A Note on q-Bernoulli–Euler Polynomials

Subuhi Khan and Mumtaz Riyasat

Abstract In this article, a mixed family of q-Bernoulli–Euler polynomials is introduced by means of generating function, series definition, and determinantal definition. Further, the numbers related to the q-Bernoulli–Euler polynomials are considered, and the graph of the q-Bernoulli–Euler polynomials is also drawn for index $n = 3$ and $q = 1/2$.

Keywords q-Bernoulli polynomials · q-Euler polynomials · Determinantal definition

1 Introduction and Preliminaries

Recently, there is a significant increase of research activities in the area of q-calculus due to its applications in various fields such as mathematics, physics, and engineering. By using q-analysis and umbral calculus, many special polynomials have been studied; see for example [1–4].

We review certain definitions and concepts of q-calculus.

Throughout this work, we apply the following notations: N indicates the set of natural numbers, $N0$ indicates the set of nonnegative integers, R indicates set of all real numbers, and C denotes the set of complex numbers. We refer the readers to [5] for all the following q-standard notations.

The q-analogues of a complex number a and of the factorial function are defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad q \in \mathbb{C} - \{1\}; \quad a \in \mathbb{C}. \tag{1.1}$$

S. Khan · M. Riyasat (✉)
Department of Mathematics, Aligarh Muslim University, Aligarh, India
e-mail: mumtazrst@gmail.com

S. Khan
e-mail: subuhi2006@gmail.com

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$$[n]_q! = \prod_{k=1}^{n} [k]_q = [1]_q[2]_q \ldots [n]_q, \quad n \in N; \quad [0]_q! = 1; \; q \in C. \tag{1.2}$$

The Gauss q-polynomial coefficient $\binom{n}{k}_q$ is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{\langle 1;q\rangle_n}{\langle 1;q\rangle_k \langle 1;q\rangle_{n-k}}, \quad k = 0, 1, \ldots, n, \tag{1.3}$$

where $\langle a;q\rangle_n$ are the q-shifted factorial.

The q-exponential function $e_q(x)$ is defined by

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \prod_{j=0}^{\infty} \frac{1}{(1-(1-q)q^j x)}, \quad 0<|q|<1; \; |x|<|1-q|^{-1}. \tag{1.4}$$

Al-Salaam, in [6], introduced the family of q-Appell polynomials $\{A_{n,q}(x)\}_{n\geq 0}$ and studied some of their properties. The n-degree polynomials $A_{n,q}(x)$ are called q-Appell provided they satisfy the following q-differential equation:

$$D_{q,x}\{A_{n,q}(x)\} = [n]_q A_{n-1,q}(x), \quad n = 0, 1, 2, \ldots; \; q \in C; \; 0<q<1. \tag{1.5}$$

The q-Appell polynomials $A_{n,q}(x)$ are also defined by means of the following generating function [6]:

$$A_q(t)e_q(xt) = \sum_{n=0}^{\infty} A_{n,q}(x) \frac{t^n}{[n]_q!}, \quad 0<q<1, \tag{1.6}$$

where

$$A_q(t) := \sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!}, \quad A_{0,q} = 1; \; A_q(t) \neq 0. \tag{1.7}$$

Based on different selections for the function $A_q(t)$, different members belonging to the family of q-Appell polynomials can be obtained.

For $A_q(t) = \left(\frac{t}{e_q(t)-1}\right)$, the q-Appell polynomials $A_{n,q}(x)$ become the q-Bernoulli polynomials $B_{n,q}(x)$ [1, 7], which are defined by the generating function of the following form:

$$\left(\frac{t}{e_q(t)-1}\right) e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!} \tag{1.8}$$

For $A_q(t) = \left(\frac{2}{e_q(t)+1}\right)$, the q-Appell polynomials $A_{n,q}(x)$ become the q-Euler polynomials $E_{n,q}(x)$ [1, 8], which are defined by the generating function of the following form:

$$\left(\frac{2}{e_q(t)+1}\right)e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{1.9}$$

Taking $x = 0$ in the generating functions (1.10) and (1.11), we find that the q-Bernoulli numbers (qBN) $B_{n,q}$ [1] and q-Euler numbers (qEN) $E_{n,q}$ [1] are defined by the generating relations:

$$\left(\frac{t}{e_q(t)-1}\right) = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!}, \tag{1.10}$$

$$\left(\frac{2}{e_q(t)+1}\right) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{[n]_q!} \tag{1.11}$$

respectively.

