Order Sharp Estimates for Monotone Operators on Orlicz–Lorentz Classes

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Abstract We consider the monotone operator P, which maps Orlicz-Lorentz class $\Lambda_{\Phi,v}$ into some ideal space $Y = Y(R_+)$. Orlicz-Lorentz class is determined as the cone of Lebesgue-measurable functions on $R_+ = (0, \infty)$ having the decreasing rearrangements that belong to weighted Orlicz space $L_{\Phi,v}$ under some general assumptions concerning properties of functions Φ and v. We prove the reduction theorems allowing reducing the estimates of the norm of operator $P : \Lambda_{\Phi,v} \to Y$ to the estimates for its restriction on some cone of nonnegative step-functions in $L_{\Phi,v}$. Application of these results to identical operator mapping $\Lambda_{\Phi,v}$ into the weighted Lebesgue space $Y = L_1(R_+; g)$ gives the sharp description of the associate space for $\Lambda_{\Phi,v}$. The main results of this paper were announced in [20]. They develop the results of our paper [19] related to the case of N-functions.

Keywords Monotone operators · Orlic–Lorentz classes

1 Some Properties of General Weighted Orlicz Spaces

This section contains the description of needed general properties of weighted Orlicz spaces. Some of them (not all) are presented in different forms in the literature; see for example the books of Krasnoselskii and Rutickii [1], Maligranda [2], Krein et al. [3], and Bennett and Sharpley [11].

Definition 1 We denote as Θ *a* class of functions $\Phi : [0, \infty) \to [0, \infty]$ with the following properties: $\Phi(0) = 0$; Φ is increasing and left continuous on R_+, $\Phi(+\infty) = \infty$; Φ is neither identically zero nor identically infinite on R_+.

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For $\Phi \in \Theta$ we introduce

$$t_0 = \sup\{t \in [0,\infty) : \Phi(t) = 0\}; \tag{1}$$

$$t_\infty = \inf\{t \in R_+ : \Phi(t) = \infty\} \tag{2}$$

($t_\infty = \infty$ is assumed if $\Phi(t) < \infty, t \in R_+$). Then,

$$t_0 \in [0,\infty); \quad t_\infty \in (0,\infty]; \quad t_0 \leqslant t_\infty, \tag{3}$$

$$\Phi(t) = 0, \quad t \in [0, t_0], \quad \Phi(t) = \infty, \quad t > t_\infty \tag{4}$$

(the last in the case $t_\infty < \infty$).

Everywhere below we assume that

$$\Phi \in \Theta, \quad v \in M, \quad v > 0 \quad \text{almost everywhere in } R_+. \tag{5}$$

Here, $M = M(R_+)$ is the set of all Lebesgue-measurable functions on R_+. For $\lambda > 0$, $f \in M$ we denote

$$J_\lambda(f) := \int_0^\infty \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx, \tag{6}$$

$$\|f\|_{\Phi,v} = \inf\{\lambda > 0 : J_\lambda(f) \leqslant 1\}. \tag{7}$$

Orlicz space $L_{\Phi,v}$ is defined as the set of functions $f \in M : \|f\|_{\Phi,v} < \infty$.

Note that general concept of Orlicz–Lorentz spaces was developed by Kaminska and Raynaud [12]. In this article there is a general definition of Orlicz-Lorentz spaces, even with two weights, generated by an increasing function Φ. The necessary and sufficient conditions are discussed there for the Minkowski functional to be a norm, quasi-norm or the space to be linear.

The goal of this Section is to describe some needed general properties of Orlicz spaces $L_{\Phi,v}$. In particular, we would like to answer the following question. Let $c \in R_+$; $f_1 \in M$, $f_2 \in L_{\Phi,v}$. What are the conditions on $\Phi \in \Theta$ such that the estimate

$$J_\lambda(f_1) \leqslant c J_\lambda(f_2), \quad \lambda > d\|f_2\|_{\Phi,v}, \tag{8}$$

implies that $f_1 \in L_{\Phi,v}$, and

$$\|f_1\|_{\Phi,v} \leqslant d\|f_2\|_{\Phi,v} \tag{9}$$

with some constant $d = d(c) \in R_+$ not depending of f_1, f_2.

Remark 1 Let $\Phi \in \Theta$, $c = d = 1$ in the estimate (8). Then (9) is valid with $d = 1$. Indeed, we have $J_\lambda(f_2) \leqslant 1$ for every $\lambda \geqslant \|f_2\|_{\Phi,v}$, so that (8) $\Rightarrow J_\lambda(f_1) \leqslant 1$. Therefore, $\lambda \geqslant \|f_1\|_{\Phi,v}$. Thus, (9) follows with $d = 1$. So we have $d = d(1) = 1$ in (8), (9).

Our nearest considerations will be devoted to the justification of this estimate for $c \in (0,1)$, which makes possible to obtain (9) with some $d \in (0,1)$. To consider the case $c \in (1,\infty)$ we need some additional conditions on function $\Phi \in \Theta$.

For $c \in (0,1)$ we assume that $t_0 = 0$; $t_\infty = \infty$ in (1), and in (2). Let us denote

$$d(c) = \inf\{d \in (0,1] : \ \Phi(dt) \geqslant c\Phi(t),\ t \in (0,\infty)\},\quad c \in (0,1). \tag{10}$$

For $c \in (1,\infty)$ we assume that

$$t_0 t_\infty^{-1} = 0. \tag{11}$$

It means that at least one of the conditions $t_0 = 0$; $t_\infty = \infty$ is fulfilled. We denote by

$$d(c) = \inf\left\{d > 1 : \ \Phi(dt) \geqslant c\Phi(t),\ t \in \left(t_0, d^{-1}t_\infty\right)\right\},\quad c \in (1,\infty) \tag{12}$$

(under assumption (11), we have $t_0 < d^{-1}t_\infty$ for any $d > 1$). It is clear that

$$c \in (0,1] \Rightarrow d(c) \in [0,1];\quad c \in (1,\infty) \Rightarrow d(c) \in [1,\infty].$$

For $c \in (1,\infty)$ we denote by

$$\Theta_c = \{\Phi \in \Theta : \ d(c) < \infty\}. \tag{13}$$

Theorem 1 *Let Φ and v to satisfy the conditions (5), and $c \in R_+$. If $c \in (0,1)$ we require that $t_0 = 0$; $t_\infty = \infty$ in (1), (2); if $c \in (1,\infty)$ then (11), and the condition $\Phi \in \Theta_c$ have to be fulfilled. Let $d(1) = 1$, and $d(c)$ being determined by (10), (12) for $c \neq 1$. Then the inequality,*

$$J_\lambda(f_1) \leqslant cJ_\lambda(f_2),\quad \lambda > d(c)\|f_2\|_{\Phi,v}, \tag{14}$$

for functions $f_1 \in M$, $f_2 \in L_{\Phi,v}$ implies

$$f_1 \in L_{\Phi,v},\quad \|f_1\|_{\Phi,v} \leqslant d(c)\|f_2\|_{\Phi,v}. \tag{15}$$

Corollary 1 *Let $0 < c_1 \leqslant c_2 < \infty$; and the conditions (5) and (11) be fulfilled. Moreover, if $c_0 = \min\left\{c_1^{-1}, c_2\right\} \in (0,1)$, we require that $t_0 = 0$; $t_\infty = \infty$; if $c = \max\left\{c_1^{-1}, c_2\right\} > 1$, then $\Phi \in \Theta_c$ is assumed. If*

$$J_\lambda(f_2) \leqslant c_1 J_\lambda(f_1) \leqslant c_2 J_\lambda(f_2), \tag{16}$$

for every $\lambda > 0$*, then*

$$f_1 \in L_{\Phi,v} \Leftrightarrow f_2 \in L_{\Phi,v}; \quad d_1 \|f_1\|_{\Phi,v} \leqslant \|f_2\|_{\Phi,v} \leqslant d_2 \|f_1\|_{\Phi,v}, \tag{17}$$

where

$$d_1 = d\left(c_1^{-1}\right)^{-1}, \quad d_2 = d\left(c_2\right). \tag{18}$$

see (10), (12).

We need some lemmas for the proof of Theorem 1.
Let $f \in L_{\Phi,v}$, $f \neq 0$. For $c \in R_+$ we define

$$\Lambda_f(c) = \{\lambda > 0: \quad cJ_\lambda(f) \leqslant 1\}. \tag{19}$$

It follows from (6), and from the properties of $\Phi \in \Theta$ that $J_\lambda(f)$ decreases, and it is right continuous as function of λ. Therefore,

$$\Lambda_f(c) \neq \emptyset \Rightarrow \Lambda_f(c) = \left[\lambda_f(c), \infty\right), \quad \lambda_f(c) = \inf \Lambda_f(c). \tag{20}$$

We have for $c \in (0, 1]$

$$\Lambda_f(c) \supset \Lambda_f(1) = \{\lambda > 0: \quad J_\lambda(f) \leqslant 1\} = \left[\|f\|_{\Phi,v}, \infty\right), \tag{21}$$

so that $\Lambda_f(c) \neq \emptyset$. The following lemma gives more general nonempty—conditions for $\Lambda_f(c)$.

Lemma 1 *Let the conditions* (5) *be fulfilled, let* $f \in L_{\Phi,v}$, $f \neq 0$*. Then, the following conclusions hold:*

(1) if $\Phi(+0) = 0$*, then* $\Lambda_f(c) \neq \emptyset$ *for every* $c \in R_+$*;*
(2) if $\Phi(+0) > 0$*, then*

$$c > \left[\Phi(+0) \int_{E(f)} v dx\right]^{-1} \Rightarrow \Lambda_f(c) = \emptyset, \tag{22}$$

$$c < \left[\Phi(+0) \int_{E(f)} v dx\right]^{-1} \Rightarrow \Lambda_f(c) \neq \emptyset, \tag{23}$$

where

$$E(f) = \{x \in R_+ : 0 < |f(x)| < \infty\}.$$

Remark 2 In the conditions of Lemma 1 we have,

$$0 \leqslant J_\lambda(f) \leqslant 1, \quad \lambda \in \left[\|f\|_{\Phi,v}, \infty\right), \quad J_\lambda(f) \downarrow (\lambda \uparrow). \tag{24}$$

Therefore, the following limit exists

$$0 \leqslant J_\infty(f) = \lim_{\lambda \to +\infty} J_\lambda(f) \leqslant 1. \tag{25}$$

In the proof of this lemma we particularly establish that

$$0 \leqslant J_\infty(f) = \Phi(+0) \int_{E(f)} v dx \leqslant 1. \tag{26}$$

Moreover, we will show that $\mu(E(f)) = \infty$, and

$$\Phi(+0) > 0 \Rightarrow 0 < \int_{E(f)} v dx \leqslant \Phi(+0)^{-1}, \tag{27}$$

because $v > 0$ almost everywhere.

Proof (*of Lemma* 1)
1. Denote

$$E_0(f) = \{x \in R_+ : |f(x)| = 0\}, \quad E_\infty(f) = \{x \in R_+ : |f(x)| = \infty\}.$$

Then,

$$R_+ = E_0(f) \cup E(f) \cup E_\infty(f). \tag{28}$$

For $\lambda \in \left[\|f\|_{\Phi,v}, \infty\right)$ we have,

$$J_\lambda(f) := \int_0^\infty \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\, dx \leqslant 1. \tag{29}$$

It means that almost everywhere

$$\Phi\left(\lambda^{-1}|f(x)|\right) v(x) < \infty \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) < \infty \Rightarrow |f(x)| < \infty. \tag{30}$$

In the first implication, we take into account that $v(x) > 0$ almost everywhere, and in the second one, we use the condition $\Phi(+\infty) = \infty$. From (30), it follows that

$$\mu\left(E_{\infty}\left(f\right)\right)=0. \tag{31}$$

Moreover, $f\neq 0\Rightarrow\mu\left(E_{0}\left(f\right)\right)<\infty$.

From here, and from (28) we see that $\mu\left(E\left(f\right)\right)=\infty$, and

$$J_{\lambda}\left(f\right)=\int\limits_{E_{0}(f)}\Phi\left(\lambda^{-1}\left|f\left(x\right)\right|\right)v\left(x\right)dx+\int\limits_{E(f)}\Phi\left(\lambda^{-1}\left|f\left(x\right)\right|\right)v\left(x\right)dx. \tag{32}$$

For $x\in E_{0}\left(f\right)$ we have $\lambda^{-1}\left|f\left(x\right)\right|=0\Rightarrow\Phi\left(\lambda^{-1}\left|f\left(x\right)\right|\right)=0$ (recall that $\Phi\left(0\right)=0$).

Therefore,

$$J_{\lambda}\left(f\right)=\int\limits_{E(f)}\Phi\left(\lambda^{-1}\left|f\left(x\right)\right|\right)v\left(x\right)dx. \tag{33}$$

We see that

$$\lambda\in\left[\left\|f\right\|_{\Phi,v},\infty\right)\Rightarrow\Phi\left(\lambda^{-1}\left|f\left(x\right)\right|\right)v\left(x\right)\leqslant\Phi\left(\left\|f\right\|_{\Phi,v}^{-1}\left|f\left(x\right)\right|\right)v\left(x\right)\in L_{1}\left(R_{+}\right),$$

and $\lambda\rightarrow+\infty$ implies

$$0<\lambda^{-1}\left|f\left(x\right)\right|\rightarrow 0\Rightarrow\Phi\left(\lambda^{-1}\left|f\left(x\right)\right|\right)v\left(x\right)\rightarrow\Phi\left(+0\right)v\left(x\right).$$

Therefore, we have by Lebesgue majored convergence theorem

$$J_{\infty}\left(f\right)=\lim_{\lambda\rightarrow+\infty}J_{\lambda}\left(f\right)=\Phi\left(+0\right)\int\limits_{E(f)}vdx.$$

It proves (26).

2. If $\Phi\left(+0\right)=0$ then, $\lim\limits_{\lambda\rightarrow+\infty}J_{\lambda}\left(f\right)=0$, so that for every $c\in R_{+}$ we can find $\lambda\left(c\right)\in R_{+}$, with $J_{\lambda}\left(f\right)\leqslant c^{-1}$, $\lambda\geqslant\lambda\left(c\right)$. It means that $\Lambda_{f}\left(c\right)\neq\emptyset$.

3. Now, let $\Phi\left(+0\right)>0$. Note that $J_{\lambda}\left(f\right)$ decreases in λ, therefore we have for every $\lambda>0$ by (26) and (22),

$$cJ_{\lambda}\left(f\right)\geqslant cJ_{\infty}\left(f\right)=c\Phi\left(+0\right)\int\limits_{E(f)}vdx>1\Rightarrow\Lambda_{f}\left(c\right)=\emptyset.$$

By the conditions (23) with $\lambda\rightarrow+\infty$, we have

$$\lim_{\lambda\rightarrow+\infty}cJ_{\lambda}\left(f\right)=c\Phi\left(+0\right)\int\limits_{E(f)}vdx<1,$$

so that

$$\exists\lambda(c) > 0: \ cJ_\lambda(f) \leqslant 1, \quad \lambda \geqslant \lambda(c) \Rightarrow \Lambda_f(c) \neq \emptyset.$$

Remark 3 Let $c \in (0, 1]$ in the conditions of Lemma 1. Then, $\Lambda_f(c) \neq \emptyset$. Indeed, by (26),

$$\left[\Phi(+0) \int\limits_{E(f)} v dx\right]^{-1} \geqslant 1,$$

so that the assertions (23) are fulfilled for $c \in (0, 1)$. If $c = 1$ we also obtain $\Lambda_f(c) \neq \emptyset$ (see Remark 1).

