Singularity-Free Path Following Control for Miniature Unmanned Helicopters

Xujie Ma and Wei Huo

Abstract A singularity-free nonlinear controller is presented for the miniature unmanned helicopter to follow a reference path described by implicit expressions. Based on the time-scale separation principle, the controller is designed with hierarchical inner-outer loop structure. The outer-loop position controller is constructed with hyperbolic tangent function, and temporary-path generation method is developed to keep the control matrix invertible and obviate large control energy. The desired command attitude can be derived from position controller without singularity by choosing appropriate controller parameters. The inner-loop attitude controller is designed with the unit-quaternion attitude representation and backstepping technique to achieve attitude tracking. Numerical simulation is provided to verify the theoretical results.

Keywords Miniature unmanned helicopter ⋅ Path following ⋅ Singularity-free control ⋅ Quaternion

1 Introduction

Tracking tasks for aircrafts can be classified as two categories [\[1\]](#page-14-0): trajectory tracking and path following. In the first case, the aircraft is required to track a time-varying reference trajectory at every transient. While in the second case it is required to fly along a reference path at a desired speed. Unlike trajectory tracking, there is no temporal requirements on path following. Besides the path following has some control performances that can not be obtained from trajectory tracking in some specific cases [\[2](#page-14-1)].

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Various linear and nonlinear controllers have been proposed for path following control of miniature helicopters. The linear control methods are simple and reliable, such as PID $\lceil 3 \rceil$ and LO control $\lceil 4, 5 \rceil$ $\lceil 4, 5 \rceil$ $\lceil 4, 5 \rceil$, but have the limitation to realize full envelope flight. To this end, some nonlinear control methods such as backstepping approach [\[6\]](#page-14-5), feedback linearization technique [\[7](#page-14-6)] or hybrid control methods [\[8](#page-14-7), [9](#page-14-8)] have been applied. Due to the under-actuated property, the helicopter model is always simplified to position outer-loop and attitude inner-loop structure [\[10,](#page-14-9) [11](#page-14-10)]. However, the controller designed based on the hierarchical structure may suffer from singularity when deriving the desired attitude from position controller. So far, only a few literatures [\[6\]](#page-14-5) take into account this problem, and most reference paths are parameterized curves.

The parameterized path following [\[6,](#page-14-5) [8,](#page-14-7) [9](#page-14-8)] is the most commonly used problem formulation. The path is described with a time-varying parameter, and the task is to design control law and parameter timing law such that helicopter can keep up with the moving point determined by the parameter. Another problem formulation is based on implicit expressions $\begin{bmatrix} 1, 7 \end{bmatrix}$. The path is given by the intersection of two manifolds. Unlike parameterized path following which turns path following problem to point-tracking problem, the task for implicit path following is to follow the entire path and the helicopter will enter an invariant set around the reference path. However, controller design for implicit path following always relates to control matrix of the closed-loop system, and it suffers from singularity when the matrix is not invertible.

In this paper a singularity-free implicit path following controller for miniature helicopters is presented. The control design is based on the hierarchical structure. The outer-loop position controller is constructed with the hyperbolic tangent function to realize path following, and a temporary path is planned to guarantee the control matrix invertible. From the position controller, the desired command attitude can be derived without singularity by choosing appropriate controller parameters. The inner-loop attitude controller is designed with the unit-quaternion representation to realize tracking for the command attitude.

2 Preliminaries

In following sections, *c*(⋅) and *s*(⋅) are shorts of cos(⋅) and sin(⋅). | ⋅ |denotes absolute value of a real number, $\|\cdot\|$ denotes Euclidean norm for a vector or induced Euclid-Figure 1.00 and the command attitude.
 Preliminaries
 In following sections, $c(\cdot)$ **and** $s(\cdot)$ **are shorts of cos(·) and sin(·). | · |denotes absolute

value of a real number,** $\|\cdot\|$ **denotes Euclidean norm for a vector 2** Preliminaries

In following sections, $c(\cdot)$ and $s(\cdot)$ are show

value of a real number, $\|\cdot\|$ denotes Eucean norm for a matrix. $\bar{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ d

values, respectively. For $\mathbf{x} = [x_1, \dots, x_n]$

vector values, respectively. For $x = [x_1, ..., x_n]^T \in \mathbb{R}^n$, define hyperbolic tangent function **2 Preliminaries**
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ean norm f **2** Preliminaries

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continuously differentiable vector function $f(x) = [f_1(x), ..., f_m(x)]^T$, define $\partial_x f = [\partial_{f_1} \partial_{f_1}]$ ⎡ ⎢ ⎢ $\mathsf I$ *f*1 x_1 ∂x_n continuously differentiable vector function $f(x) = [f_1(x), \dots, f_m(x)]^T$, define $\partial_x f =$ ⋮⋱⋮ *fm x*₁ *m <i>i***₁**</sup> *i*_{*x*₁} *i*_{*x*₁</sup> *i*_{*x*₁} *i*_{*x*₁} *i*_{*x*₁} *i*_{*x*₁</sup> *i*_{*x*₁} *i*_{*x*₁</sup> *i*_{*x*₁} *<i>i*_{*x*₁</sup> *d*_{*x*_{*x*₁} *i d*_{*x*_{*x*₁} *i d*_{*x*_{*x*</sup>}}}}}}}} ⎤ ⎥ ⎥ $\overline{}$ e Path Following Cordifferentiable vectors and $\partial_x^2 f = \partial_x(\partial_x f)$. Singularity-Free Path Following Control for Miniature Unmanned Helicopters

continuously differentiable vector function $f(x) = [f_1(x), ..., f_m(x)]^T$, define $\partial_x f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & ... & \$ Singularity-Free Path Following Control for Mini

continuously differentiable vector functio
 $\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$ and $\partial_x^2 f = \partial_x(\partial_x f)$.

Lemma 1 ([12] **Lemma 1** ([12]) *For* $x \in \mathbb{R}^n$, *if* $||x|| < \bar{x} < \infty$, *fm*(*x*)]^{*T*}, define $\partial_x f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$ and $\partial_x^2 f = \partial_x(\partial_x f)$.
 Lemma 1 ([12]) $\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$ and $\partial_x^2 f = \partial_x(\partial_x)$
 Lemma 1 ([12]) For $x \in \mathbb{R}$ ²

(0, 1) satisfying $\chi ||x|| \le ||\text{tan}$
 Lemma 2 ([13]) For $x \in \mathbb{R}$

0.2

Lemma 1 ([12]) *For* $x \in \mathbb{R}^n$, if $||x|| < \overline{x} < \infty$, then there exists a constant $\chi(\overline{x}) \in$

(0, 1) *satisfying* $\chi ||x|| \le ||\tanh(x)|| \le ||x||$.
 Lemma 2 ([13]) *For* $x \in \mathbb{R}$ and $\varepsilon > 0$, $0 \le |x| - x\tanh(x/\varepsilon) \le k_q \varepsilon$, w

Example 1 ([12]) For $x \in \mathbb{R}^n$, if $||x|| < \bar{x} < \infty$, then there exists a constant $\chi(\bar{x}) \in$

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(0, 1) *satisfying* $\chi ||x|| \le ||\tanh(x)|| \le ||x||$.
Lemma 2 ([13]) *For* $x \in \mathbb{R}$ and $\varepsilon > 0$, $0 \le |x| - x \tanh(x/\varepsilon) \le k_q$
0.2785 *satisfies* $k_q = e^{-(k$

3 Problem Statement

Mathematical modeling Figure [1](#page-2-0) shows the helicopter configuration, it is modeled in two frames: the earth frame $I = \{Oxyz\}$ and fuselage frame $B = \{O_b, V_b\}$.
 3 Problem Statement

Mathematical modeling Figure 1 shows the helicopter configuration, it is modeled

in two frames: the earth frame $I = \{Oxyz\$ Frame I is fixed to the earth and its origin locates on the ground. Frame B is fixed to the helicopter body and its origin locates at the helicopter center of gravity $(c.g.).$ The rotation matrix $R[1]$ $R[1]$, which defines the rotation from β to $\mathcal I$ around an unit **3 Problem Statement**
Mathematical modeling Figure 1 shows the helicopter configuration, it is modeled
in two frames: the earth frame $\mathcal{I} = \{Oxyz\}$ and fuselage frame $B = \{O_bx_by_bz_b\}$.
Frame \mathcal{I} is fixed to the e **3** Problem Statement

Mathematical modeling Figure 1 shows the helicopter configuration, it is modeled

in two frames: the earth frame $I = \{Oxyz\}$ and fuselage frame $B = \{O_bx_by_bz_b\}$.

Frame *I* is fixed to the earth and **Mathematical modeling** Figure 1 shows the helicopter configuration, it is modeled
in two frames: the earth frame $\mathcal{I} = \{Oxyz\}$ and fuselage frame $B = \{O_bx_by_bz_b\}$.
Frame \mathcal{I} is fixed to the earth and its origin lo satisfy following relation *R*(*Q*) = (*x*) *R*(*x*) = *q*) *R*(*x*) =

$$
R(Q) = (\mu^2 - q^T q)I_3 + 2qq^T + 2\mu S(q)
$$
 (1)

Fig. 1 Helicopter model and frames

where I_3 is 3×3 identity matrix, $S(q)$ = ⎡ ⎢ ⎢ ⎣ 0 −*q*³ *q*² *q*³ 0 −*q*¹ −*q*² *q*¹ 0 $\overline{}$ ⎥ \rfloor . Given Q_1 = \overline{a} 1 *q*] and $Q_2 =$ \overline{a} 2 *q* the quaternion multiplication $Q_1 \otimes Q_2 = \begin{bmatrix} \mu_1 \mu_2 - q_1^T q_2 \\ \mu_2 \mu_3 + \mu_4 q_2 + S(q_1 + q_2) \end{bmatrix}$ x

s 3 × 3 identity matrix, $S(q) = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$. Given Q

diven $Q_1 \otimes Q_2 = \begin{bmatrix} \mu_1 \mu_2 - q_1 \\ \mu_2 \mu_3 + \mu_4 q_4 + \mu_5 \mu_6 \end{bmatrix}$. *X*. Ma and W. Huo

¹^q + 2^{*q*}₁</sub> $\begin{bmatrix} -q_1 \\ q_1 \end{bmatrix}$ and
 $\begin{bmatrix} \mu_1 \mu_2 - q_1^T q_2 \\ \mu_1 q_2 + \mu_2 q_1 + S(q_1) q_2 \end{bmatrix}$. The
 T and satisfies *Q*^{−1} ⊗ *Q* = $\overline{1}$. The 672

where I_3 is 3 × 3 identity matrix, $S(\boldsymbol{q}) = \begin{bmatrix} 0 & -q \\ q_3 & 0 \\ -q_2 & q_1 \end{bmatrix}$
 $Q_2 = \begin{bmatrix} \mu_2 \\ \boldsymbol{q}_2 \end{bmatrix}$, the quaternion multiplication $Q_1 \otimes Q_2 =$

inverse of *Q* is defined as $Q^{-1} = [\mu, -\boldsymbol{q}^T]$

[1, 0 *p*^{*p*}_{*p*}^{*n*}_{*n}* on multiplication $Q_1 \otimes Q_2 = \begin{bmatrix} \mu_1 \mu_2 - q_1^T q_2 \\ \mu_1 q_2 + \mu_2 q_1 + S(q_1) q_2 \end{bmatrix}$. The
efined as $Q^{-1} = [\mu, -q^T]^T$ and satisfies $Q^{-1} \otimes Q =$
n-Euler equations, the dynamic model of the helicopter can
 \vdots
 $\dot{p} = v$, m

Based on the Newton–Euler equations, the dynamic model of the helicopter can be derived as follows [\[1](#page-14-0)] [1, 0, 0, 0]^T.
Based on the N
be derived as follow
where $p = [x, y, z]$
in $\mathcal I$ *m* is the helic **Figure 4** and *v* = *P*,
 $\dot{Q} = \frac{1}{2} \Theta(Q)$
 T and $v = [u, v, w]$

conter mass $g = -1$

$$
\dot{p} = v, \quad m\dot{v} = -mg_3 + R(Q)f \tag{2}
$$

$$
\dot{Q} = \frac{1}{2}\Theta(Q)\omega, \quad J\dot{\omega} = -\omega \times J\omega + \tau \tag{3}
$$

^T are position and velocity of helicopter denoted Based on the Newton-Euler equations,
be derived as follows [1]
 $\dot{p} = v, m\dot{v} = -\frac{\dot{Q}}{2} = \frac{1}{2}\Theta(Q)\omega, J\dot{\alpha}$
where $p = [x, y, z]^T$ and $v = [u, v, w]^T$ are poin *I*, *m* is the helicopter mass, $g_3 = [0, 0, g]$ \overline{a} and **g** is gravitational acceleration, $\omega =$ **(2)
** $**Q** = \frac{1}{2}Θ(Q)ω, *J*ω = -ω × *J*ω + τ$ **(3)

where** $**p** = [x, y, z]^T$ **and** $**v** = [u, v, w]^T$ **are position and velocity of helicopter denoted
** in *I*, *m* is the helicopter mass, $g_3 = [0, 0, g]^T$ and *g* is gravitational acceleration, $\omega =$ matrix, $J =$ $\int_{x}^{x} \int_{0}^{x} -J_{xz}$ 0 *Jy* 0 −*Jxz* 0 *Jz*] is the inertial matrix denoted in B . The applied force f and matrix, $J = \begin{bmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{bmatrix}$ is the inertial
torque τ denoted in \vec{B} are given by [\[1\]](#page-14-0) *,z* e position and velocity of helice $,g]^{T}$ and g is gravitational acce
in B, $\Theta(Q) = \begin{bmatrix} -q^{T} \\ \mu I_{3} + S(q) \end{bmatrix}$ is attitudinatrix denoted in B. The applie
 $T_{m}h_{m}sb_{s} + L_{b}b_{s} + T_{t}h_{t} + \tau_{m}sa_{s}$ are position and velocity of helicopte
 J, 0, g]^{*T*} and *g* is gravitational accelerated in *B*, $\Theta(Q) = \begin{bmatrix} -q^T \\ \mu I_3 + S(q) \end{bmatrix}$ is attitude
 *I*l matrix denoted in *B*. The applied for
 T_mh_msb_s + *L_bb_* T_1^T and g is gravitational according to B , $\Theta(Q) = \begin{bmatrix} -q^T \\ \mu I_3 + S(q) \end{bmatrix}$ is attically trix denoted in B. The applitude trivial $m_1 h_m s b_s + L_b b_s + T_i h_i + \tau_m s$
 $T_m h_m s a_s + M_a a_s + \tau_t - \tau_m I_m s b_s - T_t l_t + \tau_m c a_s c b_s$ matrix, $J = \begin{bmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{bmatrix}$ is the inertial matrix denoted in *B*. The applied force *f* and
torque τ denoted in *B* are given by [1]
 $f = \begin{bmatrix} T_m s a_s c b_s \\ -T_m s b_s c a_s + T_t \\ T_m c a_s c b_s \end{bmatrix}, \tau = \begin{bmatrix} T$