Consequently, from Eqs. (1.10), (1.11) and generating functions (1.8), (1.9), we have

$$B_{n,q} := B_{n,q}(0); \quad E_{n,q} := E_{n,q}(0). \tag{1.12}$$

The determinantal definition for the q-Appell polynomials is considered in [9]. Further, the determinantal definition for the q-Bernoulli polynomials $B_{n,q}(x)$ and q-Euler polynomials $E_{n,q}(x)$ are considered in [10]. The determinantal definition for a mixed family of q-Bernoulli and Euler polynomials can also be considered.

In this article, the q-Bernoulli and q-Euler polynomials are combined to introduce the family of q-Bernoulli–Euler polynomials by means of generating function, series definition, and determinantal definition. Further, the numbers related to the q-Bernoulli–Euler polynomials are considered, and the graph for these polynomials is also drawn for particular values of n and q.

2 q-Bernoulli–Euler Polynomials

The q-Bernoulli–Euler polynomials (qBEP) are introduced by means of generating function and series definition. In order to derive the generating function for the qBEP, we prove the following result:

Theorem 2.1 *The qBEP are defined by the following generating function*:

$$\frac{(2t)}{(e_q(t)-1)(e_q(t)+1)}e_q(xt)=\sum_{n=0}^{\infty}{}_BE_{n,q}(x)\frac{t^n}{[n]_q!},\quad 0<q<1. \tag{2.1}$$

Proof Expanding the q-exponential function $e_q(xt)$ in the l.h.s. of Eq. (1.9) and then replacing the powers of x, i.e., $x^0, x^1, x^2, \ldots, x^n$ by the corresponding polynomials $B_{0,q}(x), B_{1,q}(x), \ldots, B_{n,q}(x)$ in both sides of the resultant equation, we have

$$\left(\frac{2}{e_q(t)+1}\right)\left[1+B_{1,q}(x)\frac{t}{[1]_q!}+B_{2,q}(x)\frac{t^2}{[2]_q!}+\cdots+B_{n,q}(x)\frac{t^n}{[n]_q!}+\cdots\right]$$
$$=\sum_{n=0}^{\infty}E_{n,q}\{B_{1,q}(x)\}\frac{t^n}{[n]_q!}. \tag{2.2}$$

Summing up the series in l.h.s. and then using Eq. (1.8) and denoting the resultant qBEP in the r.h.s. by ${}_BE_{n,q}(x)=E_{n,q}\{B_{1,q}(x)\}=E_{n,q}\left\{x-\frac{1}{1+q}\right\}$, we are led to assertion (2.1).

Remark 2.1 We have derived the generating function (2.1) for the qBEP ${}_BE_{n,q}(x)$ by replacing the powers of x by the polynomials $B_{n,q}(x)$ (n = 0,1,...) in generating function (1.9) of the q-Euler polynomials $E_{n,q}(x)$. If we replace the powers of x by the polynomials $E_{n,q}(x)$ (n = 0,1,...) in generating function (1.8) of the q-Bernoulli polynomials $B_{n,q}(x)$, we get the same generating function. Thus, if we denote the resultant q-Euler–Bernoulli polynomials (qEBP) by ${}_EB_{n,q}(x)$, we have

$${}_BE_{n,q}(x)\equiv{}_EB_{n,q}(x). \tag{2.3}$$

Theorem 2.2 *The qBEP* ${}_BE_{n,q}(x)$ *are defined by the following series*:

$${}_BE_{n,q}(x)=\sum_{k=0}^{n}\binom{n}{k}_q E_{k,q}B_{n-k,q}(x). \tag{2.4}$$

Proof Using Eqs. (1.8) and (1.11) in the l.h.s. of generating function (2.1) and then using Cauchy's product rule in the l.h.s. of resultant equation, we find

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_q E_{k,q}B_{n-k,q}(x)\frac{t^n}{[n]_q!}=\sum_{n=0}^{\infty}{}_BE_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{2.5}$$

Equating the coefficients of same powers of t in both sides of Eq. (2.5), we are led to assertion (2.4).