Remark 4 Under assumptions of Lemma 1 let

$$\Phi(+0) > 0; c = \left[\Phi(+0) \int\limits_{E(f)} v dx\right]^{-1} \in (1, \infty) \tag{34}$$

(see (25) and (26)). Then both variants of the answer are possible. Let us give the examples.

1. If $\Phi(t) > \Phi(+0)$, $t \in R_+$ then we have $E(f_0) = E$; for function $f_0 = \chi_E$ where $E \subset R_+$, $0 < \mu(E) < \infty$, and therefore

$$cJ_\lambda(f_0) = c\Phi\left(\lambda^{-1}\right) \int\limits_E v(x)\, dx > c\Phi(+0) \int\limits_E v(x)\, dx = 1.$$

It means that $\Lambda_{f_0}(c) = \emptyset$.

2. Let $\exists\delta > 0: \ \Phi(t) = \Phi(+0), t \in (0, \delta)$.

Then we have $\Lambda_f(c) \neq \emptyset$ for every bounded function f. Indeed, let $|f(x)| \leqslant M$ almost everywhere. Then, $\lambda > M\delta^{-1} \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) \leqslant \Phi\left(\lambda^{-1}M\right) = \Phi(+0)$,

$$cJ_\lambda(f) \leqslant c\Phi(+0) \int\limits_{E(f)} vdx = 1 \Rightarrow \Lambda_f(c) \supset \left(M\delta^{-1}, \infty\right).$$

Let the conditions (5) be fulfilled, and $f \in L_{\Phi,v}$, $f \neq 0$. Denote

$$\lambda(f; d) = \inf\{\lambda > 0: \ J_\lambda(df) < \infty\}. \tag{35}$$

We have

$$\lambda \in \left[d\|f\|_{\Phi,v}, \infty\right) \Rightarrow J_\lambda(df) \leqslant 1, \tag{36}$$

so that

$$\lambda(f; d) \leqslant d\|f\|_{\Phi,v} \tag{37}$$

Lemma 2 *Let the conditions* (5) *be fulfilled, and* $c \in (0, 1)$*;* $t_0 = 0$*,* $t_\infty = \infty$ *in (1), (2). Let* $d(c)$ *be defined by (10). Then the following estimate holds for function* $f \in L_{\Phi,v}$*,* $f \neq 0$

$$cJ_\lambda(f) \leqslant J_\lambda(df), \quad \lambda \in [\lambda(f; d), \infty). \tag{38}$$

with any $d > d(c)$*.*

Proof We use formula (33). For $x \in E(f)$, $d > d(c)$ we have by definition (10)

$$0 < \lambda^{-1}|f(x)| < \infty \Rightarrow c\Phi\left(\lambda^{-1}|f(x)|\right) \leqslant \Phi\left(\lambda^{-1}|df(x)|\right),$$

so that

$$cJ_\lambda(f) = \int_{E(f)} c\Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx \leqslant \int_{E(f)} \Phi\left(\lambda^{-1}|df(x)|\right) v(x)\,dx \leqslant J_\lambda(df).$$

Corollary 2 *From (36)–(38), it follows that* $\lambda \in \left[d\|f\|_{\Phi,v}, \infty\right) \Rightarrow cJ_\lambda(f) \leqslant 1$*, so that*

$$\Lambda_f(c) \supset \left[d\|f\|_{\Phi,v}, \infty\right) \neq \emptyset, \quad \forall d > d(c).$$

Thus,

$$\Lambda_f(c) \supset \left[d(c)\|f\|_{\Phi,v}, \infty\right). \tag{39}$$

Lemma 3 *Let the conditions (5) and (11) be fulfilled, and* $c \in (1, \infty)$*,* $d(c)$ *being defined by (12) and* $\Phi \in \Theta_c$*. Then, estimate (38) holds for function* $f \in L_{\Phi,v}$*,* $f \neq 0$*, with any* $d > d(c)$*.*

Proof For $\lambda > 0$, $d > d(c)$ we define

$$G_0(f) \equiv G_0(f; \lambda) = \left\{x \in R_+ : \lambda^{-1}|f(x)| \leqslant t_0\right\}, \tag{40}$$

$$G(f) \equiv G(f; \lambda) = \left\{x \in R_+ : t_0 < \lambda^{-1}|f(x)| < \infty\right\}, \quad t_\infty = \infty; \tag{41}$$

$$G(f) \equiv G(f; \lambda, d) = \left\{x \in R_+ : t_0 < \lambda^{-1}|f(x)| \leqslant d^{-1}t_\infty\right\}, \quad t_\infty < \infty; \tag{42}$$

$$G_\infty(f) = \{x \in R_+ : |f(x)| = \infty\}, \quad t_\infty = \infty; \tag{43}$$

$$G_\infty(f) \equiv G_\infty(f; \lambda, d) = \left\{x \in R_+ : \lambda^{-1}|f(x)| > d^{-1}t_\infty\right\}, \quad t_\infty < \infty. \tag{44}$$

Then,

$$R_+ = G_0(f) \cup G(f) \cup G_\infty(f). \tag{45}$$

We have according to (40) and (4),

$$x \in G_0(f) \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) = 0 \Rightarrow \int\limits_{G_0(f)} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx = 0. \quad (46)$$

Further, $\lambda > \lambda(f; d)$ implies $J_\lambda(df) < \infty$. Therefore, almost everywhere

$$\Phi\left(\lambda^{-1}|df(x)|\right) v(x) < \infty \Rightarrow \Phi\left(\lambda^{-1}|df(x)|\right) < \infty. \quad (47)$$

Here we take into account that $v(x) > 0$ almost everywhere. Now, if $t_\infty = \infty$ then $\Phi(+\infty) = \infty$, and if $t_\infty < \infty$ then $\Phi(t) = \infty$, $t > t_\infty$. Therefore, in both cases

$$x \in G_\infty(f) \Rightarrow \Phi\left(\lambda^{-1}|df(x)|\right) = \infty. \quad (48)$$

From here, and from (47), it follows that

$$\mu(G_\infty(f)) = 0 \Rightarrow \int\limits_{G_\infty(f)} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx = 0. \quad (49)$$

Now, (45), (46), and (49) imply

$$J_\lambda(f) = \int\limits_{G(f)} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx. \quad (50)$$

For $x \in G(f)$ we have $t = \lambda^{-1}|f(x)| \in (t_0, \infty)$, if $t_\infty = \infty$, or $t \in \left(t_0, d^{-1}t_\infty\right]$ if $t_\infty < \infty$. By (12) we have for $d > d(c)$

$$c\Phi(t) \leqslant \Phi(dt), \quad t \in \left(t_0, d^{-1}t_\infty\right). \quad (51)$$

If $t_\infty < \infty$, this inequality is extended onto $\left(t_0, d^{-1}t_\infty\right]$ by the limiting passage with $t \to d^{-1}t_\infty$ (let us recall that Φ is left continuous). Therefore,

$$c\Phi\left(\lambda^{-1}|f(x)|\right) \leqslant \Phi\left(\lambda^{-1}|df(x)|\right), \quad x \in G(f), \quad (52)$$

so that,

$$cJ_\lambda(f) = \int\limits_{G(f)} c\Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx \leqslant \int\limits_{G(f)} \Phi\left(\lambda^{-1}|df(x)|\right) v(x)\,dx \leqslant J_\lambda(df).$$

This proves estimate (38) .

Proof (*of Theorem* 1) In the assumptions of this theorem, Remark 1 exhausts the case $= 1$. For function $f = f_2 \in L_{\Phi,v}$, $f_2 \neq 0$, we can apply Lemma 2 with $c \in (0, 1)$, or Lemma 3 with $c \in (1, \infty)$. In both cases we obtain (38) for $f = f_2$. It is true in particular for all $\lambda \in \left[d\|f_2\|_{\Phi,v}, \infty\right)$ because of (37). For such values of λ, we have inequality $J_\lambda(df_2) \leqslant 1$. Therefore, by (14), and (38),

$$J_\lambda(f_1) \leqslant cJ_\lambda(f_2) \leqslant J_\lambda(df_2) \leqslant 1, \quad \lambda \in \left[d\|f_2\|_{\Phi,v}, \infty\right).$$

It means that,

$$\|f_1\|_{\Phi,v} \leqslant d\|f_2\|_{\Phi,v}, \quad d > d(c).$$

Thus, the relations (15) follow.

Example 1 If $\Phi(t) = t^\varepsilon$, $t \in [0, \infty)$, $\varepsilon > 0$, then

$$t_0 = 0, \quad t_\infty = \infty, \quad d(c) = c^{1/\varepsilon}, \quad c \in R_+.$$

Example 2 Let $\Phi(t) = e^t - 1$, $t \in [0, \infty)$. Then,

$$t_0 = 0, \quad t_\infty = \infty, \quad c > 1 \Rightarrow d(c) = c.$$

Example 3 Let $\Phi(t) = \ln^\gamma(t + 1)$, $t \in [0, \infty)$, $\gamma > 0$. Then, $t_0 = 0$, $t_\infty = \infty$, $d(c) = \infty$ for every $c > 1$. Indeed, if $c > 1$, the inequality $\ln^\gamma(dt + 1) \geqslant c\ln^\gamma(t + 1)$ fails for every $d \in R_+$ when $t \in R_+$ is big enough, because

$$\lim_{t\to+\infty}\left[\frac{\ln^\gamma(dt + 1)}{\ln^\gamma(t + 1)}\right] = 1.$$

Example 4 Let the condition (11) be fulfilled, let $\varepsilon > 0$, and $\Phi(t)\,t^{-\varepsilon} \uparrow$ on (t_0, t_∞). Then,

$$c > 1 \Rightarrow d(c) \leqslant c^{1/\varepsilon}. \tag{53}$$

Indeed, for every $t \in \left(t_0, c^{-1/\varepsilon}t_\infty\right)$

$$\Phi\left(c^{1/\varepsilon}t\right) = \left(c^{1/\varepsilon}t\right)^\varepsilon\left[\Phi\left(c^{1/\varepsilon}t\right)\left(c^{1/\varepsilon}t\right)^{-\varepsilon}\right] \geqslant \left(c^{1/\varepsilon}t\right)^\varepsilon\left[\Phi(t)\,t^{-\varepsilon}\right] = c\Phi(t).$$

It means that $d(c) \leqslant c^{1/\varepsilon}$.

Example 5 Let the condition (11) be fulfilled, let $p \in (0, 1]$, and Φ be p-convex on $[t_0, t_\infty)$, that is for $\alpha, \beta \in (0, 1]$, $\alpha^p + \beta^p = 1$ the inequality holds

$$\Phi(\alpha t + \beta\tau) \leqslant \alpha^p\Phi(t) + \beta^p\Phi(\tau), \quad t, \tau \in [t_0, t_\infty). \tag{54}$$

If $t_\infty < \infty$, then by passage to the limit this inequality is extended on $[t_0, t_\infty]$. Thus, we have,

$$c > 1 \Rightarrow d(c) \leqslant c^{1/p}. \tag{55}$$

Indeed, (54) implies $\Phi(t)\, t^{-p} \uparrow$ on $[t_0, t_\infty)$, and the result of Example 4 is applicable here.

Example 6 (Young function) Let $\Phi : [0, \infty) \to [0, \infty]$ be the so-called Young function that is,

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \tag{56}$$

where $\varphi : [0, \infty) \to [0, \infty]$ is the decreasing and left-continuous function, and $\varphi(0) = 0$, φ is neither identically zero, nor identically infinity on $(0, \infty)$. Then, $\Phi \in \Theta$, and t_0, t_∞, being introduced for Φ by (1) and (2), are the same as their analogues for φ. We assume that (11) is satisfied. Function Φ is convex on $[t_0, t_\infty)$ because $0 \leqslant \varphi \uparrow$. Thus, we can apply the conclusions of Example 5 with $p = 1$. In particular, $c > 1 \Rightarrow d(c) \leqslant c$.

Theorem 2 *Let the conditions* (5) *and (11) be fulfilled, and Φ being p-convex on* $[t_0, t_\infty)$ *with some* $p \in (0, 1]$. *Then, the following conclusions hold.*

(1) The triangle inequality takes place in $L_{\Phi,v}$: if $f, g \in L_{\Phi,v}$ then $f + g \in L_{\Phi,v}$, and

$$\|f + g\|_{\Phi,v} \leqslant \left(\|f\|_{\Phi,v}^p + \|g\|_{\Phi,v}^p\right)^{1/p}. \tag{57}$$

(2) The quantity $\|f\|_{\Phi,v}$ is monotone quasi-norm (norm, if $p = 1$):

$$f \in M, \quad |f| \leqslant g \in L_{\Phi,v} \Rightarrow f \in L_{\Phi,v}, \quad \|f\|_{\Phi,v} \leqslant \|g\|_{\Phi,v}, \tag{58}$$

that has Fatou property:

$$f_n \in M, \quad 0 \leqslant f_n \uparrow f \Rightarrow \|f\|_{\Phi,v} = \lim_{n\to\infty} \|f_n\|_{\Phi,v}. \tag{59}$$

Conclusion. *In the conditions of Theorem 2 $L_{\Phi,v}$ forms ideal quasi-Banach space having Fatou property (Banach space if $p = 1$, in particular in the case of Young function Φ).*

Proof (of Theorem 2) **1**. Let $f, g \in L_{\Phi,v}$. Then, we have for all $\lambda \geqslant \|f\|_{\Phi,v}^p$, $\mu \geqslant \|g\|_{\Phi,v}^p$,

$$J_{\lambda^{1/p}}(f) = \int_{R_+} \Phi\left(\lambda^{-1/p} |f(x)|\right) v(x)\, dx \leqslant 1; \tag{60}$$

$$J_{\mu^{1/p}}(g) = \int\limits_{R_+} \Phi\left(\mu^{-1/p}|g(x)|\right) v(x)\,dx \leqslant 1. \tag{61}$$

Now, almost everywhere on R_+ (60), and (61) yield,

$$\Phi\left(\lambda^{-1/p}|f(x)|\right) + \Phi\left(\mu^{-1/p}|g(x)|\right) < \infty, \tag{62}$$

because $v(x) > 0$ almost everywhere on R_+. Further, for $t_\infty = \infty$ we denote

$$\tilde{E}(f) = \{x \in R_+ : |f(x)| < \infty\}, \tag{63}$$

$$\tilde{E}(g) = \{x \in R_+ : |g(x)| < \infty\}, \tag{64}$$

and for $t_\infty < \infty$ we denote

$$\tilde{E}(f) = \left\{x \in R_+ : \lambda^{-1/p}|f(x)| \leqslant t_\infty\right\}, \tag{65}$$

$$\tilde{E}(g) = \left\{x \in R_+ : \lambda^{-1/p}|g(x)| \leqslant t_\infty\right\}. \tag{66}$$

In both cases we have according to (62),

$$\Phi\left(\lambda^{-1/p}|f(x)|\right) = \infty, \quad x \in R_+ \backslash \tilde{E}(f) \Rightarrow mes\left(R_+ \backslash \tilde{E}(f)\right) = 0,$$

$$\Phi\left(\mu^{-1/p}|g(x)|\right) = \infty, \quad x \in R_+ \backslash \tilde{E}(g) \Rightarrow mes\left(R_+ \backslash \tilde{E}(g)\right) = 0.$$

