Systematic Analysis of the Concept of Equivalent Linear Behavior in Seismic Engineering

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Abstract. In seismic engineering, taking into account the non-linear behavior of the structure into the calculation of its responses against strong ground motions, which are due to plastic and/or damage, is not easy because of the need, in terms of computation time and memory, due to many iterations required at each step in order to satisfy the equilibrium. It is common that many concepts of equivalent linear behavior have been used in order to determine the maximal response of structure without performing non-linear transient calculation. In this paper, we deal with systematic and argumentative analysis in order to establish a concept of equivalent linearization by considering the equivalence criterion through the transfer function from the time domain to the frequency domain. Its idea is to identify the frequency and damping of the equivalent linear oscillator whose theoretical transfer function of response in acceleration fits the best the experimental one of the nonlinear system. This concept will be applied to elastoplastic Sdof oscillators undergoing the filtered white noise signal Clough-Penzien. As the result, this equivalent linearization reestablishes the transferred signal through a structure with the non-linearity.

1 Introduction

While undergoing strong ground motions, structures may exhibit nonlinear behavior, which is characterized by hysteretic strain-stress loops during the cyclic load. It is important to note that the stiffness of the structure seems to be reduced and its damping increased versus plastic deformation (Iwan 1980; EPRI 1994; Ravikiran 2015). In cases of strong earthquakes, this leads seismic engineers to perform nonlinear transient calculation that results in the huge timing and memory consummation and raise the convergence problem.

In the geotechnical field, the concept of equivalent linearization has been introduced by (Seed and Idriss 1970), using the curves of modulus degradation and damping ratio increasing versus soil shear strain. It enables to cope the response within non-linear behavior without performing nonlinear transient analyses.

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In civil engineering, (Iwan 1980) had established the curve of stiffness drift ratio, so called the frequency ratio, and damping ratio increasing as functions of ductile demand increasing for a series of single degree of freedom (Sdof) consisting of a combination of linear and elastoplastic or Coulomb slip type elements (Iwan 1979). According to his concept of equivalent linearization, he used the maximum of the time history response in displacement of the mass as the equivalence's criterion. That leads to establish the equivalent linear system, the equivalent frequency and damping ratio associated to one value of ductile demand, which gives the same maximum of time history response in displacement as the nonlinear one. This concept involves some problems, which are not discussed in this article.

Then, Chopra and Goel (Chopra and Goel 1999; Chopra 2001) have defined the equivalent linear behavior of an elastoplastic oscillator by using the secant stiffness, which is linked to the slope of the straight connecting the origin and the maximal deformation of the deformation-force curve during the earthquake, and the equivalent damping ratio based on the dissipated energy within the hysteretic loop corresponding to the maximal deformation. This concept has two disadvantages, at least, at the author's point of view:

1. The equivalent stiffness is equal to the secant one but the dissipated energy is performed with the unloading stiffness equal to the elastic one. Therefore, there is an inconsistency in term of model of equivalent linearization.
2. The model bases completely on the loop of the maximum of the deformation time history but there are many loops of smaller deformation appearing before this maximum.

In this article, we will deal with the equivalent linearization through the complex transfer function. The complete procedure will be described at the next sections of this article. As the result, we establish the frequency drift ratio and damping ratio increasing as functions of ductile demand increasing. Then, the pertinence of the concept of equivalent linear behavior is demonstrated by using Andersons's criteria (Anderson 2004).

2 Clough and Penzien Filtered White Noise

Thanks to the simple usage, the filtered white noise is often fabricated in seismic engineering as strong ground motions. In this article, the Clough & Penzien type with a low-pass filter is used with the Kanaï-Tajimi parameters of $f_k = 2.5$ Hz and $\xi_k = 50\%$, the Clough-Penzien parameters of $f_c = 0.125$ Hz and $\xi_c = 100\%$ and low-pass filter parameter of $f_{low} = 10$ Hz and $\xi_{low} = 100\%$. Then, the α-type envelope's curve is applied in order to avoid extreme values at the beginning of filtered signal. The central frequency is about 3.3 Hz and the range of frequencies is of 0–50 Hz. The signal lasts 20 s and the time steps is of 10 ms. Then, 1000 signals of the said random process are fabricated for this study.