matrix,
$$
J = \begin{bmatrix} 0 & J_y & 0 \\ -J_x & 0 & J_z \end{bmatrix}
$$
 is the inertial matrix denoted in *B*. The applied force *f* and
\ntorque τ denoted in *B* are given by [1]
\n
$$
f = \begin{bmatrix} T_m sa_s cb_s \\ -T_m sb_s ca_s + T_t \\ T_m ca_s cb_s \end{bmatrix}, \tau = \begin{bmatrix} T_m h_m sb_s + L_b b_s + T_t h_t + \tau_m sa_s \\ T_m l_m + T_m h_m sa_s + M_a a_s + \tau_t - \tau_m sb_s \\ -T_m l_m sb_s - T_t l_t + \tau_m ca_s cb_s \end{bmatrix}
$$
\n(4)
\nwhere T_m , τ_m , T_t , τ_t are thrust and anti-torque generated by the main and tail rotors,
\nrespectively; M_a , L_b are stiffness coefficients of main rotor; h_m , h_t , l_m , l_t denote vertical and horizontal distances between the rotor centers and helicopter c.g.; a_s , b_s

denote longitudinal and lateral flapping angles of the main rotor. The relationship $f = \begin{bmatrix} T_m s a_s c b_s \\ -T_m s b_s c a_s + T_t \\ T_m c a_s c b_s \end{bmatrix}, \tau = \begin{bmatrix} T_m h_m s b_s + L_b b_s + T_t h_t + T_m h_m s a_s + t a_s + t \\ T_m l_m s b_s - T_t l_t + \tau_m c c_s \end{bmatrix}$
where T_m , τ_m , T_t , τ_t are thrust and anti-torque generated by the m
respectively; M_a , L_b are st ere T_m , τ_n , τ_t , τ_t are thrust and anti-torque generated by the main and tail rotors,
pectively; M_a , L_b are stiffness coefficients of main rotor; h_m , h_t , l_m , l_t denote ver-
al and horizontal distance

$$
\tau_i = A_i |T_i|^{1.5} + B_i \tag{5}
$$

where A_i and B_i are aerodynamic constants.

small, thus it is reasonable to express (4) as $[11]$ $[11]$: T_i and anti-to
 f are aerodyna

imitation of *f*

reasonable to
 $f = [0, 0, T_m]$ prope τ_i ($i = m$ or t) is given by [12]
 $\tau_i = A_i |T_i|^{1.5} + B_i$ (5)

mmic constants.

elicopter physical structure, a_s , b_s , T_t and τ_t are fairly

express (4) as [11]:
 $T + f_A$, $\tau = Q_A \rho + \tau_B + \Delta_{\tau} = \tau_1 + \Delta_{\tau}$ (6) of e
*I*e to T_m
 $T_m l_m$ elicopter physical structu

express (4) as [11]:
 $T^r + f_A$, $\tau = Q_A \rho + \tau_B + I_A$
 $Q_A = \begin{bmatrix} h_t & \tau_m & T_m h_m + \sigma_A & -\tau_m \\ 0 & T_m h_m + M_a & -\tau_m \\ 0 & 0 & T_m \end{bmatrix}$ Due to the limitation of helicopter physical structure, a_s , b_s , T_t and τ_t are fairly

$$
f = [0, 0, T_m]^T + f_A, \ \tau = Q_A \rho + \tau_B + \Delta_\tau = \tau_1 + \Delta_\tau \tag{6}
$$

where $\rho = \begin{bmatrix} T_t \\ a_s \\ b_s \end{bmatrix}$ \overline{a} $f = [0,$
, $\tau_B =$ $\overline{)}$ *m* $\overline{1}$ −*lt* 0 −*Tmlm*] a_s, b_s, T_t and τ_t are fairly
 $\tau_1 + \Delta_\tau$ (6)

and det(Q_A) $\neq 0$ [\[1](#page-14-0)]; f_A and $f = [0, 0, T_m]^T + f_A$, $\tau = Q_A \rho + \tau_B + \Delta_{\tau} = \tau_1 + \Delta_{\tau}$ (6)

where $\rho = \begin{bmatrix} T_t \\ a_s \\ b_s \end{bmatrix}$, $\tau_B = \begin{bmatrix} T_m^0 \\ T_m h_m^0 \end{bmatrix}$, $Q_A = \begin{bmatrix} h_t & \tau_m & T_m h_m + L_b \\ 0 & T_m h_m + M_a & -\tau_m \\ -l_t & 0 & -T_m l_m \end{bmatrix}$ and $\det(Q_A) \neq 0$ [1]; f_A

and Δ can be rewritten as

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Control for Miniature Unmanuel Helicopters
\n
$$
\dot{v} = -g_3 + (T_m/m)r_3 + \bar{f}_4
$$
\n(7)

Singularity-Free Path Following Control for Miniat
 $\dot{v} = -g_3 + (T_m)$

where $r_3 = Re_3$, $e_3 = [0, 0, 1]^T$, $\bar{f}_A = \frac{1}{m}Rf_A$.
 Control objective The reference path P_r is

expressions, i.e. $P_r = \{[x, y, z]^T [f_1(x, y, z) =$ **Control objective** The reference path P_r is a regular curve described by implicit $\mathcal{P}_r = \{ [x, y, z]^T | f_1(x, y, z) = 0, f_2(x, y, z) = 0 \}$, and $\partial_p f_1 \times \partial_p f_2 |_{p \in \mathcal{P}_r} \neq 0$ Singularity-Free Path Following Control for Miniature Unmanned Helicopters 673
 $\dot{v} = -g_3 + (T_m/m)r_3 + \bar{f}_4$ (7)

where $r_3 = Re_3$, $e_3 = [0, 0, 1]^T$, $\bar{f}_4 = \frac{1}{m}Rf_4$.
 Control objective The reference path P_r is a r *F*
 F Re₃, *R*₂ = [0, 0, 1]^{*T*}, $\bar{f}_A = \frac{1}{m} R f_A$.
 Control objective The reference path P_r is a regular curve described by implicit

expressions, i.e. $P_r = \{ [x, y, z]^T [f_1(x, y, z) = 0, f_2(x, y, z) = 0 \}$, and ∂ **ntrol objective** The reference path P_r is a regular curve described by implicity pressions, i.e. $P_r = \{[x, y, z]^T | f_1(x, y, z) = 0, f_2(x, y, z) = 0\}$, and $\partial_p f_1 \times \partial_p f_2 |_{p \in P_r} \neq$ where $f_1, f_2 \in C^\infty$; $\|\partial_p f_1\|$, $\|\partial_p f_2$

neighbourhood of *r*.

copter trajectory converges to the reference path ultimately and the magnitude of its expressions, i.e. $\mathcal{P}_r = \{ [x, y, z]^T | f_1(x, y, z) = 0, f_2(x, y, z) = 0 \}$, and $\partial_p f_1 \times \partial_p f_2 |_{p \in \mathcal{P}_r} \neq 0$, where $f_1, f_2 \in C^\infty$; $\|\partial_p f_1\|$, $\|\partial_p f_2\|$ are bounded on \mathcal{P}_r . In this paper the manifold $f_1 = 0$ is 0, where $f_1, f_2 \in C^{\infty}$; $||\partial_p f_1||$, $||\partial_p f_2||$ are bo $f_1 = 0$ is specified as a plane $f_1 = ax + by +$
Remark 1 From $f_1, f_2 \in C^{\infty}$ it follows that neighbourhood of P_r .
The control object is: (*i*) designing cont co *Remark 1* From $f_1, f_2 \text{ } \in C^\infty$ it follows that $\partial_p f_1 \times \partial_p f_2 \in C^\infty$ and $\partial_p f_1 \times \partial_p f_2 \neq 0$ in neighbourhood of \mathcal{P}_r .
The control object is: (*i*) designing control inputs T_m, T_t, a_s, b_s such that the helico *i*) des to t

s to t

erend
 $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ *p*_{*f*</sup> is equal to the reference path to o_3 such the v, z $| < o_2$ $| p_1 f_1 \times o_p f_2 |$} des to t

and θ_2 ,
 x, θ_1
 θ_2

\n object is: \n \n- (i) designing control inputs
$$
T_m, T_t, a_s, b_s
$$
 such that the heliconverges to the reference path ultimately and the magnitude of its on on reference path tends to desired speed $v_r > 0$, i.e. there exist not at $0_1, 0_2, 0_3$ such that

\n\n\n $\lim_{t \to \infty} |f_1(x, y, z)| < o_1, \quad \lim_{t \to \infty} |f_2(x, y, z)| < o_2$ \n

\n\n $\lim_{t \to \infty} \left| \left(\frac{\partial_p f_1 \times \partial_p f_2}{\|\partial_p f_1 \times \partial_p f_2\|} \right)^T v - v_r \right| < o_3$ \n

(*ii*) No singularity occurs in control process.

4 Controller Design

Define the path-following and velocity errors

ity occurs in control process.

\n**Design**

\nllowing and velocity errors

\n
$$
\zeta_1 = f_1(x, y, z), \quad \zeta_2 = f_2(x, y, z), \quad \zeta_3 = \eta^T v - v_r
$$

\n
$$
\zeta_2 \frac{\partial \phi f_2}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_1}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \frac{\partial \zeta_2}{\partial \eta} \frac{\partial \zeta_3}{\partial \eta} \
$$

Fine the path-follow
 ζ_1 :
where $\eta = \frac{\partial_p f_1 \times \partial_p f_2}{\partial p_1 \times \partial p_2 f_1}$ $h -$
 $\frac{\partial}{\partial \theta}$ following and velocity
 $\varsigma_1 = f_1(x, y, z), \varsigma_2 =$
 $\frac{\rho f_1 \times \partial_p f_2}{\rho f_1 \times \partial_p f_2}$, we can derive լ−ք
|-
|∂ր
|ζ $\begin{aligned} \n\text{in} \quad & \text{for } t_1 \times \text{d} \\ \n\text{in} \quad & \text{for } t_1 \times \text{d} \\ \n\text{in} \quad & \text{for } t_1 \times \text{d} \\ \n\text{in} \quad & \text{for } t_1 \times \text{d} \end{aligned}$ 2^{p}
 2^{p}
 $\frac{1}{p} f_2$
 $\frac{1}{p} f_2$
 $\frac{1}{p} f_2$
 $\frac{1}{p} f_2$

$$
\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3
$$

\n
$$
\mathcal{E}_1 = f_1(x, y, z), \quad \mathcal{E}_2 = f_2(x, y, z), \quad \mathcal{E}_3 = \eta^T \mathbf{v} - \mathbf{v}_r
$$

\n
$$
\frac{\partial_p f_1 \times \partial_p f_2}{\partial_p f_1 \times \partial_p f_2}
$$
, we can derive
\n
$$
[\xi_1, \xi_2, \xi_3]^T = G\mathbf{v} - [0, 0, \mathbf{v}_r]^T, \quad [\xi_1, \xi_2, \xi_3]^T = \mathbf{h} + G\mathbf{v}
$$
 (10)

Define the path-following and velocity
 $\varsigma_1 = f_1(x, y, z), \ \varsigma_2 =$

where $\eta = \frac{\partial_p f_1 \times \partial_p f_2}{\|\partial_p f_1 \times \partial_p f_2\|}$, we can derive
 $[\dot{\varsigma}_1, \dot{\varsigma}_2, \varsigma_3]^T = Gv - [0, 0]$

where $\boldsymbol{h} = [0, v^T(\partial_p^2 f_2)v, v^T(\partial_p \eta)v - \dot{v}_r]$

Fro *T F*₂ = $f_2(x, y, z)$, $c_3 = \eta^T v - v_r$ (9)
 T T + [0, 0, v_r]^{*T*}, $[\ddot{c}_1, \ddot{c}_2, \dot{c}_3]^T = h + G\dot{v}$ (10)
 $\eta v - \dot{v}_r$]^{*T*}, $G = [\partial_p f_1, \partial_p f_2, \eta]^T$ is the control matrix.
 $T = h + G[-g_3 + (T_m/m)r_3] + \Delta_f$ (11) From (10) and (7) we know $\int_{p}^{p} \frac{p^{f_2}}{2}, \text{ we}$
 $\int_{2}^{p} \frac{p^{f_2}}{2} \, dx$, we $\int_{2}^{p} \frac{p^{f_2}}{2}, \text{ } \frac{p^{f_2}}$ $[\xi_1, \xi_2, \xi_3]^T = Gv - [0, 0, v_r]^T$, $[\xi_1, \xi_2, \xi_3]^T = h + Gv$

where $h = [0, v^T(\partial_p^2 f_2)v, v^T(\partial_p \eta)v - v_r]^T$, $G = [\partial_p f_1, \partial_p f_2, \eta]^T$ is the controller