Therefore,

$$mes\left(R_+ \backslash \left[\tilde{E}(f) \cap \tilde{E}(g)\right]\right) = 0, \tag{67}$$

$$J_{\lambda^{1/p}}(f) = \int\limits_{\tilde{E}(f)\cap\tilde{E}(g)} \Phi\left(\lambda^{-1/p}|f(x)|\right) v(x)\,dx, \tag{68}$$

$$J_{\mu^{1/p}}(g) = \int\limits_{\tilde{E}(f)\cap\tilde{E}(g)} \Phi\left(\mu^{-1/p}|g(x)|\right) v(x)\,dx, \tag{69}$$

$$J_{(\lambda+\mu)^{1/p}}(f+g) = \int\limits_{\tilde{E}(f)\cap\tilde{E}(g)} \Phi\left((\lambda+\mu)^{-1/p}|f(x)+g(x)|\right) v(x)\,dx. \tag{70}$$

For $\Phi \in \Theta$ the following inequality holds

$$\Phi\left((\lambda+\mu)^{-1/p}\left|f(x)+g(x)\right|\right) \leqslant \\ \leqslant \Phi\left((\lambda+\mu)^{-1/p}\left|f(x)\right|+(\lambda+\mu)^{-1/p}\left|g(x)\right|\right). \quad (71)$$

We define

$$\alpha=\lambda^{1/p}(\lambda+\mu)^{-1/p}, \quad \beta=\mu^{1/p}(\lambda+\mu)^{-1/p};$$

$$t=\lambda^{-1/p}\left|f(x)\right|, \quad \tau=\mu^{-1/p}\left|g(x)\right|.$$

In this case $\alpha^p+\beta^p=1$, and we have for $x \in \tilde{E}(f) \cap \tilde{E}(g)$

$$t,\tau \in [0,\infty), \quad t_\infty=\infty; \quad t,\tau \in [0,t_\infty], \quad t_\infty<\infty.$$

Therefore, the estimate (54) is applicable for the right-hand side of (71). As the result,

$$\Phi\left((\lambda+\mu)^{-1/p}\left|f(x)+g(x)\right|\right) \leqslant \\ \leqslant \frac{\lambda}{\lambda+\mu}\Phi\left(\lambda^{-1/p}\left|f(x)\right|\right)+\frac{\mu}{\lambda+\mu}\Phi\left(\mu^{-1/p}\left|g(x)\right|\right).$$

We integrate this inequality over the set $\tilde{E}(f) \cap \tilde{E}(g)$, and take into account formulas (68)–(70). Then,

$$J_{(\lambda+\mu)^{1/p}}(f+g) \leqslant \frac{\lambda}{\lambda+\mu}J_{\lambda^{1/p}}(f)+\frac{\mu}{\lambda+\mu}J_{\mu^{1/p}}(g). \quad (72)$$

From (72), (60), and (61), it follows that

$$J_{(\lambda+\mu)^{1/p}}(f+g) \leqslant \frac{\lambda}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}=1.$$

Thus,

$$\left\|f+g\right\|_{\Phi,v} \leqslant (\lambda+\mu)^{1/p}.$$

This inequality holds for all λ, μ, satisfying the conditions $\lambda \geqslant \left\|f\right\|_{\Phi,v}^p$, $\mu \geqslant \left\|g\right\|_{\Phi,v}^p$. Therefore, estimate (57) is valid.

2. Let us check the properties of quasi-norm.

For $c=0$ it is obvious that $J_\lambda(cf)=J_\lambda(0)=0$, $\forall\lambda>0$, so that

$$\left\|cf\right\|_{\Phi,v}=\inf\left\{\lambda>0:\ J_\lambda(cf)\leqslant 1\right\}=0=|c|\left\|f\right\|_{\Phi,v}.$$

For $c \neq 0$ we have,

$$\|cf\|_{\Phi,v} = \inf\{\lambda > 0 : J_\lambda(cf) \leqslant 1\} = \inf\left\{\lambda > 0 : J_{\lambda/|c|}(f) \leqslant 1\right\}$$
$$= \inf\left\{|c|\,\mu > 0 : J_\mu(f) \leqslant 1\right\} = |c|\,\|f\|_{\Phi,v}.$$

Thus, we have $\|cf\|_{\Phi,v} = |c|\,\|f\|_{\Phi,v}$ for all $\in R$.

Moreover, it is evident that $f = 0 \Rightarrow \|f\|_{\Phi,v} = 0$. Let us show the inverse. Let $\|f\|_{\Phi,v} = 0$. Then,

$$\|f\|_{\Phi,v} = \inf\{\lambda > 0 : J_\lambda(f) \leqslant 1\} = 0 \Rightarrow J_\lambda(f) \leqslant 1, \forall\lambda > 0. \tag{73}$$

Let us suppose that f is not equivalent to zero. Then,

$$\exists\varepsilon > 0, E \subset R_+ : mes E > 0; |f(x)| \geqslant \varepsilon, \quad x \in E.$$

It means that for every $\lambda > 0$

$$J_\lambda(f) \geqslant \int_E \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx \geqslant \Phi\left(\lambda^{-1}\varepsilon\right)\int_E v(x)\,dx. \tag{74}$$

We know that $v(x) > 0$ almost everywhere, and $mes E > 0$. Then, $\int_E v(x)\,dx > 0$. Moreover, $\Phi\left(\lambda^{-1}\varepsilon\right) \uparrow \infty\ (\lambda \downarrow 0)$. Thus, the right-hand side in (74) tends to $+\infty$ if $\lambda \downarrow 0$, that prevents to (73). Therefore, the above assumption fails, that is $f = 0$ almost everywhere on R_+. These assertions together with triangle inequality (57) show that the quantity $\|f\|_{\Phi,v}$ has all properties of quasi-norm (norm if $p = 1$).

3. Let us prove the property of monotonicity for quasi-norm. The increasing of function $\Phi \in \Theta$ implies that

$$|f| \leqslant g \Rightarrow J_\lambda(f) \leqslant J_\lambda(g), \quad \forall\lambda > 0.$$

We have inequality $J_\lambda(g) \leqslant 1$ when$\lambda \geqslant \|g\|_{\Phi,v}$, $g \in L_{\Phi,v}$. Then,

$$J_\lambda(f) \leqslant 1, \quad \forall\lambda \geqslant \|g\|_{\Phi,v} \Rightarrow \|f\|_{\Phi,v} \leqslant \|g\|_{\Phi,v}. \tag{75}$$

4. Now, we prove the Fatou property. Let $f_n \in M_+$, $f_n \uparrow f$. Function $\Phi \in \Theta$ is increasing and left continuous, therefore $\Phi\left(\lambda^{-1}|f_n(x)|\right) \uparrow \Phi\left(\lambda^{-1}|f(x)|\right)$ almost everywhere. We can apply B. Levy monotone convergence theorem for every $\lambda > 0$:

$$J_\lambda(f_n) = \int_{R_+} \Phi\left(\lambda^{-1}|f_n(x)|\right) v(x)\,dx \uparrow \int_{R_+} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx = J_\lambda(f).$$

(this conclusion is valid as well in the case $J_\lambda(f) = \infty$). Then,

$$J_\lambda(f_n) \leqslant J_\lambda(f), \quad n \in N \Rightarrow \|f_n\|_{\Phi,v} \leqslant \|f\|_{\Phi,v}, \quad n \in N.$$

Denote

$$B_f = \sup_{n\in N} \|f_n\|_{\Phi,v} = \lim_{n\to\infty} \|f_n\|_{\Phi,v}$$

Let us show that $B_f = \|f\|_{\Phi,v}$. It is clear that $B_f \leqslant \|f\|_{\Phi,v}$. Suppose that $B_f < \|f\|_{\Phi,v}$. For any $\lambda \in \left(B_f, \|f\|_{\Phi,v}\right)$ we have

$$\lambda < \|f\|_{\Phi,v} = \inf\left\{\mu > 0 : J_\mu(f) \leqslant 1\right\} \Rightarrow J_\lambda(f) > 1.$$

At the same time, for every $n \in N$

$$\lambda > \|f_n\|_{\Phi,v} \Rightarrow J_\lambda(f_n) \leqslant 1.$$

Thus,

$$J_\lambda(f) = \lim_{n\to\infty} J_\lambda(f_n) \leqslant 1.$$

This contradiction shows that the above assumption was wrong. Thus, $B_f = \|f\|_{\Phi,v}$.

The following result is useful by the calculation of the norm of operator over Orlicz space $L_{\Phi,v}$.

Lemma 4 *Let the condition* (5) *be fulfilled. Then, the following equivalence takes place for* $f \in M$*,*

$$\|f\|_{\Phi,v} \leqslant 1 \Leftrightarrow J_1(f) = \int_0^\infty \Phi(|f(x)|)v(x)\,dx \leqslant 1. \tag{76}$$

Proof Obviously,

$$J_1(f) \leqslant 1 \Rightarrow \|f\|_{\Phi,v} \leqslant 1. \tag{77}$$

From the other side, we have

$$J_1(f) = \lim_{\lambda\downarrow 1} J_\lambda(f). \tag{78}$$

Indeed, $\lambda \downarrow 1 \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) \uparrow \Phi(|f(x)|)$ almost everywhere because of increasing and left-continuity of function $\Phi \in \Theta$. Then, by B. Levy monotone convergence theorem

$$\int_0^\infty \Phi(|f(x)|)v(x)\,dx = \lim_{\lambda\downarrow 1}\int_0^\infty \Phi\left(\lambda^{-1}|f(x)|\right)v(x)\,dx,$$

which gives (78). Consequently, if $J_1(f) > 1$, we can find $\lambda_0 > 1$, such that $J_{\lambda_0}(f) > 1$. Then, $J_\lambda(f) \leqslant 1 \Rightarrow \lambda > \lambda_0$ (because of decreasing of $J_\lambda(f)$ by λ). Therefore,

$$\|f\|_{\Phi,v} = \inf\{\lambda > 0 : \ J_\lambda(f) \leqslant 1\} \geqslant \lambda_0 > 1.$$

Finally,

$$J_1(f) > 1 \Rightarrow \|f\|_{\Phi,v} > 1.$$

Together with (77), it implies the equivalence (76).

For the completeness, we formulate the results in the case of failure of the conditions (11), namely when

$$t_0^{-1}t_\infty < \infty \Leftrightarrow 0 < t_0 \leqslant t_\infty < \infty. \tag{79}$$

Lemma 5 *In the conditions* (5) *the following estimates hold for function* $f \in M$,

$$t_0\|f\|_{\Phi,v} \leqslant \|f\|_{L_\infty}; \quad \|f\|_{L_\infty} \leqslant t_\infty\|f\|_{\Phi,v}. \tag{80}$$

Proof Let $t_0 > 0$, $\|f\|_{L_\infty} < \infty$. Then, we have for any $\lambda \geqslant t_0^{-1}\|f\|_{L_\infty}$ that

$$|f(x)| \leqslant \|f\|_{L_\infty} \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) \leqslant \Phi\left(\lambda^{-1}\|f\|_{L_\infty}\right) = 0,$$

almost everywhere by the property (4). Therefore, $\lambda \geqslant t_0^{-1}\|f\|_{L_\infty} \Rightarrow J_\lambda(f) = 0$, that is

$$\|f\|_{\Phi,v} = \inf\{\lambda > 0 : \ J_\lambda(f) \leqslant 1\} \leqslant t_0^{-1}\|f\|_{L_\infty}.$$

It gives the first estimate in (80). Further, let $t_\infty < \infty$, $\|f\|_{\Phi,v} < \infty$. For any $\lambda \geqslant \|f\|_{\Phi,v}$ we have $J_\lambda(f) < \infty$. Then, by analogy with the proof of (29), and (30) we obtain that $\Phi\left(\lambda^{-1}|f(x)|\right) < \infty$ almost everywhere. Thus, by (4) we conclude that $\left\{x \in R_+ : \ \lambda^{-1}|f(x)| > t_\infty\right\}$ is set of measure zero. It means that $\lambda^{-1}|f(x)| \leqslant t_\infty$ almost everywhere, and

$$J_\lambda(f) < \infty \Rightarrow \|f\|_{L_\infty} \leqslant \lambda t_\infty. \tag{81}$$

It gives the second estimate in (80).

Corollary 3 *Let the conditions* (5) *and (79) be fulfilled. Then the two-sided estimate takes place for every function* $f \in M$

$$t_0 \left\| f \right\|_{\Phi,v} \leqslant \left\| f \right\|_{L_\infty} \leqslant t_\infty \left\| f \right\|_{\Phi,v}, \tag{82}$$

showing that $L_{\Phi,v} = L_\infty$ *with the equivalence of the norms. Here* $L_\infty = L_\infty(R_+)$ *is the space of all essentially bounded functions.*

The above corollary shows that we lose the specific of Orlicz spaces in its conditions.

Nevertheless, we formulate in this case the answer on the above posed question.

Lemma 6 *Let the conditions* (5) *and (79) be fulfilled, and* $f_1 \in M$, $f_2 \in L_{\Phi,v}$. *If for every* $\lambda > \left\| f_2 \right\|_{\Phi,v}$ *we have* $J_\lambda(f_1) < \infty$, *then* $f_1 \in L_{\Phi,v}$, *and*

$$\left\| f_1 \right\|_{\Phi,v} \leqslant t_0^{-1} t_\infty \left\| f_2 \right\|_{\Phi,v}. \tag{83}$$

Proof We have $J_\lambda(f_1) < \infty$ for every $\lambda > \left\| f_2 \right\|_{\Phi,v}$ so that we obtain inequality $\left\| f_1 \right\|_{L_\infty} \leqslant t_\infty \lambda$ similarly as it was made in (81). Therefore, $\left\| f_1 \right\|_{L_\infty} \leqslant t_\infty \left\| f_2 \right\|_{\Phi,v}$. Together with the first estimate in (80), it gives (83).

2 Discrete Weighted Orlicz Spaces

2.1. Here, we consider the discrete variants of Orlicz spaces. For it, we assume that

$$\Phi \in \Theta; \quad \beta = \{\beta_m\}, \quad \beta_m \in R_+, \quad m \in Z = \{0, \pm 1, \pm 2, ...\}. \tag{84}$$

Denote

$$l_{\Phi,\beta} = \left\{ \alpha = \{\alpha_m\}, \ \alpha_m \in R : \ \left\| \alpha \right\|_{l_{\Phi,\beta}} < \infty \right\},$$

where

$$\left\| \alpha \right\|_{l_{\Phi,\beta}} := \inf \{\lambda > 0 : \ j_\lambda(\alpha) \leqslant 1\}, \quad j_\lambda(\alpha) = \sum_m \Phi\left(\lambda^{-1} |\alpha_m|\right)\beta_m. \tag{85}$$

Let us formulate some discrete analogues of the results of Sect. 1. An analogue of Theorem 1 is as follows.

Theorem 3 *Let the conditions (84) be fulfilled; let* $c \in R_+$, *and if* $c \in (0, 1)$, *then* $t_0 = 0$; $t_\infty = \infty$ *in (1), (2); if* $c \in (1, \infty)$ *the (11) is fulfilled. Let* $d(1) = 1$; $d(c)$ *is determined by (10), and (12) for* $c \neq 1$, *moreover, for* $c \in (1, \infty)$ *we assume that*

$\Phi \in \Theta_c$. Let the following estimate holds for sequences $\alpha = \{\alpha_m\}$, $\gamma = \{\gamma_m\}$, where $\gamma \in l_{\Phi,v}$:

$$j_\lambda(\alpha) \leqslant c j_\lambda(\gamma), \quad \lambda \geqslant d(c) \left\|\gamma\right\|_{l_{\Phi,\beta}}. \tag{86}$$

Then, $\alpha \in l_{\Phi,v}$, and the inequality holds

$$\left\|\alpha\right\|_{l_{\Phi,\beta}} \leqslant d(c) \left\|\gamma\right\|_{l_{\Phi,\beta}} \tag{87}$$

Corollary 4 *Let the conditions (84) and (11) be fulfilled, let $0 < c_1 \leqslant c_2 < \infty$, and $\alpha = \{\alpha_m\}$, $\gamma = \{\gamma_m\}$. Moreover, if $c_0 = \min\left\{c_1^{-1}, c_2\right\} \in (0,1)$, then we require $t_0 = 0$; $t_\infty = \infty$; if $c = \max\left\{c_1^{-1}, c_2\right\} > 1$, then we require $\Phi \in \Theta_c$. Let*

$$c_1 j_\lambda(\gamma) \leqslant j_\lambda(\alpha) \leqslant c_2 j_\lambda(\gamma), \tag{88}$$

for every $\lambda > 0$. Then the following estimates hold

$$d_1 \left\|\gamma\right\|_{l_{\Phi,\beta}} \leqslant \left\|\alpha\right\|_{l_{\Phi,\beta}} \leqslant d_2 \left\|\gamma\right\|_{l_{\Phi,\beta}}, \tag{89}$$

with $d_1 = d\left(c_1^{-1}\right)^{-1}$, $d_2 = d(c_2)$, see (10), (12).