3 Response of Elastoplastic Oscillator Using Beta-Newmark Scheme

The elastoplastic oscillator studied in this article consists of a combination of elastic springs ($\sigma = E^*\varepsilon$), Newtonian damper ($\sigma = C * \dot{\varepsilon}$) and Saint-Venant sliding element (σ_y). In fact, only the sdof is considered in this article so the stress σ and strain ε are converted into the force f and the displacement u. Modulus E_1 and E_1 will be replaced by k_1 and k_2, which are the stiffness of the elastic springs. Yielding stress σ_y is translated by the yielding force f_y beyond which the Saint-Venant sliding element slides without any resistance. E_T is the hardening modulus so replaced by hardening stiffness k_p.

The dynamic equation of the motion of the mass of this elastoplastic oscillator is written as following:

$$\ddot{u} + 2\xi_0\omega_0\dot{u} + f(u)/m = -\ddot{u}_g \tag{1}$$

where:

u : Relative displacement of the mass m;

m : Mass of the elastoplastic oscillator;

ω_0: Angular velocity corresponding to the elastic frequency of the oscillator $\omega_0 = 2\pi f_0$;

ξ_0 : Damping ratio corresponding to the angular velocity and the Newtonian viscosity $\xi_0 = \frac{C}{2m\omega_0}$;

u_g : Ground motion displacement applied to the elastoplastic oscillator

By using the β-newmark ($\beta = ¼$ and $\gamma = ½$), we can solve the response of elastoplastic which raise the problem of timing and memory consummation and convergence.

4 Equivalent Linear Oscillator's Parameters Identification by Minimization in Frequency Domain

As mentioned at the introduction that the frequency shift and damping ratio increasing are observed by many authors (Ravikiran 2015; Labbé 2012; EPRI 1994)… as function of increasing plasticity or damage in the structure. In this paper, we execute an experimental plan in order to establish the relationship between the equivalent linear oscillator's parameters versus plasticity parameter. The damage phenomenon is not considered in this article then the ductile demand of structure is chosen as plasticity parameter. It is important to note that the ductile demand is directly and only linked to the maximal value of time history response in displacement. Nevertheless, in some seismic standards (EURO 2005), the non-linear behavior of structure is always based on the value of ductile capacity.

According to the initial parameters of the elastoplastic oscillators, some studies have showed that maximum of time history responses in displacement of elastoplastic structure depend on the relative position of initial frequency f_0 in respect to the central frequency of strong ground motion fc. Therefore, in this study, we examine many oscillators whose initial frequency f_0 goes from 0.1 * fc to 10 * fc, which correspond successively to the "very low" frequency and the high cutting frequency of filtered white noise Clough-Penzien and damping ratio ξ_0 is arbitrarily fixed at 5%.

In addition, we analyze also the influence of the hardening on the equivalent linearization. Therefore, an hardening measurement is defined by the ratio of hardening stiffness against the initial stiffness, so denoted α_p. For numerical application, α_p takes values of 0%, 10%, 20%. By definition, the case of α_p equal to 0 corresponds to the elastic perfectly plastic behavior. The dependence of the ductile demand (μ) on the level of signal (λ) is shown in the Fig. 1.

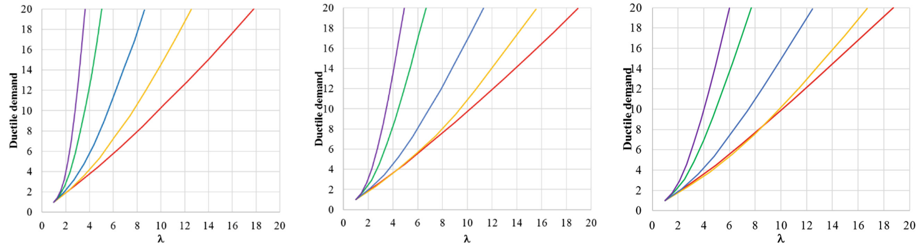


Fig. 1. (a): For $\alpha_p = 0\%$ (b): For $\alpha_p = 10\%$ (c): For $\alpha_p = 20\%$

