Adaptive Generalized Function Projective Synchronization of Colored Networks in Finite Time

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1 Introduction

Various control and synchronization of complex networks have been widely studying, which have many potential applications in many areas such as biology system, physics, communication, traffic and so on [1–3]. In view of the ubiquitous synchronization phenomena, the studies of synchronization and control have been attracted increasing attention [4–6]. Many works mainly study the outer relationship between the nodes. However the inner relationship ignored in many literatures plays important roles for the study of the whole networks.

In many realistic systems, another relationship may exist in the social networks consisting of N individuals, e.g., schoolmates, relatives and collaborative relationship. For individuals i and j, they may be either schoolmate or relatives but have no collaborative relationship, while for individual i and k ($k \neq j$), they may only have collaborative relationship. To depict this phenomenon more clearly, the graph theory in mathematics is introduced to solve such problem [7, 8]. In the colored networks, nodes with different color signify that they have different properties, and a pair of nodes connected by different color edges means that they have different mutual interactions. In particular, networks of coupled nonidentical dynamical systems with identical inner coupling matrixes can be deem as node colored networks, while networks of coupled identical dynamical systems with nonidentical inner coupling matrixes can be regarded as edge colored networks. Figure 1 shows a colored networks consisting of 6 colored nodes and 9 colored edges.

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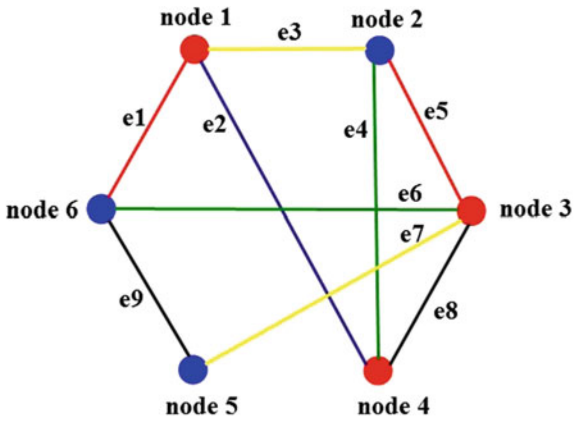


Fig. 1 A colored networks consisting of 6 colored nodes and 9 colored edges

To our knowledge, the research of colored networks synchronization was concerned with asymptotical or exponential synchronization of networks through impulsive control, intermittent control to reduce the synchronization time [9, 10]. However, in reality, it needs a faster rate to achieve in engineering area. For achieving synchronization quickly, an effective method is to use finite time synchronization control technique [11, 12]. There are few works involved in the synchronization of colored networks in finite time.

This paper investigates the problems of adaptive generalized function projective synchronization of colored networks in finite time. By using finite time synchronization control technique, one considers a complex networks consisting of N linearly and diffusively coupled identical nodes, which in many papers only consider the general synchronization of drive-response networks. Particularly, the parameter identification is considered in this paper, and the finite time considers the unknown parameters identification. Based on Lyapunov stability theory, sufficient conditions for ensuring the synchronization of colored networks are derived through designing appropriate controllers.

This paper is organized as follow: In Sect. 2, a general colored networks consisting of N linearly and diffusively coupled identical nodes is considered. At the same time, assumption and lemma are stated. In order to reach the generalized function projective synchronization with general colored networks, a sufficient criterion is presented in Sect. 3. In Sect. 4, several simulations are illustrated to verify the effectiveness of the theory proposed. Finally, conclusions are gained in Sect. 5.

