}^{n} \frac{\upsilon_R(N)}{n} + \frac{1}{2^{n-2}}\sum_{i=1}^{n} a_{Ri}(\upsilon) - \frac{1}{2^{n-2}}\sum_{j\in N} a_{Rj}(\upsilon)] \\ &= [\upsilon_L(N), \upsilon_R(N)],\end{aligned}$$

i.e., $\sum_{i=1}^{n} \bar{x}_i^{*E} = \bar{\upsilon}(N)$. Thus, we have completed the proof of Theorem 3.

Theorem 4. Assume that $\bar{\upsilon} \in \bar{G}^n$ and $\bar{\nu} \in \bar{G}^n$ are any interval-valued cooperative games. Then, $\bar{\boldsymbol{x}}^{*E}(\bar{\upsilon} + \bar{\nu}) = \bar{\boldsymbol{x}}^{*E}(\bar{\upsilon}) + \bar{\boldsymbol{x}}^{*E}(\bar{\nu})$.

Proof. It is easily derived from Eq. (27) that

$$x_{Li}^{*\mathrm{E}}(\bar{\upsilon}+\bar{v}) = \frac{\upsilon_L(N)+v_L(N)}{n} + \frac{1}{n2^{n-2}}[n(a_{Li}(\upsilon)+a_{Li}(v)) - \sum_{j\in N}(a_{Lj}(\upsilon)+a_{Lj}(v))]$$
$$= \frac{\upsilon_L(N)}{n} + \frac{1}{n2^{n-2}}(na_{Li}(\upsilon) - \sum_{j\in N}a_{Lj}(\upsilon)) + \frac{v_L(N)}{n} + \frac{1}{n2^{n-2}}(na_{Li}(v) - \sum_{j\in N}a_{Lj}(v))$$
$$= x_{Li}^{*\mathrm{E}}(\bar{\upsilon}) + x_{Li}^{*\mathrm{E}}(\bar{v}),$$

i.e., $x_{Li}^{*\mathrm{E}}(\bar{\upsilon}+\bar{v}) = x_{Li}^{*\mathrm{E}}(\bar{\upsilon}) + x_{Li}^{*\mathrm{E}}(\bar{v})$.

Analogously, according to Eq. (28), we can easily prove that $x_{Ri}^{*\mathrm{E}}(\bar{\upsilon}+\bar{v}) = x_{Ri}^{*\mathrm{E}}(\bar{\upsilon}) + x_{Ri}^{*\mathrm{E}}(\bar{v})$. Hence, according to Eq. (1), we have

$$\bar{x}_i^{*\mathrm{E}}(\bar{\upsilon}+\bar{v}) = \bar{x}_i^{*\mathrm{E}}(\bar{\upsilon}) + \bar{x}_i^{*\mathrm{E}}(\bar{v}) \; (i=1,2,\cdots,n),$$

i.e., $\bar{\boldsymbol{x}}^{*\mathrm{E}}(\bar{\upsilon}+\bar{v}) = \bar{\boldsymbol{x}}^{*\mathrm{E}}(\bar{\upsilon}) + \bar{\boldsymbol{x}}^{*\mathrm{E}}(\bar{v})$, which implies that Theorem 4 is valid.

Theorem 5. If players $i \in N$ and $k \in N$ $(i \neq k)$ are symmetric in an interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, then $\bar{x}_i^{*\mathrm{E}} = \bar{x}_k^{*\mathrm{E}}$.

Proof. For the players $i \in N$ and $k \in N$ $(i \neq k)$, it is easily derived from Eq. (27) that

$$x_{Li}^{*\mathrm{E}} = \frac{\upsilon_L(N)}{n} + \frac{1}{n2^{n-2}}(na_{Li}(\upsilon) - \sum_{j\in N}a_{Lj}(\upsilon)) \tag{29}$$

and

$$x_{Lk}^{*\mathrm{E}} = \frac{\upsilon_L(N)}{n} + \frac{1}{n2^{n-2}}(na_{Lk}(\upsilon) - \sum_{j\in N}a_{Lj}(\upsilon)). \tag{30}$$

It is easily derived from the symmetric players' assumption [2] that

$$a_{Li}(\upsilon) = a_{Lk}(\upsilon) \; (i \in N, \, k \in N, \, i \neq k),$$

which easily follows from Eqs. (29) and (30) that $x_{Li}^{*\mathrm{E}} = x_{Lk}^{*\mathrm{E}}$.

In the same way, using Eq. (28), we can easily prove $x_{Ri}^{*\mathrm{E}} = x_{Rk}^{*\mathrm{E}}$. Combining with the aforementioned conclusion and Eq. (1), we can obtain

$$[x_{Li}^{*\mathrm{E}}, x_{Ri}^{*\mathrm{E}}] = [x_{Lk}^{*\mathrm{E}}, x_{Rk}^{*\mathrm{E}}],$$

i.e., $\bar{x}_i^{*\mathrm{E}} = \bar{x}_k^{*\mathrm{E}}$. Accordingly, we have completed the proof of Theorem 5.

Theorem 6. Assume that $\bar{\upsilon} \in \bar{G}^n$ is any interval-valued cooperative game. For any permutation σ on the set N, then $\bar{x}_{\sigma(i)}^{*\mathrm{E}}(\bar{\upsilon}^{\sigma}) = \bar{x}_i^{*\mathrm{E}}(\bar{\upsilon})$.

Proof. It can be easily proven according to Eqs. (27) and (28) (omitted).

Obviously, if all coalitions' values $\bar{\upsilon}(S)$ degenerate to real numbers, i.e., $\upsilon(S) = \upsilon_L(S) = \upsilon_R(S)$ for any coalition $S \subseteq N$, then it easily follows from Eqs. (27) and (28) that $x_i^{*E} = x_{Li}^{*E} = x_{Ri}^{*E}$ $(i \in N)$, i.e., Eqs. (27) and (28) are identical. Namely, either Eq. (27) or Eq. (28) is applicable to the classical cooperative games. Thus, the model and method developed in this section may be regarded as an extension of that for the classical cooperative games when uncertainty and imprecision are taken into account.

5 Algorithms for Least Square Interval-Valued Nucleoli of Interval-Valued Cooperative Games

Equation (18) is used to compute the least square interval-valued prenucleolus of any interval-valued cooperative game. However, the least square interval-valued prenucleolus is usually not an imputation because it possibly fails to satisfy the individual rationality. Hereby, we can construct the quadratic programming model of the least square interval-valued nucleolus for the interval-valued cooperative game $\bar{\upsilon}$ as follows:

$$\begin{aligned} \min\{ \sum_{S\subseteq N, S\neq\varnothing} [\upsilon_L(S) - x_L(S) - \tfrac{1}{2^n-1}(\sum_{S\subseteq N, S\neq\varnothing} \upsilon_L(S) - 2^{n-1}\upsilon_L(N))]^2 \\ + \sum_{S\subseteq N, S\neq\varnothing} [\upsilon_R(S) - x_R(S) - \tfrac{1}{2^n-1}(\sum_{S\subseteq N, S\neq\varnothing} \upsilon_R(S) - 2^{n-1}\upsilon_R(N))]^2 \} \end{aligned} \tag{31}$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{n} x_{Li} = \upsilon_L(N) \\ \sum_{i=1}^{n} x_{Ri} = \upsilon_R(N) \\ x_{Li} \geq \upsilon_L(i) \ (i = 1, 2, \cdots, n) \\ x_{Ri} \geq \upsilon_R(i) \ (i = 1, 2, \cdots, n). \end{cases}$$

For discussion concision and convenience, we firstly prove the following conclusion: for any interval-valued cooperative game $\bar{\upsilon} \in \bar{G}^n$, if we use any constants m_L and m_R to replace $\bar{e}_L(S, \bar{x})$ and $\bar{e}_R(S, \bar{x})$ of the objective function in Eq. (12) (or Eq. (15)), respectively, then Eq. (12) (or Eq. (15)) remains the identical optimal solution. In fact, assume that $\bar{\boldsymbol{x}}$ is any interval-valued payoff vector which satisfies the efficiency, then for any constants m_L and m_R, we have

$$\begin{aligned} \sum_{S\subseteq N, S\neq\varnothing} [(e_L(S, \bar{x}) - m_L)^2 + (e_R(S, \bar{x}) - m_R)^2] = \sum_{S\subseteq N, S\neq\varnothing} (e_L(S, \bar{x}) - m_L)^2 + \sum_{S\subseteq N, S\neq\varnothing} (e_R(S, \bar{x}) - m_R)^2 \\ = \sum_{S\subseteq N, S\neq\varnothing} e_L(S, \bar{x})^2 + (2^n - 1)m_L^2 - 2m_L \sum_{S\subseteq N, S\neq\varnothing} e_L(S, \bar{x}) + \sum_{S\subseteq N, S\neq\varnothing} e_R(S, \bar{x})^2 \\ + (2^n - 1)m_R^2 - 2m_R \sum_{S\subseteq N, S\neq\varnothing} e_R(S, \bar{x}). \end{aligned} \tag{32}$$

It is easily derived from Eqs. (16) and (17) that the objective function of Eq. (12) (or Eq. (15)) replaced with Eq. (32) remains the same optimal solution as Eq. (12) (or Eq. (15)) while only their optimal objective values have a difference of constants.

In particular, for $m_L = m_R = 0$, it is obvious that the optimal solution of Eq. (12) is the same as that of the quadratic programming model as follows:

$$\min\{\sum_{S\subseteq N,S\neq\varnothing}[(\upsilon_L(S)-x_L(S))^2+(\upsilon_R(S)-x_R(S))^2]\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^{n}x_{Li}=\upsilon_L(N)\\ \sum_{i=1}^{n}x_{Ri}=\upsilon_R(N),\end{cases}\tag{33}$$

and the optimal solution of Eq. (15) is the same as that of the quadratic programming model as follows:

$$\min\{\sum_{S\subseteq N,S\neq\varnothing}[(\upsilon_L(S)-x_L(S))^2+(\upsilon_R(S)-x_R(S))^2]\}$$
$$\text{s.t.}\begin{cases}\sum_{i=1}^{n}x_{Li}=\upsilon_L(N)\\ \sum_{i=1}^{n}x_{Ri}=\upsilon_R(N)\\ x_{Li}\geq\upsilon_L(i)\ (i=1,2,\cdots,n)\\ x_{Ri}\geq\upsilon_R(i)\ (i=1,2,\cdots,n).\end{cases}\tag{34}$$

Stated as earlier, computing the least square interval-valued nucleolus of any interval-valued cooperative game can be equivalently converted into solving the optimal solution of Eq. (34). Therefore, combining with the optimal solution of Eq. (33), i.e., the least square interval-valued prenucleolus of any interval-valued cooperative game, we mainly propose simple and effective algorithms for solving the least square interval-valued nucleolus.

Without loss of generality, assume that we are considering any interval-valued cooperative game $\bar{\upsilon}$ with $\bar{\upsilon}(i)=[0,0]$ for all $i\in N$. In the following, we summarize the algorithms for solving the lower and upper bound of the least square interval-valued nucleolus of the interval-valued cooperative game $\bar{\upsilon}$ as follows.

We propose Algorithm 1 for determining nonnegativity of the lower bounds of the least square interval-valued nucleolus of the interval-valued cooperative game $\bar{\upsilon}$ as follows:

Step 1: Set $k=1$. Let $\boldsymbol{x}_L^k=\boldsymbol{x}_L^{*\mathrm{E}}$, where $\boldsymbol{x}_L^{*\mathrm{E}}=(x_{L1}^{*\mathrm{E}},x_{L2}^{*\mathrm{E}},\cdots,x_{Ln}^{*\mathrm{E}})^{\mathrm{T}}$ is the lower bound vector of the least square interval-valued prenucleolus of the interval-valued cooperative game $\bar{\upsilon}$, which is given by Eq. (27) (or generated by solving Eq. (33)). Let $M_L^k=\{j\in N|x_{Lj}^k<0\}$, which is the set of the players who have negative lower bounds of the interval-valued payoff vector $\boldsymbol{x}_L^k$.

Step 2: Compute

$$x_{Lj}^{k+1} = \begin{cases} x_{Lj}^{k} + \frac{x_L^k(M_L^k)}{n-m_L^k} & (j \notin M_L^k) \\ 0 & (j \in M_L^k), \end{cases}$$

where m_L^k is the cardinality of the player set M_L^k, i.e., $m_L^k = |M_L^k|$.
Step 3: Let $M_L^{k+1} = M_L^k \cup \{j \in N | x_{Lj}^{k+1} < 0\}$, which is the new set of the players who have negative lower bounds of the interval-valued payoff vector $\boldsymbol{x}_L^{k+1}$.
Step 4: If $M_L^{k+1} \supset M_L^k$, then set $k = k + 1$ and return to Step 2; If $M_L^{k+1} = M_L^k$, then the solving process stops, hereby we can obtain the lower bounds of the least square interval-valued nucleolus of the interval-valued cooperative game $\bar{\upsilon}$, depicted as in Fig. 1.

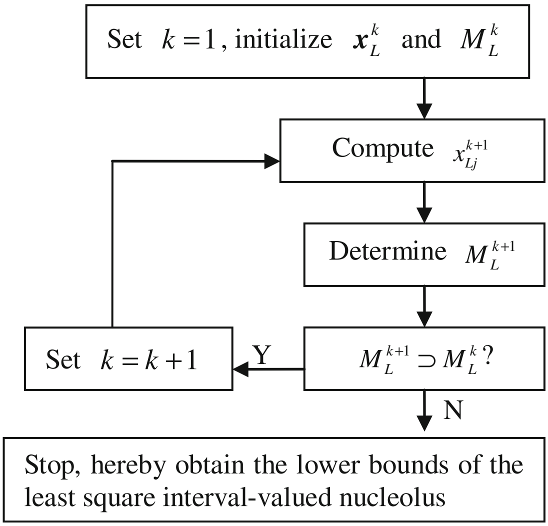

Fig. 1. Algorithm for determining nonnegativity of the lower bounds of the least square interval-valued nucleolus

Analogously, we can propose Algorithm 2 for determining nonnegativity of the upper bounds of the least square interval-valued nucleolus of the interval-valued cooperative game $\bar{\upsilon}$ as follows:

Step 1: Set $k = 1$. Let $\boldsymbol{x}_R^k = \boldsymbol{x}_R^{*E}$, where $\boldsymbol{x}_R^{*E} = (x_{R1}^{*E}, x_{R2}^{*E}, \cdots, x_{Rn}^{*E})^T$ is the upper bound vector of the least square interval-valued prenucleolus of the interval-valued cooperative game $\bar{\upsilon}$, which is given by Eq. (28) (or generated by solving Eq. (33)).

Let $M_R^k = \{j \in N | x_{Rj}^k < 0\}$, which is the set of the players who have negative upper bounds of the interval-valued payoff vector $\boldsymbol{x}_R^k$.
Step 2: Compute

$$x_{Rj}^{k+1} = \begin{cases} x_{Rj}^k + \frac{x_R^k(M_R^k)}{n - m_R^k} & (j \notin M_R^k) \\ 0 & (j \in M_R^k), \end{cases}$$

where m_R^k is the cardinality of the player set M_R^k, i.e., $m_R^k = |M_R^k|$.
Step 3: Let $M_R^{k+1} = M_R^k \cup \{j \in N | x_{Rj}^{k+1} < 0\}$, which is the new set of the players who have negative upper bounds of the interval-valued payoff vector $\boldsymbol{x}_R^{k+1}$.
Step 4: If $M_R^{k+1} \supset M_R^k$, then set $k = k + 1$ and return to Step 2; If $M_R^{k+1} = M_R^k$, then the solving process stops, hereby we can obtain the upper bounds of the least square interval-valued nucleolus of the interval-valued cooperative game $\bar{\upsilon}$, depicted as in Fig. 2.

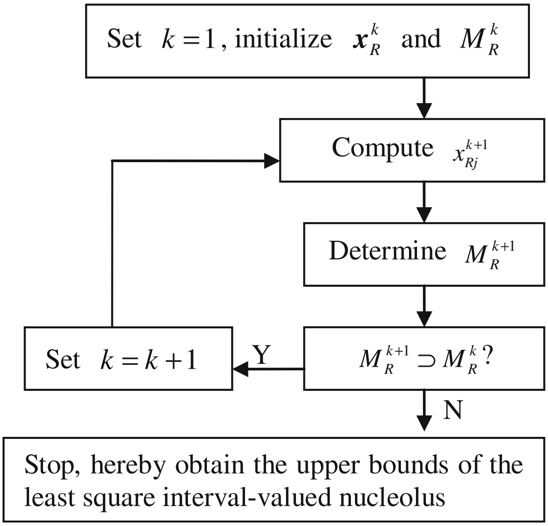

Fig. 2. Algorithm for determining nonnegativity of the upper bounds of the least square interval-valued nucleolus

From the above discussion, we can propose Algorithm 3 for computing the least square interval-valued nucleolus of any interval-valued cooperative game $\bar{\upsilon}$, depicted as in Fig. 3.

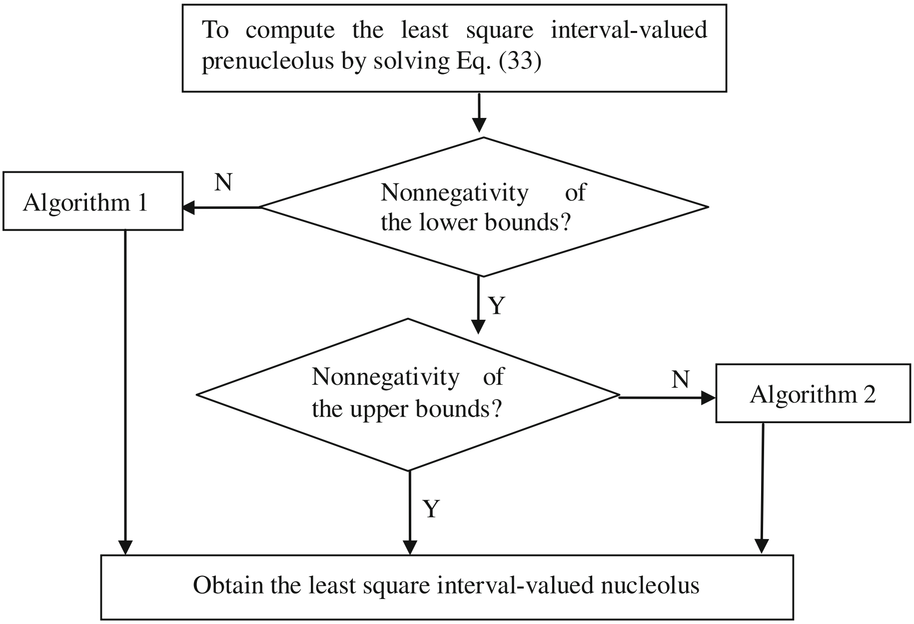

Fig. 3. Algorithm 3 for computing the least square interval-valued nucleolus

In the following, we discuss some important and useful properties of the least square interval-valued nucleolus of any interval-valued cooperative game.

Theorem 7. Assume that $\bar{\upsilon}$ is any interval-valued cooperative game. Then, there exists a unique least square interval-valued nucleolus, which satisfies the efficiency, individual rationality, additivity, symmetry, and anonymity.

Proof. It is easy to prove Theorem 7 in a similar way to Theorems 2–6 and combining with Algorithms 1 and 2 (omitted).

6 A Numerical Example of Joint Production Problems

The following is an example how interval-valued cooperative games are applied to solve joint production problems.

Let us consider a joint production problem in which five decision makers actively cooperate with one another to develop new products. The five decision makers are named players 1, 2, 3, 4, and 5, respectively. Denoted the set of players by $N' = \{1, 2, 3, 4, 5\}$. Before the cooperation starts, it is necessary for the five players (i.e., decision makers) to evaluate the revenue of the joint production project in order to decide whether the coalitions can be formed. However, the cooperative profit is dependent on many factors such as cost of human resources, product price, supply, and demand. Usually, players may estimate ranges of their profits instead of precisely forecasting their profits. Namely, the profit of a coalition $S \subseteq N'$ of the players may be expressed with an interval $\bar{\upsilon}'(S)=[\upsilon'_L(S),\upsilon'_R(S)]$. In this case, the optimal allocation of profits for the five decision makers may be regarded as an interval-valued cooperative game $\bar{\upsilon}'$ in which the interval-valued characteristic function is equal to $\bar{\upsilon}'(S)$ for any coalition $S\subseteq N'$.

For example, let us consider a specific interval-valued cooperative game $\bar{\upsilon}'$ which is defined as follows: $\bar{\upsilon}'(2,3)=\bar{\upsilon}'(2,4)=\bar{\upsilon}'(3,4)=\bar{\upsilon}'(3,5)=\bar{\upsilon}'(4,5)=[100,200]$, $\bar{\upsilon}'(1,3,4)=\bar{\upsilon}'(1,3,5)=\bar{\upsilon}'(1,4,5)=\bar{\upsilon}'(2,3,5)=\bar{\upsilon}'(2,4,5)=[100,200]$, $\bar{\upsilon}'(2,3,4)=[120,240]$, $\bar{\upsilon}'(3,4,5)=[175,300]$, $\bar{\upsilon}'(1,2,3,4)=[175,350]$, $\bar{\upsilon}'(1,2,3,5)=[100,220]$, $\bar{\upsilon}'(1,2,4,5)=[100,250]$, $\bar{\upsilon}'(1,3,4,5)=[200,380]$, $\bar{\upsilon}'(2,3,4,5)=[200,400]$, $\bar{\upsilon}'(1,2,3,4,5)=[200,600]$, and otherwise $\bar{\upsilon}'(S)=0$..

Using Eq. (27), we can obtain the lower bounds of the least square interval-valued prenucleolus of the interval-valued cooperative game $\bar{\upsilon}'$ as follows:

$$x_{L1}^{*\mathrm{E}}=\frac{\upsilon_L(N')}{5}+\frac{1}{5\times 2^{5-2}}\left(5a_{L1}(\upsilon)-\sum_{j\in N'}a_{Lj}(\upsilon)\right)=\frac{200}{5}+\frac{1}{5\times 8}(5\times 1075-7485)$$
$$=-12.75,$$

$$x_{L2}^{*\mathrm{E}}=\frac{\upsilon_L(N')}{5}+\frac{1}{5\times 2^{5-2}}\left(5a_{L2}(\upsilon)-\sum_{j\in N'}a_{Lj}(\upsilon)\right)=\frac{200}{5}+\frac{1}{5\times 8}(5\times 1295-7485)$$
$$=14.75,$$

$$x_{L3}^{*\mathrm{E}}=\frac{\upsilon_L(N')}{5}+\frac{1}{5\times 2^{5-2}}\left(5a_{L3}(\upsilon)-\sum_{j\in N'}a_{Lj}(\upsilon)\right)=\frac{200}{5}+\frac{1}{5\times 8}(5\times 1770-7485)$$
$$=74.125,$$

$$x_{L4}^{*\mathrm{E}}=\frac{\upsilon_L(N')}{5}+\frac{1}{5\times 2^{5-2}}\left(5a_{L4}(\upsilon)-\sum_{j\in N'}a_{Lj}(\upsilon)\right)=\frac{200}{5}+\frac{1}{5\times 8}(5\times 1770-7485)$$
$$=74.125,$$

and

$$x_{L5}^{*\mathrm{E}}=\frac{\upsilon_L(N')}{5}+\frac{1}{5\times 2^{5-2}}\left(5a_{L5}(\upsilon)-\sum_{j\in N'}a_{Lj}(\upsilon)\right)=\frac{200}{5}+\frac{1}{5\times 8}(5\times 1575-7485)$$
$$=49.75,$$

respectively.

According to Algorithm 1, it is obvious that

$$\boldsymbol{x}_L^1=\boldsymbol{x}_L^{*\mathrm{E}}=(x_{L1}^{*\mathrm{E}},x_{L2}^{*\mathrm{E}},x_{L3}^{*\mathrm{E}},x_{L4}^{*\mathrm{E}},x_{L5}^{*\mathrm{E}})^{\mathrm{T}}=(-12.75,14.75,74.125,74.125,49.75)^{\mathrm{T}}.$$

Then, we give 0 to player 1 and divide -12.75 equally among players 2, 3, 4, and 5. Hereby, we obtain

$$\boldsymbol{x}_L^2 = (x_{L1}^2, x_{L2}^2, x_{L3}^2, x_{L4}^2, x_{L5}^2)^{\mathrm{T}} = (0, 11.5625, 70.9375, 70.9375, 46.5625)^{\mathrm{T}}$$

Thus, we finally obtain the lower bounds of the least square interval-valued nucleolus for the interval-valued cooperative game $\bar{v}'$, i.e.,

$$\boldsymbol{x}_L^{*\mathrm{n}} = (0, 11.5625, 70.9375, 70.9375, 46.5625)^{\mathrm{T}}.$$

Likewise, according to Eq. (28), we can obtain the upper bounds of the least square interval-valued prenucleolus of the interval-valued cooperative game $\bar{v}'$ as follows:

$$\boldsymbol{x}_R^{*\mathrm{E}} = (19.5, 77, 180.75, 184.5, 138.25)^{\mathrm{T}}.$$

Then, using Algorithm 2, we can obtain

$$\boldsymbol{x}_R^1 = \boldsymbol{x}_R^{*\mathrm{E}} = (19.5, 77, 180.75, 184.5, 138.25)^{\mathrm{T}}.$$

Owing to the fact that all x_{Ri}^1 $(i \in N')$ are nonnegative, we directly have

$$\boldsymbol{x}_R^{*\mathrm{n}} = \boldsymbol{x}_R^1 = (19.5, 77, 180.75, 184.5, 138.25)^{\mathrm{T}},$$

which is the upper bounds of the least square interval-valued nucleolus for the interval-valued cooperative game $\bar{v}'$.

Therefore, we can obtain the least square interval-valued nucleolus of the interval-valued cooperative game $\bar{v}'$ as follows:

$$\bar{\boldsymbol{x}}^{*\mathrm{n}} = ([0, 19.5], [11.5625, 77], [70.9375, 180.75], [70.9375, 184.5], [46.5625, 138.25])^{\mathrm{T}},$$

which may be interpreted as follows: player 1 can obtain at least 0 and at most 19.5, i.e., the interval $[0, 19.5]$, which is almost greater than the interval $\bar{v}'(1) = [0, 0]$ obtained by itself alone. Analogously, player 2 can obtain at least 11.5625 and at most 77, i.e., the interval $[11.5625, 77]$, which is obviously greater than the interval $\bar{v}'(2) = [0, 0]$ obtained by itself alone. Player 3 can obtain at least 70.9375 and at most 180.75, i.e., the interval $[70.9375, 180.75]$, which is remarkably greater than the interval $\bar{v}'(3) = [0, 0]$ obtained by itself alone. The similar explanation can be done for players 4 and 5. In other words, the optimal allocations of all the five players i $(i \in N')$ satisfy the individual rationality of interval-valued payoff vectors according to Eq. (3), which is the Moore's order relation over intervals [8].

Obviously, we have

$$\sum_{i=1}^{5} x_{Li}^{*\mathrm{n}} = 0 + 11.5625 + 70.9375 + 70.9375 + 46.5625 = 200$$

and

$$\sum_{i=1}^{5} x_{Ri}^{*\mathrm{n}} = 19.5 + 77 + 180.75 + 184.5 + 138.25 = 600.$$

Hence,

$$\sum_{i=1}^{5} x_{i}^{*\mathrm{n}} = \bar{\upsilon}'(N'),$$

which implies that the least square interval-valued nucleolus $\bar{\boldsymbol{x}}^{*\mathrm{n}}$ satisfies the efficiency of interval-valued payoff vectors as expected.

