Chapter 6
Additive Results for the Drazin Inverse

The Drazin inverse, introduced in [1], named after Michael P. Drazin in 1958 in the setting of an abstract ring, is a kind of generalized inverse of a matrix. Many interesting spectral properties of the Drazin inverse make it as a concept that is extremely useful in various considerations in topics such as Markov chains, multibody system dynamics, singular difference and differential equations, differential-algebraic equations and numerical analysis ([1–6]).

In this chapter we will focus our attention on the behavior of the Drazin inverse of a sum of two Drazin invertible elements in the setting of matrices as well as in Banach algebras, where we will also consider the concept of the generalized Drazin inverse. In 1958, while considering the question of Drazin invertibility of a sum of two Drazin invertible elements of a ring Drazin proved that

$$(A + E)^{\mathrm{D}} = A^{\mathrm{D}} + E^{\mathrm{D}}$$

provided that $AE = EA = 0$. After that this topic received considerable interest with many authors working on this problem [4, 7–10], which in turn lead to a number of different formulae for the Drazin inverse $(A + E)^{\mathrm{D}}$ as a function of A, E, A^{D} and E^{D}.

6.1 Additive Results for the Drazin Inverse

Although it was already even in 1958 that Drazin [1] pointed out that computing the Drazin inverse of a sum of two elements in a ring was not likely to be easy, this problem remains open to this day even for matrices. It is precisely this problem when considered in rings of matrices that will be the subject of our interest in this section, i.e., under various conditions we will compute $(P + Q)^{\mathrm{D}}$ as a function of P, Q, P^{D} and Q^{D}. We will extend Drazin's result in the sense that only one of the conditions

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$PQ = 0$ or $PQ = QP$ is assumed. The results obtained will be then used to analyze a special class of perturbations of the type $A - X$.

Throughout the section, we shall assume familiarity with the theory of Drazin inverses (see [11]). Also, for $A \in \mathbb{C}^{n\times n}$, we denote $Z_A = I - AA^{\mathrm{D}}$.

First, we will give a representation of $(P+Q)^{\mathrm{D}}$ under the condition $PQ = 0$ which was considered in [10, Theorem 2.1]:

Theorem 6.1 *Let* $P, Q \in \mathbb{C}^{n\times n}$. *If* $PQ = 0$, *then*

$$\begin{aligned}(P+Q)^{\mathrm{D}} = (I - QQ^{\mathrm{D}})[I + QP^{\mathrm{D}} + \cdots + Q^{k-1}(P^{\mathrm{D}})^{k-1}]P^{\mathrm{D}} \\ + Q^{\mathrm{D}}[I + Q^{\mathrm{D}}P + \cdots + (Q^{\mathrm{D}})^{k-1}P^{k-1}](I - PP^{\mathrm{D}}),\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}(P+Q)(P+Q)^{\mathrm{D}} = (I - QQ^{\mathrm{D}})[I + QP^{\mathrm{D}} + \cdots + Q^{k-1}(P^{\mathrm{D}})^{k-1}]PP^{\mathrm{D}} \\ + QQ^{\mathrm{D}}[I + Q^{\mathrm{D}}P + \cdots + (Q^{\mathrm{D}})^{k-1}P^{k-1}](I - PP^{\mathrm{D}}) + QQ^{\mathrm{D}}PP^{\mathrm{D}},\end{aligned} \tag{6.2}$$

where $\max\{\mathrm{Ind}(P), \mathrm{Ind}(Q)\} \le k \le \mathrm{Ind}(P) + \mathrm{Ind}(Q)$.

Proof Under the assumption $PQ = 0$, we have

$$P^{\mathrm{D}}Q = PQ^{\mathrm{D}} = 0, \quad Z_P Q = Q \quad \text{and} \quad PZ_Q = P. \tag{6.3}$$

Using Cline's Formula [12], $(AB)^{\mathrm{D}} = A[(BA)^{\mathrm{D}}]^2 B$, we have

$$(P+Q)^{\mathrm{D}} = \left([I, Q]\begin{bmatrix} P \\ I \end{bmatrix}\right)^{\mathrm{D}} = [I, Q]\left(\begin{bmatrix} P & PQ \\ I & Q \end{bmatrix}^{\mathrm{D}}\right)^2 \begin{bmatrix} P \\ I \end{bmatrix}.$$

Now, by Theorem 1 of [4], we have that

$$\begin{bmatrix} P & 0 \\ I & Q \end{bmatrix}^{\mathrm{D}} = \begin{bmatrix} P^{\mathrm{D}} & 0 \\ R & Q^{\mathrm{D}} \end{bmatrix},$$

for

$$R = -Q^{\mathrm{D}}P^{\mathrm{D}} + Z_Q Y_k (P^{\mathrm{D}})^{k+1} + (Q^{\mathrm{D}})^{k+1} Y_k Z_P$$

and

$$Y_k = Q^{k-1} + Q^{k-2}P + \cdots + QP^{k-2} + P^{k-1},$$

where $\max\{\mathrm{Ind}(P), \mathrm{Ind}(Q)\} \le k \le \mathrm{Ind}(P) + \mathrm{Ind}(Q)$.

Hence

$$(P+Q)^{\mathrm{D}} = [I, Q]\left(\begin{bmatrix} P^{\mathrm{D}} & 0 \\ R & Q^{\mathrm{D}} \end{bmatrix}\right)^2 \begin{bmatrix} P \\ I \end{bmatrix} = P^{\mathrm{D}} + QRPP^{\mathrm{D}} + QQ^{\mathrm{D}}RP + Q^{\mathrm{D}}.$$

Substituting R in the above equality, we get (6.1). It is straightforward to prove (6.2) from (6.1) and (6.3). □

Now we list some special cases of the previous result:

Corollary 6.1 *Let $P, Q \in \mathbb{C}^{n\times n}$ be such that $PQ = 0$ and let k be such that* $\max\{\operatorname{Ind}(P), \operatorname{Ind}(Q)\} \le k \le \operatorname{Ind}(P) + \operatorname{Ind}(Q)$.

(i) *If Q is nilpotent, then $(P+Q)^{\mathrm{D}} = P^{\mathrm{D}} + Q(P^{\mathrm{D}})^2 + \cdots + Q^{k-1}(P^{\mathrm{D}})^k$.*
(ii) *If $Q^2 = 0$, then $(P+Q)^{\mathrm{D}} = P^{\mathrm{D}} + Q(P^{\mathrm{D}})^2$.*
(iii) *If P is nilpotent, then $(P+Q)^{\mathrm{D}} = Q^{\mathrm{D}} + (Q^{\mathrm{D}})^2 P + \cdots + (Q^{\mathrm{D}})^k P^{k-1}$.*
(iv) *If $P^2 = 0$, then $(P+Q)^{\mathrm{D}} = Q^{\mathrm{D}} + (Q^{\mathrm{D}})^2 P$.*
(v) *If $P^2 = P$, then $(P+Q)^{\mathrm{D}} = (I - QQ^{\mathrm{D}})(I + Q + \cdots + Q^{k-1})P + Q^{\mathrm{D}}(I-P)$, and*
$(P+Q)^{\mathrm{D}}(I-P) = Q^{\mathrm{D}}(I-P)$.
(vi) *If $Q^2 = Q$, then $(P+Q)^{\mathrm{D}} = (I-Q)P^{\mathrm{D}} + Q(I + P + \cdots + P^{k-1})(I - PP^{\mathrm{D}})$, and*
$(I-Q)(P+Q)^{\mathrm{D}} = (I-Q)P^{\mathrm{D}}$.
(vii) *If $PR = 0$, then $(P+Q)^{\mathrm{D}}R = (I - QQ^{\mathrm{D}})P^{\mathrm{D}}R + Q^{\mathrm{D}}R = Q^{\mathrm{D}}R$.*

Theorem 6.1 may be used to obtain several additional perturbation results concerning the matrix $\Gamma = A - X$. Needless to say these are rather special, since addition and inversion rarely mix. First a useful result.

Lemma 6.1 *Let $A, F, X \in \mathbb{C}^{n\times n}$. If $AF = FA$ and $FX = X$, then*

$$(AF - X)^k X = (A - X)^k X, \quad \textit{for all} \quad k \in \mathbb{N}. \tag{6.4}$$

Proof Since $AF = FA$ and $(I - F)X = 0$, we have that

$$(I - F)(A - X)^k X = 0. \tag{6.5}$$

Now the assertion is proved by induction. The case $k = 1$ is trivial. Suppose $(AF - X)^k X = (A - X)^k X$. Then by (6.5),

$$\begin{aligned}(AF - X)^{k+1}X &= (AF - X)(A - X)^k X = AF(A - X)^k X - X(A - X)^k X \\ &= A(A - X)^k X - X(A - X)^k X = (A - X)^{k+1}X. \square\end{aligned}$$

Now we present a perturbation result.

Corollary 6.2 *Let $A, F, X \in \mathbb{C}^{n\times n}$ and let F be an idempotent matrix which commutes with A. Let $\Gamma = A - X$ and let* $\max\{\operatorname{Ind}(A), \operatorname{Ind}(X)\} \le k \le \operatorname{Ind}(A) + \operatorname{Ind}(X)$. *If $FX = X$ and $R = \Gamma F = AF - XF$, then*

$$(A - X)^D = R^D - \sum_{i=0}^{k-1} (R^D)^{i+2} X(I - F)A^i(I - AA^D)$$

$$+ (I + R^D X)(I - F)A^D - (I - RR^D) \sum_{i=0}^{k-2} (A - X)^i X(I - F)(A^D)^{i+2}. \quad (6.6)$$

Proof Let $\Gamma = A - X = P + Q$, where $P = A(I - F)$ and $Q = AF - FX$. Since $F^2 = F$ we have that $(I - F)^2 = I - F$ and $(I - F)^D = (I - F)$. Since

$$PQ = A(I - F)(AF - FX) = A(I - F)AF = A^2[(I - F)F] = 0,$$

after applying Theorem 6.1 we get

$$(P + Q)^D = (I - QQ^D)V + W(I - PP^D) = T_1 + T_2,$$

where $V = [P^D + Q(P^D)^2 + \cdots + Q^{k-1}(P^D)^k]$ and $W = [Q^D + (Q^D)^2 P + \cdots + (Q^D)^k P^{k-1}]$. Put $T_1 = (I - QQ^D)V$ and $T_2 = W(I - PP^D)$. So we see that we need to compute Q^D and P^D. The latter is easily found because A and F commute:

$$P^D = [A(I - F)]^D = (I - F)A^D, \quad PP^D = (I - F)AA^D.$$

On the other hand, in order to compute Q^D, we split Q further as

$$Q = R - S,$$

where $R = (A - X)F = AF - FXF$ and $S = FX(I - F)$. Since

$$SR = FX(I - F)(FA - FXF) = FX[(I - F)F](A - XF) = 0,$$

and $S^2 = FX[(I - F)F]X(I - F) = 0$, by (iv) of Corollary 6.1 we get

$$Q^D = (-S + R)^D = R^D - (R^D)^2 S, \quad QQ^D = (R - S)[R^D - (R^D)^2 S].$$

Since $SR^D = SR = 0$, it follows that $QQ^D = RR^D - R^D S$. Also, $R^D P = 0$, because

$$RP = (AF - FXF)A(I - F) = (A - FX)[F(I - F)]A = 0.$$

So $Q^D P = -(R^D)^2 SP = -(R^D)^2 XP$. Similarly, since $SR^D = 0$, we get

$$(Q^D)^2 P = [R^D - (R^D)^2 S][-(R^D)^2 XP] = -(R^D)^3 XP.$$

Repeating the process, we obtain

$$(Q^{\mathrm{D}})^{t+1}P^t = -(R^{\mathrm{D}})^{t+2}XP^t, \quad t = 1, 2, \ldots$$

which when substituted yields the second term:

$$\begin{aligned}T_2 &= W(I - PP^{\mathrm{D}}) = [R^{\mathrm{D}} - (R^{\mathrm{D}})^2 S - (R^{\mathrm{D}})^3 XP - \cdots - (R^{\mathrm{D}})^{k+1}XP^{k-1}](I - PP^{\mathrm{D}})\\ &= [R^{\mathrm{D}} - (R^{\mathrm{D}})^2 X(I-F) - (R^{\mathrm{D}})^3 XA(I-F)\cdots - (R^{\mathrm{D}})^{k+1}XA^{k-1}(I-F)]\\ &\quad - [R^{\mathrm{D}} - (R^{\mathrm{D}})^2 X(I-F) - (R^{\mathrm{D}})^3 XA(I-F)\cdots - (R^{\mathrm{D}})^{k+1}XA^{k-1}(I-F)](I-F)AA^{\mathrm{D}}\\ &= R^{\mathrm{D}} - \sum_{i=0}^{k-1}(R^{\mathrm{D}})^{i+2}X(I-F)A^i(I-AA^{\mathrm{D}}).\end{aligned}$$