2 Problem Formulation and Preliminaries

In this section, one considers a general colored networks consisting of N linearly and diffusively coupled identical nodes described as follows:

$$\dot{x}_i(t) = F_i(t, x_i(t), \alpha_i) + \varepsilon \sum_{j=1, j\neq i}^{N} a_{ij} H_{ij}\big(x_j(t) - x_i(t)\big), \quad i = 1, 2, \ldots, N. \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in R^n$ is the state variable of the ith node, $F_i(t, x_i(t), \alpha_i)$ representing the local dynamic of node i, which is continuous differentiable, can be rewritten in the following form: $F_i(t, x_i(t), \alpha_i) = f_i(t, x_i(t)) + g_i(t, x_i(t)) \cdot \alpha_i$, $f_i(\cdot)$ and $g_i(t, x_i(t))$:$R^n \to R^n$ is a nonlinear vector-valued function. The matrix $A = (a_{ij})_{N\times N}$ is outer-coupling matrix, which denotes the networks topology. If there is a connection between node i and node j $(i \neq j)$, then $a_{ij} > 0$, otherwise $a_{ij} = 0$, and the entire diagonal element $a_{ii} = 0$. $H_{ij} = \mathrm{diag}(h_{ij}^1, h_{ij}^2, \ldots, h_{ij}^n)$ is the inner coupling matrix, which represents the mutual interactions between nodes i and j, which is defined as the following: if the ζth component of node i is affected by that of node j, then $h_{ij}^{\xi} \neq 0$, otherwise $h_{ij}^{\xi} = 0$.

Figure 1 indicates that $F_1 = F_3 = F_4$, $F_2 = F_5 = F_6$, $H_{16} = H_{23}$, $H_{12} = H_{35}$, $H_{24} = H_{36}$. When $n = 3$, and $H_{16} = \mathrm{diag}\{1,1,0\}$ and $H_{56} = \mathrm{diag}\{1,0,1\}$, then the first and second components of node 1 are affected by those of node 6, and the first and third components of node 6 are affected by that of node 5, which is shown by Fig. 2.

Let $c_{ij} = diag(c_{ij}^1, c_{ij}^2, \ldots, c_{ij}^n)$, where $c_{ij}^k = a_{ij} h_{ij}^k$ for $i \neq j$ and $c_{ii}^k = -\sum_{j=1, j\neq i}^{N} c_{ij}^k$,

Then, the colored networks (1) can be rewritten as follows:

$$\dot{x}_i(t) = F_i(t, x_i(t), \alpha_i) + \varepsilon \sum_{j=1}^{N} c_{ij} x_j(t), \quad i = 1, 2, \ldots, N \tag{2}$$

Let $C_{\xi} = (c_{ij}^{\xi}) \in R^{N\times N}$, $\xi = 1, 2, \ldots, N$, then we regard the colored networks (2) as a combination of n component sub-networks with a topology determined by C_{ξ},$\xi = 1, 2, \ldots, n$.

A general response colored networks, which can be response edged-colored networks or colored networks, is given to achieve adaptive generalized function projective synchronization with the colored networks (2), which is shown as

$$\dot{y}_i(t) = F_i(t, y_i(t), \alpha_i) + \varepsilon \sum_{j=1}^{N} c_{ij} y_j(t) + u_i(t), \quad i = 1, 2, \ldots, N \tag{3}$$

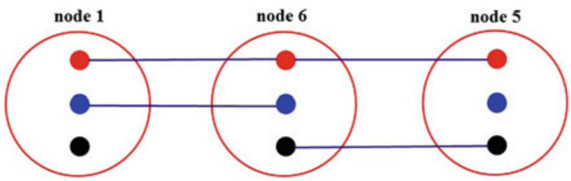


Fig. 2 The *red*, *blue* and *black* stand for the first, second, and third components of each individual node, respectively

where $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^{\mathrm{T}} \in R^n$ stands for the state vector of the ith node, $f_i(t, y_i(t))$ and $g_i(t, y_i(t))$:$R^n \to R^n$ is a nonlinear vector-valued function, u_i is the adaptive controller.

Next, one introduces definitions assumptions, lemmas that will be required in this paper.