7 Conclusions

We propose the quadratic programming model and algorithms for solving the least square interval-valued nucleoli of interval-valued cooperative games and effectively avoid the magnification of uncertainty resulted from the Moore's interval subtraction. The developed model and algorithms are simple and effective from the viewpoint of computational complexity. In addition, it is easy to see that the least square interval-valued prenucleoli and nucleoli of interval-valued cooperative games are generalizations of the least square prenucleoli and nucleoli for classical cooperative games.

However, only interval uncertainty is taken into consideration in coalition's values in this paper. In fact, uncertainty of coalition's values may be described by other types of data such as fuzzy numbers [17] and intuitionistic fuzzy numbers [18, 19]. Therefore, cooperative games with coalition values expressed by fuzzy numbers and intuitionistic fuzzy numbers will be hot topics in further research. What is more, the axiomatic characterizations of these types of cooperative games will also become hot issues of research.

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Interval-Valued Least Square Prenucleolus of Interval-Valued Cooperative Games with Fuzzy Coalitions

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Abstract. In this paper, an important solution concept of interval-valued (IV) cooperative games with fuzzy coalitions, called the IV least square prenucleolus, is proposed. Firstly, we determine the fuzzy coalitions' values by using Choquet integral and hereby obtain the IV cooperative games with fuzzy coalitions in Choquet integral forms. Then, we develop a simplified method to compute the IV least square prenucleolus of a special subclass of IV cooperative games with fuzzy coalitions in Choquet integral forms. In this method, we give some weaker coalition size monotonicity-like conditions, which can always ensure that the least square prenucleolus of our defined cooperative games with fuzzy coalitions in Choquet integral form are monotonic and non-decreasing functions of fuzzy coalitions' values. Hereby, the lower and upper bounds of the proposed IV least square prenucleolus can be directly obtained via utilizing the lower and upper bounds of the IV coalitions values, respectively. In addition, we investigate some important properties of the IV least square prenucleolus. The feasibility and applicability of the method proposed in this paper are illustrated with numerical examples.

Keywords: Game theory · Interval-valued cooperative game · Fuzzy game · Least square prenucleolus · Choquet integral

1 Introduction

The cooperative game with transferable utility (Driessen and Radzik 2002), often called the cooperative game for short, is an important part of game theory. In crisp (or classical) cooperative games, the coalitions' values (or payoffs) of players are expressed with real numbers and the rates of the players' participation in a coalition are either 0 or 1, i.e., the players fully participate in the coalition or do not participate in. However, in some real cases, due to complexity and uncertainty we only can predict the ranges of the coalitions' values rather than obtain exact values. Hence, it is more suitable to use intervals to estimate coalitions' values and hereby there appear IV cooperative games which were first introduced by Branzei et al. (2003). Furthermore, in order to reduce risk, players

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in some economic situations may choose to participate in a coalition to a certain extent, i.e., partly participate in the coalition rather than fully participate in. Therefore, following Zadeh (1965), Aubin defined cooperative games on fuzzy subsets of the set of n players and hence extended crisp cooperative games to cooperative games with fuzzy coalitions (Aubin 1980, 1981).

Recently, cooperative games with fuzzy coalitions have attracted much attention of researchers. Sakawa and Nishizaki (1994) proposed new lexicographical solution concepts in a cooperative game with fuzzy coalitions. Tijs et al. (2004) introduced the cores and stable sets of cooperative games with fuzzy coalitions. Yu and Zhang (2009) defined the fuzzy core of fuzzy games and investigated the nonempty condition of the fuzzy core. Butnariu (1980) explained the concepts of core and Shapley value for n-persons cooperative games with fuzzy coalitions. However, Tsurumi et al. (2001) pointed out that the class of the cooperative games with fuzzy coalitions introduced by Butnariu (1980) lack monotonicity and continuity. Hence, following Butnariu's method, Tsurumi et al. (2001) defined a new class of cooperative games with fuzzy coalitions via using the concept of Choquet integrals, which overcame the aforementioned drawbacks. What's more, some researchers have also proposed other solutions, such as Weber sets (Sagara 2015), Banzhaf value (Tan et. al. 2014), bargaining sets (Liu and Liu 2012), and the least square B-nucleolus (Lin and Zhang 2016). In addition, a fuzzy population monotonic allocation function (FPMAF) and a Shapley function of cooperative games with fuzzy coalitions and fuzzy characteristic functions are defined by Borkotokey (2008). Meng et al. (2016) studied the IV Shapley value of IV cooperative games with fuzzy coalitions in Choquet integral form based on the extended Hukuhara difference.

In recent years, solution concepts and their related properties of IV cooperative games have been discussed in many works. Branzei et al. (2010) gave a survey about cooperative interval games. They overviewed and updated the results about IV cooperative games and discussed various existing and potential applications of IV cooperative games in economic management situations. By defining a new order relation of intervals and using the Moore's subtraction, Han et al. (2012) studied the IV core and the IV Shapley-like value of IV cooperative games. Based on a partial subtraction operator, Palanci et al. (2015) focused on the IV Shapley value and its properties and also introduced the IV Banzhaf value and the IV egalitarian rule. Via discussing the IV square dominance and IV dominance imputations, Alparslan Gök (2014) used the efficiency property, symmetry property and strong monotonicity property to characterize the IV Shapley value. Alparslan Gök et al. (2011) introduced some set-valued solution concepts of IV cooperative games, such as the IV core, the IV dominance core, and the IV stable sets. Hong and Li (2016) developed a nonlinear programming approach to compute the IV cores of IV cooperative games. However, most of the aforementioned works except from Hong and Li (2016) used the partial subtraction operator or the Moore's interval subtraction (Moore 1979) which is not invertible and usually enlarges uncertainty of the resulted interval. Therefore, in this paper, we aim at solving the IV cooperative games with fuzzy coalitions without using the interval subtraction.

Ruiz et al. (1996) introduced the least square prenucleolus of a cooperative game. Later on, Li and Ye (2016) investigated the IV least square prenucleolus of a special subclass of IV cooperative games with crisp coalitions. The primary goal of this paper is to study and develop an effective and a simplified approach for IV cooperative games with fuzzy coalitions in Choquet integral forms. In this approach, through adding some coalition size monotonicity-like conditions, we proved that the least square prenucleolus of our defined cooperative game with fuzzy coalition in Choquet integral form is a monotonic and non-decreasing function of coalitions payoffs. Hereby, the lower and upper bounds of the IV least square prenucleolus proposed in this paper can be attained through utilizing the lower and upper bounds of the IV coalitions payoffs, respectively. Moreover, it is pointed out that the derived IV least square prenucleolus possess some useful and important properties as expected.

The rest of this paper is organized as follows. Section 2 briefly reviews some basic concepts of intervals, IV cooperative games, cooperative games with fuzzy coalitions, and IV cooperative games with fuzzy coalitions in Choquet integral forms. In Sect. 3, we investigate the IV least square prenucleolus of a subclass of IV cooperative games with fuzzy coalitions in Choquet integral forms. In Sect. 4, some important properties of the IV least square prenucleolus are discussed. Section 5 gives two examples to illustrate the proposed method. Conclusion is made in Sect. 6.

2 Some Basic Concepts and Notations

2.1 Interval Arithmetic Operations and Interval-Valued Cooperative Games

Consider an interval $\bar{a} = [a_L, a_R] = \{a \,|a \in \mathrm{R}, a_L \leq a \leq a_R\}$, where R is the set of real numbers. Then $a_L \in \mathrm{R}$ and $a_R \in \mathrm{R}$ are called the lower bound and the upper bound of the interval $\bar{a}$, respectively. Let $\bar{\mathrm{R}}$ be the set of intervals on the set R. Some interval arithmetic operations are given as follows (Moore 1979; Li 2016):

Definition 1. Let $\bar{a} = [a_L, a_R]$ *and* $\bar{b} = [b_L, b_R]$ *be two intervals on the set* $\bar{\mathrm{R}}$. *The interval arithmetic operations are stipulated as follows:*

(1) Equality of two intervals: $\bar{a} = \bar{b} \Leftrightarrow a_L = b_L$ *and* $a_R = b_R$*;*
(2) Addition (or sum) of two intervals: $\bar{a} + \bar{b} = [a_L + b_L, a_R + b_R]$*;*
(3) Scalar multiplication of a real number and an interval:

$$\gamma\bar{a} = \begin{cases} [\gamma a_L, \gamma a_R] & \text{if } \gamma \geq 0 \\ [\gamma a_R, \gamma a_L] & \text{if } \gamma < 0 \end{cases},$$

where $\gamma \in R$ *is any real number.*

A n-person IV cooperative game $\bar{\upsilon}$ is an ordered-pair $< N, \bar{\upsilon} >$, where $N = \{1, 2, \cdots, n\}$ is the set of players and $\bar{\upsilon}$ is the IV characteristic function of players' coalitions, and $\bar{\upsilon}(\emptyset) = [0, 0]$. Generally, for any coalition $S \subseteq N$, $\bar{\upsilon}(S)$ is denoted by the interval $\bar{\upsilon}(S) = [\upsilon_L(S), \upsilon_R(S)]$, where $\upsilon_L(S) \leq \upsilon_R(S)$. In the sequent, a n-person IV cooperative game $< N, \bar{\upsilon} >$ is simply called the IV cooperative game $\bar{\upsilon}$. The set of IV cooperative games is denoted by $\bar{G}^n$. Obviously, if $\upsilon_L(S) = \upsilon_R(S)$ for any coalition $S \subseteq N$, then the IV cooperative game $\bar{\upsilon} \in \bar{G}^n$ is reduced to a (crisp) cooperative game, denoted by $\upsilon \in G^n$. Thus, IV cooperative games may be regarded as a generalization of cooperative games. For any IV cooperative games $\bar{\upsilon} \in \bar{G}^n$ and $\bar{\nu} \in \bar{G}^n$, according to the interval addition, $\bar{\upsilon} + \bar{\nu}$ is defined as an IV cooperative game with the IV characteristic function $\bar{\upsilon} + \bar{\nu}$, where

$$(\bar{\upsilon} + \bar{\nu})(S) = \bar{\upsilon}(S) + \bar{\nu}(S),$$

for any coalition $S \subseteq N$, i.e.,

$$(\bar{\upsilon} + \bar{\nu})(S) = [\upsilon_L(S) + \nu_L(S), \upsilon_R(S) + \nu_R(S)]. \tag{1}$$

Usually, $\bar{\upsilon} + \bar{\nu}$ is called the sum of the IV cooperative games $\bar{\upsilon} \in \bar{G}^n$ and $\bar{\nu} \in \bar{G}^n$.

For any IV cooperative game $\bar{\upsilon} \in \bar{G}^n$, it is easy to see that each player should receive an IV payoff from the cooperation due to the fact that each coalition's value is an interval. Let $\bar{x}_i(\bar{\upsilon}) = [x_{Li}(\bar{\upsilon}), x_{Ri}(\bar{\upsilon})]$ be the IV payoff which is allocated to the player $i \in N$ under the cooperation that the grand coalition is reached. Denote $\bar{\boldsymbol{x}}(\bar{\upsilon}) = (\bar{x}_1(\bar{\upsilon}), \bar{x}_2(\bar{\upsilon}), \cdots, \bar{x}_n(\bar{\upsilon}))^{\mathrm{T}}$, which is the vector of the IV payoffs for all n players in the grand coalition N. For any IV cooperative game $\bar{\upsilon} \in \bar{G}^n$, the efficiency of an IV payoff vector $\bar{\boldsymbol{x}}(\bar{\upsilon}) = (\bar{x}_1(\bar{\upsilon}), \bar{x}_2(\bar{\upsilon}), \cdots, \bar{x}_n(\bar{\upsilon}))^{\mathrm{T}}$ can be expressed as $\sum_{i=1}^{n} \bar{x}_i(\bar{\upsilon}) = \bar{\upsilon}(N)$.

2.2 Cooperative Games with Fuzzy Coalitions

Consider cooperative games with fuzzy coalitions, whose set of players is $N = \{1, 2, \cdots, n\}$. A fuzzy coalition is a fuzzy subset of N, which is defined as a mapping from N to $[0, 1]$. Any fuzzy coalition $\tilde{\boldsymbol{S}}$ can be represented by $\tilde{\boldsymbol{S}} = (\tilde{S}(1), \tilde{S}(2), \cdots, \tilde{S}(n))$, where $\tilde{S}(i) \in [0, 1]$ is the membership degree of i in $\tilde{\boldsymbol{S}}$, i.e., the rate of participation of player i to the coalition $\tilde{\boldsymbol{S}}$. Here, $\tilde{S}(i) = 0$ means that player i does not participate in coalition $\tilde{\boldsymbol{S}}$, and $\tilde{S}(i) = 1$ indicates that player i fully participate in coalition $\tilde{\boldsymbol{S}}$. If all $\tilde{S}(i)$ equal to either 0 or 1, then fuzzy coalitions degenerate to crisp coalitions N. For any fuzzy coalition $\tilde{\boldsymbol{S}}$, we denote the level set by $[\tilde{\boldsymbol{S}}]_h = \{i \in N | \tilde{S}(i) \geq h\}$ for any $h \in [0, 1]$, and denote the support by $\mathrm{Supp}(\tilde{\boldsymbol{S}}) = \{i \in N | \tilde{S}(i) > 0\}$. For any fuzzy coalitions $\tilde{\boldsymbol{S}}$ and $\tilde{\boldsymbol{S}}' = (\tilde{S}'(1), \tilde{S}'(2), \cdots, \tilde{S}'(n))$, we stipulate as follow:

$$\tilde{\boldsymbol{S}} \cup \tilde{\boldsymbol{S}}' = (\max\{\tilde{S}(i), \tilde{S}'(i)\})_{1 \times n}.$$

In the sequent, we write fuzzy coalition $\tilde{\boldsymbol{S}} \cup \tilde{S}'(i)$ for any $i \notin \mathrm{Supp}(\tilde{\boldsymbol{S}})$ instead of $\tilde{\boldsymbol{S}} \cup (0, \cdots, 0, \tilde{S}'(i), 0, \cdots, 0)$. Thus, for any $i \notin \mathrm{Supp}(\tilde{\boldsymbol{S}})$, $\tilde{\boldsymbol{S}} \cup \tilde{S}'(i)$ means that player i participates in the newly formed fuzzy coalition $\tilde{\boldsymbol{S}} \cup \tilde{S}'(i)$ with the participation rate $\max\{\tilde{S}(i), \tilde{S}'(i)\}$. We denote by $F(N)$ the set of all fuzzy subsets of N, A cooperative game with fuzzy coalitions is a function $\upsilon_f : F(N) \to [0, +\infty]$ and $\upsilon_f(\emptyset) = 0$.

2.3 Interval-Valued Cooperative Games with Fuzzy Coalitions in Choquet Integral Form

In the following, we study interval-valued cooperative games with fuzzy coalitions. Following Tsurumi et al. (2001) and combining with Definition 1, the IV cooperative game with fuzzy coalitions in Choquet integral form can be defined as follow.

For any $\tilde{\boldsymbol{S}} \in F(N)$, denote $Q(\tilde{\boldsymbol{S}}) = \{\tilde{S}(i) | \tilde{S}(i) > 0, i \in N\}$, and let $q(\tilde{\boldsymbol{S}})$ be the number of $Q(\tilde{\boldsymbol{S}})$. We write the elements of $Q(\tilde{\boldsymbol{S}})$ in the increasing order as $h_1 < h_2 < \cdots < h_{q(\tilde{\boldsymbol{S}})}$. Then, $\bar{\upsilon}_c$ is called as an IV cooperative game with fuzzy coalitions in Choquet integral form if and only if the following holds:

$$\bar{\upsilon}_c(\tilde{\boldsymbol{S}}) = \sum_{l=1}^{q(\tilde{\boldsymbol{S}})} \bar{\upsilon}([\tilde{\boldsymbol{S}}]_{h_l}) \cdot (h_l - h_{l-1})(\tilde{\boldsymbol{S}} \in F(N)), \tag{2}$$

where $h_0 = 0$, $\bar{\upsilon} \in \bar{G}^n$. We denote by $\bar{G}_c^n$ the set of all IV cooperative games with fuzzy coalitions in Choquet integral form.

Obviously, according to Definition 1, Eq. (2) can be written as follows:

$$\begin{aligned}\bar{\upsilon}_c(\tilde{\boldsymbol{S}}) &= [\sum_{l=1}^{q(\tilde{\boldsymbol{S}})} \upsilon_L([\tilde{\boldsymbol{S}}]_{h_l}) \cdot (h_l - h_{l-1}), \sum_{l=1}^{q(\tilde{\boldsymbol{S}})} \upsilon_R([\tilde{\boldsymbol{S}}]_{h_l}) \cdot (h_l - h_{l-1})] \\ &= [\upsilon_{cL}(\tilde{\boldsymbol{S}}), \upsilon_{cR}(\tilde{\boldsymbol{S}})],\end{aligned}$$

i.e.,

$$\upsilon_{cL}(\tilde{\boldsymbol{S}}) = \sum_{l=1}^{q(\tilde{\boldsymbol{S}})} \upsilon_L([\tilde{\boldsymbol{S}}]_{h_l}) \cdot (h_l - h_{l-1}), \upsilon_{cR}(\tilde{\boldsymbol{S}}) = \sum_{l=1}^{q(\tilde{\boldsymbol{S}})} \upsilon_R([\tilde{\boldsymbol{S}}]_{h_l}) \cdot (h_l - h_{l-1}) \tag{3}$$

Thus, we can easily obtain the values of all fuzzy coalitions by Eq. (2). It is obvious that there is a one-to-one correspondence between an IV cooperative game $\bar{\upsilon} \in \bar{G}^n$ and an IV cooperative game $\bar{\upsilon}_c \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form.

3 Interval-Valued Least Square Prenucleolus

For an arbitrary cooperative game $\upsilon \in G^n$ stated as in the previous Sect. 2.1, its least square prenucleolus can be defined as

$$\boldsymbol{x}^*(\upsilon) = (x_1^*(\upsilon), x_2^*(\upsilon), \cdots x_n^*(\upsilon))^{\mathrm{T}},$$

whose components are given as follows (Ruiz et al. 1996):

$$x_i^*(v) = \frac{v(N)}{n} + \frac{n\sum\limits_{S:i\in S} v(S) - \sum\limits_{j\in N}\sum\limits_{S:j\in S} v(S)}{n2^{n-2}} (i = 1, 2, \cdots, n),$$

or equivalently,

$$x_i^*(v) = \frac{v(N)}{n} + \frac{\sum\limits_{S:i\in S}(n-s_0)v(S) - \sum\limits_{S:i\notin S} s_0 v(S)}{n2^{n-2}}, \tag{4}$$

where s_0 denotes the cardinality of the coalition S.

For any IV cooperative game $\bar{v}_c \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form, we can define an associated cooperative game $v_c(\alpha)$ with fuzzy coalitions in Choquet integral form, where the set of players still is $N = \{1, 2, \cdots, n\}$ and the characteristic function $v_c(\alpha)$ is defined as follows:

$$v_c(\alpha)(\tilde{\boldsymbol{S}}) = (1-\alpha)v_{cL}(\tilde{\boldsymbol{S}}) + \alpha v_{cR}(\tilde{\boldsymbol{S}})(\tilde{\boldsymbol{S}} \in F(N)), \tag{5}$$

and $v_c(\alpha)(\emptyset) = 0$.

According to Eq. (4), we can easily obtain the least square prenucleolus $\boldsymbol{x}^*(v_c(\alpha)) = (x_1^*(v_c(\alpha)),\ x_2^*(v_c(\alpha)),\ \cdots x_n^*(v_c(\alpha)))^{\mathrm{T}}$ of the cooperative game $v_c(\alpha) \in G_c^n$ with fuzzy coalitions in Choquet integral form, where

$$x_i^*(v_c(\alpha)) = \frac{v_c(\alpha)(\tilde{\boldsymbol{N}})}{n} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})}(n-s)v_c(\alpha)(\tilde{\boldsymbol{S}}) - \sum\limits_{\tilde{\boldsymbol{S}}:i\notin \mathrm{Supp}(\tilde{\boldsymbol{S}})} s v_c(\alpha)(\tilde{\boldsymbol{S}})}{n2^{n-2}}$$
$$(i = 1, 2, \cdots, n),$$

i.e.,

$$x_i^*(v_c(\alpha)) = \frac{(1-\alpha)v_{cL}(\tilde{\boldsymbol{N}}) + \alpha v_{cR}(\tilde{\boldsymbol{N}})}{n} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})}(n-s)[(1-\alpha)v_{cL}(\tilde{\boldsymbol{S}}) + \alpha v_{cR}(\tilde{\boldsymbol{S}})] - \sum\limits_{\tilde{\boldsymbol{S}}:i\notin \mathrm{Supp}(\tilde{\boldsymbol{S}})} s[(1-\alpha)v_{cL}(\tilde{\boldsymbol{S}}) + \alpha v_{cR}(\tilde{\boldsymbol{S}})]}{n2^{n-2}}$$
$$(i = 1, 2, \cdots, n), \tag{6}$$

where s denotes the cardinality of $\mathrm{Supp}(\tilde{\boldsymbol{S}})$. Obviously, $x_i^*(v_c(\alpha))(i = 1, 2, \cdots, n)$ is a continuous function of the parameter $\alpha \in [0, 1]$.

Theorem 1. *For any IV cooperative game $\bar{v}_c \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form, if the following system of inequalities*

$$v_{cR}(\tilde{\boldsymbol{N}}) - v_{cL}(\tilde{\boldsymbol{N}}) \geq \frac{\sum\limits_{\tilde{\boldsymbol{S}}:i\notin \mathrm{Supp}(\tilde{\boldsymbol{S}})} s(v_{cR}(\tilde{\boldsymbol{S}}) - v_{cL}(\tilde{\boldsymbol{S}})) - \sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})}(n-s)(v_{cR}(\tilde{\boldsymbol{S}}) - v_{cL}(\tilde{\boldsymbol{S}}))}{2^{n-2}} \tag{7}$$
$$(i = 1, 2, \cdots, n)$$

is satisfied, then the least square prenucleolus $x_i^*(\upsilon_c(\alpha))(i = 1, 2, \cdots, n)$ *of the cooperative game* $\upsilon_c(\alpha)$ *with fuzzy coalitions in Choquet integral form is a monotonic and non-decreasing function of the parameter* $\alpha \in [0, 1]$.

Proof. For any $\alpha \in [0, 1]$ and $\alpha' \in [0, 1]$, according to Eq. (6), we have

$$x_i^*(\upsilon_c(\alpha)) - x_i^*(\upsilon_c(\alpha')) = \frac{(\alpha - \alpha')(\upsilon_{cR}(\tilde{\boldsymbol{N}}) - \upsilon_{cL}(\tilde{\boldsymbol{N}}))}{n} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} (n-s)[(\alpha - \alpha')(\upsilon_{cR}(\tilde{\boldsymbol{S}}) - \upsilon_{cL}(\tilde{\boldsymbol{S}})] - \sum\limits_{\tilde{\boldsymbol{S}}:i\notin \mathrm{Supp}(\tilde{\boldsymbol{S}})} s[(\alpha - \alpha')(\upsilon_{cR}(\tilde{\boldsymbol{S}}) - \upsilon_{cL}(\tilde{\boldsymbol{S}}))]}{n2^{n-2}}$$
$$= \frac{(\alpha - \alpha')}{n}[(\upsilon_{cR}(\tilde{\boldsymbol{N}}) - \upsilon_{cL}(\tilde{\boldsymbol{N}})) + \frac{\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} (n-s)(\upsilon_{cR}(\tilde{\boldsymbol{S}}) - \upsilon_{cL}(\tilde{\boldsymbol{S}})) - \sum\limits_{\tilde{\boldsymbol{S}}:i\notin \mathrm{Supp}(\tilde{\boldsymbol{S}})} s(\upsilon_{cR}(\tilde{\boldsymbol{S}}) - \upsilon_{cL}(\tilde{\boldsymbol{S}}))}{2^{n-2}}],$$

where $i = 1, 2, \cdots, n$.

If $\alpha \geq \alpha'$, then combining with Eq. (7), we have

$$x_i^*(\upsilon_c(\alpha)) - x_i^*(\upsilon_c(\alpha')) \geq 0,$$

i.e., $x_i^*(\upsilon_c(\alpha)) \geq x_i^*(\upsilon_c(\alpha'))$ $(i = 1, 2, \cdots, n)$, which mean that the least square prenucleolus $x_i^*(\upsilon_c(\alpha))(i = 1, 2, \cdots, n)$ are monotonic and non-decreasing functions of the parameter $\alpha \in [0, 1]$. Thus, we have completed the proof of Theorem 1.

We call Eq. (7) as a size monotonicity-like condition. For any IV cooperative game $\bar{\upsilon}_c \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form, if it satisfies Eq. (7), then it is directly derived from Theorem 1 and Eq. (6) that the lower and upper bounds of $\bar{x}_i^*(\bar{\upsilon}_c)(i = 1, 2, \cdots, n)$ are given as follows:

$$x_{Li}^*(\bar{\upsilon}_c) = x_i^*(\upsilon_c(0)) = \frac{\upsilon_{cL}(\tilde{\boldsymbol{N}})}{n} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} (n-s)\upsilon_{cL}(\tilde{\boldsymbol{S}}) - \sum\limits_{\tilde{\boldsymbol{S}}:i\notin \mathrm{Supp}(\tilde{\boldsymbol{S}})} s\upsilon_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}} (i = 1, 2, \cdots, n),$$

and

$$x_{Ri}^*(\bar{\upsilon}_c) = x_i^*(\upsilon_c(1)) = \frac{\upsilon_{cR}(\tilde{\boldsymbol{N}})}{n} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} (n-s)\upsilon_{cR}(\tilde{\boldsymbol{S}}) - \sum\limits_{\tilde{\boldsymbol{S}}:i\notin \mathrm{Supp}(\tilde{\boldsymbol{S}})} s\upsilon_{cR}(\tilde{\boldsymbol{S}})}{n2^{n-2}}$$
$$(i = 1, 2, \cdots, n).$$

Thus, $\bar{x}_i^*(\bar{\upsilon}_c)$ of the players $i(i = 1, 2, \cdots, n)$ in the IV cooperative game $\bar{\upsilon}_c \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form are directly and explicitly expressed as follows:

$$\bar{x}_i^*(\bar{\upsilon}) = [\frac{\upsilon_{cL}(\tilde{\boldsymbol{N}})}{n} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} (n-s)\upsilon_{cL}(\tilde{\boldsymbol{S}}) - \sum\limits_{\tilde{\boldsymbol{S}}:i\notin \mathrm{Supp}(\tilde{\boldsymbol{S}})} s\upsilon_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}}, \frac{\upsilon_{cR}(\tilde{\boldsymbol{N}})}{n} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} (n-s)\upsilon_{cR}(\tilde{\boldsymbol{S}}) - \sum\limits_{\tilde{\boldsymbol{S}}:i\notin \mathrm{Supp}(\tilde{\boldsymbol{S}})} s\upsilon_{cR}(\tilde{\boldsymbol{S}})}{n2^{n-2}}],$$

or equivalently,

$$\bar{x}_i^*(\bar{\upsilon}_c) = [\frac{\upsilon_{cL}(\tilde{\boldsymbol{N}})}{n} + \frac{n\sum_{\tilde{\boldsymbol{S}}:i\in \text{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}}) - \sum_{j\in \text{Supp}(\tilde{\boldsymbol{N}})} \sum_{\tilde{\boldsymbol{S}}:j\in \text{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}}, \frac{\upsilon_{cR}(\tilde{\boldsymbol{N}})}{n} + \frac{n\sum_{\tilde{\boldsymbol{S}}:i\in \text{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}}) - \sum_{j\in \text{Supp}(\tilde{\boldsymbol{N}})} \sum_{\tilde{\boldsymbol{S}}:j\in \text{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}})}{n2^{n-2}}]. \tag{8}$$

4 Some Properties of the Interval-Valued Least Square Prenucleolus

Players $i \in N$ and $k \in N$ $(i \neq k)$ are said to be symmetric in the IV cooperative game $\bar{\upsilon}_c \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form, if $\bar{\upsilon}_c(\tilde{\boldsymbol{S}} \cup \tilde{S}'(i)) = \bar{\upsilon}_c(\tilde{\boldsymbol{S}} \cup \tilde{S}'(k))$ for any fuzzy coalition $\tilde{\boldsymbol{S}} \in F(N)$ with $i, k \notin \text{Supp}(\tilde{\boldsymbol{S}})$.

