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Hans-Jürgen Schmeisser *Editors*

Function Spaces and Inequalities

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Editors

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Preface

With the development in the last few decades, the theory of Function Spaces has become a powerful tool in several areas of mathematical and physical sciences, engineering etc. In particular, the concept of generalized functions (distributions) enables to study Partial Differential Equations and Boundary Value Problems in a much wider perspective. In order to deal with such problems, quite often, function spaces and mapping properties of operators connected with corresponding norm inequalities come into picture. With this in mind, an International Conference on "Function Spaces and Inequalities" was organized under the leadership of Pankaj Jain at South Asian University, New Delhi during December 11–15, 2015. The aim of the conference was to bring together experts and young researchers working in the field of Function Spaces and Inequalities to share their latest interests/investigations. The topics covered in the conference include (Variable/Grand/Small) Lebesgue Spaces, Orlicz Spaces, Lorentz Spaces, Sobolev Spaces, Morrey Spaces, Sequence spaces, Weight Theory, Integral Operators of Hardy Type, Sobolev Type Imbeddings, Function Algebras, Banach Algebras, Spaces & Algebras of, Analytic Functions, Geometry of Banach Spaces, Isometries of Function Spaces, (Weighted) Integral and Discrete Inequalities, Convexity Theory, Harmonic Analysis.

Simultaneously, it was proposed to bring out an edited volume based on the theme of the conference. This volume consists of original work as well as survey articles on topics of Function Spaces and Inequalities by distinguished mathematicians worldwide. The survey articles are self-contained and provide up-to-date knowledge in the respective area. Contributions are also from those who could not or did not attend the conference. All the articles are thoroughly refereed.

We express our deep gratitude to all the authors for their valuable contribution. Also, we are thankful to the extremely talented mathematicians who contributed in the reviewing process.

We owe a great debt of gratitude to Dr Kavita A Sharma, President, South Asian University for her encouragement and support towards the organization of the conference. We record our gratitude to National Board for Higher Mathematics

(NBHM), Department of Science and Technology (DST) and South Asian University for providing partial financial support for the conference.

New Delhi, India

Pankaj Jain
Hans-Jürgen Schmeisser

Contents

The Fundamental Function of Certain Interpolation Spaces Generated by N-Tuples of Rearrangement-Invariant Spaces 1
Fernando Cobos and Luz M. Fernández-Cabrera

Sobolev Embeddings for Herz-Type Triebel-Lizorkin Spaces 15
Douadi Drihem

Order Sharp Estimates for Monotone Operators on Orlicz–Lorentz Classes 37
Mikhail L. Goldman

Complex Interpolation of Morrey Spaces 85
Denny Ivanal Hakim and Yoshihiro Sawano

Gagliardo-Nirenberg Inequalities for Spaces with Dominating Mixed Derivatives 117
Dorothee D. Haroske and Hans-Jürgen Schmeisser

Recent Trends in Grand Lebesgue Spaces 137
Pankaj Jain, Monika Singh and Arun Pal Singh

On Certain New Method to Construct Weighted Hardy-Type Inequalities and Its Application to the Sharp Hardy-Poincaré Inequalities 161
Agnieszka Kałamajska and Iwona Skrzypczak

Intrinsic Characterization and the Extension Operator in Variable Exponent Function Spaces on Special Lipschitz Domains 175
Henning Kempka

The Boundedness of Sublinear Operators in Weighted Morrey Spaces Defined on Spaces of Homogeneous Type 193
Vakhtang Kokilashvili and Alexander Meskhi

Essentially Algebraic Composition Operators on Lorentz Sequence Spaces with a Weight 213
Romesh Kumar, Ajay K. Sharma, Sumit Dubey and Shagoon Wasir

Higher Dimensional Hardy-Type Inequalities 225
Santosh Kumari

Recent Advances on Generalized Trigonometric Systems in Higher Dimensions 241
Jan Lang and Osvaldo Méndez

Pointwise Multipliers on Musielak-Orlicz-Morrey Spaces 257
Eiichi Nakai

The Fatou Property of Function Spaces, Heat Kernels, Admissible Norms and Mapping Properties 283
Hans Triebel

A Survey on Some Variable Function Spaces 299
Dachun Yang, Wen Yuan and Ciqiang Zhuo

The Fundamental Function of Certain Interpolation Spaces Generated by N-Tuples of Rearrangement-Invariant Spaces

Fernando Cobos and Luz M. Fernández-Cabrera

Abstract In this paper we determine the fundamental function of the space obtained by applying an exact interpolation functor of exponent $\overline{\theta}$ to an N-tuple of rearrangement-invariant function spaces. Results apply to the extension of the real method studied by Yoshikawa and Sparr, and to the extension of the complex method investigated by Lions and Favini. Moreover, we also consider the case of the general real method for couples of spaces.

Keywords Rearrangement-invariant function spaces · Fundamental function · Interpolation methods for n-tuples · General real method

1 Introduction

A number of important function spaces have the property that any two equimeasurable functions have the same norm. These function spaces are called rearrangement-invariant and form a class which includes Lebesgue spaces, Lorentz spaces, Orlicz spaces and Marcinkiewicz spaces, among others (see [4, 12, 17, 18]). They have a close connection with interpolation theory. In fact, the class of rearrangement-invariant spaces coincides with the class of exact interpolation spaces between L_1 and L_∞. See [7] or [4, Theorem III.2.12].

The fundamental function φ_X of a rearrangement-invariant space X is an useful tool to study X. The function φ_X allows to characterize certain duality and separability properties of X, as well as to compare X with Lorentz and Marcinkiewicz spaces. See [4, 12, 17]. The fundamental function is also connected with the growth

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envelope function of X (see [16, 3.3]) and it plays an important role in the research on optimal embeddings for generalized Besov spaces (see [3, 20]).

It has been shown by Haroske ([15], (3.5), page 98) that if X_1 and X_2 are rearrangement-invariant spaces then the fundamental function of the real interpolation space $(X_1, X_2)_{\theta,q}$ is equivalent to $\varphi_{X_1}^{1-\theta}(t)\varphi_{X_2}^{\theta}(t)$. In the present paper we extend the result of Haroske to interpolation methods for N-tuples of Banach spaces which are exact of exponent $\bar{\theta} = (\theta_1, \ldots \theta_N)$. This class of methods includes the extension of the real method to N-tuples studied by Yoshikawa [25] and Sparr [23], and the extension of the complex method to N-tuples of Lions [19] and Favini [13]. For any method of this class we show that the fundamental function of the interpolated space is equivalent to $\prod_{j=1}^{N} \varphi_{X_j}^{\theta_j}(t)$.

As it is known, the step from two to several spaces involves considerable difficulties, to the effect that important results in interpolation theory for couples are no longer true in general for N-tuples. Nevertheless, interpolation methods for N-tuples still have important applications in function spaces as it can be seen, for example, in the papers by Sparr [23], Asekritova and Krugljak [1] or Asekritova et al. [2].

We also determine the fundamental function of the interpolation space generated from a couple of rearrangement-invariant spaces by means of the general real method [6, 21].

2 Rearrangement-Invariant Spaces and Interpolation Methods for N-Tuples

In what follows we assume that (Ω, μ) is a σ-finite measure space which is nonatomic or completely atomic with all atoms having equal measure. In other words, (Ω, μ) is a *resonant measure space* in the sense of [4, Definition II.2.3 and Theorem II.2.7].

We denote by $\mathcal{M}$ the space of all (equivalent classes of) scalar-valued measurable functions which are finite almost everywhere. The space $\mathcal{M}$ becomes a metrizable topological vector space with the topology of convergence in measure on sets of finite measure.

If $f \in \mathcal{M}$, we put f^* for its *non-increasing rearrangement* defined on $(0, \infty)$ by

$$f^*(t) = \inf\{\delta > 0 : \mu(\{x \in \Omega : |f(x)| > \delta\}) \leq t\}.$$

The *average function* f^{**} is given by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds.$$

It turns out that

$$(f+g)^*(t+s) \leq f^*(t) + f^*(s), \quad t, s > 0$$

but

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad t > 0 \tag{1}$$

(see [4, Proposition 2.1.7 and (3.10) in page 54]).

A Banach space $(X, \|\cdot\|_X)$ of functions in $\mathcal{M}$ is said to be a *rearrangement-invariant Banach function space* (shortly *r.i. space*) over Ω if the following conditions hold:

(P1) Whenever $g \in \mathcal{M}$, $f \in X$ and $|g| \le |f|$ μ-a.e. then $g \in X$ and $\|g\|_X \le \|f\|_X$.
(P2) If $0 \le f_n(\omega) \uparrow f(\omega)$ μ-a.e. then $\|f_n\|_X \uparrow \|f\|_X$. We put $\|f\|_X = \infty$ if f does not belong to X.
(P3) $\chi_E \in X$ for every $E \subseteq \Omega$ with $\mu(E) < \infty$.
(P4) For every $E \subseteq \Omega$ with $\mu(E) < \infty$ there is a constant $c_E > 0$ such that $\int_E |f| d\mu \le c_E \|f\|_X$ for every $f \in X$.
(P5) $\|f\|_X = \|g\|_X$ whenever $f^* = g^*$.

The usual Lebesgue spaces $L_p = L_p(\Omega, \mu)$, $1 \le p \le \infty$, are examples of r.i. spaces [4, Proposition II.1.8].

If X is an r.i. space over Ω then (see [4, Theorem I.1.4])

$$X \hookrightarrow \mathcal{M} \tag{2}$$

where $\hookrightarrow$ means continuous embedding. Moreover, according to [4, Theorem II.4.10], there is an r.i. space $\widetilde{X}$ over $(0, \infty)$ with the Lebesgue measure such that

$$\|f\|_X = \|f^*\|_{\widetilde{X}}. \tag{3}$$

The *fundamental function* φ_X of an r.i. space X over Ω is defined by

$$\varphi_X(t) = \|\chi_E\|_X .$$

Here t is any finite value belonging to the range of μ and E is any subset of Ω with $\mu(E) = t$.

Definition of φ_X is meaningful because given any other $F \subseteq \Omega$ such that $\mu(F) = t$, we have $\left(\chi_F\right)^* = \chi_{(0,t)} = \left(\chi_E\right)^*$. So $\|\chi_E\|_X = \|\chi_F\|_X$ by the rearrangement-invariance of X (property (P5)).

The function φ_X is increasing and continuous except perhaps at the origin, with $\varphi_X(t) = 0$ if and only if $t = 0$, and $\varphi_X(t)/t$ is decreasing [4, Corollary II.5.3].

We recall that in the case $X = L_p$ and (Ω, μ) being nonatomic, then its fundamental function is

$$\begin{aligned} &\varphi_{L_p}(t) = t^{1/p} \quad \text{if} \quad 0 \le t < \mu(\Omega) \quad \text{and} \quad 1 \le p < \infty, \\ &\varphi_{L_\infty}(0) = 0 \quad \text{and} \quad \varphi_{L_\infty}(t) = 1 \quad \text{for} \quad 0 < t < \mu(\Omega) . \end{aligned}$$

If $\Omega = \mathbb{N}$ with the counting measure then $L_p(\mathbb{N}) = \ell_p$ and

$$\varphi_{\ell_p}(n) = n^{1/p} \quad \text{for} \quad n = 0, 1, 2, \dots \quad \text{and} \quad 1 \le p < \infty,$$
$$\varphi_{\ell_\infty}(0) = 0 \quad \text{and} \quad \varphi_{\ell_\infty}(n) = 1 \quad \text{for} \quad n = 1, 2, \dots .$$

Next we recall some constructions from interpolation theory.

Let $\overline{A} = (A_1, \dots, A_N)$ be a *Banach N-tuple*, that is to say, N Banach spaces A_j all of which are continuously embedded in some Hausdorff topological vector space $\mathcal{A}$. If $N = 2$, then we say that $\overline{A} = (A_1, A_2)$ is a Banach couple.

Given any Banach N-tuple $\overline{A}$, let $\Sigma(\overline{A}) = A_1 + \cdots + A_N$ and $\Delta(\overline{A}) = A_1 \cap \cdots \cap A_N$. These spaces become Banach spaces when normed by

$$\|a\|_{\Sigma(\overline{A})} = \inf \left\{ \sum_{j=1}^{N} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, a_j \in A_j \right\}$$

and

$$\|a\|_{\Delta(\overline{A})} = \max\{\|a\|_{A_j} : 1 \le j \le N\}.$$

For any N-tuple of positive numbers $\bar{t} = (t_1, \dots, t_N)$, we define the K- *and J-functionals* by

$$K(\bar{t}, a) = K(\bar{t}, a; \overline{A}) = \inf \left\{ \sum_{j=1}^{N} t_j \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j, a_j \in A_j \right\},$$

$$J(\bar{t}, a) = J(\bar{t}, a; \overline{A}) = \max\{t_j \|a\|_{A_j} : 1 \le j \le N\}.$$

Note that each K-functional is a norm on $\Sigma(\overline{A})$ which is equivalent to $\|\cdot\|_{\Sigma(\overline{A})}$. Any J-functional is an equivalent norm to $\|\cdot\|_{\Delta(\overline{A})}$. If $N = 2$, then $K((1, t), \cdot)$ [respectively, $J((1, t), \cdot)$] coincides with the well-known Peetre's K-functional $K(t, \cdot)$ [respectively, J-functional $J(t, \cdot)$].

A Banach space A is said to be an *intermediate space* with respect to the N-tuple $\overline{A}$ if $\Delta(\overline{A}) \hookrightarrow A \hookrightarrow \Sigma(\overline{A})$.

Let $\overline{B} = (B_1, \dots, B_N)$ be another Banach N-tuple. We write $T : \overline{A} \longrightarrow \overline{B}$ to mean that T is a linear operator from $\Sigma(\overline{A})$ into $\Sigma(\overline{B})$ whose restriction to each A_j defines a bounded linear operator from A_j into B_j for $j = 1, \dots, N$. We put

$$\|T\|_{\overline{A},\overline{B}} = \max \left\{ \|T\|_{A_1,B_1}, \dots, \|T\|_{A_N,B_N} \right\}.$$

Let $\overline{A}$, $\overline{B}$ be Banach N-tuples and let A, B be intermediate spaces with respect to $\overline{A}$ and $\overline{B}$, respectively. We say that A and B are *interpolation spaces* with respect to $\overline{A}$ and $\overline{B}$ if given any $T : \overline{A} \longrightarrow \overline{B}$, the restriction $T : A \longrightarrow B$ is bounded. When $\overline{A} = \overline{B}$ and $A = B$, then we say that A is an interpolation space with respect to $\overline{A}$.

An *interpolation method* $\mathfrak{F}$ is a procedure that associates to any Banach N-tuple $\overline{A}$ an intermediate space $\mathfrak{F}(\overline{A})$ with respect to $\overline{A}$, such that given any other Banach N-tuple $\overline{B}$, we have that $\mathfrak{F}(\overline{A})$ and $\mathfrak{F}(\overline{B})$ are interpolation spaces with respect to $\overline{A}$

and $\overline{B}$. If this is the case, there exists a constant $C > 0$ such that for any $T : \overline{A} \longrightarrow \overline{B}$ it holds

$$\|T\|_{\mathfrak{F}(\overline{A}),\mathfrak{F}(\overline{B})} \leq C \max\left\{\|T\|_{A_j,B_j} : 1 \leq j \leq N\right\}.$$

If $C = 1$ we say that $\mathfrak{F}$ is an *exact* interpolation functor. If there is an N-tuple of numbers $\bar{\theta} = (\theta_1, \ldots, \theta_N)$ with $0 < \theta_j < 1$, $\sum_{j=1}^N \theta_j = 1$, such that

$$\|T\|_{\mathfrak{F}(\overline{A}),\mathfrak{F}(\overline{B})} \leq \prod_{j=1}^{N} \|T\|_{A_j,B_j}^{\theta_j}$$

for any N-tuples $\overline{A}$, $\overline{B}$ and any operator $T : \overline{A} \longrightarrow \overline{B}$, then we say that $\mathfrak{F}$ is *exact of exponent* $\bar{\theta}$.

Let $\mathfrak{F}_\Sigma(\overline{A}) = \Sigma(\overline{A})$ and $\mathfrak{F}_\Delta(\overline{A}) = \Delta(\overline{A})$. It is easy to check that

$$\mathfrak{F}_\Sigma \quad \text{and} \quad \mathfrak{F}_\Delta \quad \text{are exact interpolation functors.} \tag{4}$$

Note that, by (2), if $X_1, \ldots, X_N$ are any r.i. spaces over Ω, then $\overline{X} = (X_1, \ldots, X_N)$ is a Banach N-tuple. Besides, if X is an intermediate space with respect to the couple (L_1, L_∞), then X is an r.i. space over Ω if and only if X is an exact interpolation space with respect to the couple (L_1, L_∞) (see [4, Theorem III.2.12]).

Next we recall some important examples of functors of exponent $\bar{\theta}$. Given two N-tuples of positive numbers $\bar{t}$, $\bar{\theta}$, we write $\bar{t}^{-\bar{\theta}}$ for the product $\prod_{j=1}^N t_j^{-\theta_j}$.

Definition 1 Let $\overline{A} = (A_1, \ldots, A_N)$ be a Banach N-tuple, let $1 \leq q \leq \infty$ and $\bar{\theta} = (\theta_1, \ldots, \theta_N)$ where $0 < \theta_j < 1$ and $\sum_{j=1}^N \theta_j = 1$. The space $\overline{A}_{\bar{\theta},q;K}$ consists of all elements $a \in \Sigma(\overline{A})$ having a finite norm

$$\|a\|_{\overline{A}_{\bar{\theta},q;K}} = \left(\int_V \left(\bar{t}^{-\bar{\theta}} K(\bar{t}, a)\right)^q d\mu(\bar{t})\right)^{1/q}$$

(the integral should be replaced by the supremum if $q = \infty$). Here μ is the measure $(dt_2 dt_3 \ldots dt_N)/(t_2 t_3 \cdots t_N)$ supported on the set

$$V = \left\{(1, t_2, t_3, \ldots, t_N) | t_j > 0, 2 \leq j \leq N\right\} \subseteq \mathbb{R}_+^N.$$

The space $\overline{A}_{\bar{\theta},q;J}$ is formed by all elements $a \in \Sigma(\overline{A})$ which can be represented as

$$a = \int_V u(\bar{t}) d\mu(\bar{t}) \tag{5}$$

where $u(\bar{t})$ is a strongly measurable function with values in $\Delta(\overline{A})$ such that

$$\left(\int_V \left(\bar{t}^{-\bar{\theta}} J(\bar{t}, u(\bar{t}))\right)^q d\mu(\bar{t})\right)^{1/q} < \infty. \tag{6}$$

The norm in $\overline{A}_{\bar{\theta},q;J}$ is given by the infimum of the values (6) over all representations (5) of a satisfying (6).

Spaces $\overline{A}_{\bar{\theta},q;K}$, $\overline{A}_{\bar{\theta},q;J}$ have been studied by Sparr [23], Yoshikawa [25] and other authors (see [6, Section 4.7.1]). These spaces generalize the classical real interpolation spaces $(A_1, A_2)_{\theta,q}$ for Banach couples (see [4, 5, 24]). Indeed, given any $0 < \theta < 1$, we have

$$(A_1, A_2)_{(1-\theta,\theta),q;K} = (A_1, A_2)_{(1-\theta,\theta),q;J} = (A_1, A_2)_{\theta,q}$$

with equivalence of norms.

Working with N-tuples with $N > 2$, we only have that $\overline{A}_{\bar{\theta},q;J} \hookrightarrow \overline{A}_{\bar{\theta},q;K}$. The converse embedding fails in general as it is shown in [11, 23, 25]. However, if the N-tuple is formed by r.i. spaces X_j over Ω, then

$$(X_1, \dots, X_N)_{\bar{\theta},q;J} = (X_1, \dots, X_N)_{\bar{\theta},q;K}$$

(see [1, Lemma 1 and Theorem 1]).

It follows from [23, Theorems 4.2 and 4.4] that

$$K(\bar{\theta}, q)\text{- and } J(\bar{\theta}, q)\text{-methods are exact functors of exponent } \bar{\theta}\,. \tag{7}$$

The other main interpolation method for Banach couples is the complex method (see [5, 24]). Next we describe its extension to Banach N-tuples which was suggested by Lions [19] and studied in details by Favini [13]. We follow the presentation of [11].

Let Λ be the collection of all $z = (z_1, \dots, z_{N-1}) \in \mathbb{C}^{N-1}$ such that

$$0 < Re z_j < 1\,,\ 1 \le j \le N-1 \quad \text{and} \quad 0 < \sum_{j=1}^{N-1} Re z_j < 1\,.$$

For $1 \le j \le N-1$, put

$$\partial\Lambda_j = \left\{z \in \overline{\Lambda} : Re z_j = 1,\, Re z_k = 0,\, k \ne j\right\}$$

and

$$\partial\Lambda_N = \left\{z \in \overline{\Lambda} : Re z_k = 0,\, 1 \le k \le N-1\right\}.$$

Given any Banach N-tuple $\overline{A}$, we write $\mathcal{H}(\overline{A})$ for the space of all continuous and bounded functions $f : \overline{\Lambda} \longrightarrow \Sigma(\overline{A})$ such that

(i) f is holomorphic in Λ,
(ii) for each $1 \le j \le N$, $f : \partial\Lambda_j \longrightarrow A_j$ is continuous, bounded and vanishes at infinity.

We put

$$\|f\|_{\mathcal{H}(\overline{A})} = \sup\left\{\|f(z)\|_{A_j} : z \in \partial\Lambda_j, 1 \le j \le N\right\}.$$

Definition 2 Let $\overline{A} = (A_1, \ldots, A_N)$ be a Banach N-tuple and let $\bar{\theta} = (\theta_1, \ldots, \theta_N)$ where $0 < \theta_j < 1$ and $\sum_{j=1}^N \theta_j = 1$. The space $[\overline{A}]_{\bar{\theta}}$ consists of all elements $a \in \Sigma(\overline{A})$ such that $a = f(\theta_1, \ldots \theta_{N-1})$ for some $f \in \mathcal{H}(\overline{A})$. We endow $[\overline{A}]_{\bar{\theta}}$ with the norm

$$\|a\|_{[\overline{A}]_{\bar{\theta}}} = \inf\left\{\|f\|_{\mathcal{H}(\overline{A})} : f(\theta_1, \ldots, \theta_{N-1}) = a,\, f \in \mathcal{H}(\overline{A})\right\}.$$

According to [13, Teorema 1], we have that

$$[\,\cdot\,]_{\bar{\theta}} \text{ is an exact functor of exponent } \bar{\theta}\,. \tag{8}$$

In [13, 19, 23, 25] one can find examples of concrete interpolation spaces generated by these methods. We just recall one here. Let $0 < \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$, let $1 \le p_1, p_2, p_3 \le \infty$ and put $1/p = \theta_1/p_1 + \theta_2/p_2 + \theta_3/p_3$. Then we have with equivalence of norms

$$\begin{aligned} L_p &= (L_{p_1}, L_{p_2}, L_{p_3})_{(\theta_1,\theta_2,\theta_3),q;K} \\ &= (L_{p_1}, L_{p_2}, L_{p_3})_{(\theta_1,\theta_2,\theta_3),q;J} \\ &= [L_{p_1}, L_{p_2}, L_{p_3}]_{(\theta_2,\theta_3)} \end{aligned}$$

(see [23, Theorems 8.1 and 8.3] and [19, page 1855]).

3 The Fundamental Function

As we have pointed out in Sect. 2, given any N r.i. spaces $X_1, \ldots, X_N$ over Ω, then $\overline{X} = (X_1, \ldots, X_N)$ is a Banach N-tuple which has $\mathcal{M}$ as containing space.

Lemma 1 *Let $\overline{X} = (X_1, \ldots, X_N)$ be an N-tuple of r.i. spaces over Ω. Then for any N-tuple of positive numbers $\bar{t} = (t_1, \ldots, t_N)$ and any $E \subseteq \Omega$ with $\mu(E) = s$ we have that*

$$K(\bar{t}, \chi_E) = \min\left\{t_j \varphi_{X_j}(s) : 1 \le j \le N\right\}.$$

Proof By (P3) we have that $\chi_E \in \Delta(\overline{X})$. Hence

$$K(\bar{t}, \chi_E) \le \min\left\{t_j \|\chi_E\|_{X_j} : 1 \le j \le N\right\} = \min\left\{t_j \varphi_{X_j}(s) : 1 \le j \le N\right\}.$$

To check the converse inequality we follow the same idea as in [9, Lemma 5.1]. Let $\widetilde{X}_j$ be the r.i. space over $(0, \infty)$ which corresponds to X_j through equality (3). Take any representation $\chi_E = \sum_{j=1}^N f_j$ with $f_j \in X_j$. For any $y > 0$, using (1) we get

$$\int_0^y \chi_E^*(x)dx = \int_0^y \big(\sum_{j=1}^N f_j\big)^*(x)dx \le \sum_{j=1}^N \int_0^y f_j^*(x)dx = \int_0^y \sum_{j=1}^N f_j^*(x)dx\,.$$

Therefore, by [4, Lemma III.7.5], there is a linear operator $T : (\widetilde{X}_1, \dots \widetilde{X}_N) \longrightarrow (\widetilde{X}_1, \dots \widetilde{X}_N)$ such that $\|T\|_{\widetilde{X}_j,\widetilde{X}_j} \le 1$ for $1 \le j \le N$, $T\big(\sum_{j=1}^N f_j^*\big) = \chi_E^*$ and Tf_j^* is non-negative and decreasing for $1 \le j \le N$. Since $\chi_{(0,s)} = \chi_E^* = \sum_{j=1}^N T f_j^*$, it follows that $Tf_j^* = c_j\chi_{(0,s)}$ for some $0 \le c_j \le 1$ with $\sum_{j=1}^N c_j = 1$. Hence

$$\begin{aligned}\sum_{j=1}^N t_j c_j \varphi_{X_j}(s) &= \sum_{j=1}^N t_j \|c_j\chi_{(0,s)}\|_{\widetilde{X}_j} = \sum_{j=1}^N t_j \|Tf_j^*\|_{\widetilde{X}_j}\\ &\le \sum_{j=1}^N t_j \|f_j^*\|_{\widetilde{X}_j} = \sum_{j=1}^N t_j \|f_j\|_{X_j}\,.\end{aligned}$$

Consequently,

$$\begin{aligned}\min\big\{t_j\varphi_{X_j}(s) : 1 \le j \le N\big\} &= \inf\Big\{\sum_{j=1}^N d_j t_j \varphi_j(s) : 0 \le d_j \le 1, \sum_{j=1}^N d_j = 1\Big\}\\ &\le \sum_{j=1}^N c_j t_j \varphi_j(s) \le \sum_{j=1}^N t_j \|f_j\|_{X_j}\,.\end{aligned}$$

This yields that

$$\min\big\{t_j\varphi_{X_j}(s) : 1 \le j \le N\big\} \le K(\bar{t}, \chi_E)\,.$$

The proof is complete. □

Next we establish the main result of the paper. Recall that two functions f, g defined on $[0, \mu(\Omega)]$ are said to be equivalent (in symbols, $f \sim g$) if there are constants $c_1, c_2 > 0$ such that

$$c_1 f(s) \le g(s) \le c_2 f(s)\,, \quad s \in [0, \mu(\Omega)]\,.$$

Theorem 1 *Let $\bar{\theta} = (\theta_1, \dots \theta_N)$ be an N-tuple of numbers with $0 < \theta_j < 1$ and $\sum_{j=1}^N \theta_j = 1$. Assume that $\mathfrak{F}$ is an exact interpolation functor of exponent $\bar{\theta}$. Given any N-tuple $\overline{X} = (X_1, \dots, X_N)$ of r.i. spaces over Ω, we have that $\mathfrak{F}(\overline{X})$ is an r.i. space over Ω and its fundamental function $\varphi_{\mathfrak{F}(\overline{X})}$ is equivalent to $\prod_{j=1}^N \varphi_{X_j}^{\theta_j}(s)$.*

Proof It is easy to check that $\mathfrak{F}(\overline{X})$ is an exact interpolation space with respect to $(L_1(\Omega), L_\infty(\Omega))$. Hence, it follows from [4, Theorem III.2.12] that $\mathfrak{F}(\overline{X})$ is an r.i. space over Ω. Let us estimate its fundamental function. Let $\mathbb{K}$ be the scalar field ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Take any $a \in \Delta(\overline{X})$ and consider the operator $T\lambda = \lambda a$ with $\lambda \in \mathbb{K}$. Then

$T : (\mathbb{K}, \ldots, \mathbb{K}) \longrightarrow (X_1, \ldots, X_N)$ and $\|T\|_{\mathbb{K}, X_j} = \|a\|_{X_j}$. Since $\mathfrak{F}(\mathbb{K}, \ldots, \mathbb{K}) = \mathbb{K}$ with equivalence of norms and $\mathfrak{F}$ has exponent $\bar{\theta}$, we derive that there is a constant $c > 0$ independent of a such that $\|a\|_{\mathfrak{F}(\overline{X})} \leq c \prod_{j=1}^N \|a\|_{X_j}^{\theta_j}$. Choose now $a = \chi_E$ where $E \subseteq \Omega$ with $\mu(E) = s$. Then we obtain

$$\varphi_{\mathfrak{F}(\overline{X})}(s) \leq c \prod_{j=1}^N \varphi_{X_j}^{\theta_j}(s) \, .$$

To check the converse inequality, take any $a \in \mathfrak{F}(\overline{X})$ and take any positive numbers $t_2, \ldots, t_N$. Applying the Hahn-Banach theorem to the space $\Sigma(\overline{X})$ normed by $K((1, t_2, \ldots, t_N), \cdot)$, we can find a continuous linear functional f such that

$$f(a) = K((1, t_2, \ldots, t_N), a) \quad \text{and} \quad |f(x)| \leq K((1, t_2, \ldots, t_N), x) \, , \; x \in \Sigma(\overline{X}) \, .$$

Since $X_j \hookrightarrow \Sigma(\overline{X})$ and this embedding has norm $\leq t_j$, $2 \leq j \leq N$, it follows that $f : \overline{X} \longrightarrow (\mathbb{K}, \ldots, \mathbb{K})$ with $\|f\|_{X_j, \mathbb{K}} \leq t_j$ for $2 \leq j \leq N$ and $\|f\|_{X_1, \mathbb{K}} = 1$. Using that $\mathfrak{F}$ is exact of exponent $\bar{\theta}$ we obtain that there is a constant $C > 0$, depending only on the equivalence of norms in equality $\mathfrak{F}(\mathbb{K}, \ldots, \mathbb{K}) = \mathbb{K}$ such that

$$K((1, t_2, \ldots, t_N), a) = |f(a)| \leq C \prod_{j=2}^N t_j^{\theta_j} \|a\|_{\mathfrak{F}(\overline{X})} \, .$$

This yields that

$$\sup_{t_2, \ldots, t_N > 0} \frac{K((1, t_2, \ldots, t_N), a)}{\prod_{j=2}^N t_j^{\theta_j}} \leq C \|a\|_{\mathfrak{F}(\overline{X})} \, .$$

Now take $a = \chi_E$ where $E \subseteq \Omega$ with $\mu(E) = s$ and choose $t_j = \varphi_{X_1}(s)/\varphi_{X_j}(s)$, $2 \leq j \leq N$. Using Lemma 1 we derive

$$\begin{aligned} C\varphi_{\mathfrak{F}(\overline{X})}(s) = C\|\chi_E\|_{\mathfrak{F}(\overline{X})} &\geq \prod_{k=2}^N t_k^{-\theta_k} K((1, t_2, \ldots, t_N), \chi_E) \\ &= \prod_{k=2}^N t_k^{-\theta_k} \min\left\{\varphi_{X_1}(s) \, , \; t_j \varphi_{X_j}(s) : 2 \leq j \leq N\right\} \\ &= \prod_{j=1}^N \varphi_{X_j}^{\theta_j}(s) \, . \end{aligned}$$

This completes the proof. □

Corollary 1 *Let* $\bar{\theta} = (\theta_1, \ldots \theta_N)$ *be an* N*-tuple of numbers with* $0 < \theta_j < 1$ *and* $\sum_{j=1}^N \theta_j = 1$. *Let* $\overline{X} = (X_1, \ldots, X_N)$ *be an* N*-tuple of r.i. spaces over* Ω *and let*

$1 \le q \le \infty$. *Then the fundamental functions of the r.i. spaces* $(X_1, \dots X_N)_{\bar{\theta},q;K}$ *and* $[X_1, \dots X_N]_{\bar{\theta}}$ *are equivalent to* $\prod_{j=1}^{N} \varphi_{X_j}^{\theta_j}(s)$.

Writing down Corollary 1 for the case of the classical real method $(X_1, X_2)_{\theta,q}$ we recover the result established by Haroske [15, (3.5), page 98].

Remark 1 Cobos and Peetre [10] showed another way to extend the real method to N-tuples of Banach spaces. They introduced a J and a K interpolation functors by using a convex polygon Π in the plane, an interior point (α, β) of Π and a scalar parameter $q \in [1, \infty]$. The Banach spaces of the N-tuple should be now thought of as sitting on the vertices of Π. If $N = 3$ and Π is the simplex $\{(0, 0), (1, 0), (0, 1)\}$, then the spaces generated by the methods of Cobos and Peetre coincide with $\overline{A}_{\bar{\theta},q;J}$ and $\overline{A}_{\bar{\theta},q;K}$. But in general they are not interpolation functors of exponent $\bar{\theta}$ (see [8]). Furthermore, on the contrary to the case of spaces of Yoshikawa and Sparr, given an N-tuple of r.i. spaces over Ω, the K and the J-spaces might be different. The fundamental function of r.i. spaces generated by those methods have been studied in [14].

As we have mentioned in Sect. 2 after (4), any r.i. space X over Ω is an exact interpolation space with respect to the couple (L_1, L_∞). However, applying the real method to this couple we only obtain L_p spaces and Lorentz $L_{p,q}$ spaces. Namely, $(L_1, L_\infty)_{\theta,q} = L_{p,q}$ if $1/p = 1 - \theta$. The complex method only produce L_p spaces. In order to obtain all r.i. spaces, we should extend the definition of the real method by replacing the weighted L_q norm in Definition 1 by a more general lattice norm (see [6, 22]). Next we recall this extension in the discrete form presented in [21].

Let Γ be a Banach space of real-valued sequences with $\mathbb{Z}$ as index set. Suppose that Γ contains all sequences with only finite many non-zero coordinates, and that whenever $|\xi_m| \le |\eta_m|$ for each $m \in \mathbb{Z}$ and $(\eta_m) \in \Gamma$, then $(\xi_m) \in \Gamma$ and $\|(\xi_m)\|_\Gamma \le \|(\eta_m)\|_\Gamma$. Moreover, we assume that

$$(\min(1, 2^m)) \in \Gamma \tag{9}$$

and

$$\sup\left\{ \sum_{m=-\infty}^{\infty} \min(1, 2^{-m})|\xi_m| : \|(\xi_m)\|_\Gamma \le 1 \right\} < \infty\,. \tag{10}$$

Definition 3 Given any Banach couple (A_1, A_2), the K-space $(A_1, A_2)_{\Gamma;K}$ is formed by all those $a \in A_1 + A_2$ such that $(K(2^m, a)) \in \Gamma$. We put $\|a\|_{(A_1,A_2)_{\Gamma;K}} = \|(K(2^m, a))\|_\Gamma$.

The J-space $(A_1, A_2)_{\Gamma;J}$ consists of all elements $a \in A_1 + A_2$ which can be represented as $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_1 + A_2$) where $(u_m) \subseteq A_1 \cap A_2$ and $(J(2^m, u_m)) \in \Gamma$. We set

$$\|a\|_{(A_1,A_2)_{\Gamma;J}} = \inf\left\{ \|(J(2^m, u_m))\|_\Gamma : a = \sum_{m=-\infty}^{\infty} u_m \right\}.$$

It is not hard to check that the general J- and K-functors are exact. Hence, given any r.i. spaces X_1, X_2 over Ω, we have that $(X_1, X_2)_{\Gamma;K}$ and $(X_1, X_2)_{\Gamma;J}$ are also r.i. spaces over Ω.

If $0 < \theta < 1$, $1 \leq q \leq \infty$ and $\Gamma = \ell_q(2^{-\theta m})$, the space ℓ_q with the weight $(2^{-\theta m})$, then

$$(A_1, A_2)_{\ell_q(2^{-\theta m});K} = (A_1, A_2)_{\ell_q(2^{-\theta m});J} = (A_1, A_2)_{\theta,q}$$

(see [5, 6, 24]).

In general, $(A_1, A_2)_{\Gamma;J} \neq (A_1, A_2)_{\Gamma;K}$, but we always have that

$$(A_1, A_2)_{\Gamma;K} \hookrightarrow (A_1, A_2)_{\Gamma;J} \tag{11}$$

(see [21, Lemma 2.4]).

As a direct consequence of Lemma 1 we obtain the following

Corollary 2 *If X_1, X_2 are r.i. spaces over Ω, then the fundamental function of $(X_1, X_2)_{\Gamma;K}$ is*

$$\varphi_{(X_1,X_2)_{\Gamma;K}}(s) = \|\big(\min(\varphi_{X_1}(s), 2^m\varphi_{X_2}(s))\big)\|_\Gamma\,, \quad 0 \leq s < \mu(\Omega)\,.$$

In order to relate Corollary 2 with Theorem 1 for couples, we shall consider shift operators τ_k on Γ, which are defined by $\tau_k\xi = (\xi_{m+k})$ for $\xi = (\xi_m) \in \Gamma$.

Proposition 1 *Let X_1, X_2 be r.i. spaces over Ω. There are positive constants c_1, c_2 such that for any $0 < s < \mu(\Omega)$ if we take $k \in \mathbb{Z}$ such that $2^k \leq \varphi_{X_2}(s)/\varphi_{X_1}(s) < 2^{k+1}$, then we have*

$$\varphi_{(X_1,X_2)_{\Gamma;K}}(s) \leq c_1\varphi_{X_1}(s)\|\tau_k\|_{\Gamma,\Gamma} \quad \textit{and} \quad \varphi_{X_1}(s) \leq c_2\|\tau_{-k}\|_{\Gamma,\Gamma}\,\varphi_{(X_1,X_2)_{\Gamma;J}}(s)\,.$$

Proof By Corollary 2 and (9) we get

$$\begin{aligned}
\varphi_{(X_1,X_2)_{\Gamma;K}}(s) &= \|\big(\min(\varphi_{X_1}(s), 2^m\varphi_{X_2}(s))\big)\|_\Gamma \\
&= \varphi_{X_1}(s)\|\big(\min(1, 2^m\varphi_{X_2}(s)/\varphi_{X_1}(s))\big)\|_\Gamma \\
&\leq \varphi_{X_1}(s)\|\big(\min(1, 2^{m+k+1})\big)\|_\Gamma \\
&\leq \varphi_{X_1}(s)\|\tau_{k+1}\|_{\Gamma,\Gamma}\|\big(\min(1, 2^m)\big)\|_\Gamma \\
&\leq c_1\varphi_{X_1}(s)\|\tau_k\|_{\Gamma,\Gamma}\,.
\end{aligned}$$

On the other hand, let $E \subseteq \Omega$ with $\mu(E) = s$ and take any J-representation $\chi_E = \sum_{m=-\infty}^{\infty} f_m$ with $(f_m) \subseteq X_1 \cap X_2$ and $\big(J(2^m, f_m)\big) \in \Gamma$. By Lemma 1 we derive

$$
\begin{aligned}
\varphi_{X_1}(s) &\le \min\left(\varphi_{X_1}(s), 2^{-k}\varphi_{X_2}(s)\right) \\
&= K(2^{-k}, \chi_E) \le \sum_{m=-\infty}^{\infty} K(2^{-k}, f_m) \\
&\le \sum_{m=-\infty}^{\infty} \min(1, 2^{-m-k}) J(2^m, f_m) \\
&= \sum_{m=-\infty}^{\infty} \min(1, 2^{-m}) J(2^{m-k}, f_{m-k}) \\
&\le c_2 \|\left(J(2^{m-k}, f_{m-k})\right)\|_\Gamma
\end{aligned}
$$

where we have used (10) in the last inequality. Consequently,

$$\varphi_{X_1}(s) \le c_2 \|\tau_{-k}(J(2^m, f_m))\|_\Gamma \le c_2 \|\tau_{-k}\|_{\Gamma,\Gamma} \|(J(2^m, f_m))\|_\Gamma \,.$$

This implies that

$$\varphi_{X_1}(s) \le c_2 \|\tau_{-k}\|_{\Gamma,\Gamma}\, \varphi_{(X_1,X_2)_{\Gamma;J}}(s) \,.$$

□

It follows from (11) that there is a constant $c > 0$ such that

$$\varphi_{(X_1,X_2)_{\Gamma;J}}(s) \le c\varphi_{(X_1,X_2)_{\Gamma;K}}(s) \,, \quad 0 \le s < \mu(\Omega) \,. \tag{12}$$

Therefore, writing down Proposition 1 and (12) for the special case $\Gamma = \ell_q(2^{-\theta m})$, where $\|\tau_k\|_{\ell_q(2^{-\theta m}),\ell_q(2^{-\theta m})} = 2^{\theta k}$, we derive

$$\varphi_{(X_1,X_2)_{\ell_q(2^{-\theta m});K}} \sim \varphi_{(X_1,X_2)_{\ell_q(2^{-\theta m});J}} \sim \varphi_{X_1}^{1-\theta} \varphi_{X_2}^{\theta}$$

which coincides with the estimate of Theorem 1 for the classical real method.

We finish the paper by computing the fundamental function of an space generated by the general J-functor. We first recall that the *associate space* X' of an r.i. space X over Ω consists of all $g \in \mathcal{M}$ such that $\int_\Omega |fg| d\mu < \infty$ for every $f \in X$, equipped with the norm

$$\|g\|_{X'} = \sup\left\{ \int_\Omega |fg| d\mu : \|f\|_X \le 1 \right\}.$$

It turns out that X' is again an r.i. space over Ω. For its fundamental function we have that

$$\varphi_X(t)\varphi_{X'}(t) = t \quad \text{for any finite} \quad t \in [0, \mu(\Omega)] \tag{13}$$

(see [4, Theorem II.5.2]).

Using the same arguments as in [14, Lemmata 3.1 and 3.3] we can derive the following.

Lemma 2 *Let X_1, X_2 be r.i. spaces over Ω. Then the following equalities hold:*

$$(a) \quad J(t, g; X_1', X_2') = \sup_{f \in X_1 + X_2} \frac{\int_\Omega |fg| d\mu}{K(t^{-1}, f; X_1, X_2)}, \quad g \in X_1' \cap X_2'.$$

$$(b) \quad K(t, g; X_1', X_2') = \sup_{f \in X_1 \cap X_2} \frac{\int_\Omega |fg| d\mu}{J(t^{-1}, f; X_1, X_2)}, \quad g \in X_1' + X_2'.$$

Let Γ^* be the collection of all real-valued sequences $\eta = (\eta_m)$ such that

$$\|\eta\|_{\Gamma^*} = \sup \left\{ \sum_{m=-\infty}^{\infty} |\xi_m \eta_{-m}| : \|\xi\|_\Gamma \le 1 \right\} < \infty.$$

Proceeding as in [14, Theorem 3.4], from Lemma 2/(b) we obtain the following characterization for the associate space of the J-space.

Theorem 2 *Let X_1, X_2 be r.i. spaces over Ω. Then*

$$\left((X_1, X_2)_{\Gamma;J}\right)' = (X_1', X_2')_{\Gamma^*;K}.$$

Since $X = X''$ with equal norms (see [4, Theorem I.2.7]), we conclude from Theorem 2 and (13) that the fundamental function of $(X_1, X_2)_{\Gamma;J}$ is

$$\varphi_{(X_1,X_2)_{\Gamma;J}}(s) = \left\| \left(\min\left(\frac{1}{\varphi_{X_1}(s)}, \frac{2^m}{\varphi_{X_2}(s)} \right) \right) \right\|_{\Gamma^*}^{-1}.$$

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Sobolev Embeddings for Herz-Type Triebel-Lizorkin Spaces

Douadi Drihem

Abstract In this paper we prove the Sobolev embeddings for Herz-type Triebel-Lizorkin spaces,

$$\dot{K}_q^{\alpha_2,r} F_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} F_\beta^{s_1}$$

where the parameters α_1, α_2, s_1, s_2, s, q, r, p, β and θ satisfy some suitable conditions. An application we obtain new embeddings between Herz and Triebel-Lizorkin spaces. Moreover, we present the Sobolev embeddings for Triebel-Lizorkin spaces equipped with power weights. All these results cover the results on classical Triebel-Lizorkin spaces.

Keywords Triebel-Lizorkin spaces · Herz spaces · Sobolev embedding

1 Introduction

Function spaces have been widely used in various areas of analysis such as harmonic analysis and partial differential equations. In recent years, there has been increasing interest in a new family of function spaces which generalize the Besov spaces and Triebel-Lizorkin spaces. Some example of these spaces can be mentioned such as Herz-type Triebel-Lizorkin spaces, $\dot{K}_q^{\alpha,p} F_\beta^s$, that initially appeared in the papers of J. Xu and D. Yang [19–21]. Several basic properties were established, such as the Fourier analytical characterisation and lifting properties. When $\alpha = 0$ and $p = q$ they coincide with the usual function spaces $F_{p,q}^s$.

The interest in these spaces comes not only from theoretical reasons but also from their applications to several classical problems in analysis. In [11], Lu and Yang introduced the Herz-type Sobolev and Bessel potential spaces. They gave some

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applications to partial differential equations. Also in [17], Y. Tsutsui, studied the Cauchy problem for Navier-Stokes equations on Herz spaces and weak Herz spaces.

Since the Sobolev embedding plays an important role in theory of function spaces and PDE's, the main aim of this paper is to prove the Sobolev embedding of spaces $\dot{K}_q^{\alpha,p} F_\beta^s$. First we shall prove the Sobolev embeddings of associated sequence spaces. Then, from the so-called φ-transform characterization in the sense of Frazier and Jawerth, we deduce the main result of this paper. As a consequence, we obtain new Jawerth-Franke-type embeddings, the Sobolev embeddings for Triebel-Lizorkin spaces equipped with power weights, new embeddings between Herz and Triebel-Lizorkin spaces, and we present some remarks about the wavelet characterization of Herz-Triebel-Lizorkin spaces. All these results generalize the existing classical results on Triebel-Lizorkin spaces.

To recall the definition of these function spaces, we need some notation. For any $u > 0, k \in \mathbb{Z}$ we set $C(u) = \{x \in \mathbb{R}^n : u/2 \leq |x| < u\}$ and $C_k = C(2^k)$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. Let χ_k, for $k \in \mathbb{Z}$, denote the characteristic function of the set C_k. The expression $f \approx g$ means that $C\, g \leq f \leq c\, g$ for some independent constants c, C and non-negative functions f and g.