From (10) and (7) we know
 $[\xi_1, \xi_2, \xi_3]^T = h + G[-g_3 + (T_m/m)r_3] + \Delta_f$

where Δ_f *u*_{*s*} $\frac{2}{2}f_2$ γ , $\nu^T(\partial_p \eta)\nu - \nu$
we know
 $[\xi_1, \xi_2, \xi_3]^T = h$ -
efine the position
 $u_c = [u_{cx}, u_{cy}, u_{cz}]$

$$
[\ddot{\zeta}_1, \ddot{\zeta}_2, \dot{\zeta}_3]^T = \mathbf{h} + G[-\mathbf{g}_3 + (T_m/m)\mathbf{r}_3] + \mathbf{\Delta}_f
$$
 (11)

where $\Delta_f = G\bar{f}_\Delta$. Define the position loop controller $u_c = T_m r_3$ and design

$$
\boldsymbol{u}_c = [u_{cx}, u_{cy}, u_{cz}]^T = m(\boldsymbol{g}_3 + G^{-1}(-\boldsymbol{h} + \boldsymbol{v}_c))
$$
(12)

where v_c is a new control input to be determined. The singularity occurs when
 $\det(G) = (\partial f, \times \partial f)^T \mathbf{n} = ||\partial f, \times \partial f|| = 0$ From Remark 1 G is invertible in where v_c is
det(*G*) = (∂ $\frac{1}{p}f_1 \times \partial_p f_2$

pod of *P* $\text{m} = \frac{1}{\theta}$ neighborhood of \mathcal{P}_r . When helicopter initial position is far from \mathcal{P}_r , *G* cannot be guaranteed to be invertible. Besides, when helicopter is far from the path, the control energy will be large. A solution for the two problems is to plan a temporary path w de ne
gu tr
戸 *r* \mathbf{v}_c is a new control input to be determined. The singularity occurs when $ext(G) = (\partial_p f_1 \times \partial_p f_2)^T \mathbf{\eta} = ||\partial_p f_1 \times \partial_p f_2|| = 0$. From Remark 1, *G* is invertible in eighborhood of \mathcal{P}_r . When helicopter initial po wher
det(t
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trol $\bar{\mathcal{P}}_r$ fr
on $\bar{\mathcal{P}}$ *re* v_c is a new control input to be determined. The singularity occurs when G) = $(\partial_p f_1 \times \partial_p f_2)^T \eta$ = $||\partial_p f_1 \times \partial_p f_2||$ = 0. From Remark 1, *G* is invertible in hborhood of P_r . When helicopter initial position i 10, it is need to plan a temporary-path. where \mathbf{v}_c is a new control input to be determined. The s
det(G) = $(\partial_p f_1 \times \partial_p f_2)^T \mathbf{\eta} = ||\partial_p f_1 \times \partial_p f_2|| = 0$. From Remar
neighborhood of \mathcal{P}_r . When helicopter initial position is far
guaranteed to be invert is fixthed in the *i* $\left(\frac{\partial p}{\partial y}\right)^T$, $\left(\frac{\partial p}{\partial y}\right)^T$, $\left(\frac{\partial p}{\partial y}\right)^T$, $\left(\frac{\partial p}{\partial y}\right)^T$ of *Fi* find position is far from P_r , *G* canneguaranteed to be invertible. Besides, when helicopter is far from the path, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ be i
ill be i
ill be posed
pl **1** Δ_i for the coefficient *c* in f_r . When helicopter lind position to the treath f_r , g_r cannot be guaranteed to be invertible. Besides, when helicopter is far from the path, the control energy will be large. A solution fo

Assumption 1 $\mathbf{\Delta}_f = [\Delta_{f_x}, \Delta_{f_y}, \Delta_{f_z}]^T$ and $\mathbf{\Delta}_\tau = [\Delta_{\tau_x}, \Delta_{\tau_y}, \Delta_{\tau_z}]^T$ are bounded and sat-*^T* are gularity in deriving desired unit-quaternion. The physical meaning is that $f_1 = 0$ is not perpendicular to $x - y$ plane of \mathcal{I} . on P_r . In this paper define P_0 to be the initial position, when $\sqrt{|f_1(P_0)|^2 + 1}$
10, it is need to plan a temporary-path.
Assumption 1 $\mathbf{\Delta}_f = [\Delta_{fx}, \Delta_{fy}, \Delta_{fz}]^T$ and $\mathbf{\Delta}_\tau = [\Delta_{rx}, \Delta_{ry}, \Delta_{rz}]^T$ are bounde
isfy 10, it is need to plan a temporary-path.
 Assumption 1 $\Delta_f = [\Delta_{fx}, \Delta_{fy}, \Delta_{fz}]^T$ and $\Delta_{\tau} = [\Delta_{xx}, \Delta_{ry}, \Delta_{rz}]^T$ are bounded and sat-

isfy $| \Delta_{fi} | \leq \delta_i, | \Delta_{ri} | \leq \gamma_i$ (*i* = *x*, *y*, *z*), where $\delta = [\delta_x, \delta_y, \delta_z]^T$ *T* and and γ_y, γ_z]
 obiding
 $\text{at } f_1 =$
 $\text{at } T = \partial$ **Example 1** $\Delta_f = [A_{fx}, A_{fy}, A_{fz}]^T$ and $\Delta_{\tau} = [A_{rx}, A_{ry}, A_{rz}]^T$ are bounded and sat-

ffy $|A_{fi}| \le \delta_i$, $|A_{ri}| \le \gamma_i$ ($i = x, y, z$), where $\delta = [\delta_x, \delta_y, \delta_z]^T$ and $\gamma = [\gamma_x, \gamma_y, \gamma_z]^T$ are

nown upper bounds; The coefficient c

Path planning As illustrated in Fig. 2, the initial position $P_0 = [x_0, y_0, z_0]^T$. Choose that $|A_{fi}| \leq \delta_i$, $|A_{ri}| \leq \gamma_i$ (*i* = *x*, *y*, *z*), where $\delta = [\delta_x, \delta_y, \delta_z]^T$ and $\gamma = [\gamma_x, \gamma_y, \gamma_z]^T$ are
known upper bounds; The coefficient *c* in $f_1 = 0$ satisfies $|c| > \delta_x/g$ for avoiding sin-
gularity in deriving Firstly, passing through \mathbf{p}_0 and \mathbf{p}_d we can define a plane $\bar{f}_1 = a_1x + b_1y + c_1z + b_0$ and $\bar{f}_2 = 0$ is $\bar{f}_1 = [x_r, y_r, z_r]^T$ on P_r and compute tangent vector $\mathbf{p}_d = [x_0, y_0, z_0]^T$. Choose oint $P_d = [x_r, y$ guidarity in deriving desired dim-quaternion. The physical meaning is that $j_1 = 0$ is
not perpendicular to $x - y$ plane of T .
Path planning As illustrated in Fig. 2, the initial position $P_0 = [x_0, y_0, z_0]^T$. Choose
a **Path planning** As illustrated in Fig. 2, the initial position P_0

a point $P_d = [x_r, y_r, z_r]^T$ on P_r and compute tangent vector $\hat{\rho}_{p} f_2 |_{P_d}$. Define $\mathbf{p}_0 = \overline{P_0 P_d}$ and compute $\mathbf{p}_0 \times \mathbf{p}_d = [\bar{x}_1, \bar{y}_1, \$

following steps:
Firstly, passing through p_0 and p_d we can define a plane $\bar{f}_1 = a_1x + b_1y + c_1z +$ For *Γ*_{*d*} = [*x*_{*r}*, *y*_{*r*}, *z*_{*r*}]^{*t*} on *P_r* and compute tangent vector $p_d = [x, y, z]^t = o_p t_1 \times o_{z_1}^1$. Define $p_0 = \overline{P_0 P_d}$ and compute $p_0 \times p_d = [\bar{x}_1, \bar{y}_1, \bar{z}_1]^T$. If P_d are chosen such $t, \bar{z$ $O_p J_2|_{P_d}$. Define $p_0 = P_0 P_d$ and compute $p_0 \times p_d = [x_1, y_1, z_1]^T$. If P_d are chead that $\bar{z}_1 \neq 0$ and p_0 is not collinear with p_d , a temporary path can be pla following steps:
Firstly, passing through p_0 *y*₀ and p_d we can define a plane $\bar{f}_1 = q$
 *x*₁ = 0 and its normal vector is $p_0 \times p_d$
 *x*₁, *b*₁, *c*₁] = $k[\bar{x}_1, \bar{y}_1, \bar{z}_1]$, where *k* is a co
 $\frac{\delta_x}{|\bar{z}_1|g}$ can guarantee that $c_1 > \delta_x/g$.
 z

$$
\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} \bar{x} & \bar{y} & \bar{z} \\ a_1 & b_1 & c_1 \\ x_r - x_0 & y_r - y_0 & z_r - z_0 \end{bmatrix}^{-1} \begin{bmatrix} \bar{x}x_r + \bar{y}y_r + \bar{z}z_r \\ -d_1 \\ \frac{x_r^2 - x_0^2 + y_r^2 - y_0^2 + z_r^2 - z_0^2}{2} \end{bmatrix}
$$

Singularity-Free Path Following Control for Miniature Unmanned Helicopters 67
Since p_0 is not collinear to p_d , above inverse matrix is well defined and $[a_2, b_2, c_2]$
can be determined uniquely. Define $\|\mathbf{n}\| = r$, w Since \mathbf{p}_0 is not collinear to \mathbf{p} , above inverse matrix is well defined and $[a_0, b_0, c_0]^T$ Singularity-Free Path Following Control for Miniature Unmanned Helicopters 675

Since p_0 is not collinear to p_d , above inverse matrix is well defined and $[a_2, b_2, c_2]^T$

can be determined uniquely. Define $||p_1|| = r$, $(y - b_2)^2 + (z - c_2)^2 - r^2 = 0$, where $k_1 \neq 0$ is a constant. gularity-Free Path Following Control for Miniature Unmanned Helicopters 675

ice \mathbf{p}_0 is not collinear to \mathbf{p}_d , above inverse matrix is well defined and $[a_2, b_2, c_2]^T$

i be determined uniquely. Define $||\mathbf{p}_$ Since p_0 is n

can be det
 $(y - b_2)^2 + ($

Finally, te

are generate

its tangent lin

Remark 2 \bar{p} ince p_0 is not collinear to p_d , above inverse matrix is well defined and $[a_2, b_2, c_2]^T$
an be determined uniquely. Define $||p_1|| = r$, we have $\bar{f}_2 = \frac{1}{k_1}((x - a_2)^2 + (y - b_2)^2 + (z - c_2)^2 - r^2) = 0$, where $k_1 \neq 0$ is

are generated (bold line and normal one in Fig. [2\)](#page-5-0), the one with smaller angle from its tangent line at P_0 to helicopter head direction should be selected. Finally, temporary path \bar{P}_r is the intersection of $\bar{f}_1 = 0$ and $\bar{f}_2 = 0$. Since two paths are generated (bold line and normal one in Fig. 2), the one with smaller angle from its tangent line at P_0 to helicop

can be determined union
 $(y - b_2)^2 + (z - c_2)^2 - r^2$

Finally, temporary path

are generated (bold line are

its tangent line at P_0 to hel

Remark 2 \bar{P}_r satisfies that
 $\partial_p \bar{f}_2$ on \bar{P}_r , i.e. $\partial_p \bar{f}_1 \times \partial_p \bar{f$ matrix \bar{G} invertible near \bar{P} . $($ and α it is R ∂_{μ} m $\bar{\mathcal{P}}$ $r - b_2)^2 + (z - c_2)^2 - r^2 = 0$, where $k_1 \neq 0$ is a constant.

Finally, temporary path \bar{P}_r is the intersection of $\bar{f}_1 = 0$ and $\bar{f}_2 = 0$. Since generated (bold line and normal one in Fig. 2), the one with smaller s Finally, temporary path \bar{P}_r is the intersection of $\bar{f}_1 = 0$ and $\bar{f}_2 = 0$.
are generated (bold line and normal one in Fig. 2), the one with smallist tangent line at P_0 to helicopter head direction should be s angent line at P_0 to helicopter hea

nark 2 \bar{P}_r satisfies that 1) $\partial_p \bar{f}_1 \times \partial_q$

on \bar{P}_r , i.e. $\partial_p \bar{f}_1 \times \partial_p \bar{f}_2 \neq 0$ on \bar{P}_r .

rix \bar{G} invertible near \bar{P}_r .
 bollowing The position loop *̄*

 $\bar{\mathcal{P}}_r$ following The position loop controller design is divided to two steps: $\bar{\mathcal{P}}_r$ following and then P_r , following. Firstly define errors for the temporary path \bar{P}_r

$$
\vec{a}_{p} \vec{f}_{2} \text{ on } \vec{P}_{r}, \text{ is a constant } \vec{f} \cdot \vec{v}_{p} \text{ and } \vec{f} \text{ and } \vec{f} \text{ are a constant.}
$$
\n
$$
\vec{P}_{r} \text{ following The position loop controller design is divided to two steps: } \vec{P}_{r} \text{ following and then } P_{r} \text{ following. Firstly define errors for the temporary path } \vec{P}_{r}
$$
\n
$$
\vec{\zeta}_{1} = a_{1}x + b_{1}y + c_{1}z + d_{1}, \quad \vec{\zeta}_{2} = \frac{1}{k_{1}}((x - a_{2})^{2} + (y - b_{2})^{2} + (z - c_{2})^{2} - r^{2})
$$
\n
$$
\vec{\zeta}_{3} = \vec{\eta}^{T}v - \vec{v}_{r}
$$
\n
$$
\text{where } \vec{\eta} = \frac{\vec{\eta}_{1} \times \vec{\eta}_{2}}{\|\vec{\eta}_{1} \times \vec{\eta}_{2}\|}, \vec{\eta}_{1} = [a_{1}, b_{1}, c_{1}]^{T}, \vec{\eta}_{2} = \frac{2}{k_{1}}[x - a_{2}, y - b_{2}, z - c_{2}]^{T}. \text{ It yields}
$$
\n
$$
[\vec{\zeta}_{1}, \vec{\zeta}_{2}, \vec{\zeta}_{3}]^{T} = \vec{G}v - [0, 0, \vec{v}_{r}]^{T} \qquad (14)
$$
\n
$$
[\vec{\zeta}_{1}, \vec{\zeta}_{2}, \vec
$$

where $\bar{\pmb{\eta}} = \frac{\bar{\pmb{\eta}}_1 \times \bar{\pmb{\eta}}_2}{\|\bar{\pmb{\eta}}_1 \times \bar{\pmb{\eta}}_2\|}$, $\bar{\pmb{\eta}}_1 = [a_1, b_1, c_1]^T$, $\bar{\pmb{\eta}}_2 = \frac{2}{k_1} [x - a_2, y - b_2, z - c_2]$
 $[\dot{\bar{\zeta}}_1, \dot{\bar{\zeta}}_2, \dot{\bar{\zeta}}_3]^T = \bar{G}v - [0, 0, \bar{v}_r]^T$
 $[\ddot{\bar{\zeta}}_1, \ddot{\bar{\$ a_2 , $\frac{1}{n}$
0, $\frac{1}{n}$
 $\frac{1}{n}$, $\frac{1}{n}$ $y - \frac{1}{2}$
m)**r**
 $\frac{1}{2}, \bar{h}$

$$
[\dot{\bar{\zeta}}_1, \dot{\bar{\zeta}}_2, \bar{\zeta}_3]^T = \bar{G}v - [0, 0, \bar{v}_r]^T
$$
\n(14)

$$
[\ddot{\zeta}_1, \ddot{\zeta}_2, \dot{\zeta}_3]^T = \bar{h} + \bar{G}[-g_3 + (T_m/m)r_3] + \Delta_f
$$
 (15)