Let us next examine the first term

$$T_1 = (I - QQ^{\mathrm{D}})V = [I - (RR^{\mathrm{D}} - R^{\mathrm{D}}S)][P^{\mathrm{D}} + Q(P^{\mathrm{D}})^2 + \cdots + Q^{k-1}(P^{\mathrm{D}})^k].$$

We compute the powers $Q^i(P^{\mathrm{D}})^{i+1} = (AF - X)^i(I-F)(A^{\mathrm{D}})^{i+1}$. For $i = 1$, this becomes $(AF - X)(I - F)(A^{\mathrm{D}})^2 = -X(I-F)(A^{\mathrm{D}})^2$, while for higher powers of i we may use Lemma 6.1 to obtain

$$\begin{aligned}Q^i(P^{\mathrm{D}})^{i+1} &= (AF - X)^{i-1}(AF - X)(I-F)(A^{\mathrm{D}})^{i+1}\\ &= -(AF - X)^{i-1}X(I-F)(A^{\mathrm{D}})^{i+1} = -(A - X)^{i-1}X(I-F)(A^{\mathrm{D}})^{i+1}.\end{aligned}$$

Now

$$\begin{aligned}S(A - X)^{i-1}X &= X(I-F)(A-X)(A-X)^{i-2}X\\ &= XA(I-F)(A-X)^{i-2}X = \cdots = XA^{i-1}(I-F)X = 0\end{aligned}$$

for all i, and $R^{\mathrm{D}}(I - F) = (R^{\mathrm{D}})^2 R(I-F) = (R^{\mathrm{D}})^2(A - X)[F(I-F)] = 0$, so

$$\begin{aligned}T_1 &= (I - RR^{\mathrm{D}} + R^{\mathrm{D}}S)(I-F)A^{\mathrm{D}} + (I - RR^{\mathrm{D}} + R^{\mathrm{D}}S)[Q(P^{\mathrm{D}})^2 + \cdots + Q^{k-1}(P^{\mathrm{D}})^k]\\ &= [I + R^{\mathrm{D}}X(I-F)](I-F)A^{\mathrm{D}} - (I - RR^{\mathrm{D}})\sum_{i=1}^{k-1}(A-X)^{i-1}X(I-F)(A^{\mathrm{D}})^{i+1}\\ &= (I + R^{\mathrm{D}}X)(I-F)A^{\mathrm{D}} - (I - RR^{\mathrm{D}})\sum_{i=0}^{k-2}(A-X)^i X(I-F)(A^{\mathrm{D}})^{i+2},\end{aligned}$$

completing the proof. □

Using the previous result we will analyze some special types of perturbations of the matrix $A - X$. We shall thereby extend earlier work by several authors [13–16] and partially solve a problem posed in 1975 by Campbell and Meyer [17], who considered it difficult to establish norm estimates for the perturbation of the Drazin inverse.

In the following five special cases, we assume $FX = X$ and $R = AF - XF$.
Case (1) $XF = 0$.

Clearly $(R^{\mathrm{D}})^i = (A^{\mathrm{D}})^i F$ and $S = X$. Moreover $(A - X)^i FX = A^i X$ for $i \geq 0$. Thus (6.6) reduces to

$$(A - X)^{\mathrm{D}} = A^{\mathrm{D}} F - \sum_{i=0}^{k-1} (A^{\mathrm{D}})^{i+2} X A^i (I - AA^{\mathrm{D}}) + (I - F + A^{\mathrm{D}} X) A^{\mathrm{D}} - \sum_{i=0}^{k-2} A^i (I - AA^{\mathrm{D}}) X (A^{\mathrm{D}})^{i+2}. \quad (6.7)$$

Case (1a) $XF = 0$ and $F = AA^{\mathrm{D}}$.
If we in addition assume that $F = AA^{\mathrm{D}}$, then $XA^{\mathrm{D}} = 0$ and (6.7) is reduced to

$$(A - X)^{\mathrm{D}} = A^{\mathrm{D}} - \sum_{i=0}^{k-1} (A^{\mathrm{D}})^{i+2} X A^i. \quad (6.8)$$

Case (1b) $XF = 0$ and $F = I - AA^{\mathrm{D}}$.
In this case, $A^{\mathrm{D}} X = 0$ and (6.7) becomes

$$(A - X)^{\mathrm{D}} = A^{\mathrm{D}} - \sum_{i=0}^{k-2} A^i X (A^{\mathrm{D}})^{i+2}. \quad (6.9)$$

Case (2) $F = AA^{\mathrm{D}}$.
Now $AA^{\mathrm{D}} X = X$, $R = A^2 A^{\mathrm{D}} (I - A^{\mathrm{D}} X A A^{\mathrm{D}})$ and (6.6) simplifies to

$$(A - X)^{\mathrm{D}} = R^{\mathrm{D}} - \sum_{i=0}^{k-1} (R^{\mathrm{D}})^{i+2} X A^i (I - AA^{\mathrm{D}}). \quad (6.10)$$

If we set $U = I - A^{\mathrm{D}} X A A^{\mathrm{D}}$ and $V = I - AA^{\mathrm{D}} X A^{\mathrm{D}}$, then $UA^{\mathrm{D}} = A^{\mathrm{D}} V$ and $R = A^2 A^{\mathrm{D}} U = V A^2 A^{\mathrm{D}}$. Now if we assume that U is invertible, then so will be V and $U^{-1} A^{\mathrm{D}} = A^{\mathrm{D}} V^{-1}$. It is now easily verified that $R^{\#}$ exists and

$$R^{\#} = U^{-1} A^{\mathrm{D}} = A^{\mathrm{D}} V^{-1}.$$

In fact $RR^{\#} = A^2 A^{\mathrm{D}} U U^{-1} A^{\mathrm{D}} = AA^{\mathrm{D}} = A^{\mathrm{D}} V^{-1} V A^2 A^{\mathrm{D}} = R^{\#} R$ and $R^2 R^{\#} = RAA^{\mathrm{D}} = R$ and $R^{\#} R R^{\#} = U^{-1} A^{\mathrm{D}} A A^{\mathrm{D}} = U^{-1} A^{\mathrm{D}} = R^{\#}$. We then have two subcases.

Case (2a) $F = AA^{\mathrm{D}}$, and $U = I - A^{\mathrm{D}} X A A^{\mathrm{D}}$ is invertible.

In this case, (6.10) is just

$$(A-X)^{\mathrm{D}} = R^{\#} - \sum_{i=0}^{k-1} (R^{\#})^{i+2} X A^{i} (I - AA^{\mathrm{D}}), \tag{6.11}$$

where $R = A^2 A^{\mathrm{D}} U$, $R^{\#} = U^{-1} A^{\mathrm{D}}$. In general, $(R^{\#})^i \neq U^{-i} (A^{\mathrm{D}})^i$.

Remark 6.1.1 The matrix $U = I - A^{\mathrm{D}} X A A^{\mathrm{D}}$ is invertible if and only if $I - A^{\mathrm{D}} X$ is invertible. This result generalizes the main results from [13–16].

Case (2b) $F = I - AA^{\mathrm{D}}$.
We have $A^{\mathrm{D}} X = A^{\mathrm{D}} F = 0$, and (6.6) becomes

$$(A-X)^{\mathrm{D}} = R^{\mathrm{D}} + (I + R^{\mathrm{D}} X) A^{\mathrm{D}} - (I - RR^{\mathrm{D}}) \sum_{i=0}^{k-2} (A-X)^{i} X (A^{\mathrm{D}})^{i+2}, \tag{6.12}$$

where $R = A(I - AA^{\mathrm{D}}) - (I - AA^{\mathrm{D}}) X (I - AA^{\mathrm{D}})$.

Case (3) $AA^{\mathrm{D}} X F = X F A A^{\mathrm{D}} = X F$, $U = I - A^{\mathrm{D}} X F$ is invertible and $(AF)^{\#}$ exists.

Now $R = AF - XF = AF - AA^{\mathrm{D}} F X F = AF(I - A^{\mathrm{D}} X F) = AFU = VFA$, where $V = I - X F A^{\mathrm{D}}$. Furthermore $A^{\mathrm{D}} F V = U A^{\mathrm{D}} F$. We may now conclude that U is invertible exactly when V is, in which case $Y = U^{-1} A^{\mathrm{D}} F = A^{\mathrm{D}} F V^{-1}$.

We then have $RY = AFU(U^{-1} A^{\mathrm{D}} F) = AA^{\mathrm{D}} F = A^{\mathrm{D}} F V^{-1} (VFA) = YR$. Lastly,

$$Y^2 R = U^{-1} A^{\mathrm{D}} F (AA^{\mathrm{D}} F) = U^{-1} A^{\mathrm{D}} F = Y$$

and $R^2 Y = R A A^{\mathrm{D}} F = A^2 A^{\mathrm{D}} F - AA^{\mathrm{D}} F X F A A^{\mathrm{D}} = A^2 A^{\mathrm{D}} F - X F$.

If $(AF)^{\#}$ exists then $AF = AF(AF)^{\#} AF = AFF^{\#} A^{\mathrm{D}} AF = AFF^{\#} F A A^{\mathrm{D}} = A^2 A^{\mathrm{D}} F$, so $R^2 Y = AF - XF = R$, i.e., $Y = R^{\#}$ and (6.6) becomes

$$(A-X)^{\mathrm{D}} = R^{\#} - \sum_{i=0}^{k-1} (R^{\#})^{i+2} X (I - AA^{\mathrm{D}}) A^{i}. \tag{6.13}$$

Case (4) $FX = XF = X$.
In this case, (6.6) reduces to

$$(A-X)^{\mathrm{D}} = R^{\mathrm{D}} + (I - F) A^{\mathrm{D}}. \tag{6.14}$$

If in addition to $F = AA^{\mathrm{D}}$, the matrix $U = I - A^{\mathrm{D}} X$ is invertible, this reduces further to [15]

$$(A-X)^{\mathrm{D}} = R^{\mathrm{D}} = U^{-1} A^{\mathrm{D}}. \tag{6.15}$$

Case (5) If $X = A^2 A^{\mathrm{D}}$ then Γ is nilpotent and $\Gamma^{\mathrm{D}} = 0$.

Although Theorem 6.1 solves our problem under the assumption that $PQ = 0$, the condition can be relaxed and the result therefore generalized as follows: Since

$$\left(\begin{bmatrix} P & PQ \\ I & Q \end{bmatrix}^{\mathrm{D}}\right)^k = \left(\begin{bmatrix} P & PQ \\ I & Q \end{bmatrix}^k\right)^{\mathrm{D}} = \begin{bmatrix} P(P+Q)^{k-1} & P(P+Q)^{k-1}Q \\ (P+Q)^{k-1} & (P+Q)^{k-1}Q \end{bmatrix}^{\mathrm{D}}, \text{ for all } k \in \mathbb{N},$$

we may extend the considerations above to the case when $P(P+Q)^{k-1}Q = 0$.

In fact

$$(P+Q)^{\mathrm{D}} = [I, Q]\left(\begin{bmatrix} P & PQ \\ I & Q \end{bmatrix}^k\right)^{\mathrm{D}} \begin{bmatrix} P & PQ \\ I & Q \end{bmatrix}^{k-2} \begin{bmatrix} P \\ I \end{bmatrix} =$$

$$[I, Q]\begin{bmatrix} P(P+Q)^{k-1} & 0 \\ (P+Q)^{k-1} & (P+Q)^{k-1}Q \end{bmatrix}^{\mathrm{D}} \begin{bmatrix} P(P+Q)^{k-3} & P(P+Q)^{k-3}Q \\ (P+Q)^{k-3} & (P+Q)^{k-3}Q \end{bmatrix} \begin{bmatrix} P \\ I \end{bmatrix}.$$

This requires computation of $[P(P+Q)^{k-1}]^{\mathrm{D}}$ and $[(P+Q)^{k-1}Q]^{\mathrm{D}}$, which may actually be easier than that of $(P+Q)^{\mathrm{D}}$.

A second attempt to generalize Theorem 6.1 would be to assume only that $P^2Q = 0$. Needless to say, this is the best attempted via the block form, which in turn should give a suitable formula.