We denote by $|\Omega|$ the n-dimensional Lebesgue measure of $\Omega \subseteq \mathbb{R}^n$. For any measurable subset $\Omega \subseteq \mathbb{R}^n$ the Lebesgue space $L^p(\Omega)$, $0 < p \leq \infty$, consists of all measurable functions for which $\|f \mid L^p(\Omega)\| = \left(\int_\Omega |f(x)|^p\, dx\right)^{1/p} < \infty, 0 < p < \infty$ and $\|f \mid L^\infty(\Omega)\| = \underset{x\in\Omega}{\text{ess-sup}}\, |f(x)| < \infty$. If $\Omega = \mathbb{R}^n$ we put $L^p(\mathbb{R}^n) = L^p$ and $\|f \mid L^p(\mathbb{R}^n)\| = \|f\|_p$. The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined on locally integrable functions by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

and $\mathcal{M}_t f = \left(\mathcal{M}|f|^t\right)^{1/t}$ for any $0 < t \leq 1$. The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions φ on $\mathbb{R}^n$ and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by $\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx$. Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

Let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ with integer-valued components. If $v \in \mathbb{N}_0$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$ we denote $Q_{v,m}$ the dyadic cube in $\mathbb{R}^n$

$$Q_{v,m} = \{(x_1, ..., x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, ..., n\}.$$

By $\chi_{v,m}$ we denote the characteristic function of the cube $Q_{v,m}$.

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. We use c as a generic positive constant, i.e. a constant whose value may change from appearance to appearance.

2 Function Spaces

We start by recalling the definition and some of the properties of the homogenous Herz spaces $\dot{K}_q^{\alpha,p}$.

Definition 1 Let $\alpha \in \mathbb{R}, 0 < p, q \leq \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}$ is defined by

$$\dot{K}_q^{\alpha,p} = \{f \in L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f \mid \dot{K}_q^{\alpha,p}\| < \infty\},$$

where

$$\|f \mid \dot{K}_q^{\alpha,p}\| = \Big(\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \ \|f\chi_k\|_q^p\Big)^{1/p},$$

with the usual modifications made when $p = \infty$ and/or $q = \infty$.

The spaces $\dot{K}_q^{\alpha,p}$ are quasi-Banach spaces and if $\min(p, q) \geq 1$ then $\dot{K}_q^{\alpha,p}$ are Banach spaces. When $\alpha = 0$ and $0 < p = q \leq \infty$ then $\dot{K}_p^{0,p}$ coincides with the Lebesgue spaces L^p. Various important results have been proved in the space $\dot{K}_q^{\alpha,p}$ under some assumptions on α, p and q. The conditions $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$, $1 < q < \infty$ and $0 < p \leq \infty$ is crucial in the study of the boundedness of classical operators in $\dot{K}_q^{\alpha,p}$ spaces. This fact was first realized by Li and Yang [8] with the proof of the boundedness of the maximal function. The proof of the main result of this section is based on the following result, see Tang and Yang [13].

Lemma 1 *Let* $1 < \beta < \infty$, $1 < q < \infty$ *and* $0 < p \leq \infty$. *If* $\{f_j\}_{j=0}^{\infty}$ *is a sequence of locally integrable functions on* $\mathbb{R}^n$ *and* $-\frac{n}{q} < \alpha < n(1 - \frac{1}{q})$, *then*

$$\Big\|\Big(\sum_{j=0}^{\infty}(\mathcal{M}f_j)^{\beta}\Big)^{1/\beta} | \dot{K}_q^{\alpha,p}\Big\| \leq c\Big\|\Big(\sum_{j=0}^{\infty}|f_j|^{\beta}\Big)^{1/\beta} | \dot{K}_q^{\alpha,p}\Big\|.$$

A detailed discussion of the properties of these spaces my be found in the papers [7, 9, 10], and references therein.

Now, we present the Fourier analytical definition of Herz-type Triebel-Lizorkin spaces $\dot{K}_q^{\alpha,p}F_{\beta}^s$ and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity. Let ϕ_0 be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\phi_0(x) = 1$ for $|x| \leq 1$ and $\phi_0(x) = 0$ for $|x| \geq 2$. We put $\phi_j(x) = \phi_0(2^{-j}x) - \phi_0(2^{1-j}x)$ for $j = 1, 2, 3, \ldots$. Then $\{\phi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity, $\sum_{j=0}^{\infty}\phi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition $f = \sum_{j=0}^{\infty}\mathcal{F}^{-1}\phi_j * f$ of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We are now in a position to state the definition of Herz-type Triebel-Lizorkin spaces.

Definition 2 Let $\alpha, s \in \mathbb{R}$, $0 < p, q < \infty$ and $0 < \beta \leq \infty$. The Herz-type Triebel-Lizorkin space $\dot{K}_q^{\alpha,p}F_{\beta}^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \mid \dot{K}_q^{\alpha,p} F_\beta^s \big\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} \left| \mathcal{F}^{-1}\phi_j * f \right|^\beta \Big)^{1/\beta} \mid \dot{K}_q^{\alpha,p} \Big\| < \infty, \tag{1}$$

with the obvious modification if $\beta = \infty$.

Remark 1 Let $s \in \mathbb{R}, 0 < p, q < \infty, 0 < \beta \le \infty$ and $\alpha > -n/q$. The spaces $\dot{K}_q^{\alpha,p} F_\beta^s$ are independent of the particular choice of the smooth dyadic resolution of unity $\phi_{j\, j\in\mathbb{N}_0}$ (in the sense of equivalent quasi-norms). In particular $\dot{K}_q^{\alpha,p} F_\beta^s$ are quasi- Banach spaces and if $p, q, \beta \ge 1$, then $\dot{K}_q^{\alpha,p} F_\beta^s$ are Banach spaces. Further results,concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in [20–22, 24].

Now we give the definitions of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$.

Definition 3 (i) Let $s \in \mathbb{R}$ and $0 < p, \beta \le \infty$. The Besov space $B_{p,\beta}^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \mid B_{p,\beta}^s \big\| = \Big(\sum_{j=0}^{\infty} 2^{js\beta} \left\| \mathcal{F}^{-1}\phi_j * f \right\|_p^\beta \Big)^{1/\beta} < \infty.$$

(ii) Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \le \infty$. The Triebel-Lizorkin space $F_{p,\beta}^s$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \mid F_{p,\beta}^s \big\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} \left| \mathcal{F}^{-1}\phi_j * f \right|^\beta \Big)^{1/\beta} \Big\|_p < \infty.$$

The theory of the spaces $B_{p,\beta}^s$ and $F_{p,\beta}^s$ has been developed in detail in [14–16] but has a longer history already including many contributors; we do not want to discuss this here. Clearly, for $s \in \mathbb{R}, 0 < p < \infty$ and $0 < \beta \le \infty$,

$$\dot{K}_p^{0,p} F_\beta^s = F_{p,\beta}^s.$$

Let us consider $k_0, k \in \mathcal{S}(\mathbb{R}^n)$ and $S \ge -1$ an integer such that for an $\varepsilon > 0$

$$|\mathcal{F}k_0(\xi)| > 0 \quad \text{for} \quad |\xi| < 2\varepsilon \tag{2}$$

$$|\mathcal{F}k(\xi)| > 0 \quad \text{for} \quad \frac{\varepsilon}{2} < |\xi| < 2\varepsilon \tag{3}$$

and

$$\int_{\mathbb{R}^n} x^\alpha k(x)dx = 0 \quad \text{for any} \quad |\alpha| \le S. \tag{4}$$

Here (2) and (3) are Tauberian conditions, while (4) are moment conditions on k. We recall the notation

$$k_t(x) = t^{-n}k(t^{-1}x), \quad k_j(x) = k_{2^{-j}}(x), \quad \text{for} \quad t \in (0,\infty) \quad \text{and} \quad j \in \mathbb{N}.$$

Usually $k_j * f$ is called local mean. Let $\{f_v\}_v$ be a sequence of positive measurable functions. Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$ and $0 < \beta \leq \infty$. We set

$$\Big\| \{f_v\}_v \mid \dot{K}_q^{\alpha,p}(\ell_\beta) \Big\| = \Big\| \Big(\sum_{v=0}^{\infty} |f_v|^\beta \Big)^{1/\beta} \mid \dot{K}_q^{\alpha,p} \Big\|.$$

The following result is from [23, Theorem 1].

Theorem 1 *Let $\alpha, s \in \mathbb{R}$, $0 < p, q < \infty$, $0 < \beta \leq \infty$, $\alpha > -n/q$ and $s < S + 1$. Then*

$$\big\| f \mid \dot{K}_q^{\alpha,p} F_\beta^s \big\|' = \Big\| \{2^{vs} k_v * f\}_v \mid \dot{K}_q^{\alpha,p}(\ell_\beta) \Big\|, \tag{5}$$

is equivalent quasi-norm on $\dot{K}_q^{\alpha,p} F_\beta^s$.

We introduce the sequence spaces associated with the function spaces $\dot{K}_q^{\alpha,p} F_\beta^s$. If

$$\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\},$$

$\alpha, s \in \mathbb{R}$, $0 < p, q < \infty$ and $0 < \beta \leq \infty$, we set

$$\big\| \lambda \mid \dot{K}_q^{\alpha,p} f_\beta^s \big\| = \Big\| \Big(\sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{vs\beta} |\lambda_{v,m}|^\beta \chi_{v,m} \Big)^{1/\beta} \mid \dot{K}_q^{\alpha,p} \Big\|. \tag{6}$$

Let Φ, ψ, φ and Ψ satisfy

$$\Phi, \Psi, \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \tag{7}$$

$$\text{supp}\mathcal{F}\Phi, \text{supp}\mathcal{F}\Psi \subset \overline{B(0,2)} \text{ such that } |\mathcal{F}\Phi(\xi)|, |\mathcal{F}\Psi(\xi)| \geq c \text{ if } |\xi| \leq \frac{5}{3} \tag{8}$$

and

$$\text{supp}\mathcal{F}\varphi, \text{supp}\mathcal{F}\psi \subset \overline{B(0,2)} \backslash B(0,1/2) \text{ such that } |\mathcal{F}\varphi(\xi)|, |\mathcal{F}\psi(\xi)| \geq c, \tag{9}$$

if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$ such that

$$\overline{\mathcal{F}\Phi(-\xi)}\mathcal{F}\Psi(\xi) + \sum_{j=1}^{\infty} \overline{\mathcal{F}\varphi(-2^{-j}\xi)}\mathcal{F}\psi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n, \tag{10}$$

where $c > 0$. Recall that the φ-transform S_φ is defined by setting $(S_\varphi f)_{0,m} = \langle f, \Phi_m \rangle$ where $\Phi_m(x) = \Phi(x - m)$ and $(S_\varphi f)_{v,m} = \langle f, \varphi_{v,m} \rangle$ where $\varphi_{v,m}(x) = 2^{vn/2}\varphi(2^v x - m)$ and $v \in \mathbb{N}$. The inverse φ-transform T_ψ is defined by

$$T_\psi \lambda = \sum_{m\in\mathbb{Z}^n} \lambda_{0,m}\Psi_m + \sum_{v=1}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\psi_{v,m},$$

where $\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$, see [3].

For a sequence $\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$, $0 < r \leq \infty$ and a fixed $d > 0$, set

$$\lambda^*_{v,m,r,d} = \Big(\sum_{h\in\mathbb{Z}^n} \frac{|\lambda_{v,h}|^r}{(1 + 2^v|2^{-v}h - 2^{-v}m|)^d}\Big)^{1/r}$$

and $\lambda^*_{r,d} = \{\lambda^*_{v,m,r,d} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$.

Lemma 2 *Let* $\alpha, s \in \mathbb{R}, 0 < p \leq \infty, 0 < q < \infty, 0 < \beta \leq \infty,$ $d > n$ *and* $\alpha > -n/q$. *Then*

$$\Big\|\lambda^*_{\min(q,\frac{n}{\frac{n}{q}+\alpha},\beta),d} \mid \dot{K}^{\alpha,p}_q f^s_\beta\Big\| \approx \big\|\lambda \mid \dot{K}^{\alpha,p}_q f^s_\beta\big\|.$$

Proof Obviously,

$$\big\|\lambda \mid \dot{K}^{\alpha,p}_q f^s_\beta\big\| \leq \Big\|\lambda^*_{\min(q,\frac{n}{\frac{n}{q}+\alpha},\beta),d} \mid \dot{K}^{\alpha,p}_q f^s_\beta\Big\|.$$

From Lemma A.2 of [3], we obtain

$$\lambda^*_{v,m,\min(q,\frac{n}{\frac{n}{q}+\alpha},\beta),d} \leq c\mathcal{M}_a\Big(\sum_{h\in\mathbb{Z}^n} |\lambda_{v,h}|\chi_{v,h}\Big)(x), \quad x \in Q_{v,m},$$

where,

$$0 < a \leq r = \min(q, \frac{n}{\frac{n}{q}+\alpha}, \beta) < \infty,\ da > nr$$

and $c > 0$ depend only n and d. Let $\varepsilon = \frac{d}{n} - 1 > 0$ and $a = \frac{r}{1+\varepsilon/2}$, then $0 < a < r$ and $da > nr$. Hence

$$\Big\|\lambda^*_{\min(q,\frac{n}{\frac{n}{q}+\alpha},\beta),d} \mid \dot{K}^{\alpha,p}_q f^s_\beta\Big\|$$

$$\leq c\Big\|\Big(\sum_{v=0}^{\infty} \mathcal{M}_{a/\beta}\Big(\sum_{h\in\mathbb{Z}^n} 2^{vs\beta}|\lambda_{v,h}|^\beta\chi_{v,h}\Big)\Big)^{a/\beta} \mid \dot{K}^{\alpha a,p/a}_{q/a}\Big\|^{1/a}.$$

Observe that $\frac{\beta}{a} > 1$, $\frac{q}{a} > 1$ and $\frac{-na}{q} < \alpha a < n(1 - \frac{a}{q})$. Applying Lemma 1 to estimate the last expression by

$$c\Big\|\Big(\sum_{v=0}^{\infty}\sum_{h\in\mathbb{Z}^n} 2^{vs\beta}|\lambda_{v,h}|^\beta\chi_{v,h}\Big)^{1/\beta} \mid \dot{K}^{\alpha,p}_q\Big\| = c\big\|\lambda \mid \dot{K}^{\alpha,p}_q f^s_\beta\big\|.$$

The proof of the lemma is thus complete. □

To prove the main results of this paper we need the following theorem.

Theorem 2 *Let* $\alpha, s \in \mathbb{R}$, $0 < p < \infty$, $0 < q < \infty$, $0 < \beta \leq \infty$ *and* $\alpha > -n/q$. *Suppose that* φ *and* Φ *satisfy* (7)–(10). *The operators* $S_\varphi : \dot{K}_q^{\alpha,p} F_\beta^s \to \dot{K}_q^{\alpha,p} f_\beta^s$ *and* $T_\psi : \dot{K}_q^{\alpha,p} f_\beta^s \to \dot{K}_q^{\alpha,p} F_\beta^s$ *are bounded. Furthermore,* $T_\psi \circ S_\varphi$ *is the identity on* $\dot{K}_q^{\alpha,p} F_\beta^s$.

Proof We use the same arguments of [3, Theorem 2.2], see also [25, Theorem 2.1] and [1, Theorem 3.12]. For any $f \in \mathcal{S}'(\mathbb{R}^n)$ we put $\sup(f) = \{\sup_{v,m}(f) : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ where

$$\sup_{v,m}(f) = \sup_{y \in Q_{v,m}} |\widetilde{\varphi}_v * f(y)|$$

if $v \in \mathbb{N}$, $m \in \mathbb{Z}^n$ and

$$\sup_{0,m}(f) = \sup_{y \in Q_{0,m}} |\widetilde{\Phi} * f(y)|$$

if $m \in \mathbb{Z}^n$. For any $\gamma \in \mathbb{Z}_+$, we define the sequence $\inf_\gamma(f) = \{\inf_{v,m,\gamma}(f) : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ by setting

$$\inf_{v,m,\gamma}(f) = \sup_{h \in \mathbb{Z}^n} \{ \inf_{y \in Q_{v+\gamma,h}} |\widetilde{\varphi}_v * f(y)| : Q_{v+\gamma,h} \cap Q_{v,m} \neq \emptyset\}$$

if $v \in \mathbb{N}$, $m \in \mathbb{Z}^n$ and

$$\inf_{0,m,\gamma}(f) = \sup_{h \in \mathbb{Z}^n} \{ \inf_{y \in Q_{\gamma,h}} |\widetilde{\Phi} * f(y)| : Q_{\gamma,h} \cap Q_{0,m} \neq \emptyset\}$$

if $m \in \mathbb{Z}^n$. Here $\widetilde{\varphi_j}(x) = 2^{jn}\overline{\varphi(-2^j x)}$ and $\widetilde{\Phi}(x) = \overline{\Phi(-x)}$. As in Lemma A.5 of [3], see also [1, 25] we obtain

$$\big\| \inf_\gamma(f) \mid \dot{K}_q^{\alpha,p} f_\beta^s \big\| \leq c \big\| f \mid \dot{K}_q^{\alpha,p} F_\beta^s \big\|$$

for any $s \in \mathbb{R}$, $0 < p, q < \infty$, $0 < \beta \leq \infty$, $\alpha > -n/q$ and $\gamma > 0$ sufficiently large. Indeed, we have

$$\big\| \inf_\gamma(f) \mid \dot{K}_q^{\alpha,p} f_\beta^s \big\| = c \Big\| \Big(\sum_{m \in \mathbb{Z}^n} 2^{js} \inf_{j-\gamma,m,\gamma}(f) \chi_{j-\gamma,m} \Big)_{j \geq \gamma} \mid \dot{K}_q^{\alpha,p}(\ell_\beta) \Big\|.$$

Define a sequence $\{\lambda_{i,k}\}_{i \in \mathbb{N}_0, k \in \mathbb{Z}^n}$ by setting $\lambda_{i,k} = \inf_{y \in Q_{i,k}} |\widetilde{\varphi_{i-\gamma}} * f(y)|$ and $\lambda_{0,k} = \inf_{y \in Q_{\gamma,k}} |\widetilde{\Phi} * f(y)|$. We have

$$\inf_{j-\gamma,m,\gamma}(f) = \sup_{h \in \mathbb{Z}^n} \{\lambda_{j,h} : Q_{j,h} \cap Q_{j-\gamma,m} \neq \emptyset\}$$

and

$$\inf_{0,m,\gamma}(f) = \sup_{h \in \mathbb{Z}^n} \{\lambda_{0,h} : Q_{\gamma,h} \cap Q_{0,m} \neq \emptyset\}.$$

Let $h \in \mathbb{Z}^n$ with $Q_{j,h} \cap Q_{j-\gamma,m} \neq \emptyset$ and $j \geq \gamma$. Then

$$\lambda_{j,h} \leq c2^{\gamma d/r} \lambda^*_{j,z,r,\gamma},\ j > \gamma \quad \text{and} \quad \lambda_{0,h} \leq c2^{\gamma d/r} \lambda^*_{0,z,r,\gamma},\ j = \gamma \tag{11}$$

for any $z \in \mathbb{Z}^n$ with $Q_{j,z} \cap Q_{j-\gamma,m} \neq \emptyset$, where the constant $c > 0$ does not depend on j, h and z. Indeed, we observe

$$\lambda_{j,h} = \frac{\lambda_{j,h}}{(1 + 2^j |2^{-j}h - 2^{-j}z|)^d} (1 + 2^j |2^{-j}h - 2^{-j}z|)^d.$$

Let $x \in Q_{j,h} \cap Q_{j-\gamma,m}$ and $y \in Q_{j,z} \cap Q_{j-\gamma,m}$. We have

$$|2^{-j}h - 2^{-j}z| \leq |2^{-j}h - x| + |x - y| + |y - 2^{-j}z| \lesssim 2^{2-j} + 2^{\gamma-j}.$$

This implies (11). Hence

$$\sum_{m \in \mathbb{Z}^n} \inf_{j-\gamma,m,\gamma}(f) \chi_{j-\gamma,m} \leq c \sum_{k \in \mathbb{Z}^n} \lambda^*_{j,k,r,d} \chi_{j,k},\ \ j > \gamma$$

and

$$\sum_{m \in \mathbb{Z}^n} \inf_{0,m,\gamma}(f) \chi_{0,m} \leq c \sum_{k \in \mathbb{Z}^n} \lambda^*_{0,k,r,d} \chi_{\gamma,k},\ \ j = \gamma,$$

with $r = \min(q, \frac{n}{\frac{n}{q}+\alpha}, \beta)$ and $d > n$. Therefore,

$$\left\| \inf\nolimits_\gamma(f) \mid \dot{K}^{\alpha,p}_q f^s_\beta \right\| \leq c \left\| \Big(\sum_{k \in \mathbb{Z}^n} 2^{js} \lambda^*_{j,k,r,d} \chi_{j,k} \Big)_{j \geq \gamma} \mid \dot{K}^{\alpha,p}_q (\ell_\beta) \right\|.$$

Notice that if $j = \gamma$ we replace $\lambda^*_{j,k,r,d}\chi_{j,k}$ by $\lambda^*_{0,k,r,d}\chi_{\gamma,k}$. Applying Lemma 2 to estimate this term by

$$c \left\| \Big(\sum_{k \in \mathbb{Z}^n} 2^{js} \lambda_{j,k} \chi_{j,k} \Big)_{j > \gamma} \mid \dot{K}^{\alpha,p}_q (\ell_\beta) \right\| + c \left\| \sum_{k \in \mathbb{Z}^n} \lambda_{0,k} \chi_{\gamma,k} \mid \dot{K}^{\alpha,p}_q \right\|,$$

which is bounded by

$$c \left\| \Big(2^{js} \widetilde{\varphi_{j-\gamma}} * f \Big)_{j > \gamma} \mid \dot{K}^{\alpha,p}_q (\ell_\beta) \right\| + c \left\| \widetilde{\Phi} * f \mid \dot{K}^{\alpha,p}_q \right\|.$$

By Theorem 1, we obtain

$$\left\| \inf\nolimits_\gamma(f) \mid \dot{K}^{\alpha,p}_q f^s_\beta \right\| \leq c \left\| f \mid \dot{K}^{\alpha,p}_q F^s_\beta \right\|' \leq c \left\| f \mid \dot{K}^{\alpha,p}_q F^s_\beta \right\|,$$

where we use the characterization of Herz-type Triebel-Lizorkin spaces by local means. Applying Lemma A.4 of [3], see also Lemma 8.3 of [1], we obtain

$$\inf_\gamma(f)^*_{\min(q,\frac{n}{\frac{n}{q}+\alpha},\beta),\gamma} \approx \sup(f)^*_{\min(q,\frac{n}{\frac{n}{q}+\alpha},\beta),\gamma}.$$

Hence for $\gamma > 0$ sufficiently large we obtain by applying Lemma 2,

$$\left\|\inf_\gamma(f)^*_{\min(q,\frac{n}{\frac{n}{q}+\alpha},\beta),\gamma} \mid \dot{K}^{\alpha,p}_q f^s_\beta\right\| \approx \left\|\inf_\gamma(f) \mid \dot{K}^{\alpha,p}_q f^s_\beta\right\|$$

and

$$\left\| \sup(f)^*_{\min(q,\frac{n}{\frac{n}{q}+\alpha},\beta),\gamma} \mid \dot{K}^{\alpha,p}_q f^s_\beta\right\| \approx \left\| \sup(f) \mid \dot{K}^{\alpha,p}_q f^s_\beta\right\|$$

for any $s \in \mathbb{R}$, $0 < p, q < \infty$, $0 < \beta \leq \infty$ and $\alpha > -n/q$. Therefore,

$$\left\|\inf_\gamma(f) \mid \dot{K}^{\alpha,p}_q f^s_\beta\right\| \approx \left\| f \mid \dot{K}^{\alpha,p}_q F^s_\beta\right\| \approx \left\| \sup(f) \mid \dot{K}^{\alpha,p}_q f^s_\beta\right\|.$$

Use these estimates and repeating the proof of Theorem 2.2 in [3] or Theorem 2.1 in [25] then complete the proof of Theorem 2. □

Remark 2 From these to prove the embeddings

$$\dot{K}^{\alpha_2,r}_q F^{s_2}_\theta \hookrightarrow \dot{K}^{\alpha_1,p}_s F^{s_1}_\beta$$

we need only to prove

$$\dot{K}^{\alpha_2,r}_q f^{s_2}_\theta \hookrightarrow \dot{K}^{\alpha_1,p}_s f^{s_1}_\beta,$$

under the same restrictions on parameters $s_1, s_2, \alpha_1, \alpha_2, s, p, q, \beta, r, \theta$.

We end this section with one more lemma, which is basically a consequence of Hardy's inequality in the sequence Lebesgue space ℓ_q.

Lemma 3 *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=0}^k a^{k-j}\varepsilon_j$ and $\eta_k = \sum_{j=k}^\infty a^{j-k}\varepsilon_j$, $k \in \mathbb{N}_0$. Then there exists constant $c > 0$ depending only on a and q such that*

$$\Big(\sum_{k=0}^\infty \delta_k^q\Big)^{1/q} + \Big(\sum_{k=0}^\infty \eta_k^q\Big)^{1/q} \leq c\,\Big(\sum_{k=0}^\infty \varepsilon_k^q\Big)^{1/q}.$$

3 Sobolev Embeddings for Spaces $\dot{K}^{\alpha,p}_q F^s_\beta$

It is well-known that

$$F^{s_2}_{q,\infty} \hookrightarrow F^{s_1}_{s,\beta}$$

if $s_1 - n/s = s_2 - n/q$, where $0 < q < s < \infty$ and $0 < \beta \leq \infty$ (see e.g. [14, Theorem 2.7.1]). In this section we generalize these embeddings to Herz-type Triebel-

Lizorkin spaces. We need the Sobolev embeddings properties of the above sequence spaces.

Theorem 3 *Let* $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s, r, p, q < \infty, 0 < \beta \leq \infty, \alpha_1 > -n/s$ *and* $\alpha_2 > -n/q$. *We suppose that*

$$s_1 - n/s - \alpha_1 = s_2 - n/q - \alpha_2. \tag{12}$$

Let $0 < q < s < \infty$ *and* $\alpha_2 \geq \alpha_1$ *or* $0 < s \leq q < \infty$ *and*

$$\alpha_2 + n/q \geq \alpha_1 + n/s. \tag{13}$$

Then

$$\dot{K}_q^{\alpha_2,r} f_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} f_\beta^{s_1}, \tag{14}$$

if and only if $0 < r \leq p < \infty$, *where*

$$\theta = \begin{cases} \beta, & \text{if } 0 < s \leq q < \infty \text{ and } \alpha_2 + n/q = \alpha_1 + n/s; \\ \infty, & \text{otherwise.} \end{cases}$$

Proof We would like to mention that this embedding was proved in [2, Theorem 5.9] under the restriction $\max(0, \frac{\alpha_1 s}{q}) \leq \alpha_2 \leq \big(\alpha_1 + \frac{n}{s}\big)\frac{r}{p} - \frac{n}{q}$. Here we use a different method to omit this condition.

Step 1. Let us prove that $0 < r \leq p < \infty$ is necessary. In the calculations below we consider the 1-dimensional case for simplicity. For any $v \in \mathbb{N}_0$ and $N \geq 1$, we put

$$\lambda_{v,m}^N = \begin{cases} 2^{-(s_1 - \frac{1}{s} - \alpha_1)v} \sum_{i=1}^N \chi_i(2^{v-1}), & \text{if } \quad m = 1; \\ 0, & \text{otherwise,} \end{cases}$$

$\lambda^N = \{\lambda_{v,m}^N : v \in \mathbb{N}_0, m \in \mathbb{Z}\}$. We have

$$\big\|\lambda^N \mid \dot{K}_s^{\alpha_1,p} f_\beta^{s_1}\big\|^p = \sum_{k=-\infty}^{\infty} 2^{\alpha_1 kp} \Big\| \Big(\sum_{v=0}^{\infty} \sum_{m\in\mathbb{Z}} 2^{vs_1\beta} |\lambda_{v,m}^N|^\beta \chi_{v,m} \Big)^{1/\beta} \chi_k \Big\|_s^p.$$

We can rewrite the last statement as follows:

$$\begin{aligned} &\sum_{k=1-N}^{0} 2^{\alpha_1 kp} \Big\| \Big(\sum_{v=1}^{N} 2^{(\frac{1}{s}+\alpha_1)v\beta} \chi_{v,1} \Big)^{1/\beta} \chi_k \Big\|_s^p \\ &= \sum_{k=1-N}^{0} 2^{\alpha_1 kp} \big\| 2^{(\frac{1}{s}+\alpha_1)(1-k)} \chi_{1-k,1} \big\|_s^p = c\,N, \end{aligned}$$

where the constant $c > 0$ does not depend on N. Now

$$\big\|\lambda^N \mid \dot{K}_q^{\alpha_2,r} f_\theta^{s_2}\big\|^r = \sum_{k=-\infty}^{\infty} 2^{\alpha_2 kr}\Big\|\Big(\sum_{v=0}^{\infty} 2^{vs_2\theta}|\lambda_{v,1}^N|^\theta \chi_{v,1}\Big)^{1/\theta}\chi_k\Big\|_q^r.$$

Again we can rewrite the last statement as follows:

$$\begin{aligned}&\sum_{k=1-N}^{0} 2^{\alpha_2 kr}\Big\|\Big(\sum_{v=1}^{N} 2^{(s_2-s_1+\frac{1}{s}+\alpha_1)v\theta}\chi_{v,1}\Big)^{1/\theta}\chi_k\Big\|_q^r \\ &= \sum_{k=1-N}^{0} 2^{\alpha_2 kr}\Big\|2^{(s_2-s_1+\frac{1}{s}+\alpha_1)(1-k)}\chi_{1-k,1}\Big\|_q^r = c\,N,\end{aligned}$$

where the constant $c > 0$ does not depend on N. If the embeddings (14) holds then for any $N \in \mathbb{N}$, $N^{\frac{1}{p}-\frac{1}{r}} \leq C$. Thus, we conclude that $0 < r \leq p < \infty$ must necessarily hold by letting $N \to +\infty$.

Step 2. We consider the sufficiency of the conditions. First we consider $0 < q < s < \infty$ and $\alpha_2 \geq \alpha_1$. In view of the embedding $\ell_r \hookrightarrow \ell_p$, it is sufficient to prove that

$$\dot{K}_q^{\alpha_2,r} f_\infty^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,r} f_\beta^{s_1}.$$

By similarity, we only consider the case $\beta = 1$. Let $\lambda \in \dot{K}_q^{\alpha_2,r} f_\infty^{s_2}$. We have

$$\begin{aligned}\big\|\lambda \mid \dot{K}_s^{\alpha_1,r} f_1^{s_1}\big\| \leq &\Big(\sum_{k=-\infty}^{c_n+1} 2^{k\alpha_1 r}\Big\|\sum_{v=0}^{\infty} 2^{vs_1}\sum_{m\in\mathbb{Z}^n}\lambda_{v,m}\chi_{v,m}\chi_k\Big\|_s^r\Big)^{1/r} \\ &+\Big(\sum_{k=c_n+2}^{\infty} 2^{k\alpha_1 r}\Big\|\sum_{v=0}^{\infty} 2^{vs_1}\sum_{m\in\mathbb{Z}^n}\lambda_{v,m}\chi_{v,m}\chi_k\Big\|_s^r\Big)^{1/r}. \qquad (15)\end{aligned}$$

Here $c_n = 1 + [\log_2(2\sqrt{n}+1)]$. The first term can be estimated by

$$\begin{aligned}&c\Big(\sum_{k=-\infty}^{c_n+1} 2^{k\alpha_1 r}\Big\|\sum_{v=0}^{c_n-k+1} 2^{vs_1}\sum_{m\in\mathbb{Z}^n}\lambda_{v,m}\chi_{v,m}\chi_k\Big\|_s^r\Big)^{1/r} \\ &+c\Big(\sum_{k=-\infty}^{c_n+1} 2^{k\alpha_1 r}\Big\|\sum_{v=c_n-k+2}^{\infty} 2^{vs_1}\sum_{m\in\mathbb{Z}^n}\lambda_{v,m}\chi_{v,m}\chi_k\Big\|_s^r\Big)^{1/r} \\ &=I + II.\end{aligned}$$

Estimation of I. Let $x \in C_k \cap Q_{v,m}$ and $y \in Q_{v,m}$. We have $|x-y| \leq 2\sqrt{n}2^{-v} < 2^{c_n-v}$ and from this it follows that $|y| < 2^{c_n-v} + 2^k \leq 2^{c_n-v+2}$, which implies that y is located in some ball $B(0, 2^{c_n-v+2})$. Then

$$|\lambda_{v,m}|^t = 2^{nv}\int_{\mathbb{R}^n} |\lambda_{v,m}|^t \chi_{v,m}(y)dy \leq 2^{nv}\int_{B\left(0,2^{c_n-v+2}\right)} |\lambda_{v,m}|^t \chi_{v,m}(y)dy,$$

if $x \in C_k \cap Q_{v,m}$ and $t > 0$. Therefore for any $x \in C_k$

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|^t \chi_{v,m}(x) &\leq 2^{nv}\int_{B\left(0,2^{c_n-v+2}\right)} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|^t \chi_{v,m}(y)dy \\ &= 2^{nv}\Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\chi_{v,m}\chi_{B\left(0,2^{c_n-v+2}\right)}\Big\|_t^t.\end{aligned}$$

Hence

$$\begin{aligned}&2^{\alpha_1 k}\Big\| \sum_{v=0}^{c_n-k+1} 2^{vs_1} \sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\chi_{v,m}\chi_k\Big\|_s \\ &\leq c\, 2^{(\alpha_1+\frac{n}{s})k} \sum_{v=0}^{c_n-k+1} 2^{v(s_1+\frac{n}{t})}\Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\chi_{v,m}\chi_{B\left(0,2^{c_n-v+2}\right)}\Big\|_t.\end{aligned}$$

We may choose $t > 0$ such that $\frac{1}{t} > \max(\frac{1}{q}, \frac{1}{q} + \frac{\alpha_2}{n})$. Using (12) and Lemma 3 to estimate I^r by

$$\begin{aligned}&c\sum_{v=0}^{\infty} 2^{v(s_2-\frac{n}{q}-\alpha_2+\frac{n}{t})r}\Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\chi_{v,m}\chi_{B\left(0,2^{c_n-v+2}\right)}\Big\|_t^r \\ &\leq c\sum_{v=0}^{\infty} 2^{v(s_2-\frac{n}{q}-\alpha_2+\frac{n}{t})r}\Big(\sum_{i\leq -v}\Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\chi_{v,m}\chi_{i+c_n+2}\Big\|_t^\sigma\Big)^{r/\sigma} \\ &\leq c\sum_{v=0}^{\infty} 2^{v\frac{nr}{d}}\Big(\sum_{i\leq -v} 2^{i\frac{n\sigma}{d}+\alpha_2\sigma i}\Big\| \sup_j \sum_{m\in\mathbb{Z}^n} 2^{s_2 j}|\lambda_{j,m}|\chi_{j,m}\chi_{i+c_n+2}\Big\|_q^\sigma\Big)^{r/\sigma},\end{aligned}$$

by Hölder's inequality, with $\sigma = \min(1, t)$ and $\frac{n}{d} = \frac{n}{t} - \frac{n}{q} - \alpha_2$. Again, we apply Lemma 3 to obtain

$$I^r \leq c\sum_{i=0}^{\infty} 2^{-\alpha_2 ir}\Big\| \sup_j \sum_{m\in\mathbb{Z}^n} 2^{s_2 j}|\lambda_{j,m}|\chi_{j,m}\chi_{-i+c_n+2}\Big\|_q^r \leq c\big\|\lambda \mid \dot{K}_q^{\alpha_2,r} f_\infty^{s_2}\big\|^r.$$

Estimation of II. We see that it suffices to show that for any $k \leq c_n + 1$

$$\begin{aligned}2^{k\alpha_1}\Big\| \sum_{v=c_n-k+2}^{\infty} 2^{vs_1} \sum_{m\in\mathbb{Z}^n} \lambda_{v,m}\chi_{v,m}\chi_k\Big\|_s &\leq C^{s/q}\, 2^{k\alpha_2}\Big\| \sup_{v\geq c_n-k+2} \sum_{m\in\mathbb{Z}^n} 2^{vs_2}\lambda_{v,m}\chi_{v,m}\chi_{\widetilde{C}_k}\Big\|_q \\ &= \delta,\end{aligned}$$

where $\widetilde{C}_k = \{x \in \mathbb{R}^n : 2^{k-2} < |x| < 2^{k+2}\}$ and $C = 2\max(\frac{2^{\frac{1}{q}}}{1-2^{\frac{n}{s}-\frac{n}{q}}}, \frac{2^{\frac{n}{s}}}{2^{\frac{n}{s}}-1})$. This claim can be reformulated as showing that

$$\int_{C_k} 2^{k\alpha_1 s}\delta^{-s}\Big(\sum_{v=c_n-k+2}^{\infty} 2^{vs_1}\sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\chi_{v,m}(x)\Big)^s dx \le 1.$$

The left-hand side can be rewritten as

$$\int_{C_k} \delta^{-s}\Big(\sum_{v=c_n-k+2}^{\infty} 2^{v(\frac{n}{s}-\frac{n}{q})+(\alpha_1-\alpha_2)(v+k)+vs_2+k\alpha_2}\sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\chi_{v,m}(x)\Big)^s dx$$
$$\le \int_{C_k} \delta^{-s}\Big(\sum_{v=c_n-k+2}^{\infty} 2^{v(\frac{n}{s}-\frac{n}{q})+vs_2+k\alpha_2}\sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|\chi_{v,m}(x)\Big)^s dx = T_k,$$

since $\alpha_2 \ge \alpha_1$. Let us prove that $T_k \le 1$ for any $k \le c_n + 1$. Our estimate use partially some decomposition techniques already used in [18].

Case 1. $\sup_{v\ge c_n-k+2,m} 2^{vs_2+k\alpha_2}|\lambda_{v,m}|\chi_{v,m}(x) \le \delta$. In this case we obtain

$$T_k \le C^s \int_{C_k} \Big(\delta^{-1}\sup_{v\ge c_n-k+2}\sum_{m\in\mathbb{Z}^n} 2^{vs_2+k\alpha_2}|\lambda_{v,m}|\chi_{v,m}(x)\Big)^q dx \le 1.$$

Case 2. $\sup_{v\ge c_n-k+2,m} 2^{vs_2+k\alpha_2}|\lambda_{v,m}|\chi_{v,m}(x) > \delta$. We can distinguish two cases as follows:

- $\delta^{-1}\sup_{v\ge c_n-k+2,m} 2^{vs_2+k\alpha_2}|\lambda_{v,m}|\chi_{v,m}(x) = \infty$, then there is nothing to prove.
- $\delta < \sup_{v\ge c_n-k+2,m} 2^{vs_2+k\alpha_2}|\lambda_{v,m}|\chi_{v,m}(x) < \infty$. Let $N \in \mathbb{N}$ be such that

$$2^{\frac{nN}{q}} < \delta^{-1}\sup_{v\ge c_n-k+2}\sum_{m\in\mathbb{Z}^n} 2^{vs_2+k\alpha_2}|\lambda_{v,m}|\chi_{v,m}(x) < 2^{\frac{(N+1)n}{q}}.$$

Subcase 2.1. $k \ge c_n - N + 2$. We split the sum over $v \ge c_n - k + 2$ into two parts,

$$\sum_{v=c_n-k+2}^{\infty}\cdots = \sum_{v=c_n-k+2}^{N}\cdots + \sum_{v=N+1}^{\infty}\cdots.$$

Let $x \in C_k \cap Q_{v,m}$ and $y \in Q_{v,m}$. We have $|x-y| \le 2\sqrt{n}2^{-v} < 2^{c_n-v}$ and from this it follows that $2^{k-2} < |y| < 2^{c_n-v} + 2^k < 2^{k+2}$, which implies that y is located in $\widetilde{C}_k$. Then

$$|\lambda_{v,m}|^q = 2^{nv}\int_{\mathbb{R}^n} |\lambda_{v,m}|^q\chi_{v,m}(y)dy \le 2^{nv}\int_{\widetilde{C}_k} |\lambda_{v,m}|^q\chi_{v,m}(y)dy,$$

if $x \in C_k \cap Q_{v,m}$. But this immediately implies that

$$\sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|^q \chi_{v,m}(x) \leq 2^{nv} \int_{\widetilde{C}_k} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}|^q \chi_{v,m}(y)dy$$
$$= 2^{nv} \Big\| \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \chi_{\widetilde{C}_k} \Big\|_q^q \leq 2^{(\frac{n}{q}-s_2)qv-\alpha_2 qk} \delta^q.$$

Therefore for any $x \in C_k$

$$\delta^{-1} \sum_{v=c_n-k+2}^{N} 2^{v(\frac{n}{s}-\frac{n}{q})+vs_2+k\alpha_2} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}(x) \leq \sum_{v=c_n-k}^{N} 2^{\frac{n}{s}v} \leq C\, 2^{\frac{n}{s}N}$$

and

$$\delta^{-1} \sum_{v=N+1}^{\infty} 2^{v(\frac{n}{s}-\frac{n}{q})+vs_2+k\alpha_2} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}(x)$$
$$= 2^{(\frac{n}{s}-\frac{n}{q})N} \delta^{-1} \sum_{v=N+1}^{\infty} 2^{(v-N)(\frac{n}{s}-\frac{n}{q})+vs_2+k\alpha_2} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}(x)$$
$$\leq c\, 2^{(\frac{n}{s}-\frac{n}{q})N} \delta^{-1} \sup_{v\geq c_n-k+2} \sum_{m\in\mathbb{Z}^n} 2^{vs_2+k\alpha_2} |\lambda_{v,m}| \chi_{v,m}(x)$$
$$\leq C\, 2^{\frac{n}{s}N}.$$

Hence

$$T_k \leq C^s \int_{C_k} 2^{nN} dx \leq C^s \int_{C_k} \Big(\delta^{-1} \sup_{v\geq c_n-k+2} \sum_{m\in\mathbb{Z}^n} 2^{vs_2+k\alpha_2} |\lambda_{v,m}| \chi_{v,m}(x) \Big)^q dx \leq 1.$$

Subcase 2.2. $k < c_n - N + 2$. We use the same of arguments as in Subcase 2.1, in view of the fact that $\sum_{v=c_n-k+2}^{\infty} \cdots \leq \sum_{v=N+1}^{\infty} \cdots$.

Estimate of (15). The arguments here are quite similar to those used in the estimation of II. This complete the proof of the first case.

Now we consider the case $0 < s \leq q < \infty$ and $\alpha_2 + n/q > \alpha_1 + n/s$. We only need to estimate the part T_k. Hölder's inequality implies that

$$T_k \leq \Big\| \delta^{-1} \sum_{v=c_n-k+2}^{\infty} 2^{(\frac{n}{s}-\frac{n}{q}+\alpha_1-\alpha_2)(v+k)} 2^{vs_2+k\alpha_2} \sum_{m\in\mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m} \chi_k \Big\|_q^s$$
$$\leq \Big\| \delta^{-1} \sup_{v\geq c_n-k+2} \sum_{m\in\mathbb{Z}^n} 2^{vs_2+k\alpha_2} |\lambda_{v,m}| \chi_{v,m} \chi_k \Big\|_q^s = C^{-s/q},$$

where the last inequality follows by the fact that $\alpha_2 + n/q > \alpha_1 + n/s$. The remaining case can be easily solved. The proof is complete. □

As a corollary of Theorems 2 and 3, we have the following Sobolev embedding for spaces $\dot{K}_q^{\alpha,p}F_\beta^s$.

Theorem 4 *Let* $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s, q < \infty, 0 < r \leq p < \infty, 0 < \beta \leq \infty,$ $\alpha_1 > -n/s$ *and* $\alpha_2 > -n/q$. *We suppose that*

$$s_1 - n/s - \alpha_1 = s_2 - n/q - \alpha_2.$$

Let $0 < q < s < \infty$ *and* $\alpha_2 \geq \alpha_1$ *or* $0 < s \leq q < \infty$ *and*

$$\alpha_2 + n/q \geq \alpha_1 + n/s.$$

Then

$$\dot{K}_q^{\alpha_2,r}F_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p}F_\beta^{s_1},$$

where

$$\theta = \begin{cases} \beta, & \text{if } 0 < s \leq q < \infty \text{ and } \alpha_2 + n/q = \alpha_1 + n/s; \\ \infty, & \text{otherwise.} \end{cases}$$

Remark 3 We would like to mention that (13) and $s_1 - n/s - \alpha_1 \leq s_2 - n/q - \alpha_2$ are necessary, see [2].

From Theorem 4 and the fact that $\dot{K}_s^{0,s}F_\beta^{s_1} = F_{s,\beta}^{s_1}$ we immediately arrive at the following corollaries.

Corollary 1 *Let* $s_1, s_2 \in \mathbb{R}, 0 < s, q < \infty, s_1 - n/s = s_2 - n/q - \alpha_2,$ $0 < r \leq s < \infty$ *and* $0 < \beta \leq \infty$. *Let* $0 < q < s < \infty$ *and* $\alpha_2 \geq 0$ *or* $0 < s \leq q < \infty$ *and* $\alpha_2 + n/q \geq n/s$. *Then*

$$\dot{K}_q^{\alpha_2,r}F_\theta^{s_2} \hookrightarrow F_{s,\beta}^{s_1},$$

where

$$\theta = \begin{cases} \beta, & \text{if } 0 < s \leq q < \infty \text{ and } \alpha_2 + n/q = n/s; \\ \infty, & \text{otherwise.} \end{cases}$$

Corollary 2 *Let* $s_1, s_2 \in \mathbb{R}, 0 < s, q < \infty, s_1 - n/s - \alpha_1 = s_2 - n/q, 0 < q \leq p < \infty$ *and* $0 < \beta \leq \infty$. *Let* $0 < q < s < \infty$ *and* $\alpha_1 \leq 0$ *or* $0 < s \leq q < \infty$ *and* $n/q \geq \alpha_1 + n/s$. *Then*

$$F_{q,\theta}^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p}F_\beta^{s_1},$$

where

$$\theta = \begin{cases} \beta, & \text{if } 0 < s \leq q < \infty \text{ and } n/q = \alpha_1 + n/s; \\ \infty, & \text{otherwise.} \end{cases}$$

From the above corollaries and the fact that $\dot{K}_q^{\alpha,r}F_2^0 = \dot{K}_q^{\alpha,r}$ for $1 < r, q < \infty$ and $-\frac{n}{q} < \alpha < n - \frac{n}{q}$, see [19] we obtain the following embeddings between Herz and Triebel-Lizorkin spaces

$$\dot{K}_q^{\alpha_2,r} \hookrightarrow F_{s,\beta}^{s_1},$$

if $n/s - s_1 = n/q + \alpha_2$, $1 < r \leq s < \infty$, $0 < \beta \leq \infty$, and

$$1 < q < s < \infty \quad \text{and} \quad 0 \leq \alpha_2 < n - \frac{n}{q}$$

or

$$1 < s \leq q < \infty \quad \text{and} \quad \frac{n}{s} - \frac{n}{q} < \alpha_2 < n - \frac{n}{q}$$

or

$$1 < s \leq q < \infty, \alpha_2 = \frac{n}{s} - \frac{n}{q} \quad \text{and} \quad \beta = 2.$$

Again we obtain

$$F_{q,\theta}^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p}$$

holds if $n/s + \alpha_1 = n/q - s_2$, $0 < \max(q, 1) < p < \infty$ (or $1 < q = p < \infty$), $0 < \theta \leq \infty$ and

$$0 < \max(q, 1) < s < \infty \quad \text{and} \quad -\frac{n}{s} < \alpha_1 \leq 0.$$

or

$$1 < s \leq q < \infty \quad \text{and} \quad -\frac{n}{s} < \alpha_1 < n/q - n/s$$

or

$$1 < s \leq q < \infty, \alpha_1 = n/q - n/s \quad \text{and} \quad \theta = 2.$$

From the Jawerth-Franke embeddings we have

$$F_{t,\infty}^{s_3} \hookrightarrow B_{q,t}^{s_2} \hookrightarrow F_{s,\beta}^{s_1},$$

if $s_1, s_2, s_3 \in \mathbb{R}$, $s_1 - n/s = s_2 - n/q = s_3 - n/t$, $0 < t < q < s < \infty$ and $0 < \beta \leq \infty$, see [16, p. 60]. Using our results, we have the following useful consequences.