Now, we formulate an analogue of Theorem 2.

Theorem 4 *Let the conditions (21) and (11) be fulfilled, and Φ be p-convex on $[t_0, t_\infty)$ for $p \in (0,1]$. Then the following conclusions hold.*

(1) Triangle inequality takes place in $l_{\Phi,v}$. Namely, if $\alpha = \{\alpha_m\}$, $\gamma = \{\gamma_m\}$; $\alpha, \gamma \in l_{\Phi,\beta}$, then $\alpha + \gamma \in l_{\Phi,\beta}$, and

$$\left\|\alpha + \gamma\right\|_{l_{\Phi,\beta}} \leqslant \left(\left\|\alpha\right\|^p_{l_{\Phi,\beta}} + \left\|\beta\right\|^p_{l_{\Phi,\beta}}\right)^{1/p}. \tag{90}$$

(2) The quantity $\left\|\alpha\right\|_{l_{\Phi,\beta}}$ is monotone quasi-norm (norm for $p = 1$):

$$|\alpha_m| \leqslant \gamma_m, \quad m \in Z; \quad \gamma \in l_{\Phi,\beta} \Rightarrow \alpha \in l_{\Phi,\beta}, \quad \left\|\alpha\right\|_{l_{\Phi,\beta}} \leqslant \left\|\gamma\right\|_{l_{\Phi,\beta}},$$

that possess Fatou property: let $\alpha^n = \left\{\alpha^n_m\right\}$, $\gamma = \{\gamma_m\}$, $n \in N$, then

$$0 \leqslant \alpha^n_m \uparrow \gamma_m \, (n \uparrow \infty), \quad m \in Z \Rightarrow \left\|\gamma\right\|_{l_{\Phi,\beta}} = \lim_{n\to\infty} \left\|\alpha^n\right\|_{l_{\Phi,\beta}}.$$

Conclusion. *In the conditions of Theorem 4. $l_{\Phi,\beta}$ forms discrete ideal quasi-Banach space (Banach space for $p = 1$; particularly, when Φ Young function is) that possesses Fatou property.*

Lemma 7 *Let the condition (84) be fulfilled. Then the following equivalence takes place:*

$$\|\alpha\|_{l_{\Phi,\beta}} \leqslant 1 \Leftrightarrow j_1(\alpha) = \sum_m \Phi(|\alpha_m|)\beta_m \leqslant 1.$$

2.2. To establish these discrete analogues of the results of Sect. 1, we can introduce the sequence $\{\mu_m\}$ such that

$$\mu_m < \mu_{m+1}; \quad R_+ = \bigcup_m \Delta_m; \quad \Delta_m = \left[\mu_m, \mu_{m+1}\right). \tag{91}$$

We define the weight function $v \in M$, $v > 0$ satisfying the conditions

$$\int_{\Delta_m} v dt = \beta_m. \tag{92}$$

Then we restrict the considerations of Sect. 1 on the set of step-functions

$$\tilde{L}_{\Phi,v} = \left\{ f \in L_{\Phi,v} : \; f = \sum_m \alpha_m \chi_{\Delta_m}, \; \alpha_m \in R \right\}, \tag{93}$$

where χ_{Δ_m} is the characteristic function of interval Δ_m. For such functions, we have

$$J_\lambda(f) = j_\lambda(\alpha); \quad \|f\|_{\Phi,v} = \|\alpha\|_{l_{\Phi,\beta}}, \quad \alpha = \{\alpha_m\}. \tag{94}$$

Indeed,

$$J_\lambda(f) = \int_0^\infty \Phi\left(\lambda^{-1}|f(t)|\right)v(t)\,dt = \sum_m \int_{\Delta_m} \ldots =$$

$$= \sum_m \Phi\left(\lambda^{-1}|\alpha_m|\right) \int_{\Delta_m} v dt = \sum_m \Phi\left(\lambda^{-1}|\alpha_m|\right)\beta_m = j_\lambda(\alpha).$$

Now, all above-mentioned discrete formulas are the partial cases of corresponding formulas of Sect. 1 applied to step-functions in Orlicz space.

2.3. Here, we describe one special discretization procedure for integral assertions on the cone Ω of nonnegative decreasing functions in $L_{\Phi,v}$:

$$\Omega \equiv \left\{ f \in L_{\Phi,v} : \; 0 \leqslant f \downarrow \right\}. \tag{95}$$

We assume here that the weight function v satisfies the conditions

$$0 < V(t) := \int_0^t v d\tau < \infty, \quad \forall t \in R_+, \tag{96}$$

Moreover, we assume that V is strictly increasing, and

$$V(+\infty) = \infty. \tag{97}$$

(the case $V(+\infty) < \infty$ we will consider separately). For fixed $b > 1$ we introduce the sequence $\{\mu_m\}$ by formulas

$$\mu_m = V^{-1}\left(b^m\right) \Leftrightarrow V(\mu_m) = b^m, \quad m \in Z = \{0, \pm 1, \pm 2, \ldots\}, \tag{98}$$

where V^{-1} is the inverse function for the continuous increasing function V. Then, the condition (91) is fulfilled, because

$$0 < \mu_m \uparrow; \quad \lim_{m \to -\infty} \mu_m = 0; \quad \lim_{m \to +\infty} \mu_m = \infty. \tag{99}$$

Moreover, we introduce the cone of nonnegative step-functions

$$S \equiv L^+_{\Phi,v} \cap \tilde{L}_{\Phi,v} = \left\{ f \in L_{\Phi,v} : \ f = \sum_m \gamma_m \chi_{\Delta_m}; \ \gamma_m \geqslant 0, m \in Z \right\}; \tag{100}$$

as well as the cone of nonnegative decreasing step-functions

$$\tilde{\Omega} \equiv \Omega \cap \tilde{L}_{\Phi,v} = \left\{ f \in L_{\varphi,v} : \ f = \sum_m \alpha_m \chi_{\Delta_m}; \ 0 \leqslant \alpha_m \downarrow \right\}. \tag{101}$$

For $f \in \Omega$ we determine step-functions $f_0, f_1 \in \tilde{\Omega}$:

$$f_0 := \sum_m f(\mu_{m+1}) \chi_{\Delta_m}, \quad f_1 := \sum_m f(\mu_m) \chi_{\Delta_m}. \tag{102}$$

Then,

$$f_0 \leqslant f \leqslant f_1 \Rightarrow \| f_0 \|_{\Phi,v} \leqslant \| f \|_{\Phi,v} \leqslant \| f_1 \|_{\Phi,v} \tag{103}$$

(the left hand side inequality in (103) is valid everywhere on R_+). We use the equalities (94) for step-functions f_0 and f_1. Then,

$$\| f_0 \|_{\Phi,v} = \| \{\alpha_{m+1}\} \|_{l_{\Phi,\beta}}; \quad \| f_1 \|_{\Phi,v} = \| \{\alpha_m\} \|_{l_{\Phi,\beta}}, \quad \alpha_m := f(\mu_m). \tag{104}$$

Here, according to (92), and (98),

$$\beta_m = \int\limits_{\Delta_m} v dt = V(\mu_{m+1}) - V(\mu_m) = b^m (b-1), \quad m \in Z. \tag{105}$$

Remark 5 By the discretization (98)–(105) the shift-operators

$$T_+[\{\gamma_m\}] = \{\gamma_{m+1}\}, \quad T_-[\{\gamma_m\}] = \{\gamma_{m-1}\} \tag{106}$$

are bounded as operators in $l_{\Phi,\beta}$.

It is a partial case of the following result.

Lemma 8 *Let $b > 1$; $\Phi \in \Theta_b$; $\beta = \{\beta_m\}$; $\beta_m \in R_+$, $1 \leqslant \beta_{m+1}/\beta_m \leqslant b$, $m \in Z$. Then,*

$$\|T_+\| \leqslant 1, \quad \|T_-\| \leqslant d(b), \tag{107}$$

where $d(b)$ is the constant (12) with $c = b > 1$. If Φ is convex function, we obtain the estimates (107) with $d(b) = b$. In particular, it is true in the case of Young function Φ; see Example 6.

Proof To obtain estimates (107) let us note that for every $\lambda > 0$

$$j_\lambda(\{\gamma_{m+1}\}) \leqslant j_\lambda(\{\gamma_m\}); \quad j_\lambda(\{\gamma_{m-1}\}) \leqslant b j_\lambda(\{\gamma_m\}). \tag{108}$$

Indeed,

$$j_\lambda(\{\gamma_{m+1}\}) = \sum_{m\in Z} \varphi\left(\lambda^{-1}|\gamma_{m+1}|\right)\beta_m = \sum_{m\in Z} \varphi\left(\lambda^{-1}|\gamma_m|\right)\beta_{m-1};$$

$$j_\lambda(\{\gamma_{m-1}\}) = \sum_{m\in Z} \varphi\left(\lambda^{-1}|\gamma_{m-1}|\right)\beta_m = \sum_{m\in Z} \varphi\left(\lambda^{-1}|\gamma_m|\right)\beta_{m+1},$$

and we obtain (108) by taking into account the conditions on $\beta = \{\beta_m\}$. From (108), and (86), (87), it follows that

$$\left\|T_+[\{\gamma_m\}]\right\|_{l_{\Phi,\beta}} = \left\|\{\gamma_{m+1}\}\right\|_{l_{\Phi,\beta}} \leqslant \left\|\{\gamma_m\}\right\|_{l_{\Phi,\beta}},$$

$$\left\|T_-[\{\gamma_m\}]\right\|_{l_{\Phi,\beta}} = \left\|\{\gamma_{m-1}\}\right\|_{l_{\Phi,\beta}} \leqslant d(b)\left\|\{\gamma_m\}\right\|_{l_{\Phi,\beta}}. \tag{109}$$

If Φ is convex, then $d(b) = b$. Thus, we come to estimates (107).

Let us apply estimate (107) to the sequence $\{\gamma_m\} = \{\alpha_{m+1}\}$. Then, by (104) we have,

$$\|f_1\|_{\Phi,v} = \left\|\{\alpha_m\}\right\|_{l_{\Phi,\beta}} \leqslant d(b)\left\|\{\alpha_{m+1}\}\right\|_{l_{\Phi,\beta}} = d(b)\|f_0\|_{\Phi,v}. \tag{110}$$

Substituting of (110) into (103) implies the following conclusion.

Conclusion *Let $b > 1$; $\Phi \in \Theta_b$, weight v satisfies the conditions* (96), (97). *We realize the discretization procedure* (98)–(105) *for function $f \in \Omega$, see* (95). *Then,*

$$d(b)^{-1} \left\| f_1 \right\|_{\Phi,v} \leqslant \left\| f \right\|_{\Phi,v} \leqslant \left\| f_1 \right\|_{\Phi,v}, \tag{111}$$

where $d(b)$ was defined in (12) *with $c = b > 1$*. Here *f_1 is the step-function, determined by*, (102), *that satisfies*(104).

Remark 6 All the results of Sect. 2.1 are carried over the discrete weighted Orlicz spaces in which the condition $m \in Z = \{0, \pm 1, \pm 2, \ldots\}$ is replaced by the condition $m \in Z^- = \{0, -1, -2, \ldots\}$ in the notations (84) and below. Thus, here we consider the sequences $\alpha = \{\alpha_m\}$, $\beta = \{\beta_m\}$, $\gamma = \{\gamma_m\}$; $m \in Z^-$. The proofs for these discrete formulas are the same as in Sect. 2.2. Only, we have

$$\begin{gathered} \mu_1 = \infty; \quad \mu_m < \mu_{m+1}, \quad m \in Z^-; \\ R_+ = \bigcup_{m \in Z^-} \Delta_m; \quad \Delta_m = \left[\mu_m, \mu_{m+1}\right), \quad m \in Z^-, \end{gathered} \tag{112}$$

in (91), and assume $m \in Z^-$ in (92)–(94).

2.4. Now, let us describe the discretization procedure for the cone (95) in the case

$$0 < V(t) := \int_0^t v d\tau < \infty, \quad \forall t \in R_+, \quad V(+\infty) := \int_0^\infty v d\tau < \infty. \tag{113}$$

Without loss of generality, we will assume that

$$V(1) = 1. \tag{114}$$

We follow the considerations of Sect. 2.3 with small modifications.
According to (114) we have,

$$b = V(+\infty) > 1. \tag{115}$$

We introduce the discretizing sequence $\{\mu_m\}$by formulas

$$\mu_1 = \infty; \quad \mu_m = V^{-1}\left(b^m\right), \quad m \in Z^- = \{0, -1, -2, \ldots\}. \tag{116}$$

Here, V^{-1} is the inverse function for the increasing continuous function V, so that

$$V(\mu_m) = b^m, \quad m = 1, 0, -1, -2, \ldots \tag{117}$$

Then,

$$(0,1)=\bigcup_{m\leqslant -1}\Delta_m,\quad [1,\infty)=\Delta_0,$$
$$R_+=\bigcup_{m\in Z^-}\Delta_m,\quad \Delta_m=\left[\mu_m,\mu_{m+1}\right). \tag{118}$$

We introduce step-functions on R_+ connected with $f\in\Omega$ by the decomposition (118):

$$f_0(t)=\sum_{m\in Z^-}\alpha_{m+1}\chi_{\Delta_m}(t),$$
$$f_1(t)=\sum_{m\in Z^-}\alpha_m\chi_{\Delta_m}(t),\quad \alpha_m=f(\mu_m). \tag{119}$$

Then,

$$f_0\leqslant f\leqslant f_1\Rightarrow \|f_0\|_{\Phi,v}\leqslant\|f\|_{\Phi,v}\leqslant\|f_1\|_{\Phi,v}. \tag{120}$$

For step-functions f_0 and f_1 we have,

$$\|f_0\|_{\Phi,v}=\|\{\alpha_{m+1}\}\|_{\bar l_{\Phi,\beta}};\quad \|f_1\|_{\Phi,v}=\|\{\alpha_m\}\|_{\bar l_{\Phi,\beta}}. \tag{121}$$

Here $\beta=\{\beta_m\}_{m\in Z^-}$,

$$\beta_m=\int\limits_{\Delta_m}v\,dt=V(\mu_{m+1})-V(\mu_m)=b^m(b-1),\quad m\in Z^-, \tag{122}$$

and we denote for $\gamma=\{\gamma_m\}_{m\in Z^-}$

$$\bar j_\lambda(\{\gamma_m\})=\sum_{m\in Z^-}\Phi\left(\lambda^{-1}|\gamma_m|\right)\beta_m; \tag{123}$$

$$\|\{\gamma_m\}\|_{\bar l_{\Phi,\beta}}=\inf\left\{\lambda>0:\ \bar j_\lambda(\{\gamma_m\})\leqslant 1\right\}. \tag{124}$$

Let us mentioned that the notations (121)–(124) are slightly different from ones in Sects. 2.1–2.3 introduced by (84), (85). Now we deal with one-sided sequences.