According to the level of ground motions, for each $\{f_0$ and $\xi_0\}$, we determine the yielding displacement $X_{e,j}$ so that a filtered white noise signal of level 1, denoted by $g_{1,j}(t)$, leads the associated elastoplastic of $\{f_0, \xi_0$ and $X_{e,j}\}$ just to its yielding point. Then, we increase the level of filtered white noise by the factor λ ($\lambda\text{max} \geq \lambda > 1$) applied to the amplitude of the ground motion of level 1, then denoted by $g_{\lambda,j}(t) = *g_{1,j}(t)$. The maximum value of λ corresponds to the case of median value of ductile demand equal to 20. For each level of ground motion $g_{\lambda,j}(t)$, we perform nonlinear transient calculation by using the β-newmark scheme in order to determine: (1) the ductile demand and (2) the experimental transfer function. The ductile demand is equal to ratio of maximal displacement and yielding displacement and the experimental transfer function is the frequency-domain content of output and input acceleration.

The identification process of equivalent linear oscillator is described in the following figure (Fig. 2).

Where:

$g_{1,j}(t)$:	Ground motion of filtered white noise Clough-Penzien type with $j = 1\ldots1000$;
$X_{0,j}(t)$:	Time history response in displacement of elastic oscillator of $\{f_0$ and $\xi_0\}$ undergoing the $g_{1,j}(t)$;
$X_{e,j} = \max\lvert X_{0,j}(t)\rvert$:	Maximal displacement of elastic oscillator of $\{f_0$ and $\xi_0\}$ undergoing $g_{1,j}(t)$ the in m/s/s;

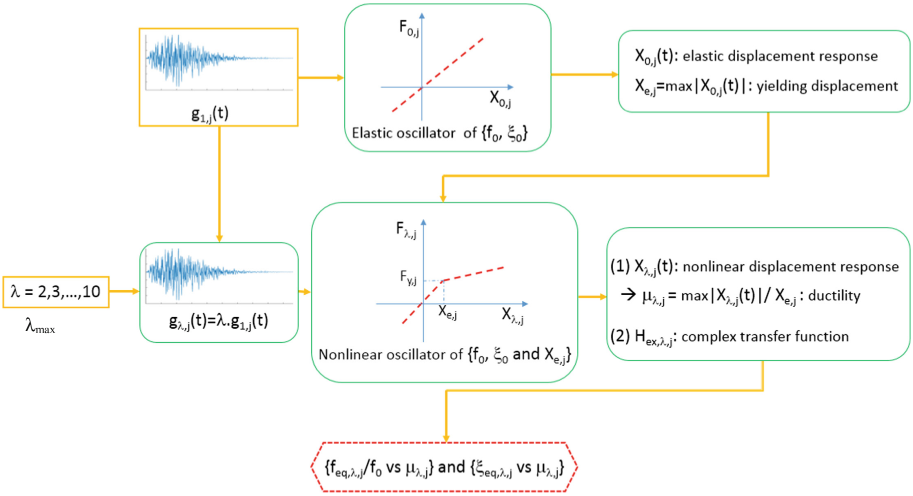


Fig. 2. Experimental plan

$\lambda = 2\ldots\lambda_{max}$:	10 values of amplification factor of the amplitude of ground motions. At this stage, it is certain to say that the $g_{1,j}(t)$ leads the elastoplastic oscillator of $\{f_0$ and $\xi_0\}$ and $X_{e,j} = \max|X_{0,j}(t)|$ just to the yielding point, no plastic deformation is observed. Then, $g_{\lambda,j}(t) = \lambda * g_{1,j}(t)$ makes elastoplastic oscillator to exhibit plastic deformation;
$u_{\lambda,j}(t)$:	Displacement time history of elastoplastic oscillator of $\{f_0; \xi_0$ and $X_{e,j}\}$ undergoing the $g_{\lambda,j}(t) = \lambda * g_{1,j}(t)$, performed by using the β-Newmark scheme. Then, we define the ductile demand as: $\mu\lambda_{,j} = \max|u_{\lambda,j}(t)|/X_{e,j}$;
$H_{ex,\lambda,j}$:	Experimental complex transfer function defined as $H_{ex,\lambda,j} = \mathrm{fft}(a_{\lambda,j})/\mathrm{fft}(g_{\lambda,j})$ where $a_{\lambda,j}$ is the output acceleration and fft(.) assigns the Fast Fourier Transform;
$\{f_{eq,\lambda,j}$ and $\xi_{eq,\lambda,j}\}$:	Equivalent linear frequency and damping ratio defined by the procedure of equivalent linearization by minimization in the frequency domain (Fig. 3).