Corollary 3 *Let* $s_1, s_2, s_3 \in \mathbb{R}$, $0 < s, q, t < \infty$, $s_1 - n/s = s_2 - n/q = s_3 - n/t$ *and* $0 < \beta \leq \infty$. *Then*

$$F_{t,\infty}^{s_3} \hookrightarrow \dot{K}_q^{0,s} F_\infty^{s_2} \hookrightarrow F_{s,\beta}^{s_1}, \quad 0 < t \leq q < s < \infty.$$

To prove this it is sufficient to take in Corollary 1, $r = s$ and $\alpha_2 = 0$. However the desired embeddings are an immediate consequence of the fact that

$$F_{t,\infty}^{s_3} \hookrightarrow F_{q,\infty}^{s_2} = \dot{K}_q^{0,q} F_\infty^{s_2} \hookrightarrow \dot{K}_q^{0,s} F_\infty^{s_2}.$$

Corollary 4 *Let* $s_1, s_2 \in \mathbb{R}$, $0 < s, q < \infty$, $s_1 - n/s = s_2 - n/q$ *and* $0 < \beta \leq \infty$. *Then*

$$F^{s_2}_{q,\infty} \hookrightarrow \dot{K}^{0,q}_s F^{s_1}_\beta \hookrightarrow F^{s_1}_{s,\beta}, \quad 0 < q < s < \infty.$$

To prove this it is sufficient to take in Corollary 2, $p = q$ and $\alpha_1 = 0$. Then the desired embeddings are an immediate consequence of the fact that

$$F^{s_2}_{q,\infty} \hookrightarrow \dot{K}^{0,q}_s F^{s_1}_\beta \hookrightarrow \dot{K}^{0,s}_s F^{s_1}_\beta = F^{s_1}_{s,\beta}.$$

4 Applications

In this section, we give a simple application of Theorems 3 and 4.

Theorem 5 *Let $s \in \mathbb{R}$, $0 < p, q, \beta < \infty$ and $\alpha > -n/q$. Then there exists a linear isomorphism T which maps $\dot{K}^{\alpha,p}_q F^s_\beta$ onto $\dot{K}^{\alpha,p}_q f^s_\beta$. Moreover, there is an unconditional basis in $\dot{K}^{\alpha,p}_q F^s_\beta$.*

The mapping T is generated by an appropriate wavelet system. A proof of this theorem can be found in Xu [24] for the non-homogeneous Herz-type Triebel-Lizorkin spaces and $\alpha > 0$. This result is also true for the spaces $\dot{K}^{\alpha,p}_q F^s_\beta$, with $\alpha > -n/q$. Indeed, the problem can be reduced to prove the $\dot{K}^{\alpha,p}_q f^s_\beta$-version of Lemma 3.5 in Xu [24]. Therefore we need to recall the definition of molecules.

Definition 4 Let $K, L \in \mathbb{N}_0$ and let $M > 0$. A K-times continuously differentiable function $a \in C^K(\mathbb{R}^n)$ is called $[K, L, M]$-molecule concentrated in $Q_{v,m}$, if for some $v \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$

$$|D^\alpha a(x)| \le 2^{v|\alpha|}(1 + 2^v|x - 2^{-v}m|)^{-M}, \quad \text{for} \quad 0 \le |\alpha| \le K, x \in \mathbb{R}^n \tag{16}$$

and if

$$\int_{\mathbb{R}^n} x^\alpha a(x)dx = 0, \quad \text{for} \quad 0 \le |\alpha| < L \text{ and } v \ge 1. \tag{17}$$

If the molecule a is concentrated in $Q_{v,m}$, that means if it fulfills (16) and (17), then we will denote it by a_{vm}. For $v = 0$ or $L = 0$ there are no moment conditions (17) required.

Now, we prove the $\dot{K}^{\alpha,p}_q f^s_\beta$-version of Lemma 3.5 in Xu [24].

Lemma 4 *Let $s \in \mathbb{R}$, $0 < p, q < \infty$, $0 < \beta \le \infty$ and $\alpha > -n/q$. Furthermore, let $K, L \in \mathbb{N}_0$ and let $M > 0$ with*

$$L > n\left(\frac{1}{\min(1, q, \beta)} - 1\right) - 1 - s, \quad K \text{ arbitrary and } M \text{ large enough.}$$

If a_{vm} *are* $[K, L, M]$*-molecules concentrated in* $Q_{v,m}$ *and* $\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\} \in \dot{K}_q^{\alpha,p} f_\beta^s$*, then the sum*

$$\sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} a_{vm} \tag{18}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$.

Proof We use the arguments of [16], see also [5]. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We get from the moment conditions (17) for fixed $v \in \mathbb{N}_0$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} a_{vm}(y) \varphi(y) dy \\
&= \int_{\mathbb{R}^n} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} 2^{-v(L+1)} a_{vm}(y) \Big(\varphi(y) - \sum_{|\beta|<L} (y - 2^{-v} m)^\beta \frac{D^\beta \varphi(2^{-v} m)}{\beta!} \Big) 2^{v(L+1)} dy \\
&= \sum_{i=-\infty}^{\infty} \int_{C_i} \cdots dy,
\end{aligned}$$

where $C_i = \{y \in \mathbb{R}^n : 2^{i-1} \leq |y| < 2^i\}$ for any $i \in \mathbb{Z}$. Let us estimate the sum $\sum_{i=-\infty}^{0} \cdots$. We use the Taylor expansion of φ up to order $L - 1$ with respect to the off-points $2^{-v} m$, we obtain

$$\varphi(y) - \sum_{|\beta|<L} (y - 2^{-v} m)^\beta \frac{D^\beta \varphi(2^{-v} m)}{\beta!} = \sum_{|\beta|=L} (y - 2^{-v} m)^\beta \frac{D^\beta \varphi(\xi)}{\beta!},$$

with ξ on the line segment joining y and $2^{-v} m$. Since $1 + |y| \leq (1 + |\xi|) \left(1 + \left|y - 2^{-v} m\right|\right)$, we estimate

$$\begin{aligned}
\Big| \sum_{|\beta|=L} (y - 2^{-v} m)^\beta \frac{D^\beta \varphi(\xi)}{\beta!} \Big| &\leq \left(1 + \left|y - 2^{-v} m\right|\right)^L \sum_{|\beta|=L} \frac{|D^\beta \varphi(\xi)|}{\beta!} \\
&\leq \left(1 + \left|y - 2^{-v} m\right|\right)^L (1 + |\xi|)^{-S} \|\varphi\|_{S,L} \\
&\leq c \,(1 + |y|)^{-S} \left(1 + \left|y - 2^{-v} m\right|\right)^{L+S},
\end{aligned}$$

where $S > 0$ is at our disposal. Let $0 < t < \min(1, q) = 1 + q - \frac{q}{\min(1,q)}$ and $h = s + \frac{n}{q}(t - 1)$ be such that $n(1 - \frac{1}{\min(1,q)}) + s > h > -1 - L$. Since a_{vm} are $[K, L, M]$-molecules, then $2^{-v(L+1)} |a_{vm}(y)| \leq 2^{hv} 2^{-v(L+1+h)} \left(1 + 2^v \left|y - 2^{-v} m\right|\right)^{-M}$. Therefore, The sum $\sum_{i=-\infty}^{0} \cdots$ can be estimated by

$$c\,2^{-v(L+1+h)}\sum_{i=-\infty}^{0}\int_{C_i}\sum_{m\in\mathbb{Z}^n}2^{hv}\left|\lambda_{v,m}\right|\left(1+2^v\left|y-2^{-v}m\right|\right)^{L+S-M}(1+|y|)^{-S}dy. \tag{19}$$

Since M can be taken large enough, by Lemma 4 in [5] we obtain

$$\sum_{m\in\mathbb{Z}^n}\left|\lambda_{v,m}\right|\left(1+2^v\left|y-2^{-v}m\right|\right)^{L+S-M}\leq c\mathcal{M}\Big(\sum_{m\in\mathbb{Z}^n}\left|\lambda_{v,m}\right|\chi_{v,m}\Big)(y)$$

for any $y\in C_i\cap Q_{v,l}$ with $l\in\mathbb{Z}^n$. We split S into $R+T$ with $R+\alpha<0$ and T large enough such that $T>\max(-R,\frac{n(q-t)}{q})$. Then (19) is bounded by

$$c\,2^{-v(L+1+h)}\sum_{i=-\infty}^{0}2^{-iR}\int_{C_i}\mathcal{M}\Big(\sum_{m\in\mathbb{Z}^n}2^{vh}\left|\lambda_{v,m}\right|\chi_{v,m}\Big)(y)(1+|y|)^{-T}dy.$$

Since we have in addition the factor $(1+|y|)^{-T}$, it follows by Hölder's inequality that this expression is bounded by

$$\begin{aligned}
&c\,2^{-v(L+1+h)}\sum_{i=-\infty}^{0}2^{-iR}\Big\|\mathcal{M}\Big(\sum_{m\in\mathbb{Z}^n}2^{hv}\left|\lambda_{v,m}\right|\chi_{v,m}\Big)\chi_i\Big\|_{q/t}\\
&\leq c\,2^{-v(L+1+h)}\sum_{i=-\infty}^{0}2^{-i(\alpha+R)}\Big\|\sum_{m\in\mathbb{Z}^n}2^{hv}\left|\lambda_{v,m}\right|\chi_{v,m}|\dot{K}_{q/t}^{\alpha,\infty}\Big\|\\
&\leq c\,2^{-v(L+1+h)}\Big\|\lambda|\dot{K}_{q/t}^{\alpha,p}f_\infty^h\Big\|,
\end{aligned}$$

where the first inequality follows by the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$ on $\dot{K}_{q/t}^{\alpha,\infty}$. Using a combination of the arguments used above, the sum $\sum_{i=1}^{\infty}\cdots$ can be estimated by $c\,2^{-v(L+1+h)}\big\|\lambda|\dot{K}_{q/t}^{\alpha,p}f_\infty^h\big\|$. Since $L+1+h>0$, the convergence of (18) is now clear by the embeddings

$$\dot{K}_q^{\alpha,p}f_\infty^s\hookrightarrow\dot{K}_{q/t}^{\alpha,p}f_\infty^h,$$

see Theorem 3. The proof is completed. □

Let w denote a positive, locally integrable function and $0<p<\infty$. Then the weighted Lebesgue space $L^p(\mathbb{R}^n,w)$ contains all measurable functions such that

$$\big\|f\mid L^p(\mathbb{R}^n,w)\big\|=\Big(\int_{\mathbb{R}^n}|f(x)|^p\,w(x)dx\Big)^{1/p}<\infty.$$

For $\varrho\in[1,\infty)$ we denote by $\mathcal{A}_\varrho$ the Muckenhoupt class of weights, and $\mathcal{A}_\infty=\cup_{\varrho\geq1}\mathcal{A}_\varrho$. We refer to [4] for the general properties of these classes. Let $w\in\mathcal{A}_\infty$,

$s \in \mathbb{R}$, $0 < \beta \leq \infty$ and $0 < p < \infty$. We define weighted Triebel-Lizorkin spaces $F^s_{p,\beta}(\mathbb{R}^n, w)$ to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\| f \mid F^s_{p,\beta}(\mathbb{R}^n, w) \big\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} \left| \mathcal{F}^{-1}\varphi_j * f \right|^{\beta} \Big)^{1/\beta} \mid L^p(\mathbb{R}^n, w) \Big\|$$

is finite. In the limiting case $\beta = \infty$ the usual modification is required. The spaces $F^s_{p,\beta}(\mathbb{R}^n, w) = F^s_{p,\beta}(w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_{j \in \mathbb{N}_0}$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, q \geq 1$), and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{p,\beta}(w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Moreover, for $w \equiv 1 \in \mathcal{A}_\infty$ we obtain the usual (unweighted) Triebel-Lizorkin spaces. Let w_γ be a power weight, i.e., $w_\gamma(x) = |x|^\gamma$ with $\gamma > -n$. Then in view of the fact that $L^p = \dot{K}^{0,p}_p$, we have

$$\big\| f \mid F^s_{p,\beta}(w_\gamma) \big\| \approx \big\| f \mid \dot{K}^{\frac{\gamma}{p},p}_p F^s_\beta \big\|.$$

Applying Corollary 4 in some particular cases yields the following embeddings.

Corollary 5 *Let $s_1, s_2 \in \mathbb{R}$, $0 < q < s < \infty$, $0 < \beta \leq \infty$ and $w_{\gamma_1}(x) = |x|^{\gamma_1}$, $w_{\gamma_2}(x) = |x|^{\gamma_2}$, with $\gamma_1 > -n$ and $\gamma_2 > -n$. We suppose that*

$$s_1 - \frac{n + \gamma_1}{s} = s_2 - \frac{n + \gamma_2}{q}$$

and

$$\gamma_2 / q \geq \gamma_1 / s.$$

Then

$$F^{s_2}_{q,\infty}(w_{\gamma_2}) \hookrightarrow F^{s_1}_{s,\beta}(w_{\gamma_1}).$$

Remark 4 We refer the reader to the recent paper [6] for further results about Sobolev embeddings for weighted spaces of Besov type where the weight belongs to some Muckenhoupt $\mathcal{A}_\varrho$ class. Notice that these results are given in [12, Theorem 1.2] but under the restrictions $1 < q < s < \infty$, $1 \leq \beta \leq \infty$.

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Order Sharp Estimates for Monotone Operators on Orlicz–Lorentz Classes

Mikhail L. Goldman

Abstract We consider the monotone operator P, which maps Orlicz-Lorentz class $\Lambda_{\Phi,v}$ into some ideal space $Y = Y(R_+)$. Orlicz-Lorentz class is determined as the cone of Lebesgue-measurable functions on $R_+ = (0, \infty)$ having the decreasing rearrangements that belong to weighted Orlicz space $L_{\Phi,v}$ under some general assumptions concerning properties of functions Φ and v. We prove the reduction theorems allowing reducing the estimates of the norm of operator $P : \Lambda_{\Phi,v} \to Y$ to the estimates for its restriction on some cone of nonnegative step-functions in $L_{\Phi,v}$. Application of these results to identical operator mapping $\Lambda_{\Phi,v}$ into the weighted Lebesgue space $Y = L_1(R_+; g)$ gives the sharp description of the associate space for $\Lambda_{\Phi,v}$. The main results of this paper were announced in [20]. They develop the results of our paper [19] related to the case of N-functions.

Keywords Monotone operators · Orlic–Lorentz classes

1 Some Properties of General Weighted Orlicz Spaces

This section contains the description of needed general properties of weighted Orlicz spaces. Some of them (not all) are presented in different forms in the literature; see for example the books of Krasnoselskii and Rutickii [1], Maligranda [2], Krein et al. [3], and Bennett and Sharpley [11].

Definition 1 We denote as Θ *a* class of functions $\Phi : [0, \infty) \to [0, \infty]$ with the following properties: $\Phi(0) = 0$; Φ is increasing and left continuous on R_+, $\Phi(+\infty) = \infty$; Φ is neither identically zero nor identically infinite on R_+.

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For $\Phi \in \Theta$ we introduce

$$t_0 = \sup\{t \in [0,\infty) :\ \Phi(t) = 0\}; \tag{1}$$

$$t_\infty = \inf\{t \in R_+ :\ \Phi(t) = \infty\} \tag{2}$$

($t_\infty = \infty$ is assumed if $\Phi(t) < \infty$, $t \in R_+$). Then,

$$t_0 \in [0,\infty); \quad t_\infty \in (0,\infty]; \quad t_0 \leqslant t_\infty, \tag{3}$$

$$\Phi(t) = 0, \quad t \in [0, t_0], \quad \Phi(t) = \infty, \quad t > t_\infty \tag{4}$$

(the last in the case $t_\infty < \infty$).

Everywhere below we assume that

$$\Phi \in \Theta, \quad v \in M, \quad v > 0 \quad \text{almost everywhere in } R_+. \tag{5}$$

Here, $M = M(R_+)$ is the set of all Lebesgue-measurable functions on R_+. For $\lambda > 0$, $f \in M$ we denote

$$J_\lambda(f) := \int_0^\infty \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx, \tag{6}$$

$$\|f\|_{\Phi,v} = \inf\{\lambda > 0 : J_\lambda(f) \leqslant 1\}. \tag{7}$$

Orlicz space $L_{\Phi,v}$ is defined as the set of functions $f \in M : \|f\|_{\Phi,v} < \infty$.

Note that general concept of Orlicz–Lorentz spaces was developed by Kaminska and Raynaud [12]. In this article there is a general definition of Orlicz-Lorentz spaces, even with two weights, generated by an increasing function Φ. The necessary and sufficient conditions are discussed there for the Minkowski functional to be a norm, quasi-norm or the space to be linear.

The goal of this Section is to describe some needed general properties of Orlicz spaces $L_{\Phi,v}$. In particular, we would like to answer the following question. Let $c \in R_+$; $f_1 \in M$, $f_2 \in L_{\Phi,v}$. What are the conditions on $\Phi \in \Theta$ such that the estimate

$$J_\lambda(f_1) \leqslant c J_\lambda(f_2), \quad \lambda > d\|f_2\|_{\Phi,v}, \tag{8}$$

implies that $f_1 \in L_{\Phi,v}$, and

$$\|f_1\|_{\Phi,v} \leqslant d\|f_2\|_{\Phi,v} \tag{9}$$

with some constant $d = d(c) \in R_+$ not depending of f_1, f_2.

Remark 1 Let $\varPhi \in \varTheta$, $c = d = 1$ in the estimate (8). Then (9) is valid with $d = 1$. Indeed, we have $J_\lambda(f_2) \leqslant 1$ for every $\lambda \geqslant \|f_2\|_{\varPhi,v}$, so that (8) $\Rightarrow J_\lambda(f_1) \leqslant 1$. Therefore, $\lambda \geqslant \|f_1\|_{\varPhi,v}$. Thus, (9) follows with $d = 1$. So we have $d = d(1) = 1$ in (8), (9).

Our nearest considerations will be devoted to the justification of this estimate for $c \in (0, 1)$, which makes possible to obtain (9) with some $d \in (0, 1)$. To consider the case $c \in (1, \infty)$ we need some additional conditions on function $\varPhi \in \varTheta$.

For $c \in (0, 1)$ we assume that $t_0 = 0$; $t_\infty = \infty$ in (1), and in (2). Let us denote

$$d(c) = \inf\{d \in (0, 1] : \varPhi(dt) \geqslant c\varPhi(t),\ t \in (0, \infty)\}, \quad c \in (0, 1). \tag{10}$$

For $c \in (1, \infty)$ we assume that

$$t_0 t_\infty^{-1} = 0. \tag{11}$$

It means that at least one of the conditions $t_0 = 0$; $t_\infty = \infty$ is fulfilled. We denote by

$$d(c) = \inf\left\{d > 1 : \varPhi(dt) \geqslant c\varPhi(t),\ t \in \left(t_0, d^{-1}t_\infty\right)\right\}, \quad c \in (1, \infty) \tag{12}$$

(under assumption (11), we have $t_0 < d^{-1}t_\infty$ for any $d > 1$). It is clear that

$$c \in (0, 1] \Rightarrow d(c) \in [0, 1]; \quad c \in (1, \infty) \Rightarrow d(c) \in [1, \infty].$$

For $c \in (1, \infty)$ we denote by

$$\varTheta_c = \{\varPhi \in \varTheta : d(c) < \infty\}. \tag{13}$$

Theorem 1 *Let $\varPhi$ and v to satisfy the conditions (5), and $c \in R_+$. If $c \in (0, 1)$ we require that $t_0 = 0$; $t_\infty = \infty$ in (1), (2); if $c \in (1, \infty)$ then (11), and the condition $\varPhi \in \varTheta_c$ have to be fulfilled. Let $d(1) = 1$, and $d(c)$ being determined by (10), (12) for $c \neq 1$. Then the inequality,*

$$J_\lambda(f_1) \leqslant cJ_\lambda(f_2), \quad \lambda > d(c)\|f_2\|_{\varPhi,v}, \tag{14}$$

for functions $f_1 \in M$, $f_2 \in L_{\varPhi,v}$ implies

$$f_1 \in L_{\varPhi,v}, \quad \|f_1\|_{\varPhi,v} \leqslant d(c)\|f_2\|_{\varPhi,v}. \tag{15}$$

Corollary 1 *Let $0 < c_1 \leqslant c_2 < \infty$; and the conditions (5) and (11) be fulfilled. Moreover, if $c_0 = \min\left\{c_1^{-1}, c_2\right\} \in (0, 1)$, we require that $t_0 = 0$; $t_\infty = \infty$; if $c = \max\left\{c_1^{-1}, c_2\right\} > 1$, then $\varPhi \in \varTheta_c$ is assumed. If*

$$J_\lambda(f_2) \leqslant c_1 J_\lambda(f_1) \leqslant c_2 J_\lambda(f_2), \tag{16}$$

for every $\lambda > 0$*, then*

$$f_1 \in L_{\Phi,v} \Leftrightarrow f_2 \in L_{\Phi,v}; \quad d_1 \| f_1 \|_{\Phi,v} \leqslant \| f_2 \|_{\Phi,v} \leqslant d_2 \| f_1 \|_{\Phi,v}, \tag{17}$$

where

$$d_1 = d\left(c_1^{-1}\right)^{-1}, \quad d_2 = d\left(c_2\right). \tag{18}$$

see (10), (12).

We need some lemmas for the proof of Theorem 1.
Let $f \in L_{\Phi,v}$, $f \neq 0$. For $c \in R_+$ we define

$$\Lambda_f(c) = \{\lambda > 0 : \quad cJ_\lambda(f) \leqslant 1\}. \tag{19}$$

It follows from (6), and from the properties of $\Phi \in \Theta$ that $J_\lambda(f)$ decreases, and it is right continuous as function of λ. Therefore,

$$\Lambda_f(c) \neq \emptyset \Rightarrow \Lambda_f(c) = \left[\lambda_f(c), \infty\right), \quad \lambda_f(c) = \inf \Lambda_f(c). \tag{20}$$

We have for $c \in (0, 1]$

$$\Lambda_f(c) \supset \Lambda_f(1) = \{\lambda > 0 : \quad J_\lambda(f) \leqslant 1\} = \left[\| f \|_{\Phi,v}, \infty\right), \tag{21}$$

so that $\Lambda_f(c) \neq \emptyset$. The following lemma gives more general nonempty — conditions for $\Lambda_f(c)$.

Lemma 1 *Let the conditions* (5) *be fulfilled, let* $f \in L_{\Phi,v}$*,* $f \neq 0$*. Then, the following conclusions hold:*

(1) if $\Phi(+0) = 0$*, then* $\Lambda_f(c) \neq \emptyset$ *for every* $c \in R_+$*;*
(2) if $\Phi(+0) > 0$*, then*

$$c > \left[\Phi(+0) \int_{E(f)} v dx\right]^{-1} \Rightarrow \Lambda_f(c) = \emptyset, \tag{22}$$

$$c < \left[\Phi(+0) \int_{E(f)} v dx\right]^{-1} \Rightarrow \Lambda_f(c) \neq \emptyset, \tag{23}$$

where

$$E(f) = \{x \in R_+ : 0 < |f(x)| < \infty\}.$$

Remark 2 In the conditions of Lemma 1 we have,

$$0 \leqslant J_\lambda (f) \leqslant 1, \quad \lambda \in \left[\| f \|_{\Phi,v}, \infty \right), \quad J_\lambda (f) \downarrow (\lambda \uparrow) . \tag{24}$$

Therefore, the following limit exists

$$0 \leqslant J_\infty (f) = \lim_{\lambda \to +\infty} J_\lambda (f) \leqslant 1. \tag{25}$$

In the proof of this lemma we particularly establish that

$$0 \leqslant J_\infty (f) = \Phi (+0) \int\limits_{E(f)} v dx \leqslant 1. \tag{26}$$

Moreover, we will show that $\mu (E (f)) = \infty$, and

$$\Phi (+0) > 0 \Rightarrow 0 < \int\limits_{E(f)} v dx \leqslant \Phi (+0)^{-1} , \tag{27}$$

because $v > 0$ almost everywhere.

Proof (*of Lemma* 1)
1. Denote

$$E_0 (f) = \{x \in R_+ : \ |f (x)| = 0\} , \quad E_\infty (f) = \{x \in R_+ : \ |f (x)| = \infty\} .$$

Then,

$$R_+ = E_0 (f) \cup E (f) \cup E_\infty (f) . \tag{28}$$

For $\lambda \in \left[\| f \|_{\Phi,v}, \infty \right)$ we have,

$$J_\lambda (f) := \int\limits_0^\infty \Phi \left(\lambda^{-1} |f (x)|\right) v (x) dx \leqslant 1. \tag{29}$$

It means that almost everywhere

$$\Phi \left(\lambda^{-1} |f (x)|\right) v (x) < \infty \Rightarrow \Phi \left(\lambda^{-1} |f (x)|\right) < \infty \Rightarrow |f (x)| < \infty. \tag{30}$$

In the first implication, we take into account that $v (x) > 0$ almost everywhere, and in the second one, we use the condition $\Phi (+\infty) = \infty$. From (30), it follows that

$$\mu\left(E_{\infty}(f)\right)=0. \tag{31}$$

Moreover, $f \neq 0 \Rightarrow \mu\left(E_0(f)\right)<\infty$.

From here, and from (28) we see that $\mu(E(f))=\infty$, and

$$J_{\lambda}(f)=\int\limits_{E_0(f)} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\, d x+\int\limits_{E(f)} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\, d x. \tag{32}$$

For $x \in E_0(f)$ we have $\lambda^{-1}|f(x)|=0 \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right)=0$ (recall that $\Phi(0)=0$).

Therefore,

$$J_{\lambda}(f)=\int\limits_{E(f)} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\, d x. \tag{33}$$

We see that

$$\lambda \in\left[\|f\|_{\Phi, v}, \infty\right) \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) v(x) \leqslant \Phi\left(\|f\|_{\Phi, v}^{-1}|f(x)|\right) v(x) \in L_1\left(R_{+}\right),$$

and $\lambda \rightarrow+\infty$ implies

$$0<\lambda^{-1}|f(x)| \rightarrow 0 \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) v(x) \rightarrow \Phi(+0) v(x).$$

Therefore, we have by Lebesgue majored convergence theorem

$$J_{\infty}(f)=\lim _{\lambda \rightarrow+\infty} J_{\lambda}(f)=\Phi(+0) \int\limits_{E(f)} v d x.$$

It proves (26).

2. If $\Phi(+0)=0$ then, $\lim\limits_{\lambda \rightarrow+\infty} J_{\lambda}(f)=0$, so that for every $c \in R_{+}$ we can find $\lambda(c) \in R_{+}$, with $J_{\lambda}(f) \leqslant c^{-1}, \lambda \geqslant \lambda(c)$. It means that $\Lambda_f(c) \neq \emptyset$.

3. Now, let $\Phi(+0)>0$. Note that $J_{\lambda}(f)$ decreases in λ, therefore we have for every $\lambda>0$ by (26) and (22),

$$c J_{\lambda}(f) \geqslant c J_{\infty}(f)=c \Phi(+0) \int\limits_{E(f)} v d x>1 \Rightarrow \Lambda_f(c)=\emptyset.$$

By the conditions (23) with $\lambda \rightarrow+\infty$, we have

$$\lim _{\lambda \rightarrow+\infty} c J_{\lambda}(f)=c \Phi(+0) \int\limits_{E(f)} v d x<1,$$

so that

$$\exists \lambda(c) > 0 : \ cJ_\lambda(f) \leqslant 1, \quad \lambda \geqslant \lambda(c) \Rightarrow \Lambda_f(c) \neq \emptyset.$$

Remark 3 Let $c \in (0, 1]$ in the conditions of Lemma 1. Then, $\Lambda_f(c) \neq \emptyset$. Indeed, by (26),

$$\left[\Phi(+0) \int\limits_{E(f)} v dx\right]^{-1} \geqslant 1,$$

so that the assertions (23) are fulfilled for $c \in (0, 1)$. If $c = 1$ we also obtain $\Lambda_f(c) \neq \emptyset$ (see Remark 1).

Remark 4 Under assumptions of Lemma 1 let

$$\Phi(+0) > 0; c = \left[\Phi(+0) \int\limits_{E(f)} v dx\right]^{-1} \in (1, \infty) \tag{34}$$

(see (25) and (26)). Then both variants of the answer are possible. Let us give the examples.

1. If $\Phi(t) > \Phi(+0)$, $t \in R_+$ then we have $E(f_0) = E$; for function $f_0 = \chi_E$ where $E \subset R_+$, $0 < \mu(E) < \infty$, and therefore

$$cJ_\lambda(f_0) = c\Phi\left(\lambda^{-1}\right) \int\limits_E v(x)\, dx > c\Phi(+0) \int\limits_E v(x)\, dx = 1.$$

It means that $\Lambda_{f_0}(c) = \emptyset$.

2. Let $\exists \delta > 0 : \ \Phi(t) = \Phi(+0), t \in (0, \delta)$.

Then we have $\Lambda_f(c) \neq \emptyset$ for every bounded function f. Indeed, let $|f(x)| \leqslant M$ almost everywhere. Then, $\lambda > M\delta^{-1} \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) \leqslant \Phi\left(\lambda^{-1}M\right) = \Phi(+0)$,

$$cJ_\lambda(f) \leqslant c\Phi(+0) \int\limits_{E(f)} v dx = 1 \Rightarrow \Lambda_f(c) \supset \left(M\delta^{-1}, \infty\right).$$

Let the conditions (5) be fulfilled, and $f \in L_{\Phi,v}$, $f \neq 0$. Denote

$$\lambda(f; d) = \inf\{\lambda > 0 : \ J_\lambda(df) < \infty\}. \tag{35}$$

We have

$$\lambda \in \left[d\|f\|_{\Phi,v}, \infty\right) \Rightarrow J_\lambda(df) \leqslant 1, \tag{36}$$

so that

$$\lambda(f; d) \leqslant d\|f\|_{\Phi,v} \tag{37}$$

Lemma 2 *Let the conditions* (5) *be fulfilled, and* $c \in (0,1)$*;* $t_0 = 0$*,* $t_\infty = \infty$ *in (1), (2). Let* $d(c)$ *be defined by (10). Then the following estimate holds for function* $f \in L_{\Phi,v}$*,* $f \neq 0$

$$cJ_\lambda(f) \leqslant J_\lambda(df), \quad \lambda \in [\lambda(f;d), \infty). \tag{38}$$

with any $d > d(c)$.

Proof We use formula (33). For $x \in E(f)$, $d > d(c)$ we have by definition (10)

$$0 < \lambda^{-1}|f(x)| < \infty \Rightarrow c\Phi\left(\lambda^{-1}|f(x)|\right) \leqslant \Phi\left(\lambda^{-1}|df(x)|\right),$$

so that

$$cJ_\lambda(f) = \int\limits_{E(f)} c\Phi\left(\lambda^{-1}|f(x)|\right)v(x)\,dx \leqslant \int\limits_{E(f)} \Phi\left(\lambda^{-1}|df(x)|\right)v(x)\,dx \leqslant J_\lambda(df).$$

Corollary 2 *From (36)–(38), it follows that* $\lambda \in \left[d\|f\|_{\Phi,v}, \infty\right) \Rightarrow cJ_\lambda(f) \leqslant 1$*, so that*

$$\Lambda_f(c) \supset \left[d\|f\|_{\Phi,v}, \infty\right) \neq \emptyset, \quad \forall d > d(c).$$

Thus,

$$\Lambda_f(c) \supset \left[d(c)\|f\|_{\Phi,v}, \infty\right). \tag{39}$$

Lemma 3 *Let the conditions (5) and (11) be fulfilled, and* $c \in (1,\infty)$*,* $d(c)$ *being defined by (12) and* $\Phi \in \Theta_c$*. Then, estimate (38) holds for function* $f \in L_{\Phi,v}$*,* $f \neq 0$*, with any* $d > d(c)$*.*

Proof For $\lambda > 0$, $d > d(c)$ we define

$$G_0(f) \equiv G_0(f;\lambda) = \left\{x \in R_+ : \lambda^{-1}|f(x)| \leqslant t_0\right\}, \tag{40}$$

$$G(f) \equiv G(f;\lambda) = \left\{x \in R_+ : t_0 < \lambda^{-1}|f(x)| < \infty\right\}, \quad t_\infty = \infty; \tag{41}$$

$$G(f) \equiv G(f;\lambda,d) = \left\{x \in R_+ : t_0 < \lambda^{-1}|f(x)| \leqslant d^{-1}t_\infty\right\}, \quad t_\infty < \infty; \tag{42}$$

$$G_\infty(f) = \{x \in R_+ : |f(x)| = \infty\}, \quad t_\infty = \infty; \tag{43}$$

$$G_\infty(f) \equiv G_\infty(f;\lambda,d) = \left\{x \in R_+ : \lambda^{-1}|f(x)| > d^{-1}t_\infty\right\}, \quad t_\infty < \infty. \tag{44}$$

Then,

$$R_+ = G_0(f) \cup G(f) \cup G_\infty(f). \tag{45}$$

We have according to (40) and (4),

$$x \in G_0(f) \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) = 0 \Rightarrow \int\limits_{G_0(f)} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx = 0. \tag{46}$$

Further, $\lambda > \lambda(f; d)$ implies $J_\lambda(df) < \infty$. Therefore, almost everywhere

$$\Phi\left(\lambda^{-1}|df(x)|\right) v(x) < \infty \Rightarrow \Phi\left(\lambda^{-1}|df(x)|\right) < \infty. \tag{47}$$

Here we take into account that $v(x) > 0$ almost everywhere. Now, if $t_\infty = \infty$ then $\Phi(+\infty) = \infty$, and if $t_\infty < \infty$ then $\Phi(t) = \infty$, $t > t_\infty$. Therefore, in both cases

$$x \in G_\infty(f) \Rightarrow \Phi\left(\lambda^{-1}|df(x)|\right) = \infty. \tag{48}$$

From here, and from (47), it follows that

$$\mu(G_\infty(f)) = 0 \Rightarrow \int\limits_{G_\infty(f)} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx = 0. \tag{49}$$

Now, (45), (46), and (49) imply

$$J_\lambda(f) = \int\limits_{G(f)} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx. \tag{50}$$

For $x \in G(f)$ we have $t = \lambda^{-1}|f(x)| \in (t_0, \infty)$, if $t_\infty = \infty$, or $t \in \left(t_0, d^{-1}t_\infty\right]$ if $t_\infty < \infty$. By (12) we have for $d > d(c)$

$$c\Phi(t) \leqslant \Phi(dt), \quad t \in \left(t_0, d^{-1}t_\infty\right). \tag{51}$$

If $t_\infty < \infty$, this inequality is extended onto $\left(t_0, d^{-1}t_\infty\right]$ by the limiting passage with $t \to d^{-1}t_\infty$ (let us recall that Φ is left continuous). Therefore,

$$c\Phi\left(\lambda^{-1}|f(x)|\right) \leqslant \Phi\left(\lambda^{-1}|df(x)|\right), \quad x \in G(f), \tag{52}$$

so that,

$$cJ_\lambda(f) = \int\limits_{G(f)} c\Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx \leqslant \int\limits_{G(f)} \Phi\left(\lambda^{-1}|df(x)|\right) v(x)\,dx \leqslant J_\lambda(df).$$

This proves estimate (38).

Proof (*of Theorem* 1) In the assumptions of this theorem, Remark 1 exhausts the case = 1. For function $f = f_2 \in L_{\Phi,v}$, $f_2 \neq 0$, we can apply Lemma 2 with $c \in (0, 1)$, or Lemma 3 with $c \in (1, \infty)$. In both cases we obtain (38) for $f = f_2$. It is true in particular for all $\lambda \in \left[d\|f_2\|_{\Phi,v}, \infty\right)$ because of (37). For such values of λ, we have inequality $J_\lambda(df_2) \leqslant 1$. Therefore, by (14), and (38),

$$J_\lambda(f_1) \leqslant cJ_\lambda(f_2) \leqslant J_\lambda(df_2) \leqslant 1, \quad \lambda \in \left[d\|f_2\|_{\Phi,v}, \infty\right).$$

It means that,

$$\|f_1\|_{\Phi,v} \leqslant d\|f_2\|_{\Phi,v}, \quad d > d(c).$$

Thus, the relations (15) follow.

Example 1 If $\Phi(t) = t^\varepsilon$, $t \in [0, \infty)$, $\varepsilon > 0$, then

$$t_0 = 0, \quad t_\infty = \infty, \quad d(c) = c^{1/\varepsilon}, \quad c \in R_+.$$

Example 2 Let $\Phi(t) = e^t - 1$, $t \in [0, \infty)$. Then,

$$t_0 = 0, \quad t_\infty = \infty, \quad c > 1 \Rightarrow d(c) = c.$$

Example 3 Let $\Phi(t) = \ln^\gamma(t+1)$, $t \in [0, \infty)$, $\gamma > 0$. Then, $t_0 = 0$, $t_\infty = \infty$, $d(c) = \infty$ for every $c > 1$. Indeed, if $c > 1$, the inequality $\ln^\gamma(dt+1) \geqslant c\ln^\gamma(t+1)$ fails for every $d \in R_+$ when $t \in R_+$ is big enough, because

$$\lim_{t\to+\infty}\left[\frac{\ln^\gamma(dt+1)}{\ln^\gamma(t+1)}\right] = 1.$$

Example 4 Let the condition (11) be fulfilled, let $\varepsilon > 0$, and $\Phi(t)\,t^{-\varepsilon} \uparrow$ on (t_0, t_∞). Then,

$$c > 1 \Rightarrow d(c) \leqslant c^{1/\varepsilon}. \tag{53}$$

Indeed, for every $t \in \left(t_0, c^{-1/\varepsilon}t_\infty\right)$

$$\Phi\left(c^{1/\varepsilon}t\right) = \left(c^{1/\varepsilon}t\right)^\varepsilon\left[\Phi\left(c^{1/\varepsilon}t\right)\left(c^{1/\varepsilon}t\right)^{-\varepsilon}\right] \geqslant \left(c^{1/\varepsilon}t\right)^\varepsilon\left[\Phi(t)\,t^{-\varepsilon}\right] = c\Phi(t).$$

It means that $d(c) \leqslant c^{1/\varepsilon}$.

Example 5 Let the condition (11) be fulfilled, let $p \in (0, 1]$, and Φ be p-convex on $[t_0, t_\infty)$, that is for $\alpha, \beta \in (0, 1]$, $\alpha^p + \beta^p = 1$ the inequality holds

$$\Phi(\alpha t + \beta\tau) \leqslant \alpha^p\Phi(t) + \beta^p\Phi(\tau), \quad t, \tau \in [t_0, t_\infty). \tag{54}$$

If $t_\infty < \infty$, then by passage to the limit this inequality is extended on $[t_0, t_\infty]$. Thus, we have,

$$c > 1 \Rightarrow d(c) \leqslant c^{1/p}. \tag{55}$$

Indeed, (54) implies $\Phi(t)\, t^{-p} \uparrow$ on $[t_0, t_\infty)$, and the result of Example 4 is applicable here.

Example 6 (Young function) Let $\Phi : [0, \infty) \to [0, \infty]$ be the so-called Young function that is,

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \tag{56}$$

where $\varphi : [0, \infty) \to [0, \infty]$ is the decreasing and left-continuous function, and $\varphi(0) = 0$, φ is neither identically zero, nor identically infinity on $(0, \infty)$. Then, $\Phi \in \Theta$, and t_0, t_∞, being introduced for Φ by (1) and (2), are the same as their analogues for φ. We assume that (11) is satisfied. Function Φ is convex on $[t_0, t_\infty)$ because $0 \leqslant \varphi \uparrow$. Thus, we can apply the conclusions of Example 5 with $p = 1$. In particular, $c > 1 \Rightarrow d(c) \leqslant c$.

Theorem 2 *Let the conditions* (5) *and (11) be fulfilled, and* Φ *being* p*-convex on* $[t_0, t_\infty)$ *with some* $p \in (0, 1]$. *Then, the following conclusions hold.*

(1) The triangle inequality takes place in $L_{\Phi,v}$*: if* $f, g \in L_{\Phi,v}$ *then* $f + g \in L_{\Phi,v}$*, and*

$$\|f + g\|_{\Phi,v} \leqslant \left(\|f\|^p_{\Phi,v} + \|g\|^p_{\Phi,v}\right)^{1/p}. \tag{57}$$

(2) The quantity $\|f\|_{\Phi,v}$ *is monotone quasi-norm (norm, if* $p = 1$*):*

$$f \in M, \quad |f| \leqslant g \in L_{\Phi,v} \Rightarrow f \in L_{\Phi,v}, \quad \|f\|_{\Phi,v} \leqslant \|g\|_{\Phi,v}, \tag{58}$$

that has Fatou property:

$$f_n \in M, \quad 0 \leqslant f_n \uparrow f \Rightarrow \|f\|_{\Phi,v} = \lim_{n\to\infty} \|f_n\|_{\Phi,v}. \tag{59}$$

Conclusion. *In the conditions of Theorem 2* $L_{\Phi,v}$ *forms ideal quasi-Banach space having Fatou property (Banach space if* $p = 1$*, in particular in the case of Young function* Φ*).*

Proof (of Theorem 2) **1**. Let $f, g \in L_{\Phi,v}$. Then, we have for all $\lambda \geqslant \|f\|^p_{\Phi,v}$, $\mu \geqslant \|g\|^p_{\Phi,v}$,

$$J_{\lambda^{1/p}}(f) = \int_{R_+} \Phi\left(\lambda^{-1/p} |f(x)|\right) v(x)\, dx \leqslant 1; \tag{60}$$

$$J_{\mu^{1/p}}(g) = \int\limits_{R_+} \Phi\left(\mu^{-1/p}|g(x)|\right) v(x)\, dx \leqslant 1. \tag{61}$$

Now, almost everywhere on R_+ (60), and (61) yield,

$$\Phi\left(\lambda^{-1/p}|f(x)|\right) + \Phi\left(\mu^{-1/p}|g(x)|\right) < \infty, \tag{62}$$

because $v(x) > 0$ almost everywhere on R_+. Further, for $t_\infty = \infty$ we denote

$$\tilde{E}(f) = \{x \in R_+ : |f(x)| < \infty\}, \tag{63}$$

$$\tilde{E}(g) = \{x \in R_+ : |g(x)| < \infty\}, \tag{64}$$

and for $t_\infty < \infty$ we denote

$$\tilde{E}(f) = \left\{x \in R_+ : \lambda^{-1/p}|f(x)| \leqslant t_\infty\right\}, \tag{65}$$

$$\tilde{E}(g) = \left\{x \in R_+ : \lambda^{-1/p}|g(x)| \leqslant t_\infty\right\}. \tag{66}$$

In both cases we have according to (62),

$$\Phi\left(\lambda^{-1/p}|f(x)|\right) = \infty, \quad x \in R_+ \backslash \tilde{E}(f) \Rightarrow mes\left(R_+ \backslash \tilde{E}(f)\right) = 0,$$

$$\Phi\left(\mu^{-1/p}|g(x)|\right) = \infty, \quad x \in R_+ \backslash \tilde{E}(g) \Rightarrow mes\left(R_+ \backslash \tilde{E}(g)\right) = 0.$$

Therefore,

$$mes\left(R_+ \backslash \left[\tilde{E}(f) \cap \tilde{E}(g)\right]\right) = 0, \tag{67}$$

$$J_{\lambda^{1/p}}(f) = \int\limits_{\tilde{E}(f)\cap\tilde{E}(g)} \Phi\left(\lambda^{-1/p}|f(x)|\right) v(x)\, dx, \tag{68}$$

$$J_{\mu^{1/p}}(g) = \int\limits_{\tilde{E}(f)\cap\tilde{E}(g)} \Phi\left(\mu^{-1/p}|g(x)|\right) v(x)\, dx, \tag{69}$$

$$J_{(\lambda+\mu)^{1/p}}(f+g) = \int\limits_{\tilde{E}(f)\cap\tilde{E}(g)} \Phi\left((\lambda+\mu)^{-1/p}|f(x)+g(x)|\right) v(x)\, dx. \tag{70}$$

For $\Phi \in \Theta$ the following inequality holds

$$\Phi\left((\lambda+\mu)^{-1/p}\left|f(x)+g(x)\right|\right) \leqslant \\ \leqslant \Phi\left((\lambda+\mu)^{-1/p}\left|f(x)\right|+(\lambda+\mu)^{-1/p}\left|g(x)\right|\right). \quad (71)$$

We define

$$\alpha=\lambda^{1/p}(\lambda+\mu)^{-1/p}, \quad \beta=\mu^{1/p}(\lambda+\mu)^{-1/p};$$

$$t=\lambda^{-1/p}\left|f(x)\right|, \quad \tau=\mu^{-1/p}\left|g(x)\right|.$$

In this case $\alpha^p+\beta^p=1$, and we have for $x \in \tilde{E}(f) \cap \tilde{E}(g)$

$$t, \tau \in [0,\infty), \quad t_\infty=\infty; \quad t, \tau \in [0, t_\infty], \quad t_\infty<\infty.$$

Therefore, the estimate (54) is applicable for the right-hand side of (71). As the result,

$$\Phi\left((\lambda+\mu)^{-1/p}\left|f(x)+g(x)\right|\right) \leqslant \\ \leqslant \frac{\lambda}{\lambda+\mu}\Phi\left(\lambda^{-1/p}\left|f(x)\right|\right)+\frac{\mu}{\lambda+\mu}\Phi\left(\mu^{-1/p}\left|g(x)\right|\right).$$

We integrate this inequality over the set $\tilde{E}(f) \cap \tilde{E}(g)$, and take into account formulas (68)–(70). Then,

$$J_{(\lambda+\mu)^{1/p}}(f+g) \leqslant \frac{\lambda}{\lambda+\mu} J_{\lambda^{1/p}}(f)+\frac{\mu}{\lambda+\mu} J_{\mu^{1/p}}(g). \quad (72)$$

From (72), (60), and (61), it follows that

$$J_{(\lambda+\mu)^{1/p}}(f+g) \leqslant \frac{\lambda}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}=1.$$

Thus,

$$\left\|f+g\right\|_{\Phi,v} \leqslant (\lambda+\mu)^{1/p}.$$

This inequality holds for all λ, μ, satisfying the conditions $\lambda \geqslant \left\|f\right\|_{\Phi,v}^p$, $\mu \geqslant \left\|g\right\|_{\Phi,v}^p$. Therefore, estimate (57) is valid.