Now, we will investigate explicit representations for the Drazin inverse $(A+E)^{\mathrm{D}}$ in the case when $AE = EA$, which was considered in [18, Theorem 2]. For $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $\mathrm{rank}(A^k) = r$, there exists an nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that

$$A = P\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}P^{-1}, \tag{6.16}$$

where $C \in \mathbb{C}^{r\times r}$ is a nonsingular matrix, N is nilpotent of index k and $\mathrm{Ind}(N) = \mathrm{Ind}(A) = k$. In that case

$$A^{\mathrm{D}} = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}, \tag{6.17}$$

If $P = I$, then the block-diagonal matrices A and A^{D} are written as $A = C \oplus N$ and $A^{\mathrm{D}} = C^{-1} \oplus 0$.

Now we state the following result which was obtained by Hartwig and Shoaf [19] and Meyer and Rose [20], since it will be used in the theorem to follow.

Theorem 6.2 *If* $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, *where* $A \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{m\times m}$ *with* $\mathrm{Ind}(A) = k$ *and* $\mathrm{Ind}(B) = l$, *then* $M^{\mathrm{D}} = \begin{bmatrix} A^{\mathrm{D}} & X \\ 0 & B^{\mathrm{D}} \end{bmatrix}$, *where*

$$X = \left[\sum_{n=0}^{l-1}(A^{\mathrm{D}})^{n+2}CB^n\right](I - BB^{\mathrm{D}}) + (I - AA^{\mathrm{D}})\left[\sum_{n=0}^{k-1}A^n C(B^{\mathrm{D}})^{n+2}\right] - A^{\mathrm{D}}CB^{\mathrm{D}}.$$

Theorem 6.3 *If* $A, E \in \mathbb{C}^{n\times n}$, $AE = EA$ *and* $\mathrm{Ind}(A) = k$, *then*

$$\begin{aligned}(A+E)^{\mathrm{D}} &= (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} + (I - AA^{\mathrm{D}})\sum_{i=0}^{k-1}(E^{\mathrm{D}})^{i+1}(-A)^i \\ &= (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} + (I - AA^{\mathrm{D}})E^{\mathrm{D}}[I + A(I - AA^{\mathrm{D}})E^{\mathrm{D}}]^{-1},\end{aligned}$$

and

$$(A+E)^{\mathrm{D}}(A+E) = (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}}(A+E) + (I - AA^{\mathrm{D}})EE^{\mathrm{D}}.$$

Proof Let $A \in \mathbb{C}^{n\times n}$ be given by (6.16). Without loss of generality, we assume that $P = I$ and $A = C \oplus N$, where C is invertible and N is nilpotent with $N^k = 0$. From $AE = EA$, we have $A^kE = EA^k$. Now $E = E_1 \oplus E_2$, $CE_1 = E_1C$ and $NE_2 = E_2N$. Hence

$$(A+E)^{\mathrm{D}} = (C + E_1)^{\mathrm{D}} \oplus (N + E_2)^{\mathrm{D}}.$$

Since C and $I + C^{-1}E_1$ commute, we get

$$(C + E_1)^{\mathrm{D}} \oplus 0 = (I + C^{-1}E_1)^{\mathrm{D}}C^{-1} \oplus 0 = (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}}.$$

Notice that $(I + T)^{-1} = \sum_{i=0}^{k-1}(-T)^i$ if $T^k = 0$. Applying Lemma 4 [19] we get $(N + E_2)^{\mathrm{D}} = E_2^{\mathrm{D}}(I + E_2^{\mathrm{D}}N)^{-1}$ and

$$\begin{aligned}0 \oplus E_2^{\mathrm{D}}(I + E_2^{\mathrm{D}}N)^{-1} &= 0 \oplus \sum_{i=0}^{k-1}(E_2^{\mathrm{D}})^{i+1}(-N)^i = (I - AA^{\mathrm{D}})\sum_{i=0}^{k-1}(E^{\mathrm{D}})^{i+1}(-A)^i \\ &= (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} + (I - AA^{\mathrm{D}})E^{\mathrm{D}}[I + A(I - AA^{\mathrm{D}})E^{\mathrm{D}}]^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}(A+E)^{\mathrm{D}} &= (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} + (I - AA^{\mathrm{D}})\sum_{i=0}^{k-1}(E^{\mathrm{D}})^{i+1}(-A)^i \\ &= (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} + (I - AA^{\mathrm{D}})E^{\mathrm{D}}[I + A(I - AA^{\mathrm{D}})E^{\mathrm{D}}]^{-1},\end{aligned}$$

and

$$
\begin{aligned}
&(A+E)^{\mathrm{D}}(A+E) \\
&= \left\{(I+A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} + (I-AA^{\mathrm{D}})\sum_{i=0}^{k-1}(E^{\mathrm{D}})^{i+1}(-A)^i\right\}(A+E) \\
&= (I+A^{\mathrm{D}}E)^{\mathrm{D}}(A+E)A^{\mathrm{D}} + (I-AA^{\mathrm{D}})E^{\mathrm{D}}A\sum_{i=0}^{k-1}(E^{\mathrm{D}})^i(-A)^i \\
&\quad +(I-AA^{\mathrm{D}})E^{\mathrm{D}}E\sum_{i=0}^{k-1}(E^{\mathrm{D}})^i(-A)^i \\
&= (I+A^{\mathrm{D}}E)^{\mathrm{D}}(A+E)A^{\mathrm{D}} + (I-AA^{\mathrm{D}})\Bigg(-\sum_{i=1}^{k}(E^{\mathrm{D}})^i(-A)^i\Bigg) \\
&\quad +(I-AA^{\mathrm{D}})\Bigg(E^{\mathrm{D}}E + \sum_{i=1}^{k-1}(E^{\mathrm{D}})^i(-A)^i\Bigg) \\
&= (I+A^{\mathrm{D}}E)^{\mathrm{D}}(A+E)A^{\mathrm{D}} + (I-AA^{\mathrm{D}})EE^{\mathrm{D}}.
\end{aligned}
$$

□

From Theorem 6.3, we can see that the generalized Schur complement $I + A^{\mathrm{D}}E$ [21] plays an important role in the representation of the Drazin inverse $(A+E)^{\mathrm{D}}$. In some special cases, it is possible to give an expression for $(I+A^{\mathrm{D}}E)^{\mathrm{D}}$.

Theorem 6.4 *Let* $A, E \in \mathbb{C}^{n\times n}$ *be such that* $AE = EA$ *and let* $\mathrm{Ind}(A) = k$ *and* $\mathrm{Ind}(E) = l$.

(1) *If* $A^{\mathrm{D}}E^{\mathrm{D}} = 0$, *then*

$$
(A+E)^{\mathrm{D}} = (I-AA^{\mathrm{D}})\sum_{i=0}^{k-1}(E^{\mathrm{D}})^{i+1}(-A)^i + \sum_{i=0}^{l-1}(-E)^i(A^{\mathrm{D}})^{i+1}(I-EE^{\mathrm{D}}).
$$

(2) *If* $A^{\mathrm{D}}E = 0$, *then* $(A+E)^{\mathrm{D}} = A^{\mathrm{D}} + (I-AA^{\mathrm{D}})\sum_{i=0}^{k-1}(E^{\mathrm{D}})^{i+1}(-A)^i$.

(3) *If* $\mathrm{Ind}(A) = 1$, *then* $(A+E)^{\mathrm{D}} = (I+A^{\#}E)^{\mathrm{D}}A^{\#} + (I-AA^{\#})E^{\mathrm{D}}$.

Proof We use the notations from the proof of Theorem 6.3.

(1) If $A^{\mathrm{D}}E^{\mathrm{D}} = 0$, then E_1 is nilpotent with $E_1^l = 0$. So we have

$$
\begin{aligned}
(I+A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} &= (I+C^{-1}E_1)^{-1}C^{-1}\oplus 0 = \sum_{i=0}^{l-1}(-E_1)^i(C)^{-(i+1)}\oplus 0 \\
&= \sum_{i=0}^{l-1}(-E)^i(A^{\mathrm{D}})^{i+1}(I-EE^{\mathrm{D}}).
\end{aligned}
$$

The result now follows from Theorem 6.3.

(2)–(3) Note that if $A^{\mathrm{D}}E = 0$, then $E_1 = 0$; if $\mathrm{Ind}(A) = 1$, then $N = 0$. The results follow directly from the proof of Theorem 6.3. □

Let $A, E \in \mathbb{C}^{n\times n}$. If there exists a nonzero idempotent matrix $P = P^2$ such that $AEP = EAP$ (or $PAE = PEA$), then A and E are partially commutative. For $A, E \in \mathbb{C}^{n\times n}$, let $A^{\pi} = I - AA^{\mathrm{D}}$ and $\mathrm{Ind}(A) = k$ and suppose $E^2 = 0$. In [22], Castro-González proved that if $A^{\pi}E = E$ and $AEA^{\pi} = 0$, then

$$(A+E)^{\mathrm{D}} = A^{\mathrm{D}} + \sum_{i=0}^{k} A^i E (A^{\mathrm{D}})^{i+2} + \sum_{i=0}^{k-1} EA^i E (A^{\mathrm{D}})^{i+3}.$$

But no representations of $(A+E)^{\mathrm{D}}$ assuming only partial commutativity are known. Under the conditions $A^{\pi}E = E$ and $AEA^{\pi} = EAA^{\pi}$, we are able to give an expression for $(A+E)^{\mathrm{D}}$.

Theorem 6.5 *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{Ind}(A) = k$ and $E \in \mathbb{C}^{n\times n}$ be nilpotent of index l. If $EA^{\mathrm{D}} = 0$ and $A^{\pi}AE = A^{\pi}EA$, then*

$$(A+E)^{\mathrm{D}} = A^{\mathrm{D}} + \sum_{i=0}^{k+l-2} (A^{\mathrm{D}})^{i+2} ET(i),$$

where $T(i) = (I - AA^{\mathrm{D}}) \sum_{j=0}^{i} \binom{j}{i} A^j E^{i-j}$.

Proof Similarly as in the proof of Theorem 6.3, let $A = C \oplus N$, where C is invertible and N is nilpotent with $N^k = 0$. It follows from $EA^{\mathrm{D}} = 0$ that E can be written as $E = \begin{bmatrix} 0 & E_1 \\ 0 & E_2 \end{bmatrix}$ with $E_2^l = 0$. Also by $A^{\pi}AE = A^{\pi}EA$, we get $E_2N = NE_2$. Thus

$$(N+E_2)^i \oplus 0 = \sum_{j=0}^{i} \binom{j}{i} N^j E_2^{i-j} \oplus 0 = (I - AA^{\mathrm{D}}) \left(\sum_{j=0}^{i} \binom{j}{i} A^j E^{i-j} \right) = T(i).$$

We observe that $N + E_2$ is nilpotent of index $k + l - 1$. From Theorem 6.2, we further obtain

$$(A+E)^{\mathrm{D}} = \begin{bmatrix} C & E_1 \\ 0 & N + E_2 \end{bmatrix}^{\mathrm{D}} = \begin{bmatrix} C^{-1} & X \\ 0 & 0 \end{bmatrix},$$

where

$$X = \sum_{i=0}^{k+l-2} C^{-(i+2)} E_1 (N + E_2)^i = \sum_{i=0}^{k+l-2} C^{-(i+2)} E_1 \left(\sum_{j=0}^{i} \binom{j}{i} N^j E_2^{i-j} \right). \quad (6.18)$$

Hence

$$A^{\mathrm{D}} + \sum_{i=0}^{k+l-2} (A^{\mathrm{D}})^{i+2} E T(i) = \begin{bmatrix} C^{-1} & \sum_{i=0}^{k+l-2} C^{-(i+2)} E_1 \left(\sum_{j=0}^{i} \binom{j}{i} N^j E_2^{i-j} \right) \\ 0 & 0 \end{bmatrix} = (A+E)^{\mathrm{D}}.$$

□

The following result generalizes Theorems 6.3 and 6.5 to the case of partial commutativity.

Theorem 6.6 *Let $A, E \in \mathbb{C}^{n\times n}$ and $\mathrm{Ind}(A) = k$. Also let $Q \in \mathbb{C}^{n\times n}$ be an idempotent matrix such that $QA = AQ$ and $EQ = 0$. If $(I-Q)AE = (I-Q)EA$, then*

$$\begin{aligned}(A+E)^{\mathrm{D}} = {} & QA^{\mathrm{D}} + (I-Q)\Psi - QA^{\mathrm{D}}E\Psi + Q(I - AA^{\mathrm{D}}) \left[\sum_{i=0}^{k-1} A^i E \Psi^{i+2} \right] \\ & + Q \left[\sum_{i=0}^{h-1} (A^{\mathrm{D}})^{i+2} E (A+E)^i \right] (I-Q)[I - (A+E)\Psi],\end{aligned} \tag{6.19}$$

where $\Psi = (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} + (I - AA^{\mathrm{D}}) \sum_{i=0}^{k-1} (E^{\mathrm{D}})^{i+1}(-A)^i$ *and* $h = \mathrm{Ind}\,[(I-Q)(A+E)]$.