Remark 7 The next shift-operator is bounded in $\bar l_{\Phi,\beta}$:

$$T_-[\{\gamma_m\}]=\{\gamma_{m-1}\}_{m\in Z^-}. \tag{125}$$

This is the partial case of the following result.

Lemma 9 *Let $b>1$; $\Phi\in\Theta_b$, and*

$$\beta=\{\beta_m\}_{m\in Z^-},\quad \beta_m>0,\quad 1\leqslant\beta_m/\beta_{m-1}\leqslant b,\quad m\in Z^-.$$

Then the following estimate holds for the norm of operator $T_{-}:\bar{l}_{\Phi,\beta}\to\bar{l}_{\Phi,\beta}$

$$\left\|T_{-}\right\|\leqslant d\left(b\right), \tag{126}$$

where $d\left(b\right)$ *is the constant (12) with* $c=b>1$. *If* Φ *is* p*-convex, we obtain estimate (126) with* $d\left(b\right)=b^{1/p}$.

Proof Note that

$$\bar{j}_{\lambda}\left(\{\gamma_{m-1}\}\right)\leqslant b\bar{j}_{\lambda}\left(\{\gamma_{m}\}\right). \tag{127}$$

Indeed,

$$\bar{j}_{\lambda}\left(\{\gamma_{m-1}\}\right)=\sum_{m\in Z^{-}}\Phi\left(\lambda^{-1}\left|\gamma_{m-1}\right|\right)\beta_{m}=\sum_{m\leqslant-1}\Phi\left(\lambda^{-1}\left|\gamma_{m}\right|\right)\beta_{m+1};$$

and we obtain (127) by taking into account the conditions on $\beta=\{\beta_m\}_{m\in Z^{-}}$. It follows from (127), and (86), (87) (see also Remark 6)

$$\left\|T_{-}\left[\{\gamma_m\}\right]\right\|_{\bar{l}_{\Phi,\beta}}=\left\|\{\gamma_{m-1}\}\right\|_{\bar{l}_{\Phi,\beta}}\leqslant d\left(b\right)\left\|\{\gamma_m\}\right\|_{\bar{l}_{\Phi,\beta}}. \tag{128}$$

If Φ is p-convex, then $d\left(b\right)=b^{1/p}$. Thus, estimate (126) holds.

We apply (126) to the sequence $\{\gamma_m\}=\{\alpha_{m+1}\}$. Then, we have according to (121),

$$\left\|f_1\right\|_{\Phi,v}=\left\|\{\alpha_m\}\right\|_{\bar{l}_{\Phi,\beta}}\leqslant d\left(b\right)\left\|\{\alpha_{m+1}\}\right\|_{\bar{l}_{\Phi,\beta}}=d\left(b\right)\left\|f_0\right\|_{\Phi,v}. \tag{129}$$

Substitution of (129) into (120) gives the following conclusion.

Proposition 1 *Let us realize the discretization procedure (113)–(129) for function* $f\in\Omega$. *Then,*

$$d\left(b\right)^{-1}\left\|f_1\right\|_{\Phi,v}\leqslant\left\|f\right\|_{\Phi,v}\leqslant\left\|f_1\right\|_{\Phi,v}, \tag{130}$$

where $d\left(b\right)$ *is determined by (12) with* $c=b>1$. *Here, the equality (121) holds for function* f_1 *(119).*

3 Estimates for the Norm of Monotone Operator on Cone Ω

3.1 The Case of Nondegenerate Weight

We preserve all the notation of Sects. 1 and 2. Let $(\mathrm{N},\Re,\eta)$ be the measure-space with non-negative full σ-finite measure η; let $L=L\left(\mathrm{N},\Re,\eta\right)$ be the set of all η-measurable functions $u:\mathrm{N}\to R$; $L^{+}=\{u\in L:u\geqslant 0\}$. Here, we assume pointwise inequalities to be fulfilled η-almost everywhere. Let $Y=Y\left(\mathrm{N},\Re,\eta\right)\subset L$ be an ideal space, that is Banach, or quasi-Banach space of measurable functions with monotone norm, or quasi-norm $\|\cdot\|_Y$ so that

$$u_1 \in L, \quad |u_1| \leqslant |u_2|, \quad u_2 \in Y \Rightarrow u_1 \in Y, \quad \|u_1\|_Y \leqslant \|u_2\|_Y. \tag{131}$$

General theory of ideal spaces in the normed case was considered in [3], one special variant of such theory was developed in [11] on the base of concept of Banach function spaces, that includes Orlicz spaces. Let $P : M^+ \to L^+$ be the so called monotone operator, i.e.,

$$f, h \in M^+, \quad f \leqslant h \quad \mu - a.e. \Rightarrow Pf \leqslant Ph \quad \eta - \text{a. e.} \tag{132}$$

We define the norms of restrictions of operator P on the cones Ω (95), and $\tilde{\Omega}$ (101):

$$\|P\|_{\Omega \to Y} = \sup \left\{ \|Pf\|_Y : \ f \in \Omega, \ \|f\|_{\Phi,v} \leqslant 1 \right\}. \tag{133}$$

$$\|P\|_{\tilde{\Omega} \to Y} = \sup \left\{ \|Pf\|_Y : \ f \in \tilde{\Omega}, \ \|f\|_{\Phi,v} \leqslant 1 \right\}. \tag{134}$$

Lemma 10 *Let the conditions (84) be fulfilled, $b > 1$; $\Phi \in \Theta_b$. We assume that weight function satisfies (96) and (97), and realize the discretization procedure (98)–(105) for function $f \in \Omega$. The following estimates take place*

$$\|P\|_{\tilde{\Omega} \to Y} \leqslant \|P\|_{\Omega \to Y} \leqslant d(b) \|P\|_{\tilde{\Omega} \to Y}, \tag{135}$$

with $d(b)$ determined in (12) for $c = b > 1$.

Proof The left-hand side inequality in (135) is obvious because of embedding $\tilde{\Omega} \subset \Omega$. From the other side, for every function $f \in \Omega$, and for f_1 in (102), we have $f \leqslant f_1 \Rightarrow Pf \leqslant Pf_1$, and $\|f_1\|_{\Phi,v} \leqslant d(b) \|f\|_{\Phi,v}$ (see the conclusion after the proof of Lemma 8). Moreover,

$$f \in \Omega \Rightarrow f_1 = \sum_m f(\mu_m) \chi_{\Delta_m} \in \tilde{\Omega}.$$

Consequently, for every $f \in \Omega$

$$\|Pf\|_Y \leqslant \|Pf_1\|_Y \leqslant \|P\|_{\tilde{\Omega} \to Y} \|f_1\|_{\Phi,v} \leqslant d(b) \|P\|_{\tilde{\Omega} \to Y} \|f\|_{\Phi,v}, \tag{136}$$

and

$$\|P\|_{\Omega \to Y} = \sup \left\{ \|Pf\|_Y : \ f \in \Omega, \|f\|_{\Phi,v} \leqslant 1 \right\} \leqslant d(b) \|P\|_{\tilde{\Omega} \to Y}.$$

Now, we consider the norm of restriction on the cone S (100):

$$\left\| P \right\|_{S \to Y} = \sup \left\{ \left\| Pf \right\|_Y : \ f \in S, \ \left\| f \right\|_{\Phi, v} \leqslant 1 \right\}. \tag{137}$$

Theorem 5 *Let the conditions of Lemma 10 be fulfilled. Then, the following two-sided estimate takes place*

$$c\,(b)^{-1} \left\| P \right\|_{S \to Y} \leqslant \left\| P \right\|_{\Omega \to Y} \leqslant d\,(b) \left\| P \right\|_{S \to Y}, \tag{138}$$

where $d\,(b)$ *is determined by (12) with* $c = b > 1$, *and*

$$c\,(b) = d\,(c_0\,(b))\,; \quad c_0\,(b) = \left[b\,(b-1)^{-1} \right] > 1. \tag{139}$$

Proof Inequality (138) follows by (135), and by the analogous inequality

$$\left\| P \right\|_{\tilde{\Omega} \to Y} \leqslant \left\| P \right\|_{S \to Y} \leqslant c\,(b) \left\| P \right\|_{\tilde{\Omega} \to Y}. \tag{140}$$

The left inequality in (140) is obvious because of inclusion $\tilde{\Omega} \subset S$. Let us prove the right one.

1. We introduce sup-operator A by formula $A\gamma = \alpha$, where $\gamma = \{\gamma_m\}_{m \in Z}$; $\alpha = \{\alpha_m\}_{m \in Z}$, and

$$\alpha_m = \sup_{k \geqslant m} |\gamma_k|\,, \quad m \in Z. \tag{141}$$

Let us prove the boundedness of operator $A : \ l_{\Phi,\beta} \to l_{\Phi,\beta}$ with corresponding estimate

$$\left\| A\gamma \right\|_{l_{\Phi,\beta}} \leqslant c\,(b) \left\| \gamma \right\|_{l_{\Phi,\beta}}. \tag{142}$$

We assume that $\gamma \in l_{\Phi,\beta}$ (otherwise is nothing to prove). Let $\lambda \geqslant \left\| \gamma \right\|_{l_{\Phi,\beta}}$. Then,

$$j_\lambda\,(\gamma) = \sum_{k \in Z} \Phi\left(\lambda^{-1} |\gamma_k| \right) \beta_k \leqslant 1. \tag{143}$$

We have $\beta_k = b^k\,(b-1) \uparrow \infty$, so that

$$(143) \Rightarrow \Phi\left(\lambda^{-1} |\gamma_k| \right) \to 0\,(k \to +\infty)\,. \tag{144}$$

Let us show that for all non-zero terms of series

$$j_\lambda\,(\alpha) = \sum_{m \in Z} \Phi\left(\lambda^{-1} \alpha_m \right) \beta_m, \tag{145}$$

the equalities hold

$$\exists k\left(m\right):\ m\leqslant k\left(m\right)<\infty,\quad \varPhi\left(\lambda^{-1}\alpha_m\right)=\varPhi\left(\lambda^{-1}\left|\gamma_{k(m)}\right|\right). \tag{146}$$

For any $\varepsilon>0$ we have

$$\exists K\left(\varepsilon\right)\in Z:\ \lambda^{-1}\left|\gamma_k\right|\leqslant t_0+\varepsilon,\quad \forall k\geqslant K\left(\varepsilon\right). \tag{147}$$

Here t_0 is determined by (1) for $\varPhi\in\varTheta$. Indeed, if (147) fails, there exist $\varepsilon_0>0$ and subsequence of numbers $k_j\to+\infty$ such that

$$\lambda^{-1}\left|\gamma_{k_j}\right|\geqslant t_0+\varepsilon_0,\quad j\in N\Rightarrow\varPhi\left(\lambda^{-1}\left|\gamma_{k_j}\right|\right)\geqslant\varPhi\left(t_0+\varepsilon_0\right)>0.$$

This contradicts to (144). Thus, (147) is valid. Moreover, for every $m\in Z$, we have $\varPhi\left(\lambda^{-1}\alpha_m\right)\neq 0\Rightarrow\lambda^{-1}\alpha_m>t_0$. Therefore, if we set $\varepsilon=\varepsilon_{m,\lambda}\equiv 2^{-1}\left(\lambda^{-1}\alpha_m-t_0\right)>0$ then,

$$\lambda^{-1}\left|\gamma_k\right|\leqslant t_0+\varepsilon=2^{-1}\left(\lambda^{-1}\alpha_m+t_0\right),\quad \forall k\geqslant K\left(\varepsilon_{m,\lambda}\right),$$

according to (147). It means that $\sup\limits_{k\geqslant K\left(\varepsilon_{m,\lambda}\right)}\left|\gamma_k\right|\leqslant 2^{-1}\left(\alpha_m+t_0\lambda\right)<\alpha_m$. Thus,

$$\alpha_m=\sup_{k\geqslant m}\left|\gamma_k\right|=\max_{m\leqslant k\leqslant K\left(\varepsilon_{m,\lambda}\right)}\left|\gamma_k\right|.$$

Therefore, $\exists k\left(m\right):\ m\leqslant k\left(m\right)\leqslant K\left(\varepsilon_{m,\lambda}\right)$, $\alpha_m=\left|\gamma_{k(m)}\right|$. It follows from (145) and (146), that

$$j_\lambda\left(\alpha\right)=\sum_{m\in Z}\varPhi\left(\lambda^{-1}\left|\gamma_{k(m)}\right|\right)\beta_m. \tag{148}$$

Moreover, all terms in (148) are finite because of (143). From (148), it follows that

$$j_\lambda\left(\alpha\right)\leqslant\sum_{m\in Z}\beta_m\sum_{k\geqslant m}\varPhi\left(\lambda^{-1}\left|\gamma_k\right|\right)=\sum_{k\in Z}\varPhi\left(\lambda^{-1}\left|\gamma_k\right|\right)\sum_{m\leqslant k}\beta_m.$$

But, $\beta_m=b^{m+1}-b^m$, so that

$$\sum_{m\leqslant k}\beta_m=b^{k+1}=c_0\left(b\right)\beta_k,\quad c_0\left(b\right)=b\left(b-1\right)^{-1}.$$

As the result, we have estimate

$$j_\lambda\left(\alpha\right)\leqslant c_0\left(b\right)\sum_{k\in Z}\varPhi\left(\lambda^{-1}\left|\gamma_k\right|\right)\beta_k=c_0\left(b\right)j_\lambda\left(\gamma\right), \tag{149}$$

for all $\lambda \geqslant \|\gamma\|_{l_{\varphi,\beta}}$. Here,$c_0(b) > 1$, so that $d(c_0(b)) \geqslant 1$, where $d(c)$ is the constant (12). It means that inequality (149) is true for $\lambda \geqslant d(c_0(b))\|\gamma\|_{l_{\varphi,\beta}}$. By Theorem 3, it implies the estimate

$$\|\alpha\|_{l_{\varphi,\beta}} \leqslant d(c_0(b))\|\gamma\|_{l_{\varphi,\beta}},$$

coinciding with (142).

2. Now, we denote $\gamma = \{\gamma_m\}$, $\gamma_m = f(\mu_m) \geqslant 0$, $m \in Z$ for every $f \in S$. Then,

$$f = f_{(\gamma)} := \sum_m \gamma_m \chi_{\Delta_m}.$$

Further, we introduce $\alpha_m = \sup\limits_{k \geqslant m} \gamma_k$, $m \in Z$, and for $\alpha = \{\alpha_m\}$ consider function

$$f_{(\alpha)} = \sum_m \alpha_m \chi_{\Delta_m}.$$

Then, $f_{(\alpha)} \in \tilde{\Omega}$, see (101), and

$$f_{(\gamma)} \leqslant f_{(\alpha)}, \quad \|f_{(\alpha)}\|_{\Phi,v} = \|\alpha\|_{l_{\Phi,\beta}} \leqslant c(b)\|\gamma\|_{l_{\Phi,\beta}} = c(b)\|f_{(\gamma)}\|_{\Phi,v}; \tag{150}$$

see (142). Therefore, for $f = f_{(\gamma)} \in S$ there exists $f_{(\alpha)} \in \tilde{\Omega}$ such that

$$Pf \leqslant Pf_{(\alpha)}; \quad \|f_{(\alpha)}\|_{\Phi,v} \leqslant c(b)\|f\|_{\Phi,v}.$$

Here, $f_{(\alpha)} \in \tilde{\Omega}$, and we obtain for every function $f \in S$

$$\|Pf\|_Y \leqslant \|Pf_{(\alpha)}\|_Y \leqslant \|P\|_{\tilde{\Omega}\to Y}\|f_{(\alpha)}\|_{\Phi,v} \leqslant c(b)\|P\|_{\tilde{\Omega}\to Y}\|f\|_{\Phi,v}.$$

This gives the second inequality in (140).