 $(b_2, z - c_2]^T$. It yields

(14)
 $[\mathbf{a}]\mathbf{a} + \mathbf{A}_f$ (15)
 $[\mathbf{a}]\mathbf{a}^T = [0, \frac{2}{k_1} \mathbf{v}^T \mathbf{v}, \mathbf{v}^T (\partial_p \bar{\mathbf{\eta}}) \mathbf{v} - \mathbf{v}^T (\partial_p \bar{\mathbf{v}}) \mathbf{v} - \mathbf{v}^T (\partial_p \bar{\mathbf{v}}) \mathbf{v} - \mathbf{v}^T (\partial_p \bar{\mathbf{v}}) \mathbf{v} - \mathbf{v}^T (\partial$ wh
wh
 $\dot{\bar{\nu}}_r$] *T* $\bar{\pmb{\eta}} = \frac{\bar{\pmb{\eta}}_1 \times \bar{\pmb{\eta}}_2}{\|\bar{\pmb{\eta}}_1 \times \bar{\pmb{\eta}}_2\|}, \bar{\pmb{\eta}}_1 = [a_1, b_1, c_1]^T, \bar{\pmb{\eta}}_1$
 $[\dot{\bar{\zeta}}_1, \dot{\bar{\zeta}}_2, \dot{\bar{\zeta}}_3]^T = \bar{\pmb{h}} +$
 F $\bar{G} = [\bar{\pmb{\eta}}_1, \bar{\pmb{\eta}}_2, \bar{\pmb{\eta}}]^T$ is control mat
 T. Def $[\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3]^T = \vec{G}\vec{v} - [0, 0, \vec{v}_r]^T$ (14)
 $[\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3]^T = \vec{h} + \vec{G}[-g_3 + (T_m/m)r_3] + \Delta_f$ (15)
 $\vec{\eta}]^T$ is control matrix, $\vec{h} = [\vec{h}_1, \vec{h}_2, \vec{h}_3]^T = [0, \frac{2}{k_1} \vec{v}^T \vec{v}, \vec{v}^T (\partial_p \vec{\eta}) \vec{v} - \$ *̄̄* $\begin{bmatrix} 3 \end{bmatrix}^T$
besi
*F*_{*k*} \overline{u} $\vec{h} = \vec{h} + \vec{G}[-g_3 + (T_m/m)r_3] + \Delta_f$

rol matrix, $\vec{h} = [\vec{h}_1, \vec{h}_2, \vec{h}_3]^T = [0]$

ign it as
 $\begin{aligned} \sum_{c_y} \vec{u}_{cz}]^T &= m(g_3 + \vec{G}^{-1}(-\vec{h} + \vec{v}_c)) \\ \sum_{i=1}^T (\tanh(\vec{\xi}_{k1}) + \tanh(\vec{\xi}_1)) - \tanh(\frac{\vec{\xi}_{k1}}{\vec{\xi}_{k1}}) \end{aligned}$ *̄* $\frac{2}{k_1}$
 δ_x

$$
\bar{u}_c = [\bar{u}_{cx}, \bar{u}_{cy}, \bar{u}_{cz}]^T = m(g_3 + \bar{G}^{-1}(-\bar{h} + \bar{v}_c))
$$
(16)

where
$$
\vec{G} = [\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}]^T
$$
 is control matrix, $\vec{h} = [\bar{h}_1, \bar{h}_2, \bar{h}_3]^T = [0, \frac{2}{k_1} v^T v, v^T (\partial_p \vec{\eta}) v - \vec{v}_r]^T$. Define $\vec{u}_c = T_m r_3$ and design it as
\n
$$
\vec{u}_c = [\vec{u}_{cx}, \vec{u}_{cy}, \vec{u}_{cz}]^T = m(g_3 + \vec{G}^{-1}(-\vec{h} + \vec{v}_c))
$$
\n
$$
\vec{v}_c = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix} = \begin{bmatrix} -\bar{k}_{11} (\tanh(\vec{\zeta}_{k1}) + \tanh(\dot{\vec{\zeta}}_1)) - \tanh(\frac{\vec{\theta}_1}{\bar{\varepsilon}_1}) \delta_x \\ -\bar{k}_{21} (\tanh(\vec{\zeta}_{k2}) + \tanh(\dot{\vec{\zeta}}_2)) - \tanh(\frac{\vec{\theta}_2}{\bar{\varepsilon}_2}) \delta_y \\ -\bar{k}_{31} \tanh(\vec{\zeta}_3) - \tanh(\frac{\vec{\zeta}_3}{\bar{\varepsilon}_3}) \delta_z \end{bmatrix}
$$
\nwhere $\vec{\zeta}_{k1} = \vec{k}_{12} \vec{\zeta}_1 + \vec{\zeta}_1$, $\vec{\zeta}_{k2} = \vec{k}_{22} \vec{\zeta}_2 + \vec{\zeta}_2$, $\vec{\theta}_1 = \tanh(\vec{\zeta}_{k1}) + \tanh(\vec{\zeta}_1) + \frac{\vec{k}_{12}}{\vec{k}_{11}} \vec{\zeta}_1$, $\vec{\theta}_2 = \tanh(\vec{\zeta}_{k2}) + \tanh(\vec{\zeta}_2) + \frac{\vec{k}_{22}}{\vec{k}_{21}} \vec{\zeta}_2$; $\vec{k}_{11}, \vec{k}_{12}, \vec{k}_{21}, \vec{k}_{22}, \vec{k}_{31}, \vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3$ are positive constants.
\nSince $||\mathbf{r}_1|| = 1$ $T = ||\vec{\mathbf{u}}||$ If $\vec{\mathbf{u}} \in \mathcal{C} = \{10, 0, u\}^T$

 $\frac{12}{11}\bar{\zeta}_1, \ \bar{\vartheta}_2 =$ 22 $\frac{1}{k}$, $\frac{1}{k}$ 21 $\bar{v}_c = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{bmatrix} = \begin{bmatrix} -\bar{k}_{11}(\tanh(\bar{\zeta}_{k1}) + \tanh(\bar{k}_{k2})) + \tanh(\bar{k}_{k3}) \\ -\bar{k}_{21}(\tanh(\bar{k}_{k2}) + \tanh(\bar{k}_{33})) - \bar{k}_{31}(\tanh(\bar{k}_{33})) - \end{bmatrix}$
where $\bar{\zeta}_{k1} = \bar{k}_{12}\bar{\zeta}_1 + \dot{\bar{\zeta}}_1, \ \bar{\zeta}_{k2} = \bar{k}_{22}\bar{\zeta}_2 + \dot{\bar{\zeta}}$ Since $||r_3|| = 1$, $T_m = ||\bar{u}_c||$. If $\bar{u}_c \notin \mathcal{L} = \{ [0, 0, u]^T, u \le 0 \}$, then the desired unit- $\bar{v}_c = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{bmatrix} = \begin{bmatrix} -\bar{k}_{21}(\tanh(\bar{\zeta}_{k2}) + \tanh(\bar{\zeta}_{2})) \\ -\bar{k}_{31}\tanh(\bar{\zeta}_{3}) - \tan \end{bmatrix}$
where $\bar{c}_{k1} = \bar{k}_{12}\bar{c}_{1} + \dot{\bar{c}}_{1}, \ \bar{c}_{k2} = \bar{k}_{22}\bar{c}_{2} + \dot{\bar{c}}_{2}, \ \bar{\theta}_{1} = \tanh(\tanh(\bar{\zeta}_{k2}) + \tanh(\dot{\bar{c}}_{2})$ quaternion $Q_c = [\mu_c, \mathbf{q}_c^T]^T$ can be derived from $\bar{\mathbf{u}}_c$ [12] $\vec{\xi}_{k2} = \vec{k}_{22}$
 $\vec{\xi}_{2}$; \vec{k}_{11} , \vec{k}_{12}
 $\|\cdot\|$. If \vec{u}_{c}

can be d
 $\frac{1}{2\|\vec{u}_{c}\| \mu_{c}}$ $\begin{aligned} k_{31}^{\dagger} \\ + \dot{\xi} \\ \bar{k}_{2}^{\dagger} \\ \mathcal{L} \\ \bar{w}_{c} \end{aligned}$ + $\dot{\bar{z}}_2$
 r, \bar{k}_{21}
 cx
 *u*_{*cy*}
 *u*_{*cy*}
 *u*_{*cx*} $\bar{\vartheta}_1$
 $\bar{\vartheta}_2$
 $\bar{\vartheta}_3$
 $\bar{\vartheta}_4$
 $\bar{\vartheta}_5$
 $\bar{\vartheta}_7$
 $\bar{\vartheta}_8$ $h(\zeta_{k1}) + ta$
 $h(\zeta_{k1}) + ta$
 $h(\zeta_{k1}) + ta$
 $h(\zeta_{k1}) + ta$
 $h(\zeta_{k1}) + \bar{u}_{c}$
 $\frac{u}{\zeta_{k1}} + \bar{u}_{c}$ $(\vec{\xi}_{k1})$ + tanh $(\dot{\vec{\xi}}_1)$ + $\frac{\vec{k}_{12}}{\vec{k}_{11}} \dot{\vec{\xi}}_1$, $\vec{\vartheta}_2$ =
 $\frac{1}{2}, \vec{\xi}_3$ are positive constants.
 ≤ 0 , then the desired unit-
 $\frac{\vec{u}_c \parallel + \vec{u}_{cz}}{2 \parallel \vec{u}_c \parallel}$ (18)

$$
\boldsymbol{q}_{c} = \frac{1}{2\|\bar{\boldsymbol{u}}_{c}\| \mu_{c}} \begin{bmatrix} \bar{u}_{cy} \\ -\bar{u}_{cx} \\ 0 \end{bmatrix}, \mu_{c} = \sqrt{\frac{\|\bar{\boldsymbol{u}}_{c}\| + \bar{u}_{cz}}{2\|\bar{\boldsymbol{u}}_{c}\|}}
$$
(18)

Theorem 1 *If initial velocity* $v(0) = 0$ *and the parameters in* [\(17\)](#page-6-0) *satisfy , ̄*

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\n**Theorem 1** If initial velocity
$$
v(0) = 0
$$
 and the parameters in (17) satisfy
\n
$$
\bar{k}_{12} \le \bar{k}_{11} < \frac{|c_1|g - \delta_x}{2}, \bar{k}_{22} \le \bar{k}_{21}, \bar{\epsilon}_1 < \frac{0.382\bar{k}_{12}}{k_q \delta_x}, \bar{\epsilon}_2 < \frac{0.382\bar{k}_{22}}{k_q \delta_y}
$$
\nThen control law (16) guarantees: (i) $\bar{\zeta}_1$, $\bar{\zeta}_1$, $\bar{\zeta}_2$, $\bar{\zeta}_3$ are bounded and converge to a small neighbourhood of origin. (ii) \bar{G} is invertible during control and no singularity

occurs in deriving desired unit-quaternion with [\(18\)](#page-6-2)*. From Proof* (*i*) *Proof* (*i*) $\bar{k}_{12} \leq \bar{k}_{11} < \frac{|c_1|g - \delta_x}{2}, \bar{k}_{22} \leq \bar{k}_{21}, \bar{\epsilon}_1 < \frac{0.382\bar{k}_1}{k_q \delta_x}$
 Proof i aw (16) guarantees: (*i*) $\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_2, \bar{\zeta}_3$ *are is mall neighbourhood of* $\hat{\tau}$ $\hat{\tau}$ *k̄* $\frac{1}{2k}$
 $\frac{\bar{k}_1}{2\bar{k}}$ *Vurs in di*
of (*i*) D
 $\dot{V} = -\bar{k}$ *small neighbourhood of origin.* (ii) \overline{G} *is invertible during control and no singularity*

̄̄

bburhood of origin. (ii) *G* is invertible during control and no singularity
eriving desired unit-quaternion with (18).

$$
V = \int_0^{\xi_{k1}} \xi_1 \, ds + \int_0^{\xi_1} \tanh(s) \, ds + \frac{\bar{k}_{12}}{2\bar{k}_{11}} \xi_1^2 \tag{20}
$$

$$
V = \int_0^{\xi_{k1}} \tanh(s) \, ds + \int_0^{\xi_1} \tanh(s) \, ds + \frac{\bar{k}_{12}}{2\bar{k}_{11}} \xi_1^2 \tag{21}
$$

$$
V = \int_0^{\xi_{k1}} \tanh(\xi_1) \, ds + \int_0^{\xi_1} \tanh(\xi_1) \, ds + \int_0^{\xi_2} \tanh(\xi_1) \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_1} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_1} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_1} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_1} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_1} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_1} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_1} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_1} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_0^{\xi_1} \xi_1 \, ds + \int_0^{\xi_2} \xi_1 \, ds + \int_
$$

Proof (i) Define
$$
\bar{\zeta} = [\bar{\zeta}_{k1}, \dot{\bar{\zeta}}_1]
$$
, choose Lyapunov function
\n
$$
V = \int_0^{\bar{\zeta}_{k1}} \tanh(s)ds + \int_0^{\dot{\bar{\zeta}}_1} \tanh(s)ds + \frac{\bar{k}_{12}}{2\bar{k}_{11}} \dot{\bar{\zeta}}_1^2
$$
\n
$$
\dot{V} = -\bar{k}_{11}[\tanh(\bar{\zeta}_{k1}) + \tanh(\dot{\bar{\zeta}}_1)]^2 - \bar{k}_{12}\dot{\bar{\zeta}}_1\tanh(\dot{\bar{\zeta}}_1) - \bar{\vartheta}_1[\tanh(\frac{\bar{\vartheta}_1}{\bar{\varepsilon}_1})\delta_x - 4_{fx}]
$$
\n
$$
\le -\tanh^T(\bar{\zeta}) \left[\frac{\bar{k}_{11}}{\bar{k}_{11}} \frac{\bar{k}_{11}}{\bar{k}_{11}} + \bar{k}_{12} \right] \tanh(\bar{\zeta}) - \bar{\vartheta}_1 \tanh(\frac{\bar{\vartheta}_1}{\bar{\varepsilon}_1})\delta_x + \bar{\vartheta}_1 A_{fx}
$$
\nBased on Lemma 2, we know $\bar{\vartheta}_1 A_{fx} \le |\bar{\vartheta}_1| \delta_x \le |\bar{\vartheta}_1 \tanh(\frac{\bar{\vartheta}_1}{\bar{\varepsilon}_1}) + k_q \bar{\varepsilon}_1] \delta_x$. Consider-
\ning $\left[\frac{\bar{k}_{11}}{\bar{k}_{11}} \frac{\bar{k}_{11}}{\bar{k}_{11}} + \frac{\bar{k}_{12}}{\bar{k}_{12}} \right] = A^T \left[\frac{\bar{k}_{11}}{0} \frac{0}{\bar{k}_{12}} \right] A$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (21) yields $\dot{V} \le k_q \bar{\varepsilon}_1 \delta_x$

ing $\begin{bmatrix} \bar{k}_{11} & \bar{k}_{11} \\ \bar{k} & \bar{k}_{11} & \bar{k}_{11} \\ \bar{k} & \bar{k} & \bar{k} \end{bmatrix} = A^T \begin{bmatrix} \bar{k}_{11} & 0 \\ 0 & \bar{k} \end{bmatrix} A$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (2) $_{11}$ $_{0}$.
Г 1 1 $V = -k_{11}[\tanh(\bar{\zeta}_{k1}) + \tanh(\bar{\zeta}_{1})]^2 - k_{12}\bar{\zeta}_1\tanh(\bar{\zeta}_1) - \vartheta_1[\tanh(\frac{1}{\bar{\epsilon}_1})\delta_x - \Delta_{fx}]$
 $\leq -\tanh^T(\bar{\zeta}) \left[\frac{\bar{k}_{11}}{\bar{k}_{11}} \frac{\bar{k}_{11}}{\bar{k}_{11} + \bar{k}_{12}}\right] \tanh(\bar{\zeta}) - \bar{\vartheta}_1 \tanh(\frac{\bar{\vartheta}_1}{\bar{\epsilon}_1})\delta_x + \bar{\vartheta}_1 \Delta_{fx}$