Proof Suppose that $Q = I_{r\times r} \oplus 0_{(n-r)\times(n-r)}$, where $r \le n$. If $QA = AQ$, $EQ = 0$ and $(I-Q)AE = (I-Q)EA$, then $A = A_1 \oplus A_2$ and $E = \begin{bmatrix} 0 & E_1 \\ 0 & E_2 \end{bmatrix}$ with $A_2E_2 = E_2A_2$. Using Theorems 6.2 and 6.3, we have

$$(A+E)^{\mathrm{D}} = \begin{bmatrix} A_1^{\mathrm{D}} & X \\ 0 & (I + A_2^{\mathrm{D}}E_2)^{\mathrm{D}}A_2^{\mathrm{D}} + (I - A_2A_2^{\mathrm{D}}) \sum_{i=0}^{k-1} (E_2^{\mathrm{D}})^{i+1}(-A_2)^i \end{bmatrix},$$

where

$$\begin{aligned} X = {} & \left[\sum_{i=0}^{h-1} (A_1^{\mathrm{D}})^{i+2} E_1 (A_2+E_2)^i \right] [I - (A_2+E_2)(A_2+E_2)^{\mathrm{D}}] \\ & + (I - A_1A_1^{\mathrm{D}}) \left[\sum_{i=0}^{k-1} A_1^i E_1 ((A_2+E_2)^{\mathrm{D}})^{i+2} \right] - A_1^{\mathrm{D}}E_1(A_2+E_2)^{\mathrm{D}}, \end{aligned}$$

and $\mathrm{Ind}(A_2+E_2) = h$.

If we write $\Psi = (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} + (I - AA^{\mathrm{D}}) \sum_{i=0}^{k-1} (E^{\mathrm{D}})^{i+1}(-A)^i$, then $(I-Q)\Psi = 0 \oplus (A_2+E_2)^{\mathrm{D}}$. We can simplify the expression for $(A+E)^{\mathrm{D}}$ using the block decomposition above. We deduce

$$\begin{aligned}\Sigma_1 &= Q\left[\sum_{i=0}^{h-1}(A^{\mathrm{D}})^{i+2}E(A+E)^i\right](I-Q)[I-(A+E)\Psi] \\ &= \begin{bmatrix} 0 & \left[\sum_{i=0}^{h-1}(A_1^{\mathrm{D}})^{i+2}E_1(A_2+E_2)^i\right][I-(A_2+E_2)(A_2+E_2)^{\mathrm{D}}] \\ 0 & 0 \end{bmatrix}, \\ \Sigma_2 &= QA^{\pi}\left[\sum_{i=0}^{k-1}A^iE\Psi^{i+2}\right] = \begin{bmatrix} 0 & A_1^{\pi}\left[\sum_{i=0}^{k-1}A_1^iE_1((A_2+E_2)^{\mathrm{D}})^{i+2}\right] \\ 0 & 0 \end{bmatrix}\end{aligned}$$

and $\Sigma_3 = QA^{\mathrm{D}}E\Psi = \begin{bmatrix} 0 & A_1^{\mathrm{D}}E_1(A_2+E_2)^{\mathrm{D}} \\ 0 & 0 \end{bmatrix}$.

Thus

$$(A+E)^{\mathrm{D}} = QA^{\mathrm{D}} + (I-Q)\Psi + \Sigma_1 + \Sigma_2 - \Sigma_3.$$

□

Now a few special cases follow immediately.

Corollary 6.3 *Let* $A, E \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(A) = k$ *and* $\mathrm{Ind}(E) = l$.

(1) *If* $EA^{\pi} = 0$ *and* $(I - A^{\pi})AE = (I - A^{\pi})EA$, *then*

$$(A+E)^{\mathrm{D}} = AA^{\mathrm{D}}\Psi + (I - AA^{\mathrm{D}})\left[\sum_{i=0}^{k-1}A^iE\Psi^{i+2}\right],$$

where $\Psi = (I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} + (I - AA^{\mathrm{D}})\sum_{i=0}^{k-1}(E^{\mathrm{D}})^{i+1}(-A)^i$.

(2) *If* E *is nilpotent,* $EA^{\pi} = E$ *and* $A^{\pi}AE = A^{\pi}EA$, *then*

$$(A+E)^{\mathrm{D}} = A^{\mathrm{D}} + \sum_{i=0}^{k+l-2}(A^{\mathrm{D}})^{i+2}E(A+E)^i.$$

Proof We adopt the notations from Theorem 6.6.

(1) Let $Q = I - AA^{\mathrm{D}}$ in Theorem 6.6 and apply $QA^{\mathrm{D}} = 0$ to (6.19).

(2) Let $Q = AA^{\mathrm{D}}$ in Theorem 6.6. Since $EA^{\pi} = E$, we obtain $EA^{\mathrm{D}} = EA^{\pi}A^{\mathrm{D}} = 0$. Thus $(A^{\mathrm{D}}E)^2 = A^{\mathrm{D}}EA^{\mathrm{D}}E = 0$ and $(I + A^{\mathrm{D}}E)^{\mathrm{D}}A^{\mathrm{D}} = (I + A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}} = A^{\mathrm{D}}$. Note that E is nilpotent so that $\Psi = A^{\mathrm{D}}$. Hence

$$E(A+E)^i(I-Q)[I-(A+E)\Psi] = E(A+E)^iA^{\pi} = E(A+E)^i, \quad for \quad i \geq 0.$$

The result follows directly from (6.19). □

Let A be an $n \times n$ complex matrix and $B = A + E$ be a perturbation of A. The classical Bauer-Fike theorem on eigenvalue perturbation gives a bound on the distance between an eigenvalue μ of B and the closest eigenvalue λ of A, which is required to be diagonalizable.

Let $A = X\Sigma X^{-1}$ be an eigendecomposition, where Σ is a diagonal matrix, and X is an eigenvector matrix. The Bauer-Fike theorem [23, Theorem IIIa] states that for any eigenvalue μ of B, there exists an eigenvalue λ of A such that $|\mu - \lambda| \le \kappa(X)\|E\|$, where $\kappa(X) = \|X\|\,\|X^{-1}\|$ is the condition number of X.

The relative perturbation version of the Bauer-Fike theorem [24, Corollary 2.2] below requires, in addition, that A be invertible. That is, if A is diagonalizable and invertible, then for any eigenvalue μ of B, there exists an eigenvalue λ of A such that

$$\frac{|\mu - \lambda|}{|\lambda|} \le \kappa(X)\|A^{-1}E\|. \tag{6.20}$$

Without the assumption of diagonalizability and invertibility of A, we refine the bound (6.20) under the condition that $AE = EA$.

Theorem 6.7 *Let $B = A + E \in \mathbb{C}^{n\times n}$ be such that A is not nilpotent and $AE = EA$. For any eigenvalue μ of B, there exists a nonzero eigenvalue λ of A such that*

$$\frac{|\mu - \lambda|}{|\lambda|} \le \rho(A^{\mathrm{D}}E), \tag{6.21}$$

where $\rho(A^{\mathrm{D}}E)$ is the spectral radius of $A^{\mathrm{D}}E$.

Proof Assume that $AE = EA$ and that A is not nilpotent. Then for any nonzero eigenvalue λ of A, there exits a common eigenvector x [25, p.250] such that

$$Ax = \lambda x, \qquad (A + E)x = \mu x.$$

Therefore

$$A^{\mathrm{D}}x = \frac{1}{\lambda}x, \qquad A^{\mathrm{D}}Ex = A^{\mathrm{D}}(\mu x - Ax) = (\mu - \lambda)A^{\mathrm{D}}x = \frac{\mu - \lambda}{\lambda}x,$$

whence

$$\frac{|\mu - \lambda|}{|\lambda|} \le \rho(A^{\mathrm{D}}E).$$

□

Recently, the perturbation of the Drazin inverse has been studied by several authors ([6, 9, 22, 26–33]). As one application of our results in Theorem 6.3, we can establish upper bounds for the relative error $\|B^{\mathrm{D}}\|$ and $\|B^{\mathrm{D}} - A^{\mathrm{D}}\|/\|A^{\mathrm{D}}\|$ under the assumption that $AE = EA$.

Theorem 6.8 *If $B = A + E \in \mathbb{C}^{n\times n}$, $AE = EA$ and* $\max\{\|A^{\mathrm{D}}E\|, \|A^{\pi}AE^{\mathrm{D}}\|\} < 1$*, then*

$$\|B^{\mathrm{D}}\| \le \frac{\|A^{\mathrm{D}}\|}{1 - \|A^{\mathrm{D}}E\|} + \frac{\|A^{\pi}E^{\mathrm{D}}\|}{1 - \|A^{\pi}AE^{\mathrm{D}}\|}$$

and

$$\frac{\|B^{\mathrm{D}}-A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \leq \frac{\|A^{\mathrm{D}}E\|}{1-\|A^{\mathrm{D}}E\|} + \frac{\|A\|\|E^{\mathrm{D}}\|}{1-\|A^{\pi}AE^{\mathrm{D}}\|}.$$

Proof Note that the assumption $\max\{\|A^{\mathrm{D}}E\|, \|A^{\pi}AE^{\mathrm{D}}\|\} < 1$ implies invertibility of $I + A^{\mathrm{D}}E$ and $I + A^{\pi}AE^{\mathrm{D}}$. It follows directly from Theorem 6.3 that

$$\begin{aligned}\|B^{\mathrm{D}}\| &\leq \|(I + A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}}\| + \|A^{\pi}E^{\mathrm{D}}[I + A^{\pi}AE^{\mathrm{D}}]^{-1}\| \\ &\leq \frac{\|A^{\mathrm{D}}\|}{1-\|A^{\mathrm{D}}E\|} + \frac{\|A^{\pi}E^{\mathrm{D}}\|}{1-\|A^{\pi}AE^{\mathrm{D}}\|},\end{aligned}$$

and

$$\begin{aligned}\|B^{\mathrm{D}}-A^{\mathrm{D}}\| &\leq \|(I + A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}} - A^{\mathrm{D}}\| + \|A^{\pi}E^{\mathrm{D}}[I + A^{\pi}AE^{\mathrm{D}}]^{-1}\| \\ &\leq \|(I + A^{\mathrm{D}}E)^{-1}A^{\mathrm{D}}EA^{\mathrm{D}}\| + \frac{\|A^{\pi}\|\|E^{\mathrm{D}}\|}{1-\|A^{\pi}AE^{\mathrm{D}}\|} \\ &\leq \frac{\|A^{\mathrm{D}}E\|\|A^{\mathrm{D}}\|}{1-\|A^{\mathrm{D}}E\|} + \frac{\|AA^{\mathrm{D}}\|\|E^{\mathrm{D}}\|}{1-\|A^{\pi}AE^{\mathrm{D}}\|} \\ &\leq \left(\frac{\|A^{\mathrm{D}}E\|}{1-\|A^{\mathrm{D}}E\|} + \frac{\|A\|\|E^{\mathrm{D}}\|}{1-\|A^{\pi}AE^{\mathrm{D}}\|}\right)\|A^{\mathrm{D}}\|.\end{aligned}$$

□

Remark 6.1.2 For any non-zero eigenvalue μ of the spectral set $\sigma(A+E)$, we can estimate its lower bound: let $\mu \in \sigma(A+E)$. We have $1/\mu \in \sigma[(A+E)^{\mathrm{D}}]$ and $|1/\mu| \leq \rho[(A+E)^{\mathrm{D}}] \leq \|(A+E)^{\mathrm{D}}\|$, i.e.,

$$|\mu| \geq 1/\|(A+E)^{\mathrm{D}}\| \geq 1/\left[\frac{\|A^{\mathrm{D}}\|}{1-\|A^{\mathrm{D}}E\|} + \frac{\|A^{\pi}E^{\mathrm{D}}\|}{1-\|A^{\pi}AE^{\mathrm{D}}\|}\right].$$

Next we will apply Theorem 6.5 to obtain a perturbation bound in terms of A^{D} and $\mathscr{E}_l = B^l - A^l$ for some positive integer l.