Let P be an arbitrary permutation of the set N. The permutation P has associated with a substitution σ which is a one-to-one function, i.e., $\sigma : N \to N$ such that for $i \in N$, then $\sigma(i) \in N$ is the corresponding element, which is changed in the permutation. For an IV cooperative game $\bar{\upsilon}_c \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form, we define an IV cooperative game $\bar{\upsilon}_c^\sigma \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form and its IV characteristic function is $\bar{\upsilon}^\sigma$, where $\bar{\upsilon}_c^\sigma(\tilde{\boldsymbol{S}}) = \bar{\upsilon}_c(\sigma^{-1}(\tilde{\boldsymbol{S}}))$ for any fuzzy coalition $\tilde{\boldsymbol{S}} \in F(N)$.

In the sequent, a theorem is given to describe some properties of the IV least square prenucleolus of IV cooperative game $\bar{\upsilon}_c \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form which satisfies Eq. (7).

Theorem 2. *For an arbitrary IV cooperative game $\bar{\upsilon}_c \in \bar{G}_c^n$ with fuzzy coalitions in Choquet integral form, if $\bar{\upsilon}_c$ satisfies Eq. (7), then there always exists a unique IV least square prenucleolus $\bar{\boldsymbol{x}}^*(\bar{\upsilon}_c)$ determined by Eq. (8), which satisfies the following properties:*

(1) Efficiency: $\sum_{i\in \text{Supp}(\tilde{\boldsymbol{N}})} \bar{x}_i^*(\bar{\upsilon}_c) = \bar{\upsilon}_c(\tilde{\boldsymbol{N}})$,

(2) Additivity: $\bar{x}_i^*(\bar{\upsilon}_c + \bar{\nu}_c) = \bar{x}_i^*(\bar{\upsilon}_c) + \bar{x}_i^*(\bar{\nu}_c)(i = 1, 2, \cdots, n)$ *for any IV cooperative game* $\bar{\upsilon}_c \in \bar{G}_c^n$ *with fuzzy coalitions in Choquet integral form which satisfies Eq. (7),*

(3) Symmetry: $\bar{x}_i^*(\bar{\upsilon}_c) = \bar{x}_k^*(\bar{\upsilon}_c)$ *for any symmetric players* $i \in N$ *and* $k \in N$ $(i \neq k)$*,*

(4) Anonymity: $\bar{x}_{\sigma(i)}^*(\bar{\upsilon}_c^\sigma) = \bar{x}_i^*(\bar{\upsilon}_c)(i = 1, 2, \cdots, n)$ *for any substitution* σ *on the set* N*.*

Proof. According to Eq. (8) and Definition 1, there always exists a unique IV least square prenucleolus $\bar{\boldsymbol{x}}^*(\bar{\upsilon}_c)$, which is determined by Eq. (8).

(1) According to Eq. (8), and combining with Definition 1, we have

$$\sum_{i\in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \bar{x}^*_{Li}(\bar{\upsilon}_c) = \sum_{i\in \mathrm{Supp}(\tilde{\boldsymbol{N}})} [\frac{\upsilon_{cL}(\tilde{\boldsymbol{N}})}{n} + \frac{n\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}}) - \sum\limits_{j\in \mathrm{Supp}(\tilde{\boldsymbol{N}})}\sum\limits_{\tilde{\boldsymbol{S}}:j\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}}]$$
$$= \upsilon_{cL}(\tilde{\boldsymbol{N}}) + \frac{n\sum\limits_{i\in \mathrm{Supp}(\tilde{\boldsymbol{N}})}\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}}) - n\sum\limits_{j\in \mathrm{Supp}(\tilde{\boldsymbol{N}})}\sum\limits_{\tilde{\boldsymbol{S}}:j\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}}$$
$$= \upsilon_{cL}(\tilde{\boldsymbol{N}}),$$

i.e., $\sum\limits_{i\in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \bar{x}^*_{Li}(\bar{\upsilon}_c) = \upsilon_{cL}(\tilde{\boldsymbol{N}})$.

Similarly, it can be easily proven that $\sum\limits_{i\in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \bar{x}^*_{Ri}(\bar{\upsilon}_c) = \upsilon_{cR}(\tilde{\boldsymbol{N}})$, Combining with the aforementioned conclusion, according to the case (1) of Definition 1, we obtain $\sum\limits_{i\in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \bar{x}^*_{i}(\bar{\upsilon}_c) = \bar{\upsilon}_c(\tilde{\boldsymbol{N}})$.

Therefore we have proved the efficiency.

(2) According to Eqs. (1) and (8), we have

$$x^*_{Li}(\bar{\upsilon}_c + \bar{\nu}_c) = \frac{\upsilon_{cL}(\tilde{\boldsymbol{N}}) + \nu_{cL}(\tilde{\boldsymbol{N}})}{n}$$
$$+ \frac{n\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} (\upsilon_{cL}(\tilde{\boldsymbol{S}}) + \nu_{cL}(\tilde{\boldsymbol{S}})) - \sum\limits_{j\in \mathrm{Supp}(\tilde{\boldsymbol{N}})}\sum\limits_{\tilde{\boldsymbol{S}}:j\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} (\upsilon_{cL}(\tilde{\boldsymbol{S}}) + \nu_{cL}(\tilde{\boldsymbol{S}}))}{n2^{n-2}}$$
$$= (\frac{\upsilon_{cL}(\tilde{\boldsymbol{S}})}{n} + \frac{n\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}}) - \sum\limits_{j\in \mathrm{Supp}(\tilde{\boldsymbol{N}})}\sum\limits_{\tilde{\boldsymbol{S}}:j\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}})$$
$$+ (\frac{\nu_{cL}(\tilde{\boldsymbol{N}})}{n} + \frac{n\sum\limits_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \nu_{cL}(\tilde{\boldsymbol{S}}) - \sum\limits_{j\in \mathrm{Supp}(\tilde{\boldsymbol{N}})}\sum\limits_{\tilde{\boldsymbol{S}}:j\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \nu_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}})$$
$$= x^*_{Li}(\bar{\upsilon}_c) + x^*_{Li}(\bar{\nu}_c),$$

i.e., $x^*_{Li}(\bar{\upsilon}_c + \bar{\nu}_c) = x^*_{Li}(\bar{\upsilon}_c) + x^*_{Li}(\bar{\nu}_c)(i = 1, 2, \cdots, n)$.

Analogously, we can easily prove that $x^*_{Ri}(\bar{\upsilon}_c+\bar{\nu}_c) = x^*_{Ri}(\bar{\upsilon}_c)+x^*_{Ri}(\bar{\nu}_c)$. Hence, we obtain

$$\bar{x}^*_i(\bar{\upsilon}_c + \bar{\nu}_c) = \bar{x}^*_i(\bar{\upsilon}_c) + \bar{x}^*_i(\bar{\nu}_c), (i = 1, 2, \cdots, n).$$

Thus we have proved the additivity.

(3) As the assumption that the players $i \in N$ and $k \in N$ $(i \neq k)$ are symmetric in the IV cooperative game $\bar{\upsilon}_c \in \bar{G}^n_c$ with fuzzy coalitions in Choquet integral form, then we know that for any fuzzy coalition $\tilde{\boldsymbol{S}} \in F(N)$ with $i, k \notin \mathrm{Supp}(\tilde{\boldsymbol{S}})$, we have

$$\bar{\upsilon}_c(\tilde{\boldsymbol{S}} \cup \tilde{S}'(i)) = \bar{\upsilon}_c(\tilde{\boldsymbol{S}} \cup \tilde{S}'(k)).$$

Namely, $\upsilon_{cL}(\tilde{\boldsymbol{S}}\cup\tilde{S}'(i)) = \upsilon_{cL}(\tilde{\boldsymbol{S}}\cup\tilde{S}'(k))$ and $\upsilon_{cR}(\tilde{\boldsymbol{S}}\cup\tilde{S}'(i)) = \upsilon_{cR}(\tilde{\boldsymbol{S}}\cup\tilde{S}'(k))$. Hence, we have

$$\sum_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}}) = \sum_{\tilde{\boldsymbol{S}}:k\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}}), \quad \sum_{\tilde{\boldsymbol{S}}:i\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}}) = \sum_{\tilde{\boldsymbol{S}}:k\in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}}).$$

According to Eq. (8), we can easily obtain that

$$\bar{x}_i^*(\bar{\upsilon}_c) = [\frac{\upsilon_{cL}(\tilde{\boldsymbol{N}})}{n} + \frac{n \sum\limits_{\tilde{\boldsymbol{S}}:i \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}}) - \sum\limits_{j \in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \sum\limits_{\tilde{\boldsymbol{S}}:j \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}}, \frac{\upsilon_{cR}(\tilde{\boldsymbol{N}})}{n}$$
$$+ \frac{n \sum\limits_{\tilde{\boldsymbol{S}}:i \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}}) - \sum\limits_{j \in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \sum\limits_{\tilde{\boldsymbol{S}}:j \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}})}{n2^{n-2}}]$$
$$= [\frac{\upsilon_{cL}(\tilde{\boldsymbol{N}})}{n} + \frac{n \sum\limits_{\tilde{\boldsymbol{S}}:k \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}}) - \sum\limits_{j \in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \sum\limits_{\tilde{\boldsymbol{S}}:j \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}}, \frac{\upsilon_{cR}(\tilde{\boldsymbol{N}})}{n}$$
$$+ \frac{n \sum\limits_{\tilde{\boldsymbol{S}}:k \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}}) - \sum\limits_{j \in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \sum\limits_{\tilde{\boldsymbol{S}}:j \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}})}{n2^{n-2}}]$$
$$= \bar{x}_k^*(\bar{\upsilon}_c),$$

i.e., $\bar{x}_i^*(\bar{\upsilon}_c) = \bar{x}_k^*(\bar{\upsilon}_c)$. Thus we have proved the symmetry.

(4) According to Eq. (8) and combining with $\bar{\upsilon}_c^\sigma(\tilde{\boldsymbol{S}})$=$\bar{\upsilon}_c(\sigma^{-1}(\tilde{\boldsymbol{S}}))$, we can obtain that

$$\bar{x}_{\sigma(i)}^*(\bar{\upsilon}_c^\sigma) = [\frac{\upsilon_{cL}^\sigma(\tilde{\boldsymbol{N}})}{n} + \frac{n \sum\limits_{\tilde{\boldsymbol{S}}:\sigma(i) \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}^\sigma(\tilde{\boldsymbol{S}}) - \sum\limits_{j \in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \sum\limits_{\tilde{\boldsymbol{S}}:j \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}^\sigma(\tilde{\boldsymbol{S}})}{n2^{n-2}}, \frac{\upsilon_{cR}^\sigma(\tilde{\boldsymbol{N}})}{n}$$
$$+ \frac{n \sum\limits_{\tilde{\boldsymbol{S}}:\sigma(i) \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}^\sigma(\tilde{\boldsymbol{S}}) - \sum\limits_{j \in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \sum\limits_{\tilde{\boldsymbol{S}}:j \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}^\sigma(\tilde{\boldsymbol{S}})}{n2^{n-2}}]$$
$$= [\frac{\upsilon_{cL}(\tilde{\boldsymbol{N}})}{n} + \frac{n \sum\limits_{\tilde{\boldsymbol{S}}:i \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}}) - \sum\limits_{j \in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \sum\limits_{\tilde{\boldsymbol{S}}:j \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cL}(\tilde{\boldsymbol{S}})}{n2^{n-2}}, \frac{\upsilon_{cR}(\tilde{\boldsymbol{N}})}{n}$$
$$+ \frac{n \sum\limits_{\tilde{\boldsymbol{S}}:i \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}}) - \sum\limits_{j \in \mathrm{Supp}(\tilde{\boldsymbol{N}})} \sum\limits_{\tilde{\boldsymbol{S}}:j \in \mathrm{Supp}(\tilde{\boldsymbol{S}})} \upsilon_{cR}(\tilde{\boldsymbol{S}})}{n2^{n-2}}]$$
$$= \bar{x}_i^*(\bar{\upsilon}_c),$$

i.e., $\bar{x}_{\sigma(i)}^*(\bar{\upsilon}_c^\sigma) = \bar{x}_i^*(\bar{\upsilon}_c)(i = 1, 2, \cdots, n)$.

Therefore, we have proved the Theorem 2.

In particular, for any fuzzy coalition $\tilde{\boldsymbol{S}} = (\tilde{S}(1), \tilde{S}(2), \cdots, \tilde{S}(n))$, a new fuzzy coalition is defined as $\tilde{\boldsymbol{S}}'' = (\tilde{S}''(1), \tilde{S}''(2), \cdots, \tilde{S}''(n))$, whose components are given as

$$\tilde{S}''(i) = \begin{cases} 1, & \text{if } 0 < \tilde{S}(i) \le 1 \\ 0, & \text{if } \tilde{S}(i) = 0 \end{cases},$$

where $i = 1, 2, \cdots, n$.

Therefore $\tilde{\boldsymbol{S}}''$ degenerates to the crisp coalition $S \subseteq N$, we regard the coalition S as the associated crisp coalition of fuzzy coalition $\tilde{\boldsymbol{S}}''$. And we have

$$Q(\tilde{\boldsymbol{S}}'') = \{1\}, q(\tilde{\boldsymbol{S}}'') = 1, h_1 = 1, h_0 = 0.$$

According to Eq. (3), we get

$$\upsilon_{cL}(\tilde{\boldsymbol{S}}'') = \sum_{l=1}^{q(\tilde{\boldsymbol{S}}'')} \upsilon_L([\tilde{\boldsymbol{S}}'']_{h_l}) \cdot (h_l - h_{l-1}) = \upsilon_L(S), \upsilon_{cR}(\tilde{\boldsymbol{S}}'') = \sum_{l=1}^{q(\tilde{\boldsymbol{S}}'')} \upsilon_R([\tilde{\boldsymbol{S}}'']_{h_l}) \cdot (h_l - h_{l-1}) = \upsilon_R(S).$$

So Eq. (8) can be rewritten as

$$\bar{x}_i^*(\bar{\upsilon}_c) = [\frac{\upsilon_L(N)}{n} + \frac{n\sum\limits_{S:i\in S}\upsilon_L(S) - \sum\limits_{j\in N}\sum\limits_{S:j\in S}\upsilon_L(S)}{n2^{n-2}}, \frac{\upsilon_R(N)}{n} + \frac{n\sum\limits_{S:i\in S}\upsilon_R(S) - \sum\limits_{j\in N}\sum\limits_{S:j\in S}\upsilon_R(S)}{n2^{n-2}}] \quad (9)$$

Furthermore, if the interval-valued payoffs $\bar{\upsilon}(S) = [\upsilon_L(S), \upsilon_R(S)]$ degenerate to crisp payoffs, i.e., $\upsilon_L(S) = \upsilon_R(S) = \upsilon(S)$, then, Eq. (9) simply becomes

$$\bar{x}_i^*(\bar{\upsilon}_c) = \frac{\upsilon(N)}{n} + \frac{n\sum\limits_{S:i\in S}\upsilon(S) - \sum\limits_{j\in N}\sum\limits_{S:j\in S}\upsilon(S)}{n2^{n-2}}.$$

Namely, if a coalition size monotonicity-like IV cooperative game with fuzzy coalitions in Choquet integral form degenerates to the associated IV cooperative game with crisp coalitions, then Eq. (8) degenerates to the IV least square prenucleolus introduced by Li and Ye (2016), and if interval values further degenerate to crisp values, then we obtain the least square prenucleolus introduced by Ruiz et al. (1996) as we expect.

5 Two Numerical Examples

Example 1. Suppose that there exist three companies (i.e., players) 1, 2, and 3, they plan to work together to complete a project. Due to the incomplete and uncertain information, they can only estimate the ranges of their profits (or gains) rather than precisely forecast their profits. Suppose that if the companies work independently, then their gains are expressed as

$$\bar{\upsilon}^0(1) = [0, 2], \bar{\upsilon}^0(2) = [1, 2.5], \bar{\upsilon}^0(3) = [1.5, 2.5].$$

If any two companies cooperatively complete the project, then their gains are

$$\bar{\upsilon}^0(1, 2) = [3, 5], \bar{\upsilon}^0(1, 3) = [2.5, 6], \bar{\upsilon}^0(2, 3) = [5, 8].$$

If three companies work together, then the gain is $\bar{\upsilon}^0(N') = [7.5, 10]$, and $\bar{\upsilon}^0(\emptyset) = 0$, where $N'=\{1, 2, 3\}$. However, in order to reduce or avoid risk, three companies are unwilling to put all their resources into one project, they may choose to provide partial resources. Suppose that each player has 100 units of resources, if player 1 supplies 30 units resources to the cooperation, then we regard the rate of participation of player 1 as 0.3=30/100. Consider a fuzzy coalition $\tilde{\boldsymbol{S}}'$, where $\tilde{S}'(1) = 0.3$, $\tilde{S}'(2) = 0.4$, $\tilde{S}'(3) = 0.5$. Hence, the problem can be regarded as a three-person IV cooperative game with fuzzy coalitions. Now compute the IV least square prenucleolus of the IV cooperative game $\bar{v}_c^0 \in \bar{G}_c^3$ with fuzzy coalitions in Choquet integral form.

Using the above values of the crisp coalitions, and according to Eq. (2), the values of the fuzzy coalitions are

$$\bar{v}_c^0(0.3,0,0) = [0,0.6],\ \bar{v}_c^0(0,0.4,0) = [0.4,1],\ \bar{v}_c^0(0,0,0.5) = [0.75,1.25], \bar{v}_c^0(0.3,0.4,0) = [1,1.75],$$
$$\bar{v}_c^0(0.3,0,0.5) = [1.05,2.3],\ \bar{v}_c^0(0,0.4,0.5) = [2.15,3.45],\ \bar{v}_c^0(0.3,0.4,0.5) = [2.9,4.05].$$

Obviously, the IV cooperative game $\bar{v}_c^0 \in \bar{G}_c^3$ with fuzzy coalitions in Choquet integral form satisfies Eq. (7). Thus, according to Eq. (8), we have

$$\begin{aligned}\bar{x}_1^*(\bar{v}_c^0) = {} & [\frac{v_{cL}^0(\tilde{\boldsymbol{N}}')}{3} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}':1\in \mathrm{Supp}(\tilde{\boldsymbol{S}}')}(3-s)v_{cL}^0(\tilde{\boldsymbol{S}}') - \sum\limits_{\tilde{\boldsymbol{S}}':1\notin \mathrm{Supp}(\tilde{\boldsymbol{S}}')} s v_{cL}^0(\tilde{\boldsymbol{S}}')}{3\times 2^{3-2}}, \frac{v_{cR}^0(\tilde{\boldsymbol{N}}')}{3}\\ & + \frac{\sum\limits_{\tilde{\boldsymbol{S}}':1\in \mathrm{Supp}(\tilde{\boldsymbol{S}}')}(3-s)v_{cR}^0(\tilde{\boldsymbol{S}}') - \sum\limits_{\tilde{\boldsymbol{S}}':1\notin \mathrm{Supp}(\tilde{\boldsymbol{S}}')} s v_{cR}^0(\tilde{\boldsymbol{S}}')}{3\times 2^{3-2}}]\\ = {} & [\frac{2.9}{3} + \frac{(0+1+1.05)-(0.4+0.75+4.3)}{6}, \frac{4.05}{3}\\ & + \frac{(1.2+1.75+2.3)-(1+1.25+6.9)}{6}] = [0.4,0.7],\end{aligned}$$

$$\begin{aligned}\bar{x}_2^*(\bar{v}_c^0) = {} & [\frac{v_{cL}^0(\tilde{\boldsymbol{N}}')}{3} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}':2\in \mathrm{Supp}(\tilde{\boldsymbol{S}}')}(3-s)v_{cL}^0(\tilde{\boldsymbol{S}}') - \sum\limits_{\tilde{\mathbf{S}}':2\notin \mathrm{Supp}(\tilde{\boldsymbol{S}}')} s v_{cL}^0(\tilde{\boldsymbol{S}}')}{3\times 2^{3-2}}, \frac{v_{cR}^0(\tilde{\boldsymbol{N}}')}{3}\\ & + \frac{\sum\limits_{\tilde{\mathbf{S}}':2\in \mathrm{Supp}(\tilde{\boldsymbol{S}}')}(3-s)v_{cR}^0(\tilde{\boldsymbol{S}}') - \sum\limits_{\tilde{\boldsymbol{S}}':2\notin \mathrm{Supp}(\tilde{\boldsymbol{S}}')} s v_{cR}^0(\tilde{\boldsymbol{S}}')}{3\times 2^{3-2}}]\\ = {} & [\frac{2.9}{3} + \frac{(0.8+1+2.15)-(0+0.75+2.1)}{6}, \frac{4.05}{3}\\ & + \frac{(2+1.75+3.45)-(0.6+1.25+4.6)}{6}] = [1.15,1.475],\end{aligned}$$

and

$$\begin{aligned}\bar{x}_3^*(\bar{v}_c^0) = {} & [\frac{v_{cL}^0(\tilde{\boldsymbol{N}}')}{3} + \frac{\sum\limits_{\tilde{\mathbf{S}}':3\in \mathrm{Supp}(\tilde{\mathbf{S}}')}(3-s)v_{cL}^0(\tilde{\mathbf{S}}') - \sum\limits_{\tilde{\boldsymbol{S}}':3\notin \mathrm{Supp}(\tilde{\boldsymbol{S}}')} s v_{cL}^0(\tilde{\boldsymbol{S}}')}{3\times 2^{3-2}}, \frac{v_{cR}^0(\tilde{\boldsymbol{N}}')}{3}\\ & + \frac{\sum\limits_{\tilde{\boldsymbol{S}}':3\in \mathrm{Supp}(\tilde{\boldsymbol{S}}')}(3-s)v_{cR}^0(\tilde{\boldsymbol{S}}') - \sum\limits_{\tilde{\boldsymbol{S}}':3\notin \mathrm{Supp}(\tilde{\boldsymbol{S}}')} s v_{cR}^0(\tilde{\boldsymbol{S}}')}{3\times 2^{3-2}}]\\ = {} & [\frac{2.9}{3} + \frac{(1.5+1.05+2.15)-(0+0.4+2)}{6}, \frac{4.05}{3}\\ & + \frac{(2.5+2.3+3.45)-(0.6+1+3.5)}{6}] = [1.35,1.875].\end{aligned}$$

Hence, we obtain the IV least square prenucleolus of the IV cooperative game with fuzzy coalitions in Choquet integral form as follows:

$$\bar{\boldsymbol{x}}^*(\bar{v}_c^0) = ([0.4, 0.7],\ [1.15, 1.475],\ [1.35, 1.875])^{\mathrm{T}}.$$

However, if we using the Moore's interval subtraction (Moore 1979), i.e., $\bar{a} - \bar{b} = [a_L - b_R, a_R - b_L]$, then we have

$$\begin{aligned}\bar{x}_1^{*\mathrm{M}}(\bar{v}_c^0) &= \frac{\bar{v}_c^0(\tilde{\boldsymbol{N}}')}{3} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}':1\in\mathrm{Supp}(\tilde{\boldsymbol{S}}')} (3-s)\bar{v}_c^0(\tilde{\boldsymbol{S}}') - \sum\limits_{\tilde{\mathbf{S}}':1\notin\mathrm{Supp}(\tilde{\boldsymbol{S}}')} s\bar{v}_c^0(\tilde{\boldsymbol{S}}')}{3\times 2^{3-2}} = \frac{[2.9, 4.05]}{3} \\ &+ \frac{(2\times[0, 0.6] + [1, 1.75] + [1.05, 2.3]) - ([0.4, 1] + [0.75, 1.25] + 2\times[2.15, 3.45])}{6} \\ &= \frac{[2.9, 4.05]}{3} + \frac{[2.05, 5.25] - [5.45, 9.15]}{6} = [-0.217, 1.317].\end{aligned}$$

Obviously, the above result is irrational due to the lower bound $-0.217 < 0$ from the realistic meaning of the profit.

Example 2. The economic situation is stated as in Example 1. We construct a new IV cooperative game $\bar{v}'' \in \bar{G}^2$, where the set of players $N'' = \{1, 2\}$. Suppose that

$$\bar{v}''(1) = [0.3, 1], \bar{v}''(2) = [2, 5], \bar{v}''(1, 2) = [5.5, 6], S''(1) = 0.3, S''(2) = 0.4.$$

Let us discuss the IV least square prenucleolus of the IV cooperative game $\bar{v}_c'' \in \bar{G}_c^2$ with fuzzy coalitions in Choquet integral form.

According to Eq. (2), we have

$$\bar{v}_c''(0.3, 0) = [0.09, 0.3], \bar{v}_c''(0, 0.4) = [0.8, 2], \bar{v}_c''(0.3, 0.4) = [1.85, 2.3].$$

Therefore, we obtain

$$\bar{v}_{cR}''(0.3, 0.4) - \bar{v}_{cL}''(0.3, 0.4) = 2.3 - 1.85 = 0.45.$$

Hereby, we have

$$\begin{aligned}&v_{cR}''(\tilde{\boldsymbol{N}}'') - v_{cL}''(\tilde{\boldsymbol{N}}'') \\ &\quad< \frac{\sum\limits_{\tilde{\boldsymbol{S}}'':1\notin\mathrm{Supp}(\tilde{\boldsymbol{S}}'')} s(v_{cR}(\tilde{\boldsymbol{S}}'') - v_{cL}(\tilde{\boldsymbol{S}}'')) - \sum\limits_{\tilde{\boldsymbol{S}}'':1\in\mathrm{Supp}(\tilde{\boldsymbol{S}}'')} (n-s)(v_{cR}(\tilde{\boldsymbol{S}}'') - v_{cL}(\tilde{\boldsymbol{S}}''))}{2} \\ &= 0.495,\end{aligned}$$

i.e., the IV cooperative game $\bar{v}_c'' \in \bar{G}_c^2$ with fuzzy coalitions in Choquet integral form does not satisfy Eq. (7). But, if Eq. (8) is used, then we can obtain

$$\bar{x}_1^*(\bar{v}_c'') = [\frac{v''_{cL}(\tilde{\boldsymbol{N}}'')}{2} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}'':1\in \mathrm{Supp}(\tilde{\boldsymbol{S}}'')}(2-s)v''_{cL}(\tilde{\boldsymbol{S}}'') - \sum\limits_{\tilde{\boldsymbol{S}}'':1\notin \mathrm{Supp}(\tilde{\boldsymbol{S}}'')} s v''_{cL}(\tilde{\boldsymbol{S}}'')}{2\times 2^{2-2}}, \frac{v''_{cR}(\tilde{\boldsymbol{N}}'')}{2} + \frac{\sum\limits_{\tilde{\boldsymbol{S}}'':1\in \mathrm{Supp}(\tilde{\boldsymbol{S}}'')}(2-s)v''_{cR}(\tilde{\boldsymbol{S}}'') - \sum\limits_{\tilde{\boldsymbol{S}}'':1\notin \mathrm{Supp}(\tilde{\boldsymbol{S}}'')} s v''_{cR}(\tilde{\boldsymbol{S}}'')}{2\times 2^{2-2}}]$$
$$= [\frac{1.85}{2} + \frac{0.09-0.8}{2}, \frac{2.3}{2} + \frac{0.3-2}{2}] = [0.57, 0.3].$$

Clearly, the above result is irrational due to $0.57 > 0.3$ from the notation of intervals. That is to say, if Eq. (7) is not satisfied, then the IV least square prenucleolus of the IV cooperative game with fuzzy coalitions in Choquet integral form given by Eq. (8) is not always reasonable and correct.