Next, we derive the determinantal definition for the qBEP ${}_BE_{n,q}(x)$. For this, we prove the following result:

Theorem 2.3 *The qBEP* ${}_BE_{n,q}(x)$ *of degree n are defined by*

$$ {}_BE_{0,q}(x) = 1, \tag{2.6}$$

$$ {}_BE_{n,q}(x) = (-1)^n \begin{vmatrix} 1 & B_{1,q}(x) & B_{1,q}(x) & \cdots & B_{n-1,q}(x) & B_{n,q}(x) \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1}_q & \cdots & \frac{1}{2}\binom{n-1}{1}_q & \frac{1}{2}\binom{n}{1}_q \\ 0 & 0 & 1 & \cdots & \frac{1}{2}\binom{n-1}{2}_q & \frac{1}{2}\binom{n}{2}_q \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{n}{n-1}_q \end{vmatrix}, \quad n = 1, 2, \cdots, \tag{2.7}$$

where $B_{n,q}(x)$ (n = 0,1,2,…) are the q-Bernoulli polynomials.

Proof We recall the following determinantal definition of the q-Euler polynomials $E_{n,q}(x)$ [10]:

$$E_{0,q}(x) = 1, \tag{2.8}$$

$$E_{n,q}(x) = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1}_q & \cdots & \frac{1}{2}\binom{n-1}{1}_q & \frac{1}{2}\binom{n}{1}_q \\ 0 & 0 & 1 & \cdots & \frac{1}{2}\binom{n-1}{2}_q & \frac{1}{2}\binom{n}{2}_q \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{n}{n-1}_q \end{vmatrix}, \quad n = 1, 2, \cdots. \tag{2.9}$$

Replacing the powers of x, i.e., $x^0, x^1, x^2, \ldots, x^n$ by the corresponding polynomials $B_{0,q}(x), B_{1,q}(x), \ldots, B_{n,q}(x)$ in both sides of Eqs. (2.8) and (2.9) and then using equation ${}_BE_{n,q}(x) = E_{n,q}, \{B_{1,q}(x)\}$ in l.h.s. of resultant equations for n = 0,1,…, we are led to assertions (2.6) and (2.7).

In the next section, we consider the numbers related to the q-Bernoulli–Euler polynomials.

3 Concluding Remarks

We consider the numbers related to the q-Bernoulli–Euler polynomials ${}_BE_{n,q}(x)$. Taking $x = 0$ in both sides of series definition (2.4) of the q-Bernoulli–Euler polynomials ${}_BE_{n,q}(x)$ and then using Eq. (1.12) in the r.h.s. and notation ${}_BE_{n,q} := {}_BE_{n,q}(0)$ in the l.h.s. of the resultant equation, we find the q-Bernoulli–Euler numbers denoted by ${}_BE_{n,q}$ are defined as:

$$ {}_BE_{n,q} = \sum_{k=0}^{n} \binom{n}{k}_q E_{k,q} B_{n-k,q}. \tag{3.1}$$

Next, we find the determinantal definition of the q-Bernoulli–Euler numbers ${}_BE_{n,q}$.

Taking $x = 0$ in both sides of Eqs. (2.6) and (2.7) and then using Eq. (1.12) in the r.h.s. and notation ${}_BE_{n,q} := {}_BE_{n,q}(0)$ in the l.h.s. of the resultant equations, we find that the q-Bernoulli–Euler numbers ${}_BE_{n,q}$ are defined by the following determinantal definition:

$$ {}_BE_{0,q} = 1, \tag{3.2}$$

$$ {}_BE_{0,q} = (-1)^n \begin{vmatrix} 1 & B_{1,q} & B_{2,q} & \cdots & B_{n-1,q} & B_{n,q} \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1}_q & \cdots & \frac{1}{2}\binom{n-1}{1}_q & \frac{1}{2}\binom{n}{1}_q \\ 0 & 0 & 1 & \cdots & \frac{1}{2}\binom{n-1}{2}_q & \frac{1}{2}\binom{n}{2}_q \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{n}{n-1}_q \end{vmatrix}, \tag{3.3}$$

$n = 1, 2, \cdots,$

where $B_{n,q}$ ($n = 0,1,2,\ldots$) are the q-Bernoulli numbers.

Further, we proceed to draw the graph of ${}_BE_{n,q}(x)$. To draw the graphs of these polynomials, we consider the values of the first four $B_{n,q}$, $E_{n,q}$ [1], $B_{n,q}(x)$, and $E_{n,q}(x)$ [10]. We list the first four $B_{n,q}$, $E_{n,q}$ in Table 1 and first four $B_{n,q}(x)$ and $E_{n,q}(x)$ in Table 2.