2. Let us check the properties of quasi-norm.

For $c=0$ it is obvious that $J_\lambda(cf)=J_\lambda(0)=0$, $\forall \lambda>0$, so that

$$\left\|cf\right\|_{\Phi,v}=\inf\left\{\lambda>0: \; J_\lambda(cf) \leqslant 1\right\}=0=|c|\left\|f\right\|_{\Phi,v}.$$

For $c \neq 0$ we have,

$$\|cf\|_{\Phi,v} = \inf\{\lambda > 0 : J_\lambda(cf) \leqslant 1\} = \inf\left\{\lambda > 0 : J_{\lambda/|c|}(f) \leqslant 1\right\}$$
$$= \inf\left\{|c|\mu > 0 : J_\mu(f) \leqslant 1\right\} = |c|\,\|f\|_{\Phi,v}.$$

Thus, we have $\|cf\|_{\Phi,v} = |c|\,\|f\|_{\Phi,v}$ for all $\in R$.

Moreover, it is evident that $f = 0 \Rightarrow \|f\|_{\Phi,v} = 0$. Let us show the inverse. Let $\|f\|_{\Phi,v} = 0$. Then,

$$\|f\|_{\Phi,v} = \inf\{\lambda > 0 : J_\lambda(f) \leqslant 1\} = 0 \Rightarrow J_\lambda(f) \leqslant 1, \forall\lambda > 0. \tag{73}$$

Let us suppose that f is not equivalent to zero. Then,

$$\exists\varepsilon > 0, E \subset R_+ : mesE > 0; |f(x)| \geqslant \varepsilon, \quad x \in E.$$

It means that for every $\lambda > 0$

$$J_\lambda(f) \geqslant \int_E \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx \geqslant \Phi\left(\lambda^{-1}\varepsilon\right) \int_E v(x)\,dx. \tag{74}$$

We know that $v(x) > 0$ almost everywhere, and $mesE > 0$. Then, $\int_E v(x)\,dx > 0$. Moreover, $\Phi\left(\lambda^{-1}\varepsilon\right) \uparrow \infty\ (\lambda \downarrow 0)$. Thus, the right-hand side in (74) tends to $+\infty$ if $\lambda \downarrow 0$, that prevents to (73). Therefore, the above assumption fails, that is $f = 0$ almost everywhere on R_+. These assertions together with triangle inequality (57) show that the quantity $\|f\|_{\Phi,v}$ has all properties of quasi-norm (norm if $p = 1$).

3. Let us prove the property of monotonicity for quasi-norm. The increasing of function $\Phi \in \Theta$ implies that

$$|f| \leqslant g \Rightarrow J_\lambda(f) \leqslant J_\lambda(g), \quad \forall\lambda > 0.$$

We have inequality $J_\lambda(g) \leqslant 1$ when$\lambda \geqslant \|g\|_{\Phi,v}$, $g \in L_{\Phi,v}$. Then,

$$J_\lambda(f) \leqslant 1, \quad \forall\lambda \geqslant \|g\|_{\Phi,v} \Rightarrow \|f\|_{\Phi,v} \leqslant \|g\|_{\Phi,v}. \tag{75}$$

4. Now, we prove the Fatou property. Let $f_n \in M_+$, $f_n \uparrow f$. Function $\Phi \in \Theta$ is increasing and left continuous, therefore $\Phi\left(\lambda^{-1}|f_n(x)|\right) \uparrow \Phi\left(\lambda^{-1}|f(x)|\right)$ almost everywhere. We can apply B. Levy monotone convergence theorem for every $\lambda > 0$:

$$J_\lambda(f_n) = \int_{R_+} \Phi\left(\lambda^{-1}|f_n(x)|\right) v(x)\,dx \uparrow \int_{R_+} \Phi\left(\lambda^{-1}|f(x)|\right) v(x)\,dx = J_\lambda(f).$$

(this conclusion is valid as well in the case $J_\lambda(f) = \infty$). Then,

$$J_\lambda(f_n) \leqslant J_\lambda(f), \quad n \in N \Rightarrow \|f_n\|_{\Phi,v} \leqslant \|f\|_{\Phi,v}, \quad n \in N.$$

Denote

$$B_f = \sup_{n \in N} \|f_n\|_{\Phi,v} = \lim_{n \to \infty} \|f_n\|_{\Phi,v}$$

Let us show that $B_f = \|f\|_{\Phi,v}$. It is clear that $B_f \leqslant \|f\|_{\Phi,v}$. Suppose that $B_f < \|f\|_{\Phi,v}$. For any $\lambda \in \left(B_f, \|f\|_{\Phi,v}\right)$ we have

$$\lambda < \|f\|_{\Phi,v} = \inf\left\{\mu > 0 : J_\mu(f) \leqslant 1\right\} \Rightarrow J_\lambda(f) > 1.$$

At the same time, for every $n \in N$

$$\lambda > \|f_n\|_{\Phi,v} \Rightarrow J_\lambda(f_n) \leqslant 1.$$

Thus,

$$J_\lambda(f) = \lim_{n \to \infty} J_\lambda(f_n) \leqslant 1.$$

This contradiction shows that the above assumption was wrong. Thus, $B_f = \|f\|_{\Phi,v}$.

The following result is useful by the calculation of the norm of operator over Orlicz space $L_{\Phi,v}$.

Lemma 4 *Let the condition* (5) *be fulfilled. Then, the following equivalence takes place for* $f \in M$*,*

$$\|f\|_{\Phi,v} \leqslant 1 \Leftrightarrow J_1(f) = \int_0^\infty \Phi(|f(x)|) v(x)\, dx \leqslant 1. \tag{76}$$

Proof Obviously,

$$J_1(f) \leqslant 1 \Rightarrow \|f\|_{\Phi,v} \leqslant 1. \tag{77}$$

From the other side, we have

$$J_1(f) = \lim_{\lambda \downarrow 1} J_\lambda(f). \tag{78}$$

Indeed, $\lambda \downarrow 1 \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) \uparrow \Phi(|f(x)|)$ almost everywhere because of increasing and left-continuity of function $\Phi \in \Theta$. Then, by B. Levy monotone convergence theorem

$$\int_0^\infty \Phi(|f(x)|)v(x)\,dx = \lim_{\lambda\downarrow 1}\int_0^\infty \Phi\left(\lambda^{-1}|f(x)|\right)v(x)\,dx,$$

which gives (78). Consequently, if $J_1(f) > 1$, we can find $\lambda_0 > 1$, such that $J_{\lambda_0}(f) > 1$. Then, $J_\lambda(f) \leqslant 1 \Rightarrow \lambda > \lambda_0$ (because of decreasing of $J_\lambda(f)$ by λ). Therefore,

$$\|f\|_{\Phi,v} = \inf\{\lambda > 0 :\ J_\lambda(f) \leqslant 1\} \geqslant \lambda_0 > 1.$$

Finally,

$$J_1(f) > 1 \Rightarrow \|f\|_{\Phi,v} > 1.$$

Together with (77), it implies the equivalence (76).

For the completeness, we formulate the results in the case of failure of the conditions (11), namely when

$$t_0^{-1}t_\infty < \infty \Leftrightarrow 0 < t_0 \leqslant t_\infty < \infty. \tag{79}$$

Lemma 5 *In the conditions (5) the following estimates hold for function $f \in M$,*

$$t_0\|f\|_{\Phi,v} \leqslant \|f\|_{L_\infty};\quad \|f\|_{L_\infty} \leqslant t_\infty\|f\|_{\Phi,v}. \tag{80}$$

Proof Let $t_0 > 0$, $\|f\|_{L_\infty} < \infty$. Then, we have for any $\lambda \geqslant t_0^{-1}\|f\|_{L_\infty}$ that

$$|f(x)| \leqslant \|f\|_{L_\infty} \Rightarrow \Phi\left(\lambda^{-1}|f(x)|\right) \leqslant \Phi\left(\lambda^{-1}\|f\|_{L_\infty}\right) = 0,$$

almost everywhere by the property (4). Therefore, $\lambda \geqslant t_0^{-1}\|f\|_{L_\infty} \Rightarrow J_\lambda(f) = 0$, that is

$$\|f\|_{\Phi,v} = \inf\{\lambda > 0 :\ J_\lambda(f) \leqslant 1\} \leqslant t_0^{-1}\|f\|_{L_\infty}.$$

It gives the first estimate in (80). Further, let $t_\infty < \infty$, $\|f\|_{\Phi,v} < \infty$. For any $\lambda \geqslant \|f\|_{\Phi,v}$ we have $J_\lambda(f) < \infty$. Then, by analogy with the proof of (29), and (30) we obtain that $\Phi\left(\lambda^{-1}|f(x)|\right) < \infty$ almost everywhere. Thus, by (4) we conclude that $\left\{x \in R_+ :\ \lambda^{-1}|f(x)| > t_\infty\right\}$ is set of measure zero. It means that $\lambda^{-1}|f(x)| \leqslant t_\infty$ almost everywhere, and

$$J_\lambda(f) < \infty \Rightarrow \|f\|_{L_\infty} \leqslant \lambda t_\infty. \tag{81}$$

It gives the second estimate in (80).

Corollary 3 *Let the conditions* (5) *and (79) be fulfilled. Then the two-sided estimate takes place for every function* $f \in M$

$$t_0 \left\| f \right\|_{\Phi,v} \leqslant \left\| f \right\|_{L_\infty} \leqslant t_\infty \left\| f \right\|_{\Phi,v}, \tag{82}$$

showing that $L_{\Phi,v} = L_\infty$ *with the equivalence of the norms. Here* $L_\infty = L_\infty(R_+)$ *is the space of all essentially bounded functions.*

The above corollary shows that we lose the specific of Orlicz spaces in its conditions.

Nevertheless, we formulate in this case the answer on the above posed question.

Lemma 6 *Let the conditions* (5) *and (79) be fulfilled, and* $f_1 \in M$, $f_2 \in L_{\Phi,v}$. *If for every* $\lambda > \left\| f_2 \right\|_{\Phi,v}$ *we have* $J_\lambda(f_1) < \infty$, *then* $f_1 \in L_{\Phi,v}$, *and*

$$\left\| f_1 \right\|_{\Phi,v} \leqslant t_0^{-1} t_\infty \left\| f_2 \right\|_{\Phi,v}. \tag{83}$$

Proof We have $J_\lambda(f_1) < \infty$ for every $\lambda > \left\| f_2 \right\|_{\Phi,v}$ so that we obtain inequality $\left\| f_1 \right\|_{L_\infty} \leqslant t_\infty \lambda$ similarly as it was made in (81). Therefore, $\left\| f_1 \right\|_{L_\infty} \leqslant t_\infty \left\| f_2 \right\|_{\Phi,v}$. Together with the first estimate in (80), it gives (83).

2 Discrete Weighted Orlicz Spaces

2.1. Here, we consider the discrete variants of Orlicz spaces. For it, we assume that

$$\Phi \in \Theta; \quad \beta = \{\beta_m\}, \quad \beta_m \in R_+, \quad m \in Z = \{0, \pm 1, \pm 2, ...\}. \tag{84}$$

Denote

$$l_{\Phi,\beta} = \left\{ \alpha = \{\alpha_m\}, \ \alpha_m \in R : \ \left\| \alpha \right\|_{l_{\Phi,\beta}} < \infty \right\},$$

where

$$\left\| \alpha \right\|_{l_{\Phi,\beta}} := \inf\{\lambda > 0 : \ j_\lambda(\alpha) \leqslant 1\}, \quad j_\lambda(\alpha) = \sum_m \Phi\left(\lambda^{-1} |\alpha_m|\right)\beta_m. \tag{85}$$

Let us formulate some discrete analogues of the results of Sect. 1. An analogue of Theorem 1 is as follows.

Theorem 3 *Let the conditions (84) be fulfilled; let* $c \in R_+$, *and if* $c \in (0, 1)$, *then* $t_0 = 0$; $t_\infty = \infty$ *in (1), (2); if* $c \in (1, \infty)$ *the (11) is fulfilled. Let* $d(1) = 1$; $d(c)$ *is determined by (10), and (12) for* $c \neq 1$, *moreover, for* $c \in (1, \infty)$ *we assume that*

$\Phi \in \Theta_c$. Let the following estimate holds for sequences $\alpha = \{\alpha_m\}$, $\gamma = \{\gamma_m\}$, where $\gamma \in l_{\Phi,v}$:

$$j_\lambda(\alpha) \leqslant c j_\lambda(\gamma), \quad \lambda \geqslant d(c) \left\|\gamma\right\|_{l_{\Phi,\beta}}. \tag{86}$$

Then, $\alpha \in l_{\Phi,v}$, and the inequality holds

$$\left\|\alpha\right\|_{l_{\Phi,\beta}} \leqslant d(c) \left\|\gamma\right\|_{l_{\Phi,\beta}} \tag{87}$$

Corollary 4 *Let the conditions (84) and (11) be fulfilled, let $0 < c_1 \leqslant c_2 < \infty$, and $\alpha = \{\alpha_m\}$, $\gamma = \{\gamma_m\}$. Moreover, if $c_0 = \min\left\{c_1^{-1}, c_2\right\} \in (0,1)$, then we require $t_0 = 0$; $t_\infty = \infty$; if $c = \max\left\{c_1^{-1}, c_2\right\} > 1$, then we require $\Phi \in \Theta_c$. Let*

$$c_1 j_\lambda(\gamma) \leqslant j_\lambda(\alpha) \leqslant c_2 j_\lambda(\gamma), \tag{88}$$

for every $\lambda > 0$. Then the following estimates hold

$$d_1 \left\|\gamma\right\|_{l_{\Phi,\beta}} \leqslant \left\|\alpha\right\|_{l_{\Phi,\beta}} \leqslant d_2 \left\|\gamma\right\|_{l_{\Phi,\beta}}, \tag{89}$$

with $d_1 = d\left(c_1^{-1}\right)^{-1}$, $d_2 = d(c_2)$, see (10), (12).

Now, we formulate an analogue of Theorem 2.

Theorem 4 *Let the conditions (21) and (11) be fulfilled, and Φ be p-convex on $[t_0, t_\infty)$ for $p \in (0,1]$. Then the following conclusions hold.*

(1) Triangle inequality takes place in $l_{\Phi,v}$. Namely, if $\alpha = \{\alpha_m\}$, $\gamma = \{\gamma_m\}$; $\alpha, \gamma \in l_{\Phi,\beta}$, then $\alpha + \gamma \in l_{\Phi,\beta}$, and

$$\left\|\alpha + \gamma\right\|_{l_{\Phi,\beta}} \leqslant \left(\left\|\alpha\right\|_{l_{\Phi,\beta}}^p + \left\|\beta\right\|_{l_{\Phi,\beta}}^p\right)^{1/p}. \tag{90}$$

(2) The quantity $\left\|\alpha\right\|_{l_{\Phi,\beta}}$ is monotone quasi-norm (norm for $p = 1$):

$$|\alpha_m| \leqslant \gamma_m, \quad m \in Z; \quad \gamma \in l_{\Phi,\beta} \Rightarrow \alpha \in l_{\Phi,\beta}, \quad \left\|\alpha\right\|_{l_{\Phi,\beta}} \leqslant \left\|\gamma\right\|_{l_{\Phi,\beta}},$$

that possess Fatou property: let $\alpha^n = \left\{\alpha_m^n\right\}$, $\gamma = \{\gamma_m\}$, $n \in N$, then

$$0 \leqslant \alpha_m^n \uparrow \gamma_m \, (n \uparrow \infty), \quad m \in Z \Rightarrow \left\|\gamma\right\|_{l_{\Phi,\beta}} = \lim_{n\to\infty} \left\|\alpha^n\right\|_{l_{\Phi,\beta}}.$$

Conclusion. *In the conditions of Theorem 4. $l_{\Phi,\beta}$ forms discrete ideal quasi-Banach space (Banach space for $p = 1$; particularly, when Φ Young function is) that possesses Fatou property.*

Lemma 7 *Let the condition (84) be fulfilled. Then the following equivalence takes place:*

$$\left\|\alpha\right\|_{l_{\Phi,\beta}} \leqslant 1 \Leftrightarrow j_1(\alpha) = \sum_m \Phi\left(|\alpha_m|\right)\beta_m \leqslant 1.$$

2.2. To establish these discrete analogues of the results of Sect. 1, we can introduce the sequence $\{\mu_m\}$ such that

$$\mu_m < \mu_{m+1}; \quad R_+ = \bigcup_m \Delta_m; \quad \Delta_m = \left[\mu_m, \mu_{m+1}\right). \tag{91}$$

We define the weight function $v \in M$, $v > 0$ satisfying the conditions

$$\int_{\Delta_m} v dt = \beta_m. \tag{92}$$

Then we restrict the considerations of Sect. 1 on the set of step-functions

$$\tilde{L}_{\Phi,v} = \left\{ f \in L_{\Phi,v} : \, f = \sum_m \alpha_m \chi_{\Delta_m}, \; \alpha_m \in R \right\}, \tag{93}$$

where χ_{Δ_m} is the characteristic function of interval Δ_m. For such functions, we have

$$J_\lambda(f) = j_\lambda(\alpha); \quad \left\|f\right\|_{\Phi,v} = \left\|\alpha\right\|_{l_{\Phi,\beta}}, \quad \alpha = \{\alpha_m\}. \tag{94}$$

Indeed,

$$J_\lambda(f) = \int_0^\infty \Phi\left(\lambda^{-1}|f(t)|\right)v(t)\,dt = \sum_m \int_{\Delta_m} \ldots =$$
$$= \sum_m \Phi\left(\lambda^{-1}|\alpha_m|\right)\int_{\Delta_m} vdt = \sum_m \Phi\left(\lambda^{-1}|\alpha_m|\right)\beta_m = j_\lambda(\alpha).$$

Now, all above-mentioned discrete formulas are the partial cases of corresponding formulas of Sect. 1 applied to step-functions in Orlicz space.

2.3. Here, we describe one special discretization procedure for integral assertions on the cone Ω of nonnegative decreasing functions in $L_{\Phi,v}$:

$$\Omega \equiv \left\{ f \in L_{\Phi,v} : \, 0 \leqslant f \downarrow \right\}. \tag{95}$$

We assume here that the weight function v satisfies the conditions

$$0 < V(t) := \int_0^t v d\tau < \infty, \quad \forall t \in R_+, \tag{96}$$

Moreover, we assume that V is strictly increasing, and

$$V(+\infty) = \infty. \tag{97}$$

(the case $V(+\infty) < \infty$ we will consider separately). For fixed $b > 1$ we introduce the sequence $\{\mu_m\}$ by formulas

$$\mu_m = V^{-1}\left(b^m\right) \Leftrightarrow V(\mu_m) = b^m, \quad m \in Z = \{0, \pm 1, \pm 2, \ldots\}, \tag{98}$$

where V^{-1} is the inverse function for the continuous increasing function V. Then, the condition (91) is fulfilled, because

$$0 < \mu_m \uparrow; \quad \lim_{m \to -\infty} \mu_m = 0; \quad \lim_{m \to +\infty} \mu_m = \infty. \tag{99}$$

Moreover, we introduce the cone of nonnegative step-functions

$$S \equiv L^+_{\Phi,v} \cap \tilde{L}_{\Phi,v} = \left\{ f \in L_{\Phi,v} : \ f = \sum_m \gamma_m \chi_{\Delta_m}; \ \gamma_m \geqslant 0, m \in Z \right\}; \tag{100}$$

as well as the cone of nonnegative decreasing step-functions

$$\tilde{\Omega} \equiv \Omega \cap \tilde{L}_{\Phi,v} = \left\{ f \in L_{\varphi,v} : \ f = \sum_m \alpha_m \chi_{\Delta_m}; \ 0 \leqslant \alpha_m \downarrow \right\}. \tag{101}$$

For $f \in \Omega$ we determine step-functions $f_0, f_1 \in \tilde{\Omega}$:

$$f_0 := \sum_m f(\mu_{m+1}) \chi_{\Delta_m}, \quad f_1 := \sum_m f(\mu_m) \chi_{\Delta_m}. \tag{102}$$

Then,

$$f_0 \leqslant f \leqslant f_1 \Rightarrow \|f_0\|_{\Phi,v} \leqslant \|f\|_{\Phi,v} \leqslant \|f_1\|_{\Phi,v} \tag{103}$$

(the left hand side inequality in (103) is valid everywhere on R_+). We use the equalities (94) for step-functions f_0 and f_1. Then,

$$\|f_0\|_{\Phi,v} = \left\|\{\alpha_{m+1}\}\right\|_{l_{\Phi,\beta}}; \quad \|f_1\|_{\Phi,v} = \left\|\{\alpha_m\}\right\|_{l_{\Phi,\beta}}, \quad \alpha_m := f(\mu_m). \tag{104}$$

Here, according to (92), and (98),

$$\beta_m = \int\limits_{\Delta_m} v dt = V(\mu_{m+1}) - V(\mu_m) = b^m(b-1), \quad m \in Z. \tag{105}$$

Remark 5 By the discretization (98)–(105) the shift-operators

$$T_+[\{\gamma_m\}] = \{\gamma_{m+1}\}, \quad T_-[\{\gamma_m\}] = \{\gamma_{m-1}\} \tag{106}$$

are bounded as operators in $l_{\Phi,\beta}$.

It is a partial case of the following result.

Lemma 8 *Let $b > 1$; $\Phi \in \Theta_b$; $\beta = \{\beta_m\}$; $\beta_m \in R_+$, $1 \leqslant \beta_{m+1}/\beta_m \leqslant b$, $m \in Z$. Then,*

$$\|T_+\| \leqslant 1, \quad \|T_-\| \leqslant d(b), \tag{107}$$

where $d(b)$ is the constant (12) with $c = b > 1$. If Φ is convex function, we obtain the estimates (107) with $d(b) = b$. In particular, it is true in the case of Young function Φ; see Example 6.

Proof To obtain estimates (107) let us note that for every $\lambda > 0$

$$j_\lambda(\{\gamma_{m+1}\}) \leqslant j_\lambda(\{\gamma_m\}); \quad j_\lambda(\{\gamma_{m-1}\}) \leqslant b j_\lambda(\{\gamma_m\}). \tag{108}$$

Indeed,

$$j_\lambda(\{\gamma_{m+1}\}) = \sum_{m\in Z} \varphi\left(\lambda^{-1}|\gamma_{m+1}|\right)\beta_m = \sum_{m\in Z} \varphi\left(\lambda^{-1}|\gamma_m|\right)\beta_{m-1};$$

$$j_\lambda(\{\gamma_{m-1}\}) = \sum_{m\in Z} \varphi\left(\lambda^{-1}|\gamma_{m-1}|\right)\beta_m = \sum_{m\in Z} \varphi\left(\lambda^{-1}|\gamma_m|\right)\beta_{m+1},$$

and we obtain (108) by taking into account the conditions on $\beta = \{\beta_m\}$. From (108), and (86), (87), it follows that

$$\left\|T_+[\{\gamma_m\}]\right\|_{l_{\Phi,\beta}} = \left\|\{\gamma_{m+1}\}\right\|_{l_{\Phi,\beta}} \leqslant \left\|\{\gamma_m\}\right\|_{l_{\Phi,\beta}},$$

$$\left\|T_-[\{\gamma_m\}]\right\|_{l_{\Phi,\beta}} = \left\|\{\gamma_{m-1}\}\right\|_{l_{\Phi,\beta}} \leqslant d(b)\left\|\{\gamma_m\}\right\|_{l_{\Phi,\beta}}. \tag{109}$$

If Φ is convex, then $d(b) = b$. Thus, we come to estimates (107).

Let us apply estimate (107) to the sequence $\{\gamma_m\} = \{\alpha_{m+1}\}$. Then, by (104) we have,

$$\|f_1\|_{\Phi,v} = \left\|\{\alpha_m\}\right\|_{l_{\Phi,\beta}} \leqslant d(b)\left\|\{\alpha_{m+1}\}\right\|_{l_{\Phi,\beta}} = d(b)\|f_0\|_{\Phi,v}. \tag{110}$$

Substituting of (110) into (103) implies the following conclusion.

Conclusion *Let* $b > 1$; $\Phi \in \Theta_b$, *weight* v *satisfies the conditions* (96), (97). *We realize the discretization procedure* (98)–(105) *for function* $f \in \Omega$, *see* (95). *Then,*

$$d(b)^{-1} \left\| f_1 \right\|_{\Phi,v} \leqslant \left\| f \right\|_{\Phi,v} \leqslant \left\| f_1 \right\|_{\Phi,v}, \tag{111}$$

where $d(b)$ *was defined in* (12) *with* $c = b > 1$. Here f_1 *is the step-function, determined by*, (102), *that satisfies*(104).

Remark 6 All the results of Sect. 2.1 are carried over the discrete weighted Orlicz spaces in which the condition $m \in Z = \{0, \pm 1, \pm 2, \ldots\}$ is replaced by the condition $m \in Z^- = \{0, -1, -2, \ldots\}$ in the notations (84) and below. Thus, here we consider the sequences $\alpha = \{\alpha_m\}, \beta = \{\beta_m\}, \gamma = \{\gamma_m\}$; $m \in Z^-$. The proofs for these discrete formulas are the same as in Sect. 2.2. Only, we have

$$\begin{gathered} \mu_1 = \infty; \quad \mu_m < \mu_{m+1}, \quad m \in Z^-; \\ R_+ = \bigcup_{m \in Z^-} \Delta_m; \quad \Delta_m = \left[\mu_m, \mu_{m+1}\right), \quad m \in Z^-, \end{gathered} \tag{112}$$

in (91), and assume $m \in Z^-$ in (92)–(94).

2.4. Now, let us describe the discretization procedure for the cone (95) in the case

$$0 < V(t) := \int_0^t v d\tau < \infty, \quad \forall t \in R_+, \quad V(+\infty) := \int_0^\infty v d\tau < \infty. \tag{113}$$

Without loss of generality, we will assume that

$$V(1) = 1. \tag{114}$$

We follow the considerations of Sect. 2.3 with small modifications.
According to (114) we have,

$$b = V(+\infty) > 1. \tag{115}$$

We introduce the discretizing sequence $\{\mu_m\}$by formulas

$$\mu_1 = \infty; \quad \mu_m = V^{-1}\left(b^m\right), \quad m \in Z^- = \{0, -1, -2, \ldots\}. \tag{116}$$

Here, V^{-1} is the inverse function for the increasing continuous function V, so that

$$V(\mu_m) = b^m, \quad m = 1, 0, -1, -2, \ldots \tag{117}$$

Then,

$$(0,1)=\bigcup_{m\leqslant -1}\Delta_m,\quad [1,\infty)=\Delta_0,$$
$$R_+=\bigcup_{m\in Z^-}\Delta_m,\quad \Delta_m=\left[\mu_m,\mu_{m+1}\right). \tag{118}$$

We introduce step-functions on R_+ connected with $f\in\Omega$ by the decomposition (118):

$$f_0(t)=\sum_{m\in Z^-}\alpha_{m+1}\chi_{\Delta_m}(t),$$
$$f_1(t)=\sum_{m\in Z^-}\alpha_m\chi_{\Delta_m}(t),\quad \alpha_m=f(\mu_m). \tag{119}$$

Then,

$$f_0\leqslant f\leqslant f_1\Rightarrow \left\|f_0\right\|_{\Phi,v}\leqslant\left\|f\right\|_{\Phi,v}\leqslant\left\|f_1\right\|_{\Phi,v}. \tag{120}$$

For step-functions f_0 and f_1 we have,

$$\left\|f_0\right\|_{\Phi,v}=\left\|\{\alpha_{m+1}\}\right\|_{\bar l_{\Phi,\beta}};\quad \left\|f_1\right\|_{\Phi,v}=\left\|\{\alpha_m\}\right\|_{\bar l_{\Phi,\beta}}. \tag{121}$$

Here $\beta=\{\beta_m\}_{m\in Z^-}$,

$$\beta_m=\int_{\Delta_m}vdt=V(\mu_{m+1})-V(\mu_m)=b^m(b-1),\quad m\in Z^-, \tag{122}$$

and we denote for $\gamma=\{\gamma_m\}_{m\in Z^-}$

$$\bar j_\lambda(\{\gamma_m\})=\sum_{m\in Z^-}\Phi\left(\lambda^{-1}|\gamma_m|\right)\beta_m; \tag{123}$$

$$\left\|\{\gamma_m\}\right\|_{\bar l_{\Phi,\beta}}=\inf\left\{\lambda>0:\ \bar j_\lambda(\{\gamma_m\})\leqslant 1\right\}. \tag{124}$$

Let us mentioned that the notations (121)–(124) are slightly different from ones in Sects. 2.1–2.3 introduced by (84), (85). Now we deal with one-sided sequences.

Remark 7 The next shift-operator is bounded in $\bar l_{\Phi,\beta}$:

$$T_-\left[\{\gamma_m\}\right]=\{\gamma_{m-1}\}_{m\in Z^-}. \tag{125}$$

This is the partial case of the following result.

Lemma 9 *Let $b>1$; $\Phi\in\Theta_b$, and*

$$\beta=\{\beta_m\}_{m\in Z^-},\quad \beta_m>0,\quad 1\leqslant\beta_m/\beta_{m-1}\leqslant b,\quad m\in Z^-.$$

Then the following estimate holds for the norm of operator $T_- : \bar{l}_{\Phi,\beta} \to \bar{l}_{\Phi,\beta}$

$$\left\| T_- \right\| \leqslant d\,(b)\,, \tag{126}$$

where $d\,(b)$ *is the constant (12) with* $c = b > 1$. *If* Φ *is* p*-convex, we obtain estimate (126) with* $d\,(b) = b^{1/p}$.

Proof Note that

$$\bar{j}_\lambda\,(\{\gamma_{m-1}\}) \leqslant b\,\bar{j}_\lambda\,(\{\gamma_m\})\,. \tag{127}$$

Indeed,

$$\bar{j}_\lambda\,(\{\gamma_{m-1}\}) = \sum_{m\in Z^-} \Phi\left(\lambda^{-1}\left|\gamma_{m-1}\right|\right)\beta_m = \sum_{m\leqslant -1} \Phi\left(\lambda^{-1}\left|\gamma_m\right|\right)\beta_{m+1};$$

and we obtain (127) by taking into account the conditions on $\beta = \{\beta_m\}_{m\in Z^-}$. It follows from (127), and (86), (87) (see also Remark 6)

$$\left\| T_-\left[\{\gamma_m\}\right]\right\|_{\bar{l}_{\Phi,\beta}} = \left\|\{\gamma_{m-1}\}\right\|_{\bar{l}_{\Phi,\beta}} \leqslant d\,(b)\left\|\{\gamma_m\}\right\|_{\bar{l}_{\Phi,\beta}}. \tag{128}$$

If Φ is p-convex, then $d\,(b) = b^{1/p}$. Thus, estimate (126) holds.

We apply (126) to the sequence $\{\gamma_m\} = \{\alpha_{m+1}\}$. Then, we have according to (121),

$$\left\| f_1 \right\|_{\Phi,v} = \left\|\{\alpha_m\}\right\|_{\bar{l}_{\Phi,\beta}} \leqslant d\,(b)\left\|\{\alpha_{m+1}\}\right\|_{\bar{l}_{\Phi,\beta}} = d\,(b)\left\| f_0 \right\|_{\Phi,v}. \tag{129}$$

Substitution of (129) into (120) gives the following conclusion.

Proposition 1 *Let us realize the discretization procedure (113)–(129) for function* $f \in \Omega$. *Then,*

$$d\,(b)^{-1}\left\| f_1 \right\|_{\Phi,v} \leqslant \left\| f \right\|_{\Phi,v} \leqslant \left\| f_1 \right\|_{\Phi,v}, \tag{130}$$

where $d\,(b)$ *is determined by (12) with* $c = b > 1$. *Here, the equality (121) holds for function* f_1 *(119).*

3 Estimates for the Norm of Monotone Operator on Cone Ω

3.1 The Case of Nondegenerate Weight

We preserve all the notation of Sects. 1 and 2. Let $(\mathrm{N}, \mathfrak{R}, \eta)$ be the measure-space with non-negative full σ-finite measure η; let $L = L\,(\mathrm{N}, \mathfrak{R}, \eta)$ be the set of all η-measurable functions $u : \mathrm{N} \to R$; $L^+ = \{u \in L : u \geqslant 0\}$. Here, we assume pointwise inequalities to be fulfilled η-almost everywhere. Let $Y = Y\,(\mathrm{N}, \mathfrak{R}, \eta) \subset L$ be an ideal space, that is Banach, or quasi-Banach space of measurable functions with monotone norm, or quasi-norm $\|\cdot\|_Y$ so that

$$u_1 \in L, \quad |u_1| \leqslant |u_2|, \quad u_2 \in Y \Rightarrow u_1 \in Y, \quad \|u_1\|_Y \leqslant \|u_2\|_Y. \tag{131}$$

General theory of ideal spaces in the normed case was considered in [3], one special variant of such theory was developed in [11] on the base of concept of Banach function spaces, that includes Orlicz spaces. Let $P : M^+ \to L^+$ be the so called monotone operator, i.e.,

$$f, h \in M^+, \quad f \leqslant h \quad \mu - a.e. \Rightarrow Pf \leqslant Ph \quad \eta - \text{a. e.} \tag{132}$$

We define the norms of restrictions of operator P on the cones Ω (95), and $\tilde{\Omega}$ (101):

$$\|P\|_{\Omega\to Y} = \sup\left\{\|Pf\|_Y : \ f \in \Omega, \ \|f\|_{\Phi,v} \leqslant 1\right\}. \tag{133}$$

$$\|P\|_{\tilde{\Omega}\to Y} = \sup\left\{\|Pf\|_Y : \ f \in \tilde{\Omega}, \ \|f\|_{\Phi,v} \leqslant 1\right\}. \tag{134}$$

Lemma 10 *Let the conditions (84) be fulfilled, $b > 1$; $\Phi \in \Theta_b$. We assume that weight function satisfies (96) and (97), and realize the discretization procedure (98)–(105) for function $f \in \Omega$. The following estimates take place*

$$\|P\|_{\tilde{\Omega}\to Y} \leqslant \|P\|_{\Omega\to Y} \leqslant d\,(b)\,\|P\|_{\tilde{\Omega}\to Y}, \tag{135}$$

with $d\,(b)$ determined in (12) for $c = b > 1$.

Proof The left-hand side inequality in (135) is obvious because of embedding $\tilde{\Omega} \subset \Omega$. From the other side, for every function $f \in \Omega$, and for f_1 in (102), we have $f \leqslant f_1 \Rightarrow Pf \leqslant Pf_1$, and $\|f_1\|_{\Phi,v} \leqslant d\,(b)\,\|f\|_{\Phi,v}$ (see the conclusion after the proof of Lemma 8). Moreover,

$$f \in \Omega \Rightarrow f_1 = \sum_m f\,(\mu_m)\chi_{\Delta_m} \in \tilde{\Omega}.$$

Consequently, for every $f \in \Omega$

$$\|Pf\|_Y \leqslant \|Pf_1\|_Y \leqslant \|P\|_{\tilde{\Omega}\to Y}\|f_1\|_{\Phi,v} \leqslant d\,(b)\,\|P\|_{\tilde{\Omega}\to Y}\|f\|_{\Phi,v}, \tag{136}$$

and

$$\|P\|_{\Omega\to Y} = \sup\left\{\|Pf\|_Y : \ f \in \Omega, \|f\|_{\Phi,v} \leqslant 1\right\} \leqslant d\,(b)\,\|P\|_{\tilde{\Omega}\to Y}.$$

Now, we consider the norm of restriction on the cone S (100):

$$\|P\|_{S\to Y} = \sup\left\{\|Pf\|_Y : \ f \in S, \ \|f\|_{\Phi,v} \leqslant 1\right\}. \tag{137}$$

Theorem 5 *Let the conditions of Lemma 10 be fulfilled. Then, the following two-sided estimate takes place*

$$c(b)^{-1}\|P\|_{S\to Y} \leqslant \|P\|_{\Omega\to Y} \leqslant d(b)\|P\|_{S\to Y}, \tag{138}$$

where $d(b)$ is determined by (12) with $c = b > 1$, and

$$c(b) = d(c_0(b)); \quad c_0(b) = \left[b(b-1)^{-1}\right] > 1. \tag{139}$$

Proof Inequality (138) follows by (135), and by the analogous inequality

$$\|P\|_{\tilde{\Omega}\to Y} \leqslant \|P\|_{S\to Y} \leqslant c(b)\|P\|_{\tilde{\Omega}\to Y}. \tag{140}$$

The left inequality in (140) is obvious because of inclusion $\tilde{\Omega} \subset S$. Let us prove the right one.

1. We introduce sup-operator A by formula $A\gamma = \alpha$, where $\gamma = \{\gamma_m\}_{m\in Z}$; $\alpha = \{\alpha_m\}_{m\in Z}$, and

$$\alpha_m = \sup_{k\geqslant m}|\gamma_k|, \quad m \in Z. \tag{141}$$

Let us prove the boundedness of operator $A : \ l_{\Phi,\beta} \to l_{\Phi,\beta}$ with corresponding estimate

$$\|A\gamma\|_{l_{\Phi,\beta}} \leqslant c(b)\|\gamma\|_{l_{\Phi,\beta}}. \tag{142}$$

We assume that $\gamma \in l_{\Phi,\beta}$ (otherwise is nothing to prove). Let $\lambda \geqslant \|\gamma\|_{l_{\Phi,\beta}}$. Then,

$$j_\lambda(\gamma) = \sum_{k\in Z}\Phi\left(\lambda^{-1}|\gamma_k|\right)\beta_k \leqslant 1. \tag{143}$$

We have $\beta_k = b^k(b-1) \uparrow \infty$, so that

$$(143) \Rightarrow \Phi\left(\lambda^{-1}|\gamma_k|\right) \to 0\,(k \to +\infty). \tag{144}$$

Let us show that for all non-zero terms of series

$$j_\lambda(\alpha) = \sum_{m\in Z}\Phi\left(\lambda^{-1}\alpha_m\right)\beta_m, \tag{145}$$

the equalities hold

$$\exists k(m): \ m \leqslant k(m) < \infty, \quad \Phi\left(\lambda^{-1}\alpha_m\right) = \Phi\left(\lambda^{-1}\left|\gamma_{k(m)}\right|\right). \tag{146}$$

For any $\varepsilon > 0$ we have

$$\exists K(\varepsilon) \in Z: \ \lambda^{-1}|\gamma_k| \leqslant t_0 + \varepsilon, \quad \forall k \geqslant K(\varepsilon). \tag{147}$$

Here t_0 is determined by (1) for $\Phi \in \Theta$. Indeed, if (147) fails, there exist $\varepsilon_0 > 0$ and subsequence of numbers $k_j \to +\infty$ such that

$$\lambda^{-1}\left|\gamma_{k_j}\right| \geqslant t_0 + \varepsilon_0, \quad j \in N \Rightarrow \Phi\left(\lambda^{-1}\left|\gamma_{k_j}\right|\right) \geqslant \Phi(t_0 + \varepsilon_0) > 0.$$

This contradicts to (144). Thus, (147) is valid. Moreover, for every $m \in Z$, we have $\Phi\left(\lambda^{-1}\alpha_m\right) \neq 0 \Rightarrow \lambda^{-1}\alpha_m > t_0$. Therefore, if we set $\varepsilon = \varepsilon_{m,\lambda} \equiv 2^{-1}\left(\lambda^{-1}\alpha_m - t_0\right) > 0$ then,

$$\lambda^{-1}|\gamma_k| \leqslant t_0 + \varepsilon = 2^{-1}\left(\lambda^{-1}\alpha_m + t_0\right), \quad \forall k \geqslant K\left(\varepsilon_{m,\lambda}\right),$$

according to (147). It means that $\sup\limits_{k \geqslant K\left(\varepsilon_{m,\lambda}\right)} |\gamma_k| \leqslant 2^{-1}(\alpha_m + t_0\lambda) < \alpha_m$. Thus,

$$\alpha_m = \sup_{k \geqslant m} |\gamma_k| = \max_{m \leqslant k \leqslant K\left(\varepsilon_{m,\lambda}\right)} |\gamma_k|.$$

Therefore, $\exists k(m): \ m \leqslant k(m) \leqslant K\left(\varepsilon_{m,\lambda}\right)$, $\alpha_m = \left|\gamma_{k(m)}\right|$. It follows from (145) and (146), that

$$j_\lambda(\alpha) = \sum_{m \in Z} \Phi\left(\lambda^{-1}\left|\gamma_{k(m)}\right|\right)\beta_m. \tag{148}$$

Moreover, all terms in (148) are finite because of (143). From (148), it follows that

$$j_\lambda(\alpha) \leqslant \sum_{m \in Z} \beta_m \sum_{k \geqslant m} \Phi\left(\lambda^{-1}|\gamma_k|\right) = \sum_{k \in Z} \Phi\left(\lambda^{-1}|\gamma_k|\right) \sum_{m \leqslant k} \beta_m.$$

But, $\beta_m = b^{m+1} - b^m$, so that

$$\sum_{m \leqslant k} \beta_m = b^{k+1} = c_0(b)\,\beta_k, \quad c_0(b) = b\,(b-1)^{-1}.$$

As the result, we have estimate

$$j_\lambda(\alpha) \leqslant c_0(b) \sum_{k \in Z} \Phi\left(\lambda^{-1}|\gamma_k|\right)\beta_k = c_0(b)\, j_\lambda(\gamma), \tag{149}$$

for all $\lambda \geqslant \|\gamma\|_{l_{\varphi,\beta}}$. Here,$c_0(b) > 1$, so that $d(c_0(b)) \geqslant 1$, where $d(c)$ is the constant (12). It means that inequality (149) is true for $\lambda \geqslant d(c_0(b))\|\gamma\|_{l_{\varphi,\beta}}$. By Theorem 3, it implies the estimate

$$\|\alpha\|_{l_{\varphi,\beta}} \leqslant d(c_0(b))\|\gamma\|_{l_{\varphi,\beta}},$$

coinciding with (142).

2. Now, we denote $\gamma = \{\gamma_m\}$, $\gamma_m = f(\mu_m) \geqslant 0$, $m \in Z$ for every $f \in S$. Then,

$$f = f_{(\gamma)} := \sum_m \gamma_m \chi_{\Delta_m}.$$

Further, we introduce $\alpha_m = \sup\limits_{k \geqslant m} \gamma_k$, $m \in Z$, and for $\alpha = \{\alpha_m\}$ consider function

$$f_{(\alpha)} = \sum_m \alpha_m \chi_{\Delta_m}.$$

Then, $f_{(\alpha)} \in \tilde{\Omega}$, see (101), and

$$f_{(\gamma)} \leqslant f_{(\alpha)}, \quad \|f_{(\alpha)}\|_{\Phi,v} = \|\alpha\|_{l_{\Phi,\beta}} \leqslant c(b)\|\gamma\|_{l_{\Phi,\beta}} = c(b)\|f_{(\gamma)}\|_{\Phi,v}; \tag{150}$$

see (142). Therefore, for $f = f_{(\gamma)} \in S$ there exists $f_{(\alpha)} \in \tilde{\Omega}$ such that

$$Pf \leqslant Pf_{(\alpha)}; \quad \|f_{(\alpha)}\|_{\Phi,v} \leqslant c(b)\|f\|_{\Phi,v}.$$

Here, $f_{(\alpha)} \in \tilde{\Omega}$, and we obtain for every function $f \in S$

$$\|Pf\|_Y \leqslant \|Pf_{(\alpha)}\|_Y \leqslant \|P\|_{\tilde{\Omega}\to Y}\|f_{(\alpha)}\|_{\Phi,v} \leqslant c(b)\|P\|_{\tilde{\Omega}\to Y}\|f\|_{\Phi,v}.$$

This gives the second inequality in (140).

Remark 8 Theorem 5 discovers the main goal of the discretization procedure (98)–(105). In this theorem, we reduce the estimates for the restriction of monotone operator on the cone of nonnegative decreasing functions Ω to the estimates of this operator on some set of nonnegative step-functions. In many cases, such reduction admits to apply known results for step-functions or their pure discrete analogues for obtaining needed estimates on the cone Ω. Such approach we realize, for example, in Sect. 4 in the problem of description of associate norms.

3.2 *The Case of Degenerate Weight*

We use all notation and assumptions of Sect. 2.4, see (113)–(130). Introduce the cones

$$\Omega_0 = \left\{\alpha = \{\alpha_m\}_{m\in Z^-} : \ 0 \leqslant \alpha_m \downarrow\right\}; \tag{151}$$

$$\tilde{\Omega}_0 = \left\{ f = f_\alpha : \ f_\alpha(t) = \sum_{m\in Z^-} \alpha_m \chi_{\Delta_m}(t);\ \alpha \in \Omega_0 \right\}. \tag{152}$$

Define

$$\left\| P \right\|_{\tilde{\Omega}_0\to Y} = \sup\left\{ \left\| Pf \right\|_Y : \ f \in \tilde{\Omega}_0, \left\| f \right\|_{\varphi,v} \leqslant 1 \right\}. \tag{153}$$

Lemma 11 *The following two-sided estimate holds in above notation and assumptions:*

$$\left\| P \right\|_{\tilde{\Omega}_0\to Y} \leqslant \left\| P \right\|_{\Omega\to Y} \leqslant d(b) \left\| P \right\|_{\tilde{\Omega}_0\to Y}. \tag{154}$$

Here, $d(b)$ is defined by (12) with $c = b > 1$.