Remark 8 Theorem 5 discovers the main goal of the discretization procedure (98)–(105). In this theorem, we reduce the estimates for the restriction of monotone operator on the cone of nonnegative decreasing functions Ω to the estimates of this operator on some set of nonnegative step-functions. In many cases, such reduction admits to apply known results for step-functions or their pure discrete analogues for obtaining needed estimates on the cone Ω. Such approach we realize, for example, in Sect. 4 in the problem of description of associate norms.

3.2 The Case of Degenerate Weight

We use all notation and assumptions of Sect. 2.4, see (113)–(130). Introduce the cones

$$\Omega_0 = \left\{\alpha = \{\alpha_m\}_{m\in Z^-} : \ 0 \leqslant \alpha_m \downarrow\right\}; \tag{151}$$

$$\tilde{\Omega}_0 = \left\{ f = f_\alpha : \ f_\alpha(t) = \sum_{m\in Z^-} \alpha_m \chi_{\Delta_m}(t); \ \alpha \in \Omega_0 \right\}. \tag{152}$$

Define

$$\left\| P \right\|_{\tilde{\Omega}_0 \to Y} = \sup\left\{ \left\| Pf \right\|_Y : \ f \in \tilde{\Omega}_0, \left\| f \right\|_{\varphi,v} \leqslant 1 \right\}. \tag{153}$$

Lemma 11 *The following two-sided estimate holds in above notation and assumptions:*

$$\left\| P \right\|_{\tilde{\Omega}_0 \to Y} \leqslant \left\| P \right\|_{\Omega \to Y} \leqslant d\,(b) \left\| P \right\|_{\tilde{\Omega}_0 \to Y}. \tag{154}$$

Here, $d\,(b)$ is defined by (12) with $c = b > 1$.

Proof The left hand side inequality in (154) is evident because of inclusion $\tilde{\Omega}_0 \subset \Omega$. From the other side we have $f \leqslant f_1 \Rightarrow Pf \leqslant Pf_1$, for every function $f \in \Omega$, and $\left\| f_1 \right\|_{\Phi,v} \leqslant d\,(b) \left\| f \right\|_{\Phi,v}$. Now, let us take into account that

$$f \in \Omega \Rightarrow f_1(t) = \sum_{m\in Z^-} f(\mu_m)\chi_{\Delta_m}(t) \in \tilde{\Omega}_0.$$

Therefore,

$$\left\| Pf \right\|_Y \leqslant \left\| Pf_1 \right\|_Y \leqslant \left\| P \right\|_{\tilde{\Omega}_0 \to Y} \left\| f_1 \right\|_{\Phi,v} \leqslant d\,(b) \left\| P \right\|_{\tilde{\Omega}_0 \to Y} \left\| f \right\|_{\Phi,v}. \tag{155}$$

Consequently,

$$\left\| P \right\|_{\Omega \to Y} = \sup\left\{ \left\| Pf \right\|_Y : \ f \in \Omega, \left\| f \right\|_{\Phi,v} \leqslant 1 \right\} \leqslant d\,(b) \left\| P \right\|_{\tilde{\Omega}_0 \to Y}.$$

Now, we introduce the cone of nonnegative step-functions connected with the participation in Sect. 2.4:

$$\bar{S} = \left\{ f = f_\alpha : \ f_\alpha(t) = \sum_{m\in Z^-} \alpha_m \chi_{\Delta_m}(t); \ \alpha_m \geqslant 0, \ m \in Z^- \right\}, \tag{156}$$

and consider the related norm of the restriction

$$\left\| P \right\|_{\bar{S} \to Y} = \sup\left\{ \left\| Pf \right\|_Y : \ f \in \bar{S}, \ \left\| f \right\|_{\Phi,v} \leqslant 1 \right\}. \tag{157}$$

Lemma 12 *Define*

$$c\,(b) = d\,(c_0\,(b))\,; \quad c_0\,(b) = \left[b\,(b-1)^{-1}\right] > 1,$$

see (85). The following two-sided estimate holds in the notation and assumptions of this Subsection:

$$\left\| P \right\|_{\tilde{\Omega}_0 \to Y} \leqslant \left\| P \right\|_{\bar{S} \to Y} \leqslant c\,(b) \left\| P \right\|_{\tilde{\Omega}_0 \to Y}. \tag{158}$$

Proof The left hand side inequality in (158) is evident because of inclusion $\tilde{\Omega}_0 \subset \bar{S}$. Let us prove the right one. We introduce the maximal operator B by the formula $B\gamma = \alpha$, where $\alpha = \{\alpha_m\}_{m \in Z^-}$; $\gamma = \{\gamma_m\}_{m \in Z^-}$, and

$$\alpha_m = \max_{k \in Z^-, k \geqslant m} |\gamma_k|\,, \quad m \in Z^-. \tag{159}$$

Let us show the boundedness of operator $B : \bar{l}_{\Phi,\beta} \to \bar{l}_{\Phi,\beta}$. Let $\gamma \in \bar{l}_{\Phi,\beta}$. Then, if $\lambda \geqslant \left\|\gamma\right\|_{\bar{l}_{\Phi,\beta}}$, we have $\bar{j}_\lambda\,(\gamma) = \sum\limits_{k \in Z^-} \Phi\left(\lambda^{-1}|\gamma_k|\right)\beta_k \leqslant 1$ so that $\Phi\left(\lambda^{-1}|\gamma_k|\right) < \infty$, $k \in Z^-$. Moreover, recall that $\Phi \in \Theta$ is increasing, so that

$$\Phi\left(\lambda^{-1}\alpha_m\right) = \max_{k \in Z^-, k \geqslant m} \Phi\left(\lambda^{-1}|\gamma_k|\right) \leqslant \sum_{k \in Z^-, k \geqslant m} \Phi\left(\lambda^{-1}|\gamma_k|\right).$$

Then,

$$\bar{j}_\lambda\,(\alpha) = \sum_{m \in Z^-} \Phi\left(\lambda^{-1}\alpha_m\right)\beta_m \leqslant$$
$$\leqslant \sum_{m \in Z^-} \beta_m \sum_{k \in Z^-, k \geqslant m} \Phi\left(\lambda^{-1}|\gamma_k|\right) = \sum_{k \in Z^-} \Phi\left(\lambda^{-1}|\gamma_k|\right) \sum_{m \leqslant k} \beta_m.$$

We have according to (122), $\beta_m = b^{m+1} - b^m$, and

$$\sum_{m \leqslant k} \beta_m = b^{k+1} = \beta_k c_0\,(b)\,. \tag{160}$$

Consequently,

$$\bar{j}_\lambda\,(\alpha) \leqslant c_0\,(b) \sum_{k \in Z^-} \Phi\left(\lambda^{-1}|\gamma_k|\right)\beta_k = c_0\,(b)\,\bar{j}_\lambda\,(\gamma)\,. \tag{161}$$

This inequality gives

$$\left\|\{\alpha_m\}\right\|_{\bar{l}_{\Phi,\beta}} \leqslant d\,(c_0\,(b)) \left\|\{\gamma_m\}\right\|_{\bar{l}_{\Phi,\beta}}. \tag{162}$$

Now, we denote $\gamma_m = f(\mu_m) \geqslant 0, \quad m \in Z^-$, for function $f \in \bar{S}$, so that $f = f_\gamma$. Further, we introduce, according to (159), $\alpha_m = \max\limits_{k \in Z^-, k \geqslant m} |\gamma_k|$, $m \in Z^-$. Then, $\alpha = \{\alpha_m\} \in \Omega_0$, $f_\alpha \in \tilde{\Omega}_0$, and

$$f_\alpha \geqslant f_\gamma, \quad \|f_\alpha\|_{\Phi,v} = \|\alpha\|_{\bar{l}_{\Phi,\beta}} \leqslant c(b)\|\gamma\|_{\bar{l}_{\Phi,\beta}} = c(b)\|f_\gamma\|_{\Phi,v}. \tag{163}$$

From (163) it follows that for given $f = f_\gamma \in \bar{S}$ there exits $f_\alpha \in \tilde{\Omega}_0$ such that

$$Pf \leqslant Pf_\alpha; \quad \|f_\alpha\|_{\Phi,v} \leqslant c(b)\|f\|_{\Phi,v}.$$

Consequently, for every $f \in \bar{S}$,

$$\|Pf\|_Y \leqslant \|Pf_\alpha\|_Y \leqslant \|P\|_{\tilde{\Omega}_0 \to Y}\|f_\alpha\|_{\Phi,v} \leqslant c(b)\|P\|_{\tilde{\Omega}_0 \to Y}\|f\|_{\Phi,v}.$$

This inequality gives the second estimate in (158).

4 The Associate Norm for the Cone of Nonnegative Decreasing Functions In Weighted Orlicz Space

4.1 The Case of Nondegenerate Weight

We preserve all notations of Sects. 1–3, and apply the results of Sect. 3 in the important partial case when ideal space Y coincides with the weighted Lebesgue space $L_1(R_+; g)$, $g \in M^+$, and monotone operator P is the identical operator. In this case

$$\|P\|_{\Omega \to Y} = \sup\left\{\int_0^\infty fgdt : f \in \Omega; \|f\|_{\varphi,v} \leqslant 1\right\} =$$
$$= \sup\left\{\int_0^\infty fgdt : f \in \Omega; J_1(f) \leqslant 1\right\} = \|g\|' \tag{164}$$

(see (133); let us recall the equivalence $\|f\|_{\Phi,v} \leqslant 1 \Leftrightarrow J_1(f) \leqslant 1$, see (76)). It means that the norm $\|P\|_{\Omega \to Y}$ coincides in this case with the associate norm for the cone Ω (95), equipped with the functional

$$J_1(f) = \int_0^\infty \Phi(f)vdx.$$

We have according to the results of Sect. 3, Theorem 5,

$$\left\| P \right\|_{\Omega \to Y} \cong \left\| P \right\|_{S \to Y}, \tag{165}$$

where in our case

$$\left\| P \right\|_{S \to Y} = \sup \left\{ \sum_{m \in Z} \alpha_m g_m : \ \alpha_m \geqslant 0; \ \sum_{m \in Z} \Phi\left(\alpha_m\right) \beta_m \leqslant 1 \right\}, \tag{166}$$

and

$$g_m = \int_{\Delta_m} g dt \geqslant 0; \quad \beta_m = \int_{\Delta_m} v dt = b^m \left(b - 1\right), \quad m \in Z. \tag{167}$$

Let us note that the norm (166) coincides with the discrete variant of Orlicz norm, see [2]:

$$\left\| \{g_m\} \right\|_{l'_{\Phi,\beta}} = \sup \left\{ \sum_{m \in Z} \alpha_m \left| g_m \right| : \ \alpha_m \geqslant 0; \ \sum_{m \in Z} \Phi\left(\alpha_m\right) \beta_m \leqslant 1 \right\}, \tag{168}$$

Our nearest aim is to describe explicitly the norm (168) in terms of complementary function Ψ. We restrict ourselves with the case of Young function. Thus, let as in Example 6, $\Phi : [0, \infty) \to [0, \infty]$ be Young function that is,

$$\Phi\left(t\right) = \int_0^t \varphi\left(\tau\right) d\tau, \tag{169}$$

where $\varphi : [0, \infty) \to [0, \infty]$ is the decreasing and left-continuous function, and $\varphi\left(0\right) = 0$, φ is neither identically zero, nor identically infinity on $(0, \infty)$. Let Ψ be the complementary Young function for Φ, that is

$$\begin{gathered} \Psi\left(t\right) = \int_0^t \psi\left(\tau\right) d\tau, \quad t \in [0, \infty]; \\ \psi\left(\tau\right) = \inf\left\{\sigma : \ \varphi\left(\sigma\right) \geqslant \tau\right\}, \quad \tau \in [0, \infty]. \end{gathered} \tag{170}$$

Function ψ is left inverse for the left-continuous increasing function φ. It has the same general properties as φ, so that Ψ is Young function too. Moreover, $\varphi\left(\sigma\right) = \inf\left\{\tau : \ \psi\left(\tau\right) \geqslant \sigma\right\}$, and Φ in its turn is the complementary Young function for Ψ (see [11, p. 271]). It is well-known that

$$\Psi(t) = \sup_{s \geqslant 0} [st - \Phi(s)];$$

$$st \leqslant \Phi(s) + \Psi(t), \quad s, t \in [0, \infty), \tag{171}$$

and the equality takes place in (171) if and only if $\varphi(s) = t$ or $\psi(t) = s$ (see [11, pp. 271–273]).

The next result is well-known in the theory of discrete weighted Orlicz spaces. It is valid for any positive weight sequence, and plays the crucial role for equivalent description of the Orlicz norm (168).

Theorem 6 *Let Φ, and Ψ be the complementary Young functions, let $\beta = \{\beta_m\}$; $\beta_m \in R_+$, $m \in Z$. Then, Orlicz norm (168) is equivalent to the norm*

$$\left\| \left\{ \beta_m^{-1} g_m \right\} \right\|_{l_{\Psi,\beta}}. \tag{172}$$

Namely,

$$\left\| \left\{ \beta_m^{-1} g_m \right\} \right\|_{l_{\Psi,\beta}} \leqslant \left\| \{g_m\} \right\|_{l'_{\Phi,\beta}} \leqslant 2 \left\| \left\{ \beta_m^{-1} g_m \right\} \right\|_{l_{\Psi,\beta}}. \tag{173}$$

Corresponding notations of the discrete norms we introduced in (84), (85).

Conclusion. Let us formulate some results of our considerations.

Let Φ, and Ψ be the complementary Young functions, let the conditions (96), *and* (97) *be fulfilled, and the discretization procedure* (98)–(105) *be realized. Then, the following equivalence takes place for the norm* (164)

$$\|g\|' \cong \left\| \{\rho_m\} \right\|_{l_{\Psi,\beta}}, \quad \beta = \{\beta_m\}, \quad \rho_m = \beta_m^{-1} \int_{\Delta_m} |g| \, dt. \tag{174}$$

Now, our aim is to present this answer in the integral form.

Theorem 7 *Let Φ, and Ψ be the complementary Young functions, let the conditions (96), and (97) be fulfilled. The following two-sided estimate holds for the associate norm (164) with fixed* $0 < a < 1$:

$$\|g\|' \cong \|\rho_a(g)\|_{\Psi,v} = \inf \left\{ \lambda > 0 : \int_0^\infty \Psi\left(\lambda^{-1} \rho_a(g; t)\right) v(t) \, dt \leqslant 1 \right\}, \tag{175}$$

$$\rho_a(g; t) := V(t)^{-1} \int_{\delta_a(t)}^{t} |g(\tau)| d\tau, \quad \delta_a(t) := V^{-1}(aV(t)), \quad t \in R_+. \tag{176}$$

The norms (175) are equivalent for different values $a \in (0, 1)$.