In fact, in the frequency domain, we determine the equivalent linear oscillator of $\{f_{eq}$ and $\xi_{eq}\}$ by minimization in the frequency domain through the complex transfer function. This procedure is completely done by using the nonlinear regression by least square solution. More precisely, for every single of $H_{ex,\lambda,j}$ with $j = 1\ldots1000$ and $\lambda = 2\ldots\lambda_{max}$, we determine the couple of $\{f_{eq,\lambda,j}$ and $\xi_{eq,\lambda,j}\}$ that minimize "the difference" between the experimental complex transfer function $H_{ex,\lambda,j}$ and theoretical one $H_{th,\lambda,j}$ of the elastic equivalent oscillator of $\{f_{eq,\lambda,j}$ and $\xi_{eq,\lambda,j}\}$. As numerically saying, the $\{f_{eq,\lambda,j}$ and $\xi_{eq,\lambda,j}\}$ are determined by minimizing this term:

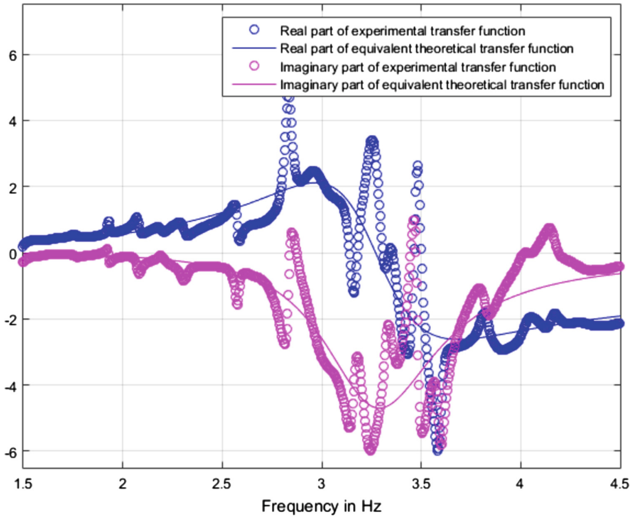


Fig. 3. Example of complex transfer function regression

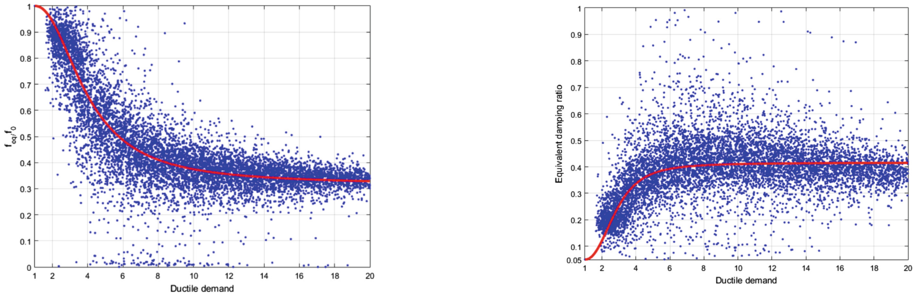


Fig. 4. (a): Scatter plot of f_{eq}/f_0 (b): Scatter plot for ξ_{eq}

$$J\left(f_{eq,\lambda,j};\xi_{eq,\lambda,j}\right)=\sum_{f_{\min}}^{f_{\max}}\left(\mathrm{Re}\left(H_{ex,\lambda,j}(f_i)\right)-\mathrm{Re}\left(H_{t,\{\mathrm{f}_{\mathrm{eq},\lambda,j},\xi_{\mathrm{eq},\lambda,j}\}}(f_i)\right)\right)^2+\left(\mathrm{Im}\left(H_{ex,\lambda,j}(f_i)\right)\right.$$
$$\left.-\mathrm{Im}\left(H_{t,\{\mathrm{f}_{\mathrm{eq},\lambda,j},\xi_{\mathrm{eq},\lambda,j}\}}(f_i)\right)\right)^2 \tag{3}$$

where Re(.) and Im(.) assign the real and imaginary part of complex number.