6 Conclusions

The least square prenucleolus is one of the important solution concepts of cooperative game. In this paper, we use Choquet integral to establish the IV fuzzy characteristic function. The main contribution of this paper is that we develop a simplified method to compute the IV least square prenucleolus for the class of IV cooperative games with fuzzy coalitions in Choquet integral forms which satisfy Eq. (7) and obtain their simplified expressions. Unlike much existing research, the method proposed in this paper uses the monotonicity rather than the Moore's interval subtraction or interval comparison. Hence, it can overcome the drawbacks of them. What's more, we give some important properties of the IV least square prenucleolus introduced in this paper. Finally, the method may make contribution to the theoretical investigation of IV cooperative games with fuzzy coalitions. In the future, we will study other solution concepts of the cooperative games under uncertain situations, such as cooperative games with interval-valued coalitions and fuzzy payoffs.

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Quadratic Programming Models and Method for Interval-Valued Cooperative Games with Fuzzy Coalitions

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Abstract. The purpose of this paper is to develop a quadratic programming method for solving interval-valued cooperative games with fuzzy coalitions. In this method, the interval-valued cooperative games with fuzzy coalitions are converted into the interval-valued cooperative games (with crisp coalitions) by using the Choquet integral. Two auxiliary quadratic programming models for solving the interval-valued cooperative games are constructed by using the least square method and distance between intervals. The proposed models and method are validated and compared with other similar methods. A numerical example is examined to demonstrate the validity, superiority and applicability of the method proposed in this paper.

Keywords and phrases: Cooperative game · Quadratic programming · Dissatisfaction function · Lagrange multiplier method · Choquet integral · Fuzzy coalition

1 Introduction

As competition and cooperation is becoming an important problem in management with economy globalization and integration, cooperative games have become an active research field in management science and operational research. According to players' knowledge about payoff values and participation levels in coalitions, cooperative games are divided into two categories: classical cooperative games and fuzzy cooperative games. In classical cooperative games, the players either fully participate in a specific coalition or fully opt out of it, which means that the participation rate of each player in a coalition is either 1 or 0. Besides, the payoff value of each coalition is expressed as a real number. However, in most real-world situations, the above assumption does not hold. Because of limited resource, ability, and the tolerant level of risk and complexity of decision making environment, players may not supply all units of resource to the formed coalition and coalitional values are not always expressed as real numbers. In this case, players can partially participate in a specific coalition (i.e., the participation rate of players in a coalition is between 0 and 1) and the coalitional

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values can be expressed as intervals, fuzzy numbers and linguistic variables. Such a cooperative game is called a fuzzy cooperative game.

Fuzzy cooperative games may be roughly divided into three categories: cooperative games with fuzzy coalitions, cooperative games with fuzzy payoff values and cooperative games with fuzzy coalitions and fuzzy payoff values (i.e., fuzzy cooperative games with fuzzy coalitions). The fuzzy cooperative games started with the work of Aubin [1] where the notions of a fuzzy cooperative game and the core of a fuzzy cooperative game were introduced. In the meantime, many solution concepts have been developed [4, 5, 10, 13, 17]. In order to incorporate fuzziness (uncertainty) in the degree of players' participation in a coalition, Butnariu [4] defined a Shapley function that maps a fuzzy cooperative game to the Shapley value from a fuzzy coalition. He furnished explicit forms of the Shapley function for a limited class of fuzzy cooperative games. However, it was later established by Tsurumi et al. [17] that most of the fuzzy cooperative games considered by Butnariu [4] are neither monotonically non-decreasing nor continuous with regard to the participation rate of the players in a coalition. Tsurumi et al. [17] conducted a study of the Shapley values for cooperative games with fuzzy coalitions, which incorporate players' rates of participation in each coalition and proposed a new class of fuzzy cooperative games using the concept of Choquet integrals. Borkotokey [2] proposed an extension of a fuzzy cooperative game with fuzzy coalitions and obtained some interesting properties. A Shapley function in the fuzzy sense was proposed as a solution concept for a class of fuzzy cooperative games. Mallozzi et al. [11] studied a core-like concept (called F-core) for cooperative games in which the worth of any coalition is given as a fuzzy interval, introduced a balancing condition and proved that the condition was necessary but not sufficient to guarantee the F-core to be non-empty. Tijs et al. [16] introduced cores and stable sets for cooperative games with fuzzy coalitions and studied relations between cores and stable sets of fuzzy cooperative games. Yu et al. [18] proposed a generalized form of fuzzy cooperative games that may be seen as an extension of the fuzzy cooperative game and gave an explicit form of the Shapley value for a new class of fuzzy cooperative games based on the Hukuhara difference [9] and the Choquet integral.

However, as far as we know, there is no investigation on how to solve interval-valued cooperative games with fuzzy coalitions. In other words, there is no specific and effective method for determining payoffs of players in interval-valued cooperative games with fuzzy coalitions. In this paper, by using the Choquet integral, the least square method, and the concepts of dissatisfaction functions and distance between intervals, we focus on developing a simple and an effective method for solving interval-valued cooperative games with fuzzy coalitions. The method proposed in this paper is remarkably different from other methods in that the former can provide analytical formulae for determining the interval-valued payoffs of all players.

The rest of this paper is organized as follows. In Sect. 2, we briefly introduce distances between two intervals and the concept of the Choquet integral. In Sect. 3, we introduce interval-valued solution concepts of interval-valued cooperative games with fuzzy coalitions and define a dissatisfaction function to measure differences between payoffs and values of coalitions. Two quadratic programming models are constructed to compute the interval-valued solution for any interval-valued cooperative game. We subsequently present an algorithm and process of solving the interval-valued cooperative games with fuzzy coalitions. In Sect. 4, a simple numerical example about optimal allocation of players in a fuzzy coalition is used to illustrate the validity and applicability of the proposed method. Further discussions on fuzzy cooperative games with fuzzy coalitions and conclusions are given in Sect. 5.

2 Distances Between Intervals and the Choquet Integral

Denote $\bar{a} = [a_L, a_R] = \{x | x \in R, a_L \leq x \leq a_R\}$, which is called an interval, where R is the set of real numbers and $a_L \in \mathrm{R}$, $a_R \in \mathrm{R}$. Obviously, if $a_L = a_R$, then the interval $\bar{a} = [a_L, a_R]$ degenerates to a real number, denoted by a, where $a = a_L = a_R$. Therefore, intervals are a generalization of real numbers and real numbers are a special case of intervals.

In the following, we give some operations of intervals such as the equality, addition and the scale multiplication as follows [8, 10, 14].

Definition 1. Let $\bar{a} = [a_L, a_R]$ and $\bar{b} = [b_L, b_R]$ be two intervals on $I(\mathrm{R})$ and $\alpha \in \mathrm{R}$ be any real number. (1) Equality of two intervals: $\bar{a} = \bar{b}$ if and only if $a_L = b_L$ and $a_R = b_R$; (2) Addition (or sum) of two intervals: $\bar{a} + \bar{b} = [a_L + b_L, a_R + b_R]$; (3) Scale multiplication: if $\alpha \geq 0$, then $\alpha\bar{a} = [\alpha a_L, \alpha a_R]$, otherwise, i.e., if $\alpha < 0$, then $\alpha\bar{a} = [\alpha a_R, \alpha a_L]$.

The concept of distance is introduced to measure differences between intervals.

Definition 2. Let $\bar{a}$, $\bar{b}$, and $\bar{c}$ be three intervals on the set $I(\mathrm{R})$. If a mapping $d : I(\mathrm{R}) \times I(\mathrm{R}) \mapsto \mathrm{R}$ satisfies the following four properties: (1) $d(\bar{a}, \bar{b}) \geq 0$, (2) $d(\bar{a}, \bar{b}) = d(\bar{b}, \bar{a})$, (3) $d(\bar{a}, \bar{b}) \leq d(\bar{a}, \bar{c}) + d(\bar{c}, \bar{b})$, (4) $d(\bar{a}, \bar{a}) = 0$, then $d(\bar{a}, \bar{b})$ is called the distance between the intervals $\bar{a}$ and $\bar{b}$.

In order to elaborate the quadratic programming models for interval-valued cooperative games based on the least square method, we define (the square of) the distance between the intervals $\bar{a}$ and $\bar{b}$ as follows:

$$D(\bar{a}, \bar{b}) = (a_L - b_L)^2 + (a_R - b_R)^2 \tag{1}$$

It is easy to validate that $D(\bar{a}, \bar{b})$ defined by Eq. (1) satisfies the four properties (1)–(4) in Definition 2. Namely, $D(\bar{a}, \bar{b})$ defined by Eq. (1) is the distance between the intervals $\bar{a}$ and $\bar{b}$.

Note that the square appears in Eq. (1), which is also the distance between two intervals. In the following, the distance between two intervals is referred to as the square of the distance given by Eq. (1) unless otherwise stated.

3 Interval-Valued Cooperative Games with Fuzzy Coalitions

The set of players is denoted by $N = \{1, 2, \cdots, n\}$, which is called the grand coalition. $\bar{\sigma}(S')$ is the interval-valued characteristic function of any (crisp) coalition $S' \subseteq N$, denoted by $\bar{\sigma}(S') = [\sigma_L(S'), \sigma_R(S')]$, where $\sigma_L(S') \leq \sigma_R(S')$. Any fuzzy coalition $\bar{S}$ is expressed as $\bar{S} = (\bar{S}(1), \bar{S}(2), \cdots, \bar{S}(n))$, where $\bar{S}(i)$ is the participation rate of the player $i \in N$ in the fuzzy coalition $\bar{S}$. Obviously, if the participation rates of all players are either 0 or 1, then the fuzzy coalition $\bar{S}$ is degenerated to a crisp coalition. Therefore, the fuzzy coalition is a generalization of the crisp coalition and crisp coalitions can be regarded as a special case of fuzzy coalitions. All fuzzy coalitions are denoted by the set $L_f(N)$. The triple $(\bar{\sigma}, N, L_f(N))$ is the interval-valued cooperative games with fuzzy coalitions.

3.1 Transformation of Interval-Valued Cooperative Games with Fuzzy Coalitions

In this section, we introduce the concept of the Choquet integral and the transformation of interval-valued cooperative games with fuzzy coalitions.

Definition 3. Let m be the capacity on X for any non-negative real function $f : X \to R^+$. The Choquet integral of f with respect to m is defined as follows [6, 7, 12, 15]:

$$\int f dm = \int_0^{\infty} m(F_\alpha) d\alpha \tag{2}$$

where $F_\alpha = \{x | f(x) \geq \alpha\}$ is called an α-cut of f, and $\alpha \in [0, \infty)$.

For the finite set $X = \{x_1, x_2, \cdots, x_n\}$, the function f can be expressed as $\{f(x_1), f(x_2), \cdots, f(x_n)\}$. We write $f(x_i)$ $(i = 1, 2, \cdots, n)$ in a monotonically non-decreasing order as $f(x_1^*) \leq f(x_2^*) \leq \cdots \leq f(x_n^*)$. According to the same monotonically non-decreasing order, the element set $\{x_1, x_2, \cdots, x_n\}$ can be rewritten as $\{x_1^*, x_2^*, \cdots, x_n^*\}$. Then, Eq. (2) is reduced to the following discrete form:

$$\int f dm = \sum_{i=1}^{n} (f(x_i^*) - f(x_{i-1}^*)) m(A_i) \tag{3}$$

where $f(x_0^*) = 0$ and $A_i = \{x_i^*, x_{i+1}^*, \cdots, x_n^*\}$ $(i = 1, 2, \cdots, n)$.

Given any fuzzy coalition $\bar{S} \in L_f(N)$, let $Q(\bar{S}) = \{\bar{S}(i) | \bar{S}(i) > 0, i \in N\}$ and $q(\bar{S}) = |Q(\bar{S})|$ be the cardinality of $Q(\bar{S})$. $\bar{S}(i) \in [0, 1]$ expresses the participation rate of the player $i \in N$ to a particular coalition $\bar{S}$. We write the elements of $Q(\bar{S})$ in a monotonically non-decreasing order as $0 < h_1 < h_2 < \cdots < h_{q(\bar{S})}$. For any fuzzy coalition $\bar{S} \in L_f(N)$, we have $h_0 = 0$ and $[\bar{S}]_{h_l} = \{i | \bar{S}(i) \geq h_l, i \in N\}$. Thus, $[\bar{S}]_{h_l}$ is a crisp coalition and defined as the coalition of players whose participation rates are no smaller than h_l.

For any fuzzy coalition $\bar{S}$, let $S = \{i | \bar{S}(i) > 0, i \in N\}$. Thus, S is the crisp coalition related to the fuzzy coalition $\bar{S}$. Then, using Eq. (3) and Definition 1, we can obtain

$$\begin{aligned}\bar{\upsilon}(S) &= \sum_{l=1}^{q(\bar{S})} \bar{\sigma}([\bar{S}]_{h_l})(h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(\bar{S})} [\sigma_L([\bar{S}]_{h_l}), \sigma_R([\bar{S}]_{h_l})](h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(\bar{S})} [\sigma_L([\bar{S}]_{h_l})(h_l - h_{l-1}), \sigma_R([\bar{S}]_{h_l})(h_l - h_{l-1})] \\ &= [\sum_{l=1}^{q(\bar{S})} \sigma_L([\bar{S}]_{h_l})(h_l - h_{l-1}), \sum_{l=1}^{q(\bar{S})} \sigma_R([\bar{S}]_{h_l})(h_l - h_{l-1})],\end{aligned} \tag{4}$$

Then, we have

$$\upsilon_L(S) = \sum_{l=1}^{q(\bar{S})} \sigma_L([\bar{S}]_{h_l})(h_l - h_{l-1})$$

and

$$\upsilon_R(S) = \sum_{l=1}^{q(\bar{S})} \sigma_R([\bar{S}]_{h_l})(h_l - h_{l-1}).$$

$\sigma_L([\bar{S}]_{h_l})$ expresses the lower bound of the characteristic function of the crisp coalition $[\bar{S}]_{h_l}$. Thus, $\upsilon_L(S)$ expresses the lower bound of the characteristic function of the crisp coalition S related to the fuzzy coalition $\bar{S}$. Similarly, $\sigma_R([\bar{S}]_{h_l})$ expresses the upper bound of the characteristic function of the crisp coalition $[\bar{S}]_{h_l}$, and $\upsilon_R(S)$ expresses the upper bound of the characteristic function of the crisp coalition S related to the fuzzy coalition $\bar{S}$, respectively.

Therefore, $\bar{\upsilon}(S)$ is the interval-valued characteristic function of the crisp coalition S related to the fuzzy coalition $\bar{S}$. That is to say, $(\bar{\upsilon}, N)$ is the interval-valued cooperative game, which is derived from the interval-valued cooperative game $(\bar{\sigma}, N, L_f(N))$ with fuzzy coalitions by using Eq. (4).

3.2 The Construction of Dissatisfaction Functions of Interval-Valued Cooperative Games

Utilizing Eq. (4), we can convert any interval-valued cooperative game with fuzzy coalitions into an interval-valued cooperative game. Then, the key of solving an interval-valued cooperative game is to obtain optimal payoffs for all players. As mentioned previously, we denote the interval-valued characteristic function of coalition S by $\bar{\upsilon}(S) = [\upsilon_L(S), \upsilon_R(S)]$. Due to the fact that each coalitional value is an interval, it is obvious that each player should receive an interval-valued payoff from cooperation. Let $\bar{x}_i = [x_{Li}, x_{Ri}]$ be the interval-valued payoff of each player $i \in N$. Denote $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^{\mathrm{T}}$, which is the vector of the interval-valued payoffs for all players

$i \in N$. $\bar{x}(S) = \sum_{i \in S} \bar{x}_i$ is the sum of the interval-valued payoffs of players in the coalition S. According to the interval operations given in Definition 1, we can express $\bar{x}(S)$ as the interval $\bar{x}(S) = [\sum_{i \in S} x_{Li}, \sum_{i \in S} x_{Ri}]$. In order to avoid the disadvantage of the interval subtraction operator, we use distances to measure the difference between $\bar{x}(S)$ and $\bar{\upsilon}(S)$. Thus, using Eq. (1), we define the square of the distance between the intervals $\bar{x}(S)$ and $\bar{\upsilon}(S)$ for the coalition S as follows:

$$D(\bar{x}(S), \bar{\upsilon}(S)) = (\sum_{i \in S} x_{Li} - \upsilon_L(S))^2 + (\sum_{i \in S} x_{Ri} - \upsilon_R(S))^2$$

In order to describe the dissatisfaction of the coalition S more intuitively, we define $e_L(S, \bar{\boldsymbol{x}}) = \sum_{i \in S} x_{Li} - \upsilon_L(S)$ and $e_R(S, \bar{\boldsymbol{x}}) = \sum_{i \in S} x_{Ri} - \upsilon_R(S)$. In this context, $e_L(S, \bar{\boldsymbol{x}})$ is called the lower bound of the excess of S on $\bar{\boldsymbol{x}}$ and $e_R(S, \bar{\boldsymbol{x}})$ is called the upper bound of the excess of S on $\bar{\boldsymbol{x}}$. Note that $e_L(S, \bar{\boldsymbol{x}})$ and $e_R(S, \bar{\boldsymbol{x}})$ can be respectively interpreted as a measure of the lower and upper bound of the dissatisfaction of the coalition S if $\bar{\boldsymbol{x}}$ were suggested as a final payoff vector. It is obvious that the less $e_L(S, \bar{\boldsymbol{x}})$ (or $e_L(S, \bar{\boldsymbol{x}})$) the more satisfactory the coalition S. However, owing to the fact that the total value of players' payoffs is limited, other players would not satisfy with the distribution result if one of the possible coalitions obtains too much. Egalitarian and utilitarian principles need to be considered. According to the excess vector and adhering to the principle of fairness and equity, we take a quite different angle to implement it. The highest excess is pushed down to flatten the excess vector. That is to say, we choose the payoff vector so that the sums of the payoffs of all players in the coalition S are as close to the coalitional values as possible.

Then, we can define the sum of the squares of the distances between $\bar{x}(S)$ and $\bar{\upsilon}(S)$ for all coalitions $S \subseteq N$ as follows:

$$L(\bar{\boldsymbol{x}}) = \sum_{S \subseteq N} D(\bar{x}(S), \bar{\upsilon}(S)) = \sum_{S \subseteq N} [(\sum_{i \in S} x_{Li} - \upsilon_L(S))^2 + (\sum_{i \in S} x_{Ri} - \upsilon_R(S))^2].$$

A interval-valued payoff vector $\boldsymbol{x}$ is said to be efficient (or a preimputation) if $\bar{x}(N) = \bar{\upsilon}(N)$. $L(\bar{\boldsymbol{x}})$ may be interpreted as a dissatisfaction function.

3.3 A Quadratic Programming Model and Its Optimal Solution

It is directly derived from the concept of dissatisfaction functions that an optimal interval-valued payoff vector of all players is the solution of the following quadratic programming model:

$$\min\{L(\bar{\boldsymbol{x}}) = \sum_{S \subseteq N} [(\sum_{i \in S} x_{Li} - \upsilon_L(S))^2 + (\sum_{i \in S} x_{Ri} - \upsilon_R(S))^2]\}. \tag{5}$$

Let partial derivatives of $L(\bar{\boldsymbol{x}})$ with respect to the variables x_{Lj} and x_{Rj} ($j \in S \subseteq N$) be equal to 0, respectively. Thus, we have

$$\sum_{S\subseteq N:j\in S}\sum_{i\in S} x_{Li} = \sum_{S\subseteq N:j\in S} \upsilon_L(S) \quad (j = 1, 2, \cdots, n) \tag{6}$$

and

$$\sum_{S\subseteq N:j\in S}\sum_{i\in S} x_{Ri} = \sum_{S\subseteq N:j\in S} \upsilon_R(S) \quad (j = 1, 2, \cdots, n). \tag{7}$$

To solve x_{Li} ($i = 1, 2, \cdots, n$) and x_{Ri} ($i = 1, 2, \cdots, n$), we rewrite Eqs. (6) and (7) as follows:

$$\begin{cases} a_{11}x_{L1} + a_{12}x_{L2} + a_{13}x_{L3} + \cdots + a_{1n}x_{Ln} = \sum\limits_{S\subseteq N:1\in S} \upsilon_L(S) \\ a_{21}x_{L1} + a_{22}x_{L2} + a_{23}x_{L3} + \cdots + a_{2n}x_{Ln} = \sum\limits_{S\subseteq N:2\in S} \upsilon_L(S) \\ \cdots \\ a_{n1}x_{L1} + a_{n2}x_{L2} + a_{n3}x_{L3} + \cdots + a_{nn}x_{Ln} = \sum\limits_{S\subseteq N:n\in S} \upsilon_L(S) \end{cases} \tag{8}$$

and

$$\begin{cases} a_{11}x_{R1} + a_{12}x_{R2} + a_{13}x_{R3} + \cdots + a_{1n}x_{Rn} = \sum\limits_{S\subseteq N:1\in S} \upsilon_R(S) \\ a_{21}x_{R1} + a_{22}x_{R2} + a_{23}x_{R3} + \cdots + a_{2n}x_{Rn} = \sum\limits_{S\subseteq N:2\in S} \upsilon_R(S) \\ \cdots \\ a_{n1}x_{R1} + a_{n2}x_{R2} + a_{n3}x_{R3} + \cdots + a_{nn}x_{Rn} = \sum\limits_{S\subseteq N:n\in S} \upsilon_R(S), \end{cases} \tag{9}$$

respectively.

Let $|S|$ be the number of all players in the coalition S. According to the knowledge on permutation and combination, for player $i \in N$, the number of coalitions S including i with $|S| = 1$ can be expressed as C_{n-1}^0. In the same way, the number of coalitions S including i with $|S| = 2$ can be expressed as C_{n-1}^1. Generally, the number of coalitions S including i with $|S| = k$ ($k = 1, 2, \cdots, n$) can be expressed as C_{n-1}^{k-1}. It is obvious that the number of coalitions S including i can be written as $C_{n-1}^0 + C_{n-1}^1 \cdots + C_{n-1}^{n-2} + C_{n-1}^{n-1}$, which is equal to 2^{n-1} by the simple observation.

In a similar way, the number of coalitions S including i and j simultaneously can be written as $C_{n-2}^0 + C_{n-2}^1 \cdots + C_{n-2}^{n-3} + C_{n-2}^{n-2}$, which is 2^{n-2}.

Then, it can be easily derived from the conclusions mentioned above that

$$a_{ij} = \begin{cases} 2^{n-1} & (i = j \text{ with } i, j \in \{1, 2, \cdots, n\}) \\ 2^{n-2} & (i \neq j \text{ with } i, j \in \{1, 2, \cdots, n\}). \end{cases}$$

Denote $\boldsymbol{X}_L = (x_{L1}, x_{L2}, \cdots, x_{Ln})^{\mathrm{T}}$, $\boldsymbol{X}_R = (x_{R1}, x_{R2}, \cdots, x_{Rn})^{\mathrm{T}}$,
$\boldsymbol{B}_L = (\sum_{S \subseteq N: 1 \in S} \upsilon_L(S), \sum_{S \subseteq N: 2 \in S} \upsilon_L(S), \cdots, \sum_{S \subseteq N: n \in S} \upsilon_L(S))^{\mathrm{T}}$,
$\boldsymbol{B}_R = (\sum_{S \subseteq N: 1 \in S} \upsilon_R(S), \sum_{S \subseteq N: 2 \in S} \upsilon_R(S), \cdots, \sum_{S \subseteq N: n \in S} \upsilon_R(S))^{\mathrm{T}}$, and

$$\boldsymbol{A} = (a_{ij})_{n \times n} = \begin{pmatrix} 2^{n-1} & 2^{n-2} & \cdots & 2^{n-2} \\ 2^{n-2} & 2^{n-1} & \cdots & 2^{n-2} \\ \vdots & \vdots & & \vdots \\ 2^{n-2} & 2^{n-2} & \cdots & 2^{n-1} \end{pmatrix}_{n \times n}.$$

Thus, Eqs. (8) and (9) can be rewritten in a matrix representation as follows:

$$\boldsymbol{A}\boldsymbol{X}_L = \boldsymbol{B}_L$$

and

$$\boldsymbol{A}\boldsymbol{X}_R = \boldsymbol{B}_R,$$

respectively.

By using matrix multiplication, we obtain the solutions of Eqs. (8) and (9) as follows:

$$\boldsymbol{X}_L = \boldsymbol{A}^{-1}\boldsymbol{B}_L \tag{10}$$

and

$$\boldsymbol{X}_R = \boldsymbol{A}^{-1}\boldsymbol{B}_R, \tag{11}$$

respectively, where

$$\mathbf{A}^{-1} = \frac{1}{2^{n-2}} \begin{pmatrix} \frac{n}{n+1} & -\frac{1}{n+1} & \cdots & -\frac{1}{n+1} \\ -\frac{1}{n+1} & \frac{n}{n+1} & \cdots & -\frac{1}{n+1} \\ \vdots & \vdots & & \vdots \\ -\frac{1}{n+1} & -\frac{1}{n+1} & \cdots & \frac{n}{n+1} \end{pmatrix}_{n \times n}.$$

Thus, we can obtain the optimal interval-valued payoffs of all players $i \in N$, which are expressed as $\bar{x}_i = [x_{Li}, x_{Ri}]$ $(i = 1, 2, \cdots, n)$.

In what follows, we discuss some useful and important properties of the optimal interval-valued solution for any interval-valued cooperative game $\bar{\upsilon}$.

Theorem 1. (Existence and Uniqueness) For an arbitrary interval-valued cooperative game $\bar{\upsilon}$, there always exists an unique optimal interval-valued solution, which is determined by Eqs. (10) and (11).

Proof. According to Eqs. (10) and (11), it is straightforward to prove Theorem 1.

Theorem 2. (Additivity) For any two interval-valued cooperative games $\bar{\upsilon}$ and $\bar{v}$, then $\bar{x}_i(\bar{\upsilon}+\bar{v}) = \bar{x}_i(\bar{\upsilon}) + \bar{x}_i(\bar{v})$ $(i = 1, 2, \cdots, n)$.

Proof. According to Eq. (10), we have

$$x_{Li}(\bar{\upsilon}) = \frac{n \sum\limits_{S \subseteq N: i \in S} \upsilon_L(S) - \sum\limits_{j=1, j \neq i}^{n} \sum\limits_{S \subseteq N: j \in S} \upsilon_L(S)}{2^{n-2}(n+1)},$$

then

$$\begin{aligned} x_{Li}(\bar{\upsilon}+\bar{v}) &= \frac{n \sum\limits_{S \subseteq N: i \in S} (\upsilon_L(S) + v_L(S)) - \sum\limits_{j=1, j \neq i}^{n} \sum\limits_{S \subseteq N: j \in S} (\upsilon_L(S) + v_L(S))}{2^{n-2}(n+1)} \\ &= \frac{n \sum\limits_{S \subseteq N: i \in S} \upsilon_L(S) - \sum\limits_{j=1, j \neq i}^{n} \sum\limits_{S \subseteq N: j \in S} \upsilon_L(S)}{2^{n-2}(n+1)} + \frac{n \sum\limits_{S \subseteq N: i \in S} v_L(S) - \sum\limits_{j=1, j \neq i}^{n} \sum\limits_{S \subseteq N: j \in S} v_L(S)}{2^{n-2}(n+1)} \\ &= x_{Li}(\bar{\upsilon}) + x_{Li}(\bar{v}), \end{aligned}$$

i.e., $x_{Li}(\bar{\upsilon}+\bar{v}) = x_{Li}(\bar{\upsilon}) + x_{Li}(\bar{v})$.