Table 1 First four $B_{n,q}$ and $E_{n,q}$

n	0	1	2	3
Bn,q	1	$-(1+q)^{-1}$	$q^2([3]_q!)^{-1}$	$(1-q)q^3([2]q)^{-1}([4]_q)^{-1}$
En,q	1	$-\frac{1}{2}$	$\frac{1}{4}(-1+q)$	$\frac{1}{8}(-1+2q+2q^2-q^3)$

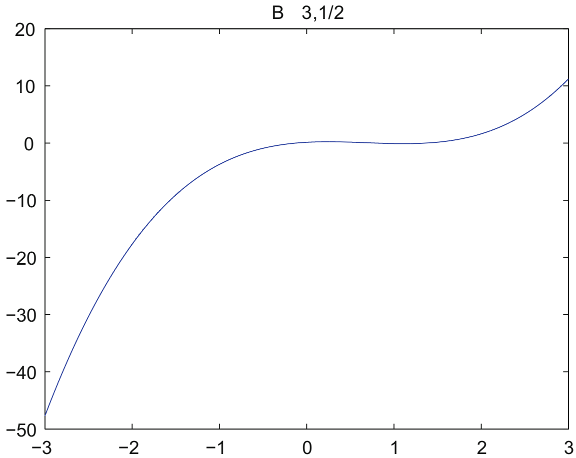
Table 2 First four $B_{n,q}(x)$ and $E_{n,q}(x)$

n	0	1	2	3
$B_{n,q}(x)$	1	$x-\frac{1}{1+q}$	$x^2-\frac{[2]_q}{1+q}x+\frac{q^2}{[3]_q[2]_q}$	$x^3-\frac{[3]_q x^2}{1+q}+\frac{q^2 x}{[2]_q}+\frac{(1-q)q^3}{[2]_q[4]_q}$
$E_{n,q}(x)$	1	$x-\frac{1}{2}$	$x^2-\frac{[2]_q}{2}x+\frac{1}{4}(-1+q)$	$x^3-\frac{[3]_q}{2}x^2+\frac{[3]_q}{4}(-1+q)x+\frac{1}{8}(-1+2q+2q^2-q^3)$

Finally, we consider the values of ${}_BE_{n,q}(x)$ for $n = 3$ and $q = 1/2$. Therefore, taking $n = 3$ and $q = 1/2$ in series definition (2.4) and then using the expressions of first four $E_{n,q}$ and $B_{n,q}(x)$ in the resultant equation and then simplifying, we find

$$ {}_BE_{3,1/2}(x) = x^3 - \frac{49}{24}x^3 + \frac{79}{96}x + \frac{379}{2880}. \tag{3.4}$$

In view of Eq. (3.4), we get the following graph:



In view of relation (2.3), we remark that the results for the qEBP ${}_EB_{n,q}(x)$ will be same as the results established for the qBEP ${}_BE_{n,q}(x)$.

References

1. Ernst, T.: q-Bernoulli and q-Euler polynomials, an umbral approach. Int. J. Diff. Equ. **1**(1), 31–80 (2006). 0973-6069
2. Ernst, T.: Examples of q-orthogonal polynomials. U.U.M.D. Report 2008:24
3. Ernst, T.: A Comprehensive Treatment of q-Calculus, pp. xvi + 491. Birkhäuser/Springer Basel AG, Basel (2012)
4. Loureiro, A.F., Maroni, P.: Around q-Appell polynomials sequences. Ramanujan J. **26**(3), 311–321 (2011)
5. Andrews, G.E., Askey, R., Roy, R.: 71th Special Functions of Encyclopedia of Mathematics and Its Applications. Cambridge University Press, Cambridge (1999)
6. Al-Salam, W.A.: q-Appell polynomials. Ann. Mat. Pura Appl. **4**(17), 31–45 (1967)
7. Al-Salam, W.A.: q-Bernoulli numbers and polynomials. Math. Nachr. **17**, 239–260 (1959)
8. Mahmudov, N.I.: On a class of q-Bernoulli and q-Euler polynomials. Adv. Differ. Equ. **108**, 11 (2013)
9. Keleshteri, M.E., Mahmudov, N.I.: A study on q-Appell polynomials from determinantal point of view. Appl. Math. Comput. **260**, 351–369 (2015)
10. Khan, S., Riyasat, M.: 2-iterated q-Appell polynomials from determinantal point of view (preprint)