Definition 1 (*GFPS*) For the colored networks (3), it is said that achieve adaptive generalized function projective synchronization (GFPS) with the colored networks (2), if there exist the continuous function $\varphi(x_i(t))$ such that

$$\lim_{t\to\infty}\|e_i(t)\| = \lim_{t\to\infty}\|y_i(t) - \varphi(x_i(t))\| = 0, \quad i = 1, 2, \ldots, N.$$

where $\varphi(x_i(t))$ are nonzero scaling functions and continuously differentiable functions.

Assumption 1 Suppose that there exist a constant $L_i > 0$ satisfying

$$\|F_i(t, y_i(t), \alpha_i) - F_i(t, x_i(t), \alpha_i)\| \le L_i\|y_i(t) - x_i(t)\| \tag{4}$$

Lemma 1 *Let* $x_1, x_2, \ldots, x_n > 0$ and $0 < r < p$. *Then*

$$\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \le \left(\sum_{i=1}^{n} x_i^r\right)^{1/r} \tag{5}$$

Lemma 2 *Cai et al.* [12]. *Assume that a continuous, positive-definite function V(t) satisfy the following differential inequality*:

$$\dot{V}(t) \le -pV^{\xi}(t) \quad \forall t \ge t_0, \; V(t_0) \ge 0 \tag{6}$$

where p > 0, 0 < ξ < 1 are two constants. Then, for any given $t_{0,}$ $V(t)$ satisfies the following inequality: $V^{1-\xi}(t) \le V^{1-\xi}(t_0) - p(1-\xi)(t - t_0)$, $t_0 \le t \le t_1$ and $V(t) = 0 \; \forall t \ge t_1$ with t_1 given by $t_1 = t_0 + \frac{V^{1-\xi}(t_0)}{p(1-\xi)}$.

3 Main Results

In this section, the colored networks (2) achieve generalized function projective synchronization with general colored networks (3).

For simplicity sake, we define $\varphi(x) = Px + Q$, where P and Q is constant matrix. The error dynamic networks can be calculated in the following:

$$\begin{aligned}\dot{e}_i(t) &= \dot{y}_i(t) - P\dot{x}_i(t) \\ &= f_i(t, y_i(t)) - Pf_i(t, x_i(t)) + (g_i(t, y_i(t)) - Pg_i(t, x_i(t))) \cdot \alpha_i \\ &\quad + \sum_{j=1}^{N} c_{ij}(y_i(t) - Px_i(t)) + u_i \quad i = 1, 2, \ldots, N\end{aligned} \tag{7}$$

For achieving the main focus, the nonlinear controllers are designed as

$$\begin{aligned}u_i &= Pf_i(t, x_i(t)) - f(t, \varphi(x_i(t))) - (g_i(t, y_i(t))) - Pg_i(t, x_i(t))\hat{\alpha} - \sum_{j=1}^{N} c_{ij}Q \\ &\quad - d_i(t)e_i - \omega sign(e_i(t))|e_i(t)|^{\theta}\end{aligned} \tag{8}$$

where $d_i(t) > 0$, $i = 1,2,\ldots,N$ are the time-varying adaptive control gains that can be suitably chosen by the generalized function projective synchronization system and satisfy the following conditions: $\dot{d}_i(t) = k_i e_i^{\mathrm{T}}(t)e_i(t) > 0$

Theorem 1 *Suppose the Assumption 1 and Lemma 2 holds. If the following condition holds:*

$$\eta = -\lambda_{\max}((L_i + C - d^*)I \otimes I) > 0 \tag{9}$$

when using the above controller (8) and the parameter identification:

$$\dot{\hat{\alpha}} = \beta_i e_i^{\mathrm{T}}(t)[g_i(y_i(t)) - Pg_i(x_i(t))] \tag{10}$$

then the drive system (2) and response system (3) can achieve synchronization in finite time $t_1 = t_0 + \frac{V(t_0)^{(1-\theta)/2}}{2\omega(1-\theta)}$ *for any given* t_0.