Similarly, according to Eq. (11), we can easily prove that $x_{Ri}(\bar{\upsilon}+\bar{v}) = x_{Ri}(\bar{\upsilon}) + x_{Ri}(\bar{v})$.

According to the aforementioned conclusion and the case (1) of Definition 1, it is obvious that

$$\bar{x}_i(\bar{\upsilon}+\bar{v}) = \bar{x}_i(\bar{\upsilon}) + \bar{x}_i(\bar{v}) \; (i = 1, 2, \cdots, n)$$

Thus, we have proven Theorem 2.

Theorem 3. (Symmetry) If $i \in N$ and $k \in N$ $(i \neq k)$ are two symmetric players in an interval-valued cooperative game $\bar{\upsilon}$, then $\bar{x}_i(\bar{\upsilon}) = \bar{x}_k(\bar{\upsilon})$.

Proof. For the players $i \in N$ and $k \in N$ $(i \neq k)$, according to Eq. (10), we obtain

$$x_{Li}(\bar{\upsilon}) = \frac{-\sum\limits_{j=1, j \neq i, j \neq k}^{n} \sum\limits_{S \subseteq N: j \in S} \upsilon_L(S) + (n \sum\limits_{S \subseteq N: i \in S} \upsilon_L(S) - \sum\limits_{S \subseteq N: k \in S} \upsilon_L(S))}{2^{n-2}(n+1)}$$

and

$$x_{Lk}(\bar{v}) = \frac{-\sum_{j=1, j\neq i, j\neq k}^{n} \sum_{S\subseteq N: j\in S} v_L(S) + (-\sum_{S\subseteq N: i\in S} v_L(S) + n\sum_{S\subseteq N: k\in S} v_L(S))}{2^{n-2}(n+1)}$$

Due to the assumption that the players i and k are symmetric in the interval-valued cooperative game $\bar{v}$, it easily follows that

$$\sum_{S\subseteq N: i\in S} v_L(S) = \sum_{S\subseteq N: k\in S} v_L(S),$$

which directly infers that

$$n\sum_{S\subseteq N: i\in S} v_L(S) - \sum_{S\subseteq N: k\in S} v_L(S) = -\sum_{S\subseteq N: i\in S} v_L(S) + n\sum_{S\subseteq N: k\in S} v_L(S)$$

Hereby, we have $x_{Li}(\bar{v}) = x_{Lk}(\bar{v})$. In the same way, we can prove $x_{Ri}(\bar{v}) = x_{Rk}(\bar{v})$.

According to the conclusion above and the case (1) of Definition 1, it is obvious that

$$[x_{Li}(\bar{v}), x_{Ri}(\bar{v})] = [x_{Lk}(\bar{v}), x_{Rk}(\bar{v})],$$

i.e., $\bar{x}_i(\bar{v}) = \bar{x}_k(\bar{v})$. Accordingly, we have completed the proof of Theorem 3.

Theorem 4. (Null player) If $i \in N$ is a null player in an interval-valued cooperative game $\bar{v}$, then $\bar{x}_i(\bar{v}) = 0$.

Proof. According to Eq. (10) and the assumption that i is a null player, we have

$$\begin{aligned} x_{Li}(\bar{v}) &= \frac{n\sum_{S\subseteq N: i\in S} v_L((S\backslash i)\cup i) - \sum_{j=1, j\neq i}^{n} \sum_{S\subseteq N: j\in S} v_L(S)}{2^{n-2}(n+1)} \\ &= \frac{n\sum_{S\subseteq N: i\in S} v_L(S\backslash i) - \sum_{j=1, j\neq i}^{n} \sum_{S\subseteq N: j\in S} v_L(S)}{2^{n-2}(n+1)}. \end{aligned}$$

Hereby, we have $x_{Li}(\bar{v}) = 0$.

Analogously, according to Eq. (11), we can prove $x_{Ri}(\bar{v}) = 0$. Thereby, we obtain $[x_{Li}(\bar{v}), x_{Ri}(\bar{v})] = 0$, i.e., $\bar{x}_i(\bar{v}) = 0$. Thus, we have proven Theorem 4.

Theorem 5. (Anonymity) For any permutation σ on the set N and an interval-valued cooperative game $\bar{v}$, then $\bar{x}_{\sigma(i)}(\bar{v}^{\sigma}) = \bar{x}_i(\bar{v})$.

Proof. It can be easily proven according to Eqs. (10) and (11) (omitted).

3.4 An Extension of the Quadratic Programming Model

In real management situations, some constraint conditions need to be considered. In this case, the quadrtic programming model (i.e., Eq. (5)) is still applicable. For example, if we consider the efficiency: $\bar{x}(N) = \bar{\upsilon}(N)$ (i.e., $[\sum_{i=1}^{n} x_{Li}, \sum_{i=1}^{n} x_{Ri}] = [\upsilon_L(N), \upsilon_R(N)]$), then Eq. (5) can be flexibly rewritten as the following quadratic programming model:

$$\min\{L(\bar{\boldsymbol{x}}) = \sum_{S\subseteq N}[(\sum_{i\in S} x_{Li} - \upsilon_L(S))^2 + (\sum_{i\in S} x_{Ri} - \upsilon_R(S))^2]\}$$
$$\text{s.t.}\begin{cases} \sum_{i=1}^{n} x_{Li} = \upsilon_L(N) \\ \sum_{i=1}^{n} x_{Ri} = \upsilon_R(N). \end{cases} \tag{12}$$

According to the Lagrange multiplier method, the Lagrange function is constructed as follows:

$$L(\bar{\boldsymbol{x}}, \lambda, \mu) = \sum_{S\subseteq N}[(\sum_{i\in S} x_{Li} - \upsilon_L(S))^2 + (\sum_{i\in S} x_{Ri} - \upsilon_R(S))^2] + \lambda(\sum_{i=1}^{n} x_{Li} - \upsilon_L(N)) + \mu(\sum_{i=1}^{n} x_{Ri} - \upsilon_R(N)).$$

Then, an optimal interval-valued payoff vector for all players (i.e., a solution of the interval-valued cooperative game $\bar{\upsilon}$) is obtained by solving the following quadratic programming model:

$$\begin{aligned} \min\{L(\bar{\boldsymbol{x}}, \lambda, \mu) = &\sum_{S\subseteq N}[(\sum_{i\in S} x_{Li} - \upsilon_L(S))^2 + (\sum_{i\in S} x_{Ri} - \upsilon_R(S))^2] + \\ &\lambda(\sum_{i=1}^{n} x_{Li} - \upsilon_L(N)) + \mu(\sum_{i=1}^{n} x_{Ri} - \upsilon_R(N))\}. \end{aligned} \tag{13}$$

Let the partial derivatives of $L(\bar{\boldsymbol{x}}, \lambda, \mu)$ with respect to the variables x_{Lj}, x_{Rj} ($j \in S\subseteq N$), λ, and μ be equal to 0, respectively. Then, we have

$$\begin{cases} \sum_{S\subseteq N: j\in S} \sum_{i\in S} x_{Li} + \frac{\lambda}{2} = \sum_{S\subseteq N: j\in S} \upsilon_L(S) \quad (j = 1, 2, \cdots, n) \\ \sum_{i=1}^{n} x_{Li} = \upsilon_L(N) \end{cases} \tag{14}$$

and

$$\begin{cases} \sum_{S\subseteq N: j\in S} \sum_{i\in S} x_{Ri} + \frac{\mu}{2} = \sum_{S\subseteq N: j\in S} \upsilon_R(S) \quad (j = 1, 2, \cdots, n) \\ \sum_{i=1}^{n} x_{Ri} = \upsilon_R(N), \end{cases} \tag{15}$$

respectively.

Denote $\boldsymbol{e} = (1, 1, \cdots, 1)^{\mathrm{T}}_{n\times 1}$ and $\boldsymbol{X}'_L = (x'_{L1}, x'_{L2}, \cdots, x'_{Ln})^{\mathrm{T}}$. Then, Eq. (14) can be rewritten as follows:

$$\boldsymbol{A}\boldsymbol{X}'_L + \frac{\lambda}{2}\boldsymbol{e} = \boldsymbol{B}_L \tag{16}$$

and

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{X}'_L = \upsilon_L(N). \tag{17}$$

It follows from Eq. (16) that

$$\boldsymbol{X}'_L = \boldsymbol{A}^{-1}\boldsymbol{B}_L - \frac{\lambda}{2}\boldsymbol{A}^{-1}\boldsymbol{e} = \boldsymbol{X}_L - \frac{\lambda}{2}\boldsymbol{A}^{-1}\boldsymbol{e}, \tag{18}$$

where $\boldsymbol{X}_L$ is given by Eq. (10). Then, the key of solving Eq. (14) is to determine λ.
Through a series of calculations, we have

$$\frac{\lambda}{2} = 2^{n-2}\frac{n+1}{n}\left(\sum_{i=1}^{n} x_{Li} - \upsilon_L(N)\right). \tag{19}$$

Thus, it can be easily derived from Eqs. (18) and (19) that

$$\begin{aligned} \boldsymbol{X}'_L &= \boldsymbol{X}_L - \frac{1}{n}\left(\sum_{i=1}^{n} x_{Li} - \upsilon_L(N)\right)\boldsymbol{e} \\ &= \boldsymbol{X}_L + \frac{1}{n}\left(\upsilon_L(N) - \sum_{i=1}^{n} x_{Li}\right)\boldsymbol{e}. \end{aligned} \tag{20}$$

Similarly, we can obtain the solution of Eq. (15) as follows:

$$\boldsymbol{X}'_R = \boldsymbol{X}_R + \frac{1}{n}\left(\upsilon_R(N) - \sum_{i=1}^{n} x_{Ri}\right)\boldsymbol{e}. \tag{21}$$

So far, we obtain the solution of Eq. (12), which consists of Eqs. (20) and (21). Thus, if the efficiency is taken into consideration, we can determine optimal interval-valued payoffs of all players (i.e., a solution of the interval-valued cooperative game), which are expressed as $\bar{x}'_i = [x'_{Li}, x'_{Ri}]$ $(i = 1, 2, \cdots, n)$, whose lower and upper bounds are given by Eqs. (20) and (21), respectively.

Theorem 6. For any interval-valued cooperative game $\bar{\upsilon}$, there always exists a unique interval-valued payoff vector (i.e., a solution of the interval-valued cooperative game $\bar{\upsilon}$ with considering the efficiency), which satisfies the efficiency, the additivity, the symmetry, and the anonymity.

3.5 A Process of Interval-Valued Cooperative Games with Fuzzy Coalitions

According to the discussions above, an algorithm and process of the interval-valued cooperative games with fuzzy coalitions are summarized as follows:

Step 1: Determine coalitional values by using the Choquet integral and players' rates of participation in a particular coalition (e.g., Eq. (4));
Step 2: Define the sum of the squares of the distances between $\bar{x}(S)$ and $\bar{\upsilon}(S)$ for all coalitions $S \subseteq N$;
Step 3: If the efficiency is taken into account, then get to Step 5; otherwise, get to Step 4;
Step 4: Compute a solution of the interval-valued cooperative game $\bar{\upsilon}$ using Eqs. (10) and (11);
Step 5: Compute an optimal interval-valued payoff vector for all players (i.e., a solution of the interval-valued cooperative game) using Eqs. (20) and (21).

4 A Numerical Example

Consider a joint production program in which three companies, named 1, 2 and 3, respectively, decide to cooperate with their resources. Suppose that each company i $(i = 1, 2, 3)$ has 1000 units of resources R_i $(i = 1, 2, 3)$ and they can cooperate freely according to their capital, human and material resources. company i can obtain profit $\upsilon(\{i\})$ by producing 1000 units of the product P_i using 1000 units of the resource R_i. In order to cope with increasingly fierce market competition and obtain more profit, each company can choose to cooperate with the other one or two companies. Valuable products can be produced by combining two or three resources from R_1, R_2, and R_3. For example, by combining one unit of the resource R_i and one unit of the resource R_j $(i < j;\ i, j \in \{1, 2, 3\})$, they can obtain one unit of the product P_{ij} and get profit $\upsilon(\{i, j\})/1000$. Here, $\upsilon(\{i, j\})$ is expressed as the coalitional value if companies i and j form a crisp coalition. In other words, if companies i and j make up a full cooperative relationship (i.e., they offer all of their 1000 units of the resources to cooperate), then they can obtain profit $\upsilon(\{i, j\})$ by producing 1000 units of the product P_{ij}. Similarly, by combining one unit of R_1, one unit of R_2 and one unit of R_3, they can produce one unit of the product P_{123} and get profit $\upsilon(\{1, 2, 3\})/1000$. That is to say, if the three companies offer all of their 1000 units of the resources to cooperate, then they can obtain profit $\upsilon(\{1, 2, 3\})$ by producing 1000 units of the product P_{123}.

However, each company needs to consider how many resources it should offer in the cooperation according to the reality. Furthermore, since there exist many uncontrollable factors under the fierce market competition, companies only know the approximate range of the coalitional values. In this paper, we use intervals to denote inherent fuzziness. Suppose that each company supplies all of its resources to cooperate (i.e., form a crisp coalition), the interval values of the crisp coalitions are given as shown in Table 1.

Table 1. The interval-valued characteristic function of the crisp coalitions

S'	$\bar{\sigma}(S')$	S'	$\bar{\sigma}(S')$
$\{1\}$	$[25, 40]$	$\{1, 3\}$	$[60, 90]$
$\{2\}$	$[15, 35]$	$\{2, 3\}$	$[40, 85]$
$\{3\}$	$[20, 36]$	$\{1, 2, 3\}$	$[120, 200]$
$\{1, 2\}$	$[50, 100]$		

As we all know, in most cases, each company may not contribute all of its resources to cooperation in real life. Thus, we have to consider an interval-valued cooperative game with fuzzy coalitions. Here, company 1 can contribute 200 units of the resource R_1, while company 2 can supply 500 units of the resource R_2, and company 3 can supply 600 units of the resource R_3. As company 1 has 1000 units of the resource R_1, we regard the participation rate of company 1 as 200/1000 = 0.2. In a similar way to the previous calculation, it is easy to see that the participation rate of company 2 is 0.5 and that of company 3 is 0.6.

Therefore, a fuzzy coalition has been defined as follows. If all three companies form a coalition to cooperate, the participation rates of companies 1, 2, and 3 can be expressed as $\bar{S}(1) = 0.2$, $\bar{S}(2) = 0.5$, and $\bar{S}(3) = 0.6$, respectively. Using the definition of fuzzy sets, the aforementioned fuzzy coalition $\bar{\boldsymbol{S}}$ can be written as $\bar{\boldsymbol{S}} = (0.2, 0.5, 0.6)$. According to the previous analysis, it is obvious that companies 1, 2, and 3 join the cooperation with 200, 500, and 600 units of the resources R_1, R_2, and R_3, respectively. In order to get the maximum profit from the cooperation mentioned above, the three companies will produce 200 units of the product P_{123}, 300 units of the product P_{23}, and 100 units of the product P_3. Thus, the interval-valued characteristic function of the crisp coalition S related to the fuzzy coalition $\bar{\boldsymbol{S}} = (0.2, 0.5, 0.6)$ can be computed using Eq. (4) as follows:

$$\begin{aligned}\bar{\upsilon}(S) &= 0.2\bar{\sigma}(\{1, 2, 3\}) + 0.3\bar{\sigma}(\{2, 3\}) + 0.1\bar{\sigma}(\{3\}) \\ &= 0.2 \times [120, 200] + 0.3 \times [40, 85] + 0.1 \times [20, 36] \\ &= [38, 69.1]\end{aligned}$$

4.1 Computational Results Obtained by Different Methods and Analysis

In order to obtain a solution of the interval-valued cooperative game with fuzzy coalitions, the first task is to estimate the interval-valued characteristic functions of all possible coalitions related to the fuzzy coalitions. The estimation results are shown in Table 2.

Table 2. Interval-valued characteristic functions of coalitions related to the fuzzy coalitions

$\bar{\boldsymbol{S}}$	$\bar{\upsilon}(S)$	$\bar{\boldsymbol{S}}$	$\bar{\upsilon}(S)$
$(0.2, 0, 0)$	$[5, 8]$	$(0.2, 0, 0.6)$	$[20, 32.4]$
$(0, 0.5, 0)$	$[7.5, 17.5]$	$(0, 0.5, 0.6)$	$[22, 46.1]$
$(0, 0, 0.6)$	$[12, 21.6]$	$(0.2, 0.5, 0.6)$	$[38, 69.1]$
$(0.2, 0.5, 0)$	$[14.5, 30.5]$		

After determining the interval-valued characteristic functions of the crisp coalitions S related to the fuzzy coalitions $\bar{S}$, we will compute interval-valued payoffs of companies 1, 2, and 3. It easily follows from Eqs. (10) and (11) that

$$\mathbf{X}_L = \mathbf{A}^{-1}\mathbf{B}_L = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 77.5 \\ 82 \\ 92 \end{pmatrix} = \begin{pmatrix} 7.3125 \\ 9.5625 \\ 14.5625 \end{pmatrix},$$

$$\mathbf{X}_R = \mathbf{A}^{-1}\mathbf{B}_R = \begin{pmatrix} \frac{3}{8} & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} 140 \\ 163.2 \\ 169.2 \end{pmatrix} = \begin{pmatrix} 10.95 \\ 22.55 \\ 25.55 \end{pmatrix},$$

respectively. Namely, $\bar{x}_1 = [x_{L1}, x_{R1}] = [7.3125, 10.95]$, $\bar{x}_2 = [x_{L2}, x_{R2}] = [9.5625, 22.55]$, and $\bar{x}_3 = [x_{L3}, x_{R3}] = [14.5625, 25.55]$, which are the optimal interval-valued payoffs of companies 1, 2, and 3, respectively.

If we take into account the efficiency condition, then it is easily derived from Eqs. (20) and (21) that

$$\begin{aligned} \boldsymbol{X}'_L &= \boldsymbol{X}_L + \frac{1}{3}\left(\upsilon_L(N) - \sum_{i=1}^{3} x_{Li}\right)\boldsymbol{e} = (7.3125, 9.5625, 14.5625)^{\mathrm{T}} + \frac{1}{3} \times (38 - 31.4375)(1,1,1)^{\mathrm{T}} \\ &= (9.5, 11.75, 16.75)^{\mathrm{T}} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{X}'_R &= \boldsymbol{X}_R + \frac{1}{3}\left(\upsilon_R(N) - \sum_{i=1}^{3} x_{Ri}\right)\boldsymbol{e} = (10.95, 22.55, 25.55)^{\mathrm{T}} + \frac{1}{3} \times (69.1 - 59.05)(1,1,1)^{\mathrm{T}} \\ &= (14.3, 25.9, 28.9)^{\mathrm{T}}, \end{aligned}$$

respectively. Namely, $\bar{x}'_1 = [x'_{L1}, x'_{R1}] = [9.5, 14.3]$, $\bar{x}'_2 = [x'_{L2}, x'_{R2}] = [11.75, 25.9]$, and $\bar{x}'_3 = [x'_{L3}, x'_{R3}] = [16.75, 28.9]$.

In the same analysis as above, the optimal interval-valued payoffs of companies 1, 2, and 3 in different fuzzy coalitions can be obtained as shown in Table 3.

Take company 1 as an example, it can receive an interval-valued payoff $[5, 8]$ if it goes alone. If it cooperates with company 2 or 3, it can receive an interval-valued payoff $[6, 10.5]$ or $[6.5, 9.4]$, respectively. Moreover, it can receive an interval-valued payoff $[9.5, 14.3]$ if it chooses to join the grand coalition consisting of all the three companies. It is easy for company 1 to decide which coalition to join. The same situation applies to the other two companies.

Table 3. The optimal interval-valued payoffs of companies in crisp coalitions related to the fuzzy coalitions

$\bar{S}$	Company 1	Company 2	Company 3
$(0.2, 0, 0)$	$[5, 8]$	0	0
$(0, 0.5, 0)$	0	$[7.5, 17.5]$	0
$(0, 0, 0.6)$	0	0	$[12, 21.6]$
$(0.2, 0.5, 0)$	$[6, 10.5]$	$[8.5, 20]$	0
$(0.2, 0, 0.6)$	$[6.5, 9.4]$	0	$[13.5, 23]$
$(0, 0.5, 0.6)$	0	$[8.75, 21]$	$[13.25, 25.1]$
$(0.2, 0.5, 0.6)$	$[9.5, 14.3]$	$[11.75, 25.9]$	$[16.75, 28.9]$

The conclusion is easily drawn from the analysis above that each company is willing to join the grand coalition formed by all the three companies, because each of them can obtain most profit from this coalition.

4.2 The Comparative Analysis and Conclusions

To compare the method proposed in this paper (i.e., Eqs. (10) and (11)) with the interval Shapley value put forward by Han et al. [8] (i.e., Eq. (4)), we firstly use Eq. (4) in Han et al. [8] to obtain the interval-valued characteristic functions for all crisp coalitions S related to the fuzzy coalitions $\bar{S}$, and then use the method proposed by Han et al. [8] with the Moore's interval subtraction [14] to solve the above numerical example. According to Eq. (4) given by Han et al. [8] and combining with the Moore's interval subtraction [14], i.e., $\bar{a} - \bar{b} = [a_L - b_R, a_R - b_L]$, where $\bar{a} = [a_L, a_R]$ and $\bar{b} = [b_L, b_R]$, we have

$$\begin{aligned}
\bar{\phi}_1^*(\bar{v}) &= \sum_{S \subseteq \{1,2,3\} \backslash \{1\}} \frac{|S|!(3-|S|-1)!}{3!}(\bar{v}(S \cup \{1\}) - \bar{v}(S)) \\
&= \frac{0!2!}{3!}(\bar{v}(1) - \bar{v}(\varnothing)) + \frac{1!1!}{3!}(\bar{v}(1,2) - \bar{v}(2)) + \\
&\quad \frac{1!1!}{3!}(\bar{v}(1,3) - \bar{v}(3)) + \frac{2!0!}{3!}(\bar{v}(1,2,3) - \bar{v}(2,3)) \\
&= \frac{0!2!}{3!}([5,8] - [0,0]) + \frac{1!1!}{3!}([14.5, 30.5] - [7.5, 17.5]) + \\
&\quad \frac{1!1!}{3!}([20, 32.4] - [12, 21.6]) + \frac{2!0!}{3!}([38, 69.1] - [22, 46.1]) \\
&= [-1.8, 25.52].
\end{aligned}$$

In the same way, we can obtain $\bar{\phi}_2^*(\bar{v}) = [5.52, 32.13]$ and $\bar{\phi}_3^*(\bar{v}) = [9.25, 36.4]$.

From the above analysis, it is easily seen that the lower bound of the interval-valued payoff of company 1 is negative when using the method given by Han et al. [8]. $\bar{\phi}_1^*(\bar{v})$ means that company 1 may get a negative profit (i.e., loss) which is not acceptable in real life. That is to say, company 1 may get worse if it cooperates with the other two companies. Obviously, the three companies will not cooperate. Moreover, it is obvious that the sums of the lower and upper bounds of the interval-valued payoffs of the three companies are 12.97 and 94.05, respectively. The value 12.97 is much smaller than 38 whereas the value 94.05 is much greater than 69.1, where 38 and 69.1 are the lower and upper bounds of the interval-valued payoff of the grand coalition formed by all the three companies, respectively.

Comparing the aforementioned modeling, methods, and computational results, we can easily draw the following conclusions.

(1) The method proposed in this paper (i.e., Eqs. (10) and (11) or Eqs. (20) and (21)) is simpler and more convenient from the viewpoint of computational complexity than other methods such as the one given by Han et al. [8]. The reason is that Eqs. (10) and (11) or Eqs. (20) and (21)) are analytical formulae.

(2) The participation rates of players are variable in real economic management, which may result in overlapping of the interval-valued characteristic functions of the fuzzy coalitions. If we use the method proposed by Han et al. [8], then we may obtain negative interval-valued Shapley values for players, which are unacceptable in real life. For instance, in the example given above, the lower bound of the interval-valued payoff of the fuzzy coalition $\bar{S} = (0.2, 0.5, 0.6)$ is smaller than the upper bound of the interval-valued payoff of the fuzzy coalition $\bar{S} = (0, 0.5, 0.6)$. If we use the method proposed by Han et al. [8], company 1 receives a negative profit because of the interval subtraction. However, the method proposed in this paper ensures that the interval-valued payoffs of companies (players) are always positive if all coalitions' values are positive (i.e., the lower bounds of the interval-valued payoffs of the coalitions are larger than 0).

(3) The magnification of uncertainty resulted from the interval subtraction such as the Moore' interval subtraction [14] is a long-standing problem which is difficult to solve. In order to overcome effectively the disadvantage of the interval subtraction [14], we adopted the distance to measure the differences between interval-valued payoffs and interval-valued payoffs of coalitions in this paper. However, the problem mentioned above still exists in the method given by Han et al. [8] with the Moore's interval subtraction [14]. Take company 1 in the aforementioned example for instance, the lengths of the interval-valued payoffs of the fuzzy coalitions $\bar{S}$ containing company 1 are no larger than 16 with an average of 10.5. However, the length of the interval-valued Shapley value of company 1 is equal to 27.32, which is greater than 10.5.

5 Conclusions

Of different types of fuzzy cooperative games, cooperative games with fuzzy payoffs have been extensively discussed. However, limited research has been carried out on cooperative games with both fuzzy coalitions and fuzzy payoffs. In this paper, based on the Choquet integral, the concepts of dissatisfaction functions and the distance between intervals, we study a class of fuzzy cooperative games taking into account not only the participation rates of players in each coalition but also the imprecision and uncertainty of the coalitional values. The interval-valued payoffs of players can be directly obtained by using the analytical formulae (i.e., Eqs. (10) and (11) or Eqs. (20) and (21)). The developed models and method have some advantages as stated previously from the aspects of the scale, solution process and computational complexity.

As stated earlier, we use intervals to describe uncertainty and imprecision of coalitional values and real numbers to denote the participation rates of players in a coalition. However, fuzzy numbers and intuitionistic fuzzy numbers are other possible tools to characterize uncertainty and imprecision in real life. Thus, we will study and develop some effective methods for solving cooperative games with participation rates and/or coalitional values expressed as fuzzy numbers and/or intuitionistic fuzzy numbers in the near future. Moreover, a general case of fuzzy cooperative games with a coalition structure where players can participate in different unions also deserves further studies.

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Cooperative Games with the Intuitionistic Fuzzy Coalitions and Intuitionistic Fuzzy Characteristic Functions

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Abstract. In this paper, the definition of the Shapley function for intuitionistic fuzzy cooperative games is given by extending the fuzzy cooperative games. Based on the extended Hukuhara difference, the specific expression of the Shapley intuitionistic fuzzy cooperative games with multilinear extension form is obtained, and its existence and uniqueness are discussed. Furthermore, the properties of the Shapley function are researched. Finally, the validity and applicability of the proposed method, as well as comparison analysis with other methods are illustrated with a numerical example.