Theorem 6.9 *Let $B = A + E \in \mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(A) = k$ *and* $\mathrm{Ind}(B) = s$*. Denote $\mathscr{E}_l = B^l - A^l$, where $l = \max\{k, s\}$. Assume that the conditions in Theorem 6.5 hold. Then*

$$\frac{\|B^{\mathrm{D}}-A^{\mathrm{D}}\|}{\|A^{\mathrm{D}}\|} \leq \|B^{\pi} - A^{\pi}\| = \|(A^{\mathrm{D}})^l\mathscr{E}_l\|. \tag{6.22}$$

Proof Since $l = \max\{k, s\}$, using the notations in the proof of Theorem 6.5, we have

$$\mathscr{E}_l = B^l - A^l = \begin{bmatrix} 0 & \sum_{i=0}^{l-1} C^{l-1-i}E_1\Big((N+E_2)^i\Big) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sum_{i=0}^{l-1} C^{l-1-i}E_1\Big(\sum_{j=0}^{i}\binom{j}{i}N^jE_2^{i-j}\Big) \\ 0 & 0 \end{bmatrix}.$$

Then

$$A^{\mathrm{D}} + (A^{\mathrm{D}})^{l+1}\mathscr{E}_l = \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (C^{-1})^{l+1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sum_{i=0}^{l-1} C^{l-1-i} E_1 \left(\sum_{j=0}^{i} \binom{j}{i} N^j E_2^{i-j} \right) \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} C^{-1} & \sum_{i=0}^{l-1} (C^{-1})^{i+2} E_1 \left(\sum_{j=0}^{i} \binom{j}{i} N^j E_2^{i-j} \right) \\ 0 & 0 \end{bmatrix} = B^{\mathrm{D}},$$

and

$$AA^{\mathrm{D}} + (A^{\mathrm{D}})^{l}\mathscr{E}_l = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (C^{-1})^{l} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sum_{i=0}^{l-1} C^{l-1-i} E_1 \left(\sum_{j=0}^{i} \binom{j}{i} N^j E_2^{i-j} \right) \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} I & \sum_{i=0}^{l-1} (C^{-1})^{i+1} E_1 \left(\sum_{j=0}^{i} \binom{j}{i} N^j E_2^{i-j} \right) \\ 0 & 0 \end{bmatrix} = BB^{\mathrm{D}}.$$

We then have

$$\|B^{\mathrm{D}} - A^{\mathrm{D}}\| = \|(A^{\mathrm{D}})^{l+1}\mathscr{E}_l\| \leq \|A^{\mathrm{D}}\| \|(A^{\mathrm{D}})^{l}\mathscr{E}_l\|,$$

and

$$\|B^{\pi} - A^{\pi}\| = \|BB^{\mathrm{D}} - AA^{\mathrm{D}}\| = \|(A^{\mathrm{D}})^{l}\mathscr{E}_l\|.$$

The proof is complete. □

Generalizations of the results of this section to linear operators on Banach spaces can be found in [9, 34–36] while their generalizations to Banach algebra elements can be found in [37] and some will also be given in the next section where the generalized Drazin inverse will be considered.

6.2 Additive Results for the Generalized Drazin Inverse in Banach Algebra

Let $\mathscr{A}$ be a complex Banach algebra with the unit 1. By $\mathscr{A}^{-1}$, $\mathscr{A}^{\mathsf{nil}}$, $\mathscr{A}^{\mathsf{qnil}}$ we denote the sets of all invertible, nilpotent and quasi-nilpotent elements in $\mathscr{A}$, respectively. Let us recall that the Drazin inverse of $a \in \mathscr{A}$ [1] is the (unique) element $x \in \mathscr{A}$ (denoted by a^{D}) which satisfies

$$xax = x, \quad ax = xa, \quad a^{k+1}x = a^k, \tag{6.23}$$

for some nonnegative integer k. The least such k is the index of a, denoted by $\mathrm{ind}(a)$. When $\mathrm{ind}(a) = 1$ then the Drazin inverse a^{D} is called the group inverse and it is denoted by $a^{\#}$. The conditions (6.23) are equivalent to

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathscr{A}^{\mathsf{nil}}. \tag{6.24}$$

The concept of the generalized Drazin inverse in a Banach algebra was introduced by Koliha [38]. The condition $a - a^2x \in \mathscr{A}^{\mathsf{nil}}$ was replaced by $a - a^2x \in \mathscr{A}^{\mathsf{qnil}}$. Hence, the generalized Drazin inverse of a is the (unique) element $x \in \mathscr{A}$ (written a^{d}) which satisfies

$$xax = x, \quad ax = xa, \quad a - a^2x \in \mathscr{A}^{\mathsf{qnil}}. \tag{6.25}$$

We mention that an alternative definition of the generalized Drazin inverse in a ring is also given in [39–41]. These two concepts of the generalized Drazin inverse are equivalent in the case when the ring is actually a complex Banach algebra with a unit. It is well known that a^{d} is unique whenever it exists [38]. The set $\mathscr{A}^{\mathsf{d}}$ consists of all $a \in \mathscr{A}$ such that a^{d} exists. For many interesting properties of the Drazin inverse see [1, 38, 42].

This section is a continuation of the previous one with the difference that here we investigate additive properties of the generalized Drazin inverse in a Banach algebra and find explicit expressions for the generalized Drazin inverse of the sum $a + b$ under various conditions.

Hartwig et al. [10] for matrices and Djordjević and Wei [9] for operators used the condition $AB = 0$ to derive a formula for $(A + B)^{\mathsf{d}}$. After that Castro and Koliha [43] relaxed this hypothesis by assuming the following complimentary condition symmetric in $a, b \in \mathscr{A}^{\mathsf{d}}$,

$$a^{\pi}b = b, \quad ab^{\pi} = a, \quad b^{\pi}aba^{\pi} = 0 \tag{6.26}$$

thus generalizing the results from [9]. It is easy to see that $ab = 0$ implies (6.26), but the converse is not true (see [43, Example 3.1]).

In the first part of the section we will find some new conditions, which are not equivalent with the conditions from [43], allowing for the generalized Drazin inverse of $a + b$ to be expressed in terms of $a, a^{\mathsf{d}}, b, b^{\mathsf{d}}$. It is interesting to note that in some cases the same expression for $(a + b)^{\mathsf{d}}$ are obtained as in [43]. In the rest of the section we will generalize some recent results from [43].

Let $a \in \mathscr{A}$ and let $p \in \mathscr{A}$ be an idempotent ($p = p^2$). Then we can write

$$a = pap + pa(1 - p) + (1 - p)ap + (1 - p)a(1 - p)$$

and use the notations

$$a_{11} = pap, \quad a_{12} = pa(1 - p), \quad a_{21} = (1 - p)ap, \quad a_{22} = (1 - p)a(1 - p).$$

Every idempotent $p \in \mathscr{A}$ induces a representation of an arbitrary element $a \in \mathscr{A}$ given by the following matrix

$$a = \begin{bmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{bmatrix}_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p . \tag{6.27}$$

Let a^{π} be the spectral idempotent of a corresponding to $\{0\}$. It is well known that $a \in \mathscr{A}^{\mathsf{d}}$ can be represented in the matrix form:

$$a = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_p ,$$

relative to $p = aa^{\mathsf{d}} = 1 - a^{\pi}$, where a_{11} is invertible in the algebra $p\mathscr{A}p$ and a_{22} is quasi-nilpotent in the algebra $(1-p)\mathscr{A}(1-p)$. Then the generalized Drazin inverse is given by

$$a^{\mathsf{d}} = \begin{bmatrix} a_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p .$$

The following result is proved in [4, 20] for matrices, extended in [44] for a bounded linear operator and in [43] for arbitrary elements in a Banach algebra.

Theorem 6.10 *Let $x, y \in \mathscr{A}$ and*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p , \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{(1-p)}$$

relative to the idempotent $p \in \mathscr{A}$.

(1) *If $a \in (p\mathscr{A}p)^{\mathsf{d}}$ and $b \in ((1-p)\mathscr{A}(1-p))^{\mathsf{d}}$, then x and y are Drazin invertible and*

$$x^{\mathsf{d}} = \begin{bmatrix} a^{\mathsf{d}} & u \\ 0 & b^{\mathsf{d}} \end{bmatrix}_p , \quad y^{\mathsf{d}} = \begin{bmatrix} b^{\mathsf{d}} & 0 \\ u & a^{\mathsf{d}} \end{bmatrix}_{(1-p)} \tag{6.28}$$

where $u = \sum_{n=0}^{\infty} (a^{\mathsf{d}})^{n+2} c b^n b^{\pi} + \sum_{n=0}^{\infty} a^{\pi} a^n c (b^{\mathsf{d}})^{n+2} - a^{\mathsf{d}} c b^{\mathsf{d}}$.

(2) *If $x \in \mathscr{A}^{\mathsf{d}}$ and $a \in (p\mathscr{A}p)^{\mathsf{d}}$, then $b \in ((1-p)\mathscr{A}(1-p))^{\mathsf{d}}$ and $x^{\mathsf{d}}, y^{\mathsf{d}}$ are given by (6.28).*

We will need the following auxiliary result.

Lemma 6.2 *Let $a, b \in \mathscr{A}^{\mathsf{qnil}}$. If $ab = ba$ or $ab = 0$, then $a + b \in \mathscr{A}^{\mathsf{qnil}}$.*

Proof If $ab = ba$, we have that

$$\rho(a + b) \leq \rho(a) + \rho(b),$$

which gives $a + b \in \mathscr{A}^{\mathsf{qnil}}$. The case when $ab = 0$ follows from the equation

$$(\lambda - a)(\lambda - b) = \lambda(\lambda - (a + b))$$

□

In view of the previous lemma, the first approach to the problem addressed in this section was to replace the condition $ab = 0$ used in [9, 10] by $ab = ba$. As expected, this alone was not enough to derive a formula for $(a + b)^{\mathsf{d}}$. We will thus impose the following three conditions on $a, b \in \mathscr{A}^{\mathsf{d}}$:

$$a = ab^{\pi}, \quad b^{\pi}ba^{\pi} = b^{\pi}b, \quad b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab. \tag{6.29}$$

Instead of the condition $ab = ba$ we are thus assuming the weaker condition $b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab$. Notice that

$$a = ab^{\pi} \Leftrightarrow ab^{\mathsf{d}} = 0 \Leftrightarrow \mathscr{A}a \subseteq \mathscr{A}b^{\pi}, \tag{6.30}$$

$$b^{\pi}ba^{\pi} = b^{\pi}b \Leftrightarrow b^{\pi}ba^{\mathsf{d}} = 0 \Leftrightarrow \mathscr{A}b^{\pi}b \subseteq \mathscr{A}a^{\pi}, \tag{6.31}$$

$$b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab \Leftrightarrow (ba - ab)\mathscr{A} \subseteq (b^{\pi}a^{\pi})^{\circ}, \tag{6.32}$$

where for $u \in \mathscr{A}$, $u^{\circ} = \{x \in \mathscr{A} \ : \ ux = 0\}$.

For matrices and bounded linear operators on a Banach space the conditions (6.30)–(6.32) are equivalent to

$$\mathscr{N}(b^{\pi}) \subseteq \mathscr{N}(a), \quad \mathscr{N}(a^{\pi}) \subseteq \mathscr{N}(b^{\pi}b), \quad \mathscr{R}(ba - ab) \subseteq \mathscr{N}(b^{\pi}a^{\pi}).$$

Remark that, unlike the conditions (3.1) from [43], the conditions (6.29) are not symmetric in a, b so our expression for $(a + b)^{\mathsf{d}}$ will not be symmetric in a, b.

In the next theorem, under the assumption that (6.29) holds, we can give an expression for $(a + b)^{\mathsf{d}}$ as follows.

Theorem 6.11 *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ be such that (6.29) is satisfied. Then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = (b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n)a^{\pi} \tag{6.33}$$
$$- \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n(a^{\mathsf{d}})^{k+2}b(a+b)^{k+1}$$
$$+ \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n a^{\mathsf{d}} b - \sum_{n=0}^{\infty} b^{\mathsf{d}} a (a^{\mathsf{d}})^{n+2} b(a+b)^n$$

Before proving Theorem 6.11, we first have to prove the special case of it given below.