Here and below, we use the notation

$$A \cong B \Leftrightarrow \exists c = c\,(a) \in [1, \infty) : \ c^{-1} \leqslant A/B \leqslant c. \tag{177}$$

Remark 9 Let us assume additionally that function Φ in Theorem 7 satisfies Δ_2-condition, that is

$$\exists C \in (1, \infty) : \ \Phi\,(2t) \leqslant C\Phi\,(t)\,, \quad \forall t \in R_+. \tag{178}$$

Then,

$$\left\|g\right\|' \cong \left\|V\,(t)^{-1}\int\limits_0^t |g\,(\tau)|d\tau\right\|_{\Psi,v}. \tag{179}$$

Proof (*of Theorem* 7) We use the description (174) with $b = a^{-1/2} > 1$. Then, $a = b^{-2}$, and

$$\rho'_m \leqslant \rho_a\,(g;t) = V\,(t)^{-1}\int\limits_{V^{-1}(aV(t))}^{t} |g|\,d\tau \leqslant \rho''_m, \quad t \in \Delta_m, \tag{180}$$

where

$$\rho'_m = b^{-(m+1)}\int\limits_{\mu_{m-1}}^{\mu_m} |g|\,d\tau; \quad \rho''_m = b^{-m}\int\limits_{\mu_{m-2}}^{\mu_{m+1}} |g|\,d\tau. \tag{181}$$

Therefore,

$$F_0\,(t) \leqslant \rho_a\,(g;t) \leqslant F_1\,(t)\,, \quad t \in R_+, \tag{182}$$

where F_0, F_1 are step-functions

$$F_0\,(t) = \sum_m \rho'_m \chi_{\Delta_m}\,(t), \quad F_1\,(t) = \sum_m \rho''_m \chi_{\Delta_m}\,(t),$$

and

$$\left\|F_0\right\|_{\Psi,v} = \left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}}, \quad \left\|F_1\right\|_{\Psi,v} = \left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}},$$

so that

$$\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}} \leqslant \left\|\rho_a\,(g)\right\|_{\Psi,v} \leqslant \left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}}. \tag{183}$$

Thus, needed result (175) follows from the equivalence

$$\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}} \cong \left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}} \cong \left\|\left\{\rho_m\right\}\right\|_{l_{\Psi,\beta}}. \tag{184}$$

It remains to prove (184). The equalities (174) and (181) show that

$$\rho'_m = b^{-2}(b-1)\,\rho_{m-1}; \tag{185}$$

$$\rho''_m = \rho'_{m-1} + b\rho'_m + (b-1)\,\rho_m. \tag{186}$$

Consequently,

$$\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}} = b^{-2}(b-1)\left\|\{\rho_{m-1}\}\right\|_{l_{\Psi,\beta}} \leqslant b^{-1}(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}. \tag{187}$$

$$\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}} = b^2(b-1)^{-1}\left\|\left\{\rho'_{m+1}\right\}\right\|_{l_{\Psi,\beta}} \leqslant b^2(b-1)^{-1}\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}}. \tag{188}$$

In the last inequality, we take into account the boundedness of shift-operators in $l_{\Psi,\beta}$ with Young function Ψ, and $\beta = \{\beta_m\}$ in (105), see Remark 5 and Lemma 8. Thus,

$$\left\|\{\rho_{m-1}\}\right\|_{l_{\Psi,\beta}} \leqslant b\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}, \quad \left\|\left\{\rho'_{m+1}\right\}\right\|_{l_{\Psi,\beta}} \leqslant \left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}}.$$

We have by (186),

$$(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}} \leqslant \left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}}; \tag{189}$$

$$\left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}} \leqslant \left\|\left\{\rho'_{m-1}\right\}\right\|_{l_{\Psi,\beta}} + b\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}} + (b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}. \tag{190}$$

Like (187), the estimate is valid

$$\left\|\left\{\rho'_{m-1}\right\}\right\|_{l_{\Psi,\beta}} \leqslant b\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}}.$$

We substitute this estimate into (190), take into account the inequality (187) and obtain

$$\left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}} \leqslant 3(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}.$$

Consequently,

$$(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}} \leqslant \left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}} \leqslant 3(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}. \tag{191}$$

The estimates (187), (188), and (191) give the needed equivalence (184).

5 The Case of Degenerated Weight Function

We use the results of Sect. 3.2 to estimate the norm of restriction of monotone operator on the cone Ω in the case of degenerated weight. According to Lemmas 11, and 12, the following two-sided estimate holds

$$\|P\|_{\Omega\to Y} \cong \|P\|_{\bar{S}\to Y}. \tag{192}$$

We apply these results in the special case, when the ideal space Y coincides with the weighted Lebesgue space $L_1(R_+; g)$, $g \in M^+$, and the monotone operator P is identical operator. Recall that in this case $\|P\|_{\Omega\to Y}$ coincides with the associate norm to the cone Ω, equipped with the functional

$$J_1(f) = \int_0^\infty \Phi(f) v\,dt,$$

and the following equality holds for $\|P\|_{\bar{S}\to Y}$:

$$\|P\|_{\bar{S}\to Y} = \sup\left\{\sum_{m\in Z^-} \alpha_m g_m : \alpha_m \geqslant 0; \sum_{m\in Z^-} \Phi(\alpha_m)\beta_m \leqslant 1\right\}. \tag{193}$$

Here,

$$g_m = \int_{\Delta_m} g\,dt \geqslant 0; \quad \beta_m = \int_{\Delta_m} v\,dt = b^{-m}(b-1), \quad m \in Z^-. \tag{194}$$

Note that the norm (193) coincides with the discrete variant of Orlicz norm; see [2]:

$$\|\{g_m\}\|_{\bar{l}'_{\Phi,\beta}} = \sup\left\{\sum_{m\in Z^-} \alpha_m |g_m| : \quad \alpha_m \geqslant 0; \sum_{m\in Z^-} \Phi(\alpha_m)\beta_m \leqslant 1\right\}, \tag{195}$$

Our nearest aim is to give the explicit description of the norm (195) in terms of complementary Young function. Thus, let Φ be Young function, and Ψ be its complementary Young function.

We apply corresponding variant of Theorem 6, and obtain the equivalence of Orlicz norm (195) to the norm

$$\|\{\rho_m\}\|_{\bar{l}_{\Psi,\beta}}; \quad \rho_m = \beta_m^{-1} g_m. \tag{196}$$

Namely,

$$\|\{\rho_m\}\|_{\bar{l}_{\Psi,\beta}} \leqslant \|\{g_m\}\|_{\bar{l}'_{\Phi,\beta}} \leqslant 2\|\{\rho_m\}\|_{\bar{l}_{\Psi,\beta}}. \tag{197}$$

Here,

$$\left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}} = \inf\left\{\lambda > 0 : \ \bar{j}_\lambda\left(\{\rho_m\}\right) \leqslant 1\right\}; \tag{198}$$

$$\bar{j}_\lambda\left(\{\rho_m\}\right) = \sum_{m\in Z^-} \Psi\left(\lambda^{-1}\left|\rho_m\right|\right)\beta_m; \tag{199}$$

See the relating notations in (121)–(124).

Conclusions. Let us formulate some results of our considerations.

We introduce the discretizing sequence $\{\mu_m\}_{m\in Z^-}$ by formulas

$$V\left(\mu_m\right) = b^m, \quad m \in Z^- = \{0, -1, -2, \ldots\} \tag{200}$$

for fixed $b > 1$, and function V with the properties described in Sect. 2.4.

We set $\mu_1 = \infty$, and determine

$$\Delta_m = \left[\mu_m, \mu_{m+1}\right), \quad m \in Z^-; \tag{201}$$

$$\beta_m = \int_{\Delta_m} v dt = b^m\left(b-1\right); \quad \rho_m = \beta_m^{-1}\int_{\Delta_m} |g|\, dt. \tag{202}$$

Further, we have the equivalence for the associate norm $\left\|g\right\|'$ (164)

$$\left\|g\right\|' \cong \left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}}, \quad \beta = \{\beta_m\}, \tag{203}$$

where Ψ is the complementary function for Young function Φ.

Now, our aim is to present this description in integral form.

Theorem 8 *Let Ψ be the complementary function for Young function Φ, and weight satisfies the conditions of Sect. 2.4, in particular,*

$$V\left(+\infty\right) < \infty. \tag{204}$$

Denote

$$b = V\left(+\infty\right)/V\left(1\right) > 1, a = b^{-2}. \tag{205}$$

Then, in the notation (176),

$$\left\|g\right\|' \cong \left\|\rho_a\left(g\right)\chi_{(0,1)}\right\|_{\Psi,v} + \int_{V^{-1}(aV(+\infty))}^{\infty} |g| dt. \tag{206}$$

Proof Let us note that

$$\rho_m' \leqslant \rho_a\left(g;t\right)\chi_{(0,1)}\left(t\right) \leqslant \rho_m'', \quad t \in \Delta_m, \quad m \in Z^-. \tag{207}$$

Here, $\rho_0' = \rho_0'' = 0$, and for $m \leqslant -1$

$$\rho_m' = b^{-(m+1)} \int\limits_{\mu_{m-1}}^{\mu_m} |g|\, d\tau; \quad \rho_m'' = b^{-m} \int\limits_{\mu_{m-2}}^{\mu_{m+1}} |g|\, d\tau. \tag{208}$$

Then,

$$F_0(t) \leqslant \rho_a(g;t)\, \chi_{(0,1)}(t) \leqslant F_1(t), \quad t \in R_+, \tag{209}$$

where F_0, F_1 are step-functions

$$F_0(t) = \sum_{m \in Z^-} \rho_m' \chi_{\Delta_m}(t), \quad F_1(t) = \sum_{m \in Z^-} \rho_m'' \chi_{\Delta_m}(t),$$

and

$$\|F_0\|_{\Psi,v} = \|\{\rho_m'\}\|_{\bar{l}_{\Psi,\beta}}, \quad \|F_1\|_{\Psi,v} = \|\{\rho_m''\}\|_{\bar{l}_{\Psi,\beta}},$$

so that

$$\left\|\{\rho_m'\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant \left\|\rho_a(g)\,\chi_{(0,1)}\right\|_{\Psi,v} \leqslant \left\|\{\rho_m''\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{210}$$

Moreover,

$$\{\rho_m\}_{m \in Z^-} = \{\bar{\rho}_m\}_{m \in Z^-} + \{\hat{\rho}_m\}_{m \in Z^-},$$

where

$$\bar{\rho}_m = \rho_m, m \leqslant -1, \bar{\rho}_0 = 0; \quad \hat{\rho}_m = 0, m \leqslant -1, \hat{\rho}_0 = \rho_0. \tag{211}$$

Consequently,

$$\left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}} \cong \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} + \left\|\{\hat{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{212}$$

Introduce

$$A_m(g) = \frac{\rho_m}{\Psi^{-1}(1/\beta_m)} = \frac{1}{\beta_m \Psi^{-1}(1/\beta_m)} \int\limits_{\Delta_m} |g| dt, \quad m \in Z^-. \tag{213}$$

Note that,

$$\left\|\{\hat{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} = \inf\left\{\lambda > 0 : \ \Psi\left(\lambda^{-1}\rho_0\right)\beta_0 \leqslant 1\right\} = A_0(g) =$$
$$= \frac{1}{(b-1)\,\Psi^{-1}\left((b-1)^{-1}\right)} \int\limits_1^\infty |g| dt.$$

According to (210),

$$\left\|\left\{\rho_m'\right\}\right\|_{\bar{l}_{\Psi,\beta}} + A_{-1}(g) \leqslant \left\|\rho_a(g)\,\chi_{(0,1)}\right\|_{\Psi,v} + A_{-1}(g) \leqslant \\ \leqslant \left\|\left\{\rho_m''\right\}\right\|_{\bar{l}_{\Psi,\beta}} + A_{-1}(g). \tag{214}$$

Further, we will prove the equivalence

$$\left\|\left\{\rho_m'\right\}\right\|_{\bar{l}_{\Psi,\beta}} + A_{-1}(g) \cong \left\|\left\{\rho_m''\right\}\right\|_{\bar{l}_{\Psi,\beta}} \cong \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{215}$$

Then, both parts of (214) will be equivalent to $\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}$ (the second term in the right hand side of (214) is subordinate to the first one). Consequently, we obtain

$$\left\|\rho_a(g)\,\chi_{(0,1)}\right\|_{\Psi,v} + A_{-1}(g) \cong \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}.$$

Now, we take into account the estimate (212), and obtain the equivalence

$$\left\|\rho_a(g)\,\chi_{(0,1)}\right\|_{\Psi,v} + A_{-1}(g) + A_0(g) \cong \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} + A_0(g) \cong \left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}}.$$

According to (203), this is the needed estimate (206).

Thus, it remains to prove (215). We recall that $\rho_0' = \rho_0'' = 0$. For $m \leqslant -1$ the equalities (202), and (208) show that

$$\rho_m' = b^{-2}(b-1)\,\bar{\rho}_{m-1}; \tag{216}$$

$$\rho_m'' = \rho_{m-1}' + b\rho_m' + (b-1)\,\bar{\rho}_m. \tag{217}$$

From (216) it follows,

$$\left\|\left\{\rho_m'\right\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant b^{-2}(b-1)\left\|\{\bar{\rho}_{m-1}\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant b^{-1}(b-1)\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{218}$$

$$\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} \cong A_{-1}(g) + \left\|\left\{\rho_m'\right\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{219}$$

In (218) we take into account the boundedness of shift operator in the space $\bar{l}_{\Psi,\beta}$ with Young function Ψ, and $\beta = \{\beta_m\}$ from (202); see Lemma 9. Therefore,

$$\left\|\{\rho_{m-1}\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant b\left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}}.$$

To prove (219) we use the following chain of equalities (recall that $\bar{\rho}_0 = \rho_0' = 0$)

$$\bar{j}_\lambda(\{\bar{\rho}_m\}) = \sum_{m\in Z^-} \Psi\left(\lambda^{-1}\bar{\rho}_m\right)\beta_m = \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} + \sum_{m\leqslant -2} \Psi\left(\lambda^{-1}\bar{\rho}_m\right)\beta_m.$$

In the second term we use the equality $\bar{\rho}_m = b^2 (b-1)^{-1} \rho'_{m+1}$, $m \leqslant -2$ (see (216)), so that

$$\begin{aligned}\sum_{m\leqslant -2} \Psi\left(\lambda^{-1}\bar{\rho}_m\right)\beta_m &= \sum_{m\leqslant -2} \Psi\left(\lambda^{-1}b^2 (b-1)^{-1} \rho'_{m+1}\right)\beta_m = \\ &= \sum_{m\leqslant -1} \Psi\left(\lambda^{-1}b^2 (b-1)^{-1} \rho'_{m}\right)\beta_{m-1} = \\ &= b^{-1}\sum_{m\leqslant -1} \Psi\left(\lambda^{-1}b^2 (b-1)^{-1} \rho'_{m}\right)\beta_{m} = \\ &= b^{-1}\sum_{m\in Z^-} \Psi\left(\lambda^{-1}b^2 (b-1)^{-1} \rho'_{m}\right)\beta_{m}.\end{aligned}$$

As the result we obtain,

$$\bar{j}_\lambda(\{\bar{\rho}_m\}) = \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} + b^{-1}\bar{j}_{(b-1)b^{-2}\lambda}\left(\left\{\rho'_m\right\}\right). \tag{220}$$

Let $\lambda = \max\{\lambda_1, \lambda_2\}$, where

$$\lambda_1 = \inf\left\{\lambda > 0 : \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} \leqslant 1 - b^{-1}\right\} = \bar{\rho}_{-1}/\Psi^{-1}(1),$$

$$\lambda_2 = \inf\left\{\lambda > 0 : \bar{j}_{(b-1)b^{-2}\lambda}\left(\left\{\rho'_m\right\}\right) \leqslant 1\right\} = b^2 (b-1)^{-1} \left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}}.$$

Then, $\bar{j}_\lambda(\{\bar{\rho}_m\}) \leqslant 1$, and (220) implies

$$\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant \lambda = \max\left\{\bar{\rho}_{-1}/\Psi^{-1}(1), \quad b^2 (b-1)^{-1}\left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}}\right\}. \tag{221}$$

From the other side, we see by (220), that

$$\begin{aligned}&\bar{j}_\lambda(\{\bar{\rho}_m\}) \geqslant \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} \Rightarrow \\ &\qquad \Rightarrow \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} \geqslant \inf\left\{\lambda > 0 : \ \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} \leqslant 1\right\} = A_{-1}(g).\end{aligned}$$

Together with (218), it gives inequality

$$\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} \geqslant \max\left\{A_{-1}(g), b(b-1)^{-1}\left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}}\right\}. \tag{222}$$

Inequalities (221) and (222) imply the two-sided estimate (219) with constants depending on b, because $\bar{\rho}_{-1}/\Psi^{-1}(1) \cong A_{-1}(g)$.