Keywords: Intuitionistic fuzzy cooperative games · Shapley function · Multilinear extension

1 Introduction

Stated as earlier, the cooperative games have been successfully applied in several areas, such as enterprise management and economics [1, 2]. Cooperative game is used to study how to fairly and reasonably determine the distribution scheme and meet certain rational behavior. In response to this issue, many scholars gave various forms of the solution, which include the core, stable set of solution and so on. However, in real situation, the coalitions' values of the player may be imprecise and vague due to the uncertainty of information and the complexity of player's behavior. That is to say, the players may partly participate in a specific coalition (i.e., the rate of the participation of players in a coalition is between 0 and 1). As a result, the theory of fuzzy coalition cooperative games started with the work of Aubin [3] in which the notions of a fuzzy game and the core of a fuzzy game were introduced. In the meantime, many solution concepts have been developed (Li [4–6]; Adler [7]; Butnariu [8]; Molina and Tejada [9]; Sheremetov [10]; Butnariu and Klement [11]; Bumariu and Kroupa [12]; Sakawa and Nishizaki [13]; Butnariu [14]; Mareš [15]; Yu and Zhang [16]; Radzik [17]; Nishizaki and Sakawa [18]; Tsurumi et al. [19]). Mareš [15] and Mareš and Vlach [20] concerned the uncertainty of the coalition values. In their model, the coalitions are crisp, namely, all players fully participate in cooperation. But the coalition values of the players are fuzzy numbers. Since Shapley [21] proposed Shapley value as a cooperative

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game local income of payment solution concept, it has been widely applied and deeply studied. Based on the Hukuhara difference (Banks and Jacobs [22]), Yu and Zhang [23] researched the Shapley function for the model given in (Mareš [15, 20]) and studied a special case (Tsurumi et al. [19]). Meng et al. [24] considered the Shapley function for fuzzy games with fuzzy characteristic functions. Meng and Jiang [25] studied the Shapley function for fuzzy games on augmenting systems with fuzzy characteristic functions.

All above researches only consider the situation that the coalition values are fuzzy set or characteristic functions are fuzzy numbers. However, these are unrealistic, because many uncertain factors exist during the process of negotiation and coalition forming. As a result, the players can only know imprecise information regarding the real outcome of cooperation. Fuzzy set theory can't express the "participation", "no participation" and "hesitation" three levels of the players. Besides, in real situations, whether players will participate in the league with certain hesitation degree. The theory of intuitionistic fuzzy set given Atanassov [26, 27] can effectively describe the affirmative, negative and hesitance three states information. Moreover, intuitionistic fuzzy sets have been researched in multiattribute decision [28–30] and non-cooperative game [31–33]. However, there exists little investigation on the intuitionistic fuzzy sets to express the uncertain information in cooperative game. Elena Mielcová [34] introduced formalization of the n-person transferable utility games in the case when expected utilities are intuitionistic fuzzy values. To address this issue, we study the Shapley function for cooperative games with the intuitionistic fuzzy coalitions and intuitionistic fuzzy characteristic functions. The difference of intuitionistic fuzzy numbers is important for the Shapley values, then, we propose the concept of the extended Hukuhara difference, and analyze the relationship between the Hukuhara difference and the extended Hukuhara difference. The intuitionistic fuzzy coalition is defined. Some basic concepts of cooperative games with the intuitionistic fuzzy coalitions and intuitionistic fuzzy characteristic functions are given. Some new axioms for the Shapley function that are based on the fuzzy case are presented, and some properties are discussed. Finally, an illustrative example is offered to confirm theory's effectiveness.

2 Preliminaries

The concept of an IFS was firstly introduced by Atanassov (1986, 1999).

Definition 2.1 [26, 27]. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universal set. An IFS $\tilde{A}$ in X may be mathematically expressed as $\tilde{A} = \{\langle x_l, \mu_{\tilde{A}}(x_l), \nu_{\tilde{A}}(x_l)\rangle | x_l \in X\}$, where $\mu_{\tilde{A}} : X \mapsto [0,1]$ and $\nu_{\tilde{A}} : X \mapsto [0,1]$ are the membership degree and the non-membership degree of an element $x_l \in X$ to the set $\tilde{A} \subseteq X$, respectively, such that they satisfy the following condition: $0 \le \mu_{\tilde{A}}(x_l) + \nu_{\tilde{A}}(x_l) \le 1$ for all $x_l \in X$.

Let $\pi_{\tilde{A}}(x_l) = 1 - \mu_{\tilde{A}}(x_l) - \nu_{\tilde{A}}(x_l)$, which is called the intuitionistic index (or hesitancy degree) of an element x_l to the set $\tilde{A}$. It is the degree of indeterminacy membership of the element x_l to the set $\tilde{A}$. Obviously, $0 \le \pi_{\tilde{A}}(x_l) \le 1$.

If an IFS $\tilde{C}$ in X is a singleton set, i.e., $\tilde{C} = \{\langle x_k, \mu_{\tilde{C}}(x_k), \nu_{\tilde{C}}(x_k)\rangle\}$, then it is usually denoted by $\tilde{C} = \langle \mu_{\tilde{C}}(x_k), \nu_{\tilde{C}}(x_k)\rangle$ for short.

Definition 2.2 [26]. Let $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle\}$, $\tilde{B} = \{\langle x, \mu_{\tilde{B}}(x), \nu_{\tilde{B}}(x)\rangle\}$ be two intuitionistic fuzzy sets on U, its union and intersection are defined as follows:

(1) $\tilde{A} \cap \tilde{B} = \{\langle x, \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}\rangle\}$;
(2) $\tilde{A} \cup \tilde{B} = \{\langle x, \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \min\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\}\rangle\}$.

The intuitionistic fuzzy number is a special kind of intuitionistic fuzzy set, especially the triangular intuitionistic fuzzy number (TIFN), which is easier to express characteristic functions in the cooperative game. Then the triangular intuitionistic fuzzy numbers is defined as follows:

Definition 2.3 [35, 36]. Let $\tilde{a} = \langle(\underline{a}_1, a, \bar{a}_1); (\underline{a}_2, a, \bar{a}_2)\rangle$ be a TIFN on the real number set R, whose membership and non-membership function are defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & (x < \underline{a}_1, x > \bar{a}_1) \\ (x - \underline{a}_1)/(a - \underline{a}_1) & (\underline{a}_1 \le x < a) \\ 1 & (x = a) \\ (\bar{a}_1 - x)/(\bar{a}_1 - a) & (a \le x < \bar{a}_1) \end{cases},$$

$$\nu_{\tilde{A}}(x) = \begin{cases} 1 & (x < \underline{a}_2, x > \bar{a}_2) \\ (a - x)/(a - \underline{a}_2) & (\underline{a}_2 \le x < a) \\ 0 & (x = a) \\ (x - a)/(\bar{a}_2 - a) & (a \le x < \bar{a}_2) \end{cases},$$

respectively, depicted as in Fig. 1.

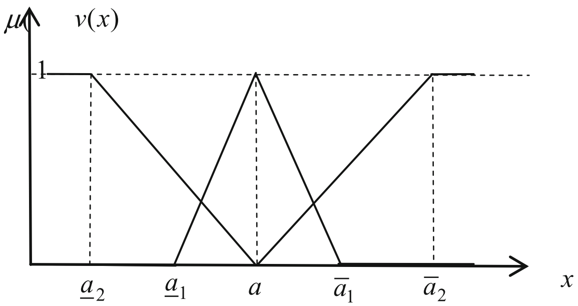

Fig. 1. Triangular intuitionistic fuzzy number

The set of all TIFNs is denoted by $\tilde{\Re}$.

Definition 2.4 [35, 36]. Let $\tilde{a} = \langle (\underline{a}_1, a, \bar{a}_1); (\underline{a}_2, a, \bar{a}_2) \rangle$, $\tilde{b} = \langle (\underline{b}_1, b, \bar{b}_1); (\underline{b}_2, b, \bar{b}_2) \rangle$ be TIFNs, and λ be a real number, the arithmetic operations of TIFNs are defined as follows:

(1) $\tilde{a} + \tilde{b} = \langle (\underline{a}_1 + \underline{b}_1, a + b, \bar{a}_1 + \bar{b}_1); (\underline{a}_2 + \underline{b}_2, a + b, \bar{a}_2 + \bar{b}_2) \rangle$;

(2) $\tilde{a} - \tilde{b} = \langle (\underline{a}_1 - \underline{b}_1, a - b, \bar{a}_1 - \bar{b}_1); (\underline{a}_2 - \underline{b}_2, a - b, \bar{a}_2 - \bar{b}_2) \rangle$;

(3) $\lambda\tilde{a} = \begin{cases} \langle (\lambda\underline{a}_1, \lambda a, \lambda\bar{a}_1); (\lambda\underline{a}_2, \lambda a, \lambda\bar{a}_2) \rangle \ (\lambda \geq 0) \\ \langle (\lambda\bar{a}_1, \lambda a, \lambda\underline{a}_1); (\lambda\bar{a}_2, \lambda a, \lambda\underline{a}_2) \rangle \ (\lambda < 0) \end{cases}$.

Definition 2.5. Let $\tilde{a} = \langle (\underline{a}_1, a, \bar{a}_1); (\underline{a}_2, a, \bar{a}_2) \rangle$ and $\tilde{b} = \langle (\underline{b}_1, b, \bar{b}_1); (\underline{b}_2, b, \bar{b}_2) \rangle$ be TIFNs,

where

$$S_\lambda(\tilde{a}) = \lambda(\underline{a}_1 + 2a + \bar{a}_1)/4 + (1 - \lambda)(\underline{a}_2 + 2a + \bar{a}_2)/4,$$

$$S_\lambda(\tilde{b}) = \lambda(\underline{b}_1 + 2b + \bar{b}_1)/4 + (1 - \lambda)(\underline{b}_2 + 2b + \bar{b}_2)/4$$

are λ weighted mean-areas of $\tilde{a}$ and $\tilde{b}$ respectively, $\lambda \in [0, 1]$, then

(1). if $S_\lambda(\tilde{a}) > S_\lambda(\tilde{b})$, then $\tilde{a} >_{IF} \tilde{b}$;

(2). if $S_\lambda(\tilde{a}) < S_\lambda(\tilde{b})$, then $\tilde{a} <_{IF} \tilde{b}$;

(3). if $S_\lambda(\tilde{a}) = S_\lambda(\tilde{b})$, then $\tilde{a} =_{IF} \tilde{b}$.

The symbol "$<_{IF}$" is an intuitionistic fuzzy version of the order relation "$<$" in the real number set and has the linguistic interpretation "essentially less than". The symbols "$>_{IF}$" and "$=_{IF}$" are explained similarly.

Since the Shapley function of cooperative game involves the subtraction of intuitionistic fuzzy numbers, as the from the point of view Definitions 2.4, we can not have $\tilde{a} + \tilde{b} - \tilde{b} = \tilde{a}$ for all TIFNs $\tilde{a}$ and $\tilde{b}$ in general. The Hukuhara difference of intervals (Banks and Jacobs [22]), the generalized Hukuhara difference of intervals (Stefanini (2010)) and the extended Hukuhara difference of intervals (Meng 2016) can well cope with this issue. As showed in (Meng 2016), the extended Hukuhara difference can be applied in more interval games than the Hukuhara difference and the generalized Hukuhara difference.

Similarly the extended Hukuhara difference of intervals, we give the extended Hukuhara difference of intuitionistic fuzzy numbers.

Definition 2.6. Let $\tilde{a} = \langle (\underline{a}_1, a, \bar{a}_1); (\underline{a}_2, a, \bar{a}_2) \rangle$, $\tilde{b} = \langle (\underline{b}_1, b, \bar{b}_1); (\underline{b}_2, b, \bar{b}_2) \rangle$ and $\tilde{c} = \langle (\underline{c}_1, c, \tilde{c}_1); (\underline{c}_2, c, \tilde{c}_2) \rangle$ be TIFNs. If $\underline{b}_i - \underline{a}_i > \bar{b}_i - \bar{a}_i, (i = 1, 2)$, $\tilde{c} = \tilde{a} -_H \tilde{b}$ is said to the "imaginary" Hukuhara difference. The extended Hukuhara difference is defined

$$\tilde{a} -_{eH} \tilde{b} = \left\langle (\underline{a_1} - \underline{b_1}, a - b, \bar{a}_1 - \bar{b}_1); (\underline{a_2 - b_2}, a - b, \bar{a}_2 - \bar{b}_2) \right\rangle.$$

For example, let $\tilde{a} = \langle (4, 6, 8); (3, 6, 10) \rangle$, $\tilde{b} = \langle (5, 6, 8); (3, 6, 10) \rangle$, becsuse of $5 - 4 > 8 - 8$ then $\tilde{c} = \tilde{a} -_H \tilde{b}$ is said to the "imaginary" Hukuhara difference between $\tilde{a}$ and $\tilde{b}$.

Let $\tilde{a} = \langle (4, 6, 8); (3, 6, 10) \rangle$, $\tilde{b} = \langle (1, 2, 3); (0, 2, 3) \rangle$, then

$$\tilde{a} -_{eH} \tilde{b} = \langle (3, 4, 5); (3, 4, 7) \rangle.$$

3 Cooperative Games with the Intuitionistic Fuzzy Coalitions and Intuitionistic Fuzzy Characteristic Functions

3.1 Intuitionistic Fuzzy Coalitions and Intuitionistic Fuzzy Characteristic Functions

In this section, the fuzzy cooperative game is extended. We consider cooperative games with intuitionistic fuzzy coalitions and intuitionistic fuzzy characteristic functions, which is called intuitionistic fuzzy cooperative games for short.

For the cooperative game with the set of players $N = \{1, 2, \cdots, n\}$, an intuitionistic fuzzy coalition is an intuitionistic fuzzy subset on N, which is identified with a function $f : N \mapsto [0, 1] \times [0, 1]$. Then, for an intuitionistic fuzzy coalition

$$S_T = (S_T(1), S_T(2), \cdots, S_T(n))$$

and $S_T(i) = \langle \mu_T(i), \nu_T(i) \rangle$,$(\mu_T(i) \in [0, 1]$, $\nu_T(i) \in [0, 1], 0 \le \mu_{\mathrm{T}}(i) + \nu_T(i) \le 1)$, where $\mu_T(i)$ is membership degree of the player i in the intuitionistic fuzzy coalition S_T, and $\nu_T(i)$ is non-membership degree of the player i in the intuitionistic fuzzy coalition. $\pi_T(i) = 1 - \mu_T(i) - \nu_T(i)$ is hesitation degree of the player i in the intuitionistic fuzzy coalition.

Specially, let $e^{\phi} = (\langle 1, 0 \rangle, \langle 1, 0 \rangle, \cdots, \langle 1, 0 \rangle)$ be the empty alliance and $e^N = (\langle 1, 0 \rangle, \langle 1, 0 \rangle, \cdots, \langle 1, 0 \rangle)$ be the grand coalition. Obviously, if $\mu_{\mathrm{T}}(i) + \nu_T(i) = 1$, the intuitionistic fuzzy coalition becomes fuzzy coalition. The set of all intuitionistic fuzzy coalition on N is denoted by $IF(N)$.

The support is denoted by $Supp(S_T) = (i \in N | \mu_T(i) > 0, 1 - \nu_T(i) > 0)$, and the cardinality is written as $|Supp(S_T)|$. For any $S_T, S_M \in IF(N)$, the notation $S_T \subseteq S_M$, if and only if $\langle \mu_T(i), \nu_T(i) \rangle = \langle \mu_M(i), \nu_M(i) \rangle$, and $\langle \mu_T(i), \nu_T(i) \rangle = \langle 0, 1 \rangle$ for any $i \in N$. For any $S_T, S_M \in IF(N)$, its union and intersection are defined as, i.e.,

$$\begin{aligned}(S_M \cup S_T)(i) &= S_M(i) \cup S_T(i) = \langle \mu_M(i), \nu_M(i) \rangle \cup \langle \mu_T(i), \nu_T(i) \rangle \\ &= \langle \max\{\mu_M(i), \mu_T(i)\}, \min\{\nu_M(i), \nu_T(i)\} \rangle\end{aligned}$$

$$(S_M \cap S_T)(i) = S_M(i) \cap S_T(i) = \langle \mu_M(i), \nu_M(i)\rangle \cap \langle \mu_T(i), \nu_T(i)\rangle$$
$$= \langle \min\{\mu_M(i), \mu_T(i)\}, \max\{\nu_M(i), \nu_T(i)\}\rangle$$

Namely, $i \in Supp(S_T \cup S_M)$ if and only if $i \in Supp(S_T) \cup Supp(S_M)$. $i \in Supp(S_T \cap S_M)$ if and only if $i \in Supp(S_T) \cap Supp(S_M)$.

The crisp coalitions in N are denoted by $S_0, P_0, \cdots$. The power set of all crisp subsets on N is denoted by $P(N)$ for all $S_0 \subseteq P(N)$, the cardinality of S_0 is denoted by $|S_0|$. A function

$\tilde{v}_0 : P(N) \to \tilde{\Re}$ satisfying $\tilde{v}_0(\phi) = \tilde{0}$, is called an intuitionistic fuzzy characteristic function. The set of all games with intuitionistic fuzzy characteristic function on $P(N)$ is denoted by $\tilde{G}_0(N)$.

Let $\tilde{v}(P_0)$ indicate the intuitionistic fuzzy characteristic function value for any $P_0 \subseteq P(N)$, where $\tilde{v}(P_0) = \langle(\underline{v}_1(P_0), v(P_0), \bar{v}_1(P_0)); (\underline{v}_2(P_0), v(P_0), \bar{v}_2(P_0))\rangle$ is TIFN.

Similarly, function $\tilde{v} : IF(N) \to \tilde{\Re}$, satisfying $\tilde{v}(\phi) = \tilde{0}$, is called an intuitionistic fuzzy characteristic function. The set of all games with intuitionistic fuzzy characteristic function on $IF(N)$ is denoted by $\tilde{G}(N)$.

Theorem 3.1. Let $S_N \in IF(N)$ and $\tilde{v} \in \tilde{G}(N)$, the function $f : IF(N) \to R^{IF(N)}$ is called a Shapley function on $\tilde{G}(N)$ if it satisfies the following three axioms.

Axiom 1 (Efficiency). If S_T is an intuitionistic fuzzy carrier for $\tilde{v}$ in S_N.

$$\sum_{i \in Supp(S_T)} \tilde{f}_i(S_N, \tilde{v}) = \tilde{v}(S_T).$$

Axiom 2 (Symmetry). For any $i,j \in Supp(S_N)$ and any $S_T \subseteq S_N$ with $i,j \notin Supp(S_T)$, we have $\tilde{v}(S_T \cup S_N(i))=\tilde{v}(S_T \cup S_N(j))$, then $\tilde{f}_i(S_N, \tilde{v}) = \tilde{f}_j(S_N, \tilde{v})$.

Axiom 3 (Additivity). Let $\tilde{v}, \tilde{w} \in \tilde{G}(N)$, if there exists $\tilde{v} + \tilde{w} \in \tilde{G}(N)$ such that $(\tilde{v} + \tilde{w})(S_T) = \tilde{v}(S_T) + \tilde{w}(S_T)$ for all $S_T \subseteq S_N$, then

$\tilde{f}_i(S_N, \tilde{v} + \tilde{w}) = \tilde{f}_i(S_N, \tilde{v}) + \tilde{f}_i(S_N, \tilde{w})$, for all $i \in Supp(S_N)$.

Theorem 3.2. Let $S_N \in IF(N)$ and $\tilde{v} \in \tilde{G}(N)$. The function $\varphi : IF(N) \to R^{IF(N)}$ is defined by

$$\varphi_i(S_N, \tilde{v}) = \sum_{T_0 \subseteq Supp(S_N)\setminus\{i\}} \frac{(|Supp(S_N)| - |T_0|)!(|T_0| - 1)!}{(|Supp(S_N)|)!} [\tilde{v}(S_{T_0 \cup \{i\}}) -_{eH} \tilde{v}(S_{T_0})],$$
$$\forall i \in Supp(S_N).$$

Then φ is the unique Shapley function for $\tilde{v} \in IFG(N)$ in S_N.Where ,

Proof. From the concept of the extended Hukuhara difference and the Shapley function, one can obtain the conclusion.

3.2 Intuitionistic Fuzzy Cooperative Games with Multilinear Extension Form

In this section, extending the fuzzy cooperative game proposed by Owen [2]. The characteristic function values of intuitionistic fuzzy cooperative games are calculated as follows:

$$\tilde{v}(S_N) = \sum_{P \subseteq Supp(S_N)} \left(\prod_{i \in P} (\lambda \pi_N(i) + \mu_N(i)) \prod_{i \in Supp(S_N) \setminus P} (v_N(i) + (1-\lambda)\pi_N(i)) \right) \cdot \tilde{v}(P_0),$$

where $S_N \in IF(N)$ and $\tilde{v}(P_0)$ indicate the value of $P_0 \subseteq P(N)$.

The value of $S_T \subseteq S_N$ is expressed by

$$\tilde{v}(S_T) = \sum_{P \subseteq Supp(S_T)} \left(\prod_{i \in P} (\lambda \pi_N(i) + \mu_N(i)) \prod_{i \in Supp(S_N) \setminus P} (v_N(i) + (1-\lambda)\pi_N(i)) \right) \cdot \tilde{v}(P_0).$$

Namely the players have an influence on the values of other players.

Definition 3.1. $\tilde{v} \in \tilde{G}(N)$ is said to be an intuitionistic fuzzy convex if it satisfies

$$\tilde{v}(S_T \cup S_K) + \tilde{v}(S_T \cap S_K) \geq \tilde{v}(S_T) + \tilde{v}(S_K)$$

for all $S_T, S_K \subseteq S_N$.

Definition 3.2. $\tilde{v} \in \tilde{G}(N)$ is said to be an intuitionistic fuzzy supperadditivity if it satisfies

$$\tilde{v}(S_T \cup S_K) \geq \tilde{v}(S_T) + \tilde{v}(S_K)$$

For all $S_T, S_K \subset S_N, S_T \cap S_K = \phi$.

Definition 3.3. Let $S_N \in IF(N)$ and $\tilde{v} \in \tilde{G}(N)$, $S_T \subseteq S_N$ is called an intuitionistic fuzzy carrier for $\tilde{v}$ on S_N if

$$\tilde{v}(S_T \cup S_K) = \tilde{v}(S_K)$$

for all $S_K \subseteq S_N$.

Definition 3.4. Let $S_N \in IF(N)$ and $\tilde{v} \in \tilde{G}(N)$, if
$\tilde{v}(S_T \cup S_N(i)) - \tilde{v}(S_T) = \tilde{v}(S_N(i))$ for all $S_T \subseteq S_N$ with $i \notin Supp(S_T)$,
then i is called an intuitionistic fuzzy dummy player for $\tilde{v}$ on S_N.

Definition 3.5. Let $S_N \in IF(N)$ and $\tilde{v} \in \tilde{G}(N)$, if $\tilde{v}(S_T \cup S_N(i)) = \tilde{v}(S_T)$ for all $S_T \subseteq S_N$ with $i \notin Supp(S_T)$, then i is called an intuitionistic fuzzy null player for $\tilde{v}$ on S_N.

3.3 The Shapley Function for Intuitionistic Fuzzy Cooperative Games

The Shapley value is a well-known solution concept in cooperative game theory. In this section, extending the Shapley value of fuzzy cooperative game, the Shapley value of intuitionistic fuzzy cooperative games is studied.

Theorem 3.3. Let $S_N \in IF(N)$ and $\tilde{v} \in \tilde{G}(N)$. The function $\varphi : IF(N) \to R^{IF(N)}$ is defined by

$$\varphi_i(S_N, \widetilde{v}) = \sum_{T_0 \subseteq Supp(S_N)\backslash\{i\}} \frac{(|Supp(S_N)| - |T_0|)!(|T_0| - 1)!}{(|Supp(S_N)|)!}$$

$$\Bigg[\sum_{P_0 \subseteq T_0 \cup \{i\}} (\prod_{j \in P_0} (\lambda \pi_N(j) + \mu_N(j)) \prod_{j \in Supp(S_N)\backslash P_0} ((1-\lambda)\pi_N(j) + \nu_N(j))) \cdot \widetilde{v}(P_0) \tag{1}$$

$$-_{eH} \sum_{P_0 \subseteq T_0} (\prod_{j \in P_0} (\lambda \pi_N(j) + \mu_N(j)) \prod_{j \in Supp(S_N)\backslash P_0} (((1-\lambda)\pi_N(j) + \nu_N(j)) \cdot \widetilde{v}(P_0))\Bigg],$$

$$\forall i \in Supp(S_N).$$

Then φ is the unique Shapley function for $\tilde{v} \in IFG(N)$ in S_N.

Proof Existence. From Theorem 3.2, one can easily obtain existence.

Uniqueness. Similar the uniqueness proof of the Shapley function in classical case and fuzzy case, we give the following process.

For any $S_T \subseteq S_N, T \neq \phi$, define the unanimity game u_T on S_T as follows:

$$u_T(S_R) = \begin{cases} 1 & S_T \subseteq S_R \subseteq S_N \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

For any $0 \leq c \in R$, Let $S_M \in IF(N)$. Given $c \in R_+$ and $S_T \subseteq S_M$, it is obvious that the game $c\mu_T \in \tilde{G}(N)$ and S_T is a carrier for game $c\mu_T$, From Definition 2.6, Theorems 3.1 and 3.2, we have that

$$\sum_{i \in Supp(S_M)} \varphi_i(c \cdot u_T)(S_M) + c \cdot u_T(S_M) = c = c \cdot u_T = \sum_{i \in Supp(S_T)} \varphi_i(c \cdot u_T)(S_M)$$

$$\sum_{i \in Supp(S_M)\backslash Supp(S_T)} \varphi_i(c \cdot u_T)(S_M) = 0.$$

For any $k \in Supp(S_M)\backslash Supp(S_T)$, it can be seen that $Supp(S_T) \cup \{k\}$ is also a carrier for game $c\mu_T$, so

$$\sum_{i \in Supp(S_T)} \varphi_i(c \cdot u_T)(S_M) + \varphi_k(c \cdot u_T)(S_M) = c \cdot u_T(Supp(S_T) \cup \{k\})$$

$$= c = \sum_{i \in Supp(S_T)} \varphi_i(c \cdot u_T)(S_M)$$

Therefore, we have that $\varphi_k\big(c \cdot u_T\big)\big(S_M\big) = 0$ for $k \notin Supp(S_M) \backslash Supp(S_T)$. Given any $i, j \in Supp(S_T)$, we can see that

$c \cdot u_T(Supp(S_W) \cup \{j\}) = c \cdot u_T(Supp(S_W) \cup \{i\}) = 0$ for $S_W \in S_M / \{i, j\}$, it is apparent that $\phi_i\big(c \cdot u_T\big)\big(S_M\big) = c / |Supp\big(S_T\big)|$ for any $i \in Supp(S_T)$. Therefore, we get

$\varphi_i(c \cdot u_T)(S_M) = \begin{cases} \frac{c}{|Supp(S_T)|} & i \in Supp(S_T), \\ 0 & \text{otherwise.} \end{cases}$

We obtain

$$\varphi_i(c \cdot u_T)(S_M) = \begin{cases} \frac{c}{|Supp(S_T)|} & i \in Supp(S_T) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In the following, $\tilde{v} \in IFG(N)$ can be expressed by

$$\tilde{v} = \sum_{S_N \supseteq S_T \neq \phi} c_T u_T, , \tag{4}$$

where $c_T = \sum_{S_K \subseteq S_T} (-1)^{|Supp(S_T)| - |Supp(S_K)|} \tilde{v}(S_K)$, and u_T is expressed as Eq. (2).