Theorem 6.12 *Let $a \in \mathscr{A}^{\mathsf{qnil}}$, $b \in \mathscr{A}^{\mathsf{d}}$ satisfy $b^{\pi}ab = b^{\pi}ba$ and $a = ab^{\pi}$. Then (6.29) is satisfied, $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n. \tag{6.34}$$

Proof First, suppose that $b \in \mathscr{A}^{\mathsf{qnil}}$. Then $b^{\pi} = 1$ and from $b^{\pi}ab = b^{\pi}ba$ we obtain $ab = ba$. Using Lemma 6.2, $a + b \in \mathscr{A}^{\mathsf{qnil}}$ and (6.28) holds. Now, we assume that b is not quasi-nilpotent and consider the matrix representations of a and b relative to $p = 1 - b^{\pi}$. We have

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $b_1 \in (p\mathscr{A}p)^{-1}$ and $b_2 \in ((1-p)\mathscr{A}(1-p))^{\mathsf{qnil}} \subset \mathscr{A}^{\mathsf{qnil}}$. From $a = ab^{\pi}$, it follows that $a_{11} = 0$ and $a_{21} = 0$. We denote $a_1 = a_{12}$ and $a_2 = a_{22}$. Hence

$$a + b = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}_p.$$

The condition $b^{\pi}ab = b^{\pi}ba$ implies that $a_2b_2 = b_2a_2$. Hence, using Lemma 6.2, we get $a_2 + b_2 \in ((1-p)\mathscr{A}(1-p))^{\mathsf{qnil}}$. Now, by Theorem 6.10, we obtain $a + b \in \mathscr{A}^{\mathsf{d}}$ and

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} b_1^{-1} & \sum_{n=0}^{\infty} b_1^{-(n+2)}a_1(a_2+b_2)^n \\ 0 & 0 \end{bmatrix}_p$$
$$= b^{\mathsf{d}} + \sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+2}a(a+b)^n.$$

□

Let us observe that the expression for $(a+b)^{\mathsf{d}}$ in (6.28) and that in (3.6) of Theorem 3.3 in [43] are exactly the same. If we assume that $ab = ba$ instead of $b^{\pi}ab = b^{\pi}ba$, we get a much simpler expression for $(a+b)^{\mathsf{d}}$.

Corollary 6.4 *Suppose* $a \in \mathscr{A}^{\mathsf{qnil}}$, $b \in \mathscr{A}^{\mathsf{d}}$ *satisfy* $ab = ba$ *and* $a = ab^{\pi}$. *Then* $a + b \in \mathscr{A}^{\mathsf{d}}$ *and*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}}.$$

Proof From $a = ab^{\pi}$, as we mentioned before, it follows that $ab^{\mathsf{d}} = 0$. Because the Drazin inverse b^{d} is a double commutant of a, we have

$$(b^{\mathsf{d}})^{n+2}a(a+b)^n = a(b^{\mathsf{d}})^{n+2}(a+b)^n = 0.$$

□

Proof of Theorem 6.11: If b is quasi-nilpotent we can apply Theorem 6.12. Hence, we assume that b is neither invertible nor quasi-nilpotent and consider the matrix representations of a and b relative to $p = 1 - b^{\pi}$:

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $b_1 \in (p\mathscr{A}p)^{-1}$ and $b_2 \in ((1-p)\mathscr{A}(1-p))^{\mathsf{qnil}}$. As in the proof of Theorem 6.12, from $a = ab^{\pi}$ it follows that

$$a = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_p, \quad a+b = \begin{bmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{bmatrix}_p.$$

From the conditions $b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab$ and $b^{\pi}ba^{\pi} = b^{\pi}b$, we obtain $a_2^{\pi}b_2a_2 = a_2^{\pi}a_2b_2$ and $b_2 = b_2a_2^{\pi}$. Now, from Theorem 6.12 it follows that $(a_2+b_2) \in ((1-p)\mathscr{A}(1-p))^{\mathsf{d}}$ and

$$(a_2+b_2)^{\mathsf{d}} = a_2^{\mathsf{d}} + \sum_{n=0}^{\infty}(a_2^{\mathsf{d}})^{n+2}b_2(a_2+b_2)^n. \tag{6.35}$$

By Theorem 6.10, we get

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} b_1^{-1} & u \\ 0 & (a_2+b_2)^{\mathsf{d}} \end{bmatrix}_p,$$

where $u = \sum_{n=0}^{\infty} b_1^{-(n+2)}a_1(a_2+b_2)^n(a_2+b_2)^{\pi} - b_1^{-1}a_1(a_2+b_2)^{\mathsf{d}}$ and b_1^{-1} is the inverse of b_1 in the algebra $p\mathscr{A}p$. Using (6.35), we have

$$u = \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1 (a_2 + b_2)^n = a_2^{\pi} - \sum_{n=0}^{\infty} b_1^{-(n+2)} a_1 (a_2 + b_2)^n a_2^{\mathsf{d}} b_2$$
$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (b_1)^{-(n+2)} a_1 (a_2 + b_2)^n (a_2^{\mathsf{d}})^{k+2} b_2 (a_2 + b_2)^{k+1} - b_1^{-1} a_1 a_2^{\mathsf{d}}$$
$$- \sum_{n=0}^{\infty} b_1^{-1} a_1 (a_2^{\mathsf{d}})^{n+2} b_2 (a_2 + b_2)^n.$$

By a straightforward manipulation, (6.33) follows. □

Corollary 6.5 *Suppose $a, b \in \mathscr{A}^{\mathsf{d}}$ are such that $ab = ba$, $a = ab^{\pi}$ and $b^{\pi} = ba^{\pi} = b^{\pi} b$. Then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a + b)^{\mathsf{d}} = b^{\mathsf{d}}.$$

If a is invertible and b is group invertible, then conditions (6.31) and (6.32) are satisfied, so we only have to assume $a = ab^{\pi}$. In the remaining case when b is invertible we get $a = 0$.

It is interesting to remark that conditions (6.26) and (6.29) are independent, i.e., neither of them implies the other, but in some cases the same expressions for $(a + b)^{\mathsf{d}}$ are obtained.

If we consider the algebra $\mathscr{A}$ of all complex 3×3 matrices and $a, b \in \mathscr{A}$ which are given in the Example 3.1 [43], we can see that condition (6.26) is satisfied, whereas condition (6.29) fails. In the following example we have the opposite case. We construct a, b in the algebra $\mathscr{A}$ of all complex 3×3 matrices such that (6.29) is satisfied but (6.26) is not. If we assume that $ab = ba$ in Theorem 6.11 the expression for $(a + b)^{\mathsf{d}}$ will be exactly the same as that in [43, Theorem 3.5] (which is Corollary 6.7 there).

Example 6.1 Let

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$a^{\pi} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $b^{\pi} = 1$. We can see that $a = ab^{\pi}$, $a^{\pi} ab = a^{\pi} = ba$ and $ba^{\pi} = b$, i.e., (6.29) holds. Also, $a^{\pi} b = 0 \neq b$, so (6.26) is not satisfied.

In the rest of the section, we present a generalization of the results from [43]. We use some weaker conditions than those in [43]. For example in the next theorem, which generalizes [43, Theorem 3.3], we assume that $e = (1 - b^{\pi})(a +$

$b)(1-b^\pi) \in \mathscr{A}^{\mathsf{d}}$ instead of $ab^\pi = a$. If $ab^\pi = a$, then $e = (1-b^\pi)b = \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}_p$ for $p = 1-b^\pi$ and $e^{\mathsf{d}} = b^{\mathsf{d}}$.

Theorem 6.13 *Let $b \in \mathscr{A}^{\mathsf{d}}$, $a \in \mathscr{A}^{\mathsf{qnil}}$ be such that*

$$e = (1-b^\pi)(a+b)(1-b^\pi) \in \mathscr{A}^{\mathsf{d}}, \quad b^\pi ab = 0.$$

Then $a+b \in \mathscr{A}^{\mathsf{d}}$ and

$$(a+b)^{\mathsf{d}} = e^{\mathsf{d}} + \sum_{n=0}^{\infty} (e^{\mathsf{d}})^{n+2} ab^\pi (a+b)^n.$$

Proof The case when $b \in \mathscr{A}^{\mathsf{qnil}}$ follows from Lemma 6.2. Hence, we assume that b is not quasi-nilpotent. Then

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \quad a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $p = 1-b^\pi$. From $b^\pi ab = 0$ we have $b^\pi a(1-b^\pi) = 0$, i.e., $a_{21} = 0$. Put $a_1 = a_{11}$, $a_{22} = a_2$ and $a_{12} = a_3$. Then,

$$a+b = \begin{bmatrix} a_1+b_1 & a_3 \\ 0 & a_2+b_2 \end{bmatrix}_p.$$

Also, $b^\pi ab = 0$ implies that $a_2 b_2 = 0$, so $a_2 + b_2 \in ((1-p)\mathscr{A}(1-p))^{\mathsf{qnil}}$, according to Lemma 6.2. Applying Theorem 6.10, we obtain

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} (a_1+b_1)^{\mathsf{d}} & u \\ 0 & 0 \end{bmatrix}_p,$$

where $u = \sum_{n=0}^{\infty} ((a_1+b_1)^{\mathsf{d}})^{n+2} a_3 (a_2+b_2)^n$.

By direct computation, we verify that

$$(a+b)^{\mathsf{d}} = e^{\mathsf{d}} + \sum_{n=0}^{\infty} (e^{\mathsf{d}})^{n+2} ab^\pi (a+b)^n.$$

□

Now, as a corollary we obtain Theorem 3.3 from [43].

Corollary 6.6 *Let $b \in \mathscr{A}^{\mathsf{d}}$, $a \in \mathscr{A}^{\mathsf{qnil}}$ and $ab^\pi = a$, $b^\pi ab = 0$. Then $a+b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a (a+b)^n.$$

The next result is a generalization of [43, Theorem 3.5]. For simplicity, we use the following notation:

$$\begin{aligned} e &= (1-b^{\pi})(a+b)(1-b^{\pi}) \in \mathscr{A}^{\mathsf{d}}, \\ f &= (1-a^{\pi})(a+b)(1-a^{\pi}), \\ \mathscr{A}_1 &= (1-a^{\pi})\mathscr{A}(1-a^{\pi}), \\ \mathscr{A}_2 &= (1-b^{\pi})\mathscr{A}(1-b^{\pi}), \end{aligned}$$

for given $a, b \in A^{\mathsf{d}}$.

Theorem 6.14 *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ be such that $(1-a^{\pi})b(1-a^{\pi}) \in \mathscr{A}^{\mathsf{d}}$, $f \in \mathscr{A}_1^{-1}$ and $e \in \mathscr{A}_2^{\mathsf{d}}$. If*

$$(1-a^{\pi})ba^{\pi} = 0, \quad b^{\pi}aba^{\pi} = 0, \quad a^{\pi} = a(1-b^{\pi})a^{\pi} = 0,$$

then $a+b \in \mathscr{A}^{\mathsf{d}}$ and

$$\begin{aligned} (a+b)^{\mathsf{d}} = {} & (b^{\mathsf{d}} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a (a+b)^n) a^{\pi} + \sum_{n=0}^{\infty} b^{\pi} (a+b)^n a^{\pi} b (f)_{\mathscr{A}_1}^{-(n+2)} \\ & - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (b^{\mathsf{d}})^{k+1} a (a+b)^{n+k} a^{\pi} b (f)_{\mathscr{A}_1}^{-(n+2)} - b^{\mathsf{d}} a^{\pi} b (f)_{\mathscr{A}_1}^{-1} \\ & - \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a (a+b)^n a^{\pi} b (f)_{\mathscr{A}_1}^{-1} + (f)_{\mathscr{A}_1}^{-1}, \end{aligned}$$

where by $(f)_{\mathscr{A}_1}^{-1}$ we denote the inverse of f in $\mathscr{A}_1$.