(21)

assed on tanh $n^T(\bar{\xi}$
mma
 $\frac{1}{\bar{k}}$ ₁₂
i₁₁, \bar{k} $\leq -\tanh^{T}(\bar{\zeta}) \left[\frac{k_{11}}{k_{11}} \frac{k_{11}}{k_{11} + \bar{k}_{12}} \right] \tan$

Based on Lemma 2, we know $\bar{\vartheta}_{1} \Delta_{f}$

ing $\left[\frac{\bar{k}_{11}}{\bar{k}_{11}} \frac{\bar{k}_{11}}{\bar{k}_{11} + \bar{k}_{12}} \right] = A^{T} \left[\frac{\bar{k}_{11}}{0} \frac{0}{\bar{k}_{12}} \right] A$.
 $\frac{\lambda}{\Lambda} (\Lambda^{T} A) \min{\{\bar{k}_{11$ √ $\bar{\vartheta}_1 | \delta$,
ere θ
 $\bar{k}_q \bar{\varepsilon}_1 \delta$, $(\vec{\theta}_1 \cdot \vec{\theta}_1) = \vec{\theta}_1 \tanh(\frac{\vec{\theta}_1}{\vec{\epsilon}_1}) \delta_x + \vec{\theta}_1 \Delta_{fx}$
 $|\vec{\theta}_1| \delta_x \leq [\vec{\theta}_1 \tanh(\frac{\vec{\theta}_1}{\vec{\epsilon}_1}) + k_q \vec{\epsilon}_1] \delta_x$. Consider-

here $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (21) yields $\dot{V} \leq k_q \vec{\epsilon}_1 \delta_x - \frac{\kappa - 0.382 \vec{k}_{12}}{k_q \vec{\epsilon}_1$ Eased on Lemma 2, we know $\bar{\vartheta}_1 A_{fx} \leq |\bar{\vartheta}_1| \bar{\vartheta}_x \leq |\bar{\vartheta}_1 \tanh(\frac{\bar{\vartheta}_1}{\bar{\epsilon}_1}) + k_q \bar{\epsilon}_1] \bar{\vartheta}_x$ $\bar{\vartheta}_1 A_{fx} \leq |\bar{\vartheta}_1| \bar{\vartheta}_x \leq |\bar{\vartheta}_1 \tanh(\frac{\bar{\vartheta}_1}{\bar{\epsilon}_1}) + k_q \bar{\epsilon}_1] \bar{\vartheta}_x$ $\bar{\vartheta}_1 A_{fx} \leq |\bar{\vartheta}_1| \bar{\vartheta}_x \leq |\bar{\vartheta}_1 \tanh(\frac{\bar{\vartheta}_1}{\bar{\epsilon}_1}) + k_q \bar{\epsilon}_1] \bar{\vartheta}_x$. Con:

ing $\begin{bmatrix} \bar{k}_{11} & \bar{k}_{11} \\ \bar{k}_{11} & \bar{k}_{11} + \bar{k}_{12} \end{bmatrix} = A^T \begin{bmatrix} \bar{k}_{11} & 0 \\ 0 & \bar{k}_{12} \end{bmatrix} A$, where $A = \begin{$ *V*_{*V*} $\delta_1 A_{fx} \le |\delta_1| \delta_x \le [\delta_1 \tanh(\frac{\sigma_1}{\epsilon_1}) + k_q \bar{\epsilon}_1] \delta_x$. Consider-
 $\begin{bmatrix} \bar{k}_{11} & 0 \\ 0 & \bar{k}_{12} \end{bmatrix}$ *A*, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (21) yields $\dot{V} \le k_q \bar{\epsilon}_1 \delta_x - h(\bar{\epsilon}) ||^2 = k_q \bar{\epsilon}_1 \delta_x - 0.382 \bar{k}_{12} ||$ $\leq |\theta|$
wher
 $\frac{1}{2} \delta_x - \sqrt{\frac{k_q}{0.3}}$
 $\int \tan^{-1} 12x^2$ $\frac{d}{dt} [\bar{k}_{11} \bar{k}_{11} + \bar{k}_{12}]$ = Λ [0 \bar{k}_{12}] Λ , where $\Lambda = [0 \ 1]$, (21) Jielas $V \ge \frac{\kappa_q \epsilon_1 \sigma_x}{2}$
 $\frac{d}{dt} (\Lambda^T \Lambda) \min{\{\bar{k}_{11}, \bar{k}_{12}\}}$ ||tanh $(\bar{\zeta})$ || $^2 = k_q \bar{\epsilon}_1 \delta_x - 0.382 \bar{k}_{12}$ ||tanh $(\bar{\zeta})$ || 2 . From *k̄* from L_0 $\frac{\delta_x}{\overline{k}_{12}}$ < 1. When $\|\tanh(\bar{\zeta})\|$ > $\sqrt{\frac{\delta_x}{\overline{k}_{12}}}$

Lemma 1 we have $\chi \|\bar{\zeta}\| \le \|\tanh(\bar{x})\|$
 $\dot{V} \le -0.382\bar{k}_{12}\lambda$

om (20) we have $\frac{\chi}{2} \|\bar{\zeta}\|^2 \le V \le \lambda$
 $V + k_q \bar{\epsilon}_1 \delta_x$. Integrating it yields

$$
\dot{V} \le -0.382\bar{k}_{12}\chi^2 \|\bar{\xi}\|^2 + k_q \bar{\varepsilon}_1 \delta_x \tag{22}
$$

 $rac{1}{2}(1+\frac{\kappa_{12}}{\bar{k}_{11}})$ 11 Then f
Beside
 $-\frac{0.382\bar{k}}{k}$ rom
 s, f_1 2 $\dot{V} \le -0.$

20) we have $\frac{\chi}{2} ||\vec{\xi}||^2$
 $\dot{V} + k_q \bar{\varepsilon}_1 \delta_x$. Integrating
 $\frac{\chi}{2} ||\vec{\xi}||^2 \le V \le [V(0) - \frac{1}{2}||\vec{\xi}||^2$ \leq ||tanht
 $\frac{3}{2k_{12}}\chi^2$ ||
 $V \leq \chi_2$ |

yields
 $k_q \bar{\varepsilon}_1 \delta_x \chi_2$ $0.82\bar{k}_{12}\lambda$
 $\leq V \leq$
 $\frac{k_q\bar{\varepsilon}_1\delta}{0.382\bar{k}}$ 122 −0*.*382*k̄* $\frac{12x^2}{x^2}$ Ί χ < 1, it
 $\frac{1}{2}(1 + \frac{\overline{k}}{\overline{k}})$
 $k_q \bar{\varepsilon}_1 \delta_x \chi_2$ (22)
 $=\frac{1}{2}(1+\frac{\bar{k}_{12}}{\bar{k}_{11}})$. Then $\dot{V} \leq$
 $\frac{k_q \bar{\epsilon}_1 \delta_x \chi_2}{0.382 \bar{k}_{12} \chi^2}$ (23) Besides, from (20) we have $\frac{x}{2} ||\xi||^2 \le V \le \chi_2 ||\xi||^2$, where $\chi_2 = \frac{1}{2}(1 + \frac{\bar{k}_{12}}{\bar{k}_{11}})$. Then $\dot{V} \le -\frac{0.382\bar{k}_{12}\chi^2}{\chi_2}V + k_q\bar{\epsilon}_1\delta_x$. Integrating it yields
 $\frac{\chi}{2} ||\xi||^2 \le V \le [V(0) - \frac{k_q\bar{\epsilon}_1\delta_x\chi_2}{0.$

$$
\frac{\chi}{2} ||\vec{\xi}||^2 \le V \le [V(0) - \frac{k_q \bar{\epsilon}_1 \delta_x \chi_2}{0.382 \bar{k}_{12} \chi^2}] e^{\frac{-0.382 \bar{k}_{12} \chi^2}{\chi_2}} t + \frac{k_q \bar{\epsilon}_1 \delta_x \chi_2}{0.382 \bar{k}_{12} \chi^2}
$$
(23)
entially converges to the set $\mathbb{Z}_{v1} = {\{\vec{\xi} | ||\vec{\xi}|| \le \sqrt{\frac{0.73 \bar{\epsilon}_1 \delta_x}{\chi^3} (\frac{1}{\bar{k}_{12}} + \frac{1}{\bar{k}_{11}}) }\}}.$ Due

the *F*₁ also converges to the set \mathbb{Z}_{ν} ₁ = { $\bar{\zeta}$ || $\bar{\zeta}$ | = − $\bar{k}_1 \bar{k}_2$ ² and $|\bar{\zeta}_1|^2 \le V \le V$ = $\frac{1}{2}$ (1 + $\frac{1}{k_1}$). Then $V \le \frac{1}{2}$
 $-\frac{0.382\bar{k}_{12} \chi^2}{\chi_2}$ *V* + $k_q \bar{\epsilon}_1 \delta_x$. Integr $-\frac{0.382\bar{k}_{12}\chi^2}{\chi_2}V + k_q\bar{\varepsilon}_1\delta_x.$ Int
 $\frac{\chi}{2}||\bar{\varsigma}||^2 \le V \le$

So $\bar{\varsigma}$ exponentially converthat $|\dot{\bar{\varsigma}}_1| \le ||\bar{\varsigma}||, \dot{\bar{\varsigma}}_1$ also concerning $\dot{\bar{\varsigma}}_1$ yields

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Singularity-Free Path Following Control for Miniature Unmand Helicopters
\n
$$
|\bar{\zeta}_1| \le e^{-\bar{k}_{12}t} (|\bar{\zeta}_1(0)| - \sqrt{\frac{2k_q \bar{\epsilon}_1 \delta_x \chi_2}{0.382 \bar{k}_{12}^3 \chi^3}}) + \sqrt{\frac{2k_q \bar{\epsilon}_1 \delta_x \chi_2}{0.382 \bar{k}_{12}^3 \chi^3}}
$$
\n(24)
\nThus $\bar{\zeta}_1$ converges to set $\mathbb{Z}_{p1} = \{|\bar{\zeta}_1| \le \frac{1}{k_{12}} \sqrt{\frac{0.73 \bar{\epsilon}_1 \delta_x}{\chi^3} (\frac{1}{k_{12}} + \frac{1}{k_{11}})}\}$. Similarly, $\bar{\zeta}_2$ and $\bar{\zeta}_2$

Singularity-Free Path Following Control for Miniature Unmanned Helicopters
 $|\bar{\zeta}_1| \le e^{-\bar{k}_{12}t} (|\bar{\zeta}_1(0)| - \sqrt{\frac{2k_q \bar{\varepsilon}_1 \delta_x \chi_2}{0.382 \bar{k}_{12}^3 \chi^3}}) + \sqrt{\frac{2k_q \bar{\varepsilon}_1 \delta_x \chi_2}{0.382 \bar{k}_{12}^3 \chi^3}}$ (24)

Thus $\bar{\zeta}_1$ *k k k k k k k*^{*I*} 2
k^{*I*} *k* $\frac{2}{\bar{k}}$ $|\bar{\zeta}_1| \leq e^{-\bar{k}_{12}t}(|\bar{\zeta}_1(0)| - \sqrt{\frac{2k_q\bar{\epsilon}}{0.382}}$

Thus $\bar{\zeta}_1$ converges to set $\mathbb{Z}_{p1} = {|\bar{\zeta}_1| \leq \frac{1}{k_{12}}} \sqrt{\frac{2k_q\bar{\epsilon}_1}{k_{21}} \sqrt{\frac{2k_q\bar{\epsilon}_2}{k_{12}} \left(\frac{1}{k_{21}} + \frac{1}{k_{21}}\right)}}$
 ${\frac{2}{k_1}}$
 ${\frac{2}{k_2}}$
 ${\frac$ √**k**^{*z*} \overline{k} $\frac{1}{\bar{k}}$ to $ve = \frac{1}{2} \delta$
 V
 V $\int \csc^2 \vec{Z}_{p1} = \{ |\vec{\zeta}_1| \leq \frac{1}{\bar{k}_{12}} \sqrt{\frac{0.73\bar{\varepsilon}_1 \delta_x}{\chi^3} (\frac{1}{\bar{k}_{12}} + \frac{1}{\bar{k}_{11}})} \}$. Similarly, $\bar{\zeta}_2$ and $\dot{\bar{\zeta}}_2$
 $\int \csc^2 \vec{Z}_{p2} = \{ \bar{\zeta}_2 | |\vec{\zeta}_2| \leq \frac{1}{\bar{k}_{22}} \sqrt{\frac{0.73\bar{\varepsilon}_2 \delta_y}{\chi^3} (\frac{1}{\$

 $\frac{\varepsilon_2 o_y}{\bar{k}_{22}}(\frac{1}{\bar{k}_{22}}+\frac{1}{\bar{k}_{21}})\},$ respectively. $\{\dot{\bar{\zeta}}_2 | \|\dot{\bar{\zeta}}_2\| \leq \sqrt{\frac{0.73\bar{\varepsilon}_2 \delta_y}{\chi^3} (\frac{1}{\bar{k}_{22}} + \frac{1}{\bar{k}_{21}})}\}$, respective
For $\bar{\zeta}_3$, choose Lyapunov function $V_1 = \dot{V}_1 = -\bar{k}_{31} \bar{\zeta}_3 \tanh(\bar{\zeta}_3) - \bar{\zeta}_3$
From Lemma [2,](#page-2-1) $\bar{\zeta}_3 \Delta_{fz} \leq |\bar{\zeta$