For any $S_K \subseteq S_N, S_K \neq \phi$, we have

$$\left(\sum_{S_N \supseteq S_T \neq \phi} c_T(\lambda \pi_T + u_T) \right)(S_K) = \sum_{S_N \supseteq S_T \neq \phi} c_T(\lambda \pi_T + u_T)(S_K) = \sum_{S_K \supseteq S_T \neq \phi} c_T$$
$$= \sum_{S_K \supseteq S_T \neq \phi} \sum_{S_W \subseteq S_T} (-1)^{|Supp(S_T)| - |Supp(S_W)|} \cdot \tilde{v}(S_W)$$
$$= \sum_{S_W \supseteq S_K} \left(\sum_{S_K \subseteq S_T \neq \phi} (-1)^{|Supp(S_T)| - |Supp(S_W)|} \right) \cdot \tilde{v}(S_W)$$
$$= \sum_{S_W \supseteq S_K} \left(\sum_{|Supp(S_T)| = |Supp(S_W)|}^{|Supp(S_K)|} (-1)^{|Supp(S_T)| - |Supp(S_W)|} \begin{pmatrix} |Supp(S_K)| - |Supp(S_W)| \\ |Supp(S_T)| - |Supp(S_W)| \end{pmatrix} \right) \cdot \tilde{v}(S_W).$$

Since $\sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} = 0$ for any $n \in N$, we have

$$\sum_{S_W \supseteq S_K} \left(\sum_{|Supp(S_T)|=|Supp(S_W)|}^{|Supp(S_K)|} (-1)^{|Supp(S_T)|-|Supp(S_W)|} \binom{|Supp(S_K)| - |Supp(S_W)|}{|Supp(S_T)| - |Supp(S_W)|} \right) = 0, \forall S_W \subset S_K.$$

Hence, $\widetilde{v}(S_K) = \left(\sum_{\varphi \neq S_T \subseteq S_N} c_T(\lambda \pi_T + u_T) \right)(S_K)$ holds. From Eqs. (3), (4) and additivity, we know the function φ is uniquely determined by S_N and $\tilde{v} \in IFG(N)$. This completes the proof.

Property 3.1. Let $S_N \in IF(N)$ and $\tilde{v} \in \tilde{G}(N)$, if $i \in Supp(S_N)$ is an intuitionistic fuzzy dummy player for $\tilde{v}$ in S_N, then

$$\varphi_i(S_N, \tilde{v}) = \tilde{v}(i)(\lambda \pi_N(i) + \mu_N(i)) \cdot \prod_{j \in Supp(S_N) \setminus \{i\}} ((1-\lambda)\pi_N(i) + v_N(i)).$$

Proof. For all $S_K \subseteq S_N, i \notin Supp(S_K)$, we have

$$\tilde{v}(S_K \cup S_N(i)) - \tilde{v}(S_K) = \tilde{v}(S_N(i)).$$

Since,

$$\tilde{v}(\mu_N(i)) = \tilde{v}(i) \cdot (\lambda \pi_N(i) + \mu_N(i)) \cdot \prod_{j \in Supp(S_N) \setminus \{i\}} ((1-\lambda)\pi_N(j) + v_N(j)),$$

$$\tilde{v}(S_K) = \sum_{T_0 \subseteq Supp(S_K)} \left(\prod_{j \in T_0} (\lambda \pi_N(j) + \mu_N(j)) \cdot \prod_{j \in Supp(S_N) \setminus T_0} ((1-\lambda)\pi_N(j) + v_N(j)) \right) \cdot \tilde{v}(T_0).$$

$$\tilde{v}(S_K \cup \mu_N(i)) = \sum_{T_0 \subseteq Supp(S_K \cup \mu_N(i))} \left(\prod_{j \in T_0} (\lambda \pi_N(j) + \mu_N(j)) \cdot \prod_{j \in Supp(S_N) \setminus T_0} ((1-\lambda)\pi_N(j) + v_N(j)) \right) \cdot \tilde{v}(T_0).$$

Thus, for any $T_0 \subseteq Supp(S_K \cup S_N(i))$ with $i \in T_0, T_0 \neq \{i\}$, we have

$$\left(\prod_{j \in T_0} (\lambda \pi_N(j) + \mu_N(j)) \cdot \prod_{j \in Supp(S_N) \setminus T_0} ((1-\lambda)\pi_N(j) + v_N(j)) \right) \cdot \tilde{v}(T_0) = 0.$$

From Eq. (1), we get

$$\varphi_i(S_N, \tilde{v}) = \sum_{i \in T_0 \subseteq Supp(S_N)} \frac{(|Supp(S_N)| - |T_0|)!(|T_0| - 1)!}{(|Supp(S_N)|)!} \cdot$$

$$\left(\tilde{v}(i) \cdot (\lambda \pi_N(j) + \mu_N(i)) \cdot \prod_{j \in Supp(S_N) \backslash \{i\}} ((1-\lambda)\pi_N(j) + v_N(j)) \right)$$

$$= \left(\tilde{v}(i) \cdot (\lambda \pi_N(j) + \mu_N(i)) \cdot \prod_{j \in Supp(S_N) \backslash \{i\}} ((1-\lambda)\pi_N(j) + v_N(j)) \right)$$

Property 3.2 [37]. Let $\tilde{v}, \tilde{w} \in IFG(N)$. A method satisfies coalitional monotonicity, if an increase in the value of a particular coalition implies, ceteris paribus, no decrease in the allocation to any member of that coalition:

$\tilde{v}(S_T) \geq \tilde{w}(S_T)$ for some S_T and $\tilde{v}(S_K) = \tilde{w}(S_K)$ for all $S_T \neq S_K$,
implies $\varphi_i(S_N, \tilde{v}) \geq \varphi_i(S_N, \tilde{w})$ for all $i \in Supp(S_T)$.

4 Analysis of Example and Computational Result Comparison

There are many applications of the classical cooperative game theory about real decision problems in finance, management, business, investment, and economics. The following example is an intuitionistic fuzzy cooperative game, which is applied to determine optimal allocation strategies of enterprises (or factories).

Suppose that there are three factories (i.e., players) 1, 2, and 3, who have the ability to produce separately. Denoted the set of players by $N = \{1, 2, 3\}$. Now, they plan to work together for manufacturing a better product. As we all know, each decision maker does not need to supply all of his or her resources to cooperate in real life; it depends on individual preference. Here, decision maker 1 would supply 6 tons of R_1 to the cooperation, and would not supply 3 tons of R_1, the rest of 1 tons of R_1 hesitate to supply to the cooperation. While decision maker 2 can provide 3 tons of R_2, and would not supply 6 tons of R_2, the rest of 1 tons of R_2 hesitate to supply to the cooperation. While decision maker 3 would supply 2 tons of R_3, and would not supply 6 tons of R_3, the rest of 2 tons of R_3 hesitate to supply to the cooperation.. As decision maker 1 has 10 tons of R_1, we regard the rate of participation and non-participation of decision maker 1 as $0.6 = \frac{6}{10}$, $0.3 = \frac{3}{10}$.Similarly, we can see that the participation and non-participation of decision maker 2 is $0.3 = \frac{3}{10}$, $0.6 = \frac{6}{10}$, and that of decision maker 3 is $0.2 = \frac{2}{10}$, $0.6 = \frac{6}{10}$. Therefore, an intuitionistic fuzzy coalition S_N has been formed:

$$S_N = \{(0.6,\ 0.3),\ (0.3,\ 0.6),\ (0.2,\ 0.6)\}.$$

Otherwise, due to the incomplete and uncertain information, they cannot precisely forecast their profits (or gains), Namely, the profit of a coalition $S \subseteq N$ of the factories (i.e., players) may be expressed with an intuitionistic fuzzy number $\tilde{v}(S) = \langle (\underline{a}_1, a, \bar{a}_1), (\underline{a}_2, a, \bar{a}_2) \rangle$. In this case, the optimal allocation problem of profits for the

Table 1. Cooperative games with the clear coalitions and intuitionistic fuzzy characteristic functions

The clear coalition T	The intuitionistic fuzzy characteristic functions
$\{1\}$	$\langle(0,0,0);(0,0,0)\rangle$
$\{2\}$	$\langle(0,0,0);(0,0,0)\rangle$
$\{3\}$	$\langle(0,0,0);(0,0,0)\rangle$
$\{1,2\}$	$\langle(49,50,51);(48,50,52)\rangle$
$\{1,3\}$	$\langle(49,50,51);(48,50,52)\rangle$
$\{2,3\}$	$\langle(49,50,51);(48,50,52)\rangle$
$\{1,2,3\}$	$\langle(94,95,96);(93,95,97)\rangle$

Table 2. Cooperative games with the intuitionistic fuzzy coalitions and intuitionistic fuzzy characteristic functions

The intuitionistic fuzzy coalition S_T	The intuitionistic fuzzy characteristic functions
$S_{\{1\}}$	$\langle(0,0,0);(0,0,0)\rangle$
$S_{\{2\}}$	$\langle(0,0,0);(0,0,0)\rangle$
$S_{\{3\}}$	$\langle(0,0,0);(0,0,0)\rangle$
$S_{\{1,2\}}$	$\langle(11.1475, 11.375, 11.6025);(10.92, 11.375, 11.837)\rangle$
$S_{\{1,3\}}$	$\langle(9.555, 9.75, 9.945);(9.36, 9.75, 10.14)\rangle$
$S_{\{2,3\}}$	$\langle(5.145, 5.25, 5.355);(5.04, 5.25, 5.46)\rangle$
$S_{\{1,2,3\}}$	$\langle(22.231, 22.621, 23.012);(21.84, 22.621, 23.404)\rangle$

factories may be regarded as an intuitionistic fuzzy cooperative game $\tilde{v}$ in which the intuitionistic fuzzy characteristic function is equal to $\tilde{v}(S)$ for any coalition $S \subseteq N$. Thus, if they manufacture the product by themselves, then their profits are expressed with the intuitionistic fuzzy number. Thus, the crisp coalitions' payoffs are given as follows:

However, suppose that the rest of 1 tons of R_1 supply and half of the 1 tons of R_1 would not supply to the cooperation, the rest of 1 tons of R_2 supply and half of the 1 tons of R_2 would not supply to the cooperation, the rest of 2 tons of R_3 supply and half of the 2 tons of R_3 would not supply to the cooperation (Table 1).

From Eqs. (1), we get

$$\varphi_1(S_N, \tilde{v}) = \langle(9.146, 9.311, 9.477);(8.980, 9.311, 9.643)\rangle$$

In the same way

$$\varphi_2(S_N, \tilde{v}) = \langle(6.941, 7.061, 7.182);(6.820, 7.061, 7.303)\rangle,$$
$$\varphi_3(S_N, \tilde{v}) = \langle(6.144, 6.249, 6.353);(6.040, 6.249, 6.458)\rangle$$

$$\tilde{v}(S_N) = \langle(22.231, 22.621, 23.012);(21.84, 22.621, 23.404)\rangle$$

Table 3. Cooperative games with the clear coalitions and fuzzy characteristic functions

The clear coalition T	The fuzzy characteristic functions
$\{1\}$	$(0,0,0)$
$\{2\}$	$(0,0,0)$
$\{3\}$	$(0,0,0)$
$\{1,2\}$	$(49,50,51)$
$\{1,3\}$	$(49,50,51)$
$\{2,3\}$	$(49,50,51)$
$\{1,2,3\}$	$(94,95,96)$

Obviously, $\varphi_1(S_N,\tilde{v})+\varphi_2(S_N,\tilde{v})+\varphi_3(S_N,\tilde{v})=\tilde{v}(S_N)$, which is indicated that the Shapely value proposed in this paper satisfies and individual rationality and efficiency. so the allocation scheme is accepted by the Bureau of the 1, 2, and 3 (Table 2).

The membership function and the non membership function of player 1:

$$\mu_{\tilde{A}}(x)=\begin{cases}0 & (x<5.538, x>5.742)\\(x-5.538)/0.102 & (5.538\leq x<5.64)\\1 & (x=5.64)\\(5.742-x)/0.102 & (5.64\leq x\leq 5.742)\end{cases},$$

$$v_{\tilde{A}}(x)=\begin{cases}1 & (x<5.436, x>5.844)\\(5.64-x)/0.204 & (5.436\leq x<5.64)\\0 & (x=5.64)\\(x-5.64)/0.204 & (5.64\leq x<5.844)\end{cases}.$$

The membership function and the non membership function of player 2:

$$\mu_{\tilde{A}}(x)=\begin{cases}0 & (x<4.215, x>4.365)\\(x-4.215)/0.075 & (4.215\leq x<4.29)\\1 & (x=4.29)\\(4.365-x)/0.075 & (4.29\leq x\leq 4.365)\end{cases},$$

$$v_{\tilde{A}}(x)=\begin{cases}1 & (x<4.14, x>4.44)\\(4.14-x)/0.15 & (4.14\leq x<4.29)\\0 & (x=4.29)\\(x-4.14)/0.15 & (4.29\leq x<4.44)\end{cases}.$$

The membership function and the non membership function of player 3:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & (x<3.333, x > 3.815) \\ (x-3.333)/0.057 & (3.333 \leq x<3.39) \\ 1 & (x = 3.39) \\ (3.815-x)/0.057 & (3.39 \leq x \leq 3.815) \end{cases},$$

$$v_{\tilde{A}}(x) = \begin{cases} 1 & (x<3.276, x > 3.404) \\ (3.39-x)/0.114 & (3.276 \leq x<3.39) \\ 0 & (x = 3.39) \\ (x-3.39)/0.114 & (3.39 \leq x<3.404) \end{cases}.$$

From the above three players of the membership function and the non membership function can obtain that the income of the players, it has a very deep significance.

Obviously, if $S_N = \{(0.6,\ 0.3),\ (0.3,\ 0.6),\ (0.2,\ 0.6)\}$, the cooperative games with the intuitionistic fuzzy coalitions and intuitionistic fuzzy characteristic functions is reduced to the cooperative games with fuzzy coalitions and fuzzy characteristic functions. The fuzzy characteristic functions is shown in Table 3.

According to Eq. (2), we can get

$$\varphi_1 = (9.146, 9.311, 9.477),\ \varphi_2 = (6.941, 7.061, 7.182),\ \varphi_3 = (6.144, 6.249, 6.353).$$

Obviously, the classical cooperative game and fuzzy cooperative game are special case of intuitionistic fuzzy cooperative game.

5 The Comparison Analysis and Conclusion

This paper researches the cooperative games with the intuitionistic fuzzy coalitions and intuitionistic fuzzy characteristic functions. The Shapley function proposed in this chapter is more widely used in real life. As we all know, each decision maker does not need to supply all of his or her resources to cooperate in real life; it depends on individual preference. people participate in a coalition with hesitation in real life, however, intuitionistic fuzzy coalition is more flexible to reflect the degree of people involved in the league and non-participation in the league, it makes the alliance more general, more closer to the realistic problems. Thus we use intuitionistic fuzzy coalition to deal with uncertainty and imprecision in real life. Otherwise, due to the incomplete and uncertain information, they cannot precisely forecast their profits (or gains),in order to make the cooperative game theory is more applicable to the real problem, we use characteristic intuitionistic fuzzy numbers to deal with uncertainty and imprecision in real life. Thus the research of Yu and Zhang [23], Meng and Zhang [24] and so on are a special case of this paper, i.e., the crisp cooperative and fuzzy cooperative are a special case of intuitionistic fuzzy cooperative game.

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A Profit Allocation Model of Employee Coalitions Based on Triangular Fuzzy Numbers in Tacit Knowledge Sharing

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Abstract. We consider a profit allocation of employee coalitions in tacit knowledge sharing. Owing to the existence of uncertain factors, the allocation of profits cannot be accurately expressed among players. Triangular fuzzy numbers, which are expressed as the payoffs of coalitions, are used to give an allocation solution. Meanwhile, the allocation also addressed the influence of coalitions' importance. A quadratic programming model is built to obtain a suitable solution, which is a triangular fuzzy number distribution value of each player. Further, we add a constraint to the built model: effectiveness, and obtain the pre-allocated solution. Finally, the rationality and superiority of the proposed model are verified through a numerical example.

Keywords: Triangular fuzzy number · Coalitions' weight · Tacit knowledge · Cooperative game · Profit allocation

1 Introduction

With the rapid development of knowledge economy, capital, labor and other tangible resources as the competitive advantages in traditional strategy theory fail to meet enterprises' needs of survival and development. Knowledge has become the first factor of economic growth, which is an important source for enterprises to maintain the competitive advantages (Fuchs-Kittowski and Kohler 2002). The tacit knowledge, which accounts for 90% of total amount of knowledge, has become one of the most important strategic resources for enterprises. Therefore, tacit knowledge sharing has become the key to knowledge management in enterprises (Li and Cheng 2014).

Tacit knowledge is a concept that Michael Polanyi proposed from the field of Philosophy in 1958. It exists in employees and organizations, also in production, supply, sales, researches and decision-making activities of enterprises. It is hardly to clearly express by using language and text form. Based on nowadays market environment, enterprises are eager to achieve tacit knowledge sharing. At present, researches on tacit knowledge sharing are mainly from the perspective of employers, studying theoretical models that enterprises motivate staff to share tacit knowledge. For example, Japanese scholar Nonaka et al. (2000) put forwards a SECI model to study the transformation of tacit knowledge and explicit knowledge through socialization,

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externalization, integration and internalization. Suppiah and Sandhum (2011) used 4 indicators to realize tacit knowledge sharing, including personal interaction, organizational communication, the willingness to share knowledge and the guidance of the system. Cai et al. (2015) proposed a two-transformation methods model: decentralized and centralized, and a transformation “triple mode” which attempted to fully mobilize individuals, teams and organizations. Through constructing a variety of theoretical models, scholars have made deep researches on tacit knowledge sharing, and have achieved some results. However, there is little research from the perspective of staff to promote tacit knowledge sharing. At present, enterprises in the market do not generate a common situation of tacit knowledge sharing, which indicates employees are not completely satisfied with the current remuneration. Therefore, it is necessary to motivate employees to negotiate with enterprises actively for satisfactory results. Assuming all the employees are rational, they have willingness to pursue interests of tacit knowledge sharing. Obviously, only relying on personal strength, it is difficult to negotiate with employers to get satisfactory rewards. Employees will choose to join a coalition and then negotiate with employers. But unreasonable profit allocation has become an important resistance for coalitions. Therefore, it is of great significance to design a reasonable profit allocation method to promote the coalitions of employees, and then effectively promote tacit knowledge sharing.

Due to the importance of a fair profit allocation, Scholars have made a lot of efforts in the study of the allocation of profit. What the most representative is the Shapley model proposed by Professor Shapley (1953). It introduced some concepts to deal with such a situation. Shapley value has simple structure and is easily to put into use, so it has been widely used. However, Shapley value only considers the contribution margin of the players, ignoring the importance and uncertainty of each factor that affects the final allocation. Meanwhile, because of uncertainties in the coalitions, it is difficult to pursuit the exact allocation of profits. For example, in tacit knowledge sharing, both explicit evaluation index and dominant degree exist many uncertainties, leading to predict the benefits which the employees bring for enterprises difficultly with an exact value.

In this case, using fuzzy number like triangular fuzzy number can solve this problem. Triangular fuzzy numbers take the possible range of fuzzy numbers and the probability of each possible value into account, which can be used to express the fuzzy uncertainty (Jiang 2016; Huang and Luo 2016; Pan et al. 2015). In the previous study, Han and Li (2016) transferred cooperative games with intuitionistic fuzzy coalitions and triangular fuzzy numbers typed payoffs to cooperative games with intuitionistic fuzzy coalitions and real number typed payoffs by using the continuous ordered weighted average operator and the concept of cut sets. Yu and Zhang (2010) defined a new kind of fuzzy cores for cooperative games and give the optimistic allocation scheme based on fuzzy Shapley and the new fuzzy core. Through the analysis, we can see that in the existing profit allocation model based on triangular fuzzy numbers, coalitions’ weights are rarely considered. In the cooperative game, the importance of each coalition (coalition’s weight) is different when achieving the goals of cooperation. Therefore, in order to realize the fair and reasonable allocation, it is necessary to assign the profit depending on coalitions’ weights. According to this analysis, this paper takes coalitions’ weights into consideration in profit allocation, proposing a new and effective solution based on triangular fuzzy numbers. This solution can be better applied to the profit allocation in employees' alliance of tacit knowledge sharing.

The remaining part of the paper is organized as follows. In Sect. 2, we give the preliminary knowledge of constructing the model, including the concept of triangular fuzzy numbers, and give the definition of triangular fuzzy numbers' distance based on the least square method. Our model is given in Sect. 3. With considering the weights of the coalitions, we propose a quadratic programming model of a cooperative game whose coalitions' payoffs are expressed as triangular fuzzy numbers. Besides, by verifying the effectiveness of the cooperative game to optimize the mathematical model, we can find the most optimal solution of the coalitions. In Sect. 4, we discuss the application of the proposed model in tacit knowledge sharing, and illustrate the feasibility and effectiveness of the proposed method by a numerical example. Some conclusions and possible future work are summarized in Sect. 5.

2 Preliminaries

2.1 Triangular Fuzzy Number

Let $\tilde{a} = (a_L, a_M, a_R)$ be an arbitrary triangular fuzzy number, then the following Eq. (1)

$$\tilde{a}(x) = \begin{cases} (x - a_L)/(a_M - a_L), & a_L \leq x < a_M \\ 1, & x = a_M \\ (a_R - x)/(a_R - a_M), & a_M < x \leq a_R \\ 0, & x < a_L, x > a_M \end{cases} \tag{1}$$

is said to be the membership function (Li 2003, 2012). Here, a_L and a_R are respectively represent the lower bound and the upper bound of $\tilde{a}$, and a_M is expressed as the intermediate value, that is, the most probable value. The graph of triangular fuzzy number is shown in Fig. 1.

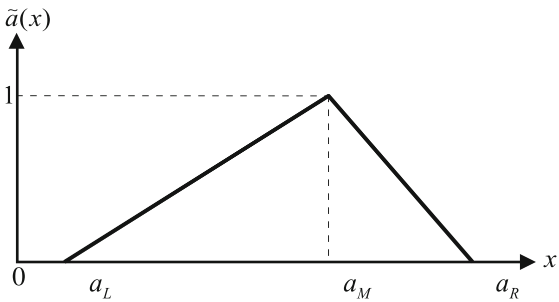

Fig. 1. Triangular fuzzy number $\tilde{a} = (a_L, a_M, a_R)$

It can be seen from Fig. 1 that when the lower bound, the intermediate value and the upper bound of the triangle fuzzy number $\tilde{a} = (a_L, a_M, a_R)$ are equal, that is, $a_L = a_M = a_R$, triangular fuzzy number $\tilde{a}$ degrades to the exact number. On the contrary, the exact number can be easily expressed in the form of triangular fuzzy numbers, that is, the exact number is a special form of triangular fuzzy numbers.

2.2 Triangular Fuzzy Numbers' Distance

Definition 1. Let $\tilde{a} = (a_L, a_M, a_R)$, $\tilde{b} = (b_L, b_M, b_R)$, and $\tilde{c} = (c_L, c_M, c_R)$ are any three triangular fuzzy numbers, if $D(\tilde{a}, \tilde{b})$ satisfies the following properties:

(1) $D(\tilde{a}, \tilde{b}) \geq 0$,
(2) $D(\tilde{a}, \tilde{b}) = D(\tilde{b}, \tilde{a})$,
(3) $D(\tilde{a}, \tilde{b}) = 0$ if and only if $\tilde{a} = \tilde{b}$,
(4) $D(a, b) \leq D(a, c) + D(c, b)$,

i.e. $D(\tilde{a}, \tilde{b})$ is the distance between $\tilde{a}$ and $\tilde{b}$.

According to the basic idea of the least square method, the formula for distance of triangular fuzzy numbers can be given as follows:

$$D(\tilde{a}, \tilde{b}) = (a_L - b_L)^2 + (a_M - b_M)^2 + (a_R - b_R)^2 \tag{2}$$

It is easily to prove that the Eq. (2) satisfies the four properties in Definition 1, so Eq. (2) can be regarded as the distance between triangular fuzzy numbers and can be used to measure the difference between triangular fuzzy numbers $\tilde{a}$ and $\tilde{b}$.

3 The Profit Allocation Model Based on Triangular Fuzzy Numbers

In this section, we provide more details regarding our model. We consider the weights of coalitions and use the triangular fuzzy number as the payoff of the cooperative game. Based on this, we give the profit allocation model. Throughout the model, we will use the following notations. The cooperative game, whose coalition's payoff is expressed as a triangular fuzzy number, can be represented by an ordered two tuples $\langle N, \tilde{\upsilon} \rangle$, $N = \{1, 2, \cdots, n\}$ is a finite set of players, and $\tilde{\upsilon}$ is a triangular fuzzy number, which represents the characteristic function of a coalition(or coalition's value), that is, $\tilde{\upsilon}(S) = (\upsilon_L(S), \upsilon_M(S), \upsilon_R(S))$, here, $\upsilon_L(S)$ and $\upsilon_R(S)$ represent the lower bound, the upper bound of the coalition's value respectively, and $\upsilon_M(S)$ is expressed as the intermediate value, that is, the most probable value. In the employee coalitions of tacit knowledge sharing, $\upsilon_L(S)$, $\upsilon_R(S)$ and $\upsilon_M(S)$ express the minimum total profit, the maximum total profit, and the most possible total profit respectively. Next, the employees will distribute the profits of the coalitions. Accordingly, the allocation of each employee has the maximum, minimum, and the most probable rewards, expressed as triangular fuzzy numbers.

The empty set $\varnothing$, as a special set, represents the coalitions with no player. We set $\tilde{\upsilon}(\varnothing) = 0$. To be clear and efficient, if without special illustration, $\tilde{\upsilon}(S\backslash\{i\})$, $\tilde{\upsilon}(S \cup \{i\})$, $\tilde{\upsilon}(\{i\})$ and $\tilde{\upsilon}(\{i,j\})$ are respectively noted as $\tilde{\upsilon}(S\backslash i)$, $\tilde{\upsilon}(S \cup i)$, $\tilde{\upsilon}(i)$ and $\tilde{\upsilon}(i,j)$. In addition, the set of all $S \subseteq N$ is noted as 2^N, $\tilde{G}^N$ is the set of cooperation of n players, whose coalition's payoff is expressed as a triangular fuzzy number.

3.1 Quadratic Programming Model with Coalitions' Weights and Solutions

In this section, we describe the model and the process of solving solutions concretely.

Let $x_i = (x_{Li}, x_{Mi}, x_{Ri})$ be an allocation for the player $i(i \in S\ )$ in our model. It is rational to expect that player i gets from a coalition at least the amount x_{Li} which he/she would obtain if played individually (Sibasis et al. 2015). We thus call x_{Li} the minimum reward. Again x_{Ri} is called the maximum reward and x_{Mi} is the most probable reward. Then we give a profit allocation model to determine the allocation value of triangular fuzzy numbers for each player in the cooperative game (Ye and Li 2016). The model innovatively takes coalitions' weights into account, which is given as follows:

$$\min\left\{L(\boldsymbol{x}) = \sum_{S\subseteq N}\omega(s)\left[\left(\sum_{i\in S}x_{Li} - \upsilon_L(S)\right)^2 + \left(\sum_{i\in S}x_{Mi} - \upsilon_M(S)\right)^2 + \left(\sum_{i\in S}x_{Ri} - \upsilon_R(S)\right)^2\right]\right\}, \tag{3}$$

where s is the number of all players in the coalition S and $\omega(s)$ is the weight of S.