Proof The left hand side inequality in (154) is evident because of inclusion $\tilde{\Omega}_0 \subset \Omega$. From the other side we have $f \leqslant f_1 \Rightarrow Pf \leqslant Pf_1$, for every function $f \in \Omega$, and $\left\| f_1 \right\|_{\Phi,v} \leqslant d(b) \left\| f \right\|_{\Phi,v}$. Now, let us take into account that

$$f \in \Omega \Rightarrow f_1(t) = \sum_{m\in Z^-} f(\mu_m)\chi_{\Delta_m}(t) \in \tilde{\Omega}_0.$$

Therefore,

$$\left\| Pf \right\|_Y \leqslant \left\| Pf_1 \right\|_Y \leqslant \left\| P \right\|_{\tilde{\Omega}_0\to Y} \left\| f_1 \right\|_{\Phi,v} \leqslant d(b) \left\| P \right\|_{\tilde{\Omega}_0\to Y} \left\| f \right\|_{\Phi,v}. \tag{155}$$

Consequently,

$$\left\| P \right\|_{\Omega\to Y} = \sup\left\{ \left\| Pf \right\|_Y : \ f \in \Omega, \left\| f \right\|_{\Phi,v} \leqslant 1 \right\} \leqslant d(b) \left\| P \right\|_{\tilde{\Omega}_0\to Y}.$$

Now, we introduce the cone of nonnegative step-functions connected with the participation in Sect. 2.4:

$$\bar{S} = \left\{ f = f_\alpha : \ f_\alpha(t) = \sum_{m\in Z^-} \alpha_m \chi_{\Delta_m}(t);\ \alpha_m \geqslant 0,\ m \in Z^- \right\}, \tag{156}$$

and consider the related norm of the restriction

$$\left\| P \right\|_{\bar{S}\to Y} = \sup\left\{ \left\| Pf \right\|_Y : \ f \in \bar{S},\ \left\| f \right\|_{\Phi,v} \leqslant 1 \right\}. \tag{157}$$

Lemma 12 *Define*

$$c(b) = d(c_0(b)); \quad c_0(b) = \left[b(b-1)^{-1}\right] > 1,$$

see (85). The following two-sided estimate holds in the notation and assumptions of this Subsection:

$$\|P\|_{\tilde{\Omega}_0 \to Y} \leqslant \|P\|_{\bar{S} \to Y} \leqslant c(b) \|P\|_{\tilde{\Omega}_0 \to Y}. \tag{158}$$

Proof The left hand side inequality in (158) is evident because of inclusion $\tilde{\Omega}_0 \subset \bar{S}$. Let us prove the right one. We introduce the maximal operator B by the formula $B\gamma = \alpha$, where $\alpha = \{\alpha_m\}_{m \in Z^-}$; $\gamma = \{\gamma_m\}_{m \in Z^-}$, and

$$\alpha_m = \max_{k \in Z^-, k \geqslant m} |\gamma_k|, \quad m \in Z^-. \tag{159}$$

Let us show the boundedness of operator $B: \bar{l}_{\Phi,\beta} \to \bar{l}_{\Phi,\beta}$. Let $\gamma \in \bar{l}_{\Phi,\beta}$. Then, if $\lambda \geqslant \|\gamma\|_{\bar{l}_{\Phi,\beta}}$, we have $\bar{j}_\lambda(\gamma) = \sum_{k \in Z^-} \Phi\left(\lambda^{-1}|\gamma_k|\right)\beta_k \leqslant 1$ so that $\Phi\left(\lambda^{-1}|\gamma_k|\right) < \infty$, $k \in Z^-$. Moreover, recall that $\Phi \in \Theta$ is increasing, so that

$$\Phi\left(\lambda^{-1}\alpha_m\right) = \max_{k \in Z^-, k \geqslant m} \Phi\left(\lambda^{-1}|\gamma_k|\right) \leqslant \sum_{k \in Z^-, k \geqslant m} \Phi\left(\lambda^{-1}|\gamma_k|\right).$$

Then,

$$\bar{j}_\lambda(\alpha) = \sum_{m \in Z^-} \Phi\left(\lambda^{-1}\alpha_m\right)\beta_m \leqslant$$
$$\leqslant \sum_{m \in Z^-} \beta_m \sum_{k \in Z^-, k \geqslant m} \Phi\left(\lambda^{-1}|\gamma_k|\right) = \sum_{k \in Z^-} \Phi\left(\lambda^{-1}|\gamma_k|\right) \sum_{m \leqslant k} \beta_m.$$

We have according to (122), $\beta_m = b^{m+1} - b^m$, and

$$\sum_{m \leqslant k} \beta_m = b^{k+1} = \beta_k c_0(b). \tag{160}$$

Consequently,

$$\bar{j}_\lambda(\alpha) \leqslant c_0(b) \sum_{k \in Z^-} \Phi\left(\lambda^{-1}|\gamma_k|\right)\beta_k = c_0(b)\, \bar{j}_\lambda(\gamma). \tag{161}$$

This inequality gives

$$\|\{\alpha_m\}\|_{\bar{l}_{\Phi,\beta}} \leqslant d(c_0(b)) \|\{\gamma_m\}\|_{\bar{l}_{\Phi,\beta}}. \tag{162}$$

Now, we denote $\gamma_m = f(\mu_m) \geqslant 0, \quad m \in Z^-$, for function $f \in \bar{S}$, so that $f = f_\gamma$. Further, we introduce, according to (159), $\alpha_m = \max\limits_{k \in Z^-, k \geqslant m} |\gamma_k|$, $m \in Z^-$. Then, $\alpha = \{\alpha_m\} \in \Omega_0$, $f_\alpha \in \tilde{\Omega}_0$, and

$$f_\alpha \geqslant f_\gamma, \quad \|f_\alpha\|_{\Phi,v} = \|\alpha\|_{\bar{l}_{\Phi,\beta}} \leqslant c(b)\,\|\gamma\|_{\bar{l}_{\Phi,\beta}} = c(b)\,\|f_\gamma\|_{\Phi,v}. \tag{163}$$

From (163) it follows that for given $f = f_\gamma \in \bar{S}$ there exits $f_\alpha \in \tilde{\Omega}_0$ such that

$$Pf \leqslant Pf_\alpha; \quad \|f_\alpha\|_{\Phi,v} \leqslant c(b)\,\|f\|_{\Phi,v}.$$

Consequently, for every $f \in \bar{S}$,

$$\|Pf\|_Y \leqslant \|Pf_\alpha\|_Y \leqslant \|P\|_{\tilde{\Omega}_0 \to Y} \|f_\alpha\|_{\Phi,v} \leqslant c(b)\,\|P\|_{\tilde{\Omega}_0 \to Y} \|f\|_{\Phi,v}.$$

This inequality gives the second estimate in (158).

4 The Associate Norm for the Cone of Nonnegative Decreasing Functions In Weighted Orlicz Space

4.1 The Case of Nondegenerate Weight

We preserve all notations of Sects. 1–3, and apply the results of Sect. 3 in the important partial case when ideal space Y coincides with the weighted Lebesgue space $L_1(R_+; g)$, $g \in M^+$, and monotone operator P is the identical operator. In this case

$$\|P\|_{\Omega \to Y} = \sup\left\{\int_0^\infty fg\,dt : \; f \in \Omega;\, \|f\|_{\varphi,v} \leqslant 1\right\} =$$
$$= \sup\left\{\int_0^\infty fg\,dt : \; f \in \Omega;\, J_1(f) \leqslant 1\right\} = \|g\|' \tag{164}$$

(see (133); let us recall the equivalence $\|f\|_{\Phi,v} \leqslant 1 \Leftrightarrow J_1(f) \leqslant 1$, see (76)). It means that the norm $\|P\|_{\Omega \to Y}$ coincides in this case with the associate norm for the cone Ω (95), equipped with the functional

$$J_1(f) = \int_0^\infty \Phi(f) v\,dx.$$

We have according to the results of Sect. 3, Theorem 5,

$$\|P\|_{\Omega\to Y} \cong \|P\|_{S\to Y}, \tag{165}$$

where in our case

$$\|P\|_{S\to Y} = \sup\left\{\sum_{m\in Z}\alpha_m g_m :\ \alpha_m \geqslant 0;\ \sum_{m\in Z}\Phi(\alpha_m)\beta_m \leqslant 1\right\}, \tag{166}$$

and

$$g_m = \int_{\Delta_m} g dt \geqslant 0; \quad \beta_m = \int_{\Delta_m} v dt = b^m(b-1), \quad m \in Z. \tag{167}$$

Let us note that the norm (166) coincides with the discrete variant of Orlicz norm, see [2]:

$$\|\{g_m\}\|_{l'_{\Phi,\beta}} = \sup\left\{\sum_{m\in Z}\alpha_m |g_m| :\ \alpha_m \geqslant 0;\ \sum_{m\in Z}\Phi(\alpha_m)\beta_m \leqslant 1\right\}, \tag{168}$$

Our nearest aim is to describe explicitly the norm (168) in terms of complementary function Ψ. We restrict ourselves with the case of Young function. Thus, let as in Example 6, $\Phi : [0,\infty) \to [0,\infty]$ be Young function that is,

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \tag{169}$$

where $\varphi : [0,\infty) \to [0,\infty]$ is the decreasing and left-continuous function, and $\varphi(0) = 0$, φ is neither identically zero, nor identically infinity on $(0,\infty)$. Let Ψ be the complementary Young function for Φ, that is

$$\begin{aligned}\Psi(t) &= \int_0^t \psi(\tau)d\tau, \quad t \in [0,\infty]; \\ \psi(\tau) &= \inf\{\sigma :\ \varphi(\sigma) \geqslant \tau\}, \quad \tau \in [0,\infty].\end{aligned} \tag{170}$$

Function ψ is left inverse for the left-continuous increasing function φ. It has the same general properties as φ, so that Ψ is Young function too. Moreover, $\varphi(\sigma) = \inf\{\tau :\ \psi(\tau) \geqslant \sigma\}$, and Φ in its turn is the complementary Young function for Ψ (see [11, p. 271]). It is well-known that

$$\Psi(t) = \sup_{s \geqslant 0} [st - \Phi(s)];$$

$$st \leqslant \Phi(s) + \Psi(t), \quad s, t \in [0, \infty), \tag{171}$$

and the equality takes place in (171) if and only if $\varphi(s) = t$ or $\psi(t) = s$ (see [11, pp. 271–273]).

The next result is well-known in the theory of discrete weighted Orlicz spaces. It is valid for any positive weight sequence, and plays the crucial role for equivalent description of the Orlicz norm (168).

Theorem 6 *Let Φ, and Ψ be the complementary Young functions, let $\beta = \{\beta_m\}$; $\beta_m \in R_+$, $m \in Z$. Then, Orlicz norm (168) is equivalent to the norm*

$$\left\| \left\{ \beta_m^{-1} g_m \right\} \right\|_{l_{\Psi,\beta}}. \tag{172}$$

Namely,

$$\left\| \left\{ \beta_m^{-1} g_m \right\} \right\|_{l_{\Psi,\beta}} \leqslant \left\| \{ g_m \} \right\|_{l'_{\Phi,\beta}} \leqslant 2 \left\| \left\{ \beta_m^{-1} g_m \right\} \right\|_{l_{\Psi,\beta}}. \tag{173}$$

Corresponding notations of the discrete norms we introduced in (84), (85).

Conclusion. Let us formulate some results of our considerations.

Let Φ, and Ψ be the complementary Young functions, let the conditions (96), *and* (97) *be fulfilled, and the discretization procedure* (98)–(105) *be realized. Then, the following equivalence takes place for the norm* (164)

$$\|g\|' \cong \left\| \{ \rho_m \} \right\|_{l_{\Psi,\beta}}, \quad \beta = \{\beta_m\}, \quad \rho_m = \beta_m^{-1} \int_{\Delta_m} |g| \, dt. \tag{174}$$

Now, our aim is to present this answer in the integral form.

Theorem 7 *Let Φ, and Ψ be the complementary Young functions, let the conditions (96), and (97) be fulfilled. The following two-sided estimate holds for the associate norm (164) with fixed* $0 < a < 1$:

$$\|g\|' \cong \left\| \rho_a(g) \right\|_{\Psi,v} = \inf \left\{ \lambda > 0 : \int_0^\infty \Psi\left(\lambda^{-1} \rho_a(g;t) \right) v(t) \, dt \leqslant 1 \right\}, \tag{175}$$

$$\rho_a(g;t) := V(t)^{-1} \int_{\delta_a(t)}^{t} |g(\tau)| d\tau, \quad \delta_a(t) := V^{-1}(aV(t)), \quad t \in R_+. \tag{176}$$

The norms (175) are equivalent for different values $a \in (0, 1)$.

Here and below, we use the notation

$$A \cong B \Leftrightarrow \exists c = c\,(a) \in [1, \infty) : \; c^{-1} \leqslant A/B \leqslant c. \tag{177}$$

Remark 9 Let us assume additionally that function Φ in Theorem 7 satisfies Δ_2-condition, that is

$$\exists C \in (1, \infty) : \; \Phi\,(2t) \leqslant C\Phi\,(t)\,, \quad \forall t \in R_+. \tag{178}$$

Then,

$$\|g\|' \cong \left\| V\,(t)^{-1} \int_0^t |g\,(\tau)| d\tau \right\|_{\Psi,v}. \tag{179}$$

Proof (*of Theorem* 7) We use the description (174) with $b = a^{-1/2} > 1$. Then, $a = b^{-2}$, and

$$\rho'_m \leqslant \rho_a\,(g; t) = V\,(t)^{-1} \int_{V^{-1}(aV(t))}^{t} |g|\, d\tau \leqslant \rho''_m, \quad t \in \Delta_m, \tag{180}$$

where

$$\rho'_m = b^{-(m+1)} \int_{\mu_{m-1}}^{\mu_m} |g|\, d\tau; \quad \rho''_m = b^{-m} \int_{\mu_{m-2}}^{\mu_{m+1}} |g|\, d\tau. \tag{181}$$

Therefore,

$$F_0\,(t) \leqslant \rho_a\,(g; t) \leqslant F_1\,(t)\,, \quad t \in R_+, \tag{182}$$

where F_0, F_1 are step-functions

$$F_0\,(t) = \sum_m \rho'_m \chi_{\Delta_m}\,(t), \quad F_1\,(t) = \sum_m \rho''_m \chi_{\Delta_m}\,(t),$$

and

$$\|F_0\|_{\Psi,v} = \left\|\{\rho'_m\}\right\|_{l_{\Psi,\beta}}, \quad \|F_1\|_{\Psi,v} = \left\|\{\rho''_m\}\right\|_{l_{\Psi,\beta}},$$

so that

$$\left\|\{\rho'_m\}\right\|_{l_{\Psi,\beta}} \leqslant \|\rho_a\,(g)\|_{\Psi,v} \leqslant \left\|\{\rho''_m\}\right\|_{l_{\Psi,\beta}}. \tag{183}$$

Thus, needed result (175) follows from the equivalence

$$\left\|\{\rho'_m\}\right\|_{l_{\Psi,\beta}} \cong \left\|\{\rho''_m\}\right\|_{l_{\Psi,\beta}} \cong \left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}. \tag{184}$$

It remains to prove (184). The equalities (174) and (181) show that

$$\rho'_m = b^{-2}(b-1)\,\rho_{m-1}; \tag{185}$$

$$\rho''_m = \rho'_{m-1} + b\rho'_m + (b-1)\,\rho_m. \tag{186}$$

Consequently,

$$\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}} = b^{-2}(b-1)\left\|\{\rho_{m-1}\}\right\|_{l_{\Psi,\beta}} \leqslant b^{-1}(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}. \tag{187}$$

$$\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}} = b^2(b-1)^{-1}\left\|\left\{\rho'_{m+1}\right\}\right\|_{l_{\Psi,\beta}} \leqslant b^2(b-1)^{-1}\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}}. \tag{188}$$

In the last inequality, we take into account the boundedness of shift-operators in $l_{\Psi,\beta}$ with Young function Ψ, and $\beta = \{\beta_m\}$ in (105), see Remark 5 and Lemma 8. Thus,

$$\left\|\{\rho_{m-1}\}\right\|_{l_{\Psi,\beta}} \leqslant b\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}, \quad \left\|\left\{\rho'_{m+1}\right\}\right\|_{l_{\Psi,\beta}} \leqslant \left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}}.$$

We have by (186),

$$(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}} \leqslant \left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}}; \tag{189}$$

$$\left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}} \leqslant \left\|\left\{\rho'_{m-1}\right\}\right\|_{l_{\Psi,\beta}} + b\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}} + (b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}. \tag{190}$$

Like (187), the estimate is valid

$$\left\|\left\{\rho'_{m-1}\right\}\right\|_{l_{\Psi,\beta}} \leqslant b\left\|\left\{\rho'_m\right\}\right\|_{l_{\Psi,\beta}}.$$

We substitute this estimate into (190), take into account the inequality (187) and obtain

$$\left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}} \leqslant 3\,(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}.$$

Consequently,

$$(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}} \leqslant \left\|\left\{\rho''_m\right\}\right\|_{l_{\Psi,\beta}} \leqslant 3\,(b-1)\left\|\{\rho_m\}\right\|_{l_{\Psi,\beta}}. \tag{191}$$

The estimates (187), (188), and (191) give the needed equivalence (184).

5 The Case of Degenerated Weight Function

We use the results of Sect. 3.2 to estimate the norm of restriction of monotone operator on the cone Ω in the case of degenerated weight. According to Lemmas 11, and 12, the following two-sided estimate holds

$$\left\| P \right\|_{\Omega \to Y} \cong \left\| P \right\|_{\bar{S} \to Y}. \tag{192}$$

We apply these results in the special case, when the ideal space Y coincides with the weighted Lebesgue space $L_1(R_+; g)$, $g \in M^+$, and the monotone operator P is identical operator. Recall that in this case $\left\| P \right\|_{\Omega \to Y}$ coincides with the associate norm to the cone Ω, equipped with the functional

$$J_1(f) = \int_0^\infty \Phi(f) v dt,$$

and the following equality holds for $\left\| P \right\|_{\bar{S} \to Y}$:

$$\left\| P \right\|_{\bar{S} \to Y} = \sup \left\{ \sum_{m \in Z^-} \alpha_m g_m : \ \alpha_m \geqslant 0; \ \sum_{m \in Z^-} \Phi(\alpha_m) \beta_m \leqslant 1 \right\}. \tag{193}$$

Here,

$$g_m = \int_{\Delta_m} g dt \geqslant 0; \quad \beta_m = \int_{\Delta_m} v dt = b^{-m}(b-1), \quad m \in Z^-. \tag{194}$$

Note that the norm (193) coincides with the discrete variant of Orlicz norm; see [2]:

$$\left\| \{g_m\} \right\|_{\bar{l}'_{\Phi,\beta}} = \sup \left\{ \sum_{m \in Z^-} \alpha_m |g_m| : \quad \alpha_m \geqslant 0; \sum_{m \in Z^-} \Phi(\alpha_m) \beta_m \leqslant 1 \right\}, \tag{195}$$

Our nearest aim is to give the explicit description of the norm (195) in terms of complementary Young function. Thus, let Φ be Young function, and Ψ be its complementary Young function.

We apply corresponding variant of Theorem 6, and obtain the equivalence of Orlicz norm (195) to the norm

$$\left\| \{\rho_m\} \right\|_{\bar{l}_{\Psi,\beta}}; \quad \rho_m = \beta_m^{-1} g_m. \tag{196}$$

Namely,

$$\left\| \{\rho_m\} \right\|_{\bar{l}_{\Psi,\beta}} \leqslant \left\| \{g_m\} \right\|_{\bar{l}'_{\Phi,\beta}} \leqslant 2 \left\| \{\rho_m\} \right\|_{\bar{l}_{\Psi,\beta}}. \tag{197}$$

Here,

$$\left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}} = \inf\left\{\lambda > 0 : \ \bar{j}_\lambda\left(\{\rho_m\}\right) \leqslant 1\right\}; \tag{198}$$

$$\bar{j}_\lambda\left(\{\rho_m\}\right) = \sum_{m \in Z^-} \Psi\left(\lambda^{-1}\left|\rho_m\right|\right)\beta_m; \tag{199}$$

See the relating notations in (121)–(124).

Conclusions. Let us formulate some results of our considerations.

We introduce the discretizing sequence $\{\mu_m\}_{m\in Z^-}$ by formulas

$$V\left(\mu_m\right) = b^m, \quad m \in Z^- = \{0, -1, -2, \ldots\} \tag{200}$$

for fixed $b > 1$, and function V with the properties described in Sect. 2.4.

We set $\mu_1 = \infty$, and determine

$$\Delta_m = \left[\mu_m, \mu_{m+1}\right), \quad m \in Z^-; \tag{201}$$

$$\beta_m = \int\limits_{\Delta_m} v dt = b^m\left(b - 1\right); \quad \rho_m = \beta_m^{-1} \int\limits_{\Delta_m} |g|\, dt. \tag{202}$$

Further, we have the equivalence for the associate norm $\left\|g\right\|'$ (164)

$$\left\|g\right\|' \cong \left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}}, \quad \beta = \{\beta_m\}, \tag{203}$$

where Ψ is the complementary function for Young function Φ.

Now, our aim is to present this description in integral form.

Theorem 8 *Let Ψ be the complementary function for Young function Φ, and weight satisfies the conditions of Sect. 2.4, in particular,*

$$V\left(+\infty\right) < \infty. \tag{204}$$

Denote

$$b = V\left(+\infty\right)/V\left(1\right) > 1, a = b^{-2}. \tag{205}$$

Then, in the notation (176),

$$\left\|g\right\|' \cong \left\|\rho_a\left(g\right)\chi_{(0,1)}\right\|_{\Psi,v} + \int\limits_{V^{-1}(aV(+\infty))}^{\infty} |g| dt. \tag{206}$$

Proof Let us note that

$$\rho_m' \leqslant \rho_a\left(g; t\right)\chi_{(0,1)}\left(t\right) \leqslant \rho_m'', \quad t \in \Delta_m, \quad m \in Z^-. \tag{207}$$

Here, $\rho_0' = \rho_0'' = 0$, and for $m \leqslant -1$

$$\rho_m' = b^{-(m+1)} \int\limits_{\mu_{m-1}}^{\mu_m} |g|\, d\tau; \quad \rho_m'' = b^{-m} \int\limits_{\mu_{m-2}}^{\mu_{m+1}} |g|\, d\tau. \tag{208}$$

Then,

$$F_0(t) \leqslant \rho_a(g; t)\, \chi_{(0,1)}(t) \leqslant F_1(t), \quad t \in R_+, \tag{209}$$

where F_0, F_1 are step-functions

$$F_0(t) = \sum_{m \in Z^-} \rho_m' \chi_{\Delta_m}(t), \quad F_1(t) = \sum_{m \in Z^-} \rho_m'' \chi_{\Delta_m}(t),$$

and

$$\left\| F_0 \right\|_{\Psi, v} = \left\| \left\{ \rho_m' \right\} \right\|_{\bar{l}_{\Psi,\beta}}, \quad \left\| F_1 \right\|_{\Psi, v} = \left\| \left\{ \rho_m'' \right\} \right\|_{\bar{l}_{\Psi,\beta}},$$

so that

$$\left\| \left\{ \rho_m' \right\} \right\|_{\bar{l}_{\Psi,\beta}} \leqslant \left\| \rho_a(g)\, \chi_{(0,1)} \right\|_{\Psi, v} \leqslant \left\| \left\{ \rho_m'' \right\} \right\|_{\bar{l}_{\Psi,\beta}}. \tag{210}$$

Moreover,

$$\{\rho_m\}_{m \in Z^-} = \{\bar{\rho}_m\}_{m \in Z^-} + \left\{ \hat{\rho}_m \right\}_{m \in Z^-},$$

where

$$\bar{\rho}_m = \rho_m, m \leqslant -1, \bar{\rho}_0 = 0; \quad \hat{\rho}_m = 0, m \leqslant -1, \hat{\rho}_0 = \rho_0. \tag{211}$$

Consequently,

$$\left\| \{\rho_m\} \right\|_{\bar{l}_{\Psi,\beta}} \cong \left\| \{\bar{\rho}_m\} \right\|_{\bar{l}_{\Psi,\beta}} + \left\| \left\{ \hat{\rho}_m \right\} \right\|_{\bar{l}_{\Psi,\beta}}. \tag{212}$$

Introduce

$$A_m(g) = \frac{\rho_m}{\Psi^{-1}(1/\beta_m)} = \frac{1}{\beta_m \Psi^{-1}(1/\beta_m)} \int\limits_{\Delta_m} |g| dt, \quad m \in Z^-. \tag{213}$$

Note that,

$$\left\| \left\{ \hat{\rho}_m \right\} \right\|_{\bar{l}_{\Psi,\beta}} = \inf \left\{ \lambda > 0 : \ \Psi\left(\lambda^{-1} \rho_0 \right) \beta_0 \leqslant 1 \right\} = A_0(g) =$$

$$= \frac{1}{(b-1)\, \Psi^{-1}\left((b-1)^{-1} \right)} \int\limits_1^\infty |g| dt.$$

According to (210),

$$\left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} + A_{-1}(g) \leqslant \left\|\rho_a(g)\,\chi_{(0,1)}\right\|_{\Psi,v} + A_{-1}(g) \leqslant$$
$$\leqslant \left\|\left\{\rho''_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} + A_{-1}(g)\,. \tag{214}$$

Further, we will prove the equivalence

$$\left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} + A_{-1}(g) \cong \left\|\left\{\rho''_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} \cong \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{215}$$

Then, both parts of (214) will be equivalent to $\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}$ (the second term in the right hand side of (214) is subordinate to the first one). Consequently, we obtain

$$\left\|\rho_a(g)\,\chi_{(0,1)}\right\|_{\Psi,v} + A_{-1}(g) \cong \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}.$$

Now, we take into account the estimate (212), and obtain the equivalence

$$\left\|\rho_a(g)\,\chi_{(0,1)}\right\|_{\Psi,v} + A_{-1}(g) + A_0(g) \cong \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} + A_0(g) \cong \left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}}.$$

According to (203), this is the needed estimate (206).

Thus, it remains to prove (215). We recall that $\rho'_0 = \rho''_0 = 0$. For $m \leqslant -1$ the equalities (202), and (208) show that

$$\rho'_m = b^{-2}(b-1)\,\bar{\rho}_{m-1}; \tag{216}$$

$$\rho''_m = \rho'_{m-1} + b\rho'_m + (b-1)\,\bar{\rho}_m. \tag{217}$$

From (216) it follows,

$$\left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant b^{-2}(b-1)\left\|\{\bar{\rho}_{m-1}\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant b^{-1}(b-1)\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{218}$$

$$\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} \cong A_{-1}(g) + \left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}}. \tag{219}$$

In (218) we take into account the boundedness of shift operator in the space $\bar{l}_{\Psi,\beta}$ with Young function Ψ, and $\beta = \{\beta_m\}$ from (202); see Lemma 9. Therefore,

$$\left\|\{\rho_{m-1}\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant b\left\|\{\rho_m\}\right\|_{\bar{l}_{\Psi,\beta}}.$$

To prove (219) we use the following chain of equalities (recall that $\bar{\rho}_0 = \rho'_0 = 0$)

$$\bar{j}_\lambda(\{\bar{\rho}_m\}) = \sum_{m\in Z^-} \Psi\left(\lambda^{-1}\bar{\rho}_m\right)\beta_m = \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} + \sum_{m\leqslant -2} \Psi\left(\lambda^{-1}\bar{\rho}_m\right)\beta_m.$$

In the second term we use the equality $\bar{\rho}_m = b^2 (b-1)^{-1} \rho'_{m+1}$, $m \leqslant -2$ (see (216)), so that

$$\begin{aligned}\sum_{m\leqslant -2} \Psi\left(\lambda^{-1}\bar{\rho}_m\right)\beta_m &= \sum_{m\leqslant -2} \Psi\left(\lambda^{-1} b^2 (b-1)^{-1} \rho'_{m+1}\right)\beta_m = \\ &= \sum_{m\leqslant -1} \Psi\left(\lambda^{-1} b^2 (b-1)^{-1} \rho'_{m}\right)\beta_{m-1} = \\ &= b^{-1}\sum_{m\leqslant -1} \Psi\left(\lambda^{-1} b^2 (b-1)^{-1} \rho'_{m}\right)\beta_{m} = \\ &= b^{-1}\sum_{m\in Z^-} \Psi\left(\lambda^{-1} b^2 (b-1)^{-1} \rho'_{m}\right)\beta_{m}.\end{aligned}$$

As the result we obtain,

$$\bar{j}_\lambda\left(\{\bar{\rho}_m\}\right) = \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} + b^{-1}\bar{j}_{(b-1)b^{-2}\lambda}\left(\left\{\rho'_m\right\}\right). \tag{220}$$

Let $\lambda = \max\{\lambda_1, \lambda_2\}$, where

$$\lambda_1 = \inf\left\{\lambda > 0 : \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} \leqslant 1 - b^{-1}\right\} = \bar{\rho}_{-1}/\Psi^{-1}(1),$$

$$\lambda_2 = \inf\left\{\lambda > 0 : \bar{j}_{(b-1)b^{-2}\lambda}\left(\left\{\rho'_m\right\}\right) \leqslant 1\right\} = b^2 (b-1)^{-1} \left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}}.$$

Then, $\bar{j}_\lambda\left(\{\bar{\rho}_m\}\right) \leqslant 1$, and (220) implies

$$\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} \leqslant \lambda = \max\left\{\bar{\rho}_{-1}/\Psi^{-1}(1), \quad b^2 (b-1)^{-1}\left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}}\right\}. \tag{221}$$

From the other side, we see by (220), that

$$\begin{aligned}&\bar{j}_\lambda\left(\{\bar{\rho}_m\}\right) \geqslant \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} \Rightarrow \\ &\qquad \Rightarrow \left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} \geqslant \inf\left\{\lambda > 0 : \ \Psi\left(\lambda^{-1}\bar{\rho}_{-1}\right)\beta_{-1} \leqslant 1\right\} = A_{-1}(g).\end{aligned}$$

Together with (218), it gives inequality

$$\left\|\{\bar{\rho}_m\}\right\|_{\bar{l}_{\Psi,\beta}} \geqslant \max\left\{A_{-1}(g), b(b-1)^{-1}\left\|\left\{\rho'_m\right\}\right\|_{\bar{l}_{\Psi,\beta}}\right\}. \tag{222}$$

Inequalities (221) and (222) imply the two-sided estimate (219) with constants depending on b, because $\bar{\rho}_{-1}/\Psi^{-1}(1) \cong A_{-1}(g)$.

Now, we will obtain the estimate (215). The equality (217) shows that

$$\left\| \left\{ \rho''_m \right\} \right\|_{\bar{l}_{\Psi,\beta}} \geqslant (b-1) \left\| \{ \bar{\rho}_m \} \right\|_{\bar{l}_{\Psi,\beta}}; \tag{223}$$

$$\left\| \left\{ \rho''_m \right\} \right\|_{\bar{l}_{\Psi,\beta}} \leqslant \left\| \left\{ \rho'_{m-1} \right\} \right\|_{\bar{l}_{\Psi,\beta}} + b \left\| \left\{ \rho'_m \right\} \right\|_{\bar{l}_{\Psi,\beta}} + (b-1) \left\| \{ \rho_m \} \right\|_{\bar{l}_{\Psi,\beta}}. \tag{224}$$

The first term in (224) is not bigger than the second one because of the estimate for the norm of shift operator. In its turn, the second term is not bigger than the third one in view of the estimate (218). As the result we obtain,

$$\left\| \left\{ \rho''_m \right\} \right\|_{\bar{l}_{\Psi,\beta}} \leqslant 3 \, (b-1) \left\| \{ \bar{\rho}_m \} \right\|_{\bar{l}_{\Psi,\beta}}. \tag{225}$$

Estimates (223) and (225) imply the equivalence

$$\left\| \left\{ \rho''_m \right\} \right\|_{\bar{l}_{\Psi,\beta}} \cong \left\| \{ \bar{\rho}_m \} \right\|_{\bar{l}_{\Psi,\beta}}.$$

Together with (219) it gives (215), thus completing the proof of Theorem.

6 Applications to Weighted Orlicz-Lorentz Classes

Recall the notion of decreasing rearrangement for measurable function. Let $M_0 = M_0 \left(R_+ \right)$ be the subspace of functions $f : R_+ \to R$, measurable with respect to Lebesgue measure μ, finite almost everywhere, and such that distribution function λ_f is not identically infinity for $f \in M_0$. Here,

$$\lambda_f \left(y \right) = \mu \left\{ x \in R_+ : \left| f \left(x \right) \right| > y \right\}, \, y \in R_+. \tag{226}$$

Then, $0 \leqslant \lambda_f \downarrow$, $\lambda_f \left(y \right) \to 0 \left(y \to +\infty \right)$. Consider the decreasing rearrangement f^* of function f,

$$f^* \left(t \right) = \inf \left\{ y \in R_+ : \lambda_f \left(y \right) \leqslant t \right\}, \, t \in R_+. \tag{227}$$

We deal with Orlicz-Lorentz class $\Lambda_{\Phi,v}$ related to Orlicz space $L_{\Phi,v}$. For $f \in M_0$ we define

$$J_\lambda \left(f^* \right) = \int\limits_0^\infty \Phi \left(\lambda^{-1} f^* \left(t \right) \right) v \left(t \right) dt, \quad \lambda > 0. \tag{228}$$

Here $v \in M^+$, integration by Lebesgue measure and weight satisfies the condition (8). Weighted Orlicz-Lorentz class $\Lambda_{\Phi,v}$ consists of functions $f \in M_0(R_+)$ such that $f^* \in L_{\varphi,v}$. We equip it by the functional

$$\left\| f^* \right\|_{\Phi,v} = \inf\left\{\lambda > 0 : \quad J_\lambda\left(f^*\right) \leqslant 1\right\}. \tag{229}$$

To deal with linear space $\Lambda_{\Phi,v}$, it would be assumed additionally that weight function V (8) satisfies Δ_2-condition, that is

$$\exists C \in R_+ : \; V(2t) \leqslant CV(t), \quad \forall t \in R_+. \tag{230}$$

It is known that such assumption is necessary for the validity of triangle inequality in Lorentz space; see for example [14]. Nevertheless, *we need not estimate (230) in our considerations.* Anyway, we can consider class $\Lambda_{\Phi,v}$ as the cone in M_0, that consists of functions having finite values of functional (229). Here, we present the analogous for the results of Sect. 3 concerning estimates of the norms of monotone operators over Orlicz-Lorentz classes. We recall some descriptions. Let $(\mathrm{N}, \Re, \eta)$ be the measure space with nonnegative σ-finite measure η; as $L = L(\mathrm{N}, \Re, \eta)$ we denote space of all η-measurable functions $u : \mathrm{N} \to R$; $L^+ = \{u \in L : u \geqslant 0\}$. Let $Y_i = Y_i(\mathrm{N}, \Re, \eta) \subset L, i = 1, 2$ be ideal spaces; $P : M_0^+(R_+) \to L^+$ be a monotone operator related to these spaces by the following condition: for $h \in \Omega$

$$\left\| Ph \right\|_{Y_2} = \sup\left\{\left\| Pf \right\|_{Y_1} : \; f \in M_0^+(R_+), f^* = h\right\}. \tag{231}$$

We illustrate these conditions by two examples.

Example 7 Let P be identical operator on $M_0^+(R_+)$,

$$Y_1 = L_1(R_+; g), \quad g \in M_0^+(R_+); \quad Y_2 = L_1\left(R_+; g^*\right).$$

Then, the equality (231) reflects the well-known extremal property of decreasing rearrangements; see [11, Sects. 2.3–2.8]):

$$\sup\left\{\int_0^\infty fg\,dt : \; f \in M_0^+, \; f^* = h\right\} = \int_0^\infty hg^*\,dt.$$

Example 8 Let Y be an ideal space, and monotone operator $P : M_0^+(R_+) \to L^+$ satisfies the condition

$$\left\| Pf \right\|_Y \leqslant \left\| Pf^* \right\|_Y, \quad f \in M_0^+(R_+). \tag{232}$$

Then, the equality (231) holds with $Y_1 = Y_2 = Y$.

Indeed, $f \in M_0^+(R_+) \Rightarrow h := f^* \in M_0^+(R_+)$, $h^* = h$, and

$$\|Ph\|_Y \leqslant \sup\left\{\|Pf\|_Y : f \in M_0^+(R_+),\ f^* = h\right\}.$$

From the other side, for every function $f \in M_0^+(R_+) : f^* = h$, we have according to (232),

$$\|Pf\|_Y \leqslant \|Pf^*\|_Y = \|Ph\|_Y \Rightarrow \sup\left\{\|Pf\|_Y : f \in M_0^+(R_+),\ f^* = h\right\} \leqslant \|Ph\|_Y.$$

Remark 10 Example 8 covers, in particular, such operator as

$$(Pf)(x) = \int_0^\infty k(x,\tau) f(\tau)\, d\tau, \quad x \in \mathrm{N}, \tag{233}$$

where k is nonnegative measurable function on $\mathrm{N} \times R_+$, and $k(x,\tau)$ is decreasing and right continuous as function of $\tau \in R_+$. Then, for $f \in M_0^+(R_+)$, and almost all $x \in \mathrm{N}$, we obtain by the well-known Hardy's lemma

$$(Pf)(x) = \int_0^\infty k(x,\tau) f(\tau)\, d\tau \leqslant \int_0^\infty k(x,\tau) f^*(\tau)\, d\tau = \left(Pf^*\right)(x).$$

Consequently, inequality (232) holds for every ideal space Y.

Proposition 2 *Let $P : M_0^+(R_+) \to L^+$ be monotone operator and equality (231) be true. We define $\Lambda^+_{\Phi,v} = \Lambda_{\Phi,v} \cap M_0^+$ and introduce the norms*

$$\|P\|_{\Lambda^+_{\Phi,v} \to Y_1} = \sup\left\{\|Pf\|_{Y_1} : f \in M_0^+(R_+),\ \|f^*\|_{\Phi,v} \leqslant 1\right\}; \tag{234}$$

$$\|P\|_{\Omega \to Y_2} = \sup\left\{\|Ph\|_{Y_2} : h \in \Omega,\ \|h\|_{\Phi,v} \leqslant 1\right\}. \tag{235}$$

Then, these norms coincide to each other:

$$\|P\|_{\Lambda^+_{\Phi,v} \to Y_1} = \|P\|_{\Omega \to Y_2}. \tag{236}$$

Proof We use the equivalence

$$f \in M_0^+;\ \|f^*\|_{\Phi,v} \leqslant 1 \Leftrightarrow h = f^* \in \Omega :\ \|h\|_{\Phi,v} \leqslant 1,$$

and obtain

$$\|P\|_{\Lambda^+_{\Phi,v} \to Y_1} = \sup\left[\sup\left\{\|Pf\|_{Y_1} : f \in M_0^+(R_+),\ f^* = h\right\} : h \in \Omega,\ \|h\|_{\Phi,v} \leqslant 1\right].$$

According to (231), the right hand side here coincides with $\|P\|_{\Omega \to Y_2}$.

Remark 11 This Proposition admits us to reduce estimates of the norm $\|P\|_{\Lambda^+_{\Phi,v}\to Y_1}$ (234) to the estimates presented in Sects. 3 and 4. In particular, by the help of Example 7, we reduce the associate norm for function $g \in M$ on Orlicz–Lorentz class to the associate norm for its decreasing rearrangement g^* on the cone Ω:

$$\|g\|'_* := \sup\left\{\int_0^\infty f\,|g|\,dt:\ f\in M_0^+;\ \|f^*\|_{\Phi,v}\leqslant 1\right\} = \|g^*\|'.$$

Then, Theorem 7 and Remark 9 lead to the following result.

Theorem 9 *Let the assumptions of Theorem 7 be fulfilled. Then,*

$$\|g\|'_* \cong \|\rho_a(g^*)\|_{\Psi,v} = \inf\left\{\lambda>0:\ \int_0^\infty \Psi\left(\lambda^{-1}\rho_a\left(g^*;t\right)\right)v(t)\,dt \leqslant 1\right\}, \quad (237)$$

where ρ_a was determined in (176). Norms (237) are equivalent for different values $a \in (0,1)$.

Remark 12 Assume additionally that function Φ satisfies Δ_2-condition in Theorem 9. Then,

$$\|g\|'_* \cong \|V(t)^{-1}\int_0^t g^*(\tau)d\tau\|_{\Psi,v}. \quad (238)$$

Remark 13 In (237) and (238), we present some modifications of the result in [18] that develop preceding results of paper [13]. Note that, in [13] formula (238) was established under restriction that both functions Φ, and Ψ satisfy Δ_2-condition. Concerning duality problems for Orlicz, Lorentz, and Orlicz-Lorentz spaces see also [2, 4, 15, 16].

Now, let us describe the modification of the above presented results.

Theorem 10 *Let $Y \subset L$ be some ideal space with quasi-norm $\|\cdot\|_Y$, let $P: M^+ \to L^+$ be a monotone operator satisfying the condition: there exists constant $C \in R_+$ such that*

$$\|Pf\|_Y \leqslant C\|Pf^*\|_Y, \quad f\in M^+(R_+). \quad (239)$$

Then,

$$\|P\|_{\Omega\to Y} \leqslant \|P\|_{\Lambda^+_{\Phi,v}\to Y} \leqslant C\|P\|_{\Omega\to Y}. \quad (240)$$

If $C = 1$ in (239), then we have equality of the norms in (240).

Corollary 5 *In the conditions of Theorem 10 we have*

$$\|P\|_{\Lambda^+_{\Phi,v}\to Y} \cong \|P\|_{S\to Y}.$$

For the proof of Theorem 10, let us note that (239) implies

$$\left\| Ph \right\|_Y \leqslant \sup \left\{ \left\| Pf \right\|_Y : \ f \in M_0^+ \left(R_+ \right), f^* = h \right\} \leqslant C \left\| Ph \right\|_Y . \tag{241}$$

Indeed, $f \in M_0^+ \left(R_+ \right) \quad \Rightarrow \quad h := f^* \in M_0^+ \left(R_+ \right), h^* = h$, *and*

$$\left\| Ph \right\|_Y \leqslant \sup \left\{ \left\| Pf \right\|_Y : \ f \in M_0^+ \left(R_+ \right), \ f^* = h \right\}.$$

From the other side, for any function $f \in M_0^+ \left(R_+ \right) : \ f^* = h$, *we have by (239),*

$$\left\| Pf \right\|_Y \leqslant C \left\| Pf^* \right\|_Y = C \left\| Ph \right\|_Y \Rightarrow$$
$$\Rightarrow \sup \left\{ \left\| Pf \right\|_Y : \ f \in M_0^+ \left(R_+ \right), \ f^* = h \right\} \leqslant C \left\| Ph \right\|_Y .$$

Moreover, (241) implies (240). Indeed, we use equivalence

$$f \in M_0; \quad \left\| f^* \right\|_{\Phi, v} \leqslant 1 \Leftrightarrow h = f^* \in \Omega : \ \left\| h \right\|_{\Phi, v} \leqslant 1,$$

and obtain

$$\left\| P \right\|_{\Lambda^+_{\Phi,v} \to Y} = \sup \left[\sup \left\{ \left\| Pf \right\|_Y : \ f \in M_0^+ \left(R_+ \right), \ f^* = h \right\} : \ h \in \Omega, \left\| h \right\|_{\Phi, v} \leqslant 1 \right].$$

Here, according to (241), the right hand side is estimated from below by

$$\sup \left[\left\| Ph \right\|_Y : \ h \in \Omega, \left\| h \right\|_{\Phi, v} \leqslant 1 \right] = \left\| P \right\|_{\Omega \to Y},$$

and, in addition, from above by the same value multiplied by C.

Example 9 Theorem 10 covers the case of Hardy–Littlewood maximal operator $\mathrm{M} : M_+ \left(R_+ \right) \to M_+ \left(R_+ \right)$, where

$$\left(\mathrm{M} f \right) (x) = \sup \left\{ |\Delta|^{-1} \int_{\Delta} f \left(\tau \right) d\tau : \ \Delta \subset R_+; x \in \Delta \right\},$$

and $Y = Y \left(R_+ \right)$ is rearrangement invariant space (shortly: RIS). Indeed, by Luxemburg representation theorem (see [11, Chap. 2, Theorem 4.10]), for every RIS Y there exists unique RIS $\tilde{Y} = \tilde{Y} \left(R_+ \right)$:

$$\left\| g \right\|_Y = \left\| g^* \right\|_{\tilde{Y}}, \quad g \in M \left(R_+ \right).$$

Note that,

$$\left(\mathrm{M}f^*\right)^*(t) = \mathrm{M}f^*(t) = t^{-1}\int_0^t f^*(\tau)d\tau, \quad t \in R_+.$$

Then, $\|\mathrm{M}f\|_Y = \|(\mathrm{M}f)^*\|_{\tilde{Y}}$, $\|\mathrm{M}f^*\|_Y = \|\mathrm{M}f^*\|_{\tilde{Y}}$.

It is known that $\exists C \in R_+$: $(\mathrm{M}f)^*(x) \leqslant C\,(\mathrm{M}f^*)(x)$; see [11, Chap. 2]. Consequently,

$$\|\mathrm{M}f\|_Y = \|(\mathrm{M}f)^*\|_{\tilde{Y}} \leqslant C\|\mathrm{M}f^*\|_{\tilde{Y}} = C\|\mathrm{M}f^*\|_Y.$$

This inequality coincides with the estimate (239) for operator $P = \mathrm{M}$. Therefore, Theorem 10 is applicable to this operator, and we come to equivalences

$$\|\mathrm{M}\|_{\Lambda^+_{\Phi,v}\to Y} \cong \|\mathrm{M}\|_{\Omega\to Y} \cong \|\mathrm{M}\|_{S\to Y}.$$

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Complex Interpolation of Morrey Spaces

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Abstract In this article, we extend the previous results on the complex interpolation of Morrey spaces to the case when $0 < q \le p < \infty$. We show that the space produced by the first complex interpolation functor $[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta$ is a subset of the Morrey space $\mathcal{M}^p_q$ and it contains some closed subspaces of $\mathcal{M}^p_q$. Meanwhile, we prove that Morrey spaces are closed under the second complex interpolation functor. We also present the complex interpolation of certain closed subspaces of Morrey spaces.

Keywords Morrey spaces · Complex interpolation

1 Introduction

We aim to show that our earlier results on the complex interpolation of Morrey space $\mathcal{M}^p_q$ [12, 13] carry over to the case when $0 < q \le p < \infty$. In our earlier results, we needed to restrict ourselves in the setting of $1 \le q \le p < \infty$. We obtain some descriptions of the first and the second Calderón's complex interpolation functors. We also show that our results carry over to the case of metric measure spaces. To this end, let us place ourselves in the setting of a separable metric measure space $(\mathcal{X}, \mu)$ equipped with a σ-finite measure μ. We present the following definition of the Morrey space $\mathcal{M}^p_q(\mathcal{X}, \mu)$ with $0 < q \le p < \infty$.

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Definition 1 For $0 < q \le p < \infty$, the Morrey space $\mathcal{M}^p_q(\mathcal{X}) = \mathcal{M}^p_q(\mathcal{X}, \mu)$ is defined to be the set of all μ-measurable functions f such that the quasi-norm

$$\|f\|_{\mathcal{M}^p_q(\mathcal{X})} := \sup_{B \subseteq \mathcal{X}} \mu(B)^{\frac{1}{p}-\frac{1}{q}} \left(\int_B |f(x)|^q \, d\mu(x) \right)^{\frac{1}{q}}$$

is finite. Here, the supremum is taken over all balls B in $\mathcal{X}$.

Based on the two complex interpolation functors by Calderón [5], we prove the following theorems. The first result is on the first complex interpolation functor whose definition we recall in Definition 3.

Theorem 1 *Suppose that we have 7 parameters* $\theta \in (0, 1)$, p_0, p_1, p, q_0, q_1, *and* q *satisfying* $0 < q_0 \le p_0 < \infty$, $0 < q_1 \le p_1 < \infty$, $0 < q \le p < \infty$,

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \text{and} \quad \frac{q}{p} = \frac{q_0}{p_0} = \frac{q_1}{p_1}.$$

If $\min(q_0, q_1) < 1$, *then we have the following inclusion:*

$$\{f \in \mathcal{M}^p_q(\mathcal{X}, \mu) : \lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0\} \subseteq [\mathcal{M}^{p_0}_{q_0}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]_\theta \subseteq \mathcal{M}^p_q(\mathcal{X}, \mu). \quad (1)$$

Remark 1 In our previous result [13], we showed that the following identity holds when $\min(q_0, q_1) \ge 1$:

$$[\mathcal{M}^{p_0}_{q_0}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]_\theta = \{f \in \mathcal{M}^p_q(\mathcal{X}, \mu) : \lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0\}. \quad (2)$$

For the second interpolation functor, whose definition we recall in Definition 4, we have the following result.