 $=\frac{1}{2}\bar{\zeta}_3^2$, its derivative is

$$
\frac{\delta_y}{\delta_z}(\frac{1}{\overline{k}_{22}} + \frac{1}{\overline{k}_{21}})\},\text{ respectively.}
$$
\n
$$
\text{Lyapunov function } V_1 = \frac{1}{2}\overline{\varsigma}_3^2, \text{ its derivative is}
$$
\n
$$
\dot{V}_1 = -\overline{k}_{31}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) - \overline{\varsigma}_3\tanh\left(\frac{\overline{\varsigma}_3}{\overline{\epsilon}_3}\right)\delta_z + \overline{\varsigma}_3\Delta_{fz} \tag{25}
$$
\n
$$
\Delta_{fz} \le |\overline{\varsigma}_3|\delta_z \le (\overline{\varsigma}_3\tanh(\frac{\overline{\varsigma}_3}{\overline{\epsilon}_3}) + k_q\overline{\varsigma}_3)\delta_z, \text{ thus } \dot{V}_1 \le -\overline{k}_{31}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{32}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{33}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{33}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{34}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{34}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{34}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{35}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{36}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{37}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{38}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{38}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{38}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{38}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{39}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{31}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{31}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3) + \overline{k}_{31}\overline{\varsigma}_3\tanh(\overline{\varsigma}_3
$$

3 $\{\dot{\bar{\zeta}}_2 | \|\dot{\bar{\zeta}}_2\| \leq \sqrt{\frac{0.75\varepsilon_2 \delta_y}{\chi^3}} (\frac{1}{\bar{k}_{22}} + \frac{1}{\bar{k}_{21}})\}\$ $\{\dot{\bar{\zeta}}_2 | \|\dot{\bar{\zeta}}_2\| \leq \sqrt{\frac{0.75\varepsilon_2 \delta_y}{\chi^3}} (\frac{1}{\bar{k}_{22}} + \frac{1}{\bar{k}_{21}})\}\$ $\{\dot{\bar{\zeta}}_2 | \|\dot{\bar{\zeta}}_2\| \leq \sqrt{\frac{0.75\varepsilon_2 \delta_y}{\chi^3}} (\frac{1}{\bar{k}_{22}} + \frac{1}{\bar{k}_{21}})\}\$, respectively.
For $\bar{\zeta}_3$, choose Lyapunov function $V_1 = \frac{1}{2}\bar{\zeta}$
 $\bar{V}_1 = -\bar{k}_{31}\bar{\zeta}_3 \tanh(\bar{\zeta}_3) - \bar{\zeta}_3 \tanh(\frac{\bar{\zeta}_3}{\bar{\zeta}_3})$
Fr $\int_{3}^{\infty} \frac{z^2}{z^3}$, its derivative is

anh $\left(\frac{\bar{\zeta}_3}{\bar{\epsilon}_3}\right) \delta_z + \bar{\zeta}_3 A_{fz}$
 $+ k_q \bar{\epsilon}_3 \delta_z$, thus $\dot{V}_1 \le -\frac{1}{31} \chi \bar{\zeta}_3^2 + k_q \bar{\epsilon}_3 \delta_z = -2\bar{k}$ (25)
 $\bar{k}_{31}\bar{\zeta}_{3}$ tanh($\bar{\zeta}_{3}$) +
 $\bar{k}_{31}\chi V_{1} + k_{q}\bar{\varepsilon}_{3}\delta_{z}$. Integrating it yields $\bar{\zeta}_3$ tanh $\left(\frac{\bar{\zeta}_3}{\bar{\epsilon}_3}\right) \delta_z + \bar{\zeta}_3 A_{fz}$
 $\frac{\bar{\zeta}_3}{\bar{\epsilon}_3}$) + $k_q \bar{\epsilon}_3$) δ_z , thus $\dot{V}_1 \le -\bar{k}_{31} \chi \bar{\zeta}_3^2 + k_q \bar{\epsilon}_3 \delta_z = -\frac{\bar{\zeta}_3}{\bar{\epsilon}_3}$
 $\frac{k_q \bar{\epsilon}_3 \delta_z}{\bar{\epsilon}_3}$]e<sup>-2 $\bar{k}_{31} \chi t + \frac{k_q \bar{\epsilon}_3 \delta_z}{\bar{\$ $\bar{\zeta}_3$ ^t
 $(\bar{\zeta}_3)$
 $(\bar{\zeta}_3)$
 $(\bar{\zeta}_4)$
 $(\bar{\zeta}_5)$ \bar{k}_{31}
 \bar{k}_{31}
 $\bar{\epsilon}_3 \delta_3$
 $\bar{k}_{31} \lambda$ \overline{c} _z =
 $\frac{k_q}{2\overline{k}}$ $3A_{fz}$
 $1 \leq$
 $5A_{fz}$
 $5B_{fz}$
 $5A_{fz}$ From Lemma 2, $\bar{\zeta}_3 A_{fz} \le |\bar{\zeta}_3| \delta_z \le (\bar{\zeta}_3 \tanh(\frac{\bar{\zeta}_3}{\bar{\epsilon}_3}) + k_q \bar{\epsilon}_3) \delta_z$, thu
 $k_q \bar{\epsilon}_3 \delta_z$. Based on Lemma 1 we have $\dot{V}_1 \le -\bar{k}_{31} \chi \bar{\zeta}_3^2 + k_q \bar{\epsilon}_3$

Integrating it yields
 $\frac{1}{2} \bar{\zeta}_3^2 = V_1(t) \le [V_$ $\vec{a}_3 \delta_z$. Based on Lemma 1 we have $\vec{V}_1 \le -\bar{k}_{31} \chi \vec{\varsigma}_3^2 + k_q \bar{\varepsilon}_3 \delta_z = -2\bar{k}_{31} \chi V_1 + k_q \bar{\varepsilon}_3 \delta_z$.

egrating it yields
 $\frac{1}{2} \bar{\varsigma}_3^2 = V_1(t) \le [V_1(0) - \frac{k_q \bar{\varepsilon}_3 \delta_z}{2\bar{k}_{31} \chi}]e^{-2\bar{k}_{31} \chi t} + \frac{k_q \bar{\varepsilon}_3 \$

Lemma 1 we have
$$
\dot{V}_1 \le -\bar{k}_{31}\chi\bar{c}_3^2 + k_q\bar{\varepsilon}_3\delta_z = -2\bar{k}_{31}\chi V_1 + k_q\bar{\varepsilon}_3\delta_z
$$

\nds
\n
$$
\frac{1}{2}\bar{c}_3^2 = V_1(t) \le [V_1(0) - \frac{k_q\bar{\varepsilon}_3\delta_z}{2\bar{k}_{31}\chi}]e^{-2\bar{k}_{31}\chi t} + \frac{k_q\bar{\varepsilon}_3\delta_z}{2\bar{k}_{31}\chi}
$$
\n(26)
\n
$$
\text{ially converges to the set } \mathbb{Z}_3 = {\bar{c}_3}||\bar{c}_3| \le \sqrt{\frac{0.2785\bar{\varepsilon}_3\delta_z}{\bar{k}_{31}\chi}}.
$$
\n(b) hence $\text{small by increasing } \bar{k}_{21}, \bar{k}_{22}, \bar{k}_{31}$ and decreasing $\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3$.

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Integrating it yields
 $\frac{1}{2}\bar{c}_3^2 = V_1(t) \leq [V_1(0) - \frac{k_q \bar{\epsilon}_3 \delta_z}{2\bar{k}_{31} \chi} e^{-2\bar{k}_{31} \chi t} + \frac{k_q \bar{\epsilon}_3 \delta_z}{2\bar{k}_{31} \chi}$ (26)

Thus \bar{c}_3 exponentially converges to the set $\mathbb{Z}_3 = {\bar{c}_3} |\bar{c}_3| \leq \sqrt{\frac{0.2785\bar{\epsilon}_3 \$ the control law helicopter keeps in the neighbourhood of \bar{P}_r .
(*ii*) From Remark 2 and proof of (*i*) it yields that \bar{G} keeps invertible. When $\bar{u}_{cx} \neq 0$ $\frac{1}{2}\bar{\zeta}_3^2 = V_1(t) \le [V_1(0) - \frac{k_q \bar{\epsilon}_3 \delta_z}{2\bar{k}_{31} \chi} e^{-2\bar{k}_{31} \chi t} + \frac{k_q \bar{\epsilon}_3 \delta_z}{2\bar{k}_{31} \chi}$ $\frac{1}{2}\bar{\zeta}_3^2 = V_1(t) \le [V_1(0) - \frac{k_q \bar{\epsilon}_3 \delta_z}{2\bar{k}_{31} \chi} e^{-2\bar{k}_{31} \chi t} + \frac{k_q \bar{\epsilon}_3 \delta_z}{2\bar{k}_{31} \chi}$ $\frac{1}{2}\bar{\zeta}_3^2 = V_1(t) \le [V_1(0) - \frac{k_q \bar{\epsilon}_3 \delta_z}{2\bar{k}_{31} \chi} e^{-2\bar{k}_{31} \chi t} + \frac{k_q \bar{\epsilon}_3 \delta_z}{2\bar{k}_{31} \chi}$ (26)

Thus $\bar{\zeta}_3$ exponentially converges to the set $\mathbb{Z}_3 = {\bar{\zeta}_3 ||\bar{\zeta}_3| \le \sqrt{\frac{0.2785\bar{\epsilon}_3 \delta_z}{\bar{k}_{31} \chi}}}$.

Abov Thus $\bar{\zeta}_3$ exponentially converges to the set $\mathbb{Z}_3 = {\bar{\zeta}_3} ||\bar{\zeta}_3| \le \sqrt{\frac{0.2785\bar{\epsilon}_3 \delta_z}{\bar{k}_{31} \chi}}$.

Thus $\bar{\zeta}_3$ exponentially converges to the set $\mathbb{Z}_3 = {\bar{\zeta}_3} ||\bar{\zeta}_3| \le \sqrt{\frac{0.2785\bar{\epsilon}_3 \delta_z}{\bar{k}_{31} \chi$ Above sets can be made small by increasing \bar{k}_{21} , \bar{k}_{22} , \bar{k}_{31} and decreasing $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$, $\bar{\varepsilon}_3$.

Thus $\bar{\zeta}_3$ exponentially converges to the set $\mathbb{Z}_3 = {\bar{\zeta}_3}|\bar{\zeta}_3| \leq \sqrt{\frac{0.2785\bar{\varepsilon}_3 \delta_z}{\bar{k}_{31} \chi}}$.

Above sets can be made small by increasing $\bar{k}_{21}, \bar{k}_{22}, \bar{k}_{31}$ and decreasing $\bar{\varepsilon}_1$

Since $p(0) \$ quaternion. So we only need to consider the singularity when $\bar{u}_c = [0, 0, \bar{u}_{c}^{\dagger}]^T$. Left-Thus $\bar{\zeta}_3$ exponentially converges to the set $\mathbb{Z}_3 = {\bar{\zeta}_3}$
Above sets can be made small by increasing \bar{k}_{21}, \bar{k}_2
Since $p(0) \in \bar{P}_r$ and $v(0) = 0$, we have $\bar{\zeta}_1(0) = \dot{\zeta}_1(0)$
the control law helicopter $(z_2, k_{31} \text{ and decreasing } \bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3,$
 $(\bar{\varepsilon}_2, \bar{\varepsilon}_3) = \bar{\zeta}_2(0) = \dot{\zeta}_2(0) = 0.$ So under

od of $\bar{\mathcal{P}}_r$.

keeps invertible. When $\bar{u}_{cx} \neq 0$

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ity when $\bar{u}_c = [0, 0,$, ϵ_2 ,

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 $_{11}+\delta_x$ Above sets can be made small by increasing $\bar{k}_{21}, \bar{k}_{22}, \bar{k}_{31}$ and decreasing $\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3$.

Since $p(0) \in \bar{\mathcal{P}}_r$ and $v(0) = \mathbf{0}$, we have $\bar{\zeta}_1(0) = \dot{\bar{\zeta}}_1(0) = \bar{\zeta}_2(0) = \dot{\bar{\zeta}}_2(0) = 0$. So und ter 1

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ich
 u_1, b_1
 $\frac{1}{11} + \delta_x$ means no singularity occurs in deriving the desired unit-quaternion with [\(18\)](#page-6-2). the control law helicopter keeps in the neighbourhood of \tilde{P}_r .

(*ii*) From Remark 2 and proof of (*i*) it yields that \bar{G} keeps invertible. When $\bar{u}_{cx} \neq 0$

or $\bar{u}_{cy} \neq 0$, $\bar{u}_c \notin \mathcal{L}$, which means n (*ii*) From Remark 2 and proof of (*i*) it yields that \bar{G} keeps invertible.

or $\bar{u}_{cy} \neq 0$, $\bar{u}_c \notin \mathcal{L}$, which means no singularity occurs in deriving th

quaternion. So we only need to consider the singular eed to consider the singularity when
 b_1, c_1] yields $\bar{u}_{cz} = m(g + \frac{\bar{v}_1}{c_1})$. From
 $\frac{\bar{v}_0}{\bar{v}_1}$). Since $\bar{k}_{11} < \frac{|c_1|g - \delta_x}{2}$, we get $\bar{u}_{cz} >$

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be helicopter pos J. \bar{u}_c (1
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gn b) δ_x b_1, c_1] yields $\bar{u}_{cz} = m(g)$
 $\frac{\delta_x}{\delta_y}$). Since $\bar{k}_{11} < \frac{|c_1|g - \delta_x}{2}$

turs in deriving the desisor be helicopter position a

trol u_c is given by (12) a
 $-k_{11}(\tanh(\zeta_{k1}) + \tanh(\zeta_{k2}) + \tanh(\zeta_{k1})$ (1
 \cdot 0
 \cdot m
 mn
 gm
 δ_x
 δ_y

when $||P - P_d|| \le \bar{\varepsilon}$, control u_c is given by (12) and v_c is designed as \mathcal{P}_r following Define P to be helicopter position and $\bar{\varepsilon}$ to be a small positive constant,

$$
-\frac{2\bar{k}_{11}+\delta_{x}}{|c_{1}|}\text{. Since }\bar{k}_{11} < \frac{|c_{1}|g-\delta_{x}}{2}\text{, we get }\bar{u}_{cz} > 0 \text{ and }\bar{u}_{c} \notin \mathcal{L}\text{, which}
$$
\nity occurs in deriving the desired unit-quaternion with (18).