Proof Construct a Lyapunov function as the following

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) + \sum_{i=1}^{N}\frac{1}{2\beta_i}(\alpha - \hat{\alpha})^2 + \sum_{i=1}^{N}\frac{1}{2k_i}(d_i - d^*)^2 \tag{11}$$

Then the derivation of $V(t)$:

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\dot{e}_i(t) - \sum_{i=1}^{N}\frac{1}{\beta_i}\dot{\hat{\alpha}}(\alpha - \hat{\alpha}) + \sum_{i=1}^{N}\frac{1}{\alpha}\dot{d}_i(d_i - d^*) \\ &= \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)[f_i(t, y_i(t)) - Pf_i(t, x_i(t)) + (g_i(t, y_i(t) - Pg_i(t, x_i(t)))) \cdot \alpha_i \\ &\quad + \left.\sum_{j=1}^{N} c_{ij}(y_i(t) - Px_i(t)) + U_i\right] + \sum_{i=1}^{N}\frac{1}{\beta_i}\dot{\hat{\alpha}}(\hat{\alpha} - \alpha) + \sum_{i=1}^{N}\frac{1}{k_i}\dot{d}_i(d_i - d^*)\end{aligned} \tag{12}$$

From the above calculations, one has

$$\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^{\mathrm{T}}\omega sign\left(e_{ij}(t)\right)\left|e_{ij}(t)\right|^{\theta} = \omega\sum_{i=1}^{N}\sum_{j=1}^{N}\left|e_i^{\mathrm{T}}(t)\right|\left|e_i(t)\right|^{\theta} = \omega\sum_{i=1}^{N}\sum_{j=1}^{N}\left|e_i(t)\right|^{\theta+1}$$

By Lemma 1, $\left(\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} e_{ij}^{1+\theta}\right)^{1/(1+\theta)}$

Hence,

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left|e_i(t)\right|^{\theta+1} \geq \left(\sum_{i=1}^{N}\sum_{j=1}^{N}\left|e_{ij}(t)\right|^2\right)^{(\theta+1)/2} = \left(\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^{\mathrm{T}}(t)e_{ij}(t)\right)^{(\theta+1)/2}. \tag{13}$$

According to (3), (5) and (6), the derivation of $V(t)$ can be calculated as follows:

$$\begin{aligned}\dot{V}(t) &\leq \sum_{i=1}^{N} e_i^{\mathrm{T}}[(L_i + C - d^*)I \otimes I]e_i - \omega\left(\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^{\mathrm{T}}(t)e_{ij}(t)\right)^{(\theta+1)/2} \\ &\leq -\eta\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) - \omega\left(\sum_{i=1}^{N}\sum_{j=1}^{N} e_{ij}^{\mathrm{T}}(t)e_{ij}(t)\right)^{(\theta+1)/2} \\ &\leq -4\omega(V(t))^{(\theta+1)/2}\end{aligned} \tag{14}$$

From Lemma 2, Theorem 1 and on the basis of the Lyapunov stability theorem, one has $e(t) \to 0$ $(t \to \infty)$, which means the drive system (1) can achieve the generalized function projective synchronization with response system (2) in finite time $t_1 = t_0 + \frac{V(t_0)^{(1-\theta)/2}}{2\omega(1-\theta)}$ for any given t_0. That completes the proof.

Based on Theorem 1, one gives the procedure of complex networks finite time's calculation methods: Firstly, according to the Theorem 1, calculate the parameters d^*. Secondly, determine the system initial values, especially the unknown parameters' ones. Next step is to calculate the $V(t_0)$ when $t_0 = 0$. At the last, in line with the equation $t_1 = t_0 + \frac{V(t_0)^{(1-\theta)/2}}{2\omega(1-\theta)}$, obtain the finite time scheme t_1.

Remark 1 It should emphasized that finite time synchronization control techniques are adopted to guarantee generalized function projective synchronization of colored networks in finite time, while little of form paper has been done about this work, which can applied to many practical areas.