Theorem 2 *Keep the same condition as Theorem 1. Then we can describe the Calderón second complex interpolation as follows:*

$$[\mathcal{M}^{p_0}_{q_0}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]^\theta = \mathcal{M}^p_q(\mathcal{X}, \mu). \quad (3)$$

The related result about the interpolation of Morrey spaces can be traced back to a certain generalization of the Riesz-Thorin interpolation theorem in [25]. Let $0 < \theta < 1$. In [8, p. 35] Cobos, Peetre, and Persson pointed out that

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta \subset \mathcal{M}^p_q$$

as long as $1 \le q_0 \le p_0 < \infty$, $1 \le q_1 \le p_1 < \infty$, and $1 \le q \le p < \infty$ satisfy

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{4}$$

A counterexample by Blasco, Ruiz, and Vega [3, 22], shows that the interpolation of linear operators on Morrey spaces might not hold if we assume the condition (4) only. Using the counterexample by Ruiz and Vega in [22], Lemarié-Rieusset [15, Theorem 3(ii)] showed that if an interpolation functor F satisfies

$$F[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}] = \mathcal{M}^p_q$$

under the condition (4), then

$$\frac{q_0}{p_0} = \frac{q_1}{p_1} \tag{5}$$

holds. Lemarié-Rieusset [16] investigated the interpolation of Morrey spaces by the second complex interpolation method and proved (3) for the case

$$\min(q_0, q_1) \geq 1 \text{ and } (X, \mu)$$

is $\mathbb{R}^n$ with the Lebesgue measure. Meanwhile, as for the interpolation result under (4) and (5) by using the first complex interpolation functor by Calderón [5], Lu, Yang, and Yuan obtained the following description:

$$[\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1}]_\theta = \overline{\mathcal{M}^{p_0}_{q_0} \cap \mathcal{M}^{p_1}_{q_1}}^{\mathcal{M}^p_q}$$

in [17, Theorem 1.2]. Their result is in the setting of a metric measure space. The generalization of the result of Lu et al. and Lemarié-Rieusset in the setting of generalized Morrey spaces and generalized Orlicz-Morrey spaces can be seen in [12]. We refer to [30] for complex interpolation of Morrey spaces, certain closed subspaces, and the interpolation of smoothness Morrey spaces considered in [28, 29]. As for the real interpolation results, Burenkov and Nursultanov obtained an interpolation result in local Morrey spaces [4]. Their results are generalized by Nakai and Sobukawa to the B_σ-setting [19]. Interpolation of variable Morrey spaces on quasi-metric spaces was investigated in [18].

The interpolation on quasi-Banach spaces can be traced back to the paper by Calderon and Zygmund [6], where the authors proved some generalizations of the Riesz-Thorin interpolation theorem in the setting Lebesgue space L^p, $p \in (0, 1)$. Extensions of Calderón's complex interpolation method in quasi-Banach space are considered in [9, 14, 27]. In [27], Yuan considered the inner complex interpolation space and proves the following result:

$$[\mathcal{M}^{p_0}_{q_0}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]^i_\theta = \overline{\mathcal{M}^{p_0}_{q_0}(\mathcal{X}, \mu) \cap \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)}^{\mathcal{M}^p_q(\mathcal{X}, \mu)},$$

where $[\cdot, \cdot]^i_\theta$ denotes the inner complex interpolation space.

We employ the following notations for some closed subspaces of the Morrey space $\mathcal{M}^p_q(\mathcal{X},\mu)$:

Definition 2 Let $0 < q \le p < \infty$. Denote by $L^0_{\rm c}(\mathcal{X},\mu)$ the set of all boundedly supported μ-measurable functions. The spaces $\widetilde{\mathcal{M}^p_q}(\mathcal{X},\mu)$, $\overset{*}{\mathcal{M}}{}^p_q(\mathcal{X},\mu)$, and $\widehat{\mathcal{M}^p_q}(\mathcal{X},\mu)$ are defined as follows:

1. [12, p.299] the tilde space $\widetilde{\mathcal{M}^p_q}(\mathcal{X},\mu)$ is defined to be
$$\widetilde{\mathcal{M}^p_q}(\mathcal{X},\mu) := \overline{L^\infty_{\rm c}(\mathcal{X},\mu)}^{\mathcal{M}^p_q(\mathcal{X},\mu)},$$
where $L^\infty_{\rm c}(\mathcal{X},\mu)$ denotes the set of all essentially bounded functions with bounded support,
2. [30, Definition 2.23] the star space $\overset{*}{\mathcal{M}}{}^p_q(\mathcal{X},\mu)$ is defined to be $\overset{*}{\mathcal{M}}{}^p_q(\mathcal{X},\mu) := \overline{L^0_{\rm c}(\mathcal{X},\mu)\cap\mathcal{M}^p_q(\mathcal{X},\mu)}^{\mathcal{M}^p_q(\mathcal{X},\mu)}$,
3. [7, (3.2)–(3.4)] the bar space $\overline{\mathcal{M}^p_q}(\mathcal{X},\mu)$ is defined to be
$$\overline{\mathcal{M}^p_q}(\mathcal{X},\mu) := \overline{L^\infty(\mathcal{X},\mu)\cap\mathcal{M}^p_q(\mathcal{X},\mu)}^{\mathcal{M}^p_q(\mathcal{X},\mu)}.$$
4. [23, Definition 4.3] the hat space $\widehat{\mathcal{M}^p_q}(\mathcal{X},\mu)$ is defined to be
$$\widehat{\mathcal{M}^p_q}(\mathcal{X},\mu) := \{f \in \mathcal{M}^p_q(\mathcal{X},\mu) \\ : \lim_{k\to\infty}\|f\chi_{E_k}\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0 \text{ as } \lim_{k\to\infty}\chi_{E_k}(x) = 0 \text{ a.e.} x \in X\}.$$

Following [13, Definition 1.5.6], we use $\overline{\mathcal{M}^p_q}(\mathcal{X},\mu)$ unlike the paper [7]. One of the relation between these subspaces is as follows.

Lemma 1 [23, Theorem 4.6] *For* $0 < q \le p < \infty$*, we have* $\widetilde{\mathcal{M}^p_q}(\mathcal{X},\mu) = \widehat{\mathcal{M}^p_q}(\mathcal{X},\mu)$.

We remark that the space $\widetilde{\mathcal{M}}{}^p_q$ is equal to the closure in Morrey spaces of the set of all finite linear combination of the characteristic functions of sets of finite measure. The setting in [23, Theorem 4.6] is $X = \mathbb{R}^n$ with the Lebesgue measure and $q > 1$.

We remark that our particular results on the first complex interpolation of closed subspaces in [13] are the following identities:

$$[\widetilde{\mathcal{M}}{}^{p_0}_{q_0}, \widetilde{\mathcal{M}}{}^{p_1}_{q_1}]_\theta = [\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}, \overset{*}{\mathcal{M}}{}^{p_1}_{q_1}]_\theta = \widetilde{\mathcal{M}}{}^p_q, \tag{6}$$

and

$$[\overline{\mathcal{M}^{p_0}_{q_0}}, \overline{\mathcal{M}^{p_1}_{q_1}}]_\theta = \left\{ f \in \overline{\mathcal{M}^p_q} : \lim_{a\to 0^+}\|\chi_{\{|f|<a\}}f\|_{\mathcal{M}^p_q} = 0 \right\}. \tag{7}$$

where $\min(q_0, q_1) \geq 1$ and the parameters q_0, q_1, q, p_0, p_1, p and θ satisfy the condition in Theorem 3. Our aim in investigating the first complex interpolation of closed subspace is to extend (6) and (7) for $\min(q_0, q_1) < 1$. More precisely, our result on the complex interpolation of closed subspaces of Morrey spaces is given as follows:

Theorem 3 *Keep the same assumption as in Theorem 1. Then we have*

(i) $[\widetilde{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]_\theta = \widetilde{\mathcal{M}^{p}_{q}}(\mathcal{X}, \mu)$;

(ii) $\left\{ f \in \overset{*}{\mathcal{M}}{}^{p}_{q}(\mathcal{X}, \mu) : \lim_{N\to\infty} \| f - \chi_{\{\frac{1}{N} \leq |f| \leq N\}} f \|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0 \right\}$
$\subseteq [\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]_\theta \subseteq \overset{*}{\mathcal{M}}{}^{p}_{q}(\mathcal{X}, \mu)$;

(iii) $\left\{ f \in \overline{\mathcal{M}^{p}_{q}}(\mathcal{X}, \mu) : \lim_{N\to\infty} \| f - \chi_{\{\frac{1}{N} \leq |f|\}} f \|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0 \right\}$
$\subseteq [\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu), \overline{\mathcal{M}^{p_1}_{q_1}}(\mathcal{X}, \mu)]_\theta \subseteq \overline{\mathcal{M}^{p}_{q}}(\mathcal{X}, \mu)$;

(iv) $[\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu), \overset{*}{\mathcal{M}}{}^{p_1}_{q_1}(\mathcal{X}, \mu)]_\theta = \widetilde{\mathcal{M}^{p}_{q}}(\mathcal{X}, \mu)$.

In Theorem 3(ii) and (iii) only one inclusion is obtained. This is because we assume $0 < q_0 \leq p_0 < \infty$ and $0 < q_1 \leq p_1 < \infty$ only. If $1 \leq q_0 \leq p_0 < \infty$ and $1 \leq q_1 \leq p_1 < \infty$, we have

$$[\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]_\theta = \overline{\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X}, \mu) \cap \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)}^{\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X},\mu)^{1-\theta} \cdot \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)^{\theta}}$$

and

$$[\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu), \overline{\mathcal{M}^{p_1}_{q_1}}(\mathcal{X}, \mu)]_\theta = \overline{\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu) \cap \overline{\mathcal{M}^{p_1}_{q_1}}(\mathcal{X}, \mu)}^{\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu)^{1-\theta} \cdot \overline{\mathcal{M}^{p_1}_{q_1}}(\mathcal{X},\mu)^{\theta}},$$

where ${X_0}^{1-\theta} \cdot {X_1}^{\theta}$ denotes the Calderón product of X_0 and X_1. Thus, we can specify $[\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]_\theta$ and $[\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu), \overline{\mathcal{M}^{p_1}_{q_1}}(\mathcal{X}, \mu)]_\theta$, see [11].

In Sect. 2, we shall recall the notion of the first and the second complex interpolation functors. We also prove several lemmas related to the complex interpolation of Lebesgue spaces in this section. The proof of Theorems 1 and 2 are given in Sect. 3. We give the proof of Theorem 3 in Sect. 4.

2 Preliminaries

2.1 Complex Interpolation of Quasi-Banach Spaces

Now, we recall the notion of complex interpolations of quasi-Banach spaces. Let $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$ and $\bar{S} := \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$. For $z_0 \in \mathbb{C}$ and $\eta > 0$, we define $\Delta(z_0, \eta) := \{z \in \mathbb{C} : |z - z_0| < \eta\}$ and $\overline{\Delta}(z_0, \eta) := \{z \in \mathbb{C} : |z - z_0| \leq \eta\}$.

Definition 3 Let X be a quasi-Banach space.

1. A map $f : S \to X$ is said to be analytic, if for any $z_0 \in S$, there exist $\eta \in (0, \infty)$ and $\{h_n\}_{n=0}^{\infty} \subset X$ such that $\overline{\Delta}(z_0, \eta) \subset S$ and that
$$f(z) = \sum_{n=0}^{\infty} h_n (z - z_0)^n$$
for all $z \in \Delta(z_0, \eta)$.
2. A map $f : S \to X$ is said to be bounded, if $f(S) = \{f(z) : z \in S\}$ is a bounded set in X.
3. A quasi-Banach space X is called analytically convex if there exists a positive constant C such that, for any analytic function $f : S \to X$ with extends to a X-valued continuous function on the closed strip $\bar{S}$,
$$\sup_{z \in S} \|f(z)\|_X \le C \sup_{z \in \bar{S} \setminus S} \|f(z)\|_X.$$

We recall the definition of the first complex interpolation functor.

Definition 4 Let $\bar{X} = (X_0, X_1)$ be a compatible couple of quasi-Banach spaces such that $X_0 + X_1$ is analytically convex.

1. The set $\mathcal{F} = \mathcal{F}(\bar{X}) = \mathcal{F}(X_0, X_1)$ is defined to be the set of all continuous functions $F : \bar{S} \to X_0 + X_1$ such that
   (a) F, restricted to S, is analytic and bounded in $X_0 + X_1$;
   (b) $F(j + it) \in X_j$ for all $j = 0, 1$ and $t \in \mathbb{R}$;
   (c) the traces $t \in \mathbb{R} \mapsto F(j + it) \in X_j$ are bounded continuous functions for all $j = 0, 1$.

   Endow $\mathcal{F}$ with the quasi-norm;
$$\|F\|_{\mathcal{F}} := \max_{j=0,1} \sup_{t \in \mathbb{R}} \|F(j + it)\|_{X_j}.$$
2. The first complex interpolation space $[X_0, X_1]_\theta$ with $\theta \in (0, 1)$ is defined by
$$[X_0, X_1]_\theta := \{F(\theta) : F \in \mathcal{F}(\bar{X})\}.$$
   The quasi-norm of $f \in [X_0, X_1]_\theta$ is given by
$$\|f\|_{[X_0, X_1]_\theta} := \inf\{\|F\|_{\mathcal{F}} : F \in \mathcal{F}(\bar{X}), F(\theta) = f\}.$$

We recall the definition of the second complex interpolation functor.

Definition 5 Let $\bar{X} = (X_0, X_1)$ be a compatible couple of quasi-Banach spaces such that $X_0 + X_1$ is analytically convex.

1. The set $\mathcal{G} = \mathcal{G}(\bar{X}) = \mathcal{G}(X_0, X_1)$ is defined to be the set of all functions $G : \bar{S} \to X_0 + X_1$ such that
   (a) $\sup_{z \in \bar{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{X_0+X_1} < \infty$,
   (b) $G(\cdot + ih) - G \in \mathcal{F}(\bar{X})$ for all $h \in \mathbb{R}$,
   (c) The quasi-norm

$$\|G\|_{\mathcal{G}} := \sup_{h \in \mathbb{R} \setminus \{0\}} \frac{\|G(\cdot + ih) - G\|_{\mathcal{F}}}{|h|}$$

   is finite.

2. The second complex interpolation space $[X_0, X_1]^\theta$ with $\theta \in (0, 1)$ is defined by

$$[X_0, X_1]^\theta := \{G'(\theta) \,:\, G \in \mathcal{G}(\bar{X})\}.$$

   The quasi-norm of $g \in [X_0, X_1]^\theta$ is given by

$$\|g\|_{[X_0,X_1]^\theta} := \inf\{\|G\|_{\mathcal{G}} \,:\, G \in \mathcal{G}(\bar{X}),\, G'(\theta) = g\}.$$

2.2 Some Lemmas on Lebesgue Spaces and Subharmonic Functions

In this section, we recall some properties of the Lebesgue space $L^p(\mathcal{X}, \mu)$ for $p \in (0, 1)$ and subharmonic functions. The latter can be seen as a replacement of the holomorphic functions which play an important role in the complex interpolation of Banach spaces. We begin with the following lemma:

Lemma 2 *Let E be a subset of $\mathcal{X}$ with finite measure and* $0 < u < q_1, q_0 < \infty$. *If $f \in L^{q_0}(\mathcal{X}, \mu) + L^{q_1}(\mathcal{X}, \mu)$, then*

$$\|f\chi_E\|_{L^u(\mathcal{X},\mu)} \lesssim \|f\|_{L^{q_0}(\mathcal{X},\mu)+L^{q_1}(\mathcal{X},\mu)}. \tag{8}$$

Proof Recall that for $f \in L^{q_0}(X, \mu) + L^{q_1}(X, \mu)$, we define

$$\|f\|_{L^{q_0}(X,\mu)+L^{q_1}(X,\mu)} := \inf(\|f_0\|_{L^{q_0}(X,\mu)} + \|f_1\|_{L^{q_1}(X,\mu)}),$$

where the couple (f_0, f_1) moves over all decomposition of f: $f = f_0 + f_1$ with $f_0 \in L^{q_0}(X, \mu)$ and $f_1 \in L^{q_1}(X, \mu)$. Let $f \in L^{q_0}(\mathcal{X}, \mu) + L^{q_1}(\mathcal{X}, \mu)$ and choose $f_0 \in L^{q_0}(\mathcal{X}, \mu)$ and $f_1 \in L^{q_1}(\mathcal{X}, \mu)$ such that $f = f_0 + f_1$. Then

$$\begin{aligned}\|f\chi_E\|_{L^u(\mathcal{X},\mu)} &\lesssim \|f_0\chi_E\|_{L^u(\mathcal{X},\mu)} + \|f_1\chi_E\|_{L^u(\mathcal{X},\mu)} \\ &\lesssim \|f_0\chi_E\|_{L^{q_0}(\mathcal{X},\mu)} + \|f_1\chi_E\|_{L^{q_1}(\mathcal{X},\mu)} \\ &\le \|f_0\|_{L^{q_0}(\mathcal{X},\mu)} + \|f_1\|_{L^{q_1}(\mathcal{X},\mu)}.\end{aligned}$$

By taking the infimum over all decompositions $f = f_0 + f_1$, we get (8). □

Now, we recall the definition and some properties of subharmonic functions.

Definition 6 [21, Definition 2.1.1] Let X be a topological space. A function $H : X \to [-\infty, \infty)$ is said to be upper semicontinuous, if the set $H^{-1}([-\infty, \lambda)) = \{x \in X : H(x) < \lambda\}$ is open in X for each $\lambda \in \mathbb{R}$.

Definition 7 [21, Definition 2.2.1] Let U be an open subset of $\mathbb{C}$. A function $H : U \to [-\infty, \infty)$ is called subharmonic, if it is upper semicontinuous and satisfies the local submean inequality, i.e., given $w \in U$, there exists $\rho = \rho_w > 0$ such that

$$H(w) \le \frac{1}{2\pi}\int_0^{2\pi} H(w + re^{it})\,dt \quad (0 \le r < \rho).$$

According to [21, Corollary 2.4.2], this property automatically yields the global submean inequality. We shall use the following properties for subharmonic functions.

Theorem 4 [21, Theorem 2.2.2] *If f is holomorphic on an open set U in $\mathbb{C}$, then* $\log|f|$ *is subharmonic on U.*

Theorem 5 [21, p. 47, Exercise 4] *Let $H : U \to [0, \infty)$ be a function on an open set U in $\mathbb{C}$. Then* $\log H$ *is subharmonic if and only if H^p is subharmonic on U for each $p > 0$.*

As a replacement of the three lines lemma, we invoke the following result for subharmonic functions:

Lemma 3 [26, p. 68, Lemma 2] [10, Lemma 1.3.8] *Let $\theta \in (0, 1)$. Let $H : \overline{S} \to [-\infty, \infty)$ be a subharmonic function on S which is continuous on $\overline{S}$, and* $\sup_{z\in\overline{S}} |H(z)| < \infty$. *Then we have*

$$H(\theta) \le \int_{-\infty}^{\infty} P_0(\theta, t)H(it)\,dt + \int_{-\infty}^{\infty} P_1(\theta, t)H(1 + it)\,dt$$

where

$$P_0(\theta, t) := \frac{\sin(\pi\theta)}{2(\cosh(\pi t) - \cos(\pi\theta))} \text{ and } P_1(\theta, t) := \frac{\sin(\pi\theta)}{2(\cosh(\pi t) + \cos(\pi\theta))}.$$

Proof For each $z \in \overline{S}$, define

$$g(z) := \frac{e^{i\pi z} - i}{e^{i\pi z} + i}.$$

Observe that $g(z) \in \Delta(0, 1)$ and $g^{-1}(z) = \frac{1}{\pi i} \log\left(\frac{i(1+z)}{1-z}\right)$. Since $\mathrm{Im}(\frac{i(1+z)}{1-z}) > 0$ for every $z \in \Delta(0, 1)$, the function $z \in \Delta(0, 1) \mapsto \log\left(\frac{i(1+z)}{1-z}\right)$ is holomorphic. Therefore, $g^{-1}(z)$ is conformal map from $\Delta(0, 1)$ to S. For each $z \in \Delta(0, 1)$, define $G(z) := H(g^{-1}(z))$. Since $H(z)$ is subharmonic on S, we see that $G(z)$ is subharmonic on $\Delta(0, 1)$. Then, for every $r \in (0, \rho)$ with $\rho < 1$ and $0 \le s \le 2\pi$, we have

$$G(re^{is}) \le \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(t-s) + r^2} G(\rho e^{it})\, dt.$$

For every $\rho \in (r/2, 1)$, we have

$$\begin{aligned} \left| \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(t-s) + r^2} G(\rho e^{it}) \right| &\le \left(\sup_{z \in \overline{S}} |H(z)|\right) \frac{\rho^2 - r^2}{\rho^2 - 2\rho r + r^2} \\ &\le \left(\sup_{z \in \overline{S}} |H(z)|\right) \frac{2 + 2r}{1 - r}. \end{aligned}$$

Taking $\rho \uparrow 1$, by the Lebesgue convergence theorem and continuity of G, we get

$$G(re^{is}) \le \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(t-s) + r^2} G(e^{it})\, dt.$$

For $\theta \in (0, 1)$, we have $g(\theta) = -i \frac{\cos(\pi\theta)}{1+\sin(\pi\theta)}$, so, the solution of

$$re^{is} = g(\theta) \quad (r \in (0, 1), s \in (0, 2\pi))$$

is $(r, s) = \begin{cases} \left(\frac{\cos(\pi\theta)}{1+\sin(\pi\theta)}, \frac{3\pi}{2}\right) & \theta \in (0, 1/2], \\ \left(-\frac{\cos(\pi\theta)}{1+\sin(\pi\theta)}, \frac{\pi}{2}\right) & \theta \in (1/2, 1). \end{cases}$ For these pairs, we have

$$H(\theta) = G(g(\theta)) \le \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(t-s) + r^2} H\left(g^{-1}(e^{it})\right)\, dt.$$

and

$$\frac{1 - r^2}{1 - 2r \cos(t-s) + r^2} = \frac{\sin(\pi\theta)}{1 + \sin t \cos(\pi\theta)},$$

so

$$H(\theta) \le \frac{1}{2\pi}\int_0^{\pi} \frac{\sin(\pi\theta)}{1+\sin t\cos(\pi\theta)} H(g^{-1}(e^{it}))\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{\sin(\pi\theta)}{1+\sin t\cos(\pi\theta)} H(g^{-1}(e^{it}))\,dt.$$

For $t \in [0, \pi]$, let $1 + iy = g^{-1}(e^{it})$. Since $e^{it} = -\tanh(\pi y) + i\,\mathrm{sech}(\pi y)$, we have

$$\frac{1}{2\pi}\int_0^{\pi} \frac{\sin(\pi\theta)}{1+\sin t\cos(\pi\theta)} H(g^{-1}(e^{it}))\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(\pi\theta)}{\cosh(\pi y)+\cos(\pi\theta)} H(1+iy)\,dy. \tag{9}$$

By substitution $iy = g^{-1}(e^{it})$ for $t \in [\pi, 2\pi]$, we also have

$$\frac{1}{2\pi}\int_{\pi}^{2\pi} \frac{\sin(\pi\theta)}{1+\sin t\cos(\pi\theta)} H(g^{-1}(e^{it}))\,dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(\pi\theta)}{\cosh(\pi y)-\cos(\pi\theta)} H(iy)\,dy. \tag{10}$$

By combining (9) and (10), we get the desired inequality. □

A direct calculation shows

$$\int_{\mathbb{R}} P_0(\theta, t)\,dt = 1-\theta \tag{11}$$

and

$$\int_{\mathbb{R}} P_1(\theta, t)\,dt = \theta. \tag{12}$$

Related to the interpolation of Lebesgue spaces, we prove the following lemmas:

Lemma 4 *Let* $0 < u < q_1 < q_0 \le \infty$ *and* $0 < \theta < 1$. *Define* q *by:*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If $F \in \mathcal{F}(L^{q_0}(X,\mu), L^{q_1}(X,\mu))$ *and* E *is a measurable set with finite* μ*-measure, then the function*

$$K(z) := \int_E |F(z,x)|^u\,d\mu(x)$$

is subharmonic on S, *continuous and bounded on* $\bar{S}$.

Proof Since $f \in L^{q_0}(X,\mu) + L^{q_1}(X,\mu) \mapsto \chi_E f \in L^u(X,\mu)$ is bounded, K is continuous and bounded on S. Thus, we need to show that

$$K(z_0) \le \frac{1}{2\pi}\int_0^{2\pi} K(z_0 + re^{i\theta})\,d\theta \tag{13}$$

as long as $(z_0, r) \in S \times (0,\infty)$ satisfies $\Delta(z_0, r) \subset \Delta(z_0, 3r) \subset S$. Since F is analytic, there exists a sequence $\{h_n\}_{n=0}^{\infty} \subset L^{q_0}(X,\mu) + L^{q_1}(X,\mu)$ such that

$$\lim_{N\to\infty}\left(\sup_{z\in\Delta(z_0,2r)}\left\|F(z) - \sum_{n=0}^{N} h_n(z-z_0)^n\right\|_{L^{q_0}(X,\mu)+L^{q_1}(X,\mu)}\right) = 0.$$

According to Theorems 4 and 5, for each fixed x we have

$$\left|\sum_{n=0}^{N} h_n(x)(z-z_0)^n\right|^u \le \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=0}^{N} h_n(x)(z-z_0+re^{i\theta})^n\right|^u d\theta.$$

Thus,

$$\begin{aligned}&\int_E \left|\sum_{n=0}^{N} h_n(x)(z-z_0)^n\right|^u d\mu(x) \\ &\le \frac{1}{2\pi}\int_E\int_0^{2\pi}\left|\sum_{n=0}^{N} h_n(x)(z-z_0+re^{i\theta})^n\right|^u d\mu(x)\,d\theta.\end{aligned} \tag{14}$$

Note that the topology of $L^{q_0}(E,\mu) + L^{q_1}(E,\mu)$ is stronger than $L^u(E,\mu)$, due to Lemma 2. Thus, we have

$$K(z) = \lim_{N\to\infty}\int_E \left|\sum_{n=0}^{N} h_n(x)(z-z_0)^n\right|^u d\mu(x) \tag{15}$$

and

$$\int_0^{2\pi} K(z_0+re^{i\theta})\,d\theta = \lim_{N\to\infty}\int_E\int_0^{2\pi}\left|\sum_{n=0}^{N} h_n(x)(z-z_0+re^{i\theta})^n\right|^u d\mu(x)\,d\theta. \tag{16}$$

Thus, we conclude (13) from (14), (15) and (16). □

Proposition 1 *Let* $0 < q_1 < q_0 \le \infty$ *and* $0 < \theta < 1$. *Define* q *by:*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If $F \in \mathcal{F}(L^{q_0}(X,\mu), L^{q_1}(X,\mu))$, then $F(\theta) \in L^q(X,\mu)$ with the estimate

$$\|F(\theta)\|_{L^q(X,\mu)} \le \|F\|_{\mathcal{F}(L^{q_0}(X,\mu), L^{q_1}(X,\mu))}. \tag{17}$$

Proof Let $u \in (0, q_1)$. Set $r_0 := \frac{q_0}{u}$, $r_1 := \frac{q_1}{u}$, and $r := \frac{q}{u}$. Then we have

$$\|F(\theta)\|^u_{L^q(X,\mu)} = \|\,|F(\theta)|^u\,\|_{L^r(X,\mu)} = \sup \int_X |F(\theta,x)|^u h(x)\,d\mu(x), \tag{18}$$

where h moves over all simple functions having $L^{r'}(X,\mu)$-norm 1. With this in mind, let us fix such an h and estimate

$$\int_X |F(\theta,x)|^u h(x)\,d\mu(x).$$

Suppose h takes the form:

$$h = \sum_{j=1}^N a_j \chi_{E_j},$$

where $\{E_j\}_{j=1}^\infty$ is a μ-measurable partition of X and $a_j \ge 0$. We set

$$\tilde{F}(z,x) = \sum_{j=1}^N \left(\frac{1}{\mu(E_j)} \int_{E_j} |F(z,x)|^u\,d\mu(x) \right) \chi_{E_j}(x) \quad (z \in \bar{S}, x \in X).$$

Then we have

$$\int_X |F(\theta,x)|^u h(x)\,d\mu(x) = \int_X \tilde{F}(\theta,x) h(x)\,d\mu(x). \tag{19}$$

Notice that $\tilde{F}(\cdot,x)$ is a subharmonic function on S and a continuous function on $\bar{S}$, since the mappings

$$z \in \bar{S} \mapsto \frac{1}{\mu(E_j)} \int_{E_j} |F(z,x)|^u\,d\mu(x)$$

enjoy the same property. By virtue of Theorem 5, $\log \tilde{F}(\cdot,x)$ is also subharmonic on S. By virtue of Lemma 3, we have

$$\log \tilde{F}(\theta,x) \le \int_{\mathbb{R}} P_0(\theta,t) \log \tilde{F}(it,x)\,dt + \int_{\mathbb{R}} P_1(\theta,t) \log \tilde{F}(1+it,x)\,dt.$$

By using Jensen's inequality as well as (11) and (12), we have

$$\tilde{F}(\theta,x) \le f_0(\theta,x)^{1-\theta} f_1(\theta,x)^\theta$$

where

$$f_0(\theta, x) := \frac{1}{1-\theta} \int_{\mathbb{R}} \tilde{F}(it, x) P_0(\theta, t)\, dt$$

and

$$f_1(\theta, x) := \frac{1}{\theta} \int_{\mathbb{R}} \tilde{F}(1 + it, x) P_1(\theta, t)\, dt.$$

By using Hölder's inequality, we have

$$\begin{aligned} \|\tilde{F}(\theta)\|_{L^r(\mathcal{X},\mu)} &\le \|f_0(\theta, \cdot)^{1-\theta} f_1(\theta, \cdot)^{\theta}\|_{L^r(\mathcal{X},\mu)} \\ &\le \|f_0(\theta, \cdot)\|_{L^{r_0}(\mathcal{X},\mu)}^{1-\theta} \|f_1(\theta, \cdot)\|_{L^{r_1}(\mathcal{X},\mu)}^{\theta}. \end{aligned} \tag{20}$$

We use Hölder's inequality to obtain

$$\frac{1}{\mu(E_j)} \int_{E_j} |F(it, y)|^u \, d\mu(y) \le \frac{1}{\mu(E_j)^{\frac{1}{r_0}}} \left(\int_{E_j} |F(it, y)|^{q_0} \, d\mu(y) \right)^{\frac{1}{r_0}},$$

so

$$\|\tilde{F}(it, \cdot)\|_{L^{r_0}(X,\mu)} \le \|F(it, \cdot)\|_{L^{q_0}(X,\mu)}^u,$$

for all $t \in \mathbb{R}$. As a consequence,

$$\begin{aligned} \|f_0\|_{L^{r_0}(\mathcal{X},\mu)} &\le \frac{1}{1-\theta} \int_{\mathbb{R}} \|\tilde{F}(it)\|_{L^{r_0}(\mathcal{X},\mu)} P_0(\theta, t)\, dt \\ &\le \frac{1}{1-\theta} \int_{\mathbb{R}} \|F(it)\|_{L^{q_0}(\mathcal{X},\mu)}^u P_0(\theta, t)\, dt. \end{aligned} \tag{21}$$

By a similar argument, we also have

$$\|f_1\|_{L^{r_1}(\mathcal{X},\mu)} \le \frac{1}{\theta} \int_{\mathbb{R}} \|F(1 + it)\|_{L^{q_1}(\mathcal{X},\mu)}^u P_1(\theta, t)\, dt. \tag{22}$$

By combining (18)–(22) together, we obtain

$$\begin{aligned} \|F(\theta, \cdot)\|_{L^q(X,\mu)}^u &\le \left(\frac{1}{1-\theta} \int_{\mathbb{R}} \|F(it)\|_{L^{q_0}(\mathcal{X},\mu)}^u P_0(\theta, t)\, dt \right)^{1-\theta} \\ &\quad \times \left(\frac{1}{\theta} \int_{\mathbb{R}} \|F(1 + it)\|_{L^{q_1}(\mathcal{X},\mu)}^u P_1(\theta, t)\, dt \right)^{\theta}. \end{aligned} \tag{23}$$

From (11) and (12) we learn

$$\|F(\theta,\cdot)\|_{L^q(X,\mu)} \le \left(\sup_{t\in\mathbb{R}}\|F(it,\cdot)\|_{L^{q_0}(X,\mu)}\right)^{1-\theta}\left(\sup_{t\in\mathbb{R}}\|F(1+it,\cdot)\|_{L^{q_1}(X,\mu)}\right)^{\theta}$$
$$\le \|F\|_{\mathcal{F}(L^{q_0}(\mathcal{X},\mu),L^{q_1}(\mathcal{X},\mu))}, \tag{24}$$

as desired. □

Remark 2 One can compare (23) with its complex-valued version in [26, p. 68, Lemma 2].

2.3 Some Inequalities in Complex Analysis

We invoke two important results from our earlier papers.

Lemma 5 [13] *For each $z \in \overline{S}$ and $N \in \mathbb{N}$, define*

$$F_N(z) := \chi_{\{\frac{1}{N}\le|f|\le N\}}\mathrm{sgn}(f)|f|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}, \tag{25}$$

$$F_{N,0}(z) := \chi_{\{|f|\le 1\}}F_N(z), \quad \text{and} \quad F_{N,1}(z) := \chi_{\{|f|>1\}}F_N(z). \tag{26}$$

Let $a := \frac{p}{p_1}-\frac{p}{p_0}$, $z\in\overline{S}$, $w\in\mathbb{C}$ with $z+w\in\overline{S}$, $j\in\{0,1\}$, and $t, t_0\in\mathbb{R}$. Then we have the following inequalities:

$$|F_{N,0}(z+w)-F_{N,0}(z)| \le |f|^{\frac{p}{p_0}}\left(e^{|w|a\log N}-1\right), \tag{27}$$

$$|F_{N,1}(z+w)-F_{N,1}(z)| \le |f|^{\frac{p}{p_0}}\left(e^{|w|a\log N}-1\right), \tag{28}$$

$$|F_N(j+it)| \le |f|^{\frac{p}{p_j}} \tag{29}$$

and

$$|F_N(j+it)-F_N(j+it_0)| \le \left(e^{\left(\frac{p}{p_0}-\frac{p}{p_1}\right)|t-t_0|\log N}-1\right)|f|^{\frac{p}{p_j}}. \tag{30}$$

Denote by $L^0(\mathcal{X},\mu)$ the set of all μ-measurable functions defined on $\mathcal{X}$. We invoke the following lemma in [12].

Lemma 6 *Let $p_0 > p_1$ and $f \in L^0(\mathcal{X}, \mu)$. Define $p : \overline{S} \to \mathbb{C}$, $F : \overline{S} \to L^0(\mathcal{X}, \mu)$ and $G : \overline{S} \to L^0(\mathcal{X}, \mu)$ by:*

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \tag{31}$$

$$F(z) := \operatorname{sgn}(f) \exp\left(\frac{p}{p(z)} \log |f|\right) \quad (z \in \overline{S}), \tag{32}$$

$$G(z) := (z-\theta) \int_0^1 F(\theta + (z-\theta)t)\, dt \quad (z \in \overline{S}), \tag{33}$$

respectively. Define $F_0, F_1, G_0, G_1 : \overline{S} \to L^0(\mathcal{X}, \mu)$ by:

$$F_0(z) := F(z)\chi_{\{|f| \le 1\}}, \quad F_1(z) := F(z)\chi_{\{|f|>1\}}, \tag{34}$$

and

$$G_0(z) := G(z)\chi_{\{|f| \le 1\}}, \quad G_1(z) := G(z)\chi_{\{|f|>1\}}. \tag{35}$$

1. *For any $z \in \overline{S}$, we have*

$$|F_0(z)| \le |f|^{\frac{p}{p_0}}, \qquad |F_1(z)| \le |f|^{\frac{p}{p_1}}, \tag{36}$$

and

$$|G_0(z)| \le (1+|z|)|f|^{\frac{p}{p_0}}, \quad |G_1(z)| \le (1+|z|)|f|^{\frac{p}{p_1}}. \tag{37}$$

2. *For any $z \in \overline{S}$ and $h \in \mathbb{C}$ with $z + h \in \overline{S}$, we have*

$$|G_0(z+w) - G_0(z)| \le |w| \cdot |f|^{\frac{p}{p_0}} \quad \text{and} \quad |G_1(z+w) - G_1(z)| \le |w| \cdot |f|^{\frac{p}{p_1}}. \tag{38}$$

3. *For $j = 0, 1$ and $t_1 < t_2$, we have*

$$\left| \frac{G(j+it_2) - G(j+it_1)}{t_2 - t_1} \right| \le |f|^{\frac{p}{p_j}}. \tag{39}$$

3 Proof of Theorems 1 and 2

In our proof, we often use the following scaling property of Morrey norms:

Lemma 7 [20] *If* $f \in \mathcal{M}^p_q(\mathcal{X}, \mu)$ *and* $\frac{p}{q} = \frac{p_0}{q_0} = \frac{p_1}{q_1}$, *then*

$$\left\| |f|^{\frac{p}{p_0}} \right\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} = \|f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} \text{ and } \left\| |f|^{\frac{p}{p_1}} \right\|_{\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} = \|f\|^{\frac{p}{p_1}}_{\mathcal{M}^p_q(\mathcal{X},\mu)}.$$

We also prove that $\mathcal{M}^p_q(\mathcal{X}, \mu) \subseteq \mathcal{M}^{p_0}_{q_0}(\mathcal{X}, \mu) + \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)$.

Lemma 8 *If* $f \in \mathcal{M}^p_q(\mathcal{X}, \mu)$, $q_0 > q_1$, *and* $\frac{p}{q} = \frac{p_0}{q_0} = \frac{p_1}{q_1}$, *then*

$$\|f\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} \le \|f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} + \|f\|^{\frac{p}{p_1}}_{\mathcal{M}^p_q(\mathcal{X},\mu)}.$$

Proof Let $f \in \mathcal{M}^p_q(\mathcal{X}, \mu)$. Define $f_0 := f\chi_{\{|f|\le 1\}}$ and $f_1 := f - f_0$. From (4), it follows that $q_0 > q > q_1$. So,

$$\begin{aligned}
\|f\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} &\le \|f_0\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} + \|f_1\|_{\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} \\
&\le \||f|^{q/q_0}\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} + \||f|^{q/q_1}\|_{\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} \\
&= \|f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} + \|f\|^{\frac{p}{p_1}}_{\mathcal{M}^p_q(\mathcal{X},\mu)},
\end{aligned}$$

as desired. □

3.1 *Proof of Theorem 1*

To prove the first inclusion in (1), we use the following lemma:

Lemma 9 *Let* $f \in \mathcal{M}^p_q(\mathcal{X}, \mu)$. *For each* $z \in \overline{S}$, *define*

$$F(z) := \operatorname{sgn}(f)|f|^{\frac{p}{p_0}(1-z)+\frac{p}{p_1}z}.$$

Then $F_{|S} : S \to \mathcal{M}^{p_0}_{q_0}(\mathcal{X}, \mu) + \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)$ *is analytic.*

Proof Let $S_\varepsilon := \{z \in \mathbb{C} : \varepsilon < \operatorname{Re}(z) < 1 - \varepsilon\}$. It suffices to show that $F_{|S_\varepsilon} : S_\varepsilon \to \mathcal{M}^{p_0}_{q_0}(X, \mu) + \mathcal{M}^{p_1}_{q_1}(X, \mu)$ is analytic for any $\varepsilon \in \left(0, \frac{1}{2}\right)$. Let

$$F_0(z) = \chi_{\{|f|\le 1\}}F(z), \quad F_1(z) = F(z) - F_0(z) \quad (z \in \bar{S}).$$

Let $z_0 \in S_\varepsilon$ be fixed. Define

$$\eta := \frac{1}{2} \min\left(\mathrm{Re}(z_0) - \varepsilon, 1 - \varepsilon - \mathrm{Re}(z_0), \mathrm{Re}(z_0), \mathrm{Re}(1 - z_0)\right).$$

Then $\overline{\Delta}(z_0, \eta) \subseteq S_\varepsilon$. Let $a := \frac{p}{p_1} - \frac{p}{p_0}$ and $u := \min(1, q_0, q_1)$. For each $n \in \mathbb{N} \cup \{0\}$, define

$$h_n := F(z_0) \frac{(a \log |f|)^n}{n!}, \quad h_n^{(0)} := h_n \chi_{\{|f| \le 1\}}, \quad \text{and} \quad h_n^{(1)} := h_n \chi_{\{|f| > 1\}}$$

Notice that $|h_0^{(0)}| \le |f|^{\frac{p}{p_0}}$. Since

$$\sup_{t \in (0,1)} |t^b \log t| = \frac{1}{be}$$

for any $b > 0$, it follows that for $n \ge 1$

$$\begin{aligned} |h_n^{(0)}| &\le a^n \chi_{\{|f| \le 1\}} |f|^{\frac{p}{p_0}} |f|^{a\mathrm{Re}(z_0)} \frac{|\log |f||^n}{n!} \\ &= a^n \chi_{\{|f| \le 1\}} |f|^{\frac{p}{p_0}} \frac{||f|^{\frac{a\mathrm{Re}(z_0)}{n}} \log |f||^n}{n!} \\ &\le \frac{n^n}{e^n (\mathrm{Re} z_0)^n n!} |f|^{\frac{p}{p_0}}. \end{aligned}$$

Therefore, $h_n^{(0)} \in \mathcal{M}_{q_0}^{p_0}$ with

$$\|h_n^{(0)}\|_{\mathcal{M}_{q_0}^{p_0}} \le \frac{n^n}{e^n (\mathrm{Re} z_0)^n n!} \|f\|_{\mathcal{M}_q^p(\mathcal{X},\mu)}^{\frac{p}{p_0}}.$$

Since

$$\|h_n^{(0)}(z - z_0)^n\|_{\mathcal{M}_{q_0}^{p_0}} \le \|h_n^{(0)}\|_{\mathcal{M}_{q_0}^{p_0}} \eta^n \le \frac{n^n}{2^n e^n n!} \|f\|_{\mathcal{M}_q^p(\mathcal{X},\mu)}^{\frac{p}{p_0}}$$

and

$$\sum_{n=0}^{\infty} \left(\frac{n^n}{2^n e^n n!}\right)^u < \infty,$$

we have $\sum_{n=0}^{\infty} \|h_n^{(0)}(z - z_0)^n\|_{\mathcal{M}_{q_0}^{p_0}}^u < \infty$, so for almost every $x \in \mathcal{X}$, we have

$$\sum_{n=0}^{\infty} h_n^{(0)}(x)(z-z_0)^n = F_0(z_0)(x) \sum_{n=0}^{\infty} \frac{(a \log |f(x)|)^n (z-z_0)^n}{n!}$$
$$= F_0(z_0)(x) e^{a(z-z_0)\log|f(x)|}$$
$$= F_0(z)(x).$$

Consequently,

$$\left\| F_0(z) - \sum_{n=0}^{K} h_n^{(0)}(z-z_0)^n \right\|_{\mathcal{M}_{q_0}^{p_0}}^{u} \le \sum_{n=K+1}^{\infty} \|h_n^{(0)}(z-z_0)^n\|_{\mathcal{M}_{q_0}^{p_0}}^{u} \to 0$$

as $K \to \infty$. By a similar argument, we also have

$$\lim_{K\to\infty} \left\| F_1(z) - \sum_{n=0}^{K} h_n^{(1)}(z-z_0)^n \right\|_{\mathcal{M}_{q_1}^{p_1}(\mathcal{X},\mu)} = 0.$$

As a result,

$$F(z) = F_0(z) + F_1(z) = \sum_{n=0}^{\infty} h_n (z-z_0)^n$$

in $\mathcal{M}_{q_1}^{p_1}(\mathcal{X},\mu) + \mathcal{M}_{q_1}^{p_1}(\mathcal{X},\mu)$. □

Lemma 10 *We have the following inclusion:*

$$\{f \in \mathcal{M}_q^p(\mathcal{X},\mu) : \lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}_q^p(\mathcal{X},\mu)} = 0\}$$
$$\subseteq [\mathcal{M}_{q_0}^{p_0}(\mathcal{X},\mu), \mathcal{M}_{q_1}^{p_1}(\mathcal{X},\mu)]_\theta.$$

Proof Suppose that $f \in \mathcal{M}_q^p(\mathcal{X},\mu)$ satisfies

$$\lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}_q^p(\mathcal{X},\mu)} = 0. \tag{40}$$

Define $F(z)$ by (32). From (36), it follows that

$$\|F(z)\|_{\mathcal{M}_{q_0}^{p_0}(\mathcal{X},\mu)+\mathcal{M}_{q_1}^{p_1}(\mathcal{X},\mu)} \le \|F_0(z)\|_{\mathcal{M}_{q_0}^{p_0}(\mathcal{X},\mu)} + \|F_1(z)\|_{\mathcal{M}_{q_1}^{p_1}(\mathcal{X},\mu)}$$
$$\le \||f|^{\frac{p}{p_0}}\|_{\mathcal{M}_{q_0}^{p_0}(\mathcal{X},\mu)} + \||f|^{\frac{p}{p_1}}\|_{\mathcal{M}_{q_1}^{p_1}(\mathcal{X},\mu)}$$
$$= \|f\|_{\mathcal{M}_q^p(\mathcal{X},\mu)}^{\frac{p}{p_0}} + \|f\|_{\mathcal{M}_q^p(\mathcal{X},\mu)}^{\frac{p}{p_1}},$$

so $\sup_{z\in\overline{S}} \|F(z)\|_{\mathcal{M}_{q_0}^{p_0}(\mathcal{X},\mu)+\mathcal{M}_{q_1}^{p_1}(\mathcal{X},\mu)} < \infty$. According to (27) and (28), we have

$$\begin{aligned}
&\|F(z+w)-F(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} \\
&\le \left(\|f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)}+\|f\|^{\frac{p}{p_1}}_{\mathcal{M}^p_q(\mathcal{X},\mu)}\right)\left(e^{|w|a\log N}-1,\right) \\
&\quad +2\left\|f-\chi_{\{\frac{1}{N}\le|f|\le N\}}f\right\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)}+2\left\|f-\chi_{\{\frac{1}{N}\le|f|\le N\}}f\right\|^{\frac{p}{p_1}}_{\mathcal{M}^p_q(\mathcal{X},\mu)}
\end{aligned}$$

and hence

$$\lim_{w\to 0}\|F_N(z+w)-F_N(z)\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}}=0.$$

By virtue of Lemma 9, we have $F : S \to \mathcal{M}^{p_0}_{q_0}(\mathcal{X})+\mathcal{M}^{p_1}_{q_1}(\mathcal{X})$ is analytic.