For the best frequency resolution, the zeros-padding method is applied. The determination of $\{f_{eq,\lambda,j}$ and $\xi_{eq,\lambda,j}\}$ is reached by using the operator *lsqcurvefit* in matlab.

For example (**Ref**. Fig. 4), we present the result of $\{f_{eq,\lambda,j}$ and $\xi_{eq,\lambda,j}\}$ determined for $\lambda = 2$ and $j = 1$. The $X_{e,1}$ is determined by the linear analysis of the elastic oscillator of $\{f_0 = 3.35$ Hz and $\xi_0 = 5\%\}$ and $\alpha_p = 10\%$ equal to 0.021 m. The equivalent oscillator is characterized by $\{f_{eq,2,1}$ and $\xi_{eq,2,1}\} = \{3.27$ Hz, $13.6\%\}$ and the ductile demand equal to 1.87.

By applying this procedure to each value of $\lambda = 2\ldots\lambda_{max}$, we obtain 10 000 couples of $\{f_{eq,\lambda,j}$ and $\xi_{eq,\lambda,j}\}$ and 10 000 values of ductile demand $\mu_{\lambda,j}$. Then, we have to process the treatment of two scatter plots of $\{f_{eq,\lambda,j}$ VS $\mu_{\lambda,j}\}$ and $\{\xi_{eq,\lambda,j}$ VS $\mu_{\lambda,j}\}$ in order to get the formula of experimental curves. Here you can find an example of scatter plots of $\{f_{eq,\lambda,j}$ VS $\mu_{\lambda,j}\}$ and $\{\xi_{eq,\lambda,j}$ VS $\mu_{\lambda,j}\}$ for the oscillator of f_0/f_c equal to 0.1 with 10% kinematic hardening in which the blue dots are 10 000 couples and red line is the regression using the regression.

It is important to note from these scatter plots that:

+ the ratio of equivalent frequency and initial frequency tends to the value of $\sqrt{\alpha}$. In this case, the elastoplastic is considered to be always in plastic regime.
+ the damping ratio seems to be inverse-proportional to the frequency

At this stage, we can propose experimental curves as following:

$$\begin{aligned} \frac{f_{eq}}{f_0} &= \sqrt{\alpha_p} + \frac{1-\sqrt{\alpha_p}}{1+\frac{(\mu-1)^d}{b}} \\ \xi_{eq} &= \xi_0 + a\left(1-\left(\frac{f_{eq}}{f_0}\right)^c\right) \end{aligned} \tag{4}$$

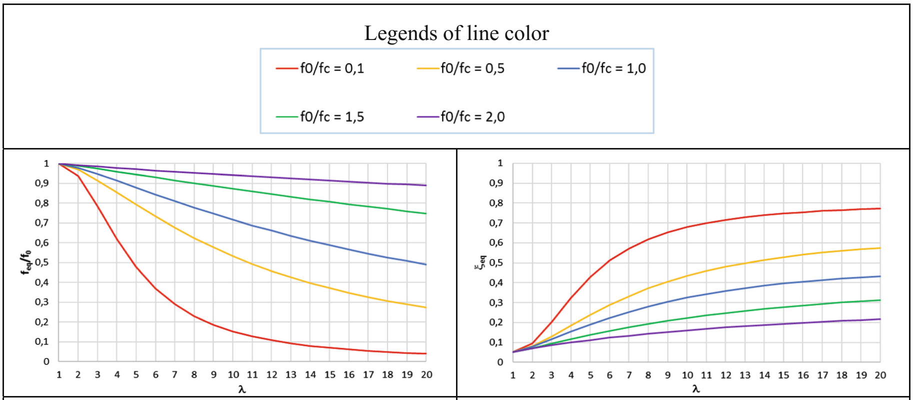
The treatment of theses scatter plots consists in adjusting these parameters {b;d} for the scatter plot of $\{f_{eq,\lambda,j}$ VS $\mu_{\lambda,j}\}$ and {a,c} for the $\{\xi_{eq,\lambda,j}$ VS $\mu_{\lambda,j}\}$.