\nne *P* to be helicopter position and $\bar{\varepsilon}$ to be a small positive constant,

\n $\bar{\varepsilon}$, control u_{c} is given by (12) and v_{c} is designed as

\n
$$
\bar{\varepsilon}
$$
\n
$$
v_{c} = \begin{bmatrix} -k_{11}(\tanh(\varsigma_{k1}) + \tanh(\xi_{1})) - \tanh(\frac{\theta_{1}}{\varepsilon_{1}})\delta_{x} \\ -k_{21}(\tanh(\varsigma_{k2}) + \tanh(\xi_{2})) - \tanh(\frac{\theta_{2}}{\varepsilon_{2}})\delta_{y} \\ -k_{31}\tanh(\varsigma_{3}) - \tanh(\frac{\varsigma_{3}}{\varepsilon_{3}})\delta_{z} \end{bmatrix}
$$
\n(27)

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where $\zeta_{k1} = k_{12}\zeta_1 + \dot{\zeta}_1$, $\zeta_{k2} = k_{22}\zeta_2 + \dot{\zeta}_2$, $\vartheta_1 = \tanh(\zeta_{k1}) + \tanh(\dot{\zeta}_1) + \frac{k_{12}}{k_{11}}\dot{\zeta}_1$, $\vartheta_2 =$ there $\varsigma_{k1} = k_{12}\varsigma_1 + \varsigma_1$, $\varsigma_{k2} = k_{22}\varsigma_2 + \varsigma_2$, $\vartheta_1 = \tanh(\varsigma_{k1}) + \tanh(\zeta_1) + \frac{k_{12}}{k_{11}}\zeta_1$, $\vartheta_2 = \tanh(\varsigma_{k2}) + \tanh(\zeta_2) + \frac{k_{22}}{k_{11}}\zeta_2$. $k_{11}, k_{12}, k_{21}, k_{22}, k_{31}, \varepsilon_1, \varepsilon_2$ and ε_3 are positi k_2) + $\frac{k_{22}}{k_{21}}$ satisfying *k*₁₂ ≤ *k*₁₁ < $\frac{k_{22}}{2}$ *k*₂₂ ≤ *z*₂ + *ξ*₂₂, *θ*₁ = tanh(*ς_{k1}*) + tanh(*ξ*₁) + $\frac{k}{k}$
 *k*₁₂ ≤ *k*₁₁ < $\frac{|c|g - \delta_x}{2}$, *k*₂₂ ≤ *k*₂₁, *ε*₁, ε₁ < $\frac{0.382k_{12}}{k_a \delta_x}$, ε_2 < $\frac{0$ $nh(\zeta, \varepsilon_2)$
382*i*
 $k_q \delta_x$ *,* $\frac{1}{2}$ and $\frac{1}{2}$ *k*_. *k*₁) +
*i*₁) +
*i*₀*sx*
k_qδ_y

$$
k_{12} \le k_{11} < \frac{|c|g - \delta_x}{2}, k_{22} \le k_{21}, \varepsilon_1 < \frac{0.382k_{12}}{k_a \delta_x}, \varepsilon_2 < \frac{0.382k_{22}}{k_a \delta_y} \tag{28}
$$

From Remark [2,](#page-6-3) ν is approximately tangent to P_r at switching instant. So under the control law heliconter keens in the neighbourhood of \mathcal{P} . Based on Remark 1 it vields control law helicopter keeps in the neighbourhood of P_r . Based on Remark [1](#page-4-2) it yields that *G* keeps invertible. The constraint for k_{11} in [\(28\)](#page-9-0) guarantees that no singularity occurs in deriving desired unit-quaternion. **Attitude controller design** Define attitude tracking error $Q_e = [\mu_e, \mathbf{q}_e^T]^T = Q_e^{-1} \otimes Q_e$
 Attitude controller design Define attitude tracking error $Q_e = [\mu_e, \mathbf{q}_e^T]^T = Q_e^{-1} \otimes Q_e$
 Attitude controller design Define *ely tangent to* P_r , at switching instant. So under the
 e neighbourhood of P_r . Based on Remark 1 it yields
 straint for k_{11} in (28) guarantees that no singularity
 uaternion.
 e attitude tracking error

[\[15\]](#page-14-14), its derivative is *c* attitude trackin
 c \dot{q}_e , $\dot{q}_e = \frac{1}{2} (\mu_e I)$
 c and be $\dot{\omega}_c$ can be $c = 2\Theta^T (Q_c) \dot{Q}$

$$
\dot{\mu}_e = -\frac{1}{2}\boldsymbol{q}_e^T\boldsymbol{\omega}_e, \ \dot{\boldsymbol{q}}_e = \frac{1}{2}(\mu_e I_3 + S(\boldsymbol{q}_e))\boldsymbol{\omega}_e \tag{29}
$$

 $\dot{\mu}_e = -\frac{1}{2} \mathbf{q}_e^T \boldsymbol{\omega}_e, \ \dot{\mathbf{q}}_e = \frac{1}{2} (\mu_e I_3 + S(\mathbf{q}_e)) \boldsymbol{\omega}_e$ (29)
where $\boldsymbol{\omega}_e = \boldsymbol{\omega} - R^T(Q_e) \boldsymbol{\omega}_e$ is the angular velocity error and $\boldsymbol{\omega}_c$ is the desired angular
velocity. From [\(3\)](#page-3-2) $\boldsymbol{\omega}_c$ and it al. $\dot{\mu}_e = -\frac{R^T (Q_e) \omega_e}{2}$ is t

(3) ω_c and its d
 $\dot{\omega}_c = 2\Theta^T (Q_c) \ddot{Q}$ $\frac{1}{2}$ $q_e^T \omega_e$, $\dot{q}_e = \frac{1}{2}$
 c he angular velocity $\dot{\omega}_c$ c:
 $\omega_c = 2\Theta^T (q_c) \dot{Q}$ $c_c(\mu_e I_3 + S(q_e))\omega_e$
 cosity error and ω_c is the posity error and ω_c is the position of Q_c) \dot{Q}_c
 $c_c = 2\Theta^T(Q_c)[\dot{Q}_c - \dot{\Theta}]$ where **ω**
velocity.
where Θ $e = \omega -$
From (3
 $\dot{\omega}_c$
 $c = \Theta(\dot{Q})$ $R^T(Q_e)\omega_c$ is the angulity $\omega_c =$
 $= 2\Theta^T(Q_c)\ddot{Q}_c + 2\dot{\Theta}^T$
 ω_c). From Lemma [3,](#page-2-3) \dot{Q} *c* and ω_c is the desired angular
 c $\dot{\omega}_c$ can be derived as
 $2\Theta^T(Q_c)\dot{Q}_c$ (30)
 $(Q_c)\dot{Q}_c = 2\Theta^T(Q_c)[\ddot{Q}_c - \dot{\Theta}_c\omega_c]$ (31)
 Q_c and \ddot{Q}_c can be obtained by a command filter

$$
\boldsymbol{\omega}_c = 2\Theta^T(\mathcal{Q}_c)\dot{\mathcal{Q}}_c \tag{30}
$$

$$
\omega_c = 2\Theta^T(Q_c)\dot{Q}_c
$$
\n
$$
\dot{\omega}_c = 2\Theta^T(Q_c)\ddot{Q}_c + 2\dot{\Theta}^T(Q_c)\dot{Q}_c = 2\Theta^T(Q_c)[\ddot{Q}_c - \dot{\Theta}_c\omega_c]
$$
\n(31)

instead of calculating them accurately. Assign Lyapunov function $\omega_c = 2\Theta^T(Q_c)\dot{Q}_c + 2\dot{\Theta}^T$
From Lemma 3, \dot{Q}_c
Ig them accurately
 $L = k_Q[(1 - k_\mu \mu_e)]$ $\dot{Q}_c = 2\Theta^T (Q_c)[\dot{Q}_c - \dot{\Theta}_c \omega_c]$ (30)
 i \dot{Q}_c can be obtained by a command filter
 ign Lyapunov function
 ign Lyapunov function
 e q_e] = 2*k*_Q(1 - *k_µµ_e*) (32) $\dot{\omega}_c = 2\Theta^T(Q_c)\dot{Q}_c + 2\dot{\Theta}^T(Q_c)\dot{Q}_c = 2\Theta^T(Q_c)[\dot{Q}_c - \dot{\Theta}_c\omega_c]$ (31)
where $\dot{\Theta}_c = \Theta(\dot{Q}_c)$. From Lemma 3, \dot{Q}_c and \ddot{Q}_c can be obtained by a command filter
instead of calculating them accurately. Assign L *Q*
gn
µ
µ

$$
L = k_Q[(1 - k_{\mu}\mu_e)^2 + \mathbf{q}_e^T \mathbf{q}_e] = 2k_Q(1 - k_{\mu}\mu_e)
$$
 (32)

 $\dot{\omega}_c = 2\Theta^T (Q_c) \dot{Q}_c + 2\dot{\Theta}^T (Q_c) \dot{Q}_c = 2\Theta^T (Q_c) [\dot{Q}_c - \dot{\Theta}_c \omega_c]$ (31)

where $\dot{\Theta}_c = \Theta(\dot{Q}_c)$. From Lemma 3, \dot{Q}_c and \ddot{Q}_c can be obtained by a command filter

instead of calculating them accurately. A punov function *e c* can be obtained by a command filter
 i Lyapunov function
 j $] = 2k_Q(1 - k_\mu \mu_e)$ (32)
 $= -1$ when $\mu_e < 0$. From (29), $\dot{L} = k_{\omega}k_{\mu}q_e$, where $k_{\omega} > 0$. Choose a Lya^r
 *f*_e *J***o**_e (33)),
։ճո
L $L = k_Q[(1 - k_\mu \mu_e)^2 + \mathbf{q}_e^T \mathbf{q}_e] = 2k_Q(1 - k_\mu \mu_e)$
 $k_\mu = 1$ when $\mu_e \ge 0$ and $k_\mu = -1$ when $\mu_e < 0$. From $n_e = [\bar{\omega}_{ex}, \bar{\omega}_{ey}, \bar{\omega}_{ez}]^T = \omega_e + k_\omega k_\mu \mathbf{q}_e$, where $k_\omega > 0$.
 $L_1 = L + \frac{1}{2} \bar{\omega}_e^T J \bar{\omega}_e$
 $L_1 = \bar{\omega}_e^T[-\$ $\kappa_{\mu} = 1$ when $\mu_{e} \ge 0$ and $\kappa_{e} = [\bar{\omega}_{ex}, \bar{\omega}_{ey}, \bar{\omega}_{ez}]^{T} = \sigma$
 $L_{1} = L + \frac{1}{2}$
 $= \bar{\omega}_{e}^{T}[-\omega \times J\omega + \tau_{1} + J(\xi_{e} + k_{\omega}k_{\mu}\dot{q}_{e}) + \Delta_{\tau}] - k_{Q}k_{\omega}q_{e}^{T}$ *e* θ , where θ *e* θ *e* θ *e* θ *e* θ

$$
L_1 = L + \frac{1}{2} \bar{\omega}_e^T J \bar{\omega}_e \tag{33}
$$

$$
\dot{L}_1 = L + \frac{1}{2} \bar{\boldsymbol{\omega}}_e^T J \bar{\boldsymbol{\omega}}_e \qquad (33)
$$
\n
$$
\dot{L}_1 = \bar{\boldsymbol{\omega}}_e^T [-\boldsymbol{\omega} \times J\boldsymbol{\omega} + \tau_1 + J(S(\boldsymbol{\omega}_e)R^T(Q_e)\boldsymbol{\omega}_c - R^T(Q_e)\boldsymbol{\omega}_c - R^T(Q_e)\boldsymbol{\omega}_c)]
$$
\n
$$
+ k_{\omega} k_{\mu} \dot{\boldsymbol{q}}_e) + \boldsymbol{\Delta}_\tau] - k_Q k_{\omega} \boldsymbol{q}_e^T \boldsymbol{q}_e + k_Q k_{\mu} \boldsymbol{q}_e^T \bar{\boldsymbol{\omega}}_e \qquad (34)
$$
\nintrol torque

\n
$$
\tau_1 = -k_{\tau} \bar{\boldsymbol{\omega}}_e + \boldsymbol{\omega} \times J\boldsymbol{\omega} - JS(\boldsymbol{\omega}_e)R^T(Q_e)\boldsymbol{\omega}_c + JR^T(Q_e)\boldsymbol{\omega}_c
$$

Design the control torque

$$
\dot{L}_1 = \bar{\omega}_e^T [-\omega \times J\omega + \tau_1 + J(S(\omega_e)R^T(Q_e)\omega_c - R^T(Q_e)\dot{\omega}_c \n+ k_{\omega}k_{\mu}\dot{q}_e) + \Delta_{\tau}]-k_{Q}k_{\omega}q_{e}^Tq_{e} + k_{Q}k_{\mu}q_{e}^T\bar{\omega}_e
$$
\nintrol torque

\n
$$
\tau_1 = -k_{\tau}\bar{\omega}_e + \omega \times J\omega - JS(\omega_e)R^T(Q_e)\omega_c + JR^T(Q_e)\dot{\omega}_c
$$
\n
$$
-k_{\omega}k_{\mu}J\dot{q}_e - \text{Tanh}(\frac{\bar{\omega}_e^T}{\epsilon_4})\gamma - k_{Q}k_{\mu}q_{e}
$$
\n(35)

where *ki* error Path Following Control for Miniature Unmanned Helicopters
 $\tau > 0$ and ϵ_4 is a small positive constant. Substituting τ_1 into [\(34\)](#page-9-2) yields *̄*

Singularity-Free Path Following Control for Miniature Unmanned Helicopters
\nwhere
$$
k_{\tau} > 0
$$
 and ϵ_4 is a small positive constant. Substituting τ_1 into (34) yields
\n
$$
\dot{L}_1 = -k_Q k_{\omega} \mathbf{q}_{e}^T \mathbf{q}_{e} - k_{\tau} \bar{\mathbf{\omega}}_e^T \bar{\mathbf{\omega}}_e - \bar{\mathbf{\omega}}_e^T \text{Tanh}(\frac{\bar{\mathbf{\omega}}_e^T}{\epsilon_4}) \mathbf{\gamma} + \bar{\mathbf{\omega}}_e^T \mathbf{\Delta}_\tau
$$
\n(36)
\n**Theorem 2** Take $k_{\omega} \leq \frac{4k_{\tau}}{\bar{\lambda}(J)}$, the attitude controller (35) guarantees that the attitude