Proof Obviously, if a is invertible, then the statement of the theorem holds. If a is quasi-nilpotent, then the result follows from Theorem 6.13. Hence, we assume that a is neither invertible nor quasi-nilpotent. As in the proof of Theorem 6.11, we have

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \quad b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_p,$$

where $p = 1 - a^{\pi}$, $a_1 \in (p\mathscr{A}p)^{-1}$ and $a_2 \in ((1-p)\mathscr{A}(1-p))^{\mathsf{qnil}}$. From $(1-a^{\pi})ba^{\pi} = 0$, we have that $b_{12} = 0$. Let $b_1 = b_{11}$, $b_{22} = b_2$ and $b_{21} = b_3$. Then,

$$a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ b_3 & a_2 + b_2 \end{bmatrix}_p.$$

The condition $a^\pi b^\pi aba^\pi = 0$ expressed in the matrix form yields

$$a^\pi b^\pi aba^\pi = \begin{bmatrix} 0 & 0 \\ 0 & b_2^\pi \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & b_2^\pi a_2 b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly, $a^\pi a(1 - b^\pi) = 0$ implies that $a_2 b_2^\pi = a_2$. From Corollary 6.6, we get $a_2 + b_2 \in \mathscr{A}^{\mathsf{d}}$ and

$$(a_2 + b_2)^{\mathsf{d}} = b_2^{\mathsf{d}} + \sum_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+2} a_2 (a_2 + b_2)^n.$$

Using Theorem 6.10, we obtain $a + b \in \mathscr{A}^{\mathsf{d}}$ and

$$(a + b)^{\mathsf{d}} = \begin{bmatrix} (a_1 + b_1)^{\mathsf{d}} & 0 \\ u & (a_2 + b_2)^{\mathsf{d}} \end{bmatrix}_p,$$

where

$$\begin{aligned} u = & \sum_{n=0}^{\infty} b_2^\pi (a_2 + b_2)^n b_3 (f)_{\mathscr{A}_1}^{-(n+2)} \\ & - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (b_2^{\mathsf{d}})^{k+1} a_2 (a_2 + b_2)^{n+k} b_3 (f)_{\mathscr{A}_1}^{-(n+2)} - b_2^{\mathsf{d}} b_3 (f)_{\mathscr{A}_1}^{-1} \\ & - \sum_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+2} a_2 (a_2 + b_2)^n b_3 (f)_{\mathscr{A}_1}^{-1}. \end{aligned}$$

By straightforward computation, the desired result follows. □

Corollary 6.7 *Suppose $a, b \in \mathscr{A}^{\mathsf{d}}$ satisfy condition (6.26). Then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$\begin{aligned} (a + b)^{\mathsf{d}} = & (b^{\mathsf{d}} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a (a + b)^n) a^\pi + \sum_{n=0}^{\infty} b^\pi (a + b)^n b (a^{\mathsf{d}})^{(n+2)} \\ & - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (b^{\mathsf{d}})^{k+1} a (a + b)^{n+k} b (a^{\mathsf{d}})^{(n+2)} + b^\pi a^{\mathsf{d}} \\ & - \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a (a + b)^n b a^{\mathsf{d}} \end{aligned}$$

Proof We have that $f = (1 - a^\pi)a$, so $(f)_{\mathscr{A}_1}^{-1} = a^{\mathsf{d}}$. □

Next we generalize the results from [45] to the Banach algebra case.

Theorem 6.15 *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ and $ab = ba$. Then $a + b \in \mathscr{A}^{\mathsf{d}}$ if and only if $1 + a^{\mathsf{d}}b \in \mathscr{A}^{\mathsf{d}}$. In this case, we have*

$$(a+b)^{\mathsf{d}} = a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}bb^{\mathsf{d}} + (1-bb^{\mathsf{d}})\left[\sum_{n=0}^{\infty}(-b)^n(a^{\mathsf{d}})^n\right]a^{\mathsf{d}} + b^{\mathsf{d}}\left[\sum_{n=0}^{\infty}(b^{\mathsf{d}})^n(-a)^n\right](1-aa^{\mathsf{d}}),$$

and

$$(a+b)(a+b)^{\mathsf{d}} = (aa^{\mathsf{d}}+ba^{\mathsf{d}})(1+a^{\mathsf{d}}b)^{\mathsf{d}}bb^{\mathsf{d}} + (1-bb^{\mathsf{d}})aa^{\mathsf{d}} + bb^{\mathsf{d}}(1-aa^{\mathsf{d}}).$$

Moreover, if $\|b\|\|a^{\mathsf{d}}\| < 1$ and $\|a\|\|b^{\mathsf{d}}\| < 1$, then we have

$$\|(a+b)^{\mathsf{d}} - a^{d}\| \leq \|bb^{\mathsf{d}}\|\|a^{\mathsf{d}}\|\left[\|(1+a^{\mathsf{d}}b)^{\mathsf{d}}\| + 1\right] + \|1-bb^{\mathsf{d}}\|\left[\sum_{n=1}^{\infty}\|(-b)^n(a^{\mathsf{d}})^n\|\right]\|a^{\mathsf{d}}\| + \|b^{\mathsf{d}}\|\left[\sum_{n=0}^{\infty}\|(b^{\mathsf{d}})^n(-a)^n\|\right]\|1-aa^{\mathsf{d}}\|,$$

and

$$\|(a+b)(a+b)^{\mathsf{d}} - aa^{d}\| \leq \left[\|aa^{\mathsf{d}}+ba^{\mathsf{d}}\|\|(1+a^{\mathsf{d}}b)^{\mathsf{d}}\| + \|1-2aa^{\mathsf{d}}\|\right]\|bb^{\mathsf{d}}\|.$$

Proof Since a is generalized Drazin invertible, and

$$a = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}_p,$$

relative to $p = 1 - a^{\pi}$, where a_{11} is invertible in the algebra $p\mathscr{A}p$ and a_{22} is a quasi-nilpotent element of the algebra $(1-p)\mathscr{A}(1-p)$. Let $b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_p$.

From $ab = ba$, we have $b_{12} = (a_{11})^{-1}_{p\mathscr{A}p}b_{12}a_{22}$ which implies that $b_{12} = (a_{11})^{-n}_{p\mathscr{A}p} b_{12}a_{22}^n$, for arbitrary $n \in \mathbb{N}$. Since a_{22} is a quasi-nilpotent, we obtain $b_{12} = 0$. Similarly, from $ab = ba$ it follows that $b_{21} = a_{22}b_{21}(a_{11})^{-1}_{p\mathscr{A}p}$, i.e., $b_{21} = 0$. Also, $a_{11}b_{11} = b_{11}a_{11}$ and $a_{22}b_{22} = b_{22}a_{22}$.

Since, $b \in \mathscr{A}^{\mathsf{d}}$ and $\sigma(b) = \sigma(b_1)_{p\mathscr{A}p} \cup \sigma(b_2)_{(1-p)\mathscr{A}(1-p)}$, using Theorem 4.2 from [38], we deduce $b_1 \in p\mathscr{A}p$ and $b_2 \in (1-p)\mathscr{A}(1-p)$, so $b_{11}, b_{22} \in \mathscr{A}^{\mathsf{d}}$ and we can represent b_{11} and b_{22} as

$$b_{11} = \begin{bmatrix} b'_{11} & 0 \\ 0 & b'_{22} \end{bmatrix}_{p_1}, \quad b_{22} = \begin{bmatrix} b''_{11} & 0 \\ 0 & b''_{22} \end{bmatrix}_{p_2},$$

where $p_1 = 1 - b_{11}^{\pi}$ and $p_2 = 1 - b_{22}^{\pi}$, b'_{11}, b''_{11} are invertible in the algebras $p_1\mathscr{A}p_1$ and $p_2\mathscr{A}p_2$ respectively, and b'_{22}, b''_{22} are quasi-nilpotent. Since b_{11} commutes with an invertible a_{11} and b_{22} with a quasi-nilpotent a_{22}, we prove as before that

$$a_{11} = \begin{bmatrix} a'_{11} & 0 \\ 0 & a'_{22} \end{bmatrix}_{p_1}, \quad a_{22} = \begin{bmatrix} a''_{11} & 0 \\ 0 & a''_{22} \end{bmatrix}_{p_2}.$$

Since $p_1 p = p p_1 = p_1$, from the fact that a_{11} is invertible in the sub-algebra $p\mathscr{A}p$, we prove that a'_{11} and a'_{22} are invertible in the algebras $p_1\mathscr{A}p_1$ and $(p - p_1)\mathscr{A}(p - p_1)$, respectively. Also, a''_{11} and a''_{22} are quasi-nilpotent, thus a'_{ii} commutes with b'_{ii} and a''_{ii} with b''_{ii}, for $i = 1, 2$.

Since a'_{22} is invertible and b'_{22} is quasi-nilpotent and they commute, we have that $(a'_{22})^{-1}_{(1-p_1)\mathscr{A}(1-p_1)} b'_{22}$ is quasi-nilpotent, so $(1 - p_1) + (a'_{22})^{-1}_{(1-p_1)\mathscr{A}(1-p_1)} b'_{22}$ is invertible in $(1 - p_1)\mathscr{A}(1 - p_1)$ and $a'_{22} + b'_{22} \in \mathscr{A}^{\mathsf{d}}$.

Similarly, we conclude that $a''_{11} + b''_{11} \in \mathscr{A}^{\mathsf{d}}$. Also, $a''_{22} + b''_{22}$ is generalized Drazin invertible.

Now, we obtain

$$a + b = a'_{11} + b'_{11} + a'_{22} + b'_{22} + a''_{11} + b''_{11} + a''_{22} + b''_{22}.$$

Since, $a'_{11} + b'_{11} \in p_1\mathscr{A}p_1$ and $b'_{22} + a''_{11} + b''_{11} + a''_{22} + b''_{22} \in (1 - p_1)\mathscr{A}(1 - p_1)$ we have

$$a + b \in \mathscr{A}^{\mathsf{d}} \Leftrightarrow \left(a'_{11} + b'_{11} \in \mathscr{A}^{\mathsf{d}}, \quad a'_{22} + b'_{22} + a''_{11} + b''_{11} + a''_{22} + b''_{22} \in \mathscr{A}^{\mathsf{d}}\right).$$

Next, we inspect generalized Drazin invertibility of $y = a'_{22} + b'_{22} + a''_{11} + b''_{11} + a''_{22} + b''_{22}$. From $p_2 y p_2 = a''_{11} + b''_{11}$ and $(1 - p_2)y(1 - p_2)y = a'_{22} + b'_{22} + a''_{22} + b''_{22}$, we conclude

$$y \in \mathscr{A}^{\mathsf{d}} \Leftrightarrow \left(a''_{11} + b''_{11} \in \mathscr{A}^{\mathsf{d}} \text{ and } a'_{22} + b'_{22} + a''_{22} + b''_{22} \in \mathscr{A}^{\mathsf{d}}\right).$$

Previously, we showed that $a''_{11} + b''_{11} \in \mathscr{A}^{\mathsf{d}}$, so $y \in \mathscr{A}^{\mathsf{d}}$ if and only if $z = a'_{22} + b'_{22} + a''_{22} + b''_{22} \in \mathscr{A}^{\mathsf{d}}$. Notice that $z = pzp + (1 - p)z(1 - p)$, where $pzp = a'_{22} + b'_{22} \in \mathscr{A}^{\mathsf{d}}$ and $(1 - p)z(1 - p) = a''_{22} + b''_{22} \in \mathscr{A}^{\mathsf{d}}$, so $z \in \mathscr{A}^{\mathsf{d}}$. Hence, $y \in \mathscr{A}^{\mathsf{d}}$ and we obtain $a + b \in \mathscr{A}^{\mathsf{d}}$ if and only if $a'_{11} + b'_{11} \in \mathscr{A}^{\mathsf{d}}$.

Now,

$$(a'_{11}+b'_{11})^{\mathsf{d}} = a'_{11}(p_1+(a'_{11})^{-1}_{p_1\mathscr{A}p_1}b'_{11})^{\mathsf{d}} = p_1pa^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}bb^{\mathsf{d}}pp_1.$$

From the first equation, we obtain

$$\begin{aligned}(a+b)^{\mathsf{d}}-a^{d} &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}bb^{\mathsf{d}}+(1-bb^{\mathsf{d}})\left[\sum_{n=0}^{\infty}(-b)^n(a^{\mathsf{d}})^n\right]a^{\mathsf{d}}\\ &\quad +b^{\mathsf{d}}\left[\sum_{n=0}^{\infty}(b^{\mathsf{d}})^n(-a)^n\right](1-aa^{\mathsf{d}})-a^{d}\\ &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}bb^{\mathsf{d}}-bb^{\mathsf{d}}a^{\mathsf{d}}+(1-bb^{\mathsf{d}})\left[\sum_{n=1}^{\infty}(-b)^n(a^{\mathsf{d}})^n\right]a^{\mathsf{d}}\\ &\quad +b^{\mathsf{d}}\left[\sum_{n=0}^{\infty}(b^{\mathsf{d}})^n(-a)^n\right](1-aa^{\mathsf{d}}).\end{aligned}$$