5 Result of Equivalent Linearization

By following the procedure described above, we apply the experimental plan for each of 5 values of f_0/fc ($f_0/fc = 0.1$; 0.5; 1.0; 1.5; 2.0) and 3 values of α_p ($\alpha_p = 0.0$; 0.1; 0.2). The output in term of curves $\{f_{eq}$ VS $\mu\}$ and $\{\xi_{eq,}$VS $\mu\}$ is presented in Fig. 5.

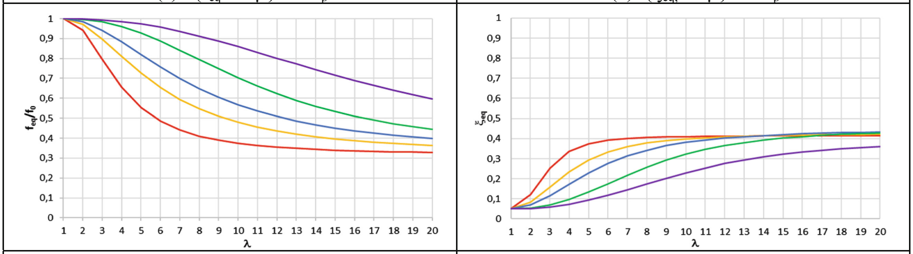
6 Comments on the Equivalent Linearization

In respect to the central frequency of input signal, the more the initial frequency is important (f_0/fc important), the more the equivalent frequency is close to the initial one (f_{eq}/f_0 approaches to 1). That is to say, when the structure is stiff (f_0/fc important), excursions in plastic domain are very violent and cause the important values of ductile demand. This strong plastic excursion is not “captured” by this concept because this excursion is of very low frequencies content, which is less than the frequency discretization for the Fast Fourier Transform. This very strong excursion can be explained by the fact that during the excursion in plastic domain, the duration of excursion is smaller than the duration of changing the sign of input signal, then, no changing of sign gets done during the plastic excursion. At the contrary for oscillators of low



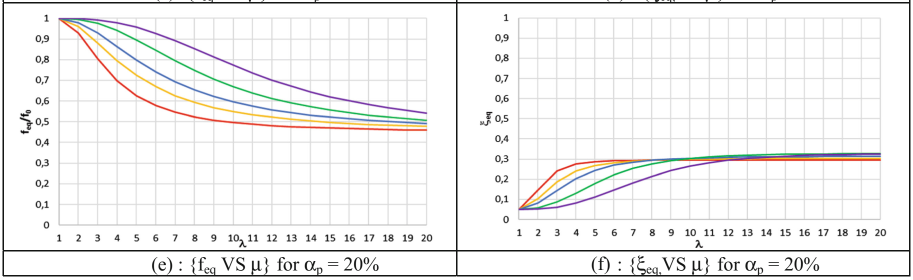
(a) : {f_{eq} VS μ} for $\alpha_p = 0\%$

(b): {$\xi_{eq,}$VS μ} for $\alpha_p = 0\%$



(c): {f_{eq} VS μ} for $\alpha_p = 10\%$

(d) : {$\xi_{eq,}$VS μ} for $\alpha_p = 10\%$



(e) : {f_{eq} VS μ} for $\alpha_p = 20\%$

(f) : {$\xi_{eq,}$VS μ} for $\alpha_p = 20\%$

Fig. 5. (a): {f_{eq} VS μ} for $\alpha_p = 0\%$ (b): {$\xi_{eq,}$VS μ} for $\alpha_p = 0\%$ (c): {f_{eq} VS μ} for $\alpha_p = 10\%$ (d): {$\xi_{eq,}$VS μ} for $\alpha_p = 10\%$ (e): {f_{eq} VS μ} for $\alpha_p = 20\%$ (f): {$\xi_{eq,}$VS μ} for $\alpha_p = 20\%$

frequencies, during the excursion, the sign changes many times so that the plastic excursion depend only on the velocity at the moment of excursion.