Theorem 2 *Take* $k_{\omega} \leq \frac{4k_{\tau}}{\lambda(J)}$, the attitude controller (35) guarantees that the attitude tracking error q_e and angular velocity error ω_e are bounded and ultimately converge to neighbourhoods of the orig *to neighbourhoods of the origin.* $\dot{L}_1 = -k_Q k_\omega \mathbf{q}_e^T \mathbf{q}_e - k_\tau \bar{\mathbf{\omega}}_e^T \bar{\mathbf{\omega}}_e - \bar{\mathbf{\omega}}_e^T \text{Tanh}(\frac{\bar{\mathbf{\omega}}_e^T}{\epsilon_4}) \mathbf{\gamma} + \bar{\mathbf{\omega}}_e^T \mathbf{\Delta}_\tau$ (36)
 Theorem 2 Take $k_\omega \leq \frac{4k_\tau}{\bar{\lambda}(J)}$, the attitude controller (35) guarantees that the a $\dot{L}_1 = -k_Q k_{\omega} \mathbf{q}_{\epsilon}^T \mathbf{q}_{\epsilon} - k_{\tau} \bar{\mathbf{\omega}}_{\epsilon}^T \bar{\mathbf{\omega}}_{\epsilon} - \bar{\mathbf{\omega}}_{\epsilon}^T \text{Tanh}(\frac{\mathbf{\omega}_{\epsilon}^T}{\epsilon_4}) \gamma + \bar{\mathbf{\omega}}_{\epsilon}^T \mathbf{\Delta}_{\tau}$
 Theorem 2 Take $k_{\omega} \leq \frac{4k_{\tau}}{\lambda(r)}$, the attitude controller (35) guarantees tha **T**
tr
to *i* 2 *Take* $k_{\omega} \leq \frac{4k_{\tau}}{\lambda(t)}$, the attitude controller (35) guarantees that the attitude
 error \mathbf{q}_{e} and angular velocity error $\mathbf{\omega}_{e}$ are bounded and ultimately converge
 ourhoods of the origin.
 L
L
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L **i**erary $\mathbf{r} = \frac{4k_r}{\lambda(t)},$ the attitude controller (35) gu
 ing error \mathbf{q}_e *and angular velocity error* $\mathbf{\omega}_e$ *are bounded*
 ighbourhoods of the origin.
 f From Lemma 2, $\mathbf{\bar{\omega}}_e^T \mathbf{\Delta}_t \le \sum_{i=x,y,z} |\mathbf$ *ee*
ui
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i
t i
k_i to neighbour
 Proof From
 $\bar{\omega}_e^T$ Tanh $(\frac{\bar{\omega}_e}{\epsilon_4})$
 $\dot{L}_1 \leq -k_\zeta$

From [\(33\)](#page-9-4), \dot{L} $\bar{\boldsymbol{\omega}}_{e}^{T} \boldsymbol{\Delta}$
re d_{τ} :
 $\tau \bar{\boldsymbol{\omega}}_{e}^{T} \bar{\boldsymbol{\omega}}$
 $\tau \bar{\boldsymbol{\omega}}_{e}^{T} \bar{\boldsymbol{\omega}}$
 $\frac{k_{\omega}}{2}, \frac{2k_{\tau}}{2(k_{\tau} - k_{\tau})}$ *̄*

$$
\text{Tanh}(\frac{\omega_e}{\epsilon_4})\mathbf{y} + d_{\tau}, \text{ where } d_{\tau} = \sum_{i=x,y,z} k_q \epsilon_4 \gamma_i. \text{ Substituting it into (36) yields}
$$
\n
$$
\dot{L}_1 \le -k_Q k_\omega \mathbf{q}_\ell^T \mathbf{q}_e - k_\tau \bar{\mathbf{\omega}}_\ell^T \bar{\mathbf{\omega}}_e + d_\tau \le -k_Q k_\omega (1 - k_\mu \mu_e) - \frac{k_\tau}{\bar{\lambda}(J)} \bar{\mathbf{\omega}}_\ell^T J \bar{\mathbf{\omega}}_e + d_\tau \quad (37)
$$
\n
$$
\text{om (33), } \dot{L}_1 \le -\min\{\frac{k_\omega}{2}, \frac{2k_\tau}{\bar{\lambda}(J)}\} L_1 + d_\tau = -\frac{k_\omega}{2} L_1 + d_\tau. \text{ Integrating it gives}
$$

 $-\frac{k_{\alpha}}{2}$ − *k*

$$
q_e^T q_e - k_\tau \bar{\omega}_e^T \bar{\omega}_e + d_\tau \le -k_Q k_\omega (1 - k_\mu \mu_e) - \frac{\kappa_\tau}{\bar{\lambda}(J)} \bar{\omega}_e^T J \bar{\omega}_e + d_\tau \quad (37)
$$

\n
$$
\le -\min\{\frac{k_\omega}{2}, \frac{2k_\tau}{\bar{\lambda}(J)}\} L_1 + d_\tau = -\frac{k_\omega}{2} L_1 + d_\tau. \text{ Integrating it gives}
$$

\n
$$
L_1 \le (L_1(0) - \frac{2d_\tau}{k_\omega})e^{-\frac{k_\omega}{2}t} + \frac{2d_\tau}{k_\omega} \le L_1(0)e^{-\frac{k_\omega}{2}t} + \frac{2d_\tau}{k_\omega} \quad (38)
$$

\nwe know that $K_Q q_\ell^T q_e \le L_1$ and $\frac{\lambda(J)}{2} \bar{\omega}_e^T \bar{\omega}_e \le L_1$. Thus q_e and $\bar{\omega}_e$

 $\dot{L}_1 \leq -k_Q k_\omega q_e^T q_e - k_\tau \bar{\omega}_e^T \bar{\omega}_e + d_\tau \leq -k_Q k_\omega (1 - k_\mu \mu_e) - \frac{\kappa_\tau}{\bar{\lambda}(J)} \bar{\omega}_e^T J \bar{\omega}_e + d_\tau$ (37)

From [\(33\)](#page-9-4), $\dot{L}_1 \leq -\min\{\frac{k_\omega}{2}, \frac{2k_\tau}{\bar{\lambda}(J)}\} L_1 + d_\tau = -\frac{k_\omega}{2} L_1 + d_\tau$. Integrating it gives
 $L_1 \leq (L_1(0) - \$ are bounded and ultimately converge to the compact sets $\mathbb{C}_q = \{q_e | ||q_e|| \le \sqrt{\frac{2a}{k_0 k}}\}$ $\frac{d}{2d_i}$ $\frac{d}{d} \frac{\bar{\omega}_e}{\frac{2d_\tau}{k_0 k_\omega}}\}$ Also from (33) we kn
are bounded and ultim
and $\bar{C}_{\omega} = {\{\bar{\omega}_e | \|\bar{\omega}_e\| \leq \pi}$ √ that
 $\frac{c}{4d}$ $\frac{2d_{\tau}}{k_{\omega}}e^{-\frac{k_{\omega}}{2}t} + \frac{2d_{\tau}}{k_{\omega}} \le L_{1}(0)e^{-\frac{k_{\omega}}{2}t} + \frac{2d_{\tau}}{k_{\omega}}$ (38)

that $K_{Q}\boldsymbol{q}_{e}^{T}\boldsymbol{q}_{e} \le L_{1}$ and $\frac{\lambda(I)}{2}\bar{\boldsymbol{\omega}}_{e}^{T}\bar{\boldsymbol{\omega}}_{e} \le L_{1}$. Thus \boldsymbol{q}_{e} and $\bar{\boldsymbol{\omega}}_{e}$

converge to th are bounded and ultimately converge to the compact :

and $\bar{C}_{\omega} = {\bar{\omega}_e} ||\bar{\omega}_e|| \le \sqrt{\frac{4d_e}{\frac{\lambda}{2}(J)k_{\omega}}}$. Since $\omega_e = \bar{\omega}_e - k_{\omega}$
 $k_{\omega} ||\bar{q}_e||$, so ω_e converges to the set $C_{\omega} = {\bar{\omega}_e} ||\bar{\omega}_e|| \le$ √ \bigcup_{q}
e, Wi
 $\frac{2k_{\omega}d_{q}}{2k_{\omega}d_{q}}$ $\frac{k_{\omega}a_{\tau}}{k_{Q}}$ + √ q_e
 ω _d
 $\frac{1}{4d}$ $\left\|\boldsymbol{\omega}_{e}\right\| \leq \frac{4d_{\tau}}{(J)k_{\omega}}.$ *Remark 3* By taking k_Q *>>k_ω*, increasing k_Q , k_w and $\frac{\Delta(U)}{2} \bar{\omega}_e^T \bar{\omega}_e \le L_1$. Thus q_e and $\bar{\omega}_e$ are bounded and ultimately converge to the compact sets $\mathbb{C}_q = \{q_e | ||q_e|| \le \sqrt{\frac{2d_t}{k_Q k_o}}\}$ and $\bar{\math$ and $\bar{C}_{\omega} = {\bar{\omega}_e} ||\bar{\omega}_e|| \le$
 $k_{\omega} || \mathbf{q}_e ||$, so ω_e converge
 Remark 3 By taking k
 \mathbb{C}_{ω} can be made small.

Remark 3 By taking k_Q >> k_ω , increasing k_Q , k_w and decreasing ε_4 , the sets \mathbb{C}_q and \mathbb{C}_ω can be made small.

5 Simulation

A simulation is performed to verify the proposed controller. The helicopter parame-
ters are as follows [12]: $m = 7.4$ kg, $I_x = 0.16$ kgm², $I_y = 0.30$ kgm², $I_z = 0.32$ kgm², $I_{xz} = 0.05$ kgm², $l_m = 0.01$ m, $h_m = 0.14$ m, $l_t = 0.95$ m, $h_t = 0.05$ m, $M_a = L_b$ 5 **Simulation**

A simulation is performed to verify the proposed controller. The helicopter parame-

ters are as follows [12]: $m = 7.4$ kg, $I_x = 0.16$ kgm², $I_y = 0.30$ kgm², $I_z = 0.32$ kgm²,
 $I_{xz} = 0.05$ kgm², I The desired reference path P_r is a circular curve determined by *f f f f <i>f*** ***<i>x <i>f f <i>x <i>f <i>f* *<i>x* *<i>f**<i>x <i>f* *<i>f* *<i>x* *<i>f**<i>f***** *<i>f**<i>f***** *<i>f***** *<i>f**<i>f**<i>f**<i>f**<i>f**<i>f**<i>f**<i>f**<i>f**<i>f**<i>f**<i>f**<i>f*

$$
f_1(x, y, z) = z - 8 = 0
$$
, $f_2(x, y, z) = \frac{1}{5}(x^2 + y^2) - 5 = 0$

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Define $P_0 = [11, 10, 0]^T$ and choose $P_d = [5, 0, 8]^T$, we can obtain the sphere center *O*^{*O*} = $[11, 10, 0]^T$ and choose $P_d = \overline{O} = [11, 0, 0]^T$ and the temporary path \overline{P} $\overline{O} = [11, 0, 0]^T$ and the temporary path \overline{P}_r is planned by *f*₁(*x*_{*y*}) = [11, 10, 0]^{*T*} and choose $P_d = [5, 0, 8]$

[11, 0, 0]^{*T*} and the temporary path \bar{P}_r is plant
 $\bar{f}_1(x, y, z) = 4x + 3z - 44 = 0$, $\bar{f}_2(x, y, z) = \frac{1}{20}$ SET $P_0 = [11, 10, 0]^T$ and choose $P_d = [5, 0, 8]^T$, we can obtain the sphere center $\overline{O} = [11, 0, 0]^T$ and the temporary path \overline{P}_r is planned by
 $\overline{f}_1(x, y, z) = 4x + 3z - 44 = 0$, $\overline{f}_2(x, y, z) = \frac{1}{20}((x - 11)^2 + y^2 + z$

$$
\bar{f}_1(x, y, z) = 4x + 3z - 44 = 0, \ \bar{f}_2(x, y, z) = \frac{1}{20}((x - 11)^2 + y^2 + z^2) - 5 = 0
$$

S80

Define $P_0 = [11, 10, 0]^T$ and choose $P_d = [5, 0, 8]^T$, we can obtain the sphere center
 $\bar{O} = [11, 0, 0]^T$ and the temporary path \bar{P}_r is planned by
 $\bar{f}_1(x, y, z) = 4x + 3z - 44 = 0$, $\bar{f}_2(x, y, z) = \frac{1}{20}((x - 11)^2 +$ *k*₂ = [1, 1, 10, 0]^{*T*} and choose P_d = [5, 0, 8]^{*T*}, we can obtain the sphere center \overline{O} = [11, 0, 0]^{*T*} and the temporary path \overline{P}_r is planned by
 $\overline{f}_1(x, y, z) = 4x + 3z - 44 = 0$, $\overline{f}_2(x, y, z) = \frac{1$ Define $P_0 = [11, 10, 0]^T$ and choose $P_d = [3, 0, 8]^T$, we can obtain the sphere center $\bar{O} = [11, 0, 0]^T$ and the temporary path \bar{P}_r is planned by
 $\bar{f}_1(x, y, z) = 4x + 3z - 44 = 0$, $\bar{f}_2(x, y, z) = \frac{1}{20}((x - 11)^2 + y^2 + z^$ $\bar{f}_1(x, y, z) = 4x + 3z - 44 = 0$

where $\delta = [1, 1, 1]^T$, $\gamma = [0.5, 0.5]$

lows: $\bar{k}_{11} = \bar{k}_{12} = 1$, $\bar{k}_{21} = \bar{k}_2$
 $k_{12} = 2$, $k_{21} = k_{22} = 3.5$, $k_{31} = 3$,
 16 , $\epsilon_4 = 0.05$; $\xi = 0.707$, $\omega_n = 10$

rems. Choose \bar where $\delta = [1, 1, 1]^T$, $\gamma = [0.5, 0.5, 0.5]^T$. The controller parameters are chosen as fol-
lows: $\bar{k}_{11} = \bar{k}_{12} = 1$, $\bar{k}_{21} = \bar{k}_{22} = 3.5$, $\bar{k}_{31} = 1$, $\bar{\epsilon}_1 = 0.3$, $\bar{\epsilon}_2 = 0.5$, $\bar{\epsilon}_3 = 0.1$; $k_{11} =$
 $k_{12} = 2$

Figure [3](#page-11-0) shows the 3-D path following result. Figures [4,](#page-11-1) [5](#page-12-0) and [6](#page-12-1) illustrate the path following, attitude and angular velocity errors respectively, which are bounded and converge to neighborhoods of origin. Figure [7](#page-12-2) shows that the speed converges to singularity occurs in deriving command attitude.

Fig. 4 Values of *ig.* 4 Values of $\frac{1}{1}$, $\frac{2}{52}$, $\frac{2}{53}$, $\frac{5}{1}$, $\frac{5}{22}$, $\frac{5}{33}$

Figures [9](#page-13-1) and [10](#page-13-2) show comparisons of spatial distance d_s , which defines the shortest distance from actual position to the path, and thrust T_m using the proposed control law and d_{s1} , T_{m1} using the control law [\[1](#page-14-0)] without temporary-path generation. Obviously, T_{ml} is large in convergence process, and it would result in control saturation if further increasing parameters to reduce the error.

Fig. 8 Control \bar{u}_{cz} and u_{cz}

Fig. 10 Comparison of T_m and T_{m1}

6 Conclusion

This paper presents a singularity-free path following controller for miniature unmanned helicopters. The reference path is defined by implicit expression. Numerical simulation demonstrates the effectiveness of proposed controller. In future research we will extend the controller to more general manifolds and consider the disturbances and parametric uncertainties.

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