Now, we will obtain the estimate (215). The equality (217) shows that

$$\left\|\left\{\rho''_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} \geqslant (b-1)\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}; \tag{223}$$

$$\left\|\left\{\rho''_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant \left\|\left\{\rho'_{m-1}\right\}\right\|_{\bar{l}_{\Psi,\beta}} + b\left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} + (b-1)\left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{224}$$

The first term in (224) is not bigger than the second one because of the estimate for the norm of shift operator. In its turn, the second term is not bigger than the third one in view of the estimate (218). As the result we obtain,

$$\left\|\left\{\rho''_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant 3(b-1)\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{225}$$

Estimates (223) and (225) imply the equivalence

$$\left\|\left\{\rho''_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} \cong \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}.$$

Together with (219) it gives (215), thus completing the proof of Theorem.

6 Applications to Weighted Orlicz-Lorentz Classes

Recall the notion of decreasing rearrangement for measurable function. Let $M_0 = M_0(R_+)$ be the subspace of functions $f : R_+ \to R$, measurable with respect to Lebesgue measure μ, finite almost everywhere, and such that distribution function λ_f is not identically infinity for $f \in M_0$. Here,

$$\lambda_f(y) = \mu\{x \in R_+ : |f(x)| > y\},\ y \in R_+. \tag{226}$$

Then, $0 \leqslant \lambda_f \downarrow$, $\lambda_f(y) \to 0\,(y \to +\infty)$. Consider the decreasing rearrangement f^* of function f,

$$f^*(t) = \inf\left\{y \in R_+ : \lambda_f(y) \leqslant t\right\},\ t \in R_+. \tag{227}$$

We deal with Orlicz-Lorentz class $\Lambda_{\Phi,v}$ related to Orlicz space $L_{\Phi,v}$. For $f \in M_0$ we define

$$J_\lambda\left(f^*\right) = \int_0^\infty \Phi\left(\lambda^{-1} f^*(t)\right) v(t)\,dt, \quad \lambda > 0. \tag{228}$$

Here $v \in M^+$, integration by Lebesgue measure and weight satisfies the condition (8). Weighted Orlicz-Lorentz class $\Lambda_{\Phi,v}$ consists of functions $f \in M_0(R_+)$ such that $f^* \in L_{\varphi,v}$. We equip it by the functional

$$\|f^*\|_{\Phi,v} = \inf\left\{\lambda > 0 : \quad J_\lambda\left(f^*\right) \leqslant 1\right\}. \tag{229}$$

To deal with linear space $\Lambda_{\Phi,v}$, it would be assumed additionally that weight function V (8) satisfies Δ_2-condition, that is

$$\exists C \in R_+ : \ V(2t) \leqslant CV(t), \quad \forall t \in R_+. \tag{230}$$

It is known that such assumption is necessary for the validity of triangle inequality in Lorentz space; see for example [14]. Nevertheless, *we need not estimate (230) in our considerations.* Anyway, we can consider class $\Lambda_{\Phi,v}$ as the cone in M_0, that consists of functions having finite values of functional (229). Here, we present the analogous for the results of Sect. 3 concerning estimates of the norms of monotone operators over Orlicz-Lorentz classes. We recall some descriptions. Let $(\mathrm{N}, \Re, \eta)$ be the measure space with nonnegative σ-finite measure η; as $L = L(\mathrm{N}, \Re, \eta)$ we denote space of all η-measurable functions $u : \mathrm{N} \to R$; $L^+ = \{u \in L : u \geqslant 0\}$. Let $Y_i = Y_i(\mathrm{N}, \Re, \eta) \subset L, i = 1, 2$ be ideal spaces; $P : M_0^+(R_+) \to L^+$ be a monotone operator related to these spaces by the following condition: for $h \in \Omega$

$$\|Ph\|_{Y_2} = \sup\left\{\|Pf\|_{Y_1} : \ f \in M_0^+(R_+), f^* = h\right\}. \tag{231}$$

We illustrate these conditions by two examples.

Example 7 Let P be identical operator on $M_0^+(R_+)$,

$$Y_1 = L_1(R_+; g), \quad g \in M_0^+(R_+); \quad Y_2 = L_1\left(R_+; g^*\right).$$

Then, the equality (231) reflects the well-known extremal property of decreasing rearrangements; see [11, Sects. 2.3–2.8]):

$$\sup\left\{\int_0^\infty fg\,dt : \ f \in M_0^+, \ f^* = h\right\} = \int_0^\infty hg^*dt.$$

Example 8 Let Y be an ideal space, and monotone operator $P : M_0^+(R_+) \to L^+$ satisfies the condition

$$\|Pf\|_Y \leqslant \|Pf^*\|_Y, \quad f \in M_0^+(R_+). \tag{232}$$

Then, the equality (231) holds with $Y_1 = Y_2 = Y$.

Indeed, $f \in M_0^+(R_+) \Rightarrow h := f^* \in M_0^+(R_+)$, $h^* = h$, and

$$\|Ph\|_Y \leqslant \sup\left\{\|Pf\|_Y : f \in M_0^+(R_+),\ f^* = h\right\}.$$

From the other side, for every function $f \in M_0^+(R_+) : f^* = h$, we have according to (232),

$$\|Pf\|_Y \leqslant \|Pf^*\|_Y = \|Ph\|_Y \Rightarrow \sup\left\{\|Pf\|_Y : f \in M_0^+(R_+),\ f^* = h\right\} \leqslant \|Ph\|_Y.$$

Remark 10 Example 8 covers, in particular, such operator as

$$(Pf)(x) = \int_0^\infty k(x,\tau) f(\tau)\, d\tau, \quad x \in \mathrm{N}, \tag{233}$$

where k is nonnegative measurable function on $\mathrm{N} \times R_+$, and $k(x,\tau)$ is decreasing and right continuous as function of $\tau \in R_+$. Then, for $f \in M_0^+(R_+)$, and almost all $x \in \mathrm{N}$, we obtain by the well-known Hardy's lemma

$$(Pf)(x) = \int_0^\infty k(x,\tau) f(\tau)\, d\tau \leqslant \int_0^\infty k(x,\tau) f^*(\tau)\, d\tau = \left(Pf^*\right)(x).$$

Consequently, inequality (232) holds for every ideal space Y.

Proposition 2 *Let $P : M_0^+(R_+) \to L^+$ be monotone operator and equality (231) be true. We define $\Lambda_{\Phi,v}^+ = \Lambda_{\Phi,v} \cap M_0^+$ and introduce the norms*

$$\|P\|_{\Lambda_{\Phi,v}^+ \to Y_1} = \sup\left\{\|Pf\|_{Y_1} : f \in M_0^+(R_+),\ \|f^*\|_{\Phi,v} \leqslant 1\right\}; \tag{234}$$

$$\|P\|_{\Omega \to Y_2} = \sup\left\{\|Ph\|_{Y_2} : h \in \Omega,\ \|h\|_{\Phi,v} \leqslant 1\right\}. \tag{235}$$

Then, these norms coincide to each other:

$$\|P\|_{\Lambda_{\Phi,v}^+ \to Y_1} = \|P\|_{\Omega \to Y_2}. \tag{236}$$

Proof We use the equivalence

$$f \in M_0^+;\ \|f^*\|_{\Phi,v} \leqslant 1 \Leftrightarrow h = f^* \in \Omega :\ \|h\|_{\Phi,v} \leqslant 1,$$

and obtain

$$\|P\|_{\Lambda_{\Phi,v}^+ \to Y_1} = \sup\left[\sup\left\{\|Pf\|_{Y_1} : f \in M_0^+(R_+),\ f^* = h\right\} : h \in \Omega,\ \|h\|_{\Phi,v} \leqslant 1\right].$$

According to (231), the right hand side here coincides with $\|P\|_{\Omega \to Y_2}$.

Remark 11 This Proposition admits us to reduce estimates of the norm $\|P\|_{\Lambda^+_{\Phi,v}\to Y_1}$ (234) to the estimates presented in Sects. 3 and 4. In particular, by the help of Example 7, we reduce the associate norm for function $g \in M$ on Orlicz–Lorentz class to the associate norm for its decreasing rearrangement g^* on the cone Ω:

$$\|g\|'_* := \sup\left\{\int_0^\infty f\,|g|\,dt :\ f \in M_0^+;\ \|f^*\|_{\Phi,v} \leqslant 1\right\} = \|g^*\|'.$$

Then, Theorem 7 and Remark 9 lead to the following result.

Theorem 9 *Let the assumptions of Theorem 7 be fulfilled. Then,*

$$\|g\|'_* \cong \|\rho_a(g^*)\|_{\Psi,v} = \inf\left\{\lambda > 0 : \int_0^\infty \Psi\left(\lambda^{-1}\rho_a\left(g^*;t\right)\right)v(t)\,dt \leqslant 1\right\}, \quad (237)$$

where ρ_a was determined in (176). Norms (237) are equivalent for different values $a \in (0,1)$.

Remark 12 Assume additionally that function Φ satisfies Δ_2-condition in Theorem 9. Then,

$$\|g\|'_* \cong \left\|V(t)^{-1}\int_0^t g^*(\tau)d\tau\right\|_{\Psi,v}. \quad (238)$$

Remark 13 In (237) and (238), we present some modifications of the result in [18] that develop preceding results of paper [13]. Note that, in [13] formula (238) was established under restriction that both functions Φ, and Ψ satisfy Δ_2-condition. Concerning duality problems for Orlicz, Lorentz, and Orlicz-Lorentz spaces see also [2, 4, 15, 16].

Now, let us describe the modification of the above presented results.

Theorem 10 *Let $Y \subset L$ be some ideal space with quasi-norm $\|\cdot\|_Y$, let $P : M^+ \to L^+$ be a monotone operator satisfying the condition: there exists constant $C \in R_+$ such that*

$$\|Pf\|_Y \leqslant C\|Pf^*\|_Y, \quad f \in M^+(R_+). \quad (239)$$

Then,

$$\|P\|_{\Omega\to Y} \leqslant \|P\|_{\Lambda^+_{\Phi,v}\to Y} \leqslant C\|P\|_{\Omega\to Y}. \quad (240)$$

If $C = 1$ in (239), then we have equality of the norms in (240).

Corollary 5 *In the conditions of Theorem 10 we have*

$$\|P\|_{\Lambda^+_{\Phi,v}\to Y} \cong \|P\|_{S\to Y}.$$

For the proof of Theorem 10, let us note that (239) implies

$$\left\| Ph \right\|_Y \leqslant \sup\left\{ \left\| Pf \right\|_Y :\ f \in M_0^+ (R_+),\, f^* = h \right\} \leqslant C \left\| Ph \right\|_Y. \tag{241}$$

Indeed, $f \in M_0^+ (R_+) \quad \Rightarrow \quad h := f^* \in M_0^+ (R_+),\, h^* = h$, *and*

$$\left\| Ph \right\|_Y \leqslant \sup\left\{ \left\| Pf \right\|_Y :\ f \in M_0^+ (R_+),\ f^* = h \right\}.$$

From the other side, for any function $f \in M_0^+ (R_+) :\ f^* = h$, *we have by (239),*

$$\left\| Pf \right\|_Y \leqslant C \left\| Pf^* \right\|_Y = C \left\| Ph \right\|_Y \Rightarrow$$
$$\Rightarrow \sup\left\{ \left\| Pf \right\|_Y :\ f \in M_0^+ (R_+),\ f^* = h \right\} \leqslant C \left\| Ph \right\|_Y.$$

Moreover, (241) implies (240). Indeed, we use equivalence

$$f \in M_0; \quad \left\| f^* \right\|_{\Phi,v} \leqslant 1 \Leftrightarrow h = f^* \in \Omega :\ \left\| h \right\|_{\Phi,v} \leqslant 1,$$

and obtain

$$\left\| P \right\|_{\Lambda^+_{\Phi,v} \to Y} = \sup\left[\sup\left\{ \left\| Pf \right\|_Y :\ f \in M_0^+ (R_+),\ f^* = h \right\} :\ h \in \Omega,\, \left\| h \right\|_{\Phi,v} \leqslant 1 \right].$$

Here, according to (241), the right hand side is estimated from below by

$$\sup\left[\left\| Ph \right\|_Y :\ h \in \Omega,\, \left\| h \right\|_{\Phi,v} \leqslant 1 \right] = \left\| P \right\|_{\Omega \to Y},$$

and, in addition, from above by the same value multiplied by C.

Example 9 Theorem 10 covers the case of Hardy–Littlewood maximal operator $\mathrm{M} : M_+ (R_+) \to M_+ (R_+)$, where

$$(\mathrm{M}f)(x) = \sup\left\{ |\Delta|^{-1} \int_\Delta f(\tau)\, d\tau :\ \Delta \subset R_+;\, x \in \Delta \right\},$$

and $Y = Y(R_+)$ is rearrangement invariant space (shortly: RIS). Indeed, by Luxemburg representation theorem (see [11, Chap. 2, Theorem 4.10]), for every RIS Y there exists unique RIS $\tilde{Y} = \tilde{Y}(R_+)$:

$$\left\| g \right\|_Y = \left\| g^* \right\|_{\tilde{Y}}, \quad g \in M(R_+).$$

Note that,

$$\left(\mathrm{M}f^{*}\right)^{*}(t) = \mathrm{M}f^{*}(t) = t^{-1}\int_{0}^{t} f^{*}(\tau)d\tau, \quad t \in R_{+}.$$

Then, $\left\|\mathrm{M}f\right\|_{Y} = \left\|(\mathrm{M}f)^{*}\right\|_{\tilde{Y}}$, $\left\|\mathrm{M}f^{*}\right\|_{Y} = \left\|\mathrm{M}f^{*}\right\|_{\tilde{Y}}$.

It is known that $\exists C \in R_{+}$: $(\mathrm{M}f)^{*}(x) \leqslant C\left(\mathrm{M}f^{*}\right)(x)$; see [11, Chap. 2]. Consequently,

$$\left\|\mathrm{M}f\right\|_{Y} = \left\|(\mathrm{M}f)^{*}\right\|_{\tilde{Y}} \leqslant C\left\|\mathrm{M}f^{*}\right\|_{\tilde{Y}} = C\left\|\mathrm{M}f^{*}\right\|_{Y}.$$

This inequality coincides with the estimate (239) for operator $P = \mathrm{M}$. Therefore, Theorem 10 is applicable to this operator, and we come to equivalences

$$\left\|\mathrm{M}\right\|_{\Lambda^{+}_{\Phi,v}\to Y} \cong \left\|\mathrm{M}\right\|_{\Omega\to Y} \cong \left\|\mathrm{M}\right\|_{S\to Y}.$$

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