Consequently, we have the estimates

$$\begin{aligned}\|(a+b)^{\mathsf{d}}-a^{d}\| &\le \|bb^{\mathsf{d}}\|\|a^{\mathsf{d}}\|\left[\|(1+a^{\mathsf{d}}b)^{\mathsf{d}}\|+1\right]\\ &\quad +\|(1-bb^{\mathsf{d}})\|\left[\sum_{n=1}^{\infty}\|(-b)^n(a^{\mathsf{d}})^n\|\right]\|a^{\mathsf{d}}\|\\ &\quad +\|b^{\mathsf{d}}\|\left[\sum_{n=0}^{\infty}\|(b^{\mathsf{d}})^n(-a)^n\|\right]\|(1-aa^{\mathsf{d}})\|,\end{aligned}$$

and

$$\begin{aligned}\|(a+b)(a+b)^{\mathsf{d}}-aa^{d}\| &= \|(aa^{\mathsf{d}}+ba^{\mathsf{d}})(1+a^{\mathsf{d}}b)^{\mathsf{d}}bb^{\mathsf{d}}-bb^{\mathsf{d}}aa^{\mathsf{d}}+bb^{\mathsf{d}}(1-aa^{\mathsf{d}})\|\\ &\le \left[\|aa^{\mathsf{d}}+ba^{\mathsf{d}}\|\|(1+a^{\mathsf{d}}b)^{\mathsf{d}}\|+\|1-2aa^{\mathsf{d}}\|\right]\|bb^{\mathsf{d}}\|.\end{aligned}$$

□

Corollary 6.8 *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ be such that $ab = ba$ and $1 + a^{\mathsf{d}}b \in \mathscr{A}^{\mathsf{d}}$.*

(1) *If b is quasi-nilpotent, then*

$$(a+b)^{\mathsf{d}} = \sum_{n=0}^{\infty}(a^{\mathsf{d}})^{n+1}(-b)^n = (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}.$$

(2) *If $b^k = 0$, then* $(a+b)^{\mathsf{d}} = \sum_{n=0}^{k-1}(a^{\mathsf{d}})^{n+1}(-b)^n = (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}.$

(3) *If* $b^k = b$ $(k \geq 3)$, *then* $b^{\mathsf{d}} = b^{k-2}$ *and*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}b^{k-1} + (1-b^{k-1})a^{\mathsf{d}} + b^{k-2}\left[\sum_{n=0}^{\infty}(b^{\mathsf{d}})^n(-a)^n\right](1-aa^{\mathsf{d}})\\ &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}b^{k-1} + (1-b^{k-1})a^{\mathsf{d}} + b^{k-2}(1+ab^{k-2})^{\mathsf{d}}(1-aa^{\mathsf{d}}).\end{aligned}$$

(4) *If* $b^2 = b$, *then* $b^{\mathsf{d}} = b$ *and*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}b + (1-b)a^{\mathsf{d}} + b\left[\sum_{n=0}^{\infty}(-a)^n\right](1-aa^{\mathsf{d}})\\ &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}b + (1-b)a^{\mathsf{d}} + b(1+a)^{\mathsf{d}}(1-aa^{\mathsf{d}}).\end{aligned}$$

(5) *If* $a^2 = a$ *and* $b^2 = b$, *then* $1+ab$ *is invertible and* $a(1+ab)^{-1}b = \frac{1}{2}ab$. *In this case,*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= a(1+ab)^{-1}b + b(1-a) + (1-b)a\\ &= a+b-\tfrac{3}{2}ab.\end{aligned}$$

Theorem 6.16 *Let* $a, b \in \mathscr{A}^{\mathsf{d}}$ *be such that* $\|a^{\mathsf{d}}b\| < 1$, $a^{\pi}ba^{\pi} = a^{\pi}b$ *and* $a^{\pi}ab = a^{\pi}ba$. *If* $a^{\pi}b \in \mathscr{A}^{\mathsf{d}}$, *then* $a+b \in \mathscr{A}^{\mathsf{d}}$. *In this case,*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}} + (1+a^{\mathsf{d}}b)^{-1}(1-aa^{\mathsf{d}})\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\\ &\quad+\left[\sum_{n=0}^{\infty}\left((1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}\right)^{n+2}b(1-aa^{\mathsf{d}})(a+b)^n\right](1-aa^{\mathsf{d}})\\ &\quad\times\left[1-(a+b)(1-aa^{\mathsf{d}})\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\right].\end{aligned}$$

Moreover, if $\|a\|\|b^{\mathsf{d}}\| < 1$, $\|b\|\|a^{\mathsf{d}}\| < 1$ *and* $\frac{\|a^{\mathsf{d}}\|\|a^{\mathsf{d}}b\|}{1-\|a^{\mathsf{d}}b\|}\|a+b\| < 1$, *then*

$$\begin{aligned}\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\| &\leq \frac{\|a^{\mathsf{d}}\|\|a^{\mathsf{d}}b\|}{1-\|a^{\mathsf{d}}b\|} + \frac{\|1-aa^{\mathsf{d}}\|}{1-\|a^{\mathsf{d}}b\|}\sum_{n=0}^{\infty}\|b^{\mathsf{d}}\|^{n+1}\|a\|^n\\ &\quad+\left[\sum_{n=0}^{\infty}\left(\frac{\|a^{\mathsf{d}}\|\|a^{\mathsf{d}}b\|}{1-\|a^{\mathsf{d}}b\|}\right)^{n+2}\|b\|\|a+b\|^n\right]\|1-aa^{\mathsf{d}}\|^2\\ &\quad+\|1-aa^{\mathsf{d}}\|^3\left[\sum_{n=0}^{\infty}\left(\frac{\|a^{\mathsf{d}}\|\|a^{\mathsf{d}}b\|}{1-\|a^{\mathsf{d}}b\|}\right)^{n+2}\|b\|\|a+b\|^{n+1}\right]\\ &\quad\times\left[\sum_{n=0}^{\infty}\|b^{\mathsf{d}}\|^{n+1}\|a\|^n\right].\end{aligned}$$

Proof Since $a \in \mathscr{A}^{\mathsf{d}}$ and $a^{\pi}b(I - a^{\pi}) = 0$, we have that for $p = 1 - a^{\pi}$

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, \quad b = \begin{pmatrix} b_1 & b_3 \\ 0 & b_2 \end{pmatrix}_p \tag{6.36}$$

where a_1 is invertible in the algebra $p\mathscr{A}p$ and a_2 is a quasi-nilpotent element of the algebra $(1-p)\mathscr{A}(1-p)$. Also from $a^{\pi}ab = a^{\pi}ba$ and the fact that $a^{\pi}b \in \mathscr{A}^{\mathsf{d}}$, we conclude that $a_2b_2 = b_2a_2$ and $b_2 \in \mathscr{A}^{\mathsf{d}}$. It follows from $\|a^{\mathsf{d}}b\| < 1$ that $1 + a^{\mathsf{d}}b$ is invertible. Now, from Theorem 6.15, we have

$$(a_2 + b_2)^d = \sum_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+1}(-a_2)^n.$$

Using Theorem 6.10, we get

$$(a+b)^{\mathsf{d}} = \begin{pmatrix} (a_1 + b_1)^{-1} & S \\ 0 & \sum\limits_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+1}(-a_2)^n \end{pmatrix}_p,$$

where

$$S = \left[\sum_{n=0}^{\infty} (a_1 + b_1)^{-n-2} b_3 (a_2 + b_2)^n\right]\left[1 - p - (a_2 + b_2)\sum_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+1}(-a_2)^n\right]$$
$$-(a_1 + b_1)^{-1} b_3 \sum_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+1}(-a_2)^n.$$

We know that

$$\begin{bmatrix} (a_1 + b_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p = (1 + a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & \sum\limits_{n=0}^{\infty} (b_2^{\mathsf{d}})^{n+1}(-a_2)^n \end{bmatrix}_p = a^{\pi}\sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+1}(-a)^n.$$

By computation we obtain

$$\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}_p = \left[\sum_{n=0}^{\infty}\left((1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}\right)^{n+2} ba^{\pi}(a+b)^n\right]a^{\pi}$$
$$\times\left[1-(a+b)a^{\pi}\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\right]$$
$$-(1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}ba^{\pi}\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n.$$

Hence, we have

$$(a+b)^{\mathsf{d}} = (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}} + (1+a^{\mathsf{d}}b)^{-1}a^{\pi}\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n$$
$$+\left[\sum_{n=0}^{\infty}\left((1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}}\right)^{n+2} ba^{\pi}(a+b)^n\right]a^{\pi}$$
$$\times\left[1-(a+b)a^{\pi}\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\right].$$

If $\|a\|\|b^{\mathsf{d}}\| < 1$, $\|b\|\|a^{\mathsf{d}}\| < 1$ and $\frac{\|a^{\mathsf{d}}\|\|a^{\mathsf{d}}b\|}{1-\|a^{\mathsf{d}}b\|}\|a+b\| < 1$, we obtain

$$\|(a+b)^{\mathsf{d}} - a^{\mathsf{d}}\| = \Big\| \sum_{n=1}^{\infty}(a^{\mathsf{d}}b)^n a^{\mathsf{d}} + \sum_{n=0}^{\infty}(a^{\mathsf{d}}b)^n(1-aa^{\mathsf{d}})\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n$$
$$+\left[\sum_{n=0}^{\infty}\left(\sum_{n=0}^{\infty}(a^{\mathsf{d}}b)^n a^{\mathsf{d}}\right)^{n+2} b(1-aa^{\mathsf{d}})(a+b)^n\right](1-aa^{\mathsf{d}})$$
$$\times\left[1-(a+b)(1-aa^{\mathsf{d}})\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\right]\Big\|$$
$$\le \frac{\|a^{\mathsf{d}}\|\|a^{\mathsf{d}}b\|}{1-\|a^{\mathsf{d}}b\|} + \frac{\|1-aa^{\mathsf{d}}\|}{1-\|a^{\mathsf{d}}b\|}\sum_{n=0}^{\infty}\|(b^{\mathsf{d}})\|^{n+1}\|(-a)\|^n$$
$$+\left[\sum_{n=0}^{\infty}\left(\frac{\|a^{\mathsf{d}}\|\|a^{\mathsf{d}}b\|}{1-\|a^{\mathsf{d}}b\|}\right)^{n+2}\|b\|\|a+b\|^n\right]\|(1-aa^{\mathsf{d}})\|^2$$
$$+\|(1-aa^{\mathsf{d}})\|^3\left[\sum_{n=0}^{\infty}\left(\frac{\|a^{\mathsf{d}}\|\|a^{\mathsf{d}}b\|}{1-\|a^{\mathsf{d}}b\|}\right)^{n+2}\|b\|\|a+b\|^{n+1}\right]$$
$$\times\left[\sum_{n=0}^{\infty}\|(b^{\mathsf{d}})\|^{n+1}\|a\|^n\right].$$

□

Corollary 6.9 *Let $a \in \mathscr{A}^{\mathsf{d}}$ and $b \in \mathscr{A}$ be such that $\|ba^{\mathsf{d}}\| < 1$, $a^{\pi}b(1-a^{\pi}) = 0$ and $a^{\pi}ab = a^{\pi}ba$,*

(1) *If $baa^{\mathsf{d}} = 0$ and b is quasi-nilpotent, then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = \sum_{n=0}^{\infty}(a^{\mathsf{d}})^{n+2}b(a+b)^n + a^{\mathsf{d}}.$$

(2) *If $a^{\pi}b = ba^{\pi}$, $\sigma(a^{\pi}b) = 0$, then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$(a+b)^{\mathsf{d}} = (1+a^{\mathsf{d}}b)^{-1}a^{\mathsf{d}} = a^{\mathsf{d}}(1+ba^{\mathsf{d}})^{-1}.$$

The following theorem is a generalization of Theorem 6.16 and Theorem 6 from [45].

Theorem 6.17 *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ and q be an idempotent such that $aq = qa$, $(1-q)bq = 0$, $(ab - ba)q = 0$ and $(1-q)(ab-ba) = 0$. If $(a+b)q$ and $(1-q)(a+b)$ are generalized Drazin invertible, then $a + b \in \mathscr{A}^{\mathsf{d}}$ and*

$$\begin{aligned}(a+b)^{\mathsf{d}} &= \sum_{n=0}^{\infty} S^{n+2}qb(1-q)(a+b)^n(1-q)\Big[1-(a+b)S\Big]\\ &\quad + \Big[1-(a+b)S\Big]q\sum_{n=0}^{\infty}(a+b)^n qb(1-q)S^{n+2}\\ &\quad + (1-Sqb)\,(1-q)S + Sq,\end{aligned}$$

where

$$\begin{aligned}S &= a^{\mathsf{d}}(1+a^{\mathsf{d}}b)^{\mathsf{d}}bb^{\mathsf{d}} + (1-bb^{\mathsf{d}})\left[\sum_{n=0}^{\infty}(-b)^n(a^{\mathsf{d}})^{n+1}\right]\\ &\quad + \left[\sum_{n=0}^{\infty}(b^{\mathsf{d}})^{n+1}(-a)^n\right](1-aa^{\mathsf{d}}).\end{aligned} \tag{6.37}$$

Proof The proof is a similar to that of Theorem 6.16. □

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