Partial derivatives of $L(\boldsymbol{x})$ with respect to the variables x_{Lj}, x_{Mj} and $x_{Rj}(j \in S \subseteq N)$ are computed as follows:

$$\frac{\partial L(\boldsymbol{x})}{\partial x_{Lj}} = 2\sum_{S\subseteq N:j\in S}\omega(s)\left(\sum_{i\in S}x^*_{Li} - \upsilon_L(S)\right)(j = 1, 2, \cdots n),$$

$$\frac{\partial L(\boldsymbol{x})}{\partial x_{Mj}} = 2\sum_{S\subseteq N:j\in S}\omega(s)\left(\sum_{i\in S}x^*_{Mi} - \upsilon_M(S)\right)(j = 1, 2, \cdots n)$$

and

$$\frac{\partial L(\boldsymbol{x})}{\partial x_{Rj}} = 2\sum_{S\subseteq N:j\in S}\omega(s)\left(\sum_{i\in S}x^*_{Ri} - \upsilon_R(S)\right)(j = 1, 2, \cdots n),$$

respectively.

Next, we discuss how to obtain the optimal solution $\tilde{\boldsymbol{x}}^* = (\tilde{x}^*_1, \tilde{x}^*_2, \cdots, \tilde{x}^*_n)^{\mathrm{T}}$ of the above model. Let the partial derivatives of $L(\boldsymbol{x})$ with respect to the variables x_{Lj}, x_{Mj} and x_{Rj} be equal to 0 respectively. Thus, we have

$$\sum_{S\subseteq N:j\in S}\omega(s)\sum_{i\in S}x_{Li}^{*}=\sum_{S\subseteq N:j\in S}\omega(s)\upsilon_{L}(S),(j=1,2,\cdots n) \tag{4}$$

$$\sum_{S\subseteq N:j\in S}\omega(s)\sum_{i\in S}x_{Mi}^{*}=\sum_{S\subseteq N:j\in S}\omega(s)\upsilon_{M}(S),(j=1,2,\cdots n) \tag{5}$$

and

$$\sum_{S\subseteq N:j\in S}\omega(s)\sum_{i\in S}x_{Ri}^{*}=\sum_{S\subseteq N:j\in S}\omega(s)\upsilon_{R}(S),(j=1,2,\cdots n). \tag{6}$$

To solve the solution $\tilde{\boldsymbol{x}}^{*}=\left(\tilde{x}_{1}^{*},\tilde{x}_{2}^{*},\cdots,\tilde{x}_{n}^{*}\right)^{\mathrm{T}}$, Eqs. (4), (5) and (6) can be rewritten as follows:

$$\begin{cases} a_{11}x_{L1}^{*}+a_{12}x_{L2}^{*}+a_{13}x_{L3}^{*}+\cdots+a_{1n}x_{Ln}^{*}=\sum\limits_{S\subseteq N:1\in S}\omega(s)\upsilon_{L}(S) \\ a_{21}x_{L1}^{*}+a_{22}x_{L2}^{*}+a_{23}x_{L3}^{*}+\cdots+a_{2n}x_{Ln}^{*}=\sum\limits_{S\subseteq N:2\in S}\omega(s)\upsilon_{L}(S) \\ \vdots \\ a_{n1}x_{L1}^{*}+a_{n2}x_{L2}^{*}+a_{n3}x_{L3}^{*}+\cdots+a_{nn}x_{Ln}^{*}=\sum\limits_{S\subseteq N:n\in S}\omega(s)\upsilon_{L}(S), \end{cases}$$

$$\begin{cases} a_{11}x_{M1}^{*}+a_{12}x_{M2}^{*}+a_{13}x_{M3}^{*}+\cdots+a_{1n}x_{Mn}^{*}=\sum\limits_{S\subseteq N:1\in S}\omega(s)\upsilon_{M}(S) \\ a_{21}x_{M1}^{*}+a_{22}x_{M2}^{*}+a_{23}x_{M3}^{*}+\cdots+a_{2n}x_{Mn}^{*}=\sum\limits_{S\subseteq N:2\in S}\omega(s)\upsilon_{M}(S) \\ \vdots \\ a_{n1}x_{M1}^{*}+a_{n2}x_{M2}^{*}+a_{n3}x_{M3}^{*}+\cdots+a_{nn}x_{Mn}^{*}=\sum\limits_{S\subseteq N:n\in S}\omega(s)\upsilon_{M}(S), \end{cases}$$

and

$$\begin{cases} a_{11}x_{R1}^{*}+a_{12}x_{R2}^{*}+a_{13}x_{R3}^{*}+\cdots+a_{1n}x_{Rn}^{*}=\sum\limits_{S\subseteq N:1\in S}\omega(s)\upsilon_{R}(S) \\ a_{21}x_{R1}^{*}+a_{22}x_{R2}^{*}+a_{23}x_{R3}^{*}+\cdots+a_{2n}x_{Rn}^{*}=\sum\limits_{S\subseteq N:2\in S}\omega(s)\upsilon_{R}(S) \\ \vdots \\ a_{n1}x_{R1}^{*}+a_{n2}x_{R2}^{*}+a_{n3}x_{R3}^{*}+\cdots+a_{nn}x_{Rn}^{*}=\sum\limits_{S\subseteq N:n\in S}\omega(s)\upsilon_{R}(S), \end{cases}$$

respectively.

According to the knowledge on the theory of permutation and combination, the following results can be obtained:

(1) If $i=j(i,j\in\{1,2,\cdots,n\})$, then

$$a_{ij}=C_{n-1}^{0}\omega(1)+C_{n-1}^{1}\omega(2)+C_{n-1}^{2}\omega(3)+\cdots+C_{n-1}^{n-1}\omega(n)$$

(2) If $i\neq j(i,j\in\{1,2,\cdots,n\})$, then

$$a_{ij} = C_{n-2}^{0}\omega(2) + C_{n-2}^{1}\omega(3) + C_{n-2}^{2}\omega(4) + \cdots + C_{n-2}^{n-2}\omega(n)$$

Let

$$a = C_{n-1}^{0}\omega(1) + C_{n-1}^{1}\omega(2) + C_{n-1}^{2}\omega(3) + \cdots + C_{n-1}^{n-1}\omega(n),$$
$$b = C_{n-2}^{0}\omega(2) + C_{n-2}^{1}\omega(3) + C_{n-2}^{2}\omega(4) + \cdots + C_{n-2}^{n-2}\omega(n),$$

then

$$a_{ij} = \begin{cases} a & (i = j \text{ with } i,j \in \{1, 2, \cdots, n\}) \\ b & (i \neq j \text{ with } i,j \in \{1, 2, \cdots, n\}) \end{cases}$$

Denote

$$\boldsymbol{X}_L^* = \left(x_{L1}^*, x_{L2}^*, \cdots, x_{Ln}^*\right)^{\mathrm{T}},$$

$$\boldsymbol{X}_M^* = \left(x_{M1}^*, x_{M2}^*, \cdots, x_{Mn}^*\right)^{\mathrm{T}},$$

$$\boldsymbol{X}_R^* = \left(x_{R1}^*, x_{R2}^*, \cdots, x_{Rn}^*\right)^{\mathrm{T}},$$

$$\boldsymbol{B}_L = \left(\sum_{S \subseteq N:1 \in S} \omega(s)\upsilon_L(S), \cdots, \sum_{S \subseteq N:n \in S} \omega(s)\upsilon_L(S)\right)^{\mathrm{T}},$$

$$\boldsymbol{B}_M = \left(\sum_{S \subseteq N:1 \in S} \omega(s)\upsilon_M(S), \cdots, \sum_{S \subseteq N:n \in S} \omega(s)\upsilon_M(S)\right)^{\mathrm{T}},$$

$$\boldsymbol{B}_R = \left(\sum_{S \subseteq N:1 \in S} \omega(s)\upsilon_R(S), \cdots, \sum_{S \subseteq N:n \in S} \omega(s)\upsilon_R(S)\right)^{\mathrm{T}}$$

and

$$\boldsymbol{A} = (a_{ij})_{n \times n} = \begin{pmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & a \end{pmatrix}_{n \times n,}$$

Accordingly, Eqs. (4), (5) and (6) can be rewritten in the matrix format as follows:

$$\boldsymbol{A}\boldsymbol{X}_L^* = \boldsymbol{B}_L,$$

$$\boldsymbol{A}\boldsymbol{X}_M^* = \boldsymbol{B}_M,$$

$$\boldsymbol{A}\boldsymbol{X}_R^* = \boldsymbol{B}_R.$$

The matrix A is reversible. After a series of elementary rows change operation, we have:

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{a+(n-2)b}{(a+(n-1)b)(a-b)} & \frac{-b}{(a+(n-1)b)(a-b)} & \cdots & \frac{-b}{(a+(n-1)b)(a-b)} \\ \frac{-b}{(a+(n-1)b)(a-b)} & \frac{a+(n-2)b}{(a+(n-1)b)(a-b)} & \cdots & \frac{-b}{(a+(n-1)b)(a-b)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-b}{(a+(n-1)b)(a-b)} & \frac{-b}{(a+(n-1)b)(a-b)} & \cdots & \frac{a+(n-2)b}{(a+(n-1)b)(a-b)} \end{pmatrix}$$

By using the multiplication of the matrix, we obtain the following solutions of Eqs. (4), (5) and (6) as follows:

$$\mathbf{X}_L^* = \mathbf{A}^{-1}\mathbf{B}_L, \tag{7}$$

$$\mathbf{X}_M^* = \mathbf{A}^{-1}\mathbf{B}_M, \tag{8}$$

$$\mathbf{X}_R^* = \mathbf{A}^{-1}\mathbf{B}_R, \tag{9}$$

respectively. Thus, we obtain the optimal allocation with considering of coalitions' weighs for player i ($i \in N$), whose components are expressed as triangular fuzzy numbers $\tilde{x}_i^* = (x_{Li}^*, x_{Mi}^*, x_{Ri}^*)(i = 1, 2, \cdots, n)$. Here, x_{Li}^* represents the minimum reward that the player i ($i \in N$) gets from the coalition. Again x_{Ri}^* is the maximum reward and x_{Mi}^* is the most probable reward.

3.2 The Pre-allocation with Coalitions' Weights and Efficiency

In Sect. 3.1, with considering the weights of coalitions, we gain the quadratic programming model's solution of the cooperative game, which is expressed as a triangular fuzzy number. However, we do not take into account the effectiveness, an important property to cooperative games. In the following, we will increase the effectiveness as a constraint in the Eq. (3), so that the final distribution of each player involved in the coalition is equal to the major coalition (Li 2016). Add the constraint conditions to Eq. (3), and then it can be flexibly rewritten as the following quadratic programming model (10):

$$\min\left\{L(\boldsymbol{x}) = \sum_{S\subseteq N}\omega(s)\left[\left(\sum_{i\in S}x_{Li} - \upsilon_L(S)\right)^2 + \left(\sum_{i\in S}x_{Mi} - \upsilon_M(S)\right)^2 + \left(\sum_{i\in S}x_{Ri} - \upsilon_R(S)\right)^2\right]\right\}$$

$$\text{s.t.}\begin{cases}\sum_{i=1}^{n}x_{Li} = \upsilon_L(N)\\ \sum_{i=1}^{n}x_{Mi} = \upsilon_M(N)\\ \sum_{i=1}^{n}x_{Ri} = \upsilon_R(N)\end{cases} \tag{10}$$

From the above analysis, it can be seen that the solution of Eq. (10) can get the pre-allocation of coalitions with weights, whose payoff is expressed as a triangular fuzzy number.

In what follows, we focus on how to solve Eq. (10). According to the Lagrange multiplier method, the Lagrange function of Eq. (10) can be constructed as follows:

$$\begin{aligned}L(\boldsymbol{x},\lambda,\gamma,\mu) = &\sum_{S\subseteq N}\omega(s)\left[\left(\sum_{i\in S}x_{Li} - \upsilon_L(S)\right)^2 + \left(\sum_{i\in S}x_{Mi} - \upsilon_M(S)\right)^2 + \left(\sum_{i\in S}x_{Ri} - \upsilon_R(S)\right)^2\right]\}\\ &+\lambda\left(\sum_{i=1}^{n}x_{Li} - \upsilon_L(N)\right) + \gamma\left(\sum_{i=1}^{n}x_{Mi} - \upsilon_M(N)\right) + \mu\left(\sum_{i=1}^{n}x_{Ri} - \upsilon_R(N)\right)\end{aligned} \tag{11}$$

where λ, γ and μ are Lagrange multipliers.

The partial derivatives of $L(\mathbf{x},\lambda,\gamma,\mu)$ with respect to the variables x_{Lj}, x_{Mj}, $x_{Rj}(j\in S\subseteq N)$, λ, γ and μ are obtained as follows:

$$\frac{\partial L(\boldsymbol{x},\lambda,\gamma,\mu)}{\partial x_{Lj}} = 2\sum_{S\subseteq N:j\in S}\omega(s)\left(\sum_{i\in S}x_{Li} - \upsilon_L(S)\right) + \lambda, \tag{12}$$

$$\frac{\partial L(\boldsymbol{x},\lambda,\gamma,\mu)}{\partial x_{Mj}} = 2\sum_{S\subseteq N:j\in S}\omega(s)\left(\sum_{i\in S}x_{Mi} - \upsilon_M(S)\right) + \lambda,$$

$$\frac{\partial L(\boldsymbol{x},\lambda,\gamma,\mu)}{\partial x_{Rj}} = 2\sum_{S\subseteq N:j\in S}\omega(s)\left(\sum_{i\in S}x_{Ri} - \upsilon_R(S)\right) + \lambda,$$

$$\frac{\partial L(\boldsymbol{x},\lambda,\gamma,\mu)}{\partial \lambda} = \sum_{i=1}^{n}x_{Li} - \upsilon_L(N), \tag{13}$$

$$\frac{\partial L(\boldsymbol{x},\lambda,\gamma,\mu)}{\partial \gamma} = \sum_{i=1}^{n}x_{Mi} - \upsilon_M(N)$$

and

$$\frac{\partial L(\boldsymbol{x},\lambda,\gamma,\mu)}{\partial \mu} = \sum_{i=1}^{n} x_{Ri} - \upsilon_R(N),$$

here, $j = 1, 2, \cdots n$.

Now continue to discuss how to solve the vector $\tilde{\boldsymbol{x}}^{E*} = \left(\tilde{x}_1^{E*}, \tilde{x}_2^{E*}, \cdots, \tilde{x}_n^{E*}\right)^{\mathrm{T}}$. We use Eqs. (12) and (13) to solve x_{Lj} as an example to illustrate the solution process.

Let the partial derivative of $L(\boldsymbol{x}, \lambda, \gamma, \mu)$ with respect to the variable $x_{Lj}(j \in S \subseteq N)$ be equal to 0. Then we obtain

$$\sum_{S\subseteq N: j\in S} \omega(s) \sum_{i\in S} x_{Li}^{E*} + \frac{1}{2}\lambda^{E*} = \sum_{S\subseteq N: j\in S} \omega(s)\upsilon_L(S) \tag{14}$$

here, $j = 1, 2, \cdots n$.

Let the partial derivative of $L(\boldsymbol{x}, \lambda, \gamma, \mu)$ with respect to the variable λ be equal to 0. Then we obtain

$$\sum_{i=1}^{n} x_{Li}^{E*} = \upsilon_L(N) \tag{15}$$

Equation (14) can be rewritten as follows:

$$\begin{cases} a_{11}x_{L1}^{E*} + a_{12}x_{L2}^{E*} + a_{13}x_{L3}^{E*} + \cdots + a_{1n}x_{Ln}^{E*} + \frac{1}{2}\lambda^{E*} = \sum\limits_{S\subseteq N:1\in S} \omega(s)\upsilon_L(S) \\ a_{21}x_{L1}^{E*} + a_{22}x_{L2}^{E*} + a_{23}x_{L3}^{E*} + \cdots + a_{2n}x_{Ln}^{E*} + \frac{1}{2}\lambda^{E*} = \sum\limits_{S\subseteq N:2\in S} \omega(s)\upsilon_L(S) \\ \vdots \\ a_{n1}x_{L1}^{E*} + a_{n2}x_{L2}^{E*} + a_{n3}x_{L3}^{E*} + \cdots + a_{nn}x_{Ln}^{E*} + \frac{1}{2}\lambda^{E*} = \sum\limits_{S\subseteq N:n\in S} \omega(s)\upsilon_L(S), \end{cases} \tag{16}$$

Let $\boldsymbol{X}_L^{E*} = \left(x_{L1}^{E*}, x_{L2}^{E*}, \cdots, x_{Ln}^{E*}\right)^{\mathrm{T}}$, $\boldsymbol{e} = (1, 1, \cdots, 1)_{n\times 1}^{\mathrm{T}}$,Then, Eqs. (15) and (16) can be rewritten as follows:

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{X}_L^{E*} = \upsilon_L(N) \tag{17}$$

and

$$\boldsymbol{A}\boldsymbol{X}_L^{E*} + \frac{1}{2}\lambda^{E*}\boldsymbol{e} = \boldsymbol{B}_L, \tag{18}$$

respectively, where the vector $\boldsymbol{B}_L$ and the matrix $\boldsymbol{A}$ are given in previous description.

Combining (17) with (18), we obtain

$$\boldsymbol{X}_L^{E*} = \boldsymbol{X}_L^* - \frac{1}{n}\left(\sum_{i=1}^{n} x_{Li}^* - \upsilon_L(N)\right)\boldsymbol{e}. \tag{19}$$

Analogously, the same method described above can be used to solve $\boldsymbol{X}_M^{E*}$, $\boldsymbol{X}_R^{E*}$.

Denote $\boldsymbol{X}_M^{E*} = \left(x_{M1}^{E*}, x_{M2}^{E*}, \cdots, x_{Mn}^{E*}\right)^{\mathrm{T}}$, $\boldsymbol{X}_R^{E*} = \left(x_{R1}^{E*}, x_{R2}^{E*}, \cdots, x_{Rn}^{E*}\right)^{\mathrm{T}}$, then we obtain the solutions as follows:

$$\boldsymbol{X}_M^{E*} = \boldsymbol{X}_M^* - \frac{1}{n}\left(\sum_{i=1}^{n} x_{Mi}^* - \upsilon_M(N)\right)\boldsymbol{e} \tag{20}$$

$$\boldsymbol{X}_R^{E*} = \boldsymbol{X}_R^* - \frac{1}{n}\left(\sum_{i=1}^{n} x_{Ri}^* - \upsilon_R(N)\right)\boldsymbol{e}, \tag{21}$$

respectively.

Therefore, for the coalitions whose payoffs are expressed as triangular fuzzy numbers, we have obtained the pre-allocation solution with considering the efficiency for player i ($i \in N$), and its component is represented as the triangular fuzzy number $\tilde{x}_i^{E*} = \left(x_{Li}^{E*}, x_{Mi}^{E*}, x_{Ri}^{E*}\right)(i = 1, 2, \cdots, n)$.

4 A Numerical Example

Tacit knowledge sharing among employees in enterprises has always been a hot issue that needs to be solved. In order to gain greater returns from tacit knowledge, rational employees will choose to cooperate to negotiate with enterprises. Employees through coalitions can achieve the integration of resources and complementary advantages, so that they can get a greater reward. However, any coalition has to deal with allocation of profits obtained from coalitions. Whether the profits can be distributed reasonably among players is the key element for coalitions' stabilization. In our numerical example, we apply a model proposed in this paper to profit allocation of employee coalitions, and verify validity of the model.

In our numerical example, we apply the model proposed in this paper to profit allocation of employee coalitions, and verify validity of the model. Let's assume that a company wants to stimulate three employees from three different departments who have core tacit knowledge to share tacit knowledge. To be clear, the three employees are called player 1, 2 and 3. Now the three players do not satisfy the reward given by their enterprise. In order to achieve complementary advantages, and to enhance the ability for negotiation, the three players would form an alliance of two or three. Because of the existence of many uncertainties, it is impossible to estimate expected benefits accurately from tacit knowledge sharing. So we can't use exact number to represent employees' reward. Based on triangular fuzzy numbers, the approximate value of the profits obtained from the coalition can be easily known.

Using $\upsilon(S)$ expresses the characteristic function (or value) of coalition S, the profits (unit: thousand) that employees choose to do alone or cooperate are as follows:

$$\upsilon(1) = (100, 120, 140), \upsilon(2) = (80, 100, 125), \upsilon(3) = (150, 160, 190),$$
$$\upsilon(1,2) = (300, 400, 500), \upsilon(1,3) = (600, 800, 1000), \upsilon(2,3) = (400, 550, 650),$$
$$\upsilon(1,2,3) = (1200, 1500, 2000).$$

The importance of each possible coalition (coalition's weight) will be given in accordance with the expert evaluation method:

if $s = 1$, $\omega(1) = \frac{1}{12}$; if $s = 2$, $\omega(2) = \frac{2}{12}$; if $s = 3$, $\omega(3) = \frac{3}{12}$.

That is to say, the weights of the coalitions $\{1\}$, $\{2\}$ and $\{3\}$ are $\frac{1}{12}$; the weights of the coalitions $\{1,2\}$, $\{2,3\}$, $\{1,3\}$ are $\frac{2}{12}$; the weights of the coalition $\{1,2,3\}$ is $\frac{3}{12}$.

Now the model proposed in Sect. 3 is used to solve the problem. In this case, a total of 3 players to participate in the cooperation, i.e., $n = 3$, and,

$$a = C_2^0\omega(1) + C_2^1\omega(2) + C_2^2\omega(3) = \frac{2}{3}, b = C_1^0\omega(2) + C_1^1\omega(3) = \frac{5}{12},$$

hereby,

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \frac{a+(n-2)b}{(a+(n-1)b)(a-b)} & \frac{-b}{(a+(n-1)b)(a-b)} & \cdots & \frac{-b}{(a+(n-1)b)(a-b)} \\ \frac{-b}{(a+(n-1)b)(a-b)} & \frac{a+(n-2)b}{(a+(n-1)b)(a-b)} & \cdots & \frac{-b}{(a+(n-1)b)(a-b)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-b}{(a+(n-1)b)(a-b)} & \frac{-b}{(a+(n-1)b)(a-b)} & \cdots & \frac{a+(n-2)b}{(a+(n-1)b)(a-b)} \end{pmatrix}_{n\times n}$$
$$= \begin{pmatrix} \frac{26}{9} & -\frac{10}{9} & -\frac{10}{9} \\ -\frac{10}{9} & \frac{26}{9} & -\frac{10}{9} \\ -\frac{10}{9} & -\frac{10}{9} & \frac{26}{9} \end{pmatrix}$$

According to Eq. (7), we have

$$\boldsymbol{X}_L^* = \boldsymbol{A}^{-1}\boldsymbol{B}_L = \begin{pmatrix} \frac{26}{9} & -\frac{10}{9} & -\frac{10}{9} \\ -\frac{10}{9} & \frac{26}{9} & -\frac{10}{9} \\ -\frac{10}{9} & -\frac{10}{9} & \frac{26}{9} \end{pmatrix} \begin{pmatrix} 458.33 \\ 423.33 \\ 479.17 \end{pmatrix} = \begin{pmatrix} 321.30 \\ 181.30 \\ 404.63 \end{pmatrix}$$

Analogously, we can calculate,

$$\mathbf{X}_M^* = \mathbf{A}^{-1}\mathbf{B}_M = \begin{pmatrix} 406.67 \\ 233.33 \\ 520.00 \end{pmatrix}, \mathbf{X}_R^* = \mathbf{A}^{-1}\mathbf{B}_R = \begin{pmatrix} 541.57 \\ 303.24 \\ 658.24 \end{pmatrix}$$

From the above results, we can see

$$
\begin{aligned}
&321.30 + 181.30 + 404.63 = 907.23 < 1200,\\
&406.67 + 233.33 + 520.00 = 1160 < 1500,\\
&541.57 + 303.24 + 658.24 = 1503.05 < 2000,
\end{aligned}
$$

the profits have not been shared by all the players, that is, the profits generated by the coalition are still left. For the players in the coalition, though the allocation scheme is fair and reasonable, it is not the optimal allocation scheme. Therefore, the following constraints are added to the problem to adjust the allocation scheme.

According to Eqs. (19) to (21), the results can be obtained sequentially:

$$
\boldsymbol{X}_L^{E*} = \boldsymbol{X}_L^* - \frac{1}{n}\left(\sum_{i=1}^{n} x_{Li}^* - \upsilon_L(N)\right)\boldsymbol{e} = \begin{pmatrix} 418.89 \\ 278.89 \\ 502.22 \end{pmatrix},
$$

$$
\boldsymbol{X}_M^{E*} = \boldsymbol{X}_M^* - \frac{1}{n}\left(\sum_{i=1}^{n} x_{Li}^* - \upsilon_M(N)\right)\boldsymbol{e} = \begin{pmatrix} 520.00 \\ 346.67 \\ 633.33 \end{pmatrix},
$$

$$
\boldsymbol{X}_R^{E*} = \boldsymbol{X}_R^* - \frac{1}{n}\left(\sum_{i=1}^{n} x_{Ri}^* - \upsilon_R(N)\right)\boldsymbol{e} = \begin{pmatrix} 707.22 \\ 468.89 \\ 823.89 \end{pmatrix}.
$$

At this point, the solutions have been obtained for all the players to meet the effectiveness of the optimal allocation schemes, which are given as follows:

$$
\begin{aligned}
\tilde{x}_1^{E*} &= \left(x_{L1}^{E*}, x_{M1}^{E*}, x_{R1}^{E*}\right) = (418.89, 520.00, 707.22),\\
\tilde{x}_2^{E*} &= \left(x_{L2}^{E*}, x_{M2}^{E*}, x_{R2}^{E*}\right) = (278.89, 346.67, 468.89),\\
\tilde{x}_3^{E*} &= \left(x_{L3}^{E*}, x_{M3}^{E*}, x_{R3}^{E*}\right) = (502.22, 633.33, 823.89).
\end{aligned}
$$

Here, $\tilde{x}_1^{E*}$, $\tilde{x}_2^{E*}$ and $\tilde{x}_3^{E*}$ are the final allocation schemes of player 1, 2 and 3 respectively. From the above results, it can be seen that the sum of the lower bound value, the intermediate value and the upper bound value of all the players are equal to the lower bound value, the intermediate value and the upper bound value of the total profits of the coalition, respectively, that is,

$$
\begin{gathered}
418.89 + 278.89 + 502.22 = 1200, 520.00 + 346.67 + 633.33 = 1500,\\
707.22 + 468.89 + 823.89 = 2000
\end{gathered}
$$

In addition, for each player, the profits that obtained from participating in the cooperation are far larger than doing alone. For further analysis, when all the players do alone in the case, player 2 obtains the profit is less than player 1 and player 3, therefore, it is intuitive to understand that the reward player 2 can be assigned from the coalition should be less than the other two, and the final allocation scheme obtained in this paper is in good agreement with the above analysis.

5 Conclusion

Tacit knowledge sharing in the enterprise is a hot issue that needs to be solved. We study tacit knowledge sharing from employees' perspective. It is different from previous studies that are only from the perspective of employers. Meanwhile, we propose a new and effective profit allocation model based on triangular fuzzy numbers with considering coalitions' weights for employee coalitions, and give a solution for each player who participants in the coalition. In addition, the profit allocation model proposed in this paper is a helpful complement to the least squares pre-kernel solution for clear cooperation game (Ruiz et al. 1996) and pre-kernel solution for interval-value cooperative game (Li and Liu 2016). It can be used to provide a new fast and effective method for cooperative game issues, whose coalitions' payoffs are expressed by triangular fuzzy numbers. Also, it can expand into other areas of cooperation, such as economy, management, politics, environment, diplomacy, etc., to provide a new research perspective and solution for the allocation of cooperative profits. The next step is to study the stability of employee coalitions in the process of negotiating with the enterprise. Because of the existence of opportunity Interest, self-interest and so on, even if the profit allocation scheme is fair and reasonable, the coalitions in tacit knowledge sharing still can be broken. Therefore, it's necessary to establish reasonable and effective punishment mechanism to ensure the stability of coalitions. Only when the profit allocation and the punishment mechanism are both reasonable and effective, can the employee coalitions be stable to promote tacit knowledge sharing rapidly.

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