4 Numerical Simulations

In this section, two illustrative examples are adopted to demonstrate the validity and reduce conservatism of the above theory.

Example 1 Consider a edge-colored networks with 10 coupled Lorenz systems.

$$f(t,x(t)) = \begin{pmatrix} 0 \\ -x_1x_3 - x_2 \\ x_1x_2 \end{pmatrix} \text{and}\, g(t,x(t)) = \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ 0 & 0 & x_1 \\ 0 & -x_3 & 0 \end{pmatrix}$$

and in view of the error system defined, one sets

$$P = \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q = (0,0,0)^{\mathrm{T}}.$$

In numerical simulation, the initial values of drive-response system are chosen as $x_i(0) = (0.3 + 0.1i, 0.3 + 0.1i, 0.3 + 0.1i)^{\mathrm{T}}$, For brevity, one always sets Γ=diag$(1,1,1)$, $\|\Gamma\|$=1, L = 1, θ=0.5, $\beta_1 = \beta_2 = \beta_3 = 1$, $\omega_1 = \omega_2 = \omega_3 = 5$ the estimated parameters have initial conditions: $\hat{a} = 0, \hat{b} = 0,\ \hat{c} = 0$. According to procedure of complex networks finite time's calculation methods, one can obtain

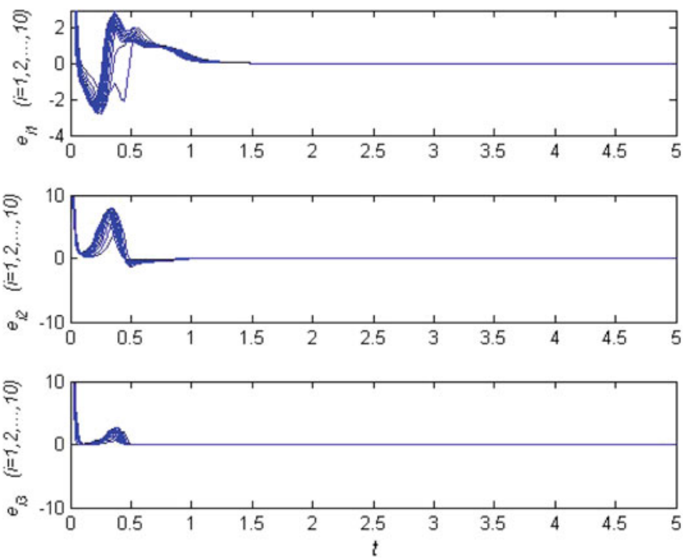
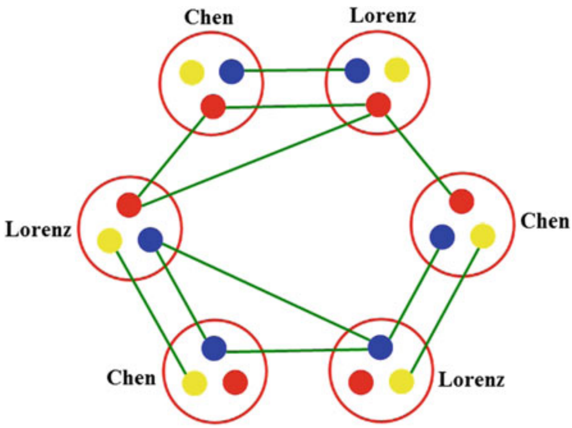


Fig. 3 Synchronization errors of the edge-colored networks coupled with 10 Lorenz systems

Fig. 4 The topology of the colored networks coupled with 3 Chen systems and 3 Lorenz systems



and give the parameter $d^{*}=5$.Then the finite time is $t_1 = t_0 + \frac{V(t_0)^{(1-\theta)/2}}{2\omega(1-\theta)} = 2.41\,\text{s}$. The Fig. 3 shows the synchronization errors of the edge-colored networks.