While λ approaches infinity, the oscillator is logically plastic so its equivalent oscillator should take the frequency associated to the hardening. Then, the ratio of f_{eq}/f_0 approaches $\sqrt{\alpha_p}$. In addition, the feq reaches the asymptotic value rapidly for the oscillator of small f_0/f_c and the contrary for oscillators of ig f_0/fc.

7 Pertinence of Equivalent Linear Oscillator Through Anderson's Criteria

Initially developed by Anderson (Anderson 2004) in the interest of quantitative measurement of the similarity between the artificial signal and real earthquake signal recorded in site, the ten criteria are essentially the comparison of duration, peaks and frequency content of time history signal. According to the duration, the first and second criteria evaluate successively the energy duration and Arias duration, then, the third and fourth one indicate the difference of total energy and total Arias of signal. In regard of peaks, the fifth, sixth and seventh criterion can compare the peak in accelerogram, in velocity and in displacement. These velocity and displacement are obtained by using the single and double integral of accelerogram. The last three criteria are based on the frequency content of signal, successively through the response spectrum, Fourier transform and cross-correlation. For each criteria, the similarity is noted from zero to ten (0 to 10). A score below 4 is a poor fit, a score of 4–6 is a fair fit, a score of 6 to 8 is a good fit, and a score over 8 is an excellent fit.

Using these criteria, we can evaluate quantitatively the similarity between the output acceleration of non-linear oscillator and the one of equivalent linear oscillator. In order to clarify the pertinence of equivalent linear oscillator, we show also the similarity between the output acceleration of non-linear oscillator and the linear oscillator with initial parameters $\{f_0$ and $\xi_0\}$. In this article, we can show an example for a simple case of $f_0/fc = 1.0$ for elastoplastic behavior with 10% of hardening ($\alpha_p = 10\%$) (Ref. Fig. 6).

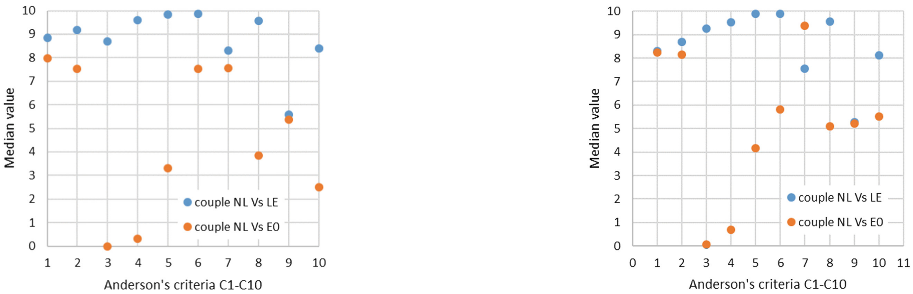


Fig. 6. (a): Example for $\lambda = 8$ and $\mu = 12.2$ (b): Example for $\lambda = 4$ and $\mu = 4.46$

As we established the curve of $\{f_{eq}/f_0$ VS $\mu)$ and (ξ_{eq} VS μ), we can expect that the score for eighth, ninth and tenth criteria are good, and that's what we observe from the Fig. 6. The energy duration and Arias duration are quietly well the same for both couple of non-linear output acceleration VS Equivalent linear output acceleration and the one of non-linear acceleration VS linear output acceleration within initial parameters. We obtain a very good for the peak of acceleration and velocity's criteria. Nevertheless, the peak in displacement is not correlated between output acceleration of nonlinear and equivalent linear system.

8 Conclusions

This study deals with the equivalent linearization for elastoplastic systems through the systematic analysis of a series of oscillators using the complex transfer function as the equivalent criterion.

The frequency ratio shifts and equivalent damping for both the numeric and experimental results are not as important as the ones corresponding to secant stiffness which is widely used in practice.

The Anderson's criteria show that the output acceleration of equivalent linear oscillator conserve well the energy and Arias duration, also the total energy of output acceleration of non-linear oscillator as well as the content in frequency domain. In addition, the peak in acceleration, velocity are well similar in respect to the value of non-linear oscillator. But, the peak in displacement of equivalent linear oscillator is different in respect to the peak of displacement of non-linear oscillator because this concept of equivalent linear behavior doesn't consider the "strong" plastic excursion.

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