Let $j \in \{0,1\}$, $t, t_0 \in \mathbb{R}$, and $a := \frac{p}{p_1}-\frac{p}{p_0}$. We use (30) to obtain

$$\begin{aligned}
\|F(j+it)-F(j+it_0)\|_{\mathcal{M}^{p_j}_{q_j}(\mathcal{X},\mu)} &\le \left(e^{a|t-t_0|\log N}-1\right)\|f\|^{\frac{p}{p_j}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} \\
&\quad +\left\|f-\chi_{\{\frac{1}{N}\le|f|\le N\}}f\right\|^{\frac{p}{p_j}}_{\mathcal{M}^p_q(\mathcal{X},\mu)}
\end{aligned}$$

so

$$\lim_{t\to t_0}\|F(j+it)-F(j+it_0)\|_{\mathcal{M}^{p_j}_{q_j}(\mathcal{X},\mu)}=0.$$

Finally, by using (29), we obtain

$$\max_{j=0,1}\sup_{t\in\mathbb{R}}\|F(j+it)\|_{\mathcal{M}^{p_j}_{q_j}(\mathcal{X},\mu)}\le\|f\|^{\frac{p}{p_j}}_{\mathcal{M}^p_q(\mathcal{X},\mu)}<\infty.$$

In total, we have shown that $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$. Since $f = F(\theta)$, we have $f \in [\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta$ as desired. □

By applying Proposition 1, we have the following inclusion:

Lemma 11 *We have the inclusion* $[\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta \subseteq \mathcal{M}^p_q(\mathcal{X},\mu)$.

Proof For $f \in [\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta$, pick $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$ such that $f = F(\theta)$ and that

$$\|F\|_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))}\lesssim\|f\|_{[\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta}. \tag{41}$$

Decompose $F(z)$ into $F(z) := F_0(z)+F_1(z)$, where $F_0(z) \in \mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)$ and $F_1(z) \in \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)$. For every ball $B \subseteq \mathcal{X}$, define

$$G_B(z) := \mu(B)^{\frac{1-z}{p_0}+\frac{z}{p_1}-\left(\frac{1-z}{q_0}+\frac{z}{q_1}\right)}\chi_B F(z), \tag{42}$$

$$G_{B,0}(z) := \mu(B)^{\frac{1-z}{p_0}+\frac{z}{p_1}-\left(\frac{1-z}{q_0}+\frac{z}{q_1}\right)}\chi_B F_0(z), \tag{43}$$

$$G_{B,1}(z) := \mu(B)^{\frac{1-z}{p_0}+\frac{z}{p_1}-\left(\frac{1-z}{q_0}+\frac{z}{q_1}\right)}\chi_B F_1(z). \tag{44}$$

Since

$$\|G_{B,0}(z)\|_{L^{q_0}(\mathcal{X},\mu)} \le \mu(B)^{\left(\frac{1}{p_1}-\frac{1}{p_0}+\frac{1}{q_0}-\frac{1}{q_1}\right)\mathrm{Re}(z)}\|F_0(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)}$$

and

$$\|G_{B,1}(z)\|_{L^{q_1}(\mathcal{X},\mu)} \le \mu(B)^{-\left(\frac{1}{p_1}-\frac{1}{p_0}+\frac{1}{q_0}-\frac{1}{q_1}\right)\mathrm{Re}(1-z)}\|F_1(z)\|_{\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)},$$

we have

$$\|G_B(z)\|_{L^{q_0}(\mathcal{X},\mu)+L^{q_1}(\mathcal{X},\mu)} \le C_B\left(\|F_0(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} + \|F_1(z)\|_{\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)}\right),$$

where

$$C_B := \max\left(\mu(B)^{\frac{1}{p_1}-\frac{1}{p_0}+\frac{1}{q_0}-\frac{1}{q_1}}, \mu(B)^{-\frac{1}{p_1}+\frac{1}{p_0}-\frac{1}{q_0}+\frac{1}{q_1}}\right).$$

Consequently,

$$\|G_B(z)\|_{L^{q_0}(\mathcal{X},\mu)+L^{q_1}(\mathcal{X},\mu)} \le C_B\|F(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)}. \tag{45}$$

Since $F : \overline{S} \to \mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu) + \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)$ is bounded and continuous by virtue of (45), we also have continuity of $G_B : \overline{S} \to L^{q_0}(\mathcal{X},\mu) + L^{q_1}(\mathcal{X},\mu)$ and

$$\sup_{z\in\overline{S}} \|G_B(z)\|_{L^{q_0}(\mathcal{X},\mu)+L^{q_1}(\mathcal{X},\mu)} < \infty.$$

Let $z_0 \in S$, choose $0 < \eta \ll 1$ and $\{f_n\}_{n=0}^{\infty}$ such that for $z \in \Delta(z_0,\eta)$, we have

$$F(z) = \sum_{n=0}^{\infty} f_n(z-z_0)^n \tag{46}$$

in $\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu) + \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)$. Write $a := \mu(B)^{\frac{1}{p_0}-\frac{1}{q_0}}$ and $b := \mu(B)^{\frac{1}{p_1}-\frac{1}{q_1}}$. For each $k, n \in \mathbb{N}\cup\{0\}$, we have

$$\left\|a^{1-z_0}b^{z_0}\frac{(\log(b/a))^k}{k!}\chi_B f_n\right\|_{L^{q_0}(\mathcal{X},\mu)+L^{q_1}(\mathcal{X},\mu)} \le C_{k,B}\|f_n\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)}.$$

Let $z \in \Delta(z_0, \eta)$ and $N \in \mathbb{N}$. Then

$$\begin{aligned}&\left\| G_B(z) - \sum_{n=0}^{N}\sum_{k=0}^{\infty} \frac{a^{1-z_0}b^{z_0}}{k!}(\log(b/a))^k \chi_B f_n(z-z_0)^{n+k} \right\|_{L^{q_0}(\mathcal{X},\mu)+L^{q_1}(\mathcal{X},\mu)} \\ &= \left\| a^{1-z}b^z\chi_B\left(F(z) - \sum_{n=0}^{N} f_n(z-z_0)^n\right)\right\|_{L^{q_0}(\mathcal{X},\mu)+L^{q_1}(\mathcal{X},\mu)} \\ &\le C_B\left\| F(z) - \sum_{n=0}^{N} f_n(z-z_0)^n\right\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)}.\end{aligned}$$

Consequently, from (46),

$$\lim_{N\to\infty}\sum_{n=0}^{N}\sum_{k=0}^{\infty} \frac{a^{1-z_0}b^{z_0}}{k!}(\log(b/a))^k \chi_B f_n(z-z_0)^{n+k} = G_B(z)$$

in $L^{q_0}(\mathcal{X},\mu) + L^{q_1}(\mathcal{X},\mu)$. Thus, $G_B : S \to L^{q_0}(\mathcal{X},\mu) + L^{q_1}(\mathcal{X},\mu)$ is analytic.

For every $t \in \mathbb{R}$ and $j \in \{0, 1\}$, we have

$$\begin{aligned}\|G_B(j+it)\|_{L^{q_j}(\mathcal{X},\mu)} &= \mu(B)^{\frac{1}{p_j}-\frac{1}{q_j}}\|\chi_B F(j+it)\|_{L^{q_0}(\mathcal{X},\mu)} \\ &\le \|F(j+it)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} \le \|F\|_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))},\end{aligned} \tag{47}$$

so, for each $j \in \{0, 1\}$, the function $t \in \mathbb{R} \mapsto G_B(j + it) \in L^{q_j}(\mathcal{X},\mu)$ is bounded.

For a fixed $t_1 \in R$ and for every $t \in \mathbb{R}$, we have

$$\begin{aligned}&\|G_B(j+it) - G_B(j+it_1)\|_{L^{q_j}(\mathcal{X},\mu)} \\ &\le \|F(j+it) - F(j+it_1)\|_{\mathcal{M}^{p_j}_{q_j}(\mathcal{X},\mu)} \\ &\quad + \|F(j+it_1)\|_{L^{q_j}(\mathcal{X},\mu)}\left|\mu(B)^{it\left(\frac{1}{p_1}-\frac{1}{q_1}+\frac{1}{q_0}-\frac{1}{p_0}\right)} - \mu(B)^{it_1\left(\frac{1}{p_1}-\frac{1}{q_1}+\frac{1}{q_0}-\frac{1}{p_0}\right)}\right| \\ &= \|F(j+it) - F(j+it_1)\|_{\mathcal{M}^{p_j}_{q_j}(\mathcal{X},\mu)} \\ &\quad + 2\|F(j+it_1)\|_{\mathcal{M}^{p_j}_{q_j}(\mathcal{X},\mu)}\left|\sin\left(\frac{(t-t_1)\log\mu(B)}{2}\left(\frac{1}{p_1}-\frac{1}{q_1}+\frac{1}{q_0}-\frac{1}{p_0}\right)\right)\right|.\end{aligned}$$

As a result,

$$\lim_{t\to t_1}\|G_B(j+it) - G_B(j+it_1)\|_{L^{q_j}(\mathcal{X},\mu)} = 0.$$

In total, we have shown that $G_B \in \mathcal{F}(L^{q_0}(\mathcal{X},\mu), L^{q_1}(\mathcal{X},\mu))$. As a consequence of (17) and (47), we have

$$\|G_B(\theta)\|_{L^q(\mathcal{X},\mu)} \le \|G_B\|_{\mathcal{F}(L^{q_0}(\mathcal{X},\mu),L^{q_1}(\mathcal{X},\mu))} \le \|F\|_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))}. \tag{48}$$

By taking the infimum over all $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$ satisfying $f = F(\theta)$, we have

$$\mu(B)^{\frac{1}{p}-\frac{1}{q}} \|f\chi_B\|_{L^q(\mathcal{X},\mu)} \le \|f\|_{[\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta}.$$

Thus, taking the supremum over all balls B, we learn

$$\|f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} \le \|f\|_{[\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta}$$

as desired. □

Remark 3 Note that by combining the argument in the proof of Lemma 11 and (23), we have

$$\|F(\theta,\cdot)\|^u_{\mathcal{M}^p_q(X,\mu)} \le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|F(it)\|^u_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} P_0(\theta,t)\,dt\right)^{1-\theta} \times \left(\frac{1}{\theta}\int_{\mathbb{R}} \|F(1+it)\|^u_{\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} P_1(\theta,t)\,dt\right)^{\theta} \tag{49}$$

for every $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$.

3.2 Proof of Theorem 2

Let

$$f \in [\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]^\theta.$$

By the definition of the second complex interpolation functor, f is realized as $f = G'(\theta)$ for some $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$ with the estimate

$$\|G(\cdot+ih) - G\|_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))} \lesssim |h| \cdot \|f\|_{[\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]^\theta} \tag{50}$$

for all $h \in \mathbb{R}$. For $z \in \overline{S}$ and $h \in \mathbb{R} \setminus \{0\}$, write $f_h(z) := \frac{G(z+ih)-G(z)}{ih}$. By virtue of Lemma 11, we have $f_h(\theta) \in \mathcal{M}^p_q(\mathcal{X},\mu)$ and combining this with (50) yield

$$\begin{aligned}\|f_h(\theta)\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} &\lesssim \frac{1}{|h|}\|G(\theta+ih) - G(\theta)\|_{[\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta} \\ &\le \|f\|_{[\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]^\theta}.\end{aligned} \tag{51}$$

Since $\lim_{h\to 0} f_h(\theta) = f$ in $\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu) + \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)$, we can choose a sequence of positive numbers $\{h_k\}_{k=1}^{\infty}$ converging to zero such that

$$\lim_{k\to\infty} f_{h_k}(\theta)(x) = f(x)$$

for almost every $x \in \mathcal{X}$. Thus, by the Fatou property of $\mathcal{M}^p_q(\mathcal{X},\mu)$ and (51), we obtain

$$\|f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} \le \liminf_{k\to\infty} \|f_{h_k}(\theta)\|_{\mathcal{M}^p_q} \lesssim \|f\|_{[\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]^\theta},$$

implying that $f \in \mathcal{M}^p_q(\mathcal{X},\mu)$.

Conversely let $f \in \mathcal{M}^p_q(\mathcal{X},\mu)$. Define $G(z)$ by (33). For each $h \in \mathbb{R}$ and $z \in \overline{S}$, define

$$H_h(z) := G(z+ih) - G(z).$$

We shall prove that $G \in \mathcal{G}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$ by showing the following chain of lemmas:

Lemma 12 *For all $z \in \overline{S}$, we have $H_h(z) \in \mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu) + \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)$. More precisely, $\sup_{z\in\overline{S}} \|H_h(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} < \infty$, and*

$$\sup_{z\in\overline{S}} \left\| \frac{G(z)}{1+|z|} \right\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} < \infty. \tag{52}$$

Proof In view of (37), we have

$$\left\| \frac{G(z)}{1+|z|} \right\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} \le \|f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} + \|f\|^{\frac{p}{p_1}}_{\mathcal{M}^p_q(\mathcal{X},\mu)}.$$

From (38), it follows that

$$\begin{aligned}\|H_h(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} &\le \|G_0(z+ih)-G_0(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} \\ &\quad + \|G_1(z+ih)-G_1(z)\|_{\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} \\ &\le |h| \left(\|f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} + \|f\|^{\frac{p}{p_1}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} \right),\end{aligned}$$

so $\sup_{z\in\overline{S}} \|H_h(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)+\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} < \infty$. □

Lemma 13 *The function $H_h : \overline{S} \to \mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$ is continuous.*

Proof Let $z \in \overline{S}$ and $w \in \mathbb{C}$ be such that $z + w \in \overline{S}$. From (38), it follows that

$$\|H_h(z+w) - H_h(z)\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}} \lesssim |w| \left(\|f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} + \|f\|^{\frac{p}{p_1}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} \right). \tag{53}$$

Consequently, $\lim_{w\to 0} \|H_h(z+w) - H_h(z)\|_{\mathcal{M}^{p_0}_{q_0}+\mathcal{M}^{p_1}_{q_1}} = 0$. □

Lemma 14 *The function $H_h : S \to \mathcal{M}^{p_0}_{q_0} + \mathcal{M}^{p_1}_{q_1}$ is analytic.*

Proof Let $a := \frac{p}{p_1} - \frac{p}{p_0}$, $\varepsilon \in (0, \frac{1}{2})$, and $S_\varepsilon := \{z \in S : \varepsilon < \mathrm{Re}(z) < 1 - \varepsilon\}$. We shall show that H_h is analytic in S_ε. For each $z_0 \in S_\varepsilon$, define

$$\eta := \frac{1}{2} \min\left(\mathrm{Re}(z_0) - \varepsilon, 1 - \varepsilon - \mathrm{Re}(z_0), \mathrm{Re}(z_0), \mathrm{Re}(1 - z_0)\right).$$

Then, $\overline{\Delta}(z_0, \eta) \subseteq S_\varepsilon$. Let $z \in \Delta(z_0, \eta)$. Since

$$\begin{aligned} G(z) &= \int_\theta^{z_0} F(w)\, dw + \int_{z_0}^{z} F(w)\, dw \\ &= G(z_0) + \mathrm{sgn}(f)|f|^{\frac{p}{p_0}} \int_{z_0}^{z} |f|^{aw}\, dw \\ &= G(z_0) + \mathrm{sgn}(f)|f|^{\frac{p}{p_0}} \frac{|f|^{az} - |f|^{az_0}}{a \log|f|} = G(z_0) + \frac{F(z) - F(z_0)}{a \log|f|} \end{aligned}$$

and

$$F(z) - F(z_0) = F(z_0)\left(|f|^{a(z-z_0)} - 1\right) = F(z_0) \sum_{n=1}^{\infty} \frac{(a \log|f|)^n}{n!} (z - z_0)^n,$$

we have

$$G(z) = G(z_0) + \sum_{n=0}^{\infty} \frac{F(z_0)(a \log|f|)^n}{(n+1)!} (z - z_0)^{n+1}.$$

For each $n \in \mathbb{N} \cup \{0\}$, set

$$h_n := \begin{cases} G(z_0), & n = 0, \\ \frac{(a \log|f|)^{n-1}}{n!} F(z_0), & n \neq 0. \end{cases}, \quad h_n^{(0)} := \chi_{\{|f| \le 1\}} h_n,$$

and

$$h_n^{(1)} := \chi_{\{|f| > 1\}} h_n.$$

Let $u = \min(1, q_0, q_1)$. Since

$$\begin{aligned}\left\| G_0(z) - \sum_{n=1}^{N} h_n^{(0)}(z - z_0)^n \right\|_{\mathcal{M}^{p_0}_{q_0}}^u &\le \sum_{n=N+1}^{\infty} (\|h_n^{(0)}\|_{\mathcal{M}^{p_0}_{q_0}} |z - z_0|^n)^u \\ &\le \sum_{n=N+1}^{\infty} \left(\frac{(n-1)^{(n-1)}}{n!(\mathrm{Re}(z_0))e^{n-1}} \|f\|_{\mathcal{M}^p_q}^{\frac{p}{p_0}} \eta^n \right)^u \\ &\lesssim \|f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)}^{\frac{uq}{q_0}} \sum_{n=N}^{\infty} \left(\frac{n^n}{n!2^n e^n} \right)^u\end{aligned}$$

and $\sum_{n=1}^{\infty} \left(\frac{n^n}{n!2^n e^n}\right)^u < \infty$, we see that

$$\lim_{N\to\infty} \left\| G_0(z) - \sum_{n=1}^{N} h_n^{(0)}(z - z_0)^n \right\|_{\mathcal{M}^{p_0}_{q_0}}^u = 0.$$

By a similar argument, we also have

$$\lim_{N\to\infty} \left\| G_1(z) - \sum_{n=1}^{N} h_n^{(1)}(z - z_0)^n \right\|_{\mathcal{M}^{p_1}_{q_1}}^u = 0.$$

Consequently, $G(z) = \sum_{n=0}^{\infty} h_n(z - z_0)^n$ in $\mathcal{M}^{p_0}_{q_0}(\mathcal{X}, \mu) + \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)$. Thus, G is analytic in S_ε. Since ε is arbitrary, we have G is analytic in S, and so is H_h. □

Lemma 15 *For each* $j = 0, 1$, *we have*

$$\sup_{t\in\mathbb{R}} \|H_h(j + it)\|_{\mathcal{M}^{p_j}_{q_j}} \le |h| \cdot \|f\|_{\mathcal{M}^p_q}^{\frac{p}{p_j}}. \tag{54}$$

Proof As a consequence of (39), for each $j = 0, 1$ and $t \in \mathbb{R}$, we have

$$\|H_h(j + it)\|_{\mathcal{M}^{p_j}_{q_j}(\mathcal{X},\mu)} \le |h| \cdot \||f|^{\frac{p}{p_j}}\|_{\mathcal{M}^{p_j}_{q_j}(\mathcal{X},\mu)} = |h| \cdot \|f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)}^{\frac{p}{p_j}}, \tag{55}$$

as desired. □

Lemma 16 *Let* $j \in \{0, 1\}$. *Then the function* $t \in \mathbb{R} \mapsto H_h(j + it)$ *is continuous.*

Proof Fix $t_0 \in \mathbb{R}$. Let $j = 0, 1$ and $t \in \mathbb{R}$. By using (39), we get

$$\|H_h(j + it) - H_h(j + it_0)\|_{\mathcal{M}^{p_j}_{q_j}} \le 2|t - t_0| \cdot \|f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)}^{\frac{p}{p_j}}. \tag{56}$$

As a result, $\lim_{t\to t_0} \|H_h(j + it) - H_h(j + it_0)\|_{\mathcal{M}^{p_j}_{q_j}} = 0$, as desired. □

4 Proof of Theorem 3

The proof of the first part of Theorem 3 is given as follows:

Proof (*of Theorem* 3 *(i)*) For any function $f \in [\widetilde{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta$, choose $F \in \mathcal{F}(\widetilde{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$ such that $f = F(\theta)$. For every $R > 0$ and some fixed $x_0 \in \mathcal{X}$, set

$$E_R := \{|f| > R\} \cup (\mathcal{X} \setminus B(x_0, R)).$$

Then $\chi_{E_R} F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$. We apply (49) to have

$$\begin{aligned}\|\chi_{E_R} f\|^u_{\mathcal{M}^p_q(\mathcal{X},\mu)} \le & \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|\chi_{E_R} F(it)\|^u_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} P_0(\theta,t)\,dt\right)^{1-\theta} \\ & \times \left(\frac{1}{\theta}\int_{\mathbb{R}} \|\chi_{E_R} F(1+it)\|^u_{\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} P_1(\theta,t)\,dt\right)^{\theta}. \end{aligned} \tag{57}$$

Since $F(it) \in \widetilde{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu) = \widehat{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu)$ and $\lim_{R\to\infty} \chi_{E_R}(x) = 0$, we have

$$\lim_{R\to\infty} \|\chi_{E_R} F(it)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X}.\mu)} = 0. \tag{58}$$

As a consequence of (57), (58), $\|\chi_{E_R} F(it)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X}.\mu)} \le \|F\|_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X}.\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X}.\mu))}$, and

$$\int_{\mathbb{R}} \|F\|^u_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X}.\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X}.\mu))} P_0(\theta,t)\,dt = (1-\theta)\|F\|^u_{\mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X}.\mu),\mathcal{M}^{p_1}_{q_1}(\mathcal{X}.\mu))},$$

which follows from (11), we have

$$\lim_{R\to\infty} \|\chi_{E_R} f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0, \tag{59}$$

and hence $f \in \widetilde{\mathcal{M}^p_q}(\mathcal{X},\mu)$.

Now, let $f \in \widetilde{\mathcal{M}^p_q}(\mathcal{X},\mu)$. By Lemma 1, we have $f \in \widehat{\mathcal{M}^p_q}(\mathcal{X},\mu)$. Consequently,

$$\lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N} \le |f| \le N\}} f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0. \tag{60}$$

For $z \in \overline{S}$, define $F(z) := \operatorname{sgn}(f)|f|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}$ and $F_0(z) := \chi_{\{|f|\le 1\}} F(z)$. By using the same argument as in the proof of Lemmas 9 and 10, we have $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}, \mathcal{M}^{p_1}_{q_1})$.

We shall show that $F \in \mathcal{F}(\widetilde{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$. For $R > 0$ and some fixed $x_0 \in \mathcal{X}$, we see that $\chi_{B(x_0,R)} F_0(z) \in L^\infty_c(\mathcal{X},\mu)$. Since $|F_0(z)| \le |f|^{\frac{p}{p_0}}$ and $f \in \widetilde{\mathcal{M}^p_q}(\mathcal{X},\mu)$, we have

$$\begin{aligned}\|F_0(z) - \chi_{B(x_0,R)} F_0(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} &\le \|\chi_{\{|f|>R\}\cup(\mathcal{X}\setminus B(x_0,R))} F_0(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} \\ &= \|f - \chi_{\{|f|\le R\}\cap B(0,R)} f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} \to 0\end{aligned}$$

as $R \to \infty$, so $F_0(z) \in \widetilde{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu)$. Combining this with

$$F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)),$$

we have $F \in \mathcal{F}(\widetilde{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$, so

$$f = F(\theta) \in [\widetilde{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta$$

as desired. □

Now, we prove the second part of Theorem 3:

Proof (*of Theorem* 3 *(ii)*) Let $f \in \overset{*}{\mathcal{M}}{}^p_q(\mathcal{X},\mu)$ be such that

$$\lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\le|f|\le N\}} f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0.$$

For every $z \in \overline{S}$, set $F(z) := \operatorname{sgn}(f)|f|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}$ and $F_0(z) := \chi_{\{|f|\le 1\}} F(z)$. Based on the proof of Lemma 10, we have $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$. Let $x_0 \in X$ be fixed and $R > 0$. Since $|F_0(z)| \le |f|^{\frac{p}{p_0}}$ and $f \in \overset{*}{\mathcal{M}}{}^p_q(\mathcal{X},\mu)$, we have

$$\begin{aligned}\|F_0(z) - \chi_{B(x_0,R)} F_0(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} &\le \|\chi_{\mathcal{X}\setminus B(x_0,R)} F_0(z)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} \\ &= \|\chi_{\mathcal{X}\setminus B(x_0,R)} f\|^{\frac{p}{p_0}}_{\mathcal{M}^p_q(\mathcal{X},\mu)} \to 0\end{aligned}$$

as $R \to \infty$, so $F_0(z) \in \overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X},\mu)$.

Combining this with

$$F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)),$$

we have

$$F \in \mathcal{F}(\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)),$$

so $f = F(\theta) \in [\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta$.

Let $f \in [\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta$ and pick $F \in \mathcal{F}(\overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$ such that $f = F(\theta)$. By applying (49), for some fixed $x_0 \in \mathcal{X}$ and for every $R > 0$, we have

$$\begin{aligned}
&\|\chi_{X\setminus B(x_0,R)} f\|^u_{\mathcal{M}^p_q(\mathcal{X},\mu)} \\
&\le \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|\chi_{\mathcal{X}\setminus B(x_0,R)} F(it)\|^u_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} P_0(\theta,t)\,dt\right)^{1-\theta} \\
&\quad\times \left(\frac{1}{\theta}\int_{\mathbb{R}} \|\chi_{\mathcal{X}\setminus B(x_0,R)} F(1+it)\|^u_{\mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)} P_1(\theta,t)\,dt\right)^{\theta}. \qquad (61)
\end{aligned}$$

Since $F(it) \in \overset{*}{\mathcal{M}}{}^{p_0}_{q_0}(\mathcal{X},\mu)$, we have $\lim_{R\to\infty} \|\chi_{X\setminus B(x_0,R)} F(it)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} = 0$, so,

$$\lim_{R\to\infty} \|\chi_{\mathcal{X}\setminus B(x_0,R)} f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0,$$

by the Lebesgue convergence theorem as we did in (59).

Therefore, $f \in \overset{*}{\mathcal{M}}{}^p_q(\mathcal{X},\mu)$. □

Proof (*of Theorem* 3*(iii)*) Let $f \in \mathcal{M}^p_q(\mathcal{X},\mu)$ be such that

$$\lim_{N\to\infty} \|f - \chi_{\{\frac{1}{N}\le|f|\le N\}} f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} = 0.$$

For every $z \in \overline{S}$, define $F(z) := \mathrm{sgn}(f)|f|^{p\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)}$ and $F_0(z) := \chi_{\{|f|\le 1\}} F(z)$. Observe that $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu))$ and $F_0(z) \in \overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu)$ for each $z \in \overline{S}$. For each $t \in \mathbb{R}$, we have

$$\begin{aligned}
\|F(it) - \chi_{\{|F(it)|\le N\}} F(it)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} &= \left\| |f|^{\frac{p}{p_0}} \left(1 - \chi_{\{|f|^{\frac{p}{p_0}}\le N\}}\right)\right\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} \\
&= \left\| f \left(1 - \chi_{\{|f|\le N^{\frac{p_0}{p}}\}}\right)\right\|^{\frac{p}{p_0}}_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} \\
&\le \left\| f - f\chi_{\left\{N^{-\frac{p_0}{p}}\le|f|\le N^{\frac{p_0}{p}}\right\}}\right\|^{\frac{p}{p_0}}_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)},
\end{aligned}$$

so

$$\lim_{N\to\infty} \|F(it) - \chi_{\{|F(it)|\le N\}} F(it)\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} = 0.$$

Therefore, $F(it) \in \overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu)$ for every $t \in \mathbb{R}$. By combining $F_0(z) \in \overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu)$, $F(it) \in \overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu)$, and $F \in \mathcal{F}(\mathcal{M}^{p_0}_{q_0}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu))$, we have

$$F \in \mathcal{F}(\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)),$$

and hence $f = F(\theta) \in [\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]_\theta$.
Now, let

$$f \in [\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu)]_\theta.$$

Then there exists $F \in \mathcal{F}(\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X}, \mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X}, \mu))$ such that $f = F(\theta)$. We use the following identity (see [7]):

$$\overline{\mathcal{M}^p_q} = \left\{ f \in \mathcal{M}^p_q : \lim_{h \downarrow 0} \sup_{E \subset \mathcal{X} : \|\chi_E\|_{\mathcal{M}^p_q} \le h} \|\chi_E f\|_{\mathcal{M}^p_q} = 0 \right\}.$$

Let $h > 0$ be arbitrary and $E \subset \mathcal{X}$ be such that $\|\chi_E\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} \le h$. Then

$$\begin{aligned}
&\|\chi_E F(\theta)\|^u_{\mathcal{M}^p_q} \\
&\le \left(\frac{1}{1-\theta} \int_{-\infty}^{\infty} \|\chi_E F(it)\|^u_{\mathcal{M}^{p_0}_{q_0}} P_0(\theta, t)\, dt \right)^{1-\theta} \\
&\quad \times \left(\frac{1}{\theta} \int_{-\infty}^{\infty} \|\chi_E F(1+it)\|^u_{\mathcal{M}^{p_1}_{q_1}} P_1(\theta, t)\, dt \right)^{\theta} \\
&\le \left(\frac{1}{1-\theta} \int_{-\infty}^{\infty} \sup_E \|\chi_E F(it)\|^u_{\mathcal{M}^{p_0}_{q_0}} P_0(\theta, t)\, dt \right)^{1-\theta} \|F\|^u_{\mathcal{F}(\overline{\mathcal{M}^{p_0}_{q_0}}, \mathcal{M}^{p_1}_{q_1})},
\end{aligned}$$

where E moves over all measurable sets such that $\|\chi_E\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} \le h^{\frac{p}{p_0}}$. Since $F(it) \in \overline{\mathcal{M}^{p_0}_{q_0}}$, we have

$$\lim_{h \downarrow 0} \left(\sup\{\|\chi_E F(it)\|_{\mathcal{M}^{p_0}_{q_0}} \ :\ E : \|\chi_E\|_{\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu)} \le h^{\frac{p}{p_0}}\} \right) = 0,$$

so by the Lebesgue convergence theorem,

$$\lim_{h \downarrow 0} \left(\sup\{\|\chi_E f\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} \ :\ E : \|\chi_E\|_{\mathcal{M}^p_q(\mathcal{X},\mu)} \le h\} \right) = 0.$$

Hence, $f \in \overline{\mathcal{M}^p_q}(\mathcal{X}, \mu)$. □

Proof (*of Theorem* 3*(iv)*) By combining

$$[\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu), \mathcal{M}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta \subseteq \overline{\mathcal{M}^p_q}(\mathcal{X},\mu)$$

and

$$[\mathcal{M}^{p_0}_{q_0}(\mathcal{X},\mu), \overset{*}{\mathcal{M}}{}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta \subseteq \overset{*}{\mathcal{M}}{}^p_q(\mathcal{X},\mu),$$

we have

$$[\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu), \overset{*}{\mathcal{M}}{}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta \subseteq \overline{\mathcal{M}^p_q}(\mathcal{X},\mu) \cap \overset{*}{\mathcal{M}}{}^p_q(\mathcal{X},\mu) = \widetilde{\mathcal{M}^p_q}(\mathcal{X},\mu).$$

Meanwhile,

$$\widetilde{\mathcal{M}^p_q}(\mathcal{X},\mu) = [\widetilde{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu), \widetilde{\mathcal{M}^{p_1}_{q_1}}(\mathcal{X},\mu)]_\theta \subseteq [\overline{\mathcal{M}^{p_0}_{q_0}}(\mathcal{X},\mu), \overset{*}{\mathcal{M}}{}^{p_1}_{q_1}(\mathcal{X},\mu)]_\theta. \qquad \square$$

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Gagliardo-Nirenberg Inequalities for Spaces with Dominating Mixed Derivatives

Dorothee D. Haroske and Hans-Jürgen Schmeisser

Abstract We study inequalities of Gagliardo-Nirenberg type for scales of function spaces with dominating mixed smoothness. This situation is more sophisticated than in the classical isotropic case. We show that satisfying results can be obtained using the concept of refined dominating mixed smoothness both in the case of Triebel-Lizorkin and Besov-type spaces.

Keywords Gagliardo-Nirenberg inequalities · Function spaces with dominating mixed derivatives · Refined logarithmic smoothness

1 Introduction

The nowadays classical Gagliardo-Nirenberg inequality says that for $1 < p, u < \infty$, $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ with $k < m$

$$\sum_{|\beta|=k} \|\mathrm{D}^\beta f|L_v(\mathbb{R}^n)\| \le c\|f|L_u(\mathbb{R}^n)\|^{1-\theta}\Big(\sum_{|\alpha|=m} \|\mathrm{D}^\alpha f|L_p(\mathbb{R}^n)\|\Big)^\theta \qquad (1)$$

for smooth functions $f \in C^\infty(\mathbb{R}^n)$ with compact support. Here $\theta = \frac{k}{m}$ and $\frac{1}{v} = \frac{1-\theta}{u} + \frac{\theta}{p}$. Using a homogeneity argument inequality (1) can be derived from

$$\|f|W_v^k(\mathbb{R}^n)\| \le c\,\|f|L_u(\mathbb{R}^n)\|^{1-\theta}\|f|W_p^m(\mathbb{R}^n)\|^\theta, \qquad (2)$$

where $W_p^m(\mathbb{R}^n)$ denotes as usual the Sobolev space and where the parameters u, p, v, θ have the above meaning (see, for example [25], Theorem 1, even in the

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Dedicated to Our Friend and Colleague Miroslav Krbec

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vector-valued case). Let us denote by $F^r_{p,q}(\mathbb{R}^n)$, $0 < p \leq \infty$, $0 < q \leq \infty$, $r \in \mathbb{R}$, the Triebel-Lizorkin space (see, for example, [28]). Then inequality (2) is a consequence of

$$\|f|F^r_{p,q}(\mathbb{R}^n)\| \leq c\,\|f|F^{r_0}_{p_0,q_0}(\mathbb{R}^n)\|^{1-\theta}\,\|f|F^{r_1}_{p_1,q_1}(\mathbb{R}^n)\|^{\theta} \tag{3}$$

which holds for all $f \in F^{r_0}_{p_0,q_0}(\mathbb{R}^n) \cap F^{r_1}_{p_1,q_1}(\mathbb{R}^n)$ if $0 < p_0,\ p_1 \leq \infty$, $0 < q_0,\ q_1,\ q \leq \infty$, $-\infty < r_0 < r_1 \leq \infty$, $0 < \theta < 1$ and

$$(r,\ 1/p) = (1-\theta)(r_0,\ 1/p_0) + \theta(r_1,\ 1/p_1). \tag{4}$$

Inequality (3) is due to Oru [21] and can be found in the paper [2] by Brezis and Mironescu. The remarkable fact in (3) is the independence of the parameters q_0, q_1 and q (disregarding the constant). In particular, one can choose $q_0 = q_1 = \infty$ which corresponds to the largest spaces with respect to the parameter q in the scale of spaces $F^r_{p,q}(\mathbb{R}^n)$. Under the above assumptions condition (4) is also necessary as has been proved in [6, 7], respectively, in the homogeneous case.

The situation is different in the case of Besov spaces $B^r_{p,q}(\mathbb{R}^n)$. Necessary and sufficient conditions for Gagliardo-Nirenberg inequalities within this scale can be found again in [6] and [7], respectively. The main result reads as follows (see [7], Theorem 4.1): Let $0 < \theta < 1$, $0 < q < \infty$, $0 < p, p_0, p_1 \leq \infty$, $-\infty < r, r_0, r_1 < \infty$ and let

$$\frac{n}{p} - r = (1-\theta)\left(\frac{n}{p_0} - r_0\right) + \theta\left(\frac{n}{p_1} - r_1\right), \tag{5}$$

$$r_0 - \frac{n}{p_0} \neq r_1 - \frac{n}{p_1}. \tag{6}$$

Then

$$\|f|B^r_{p,\,q}(\mathbb{R}^n)\| \lesssim \|f|B^{r_0}_{p_0,\,\infty}(\mathbb{R}^n)\|^{1-\theta}\,\|f|B^{r_1}_{p_1,\,\infty}(\mathbb{R}^n)\|^{\theta} \tag{7}$$

holds, if either

$$r < (1-\theta)r_0 + \theta r_1 \tag{8}$$

or

$$p_0 = p_1 \text{ and } r = (1-\theta)r_0 + r_1. \tag{9}$$

For forerunners as well as for further types of Gagliardo-Nirenberg inequalities for isotropic Besov and Triebel-Lizorkin spaces we refer, for example, to [18, 25, 30], see also [12, 33]. Some more recent results concerning further types of function spaces like (Musielak-)Orlicz spaces, Lorentz spaces, Morrey spaces, can be found in [8, 13–16, 22, 31, 34, 35], but this list of references is by no means complete.

Here we are concerned with the analogous problem for function spaces with dominating mixed smoothness. For simplicity let us consider the bivariate case. We denote by $S^r_{p,q}F = S^r_{p,q}F(\mathbb{R}^2)$, $0 < p < \infty$, $0 < q \leq \infty$, $r \in \mathbb{R}$, the Triebel-Lizorkin space with dominating mixed smoothness as defined in [26], Chap. 2. Spaces of this type turned out to be very useful in multivariate approximation. For properties, characterizations and applications we refer also to [23, 24, 32]. The counterpart of (3) reads as

$$\| f\,|S^{r}_{p,q}F\| \le c\,\| f\,|S^{r_0}_{p_0,q_0}F\|^{1-\theta}\,\| f\,|S^{r_1}_{p_1,q_1}F\|^{\theta}. \tag{10}$$

It has been observed by Hansen in [9], Proposition 6.8.1 (see also [11], Proposition 4.1), that inequality (10) holds true if, and only if, (4) is satisfied and

$$\frac{1}{q} \le \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{11}$$

In this "trivial" case (10) is a consequence of Hölder's inequality as we shall see later on. In particular, we have $q = \infty$ if $q_0 = q_1 = \infty$. The situation turns out to be different if we admit refined smoothness in the target space on the left-hand side of (10). To this end we introduce in Sect. 2 the spaces $S^{r,\alpha}_{p,q}A\ \ (A \in \{B,\ F\})$, where $\alpha \in \mathbb{R}$ refers to the exponent of refined logarithmic smoothness. First, we observe the elementary embedding

$$S^{r}_{p,\infty}F \hookrightarrow S^{r,\alpha}_{p,q}F$$

which holds if $\alpha < -1/q,\ 0 < q < \infty$. The price to pay for a better microscopic parameter q is a weaker (refined logarithmic) smoothness. The first aim of this paper is to prove the Gagliardo-Nirenberg inequality

$$\| f\,|S^{r,\alpha}_{p,q}F\| \le c\,\| f\,|S^{r_0}_{p_0,q_0}F\|^{1-\theta}\,\| f\,|S^{r_1}_{p_1,q_1}F\|^{\theta} \tag{12}$$

for all parameters $0 < q_0,\ q_1 \le \infty,\ 0 < q < \infty$ and $\alpha \le -\frac{1}{2q}$ provided that condition (4) is satisfied. This is an improvement with respect to the logarithmic smoothness in the target space.

Secondly we shall be concerned with similar inequalities within the scale of Besov spaces with dominating mixed smoothness.

The paper is organized as follows. In Sect. 2 we introduce the function spaces with dominating mixed smoothness under consideration and state some elementary embeddings. Gagliardo-Nirenberg inequalities for spaces with dominating mixed smoothness are treated in Sect. 3. After preliminary considerations in Sect. 3.1 we deal with Triebel-Lizorkin spaces in Sect. 3.2 and with Besov spaces in Sect. 3.3. The main results can be found in Theorem 11 (F-spaces) and Theorem 14 (B-spaces), respectively.

The first draft of this paper was written while the second author visited our friend and colleague Miroslav Krbec in Prague in 2011. This collaboration was stopped by his sudden death in summer 2012. We returned to the project now and feel indebted and grateful for Mirek's contribution to the results.

2 Function Spaces and Embeddings

We shall adopt the following general notation: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^n$, $n \in \mathbb{N}$, denotes the n-dimensional real Euclidean space. For a real number a let $\lfloor a \rfloor$ denote its integer part. For convenience we use the convention that $1/\infty = 0$. By c, c_1, c_2, etc. we denote positive constants independent of appropriate quantities, which may, however, depend on other parameters like smoothness, dimension, regularity. For two non-negative expressions (i.e. functions or functionals) $\mathcal{A}, \mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) means that $\mathcal{A} \le c\,\mathcal{B}$ (or $c\,\mathcal{A} \ge \mathcal{B}$). If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$, we write $\mathcal{A} \sim \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding is bounded.

Throughout the paper we shall use standard notation, e.g. $\mathcal{F}$ and $\mathcal{F}^{-1}$ for the Fourier transform and its inverse, resp., $\mathcal{S}(\mathbb{R}^n)$ for the space of rapidly decreasing C^∞ functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ for its dual, the space of tempered distributions.

The Lebesgue space $L_p = L_p(\mathbb{R}^n)$, $0 < p \le \infty$, consists of all measurable functions with finite (quasi-)norm

$$\|f\,|L_p\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{1/p}, \quad 0 < p < \infty,$$

complemented by $\|f\,|L_\infty\| = \operatorname{ess\,sup} |f(x)|$ when $p = \infty$.

We briefly recall the definitions of the spaces that we work with in the sequel. Let φ_0 be a real-valued infinitely differentiable function on $\mathbb{R}^n$ such that $0 \le \varphi_0(x) \le 1$, $\varphi_0(x) = 1$ if $|x| \le 1$, and $\varphi_0(x) = 0$ if $|x| \ge 2$. Put

$$\begin{aligned} \varphi_1(x) &= \varphi_0(x/2) - \varphi_0(x) \\ \varphi_j(x) &= \varphi_1(2^{-j+1}x), \qquad j = 2, 3, \ldots. \end{aligned} \tag{13}$$

Then

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1, \qquad x \in \mathbb{R}^n.$$

The system of functions $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is called a *smooth dyadic decomposition of the unity in* $\mathbb{R}^n$.

Now let us recall the definitions of the spaces with dominating mixed derivatives.

Definition 1 Let $0 < q \le \infty$, $-\infty < r, \alpha < \infty$, and $\{\varphi_j\}_{j\in\mathbb{N}_0}$ a smooth dyadic resolution of unity.

(i) Let $0 < p \le \infty$. The *Besov space* $S^{r,\alpha}_{p,q}B = S^{r,\alpha}_{p,q}B(\mathbb{R}^n \times \mathbb{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\|f|S^{r,\alpha}_{p,q}B\| = \left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|2^{(j+k)r}(1+j)^{\alpha}(1+k)^{\alpha}f_{j,k}|L_p\|^q\right)^{1/q}$$

is finite, where

$$f_{j,k}(x,y) = \mathcal{F}^{-1}[\varphi_j \otimes \varphi_k \mathcal{F}f](x,y), \quad x,y \in \mathbb{R}^n,\ j,k \in \mathbb{N}_0, \tag{14}$$

and in case of $q = \infty$ the usual modification is required,

$$\|f|S^{r,\alpha}_{p,\infty}B\| = \sup_{j,k\in\mathbb{N}_0}\|2^{(j+k)r}(1+j)^{\alpha}(1+k)^{\alpha}f_{j,k}|L_p\|.$$

(ii) Let $0 < p < \infty$. The *Triebel-Lizorkin space* $S^{r,\alpha}_{p,q}F = S^{r,\alpha}_{p,q}F(\mathbb{R}^n \times \mathbb{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\|f|S^{r,\alpha}_{p,q}F\| = \left\|\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|2^{(j+k)r}(1+j)^{\alpha}(1+k)^{\alpha}f_{j,k}(\cdot)|^q\right)^{1/q}\middle|L_p\right\|$$

is finite, where again in case of $q = \infty$ the usual modification is required.

Convention. We adopt the nowadays usual custom to write $S^{r,\alpha}_{p,q}A$, where $A \in \{B, F\}$, instead of $S^{r,\alpha}_{p,q}B$ or $S^{r,\alpha}_{p,q}F$, when both scales of spaces are meant simultaneously in some context.

Remark 2 The above (quasi-)norms depend on the particular choice of $\{\varphi_j\}_{j=0}^{\infty}$, but (quasi-)norms corresponding to different decompositions with the above properties (13) are equivalent. This can be proved in a similar way as in [26], Chap. 2.

If $\alpha = 0$, then the spaces coincide with the classical spaces with dominating mixed smoothness, $S^{r,0}_{p,q}A = S^{r}_{p,q}A$, see [26], Chap. 2, and the references given therein. The logarithmic refinement ($\alpha \neq 0$) has been considered in [27] (in connection with hyperbolic cross approximation for $1 < p < \infty$, $1 \le q \le \infty$) and in [29] (related to numerical integration). If the term $2^{(j+k)r}(1+j)^{\alpha}(1+k)^{\alpha}$ is replaced by some more general $b_{j,k}$, then the above spaces (for $A = B$) are special cases of spaces with generalized smoothness studied in [17].

Remark 3 We would like to point out that many of the arguments and results in this short paper can be transferred more or less immediately to the more general situation of spaces $S^{\overline{r},\overline{\alpha}}_{p,q}B(\mathbb{R}^m \times \mathbb{R}^n)$, $S^{\overline{r},\overline{\alpha}}_{p,q}F(\mathbb{R}^m \times \mathbb{R}^n)$ which consist of all $f \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n)$, $m,n \in \mathbb{N}$, such that

$$\|f|S^{\overline{r},\overline{\alpha}}_{p,q}F\| = \left\| \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left| 2^{jr_1+kr_2}(1+j)^{\alpha_1}(1+k)^{\alpha_2}\mathcal{F}^{-1}\left[\varphi_j \otimes \psi_k \mathcal{F}f\right](\cdot)\right|^q \right)^{1/q} |L_p \right\|$$

is finite, where $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\psi_j\}_{j=0}^{\infty}$ are smooth dyadic resolutions of unity in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and $0 < p < \infty$, $0 < q \leq \infty$, $\overline{r} = (r_1, r_2) \in \mathbb{R}^2$, $\overline{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. Similarly for Besov spaces $S^{\overline{r},\overline{\alpha}}_{p,q}B$, with the usual modification in case of $q = \infty$. If $\overline{\alpha} = (0, 0)$, $\overline{m} = (m_1, m_2) \in \mathbb{N}_0^2$ and $1 < p < \infty$, then

$$\begin{aligned} S^{\overline{m}}_{p,2}F(\mathbb{R}^m \times \mathbb{R}^n) &= S^{\overline{m}}_{p}W(\mathbb{R}^m \times \mathbb{R}^n) \\ &= \{f \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n) : \mathrm{D}^{\beta}_x \mathrm{D}^{\gamma}_y f \in L_p(\mathbb{R}^m \times \mathbb{R}^n),\ |\beta| \leq m_1,\ |\gamma| \leq m_2\}. \end{aligned}$$

If $\overline{r} = (r_1, r_2) \in \mathbb{R}^2$ and $1 < p < \infty$, then

$$\begin{aligned} S^{\overline{r}}_{p,2}F(\mathbb{R}^m \times \mathbb{R}^n) &= S^{\overline{r}}_{p}H(\mathbb{R}^m \times \mathbb{R}^n) \\ &= \left\{ f \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n) : \mathcal{F}^{-1}\left[(1+|\xi|^2)^{\frac{r_1}{2}}(1+|\eta|^2)^{\frac{r_2}{2}}\mathcal{F}f\right] \in L_p(\mathbb{R}^m \times \mathbb{R}^n) \right\}, \end{aligned}$$

where $S^{\overline{r}}_{p}H(\mathbb{R}^m \times \mathbb{R}^n)$ is a Sobolev space with dominating mixed derivatives of fractional order. Moreover, all spaces $S^{\overline{r}}_{p,q}A(\mathbb{R}^m \times \mathbb{R}^n)$ can be characterized by means of differences, local means, atoms or wavelets for certain ranges of parameters. For example, if $r_i > 0$, $i = 1, 2$, and $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, then the spaces $S^{\overline{r}}_{p,q}B(\mathbb{R}^m \times \mathbb{R}^n)$ coincide with spaces introduced by Nikol'skiĭ [19] ($q = \infty$) and Amanov [1] ($q < \infty$), respectively. For more recent results with respect to atomic and wavelet characterizations we refer to Hansen [9], Hansen, Sickel [10, 11] and Vybíral [32].