Example 2 Two general colored networks, whose topology coupled with 3 Chen systems and 3 Lorenz systems shown in Fig. 4, are considered.

$$f(t,x(t)) = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix} \text{ and } g(t,x(t)) = \begin{pmatrix} x_2 - x_1 & 0 & 0 \\ -x_1 & 0 & x_1 + x_2 \\ 0 & -x_3 & 0 \end{pmatrix}.$$

The initial values of drive-response system are chosen as the Example 1 except the $\omega_1 = \omega_2 = \omega_3 = 2$. So the finite time is $t_1 = t_0 + \frac{V(t_0)^{(1-\theta)/2}}{2\omega(1-\theta)} = 7.17\,\text{s}$. The synchronization error of the general colored networks is shown in Fig. 5.

Remark 2 General synchronization of colored networks has been extensively studied, in which all the nodes synchronized each other in a common manner. However, in real complex networks, different communities usually synchronize with each other in a different manner. So in this paper, one considers generalized function projective synchronization. If $P = \phi_i$, $Q = 0$, the general projective synchronization can be realized.

Remark 3 In the existing research of synchronization of the colored networks, certain networks are often considered. However, information may be not available in many practical cases. The uncertain networks (1) can be seen as the special case of the colored networks.

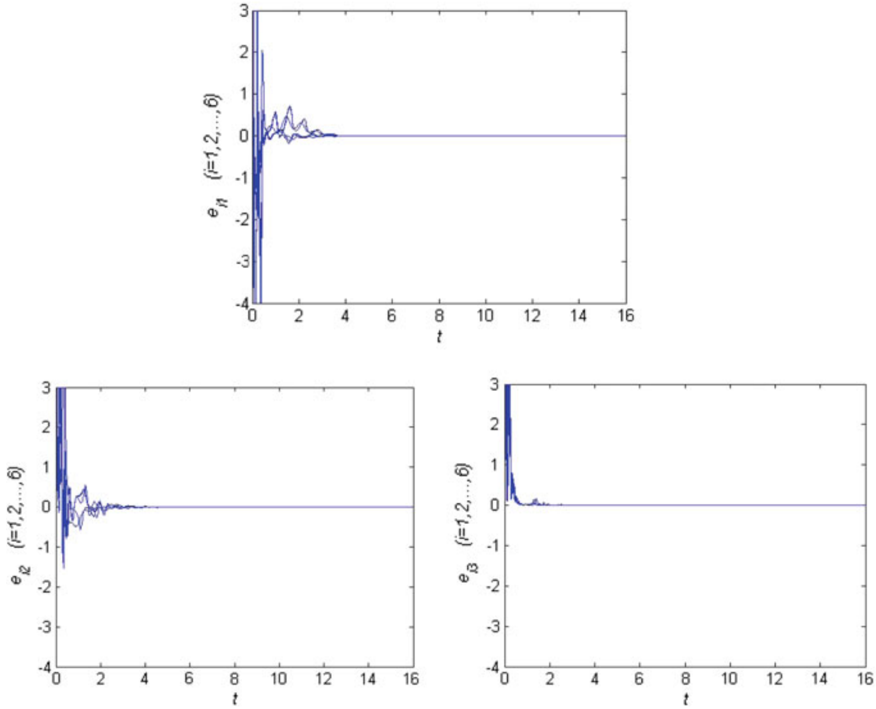


Fig. 5 Synchronization errors of the colored networks coupled with 3 Chen systems and 3 Lorenz systems

5 Conclusions

In this paper, adaptive generalized function projective synchronization of the colored networks in finite time has been investigated. A uncertain colored networks is considered as many practical cases. Specially, instead of using impulsive control, intermittent control to reduce the synchronization time, an effective method—finite time synchronization control techniques, is applied to achieve the colored networks' synchronization. Based on Lyapunov stability theorem, simple and useful criteria for the colored networks have been established. The corresponding numerical simulations have been presented to verified effectiveness and correctness of the theoretical results.

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