As already mentioned in the introduction, spaces with dominating mixed smoothness are closely related to multivariate approximation (hyperbolic cross, best m-term approximation). A survey on this topic, based on the pioneering work by Temlyakov and his co-authors, including a rather complete list of references, can be found in the recent paper [3].

Similarly to the isotropic case (see, for example, [4]) one can develop a theory of function spaces with generalized dominating mixed smoothness. A first step has been done in [17] as far as spaces of Besov type and the Fourier analytic approach is concerned. In [27] equivalent characterizations of $S^{r,\alpha}_{p,q}A$ have been proved for $1 < p, q < \infty$ and $r > 0$.

For simplicity we shall only deal with the case $m = n$, $r_1 = r_2 = r$, $\alpha_1 = \alpha_2 = \alpha$ in this paper.

Lemma 4 *Let* $0 < p < \infty$, $0 < q \le \infty$, $r, \alpha \in \mathbb{R}$, $A \in \{B, F\}$.

(i) *If* $0 < q_0 \le q_1 \le \infty$, *then*

$$S^{r,\alpha}_{p,q_0}A \hookrightarrow S^{r,\alpha}_{p,q_1}A.$$

(ii) *If* $\beta > \alpha$, *then*

$$S^{r,\beta}_{p,q}A \hookrightarrow S^{r,\alpha}_{p,q}A.$$

(iii) *If* $0 < v \le \infty$ *and* $\beta > \alpha + \frac{1}{q}$, *then*

$$S^{r,\beta}_{p,v}A \hookrightarrow S^{r,\alpha}_{p,q}A. \tag{15}$$

If $q = \infty$, $\beta = \alpha$ *is also admitted.*

Proof While assertions (i) and (ii) are obvious in view of Definition 1, we insert a short proof for (iii). In view of (i) it is sufficient to deal with $v = \infty$. Let $q < \infty$, $A = F$, recall notation (14). Then

$$\begin{aligned} \|f|S^{r,\alpha}_{p,q}F\| &= \left\| \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left|2^{(j+k)r}(1+j)^{\alpha}(1+k)^{\alpha} f_{j,k}(\cdot)\right|^q \right)^{1/q} \Big| L_p \right\| \\ &\le \left[\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (1+j)^{(\alpha-\beta)q}(1+k)^{(\alpha-\beta)q} \right]^{1/q} \|f|S^{r,\beta}_{p,\infty}F\| \end{aligned}$$

provided $(\alpha - \beta)q < -1$, that is, $\beta > \alpha + \frac{1}{q}$. The modifications in case of $q = \infty$ are clear. □

Application of Lemma 4(iii) with $\beta = 0$ immediately leads to the following useful result.

Corollary 5 *Let* $0 < p, q < \infty$, $r, \alpha \in \mathbb{R}$. *Then*

$$S^{r}_{p,\infty}A \hookrightarrow S^{r,\alpha}_{p,q}A$$

if $\alpha < -1/q$.

We finally consider an embedding with different metrics.

Lemma 6 *Let* $0 < \widetilde{p} < p \le \infty$, $-\infty < r < \widetilde{r} < \infty$, $-\infty < \alpha < \infty$, $0 < q \le \infty$ *and*

$$\widetilde{r} - \frac{n}{\widetilde{p}} = r - \frac{n}{p}. \tag{16}$$

Then

$$S^{\widetilde{r},\,\alpha}_{\widetilde{p},\,q}B \hookrightarrow S^{r,\,\alpha}_{p,\,q}B. \tag{17}$$

Proof The proof follows exactly the same line of arguments as in [26], Chap. 2, for the case $\alpha = 0$. □

3 Gagliardo-Nirenberg Inequalities

3.1 Preliminaries

We first collect some more or less immediate generalizations of the classical results and the above embeddings, before we concentrate on the interesting case of refined smoothness below.

We are interested in inequalities of type (10). First let us observe that (10) can be proved rather easily with the help of Hölder's inequality.

Proposition 7 *Let* $0 < p_i \le \infty$ *(with* $p_i < \infty$ *if* $A = F$*),* $0 < q_i \le \infty$*,* $\alpha_i, r_i \in \mathbb{R}$*,* $i = 0, 1$*, and* $0 < \eta < 1$*. If*

$$(1/p, 1/q, r, \alpha) = (1-\eta)(1/p_0, 1/q_0, r_0, \alpha_0) + \eta(1/p_1, 1/q_1, r_1, \alpha_1),$$

then

$$\| f\, | S^{r,\alpha}_{p,q}A \| \le \| f\, | S^{r_0,\alpha_0}_{p_0,q_0}A\|^{1-\eta} \| f\, | S^{r_1,\alpha_1}_{p_1,q_1}A\|^{\eta}\,. \tag{18}$$

Proof For brevity we use (14) again,

$$f_{j,k}(x) = \mathcal{F}^{-1}[\varphi_j \otimes \varphi_k \mathcal{F} f](x), \qquad x \in \mathbb{R}^n \times \mathbb{R}^n, \quad (j,k) \in \mathbb{N}_0^2,$$

and

$$\begin{aligned} |a_{jk}(x)| &= \left|2^{(j+k)r}(1+j)^{\alpha}(1+k)^{\alpha} f_{j,k}(x)\right| \\ &= \left|2^{(j+k)r_0}(1+j)^{\alpha_0}(1+k)^{\alpha_0} f_{j,k}(x)\right|^{1-\eta} \times \\ &\quad \times \left|2^{(j+k)r_1}(1+j)^{\alpha_1}(1+k)^{\alpha_1} f_{j,k}(x)\right|^{\eta} \\ &= |b_{jk}(x)|^{1-\eta} |c_{jk}(x)|^{\eta}. \end{aligned}$$

Using successively Hölder's inequality with

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}, \quad \frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1},$$

we obtain

$$\Big\|\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{jk}(\cdot)|^q\Big)^{\frac{1}{q}}\big|L_p\Big\| = \Big\|\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|b_{jk}(\cdot)|^{(1-\eta)q}|c_{jk}(\cdot)|^{\eta q}\Big)^{\frac{1}{q}}\big|L_p\Big\|$$

$$\leq \Big\|\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|b_{jk}(\cdot)|^{q_0}\Big)^{\frac{1-\eta}{q_0}}\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|c_{jk}(\cdot)|^{q_1}\Big)^{\frac{\eta}{q_1}}\big|L_p\Big\|$$

$$\leq \|\{b_{jk}(\cdot)\}|L_{p_0}(\ell_{q_0})\|^{1-\eta}\|\{c_{jk}(\cdot)\}|L_{p_1}(\ell_{q_1})\|^{\eta}.$$

This yields (18) in case of $A = F$, the proof in case of $A = B$ is similar. □

Remark 8 Note that (18) together with Corollary 5 implies

$$\|f|S^{r,\alpha}_{p,q}F\| \leq c\|f|S^{r_0}_{p_0,\infty}F\|^{1-\eta}\,\|f|S^{r_1}_{p_1,\infty}F\|^{\eta} \tag{19}$$

for $0 < q \leq \infty$, $\alpha < -1/q$, $0 < \eta < 1$, and

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}, \qquad r = (1-\eta)r_0 + \eta r_1.$$

This is an inequality of type (3) for spaces with dominating mixed smoothness. The price which one has to pay is a certain (logarithmic) loss of smoothness in the target space on the left-hand side.

The aim of the next subsection is to present an improvement of (19) with respect to the parameter α, expressing the logarithmic tuning of the smoothness.

Proposition 9 *Let* $0 < p_i, q_i \leq \infty$, $r_i \in \mathbb{R}$, $i = 0, 1$, *and* $0 < \theta < 1$. *Assume*

$$r \leq (1-\theta)r_0 + \theta r_1, \tag{20}$$

$$\frac{1}{q} \leq \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,, \tag{21}$$

$$r - \frac{n}{p} = (1-\theta)\big(r_0 - \frac{n}{p_0}\big) + \theta\big(r_1 - \frac{n}{p_1}\big). \tag{22}$$

Then

$$\|f|S^r_{p,q}B\| \leq c\,\|f|S^{r_0}_{p_0,q_0}B\|^{1-\theta}\|f|S^{r_1}_{p_1,q_1}B\|^{\theta}\,. \tag{23}$$

Proof We combine Lemma 4 and Proposition 7: It follows from (20) and (22) that

$$(1-\theta)\frac{n}{p_0} + \theta\frac{n}{p_1} - \frac{n}{p} = (1-\theta)r_0 + \theta r_1 - r = \eta \geq 0.$$

Consider

$$\frac{1}{\widetilde{p}} := \frac{1}{p} + \frac{\eta}{n} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \geq \frac{1}{p},$$

and

$$\widetilde{r} := r + \eta = (1-\theta)r_0 + \theta r_1 \geq r.$$

Then

$$\widetilde{r} - \frac{n}{\widetilde{p}} = r + \eta - n\left(\frac{1}{p} + \frac{\eta}{n}\right) = r - \frac{n}{p}.$$

Together with (17) this implies the embedding

$$S^{\widetilde{r}}_{\widetilde{p},q}B \hookrightarrow S^r_{p,q}B.$$

Taking into account (21) we find in the same way as in the proof of Proposition 7 the inequality

$$\|f|S^{\widetilde{r}}_{\widetilde{p},q}B\| \leq \|f|S^{r_0}_{p_0,q_0}B\|^{1-\theta}\|f|S^{r_1}_{p_1,q_1}B\|^{\theta}.$$

Combination with the above embedding leads to (23). □

3.2 Triebel-Lizorkin Spaces

Now we come to our main results: we obtain the counterpart of Proposition 7 for $A = F$ for arbitrary parameters q_0, q_1, but at the expense of some reduced logarithmic smoothness. Our main technical tool will be the following estimate taken from Brezis and Mironescu [2].

Lemma 10 ([2]) *Let* $0 < \theta < 1$, $0 < q < \infty$, $-\infty < r_0, r_1 < \infty$ *with* $r_0 \neq r_1$. *If* $r = (1-\theta)r_0 + \theta r_1$, *then there exists some* $c > 0$ *such that*

$$\left\|\left\{2^{kr}d_k\right\}_k|\ell_q\right\| \leq c\left\|\left\{2^{kr_0}d_k\right\}_k|\ell_\infty\right\|^{1-\theta}\left\|\left\{2^{kr_1}d_k\right\}_k|\ell_\infty\right\|^{\theta}. \tag{24}$$

Our first main result is the following.

Theorem 11 *Let* $0 < p_0, p_1 < \infty$ *with* $p_0 \neq p_1$, $0 < q < \infty$, $-\infty < r_0, r_1 < \infty$ *with* $r_0 \neq r_1$, *and* $0 < \theta < 1$. *If*

$$(r, 1/p) = (1-\theta)(r_0, 1/p_0) + \theta(r_1, 1/p_1),$$

then

$$\| f \,|S^{r,-\frac{1}{2q}}_{p,q} F\| \le c\, \| f \,|S^{r_0}_{p_0,\infty} F\|^{1-\theta} \| f \,|S^{r_1}_{p_1,\infty} F\|^{\theta} \tag{25}$$

for all $f \in S^{r_0}_{p_0,\infty}F \cap S^{r_1}_{p_1,\infty}F$.

Proof Put

$$\begin{aligned}\Bigg(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\Big|2^{(j+k)r}\ (1+j)^{-\frac{1}{2q}}(1+k)^{-\frac{1}{2q}} f_{j,k}(x)\Big|^q\Bigg)^{\frac{1}{q}} \\ &= \Bigg(\sum_{\ell=0}^{\infty} 2^{\ell r q} \sum_{j+k=\ell} \big|\, ((1+j)(1+k))^{-\frac{1}{2q}}\ f_{j,k}(x)\big|^q\Bigg)^{\frac{1}{q}} \\ &= \Bigg(\sum_{\ell=0}^{\infty} 2^{\ell r q} d_\ell(x)^q\Bigg)^{\frac{1}{q}},\end{aligned} \tag{26}$$

where

$$d_\ell(x) = \Bigg[\sum_{j+k=\ell} \big|\, ((1+j)(1+k))^{-\frac{1}{2q}}\ f_{j,k}(x)\big|^q\Bigg]^{\frac{1}{q}}, \qquad \ell \in \mathbb{N}_0.$$

Using (24) we obtain for fixed $x \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\big\|\big\{2^{\ell r q} d_\ell(x)\big\}_\ell \,|\ell_q\big\| \le c\, \big\|\big\{2^{\ell r_0} d_\ell(x)\big\}_\ell \,|\ell_\infty\big\|^{1-\theta}\ \big\|\big\{2^{\ell r_1} d_\ell(x)\big\}_\ell \,|\ell_\infty\big\|^{\theta}.$$

Furthermore, for $i = 0, 1$,

$$\begin{aligned}\big\|\big\{2^{\ell r_i} d_\ell(x)\big\}_\ell \,|\ell_\infty\big\| &= \sup_{\ell\in\mathbb{N}_0} 2^{\ell r_i}\Bigg[\sum_{j+k=\ell}\Big|(1+j)^{-\frac{1}{2q}}(1+k)^{-\frac{1}{2q}} f_{j,k}(x)\Big|^q\Bigg]^{\frac{1}{q}} \\ &\le c \sup_{\ell\in\mathbb{N}_0}\ \sup_{j+k=\ell}\ 2^{(j+k)r_i}|f_{j,k}(x)|\Bigg(\sum_{j+k=\ell}(1+j)^{-\frac{1}{2}}(1+k)^{-\frac{1}{2}}\Bigg)^{\frac{1}{q}} \\ &\le c \sup_{(j,k)\in\mathbb{N}_0^2} 2^{(j+k)r_i}|f_{j,k}(x)|,\end{aligned}$$

where we used the estimate

$$\sum_{j+k=\ell} [(1+j)(1+k)]^{-1/2} \sim 1 \tag{27}$$

with equivalence constants independent of $\ell \in \mathbb{N}_0$; this will be shown below.

It follows that

$$\left(\sum_{\ell=0}^{\infty} 2^{\ell r q} d_\ell(x)^q\right)^{\frac{1}{q}} \leq c \,\left\| \left\{2^{(j+k)r_0} f_{j,k}(x)\right\}_\ell |\ell_\infty\right\|^{1-\theta} \left\| \left\{2^{(j+k)r_1} f_{j,k}(x)\right\}_\ell |\ell_\infty\right\|^{\theta}.$$

Inserting this estimate into (26), taking the L_p-(quasi-)norms, and applying Hölder's inequality with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we obtain the desired inequality (25).

To complete the proof we establish (27), let $\ell \in \mathbb{N}$. We argue as follows,

$$\begin{aligned}
\sum_{j+k=\ell} [(1+j)(1+k)]^{-\frac{1}{2}} &= \sum_{j=0}^{\ell} [(1+j)(1+\ell-j)]^{-\frac{1}{2}} \\
&\leq 2 \sum_{j=0}^{\lfloor \ell/2 \rfloor} [(1+j)(1+\ell-j)]^{-\frac{1}{2}} \\
&\leq 2\sqrt{\frac{2}{\ell}} \sum_{j=0}^{\lfloor \ell/2 \rfloor} \frac{1}{(1+j)^{1/2}} \\
&\leq 2\sqrt{\frac{2}{\ell}} \int_0^\ell \frac{\mathrm{d}x}{\sqrt{1+x}} \leq 4\sqrt{\frac{2}{\ell}}\, \sqrt{1+\ell} \leq 8.
\end{aligned}$$

We are done. □

Remark 12 A closer look shows that we can replace $f_{j,k}(x)$ by $(1+j)^\alpha(1+k)^\alpha f_{j,k}(x)$ with arbitrary $\alpha \in \mathbb{R}$ in the above considerations. Hence under the assumptions of Theorem 11 we get

$$\left\| f | S^{r,\alpha-1/(2q)}_{p,q} F \right\| \leq c \,\left\| f | S^{r_0,\alpha}_{p_0,\infty} F \right\|^{1-\theta} \left\| f | S^{r_1,\alpha}_{p_1,\infty} F \right\|^{\theta}$$

for all $f \in S^{r_0,\alpha}_{p_0,\infty} F \cap S^{r_1,\alpha}_{p_1,\infty} F$ and any $\alpha \in \mathbb{R}$. It is a natural, but still open question, whether or not the exponent $\alpha - 1/(2q)$ is optimal within the scale of spaces with refined logarithmic smoothness $S^{r,\beta}_{p,q} F$.

3.3 Besov Spaces

We are interested in inequalities of type (7) in the sense of Theorem 11.

We begin with some preparation.

Lemma 13 *Let* $0 < p \leq \infty$, $-\infty < r < \infty$. *Then*

$$f \in S^{r,-\frac{1}{2q}}_{p,q} B \quad \textit{for all } q,\ 0 < q \leq \infty,$$

if, and only if,

$$f \in S^{r,-\frac{N}{2}}_{p,\frac{1}{N}}B \quad \textit{for all} \quad N \in \mathbb{N}_0.$$

In particular,

$$\|f|S^{r,-\frac{1}{2q}}_{p,q}B\| \le c\,\|f|S^{r,-\frac{N}{2}}_{p,\frac{1}{N}}B\|^{1-\eta}\,\|f|S^{r,0}_{p,\infty}B\|^{\eta}, \tag{28}$$

where $\frac{1}{q} = (1-\eta)N$, $0 < \eta < 1$.

Proof Inequality (28) follows from Proposition 7 for $A = B$, with $p_0 = p_1 = p$, $r_0 = r_1 = r$, $q_0 = \frac{1}{N}$, $q_1 = \infty$, $\alpha_0 = -\frac{N}{2}$, $\alpha_1 = 0$, and $1-\eta = \frac{1}{qN}$. Indeed,

$$(1-\eta)\alpha_0 + \eta\alpha_1 = -(1-\eta)\frac{N}{2} = -\frac{1}{2q},$$

and

$$(1-\eta)\frac{1}{q_0} + \eta\frac{1}{q_1} = (1-\eta)N = \frac{1}{q}.$$

□

Theorem 14 *Let* $0 < p, p_0, p_1 \le \infty$, $0 < q < \infty$, $-\infty < r, r_0, r_1 < \infty$, *and* $0 < \theta < 1$. *We assume*

$$r - \frac{n}{p} = (1-\theta)\left(r_0 - \frac{n}{p_0}\right) + \theta\left(r_1 - \frac{n}{p_1}\right), \tag{29}$$

and

$$r_0 - \frac{n}{p_0} \ne r_1 - \frac{n}{p_1}. \tag{30}$$

If either

$$p_0 = p_1 = p \quad \textit{and} \quad r = (1-\theta)r_0 + \theta r_1, \tag{31}$$

or

$$r < (1-\theta)r_0 + \theta r_1, \tag{32}$$

then

$$\|f|S^{r,-\frac{1}{2q}}_{p,q}B\| \le c\|f|S^{r_0}_{p_0,\infty}B\|^{1-\theta}\|f|S^{r_1}_{p_1,\infty}B\|^{\theta} \tag{33}$$

for all $f \in S^{r_0}_{p_0,\infty}B \cap S^{r_1}_{p_1,\infty}B$.

Proof We benefit from the lift property of the spaces $S^{r,\alpha}_{p,v}B$ which can be found in [26], Chap. 2, for the case $\alpha = 0$, but the same proof works if $\alpha \ne 0$, too. So we may choose for convenience $r_0 = 0$. We split the proof into two parts. First we consider the case $p \geq \max(p_0, p_1)$ and secondly $p < \max(p_0, p_1)$. In both cases we follow the arguments in [7], adapted to our situation with dominating mixed smoothness.

Step 1. Assume that $p_0 = p_1 = p$ and $r = \theta r_1$, or $p > \max(p_0, p_1)$ and $r < \theta r_1$. In view of Lemma 13 it is sufficient to prove (33) for $q = \frac{1}{N}$, where $N \in \mathbb{N}$ is an arbitrary natural number; we assume that $N \geq 2$. Let $f_{j,k}$ be as in Proposition 7.

We have

$$\begin{aligned} \|f|S^{r,-\frac{N}{2}}_{p,\frac{1}{N}}B\| &= \left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left[2^{(j+k)r}(1+j)^{-\frac{N}{2}}(1+k)^{-\frac{N}{2}}\|f_{j,k}|L_p\|\right]^{\frac{1}{N}}\right)^N \\ &= \left(\sum_{\ell=0}^{\infty}\sum_{j+k=\ell}\left[2^{\ell r}(1+j)^{-\frac{N}{2}}(1+k)^{-\frac{N}{2}}\|f_{j,k}|L_p\|\right]^{\frac{1}{N}}\right)^N \\ &= \left(\sum_{\ell=0}^{\infty}2^{\ell r\cdot\frac{1}{N}}\sum_{j+k=\ell}((1+j)^{-\frac{1}{2}}(1+k)^{-\frac{1}{2}}\|f_{j,k}|L_p\|^{\frac{1}{N}})^{N\cdot\frac{1}{N}}\right)^N \\ &= \left(\sum_{\ell=0}^{\infty}2^{\ell r\cdot\frac{1}{N}}d_\ell(p)^{\frac{1}{N}}\right)^N, \end{aligned} \tag{34}$$

where

$$d_\ell(p) := \left(\sum_{j+k=\ell}(1+j)^{-\frac{1}{2}}(1+k)^{-\frac{1}{2}}\|f_{j,k}|L_p\|^{\frac{1}{N}}\right)^N. \tag{35}$$

Observe, that

$$\begin{aligned} \sup_{\ell\in\mathbb{N}_0} 2^{r\ell}d_\ell(p) &= \sup_{\ell\in\mathbb{N}_0}\left(\sum_{j+k=\ell}(1+j)^{-\frac{1}{2}}(1+k)^{-\frac{1}{2}}2^{(j+k)\cdot\frac{1}{N}r}\|f_{j,k}|L_p\|^{\frac{1}{N}}\right)^N \\ &\leq \sup_{\ell\in\mathbb{N}_0}\left(\sum_{j+k=\ell}(1+j)^{-\frac{1}{2}}(1+k)^{-\frac{1}{2}}\right)^N\|f|S^r_{p,\infty}B\| \\ &\lesssim \|f|S^r_{p,\infty}B\| \end{aligned} \tag{36}$$

by equivalence (27).

Moreover, using Nikol'skiĭ's inequality we find

$$2^{\ell r}d_\ell(p) \lesssim 2^{\ell(\frac{n}{p_i}-\frac{n}{p}+r)}d_\ell(p_i), \qquad i = 0, 1. \tag{37}$$

Since $r_0 = 0$, we obtain

$$\theta\left(r_1 - r + \frac{n}{p} - \frac{n}{p_1}\right) = (1-\theta)\left(r + \frac{n}{p_0} - \frac{n}{p}\right)$$

from (29). We put $t_0 = r + \frac{n}{p_0} - \frac{n}{p}$, $t_1 = r_1 - r + \frac{n}{p} - \frac{n}{p_1}$. Then

$$\theta t_1 = (1-\theta)t_0. \tag{38}$$

We assume $t_1, t_0 > 0$; the case $t_0, t_1 < 0$ can be treated analogously. It follows from (34) that

$$\begin{aligned}
&\|f|S^{r,-\frac{N}{2}}_{r,\frac{1}{N}}B\| \\
&= \prod_{i=1}^{N}\sum_{\ell_i=0}^{\infty}[2^{\ell_i r}d_{\ell_i}(p)]^{\frac{1}{N}} \\
&= \sum_{\ell_1=0}^{\infty}\cdots\sum_{\ell_N=0}^{\infty}[2^{\ell_1 r}d_{\ell_1}(p)\cdots 2^{\ell_N r}d_{\ell_N}(p)]^{\frac{1}{N}} \\
&\lesssim \sum_{\ell_1\ge\ell_2\ge\cdots\ge\ell_N}[2^{\ell_1 r}d_{\ell_1}(p)\cdots 2^{\ell_N r}d_{\ell_N}(p)]^{\frac{1}{N^2}}[2^{\ell_1 r}d_{\ell_1}(p)\cdots 2^{\ell_N r}d_{\ell_N}(p)]^{\frac{1}{N}(1-\frac{1}{N})}.
\end{aligned} \tag{39}$$

We have $\theta N \in (0, N)$. There exists a $\kappa \in \{1, \ldots, N\}$ such that $\theta N = \kappa - 1 + a$, where $0 < a \le 1$. Using (37) and (38) we get

$$\begin{aligned}
\prod_{i=1}^{N}2^{\ell_i r}d_{\ell_i}(p) &= \left[\prod_{i=1}^{\kappa-1}2^{\ell_i r}d_{\ell_i}(p)\right]\left[2^{\ell_\kappa r\cdot a}d_{\ell_\kappa}(p)\right]\left[2^{\ell_\kappa r(1-a)}d_{\ell_\kappa}(p)^{1-a}\right] \\
&\quad\times\left[\prod_{i=\kappa+1}^{N}2^{\ell_i r}d_{\ell_i}(p)\right] \\
&\le \left[\prod_{i=1}^{\kappa-1}2^{-\ell_i t_1}2^{\ell_i r}d_{\ell_i}(p_1)\right]2^{\ell_\kappa t_1 a}2^{\ell_\kappa r_1 a}d_\kappa(p_1)^a \\
&\quad\times 2^{\ell_\kappa t_0(1-a)}d_{\ell_\kappa}(p_0)^{1-a}\times\left[\prod_{i=\kappa+1}^{N}2^{\ell_i t_0}d_{\ell_i}(p_0)\right].
\end{aligned}$$

Now we apply (36) on the right-hand side to find that

$$\prod_{i=1}^{N}2^{\ell_i r}d_{\ell_i}(p) \lesssim \Lambda(\ell_1,\ldots,\ell_N)\|f|S^{r,1}_{p_1,\infty}B\|^{\theta N}\|f|S^0_{p_0,\infty}B\|^{(1-\theta)N} \tag{40}$$

where

$$\Lambda(\ell_1,\ldots,\ell_N) = \left(\prod_{i=1}^{\kappa-1}2^{-\ell_i t_1}\right)2^{-\ell_\kappa t_1 a}\cdot 2^{\ell_\kappa t_0(1-a)}\left(\prod_{i=\kappa+1}^{N}2^{\ell_i t_0}\right). \tag{41}$$

Inserting (40) on the right-hand side of (39) we obtain

$$\begin{aligned}\|f|S^{r,-\frac{N}{2}}_{p,\frac{1}{N}}B\| &\lesssim \sum_{\ell_1\geq\ell_2\cdots\geq\ell_N}\left[\prod_{i=1}^{N}2^{\ell_i r}d_{\ell_i}(p)\right]^{\frac{1}{N^2}}\Lambda(\ell_1,\ldots,\ell_N)^{\frac{1}{N}(1-\frac{1}{N})}\times\\ &\quad\times\|f|S^{r_1}_{p_1,\infty}B\|^{\theta(1-\frac{1}{N})}\|f|S^{0}_{p_0,\infty}B\|^{(1-\theta)(1-\frac{1}{N})}\\ &\lesssim\left\{\sum_{\ell_1\geq\ell_2\geq\cdots\geq\ell_N}\Lambda(\ell_1,\ldots,\ell_N)^{\frac{1}{N}(1-\frac{1}{N})}\sum_{i=1}^{N}2^{\ell_i r\cdot\frac{1}{N}}d_{\ell_i}(p)^{\frac{1}{N}}\right\}\\ &\quad\times\|f|S^{r_1}_{p_1,\infty}B\|^{\theta(1-\frac{1}{N})}\|f|S^{0}_{p_0,\infty}B\|^{(1-\theta)(1-\frac{1}{N})}.\end{aligned}\tag{42}$$

To derive (33) for $q=\frac{1}{N}$ from (42) it suffices to prove

$$\sum_{\ell_1\geq\ell_2\geq\cdots\geq\ell_N}\Lambda(\ell_1,\ldots,\ell_N)^{\frac{1}{N}(1-\frac{1}{N})}\sum_{i=1}^{N}2^{\ell_i r\cdot\frac{1}{N}}d^{\frac{1}{N}}_{\ell_i}(p)\lesssim\|f|S^{r,-\frac{N}{2}}_{p,\frac{1}{N}}B\|^{\frac{1}{N}}.\tag{43}$$

But this can be done in exactly the same way as in the proof of formula (2.26) in [7]. If $t_0,t_1<0$, then we consider

$$\sum_{\ell_N\geq\ell_{N-1}\geq\cdots\geq\ell_1}\ldots\quad\text{in place of}\quad\sum_{\ell_1\geq\ell_2\geq\cdots\geq\ell_N}\ldots$$

and proceed in the same way.

Step 2. Now we assume $p<\max(p_0,p_1)$, $r<(1-\theta)r_0+\theta r_1$. By (29) we have

$$\frac{1}{p}<\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\qquad\text{and}\qquad\min\left(\frac{1}{p_0},\frac{1}{p_1}\right)<\frac{1}{p}<\max\left(\frac{1}{p_0},\frac{1}{p_1}\right).$$

Because of (30) we may assume without loss of generality that

$$r_0-\frac{n}{p_0}<r_1-\frac{n}{p_1}.$$

This implies (note that $0<\theta<1$)

$$r_0-\frac{n}{p_0}<r-\frac{n}{p}<r_1-\frac{n}{p_1}.$$

There exists an $\varepsilon_0>0$ such that

$$r-\frac{n}{p_0}<r-\varepsilon-\frac{n}{p}<r-\frac{n}{p}<r+\varepsilon-\frac{n}{p}<r_1-\frac{n}{p_1}$$

for all ε, $0 < \varepsilon < \varepsilon_0$. Hence we can find functions $\theta_-(\varepsilon)$ and $\theta_+(\varepsilon)$ such that

$$(1-\theta_-(\varepsilon))\left(r_0 - \frac{n}{p_0}\right) + \theta_-(\varepsilon)\left(r_1 - \frac{n}{p_1}\right) = r - \varepsilon - \frac{n}{p} \tag{44}$$

and

$$(1-\theta_+(\varepsilon))\left(r_0 - \frac{n}{p_0}\right) + \theta_+(\varepsilon)\left(r_1 - \frac{n}{p_1}\right) = r + \varepsilon - \frac{n}{p}. \tag{45}$$

By virtue of (29) we have

$$\frac{\theta_-(\varepsilon) + \theta_+(\varepsilon)}{2} = \theta \tag{46}$$

for all ε, $0 < \varepsilon < \varepsilon_0$. Moreover,

$$\lim_{\varepsilon \to 0} \theta_-(\varepsilon) = \lim_{\varepsilon \to 0} \theta_+(\varepsilon) = \theta. \tag{47}$$

We have

$$r + \rho = (1-\theta) r_0 + \theta r_1 \qquad \text{where } \rho > 0.$$

The identity (47) implies that there exists an $\varepsilon_1 > 0$ such that

$$|(1-\theta)r_0 + \theta r_1 - [(1-\theta_+(\varepsilon))r_0 + \theta_+(\varepsilon) r_1]| < \frac{\rho}{2}$$

for all $\varepsilon < \varepsilon_1$. Consequently,

$$r + \rho \le (1-\theta_+(\varepsilon))r_0 + \theta_+(\varepsilon) r_1 + \frac{\rho}{2}$$

and thus

$$r + \varepsilon \le (1-\theta_+(\varepsilon))r_0 + \theta_+(\varepsilon) r_1 \tag{48}$$

for all $\varepsilon < \min(\rho/2, \varepsilon_1)$. Similarly, there exists an ε_2 such that

$$|(1-\theta)r_0 + \theta r_1 - [(1-\theta_-(\varepsilon))r_0 + \theta_-(\varepsilon) r_1]| < \rho$$

for all $\varepsilon < \varepsilon_2$. Hence

$$r - \varepsilon < r \le (1-\theta_-(\varepsilon))r_0 + \theta_-(\varepsilon) r_1 \tag{49}$$

for all $\varepsilon < \varepsilon_2$. Now we choose $\varepsilon < \min(\varepsilon_0, \varepsilon_1, \varepsilon_2, \frac{\rho}{2})$ and formulae (44), (45), (48), and (49) are simultaneously valid.

In view of the result in Step 1 for $0 < q < \infty$, $p_0 = p_1 = p$, $r_1 = r - \varepsilon$, $r_0 = r + \varepsilon$ and $\theta = \frac{1}{2}$ we have

$$\|f|S^{r,-\frac{1}{2q}}_{p,q}B\| \lesssim \|f|S^{r-\varepsilon}_{p,\infty}B\|^{\frac{1}{2}}\,\|f|S^{r+\varepsilon}_{p,\infty}B\|^{\frac{1}{2}}. \tag{50}$$

Taking into account formulae (44), (45), (48), (49), and Proposition 9, that is, inequality (23), we conclude

$$\|f|S^{r-\varepsilon}_{p,\infty}B\| \lesssim \|f|S^{r_0}_{p_0,\infty}B\|^{1-\theta_-(\varepsilon)}\,\|f|S^{r_1}_{p_1,\infty}B\|^{\theta_-(\varepsilon)} \tag{51}$$

and

$$\|f|S^{r+\varepsilon}_{p,\infty}B\| \lesssim \|f|S^{r_0}_{p_0,\infty}B\|^{1-\theta_+(\varepsilon)}\,\|f|S^{r_1}_{p_1,\infty}B\|^{\theta_+(\varepsilon)}. \tag{52}$$

Combining (50)–(52) we get the desired inequality (33) because of (46). This completes the proof. □

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Recent Trends in Grand Lebesgue Spaces

Pankaj Jain, Monika Singh and Arun Pal Singh

Abstract The aim of this paper is two fold. Since their inception in 1992, we collect various generalizations of the grand Lebesgue spaces touching upon several of their aspects such as properties, duality, equivalent norms etc. Also, we prove certain new extrapolation results of the type of Rubio De Francia in the framework of fully measurable grand Lebesgue spaces.

Keywords Banach function norm · Rearrangement invariant · Grand Lebesgue space · Associate space · Small Lebesgue space · Extrapolation

1 Introduction

In the process of investigating the minimal hypothesis for the integrability of the Jacobian, Iwaniec and Sbordone [25] in 1992, introduced a new type of spaces called grand Lebesgue spaces, denoted by $L^{n)}(\Omega)$, $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ being a bounded domain.

Grand Lebesgue spaces have attracted many researchers during the last decade and these spaces have been considered in various different aspects. To name a few of them are: in the study of PDEs [26, 27, 53, 54], in the interpolation theory [15], boundedness of various operators, e.g., Hardy operator and maximal operator [21],

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Hilbert transform [36], singular integrals [32, 35] and fractional integral operators [47] etc. being studied on these spaces. Moreover, many other Lebesgue type spaces have been generalized to their respective *grand* versions, e.g., grand Orlicz spaces [6], grand Sobolev spaces [19], grand Bochner spaces [40] and grand Morrey spaces [38] etc.

The purpose of the present article is two fold. First, we attempt to give a systematic development of the theory of grand Lebesgue spaces, their properties and various generalizations that have taken place till recently. Secondly, we give our contribution in the study of these type of spaces by giving some extrapolation results and their applications on a recent generalized version of grand Lebesgue spaces.

The space $L^{n)}(\Omega)$ consists of measurable finite almost everywhere (a.e.) functions f defined on Ω for which

$$\sup_{1\le s<n}\left((n-s)\fint_{\Omega}|f(t)|^s dt\right)^{1/s}<\infty,$$

where $\fint_{\Omega} f(t)dt=\frac{1}{|\Omega|}\int_{\Omega} f(t)dt$.

The space $L^{n)}(\Omega)$ is a Banach space with the norm

$$\|f\|_{L^{n)}(\Omega)}:=\sup_{1\le s<n}\left((n-s)\fint_{\Omega}|f(t)|^s dt\right)^{1/s}.$$

In 1997, Greco, Iwaniec and Sbordone [24] generalized the spaces $L^{n)}(\Omega)$ to $L^{p),\theta}(\Omega)$, defined to be the collection of measurable finite a.e. functions f defined on Ω for which

$$\|f\|_{L^{p),\theta}(\Omega)}:=\sup_{0<\epsilon<p-1}\left(\varepsilon^{\theta}\fint_{\Omega}|f(t)|^{p-\varepsilon} dt\right)^{1/(p-\varepsilon)}<\infty, \tag{1}$$

where $0\le\theta<\infty$ and $1<p<\infty$. In that paper, the authors studied the existence and uniqueness results for non-homogenous n-harmonic type equations $div\mathcal{A}(x,\nabla u)=\mu$ for $\mathcal{A}$-harmonic operator with a Radon measure μ. It has been observed that in the theory of PDEs, the grand Lebesgue spaces $L^{p),\theta}$ are the appropriate spaces for the existence, uniqueness and regularity problems of various non-linear differential equations, see [18, 24].

The paper is organized as follows. In Sect. 2, we collect notations, terminology, standard definitions and results which are needed in the subsequent sections, consequently, making the text self contained as far as possible. Section 3 contains the results and theory concerning the associate space of the standard grand Lebesgue space $L^{p)}(\Omega)$, which is called the small Lebesgue space, and is obtained via an auxiliary space. However, this auxiliary space and small space are seen to be equivalent. This equivalence along with some related results are given in Sect. 4. Section 5 contains recent developments in the theory of grand Lebesgue spaces, and finally in Sect. 6, we prove some new extrapolation results for generalized grand Lebesgue spaces.

2 Preliminaries

Throughout the paper, unless otherwise specified, we shall be taking $\Omega \subseteq \mathbb{R}^n$ such that the Lebesgue measure of Ω is finite, i.e., $|\Omega| < \infty$, and using the following notations / conventions:

- $\mathbb{N}$:= set of natural numbers
- $I := (0, 1)$
- $\mathcal{M}$:= set of extended real valued measurable functions defined on Ω
- $\mathcal{M}^+$:= subset of $\mathcal{M}$, consisting of nonnegative functions
- $\mathcal{M}_0$:= set of finite a.e. measurable functions defined on Ω
- $\mathcal{M}_0^+$:= subset of $\mathcal{M}_0$, consisting of nonnegative functions
- $|E|$:= Lebesgue measure of $E,\ E \subseteq \Omega$
- χ_E := the characteristic function on $E,\ E \subseteq \Omega$
- $C_0^\infty(\Omega)$:= the space of smooth functions with compact support in Ω
- $f_n \uparrow f$ means that $\{f_n\}$ is nondecreasing sequence converging to f.
- C denotes a positive constant which may be different at different places.
- The relation $A \approx B$ means that there exist positive constants c_1 and c_2, such that $c_1 A \leq B \leq c_2 A$.
- Unless specified otherwise, our discussion will be on the set Ω and all the functions will be extended real valued measurable, defined on Ω.

In order not to disturb the flow of the text later, we collect below certain definitions and results which are essential for this paper, and can easily be found in the literature, e.g., one may refer to [3, 55].

Definition. A mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a *Banach function norm* if for all $f,\ g,\ f_n \in \mathcal{M}^+,\ n \in \mathbb{N}$ and for all measurable subsets $E \subset \Omega$, the following properties hold:

- $\rho(f) = 0$ if and only if $f = 0$ a.e. on Ω
- $\rho(\lambda f) = \lambda \rho(f)$, for all scalars $\lambda \geq 0$
- $\rho(f + g) \leq \rho(f) + \rho(g)$
- If $0 \leq g \leq f$ a.e. in Ω, then $\rho(g) \leq \rho(f)$ (lattice property)
- If $0 \leq f_n \uparrow f$ a.e. in Ω, then $\rho(f_n) \uparrow \rho(f)$ (Fatou property)
- $\rho(\chi_E) < \infty$
- $\int_E f(t)dt \leq C_E\, \rho(f)$, for some constant $C_E < \infty$, depending upon E and ρ, but independent of f.

Definition. If ρ is a Banach function norm, then the space

$$X = X(\rho) := \{f \in \mathcal{M}_0 : \rho(|f|) < \infty\}$$

is called a *Banach function space* (BFS) with the norm $\|f\|_X := \rho(|f|)$.

Definition. Let X be a BFS, then the closure in X of the set of bounded functions is denoted by X_b.

Definition. A function f in a BFS X is said to have an *absolutely continuous norm* in X if $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ of measurable sets in Ω satisfying $E_n \to \varnothing$ a.e. If all the functions in X have absolutely continuous norm, then the space X is said to have *absolutely continuous norm*. By X_a, we denote the set of all functions in the BFS X having absolutely continuous norm.

Theorem 1 *Let X be a BFS, then $X_a \subseteq X_b \subseteq X$.*

Definition. A Banach function norm ρ is said to be *rearrangement invariant* if $\rho(|f|) = \rho(|g|)$ for every pair of equimeasurable functions f, g, i.e., $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geq 0$, where $\mu_f(\lambda) := |\{t \in \Omega : |f(t)| > \lambda\}|$ is the distribution function of f.

Definition. If ρ is a rearrangement invariant Banach function norm, then $X(\rho)$ is called *rearrangement invariant BFS*.

Definition. If ρ is a Banach function norm, then its *associate norm* ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup_{\rho(f)\leq 1} \int_\Omega f(t)g(t)dt, \quad g \in \mathcal{M}^+.$$

The BFS $X(\rho')$ determined by the associate norm ρ' is called the *associate space* of the BFS $X(\rho)$, and is denoted by X'.

Theorem 2 *Every BFS X, coincides with its second associate space X''.*

Theorem 3 *The Banach space dual X^* of a BFS X, is isometrically isomorphic to the associate space X' if and only if X has absolutely continuous norm.*

Definition. Let X be a rearrangement invariant BFS. The function ϕ_X defined by $\phi_X(t) = \|\chi_E\|_X$ where $E \subset \Omega$ with $|E| = t$, $t \in (0, |\Omega|)$, is called *fundamental function* of X.

Theorem 4 *Let X be a rearrangement invariant BFS, then its associate space X' is also rearrangement invariant and $\phi_X(t)\phi_{X'}(t) = t$ for all $t \in (0, |\Omega|)$.*

Theorem 5 *A BFS X is reflexive if and only if both X and its associate space X' have absolutely continuous norm.*

Theorem 6 *Let X be a rearrangement invariant BFS, then $\lim_{t\to 0^+} \phi_X(t) = 0$ if and only if $X_b^* = X'$.*

3 Grand Lebesgue Space and its Associate Space

Grand Lebesgue spaces $L^{n)}(\Omega)$ were initially defined for $n \in \mathbb{N}$ and later generalized to $L^{p),\theta}(\Omega),\ p > 1$. However, the definition given by Greco, Iwaniec and Sbordone [24] can be taken as the standard definition of grand Lebesgue spaces.

Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the grand Lebesgue space $L^{p)}(\Omega)$ is defined to be the collection of all $f \in \mathcal{M}_0$ for which

$$\|f\|_{L^{p)}(\Omega)} := \sup_{0<\epsilon<p-1} \left(\varepsilon \fint_{\Omega} |f(t)|^{p-\varepsilon} dt\right)^{1/(p-\varepsilon)} < \infty.$$

It is a rearrangement invariant BFS and the following continuous embeddings hold:

$$L^p(\Omega) \subseteq L^{p)}(\Omega) \subseteq L^{p-\varepsilon}(\Omega), \quad \text{for } 0 < \varepsilon < p-1.$$

Remark 1 The grand Lebesgue space $L^{p)}(\Omega)$ is strictly larger than the Lebesgue space $L^p(\Omega)$. For example, one may easily check that for $\Omega = I$, the function $f(t) = t^{-\frac{1}{p}},\ t \in \Omega$ belongs to $L^{p)}(\Omega)$ but is not in $L^p(\Omega)$.

Theorem 7 ([8, 23]) *A measurable function $f \in L_b^{p)}(\Omega)$ if*

$$\lim_{\varepsilon\to 0^+} \left(\varepsilon \fint_{\Omega} |f(t)|^{p-\varepsilon} dt\right)^{1/(p-\varepsilon)} = 0.$$

Remark 2 The set of bounded functions is not dense in $L^{p)}(\Omega)$, i.e., $L_b^{p)}(\Omega) \neq L^{p)}(\Omega)$. For example, take $\Omega = I$, then the function $f(t) = t^{-\frac{1}{p}},\ t \in \Omega$ belongs to $L^{p)}(\Omega)$ but $\left(\varepsilon \fint_{\Omega} |f(t)|^{p-\varepsilon} dt\right)^{1/(p-\varepsilon)} = p^{\frac{1}{p-\varepsilon}} \nrightarrow 0$ as $\varepsilon \to 0^+$, i.e., $f \notin L_b^{p)}(\Omega)$.

Remark 3 It follows, in view of Theorem 1 and Remark 2 that $L_a^{p)}(\Omega) \subsetneq L^{p)}(\Omega)$, i.e., grand Lebesgue spaces do not have absolutely continuous norm.

Theorem 8 ([13]) *Let $\phi_{L^{p)}(\Omega)}$ be the fundamental function of the space $L^{p)}(\Omega)$. Then we have*

$$\phi_{L^{p)}(\Omega)}(t) \approx t^{\frac{1}{p}} \left(log\frac{1}{t}\right)^{-1/p} \quad as \ \ t \to 0^+.$$

Fiorenza, in [13] obtained the associate space of the space $L^{p)}(\Omega)$. In order to do so, he first introduced an intermediary space defined below:

Definition. Let $1 < p < \infty$. Then the *auxiliary space* of $L^{p)}(\Omega)$, to be denoted by $L^{(p'}(\Omega)$, is the space of all functions $g \in \mathcal{M}_0$, for which

$$\|g\|_{L^{(p'}(\Omega)} := \inf_{\substack{|g| = \sum_{k=1}^{\infty} g_k \\ g_k \in \mathcal{M}_0^+}} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\fint_{\Omega} (g_k(t))^{(p-\varepsilon)'} dt \right)^{1/(p-\varepsilon)'} < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 9 ([13]) *The space $L^{(p'}(\Omega)$ is a Banach space with lattice property.*

The following continuous embeddings hold:

$$L^{p'+\sigma}(\Omega) \subseteq L^{(p'}(\Omega) \subseteq L^{p'}(\Omega)$$

for all $\sigma > 0$. In particular, we have that $L^{\infty}(\Omega) \subseteq L^{(p'}(\Omega)$.

Definition. Let $1 < p < \infty$. The *small Lebesgue space*, denoted by $L^{p)'}(\Omega)$, is the space of all functions $g \in \mathcal{M}_0$ for which

$$\|g\|_{L^{p)'}(\Omega)} := \sup_{\substack{0 < \psi \leq |g| \\ \psi \in L^{(p'}(\Omega)}} \|\psi\|_{L^{(p'}(\Omega)} < \infty.$$

Remark 4 Since $\|\cdot\|_{L^{(p'}(\Omega)}$ has lattice property, it follows that

$$\|\psi\|_{L^{(p'}(\Omega)} \leq \|g\|_{L^{(p'}(\Omega)}$$

for all $\psi \in L^{(p'}(\Omega)$ such that $0 < \psi \leq |g|$, which implies that

$$\|g\|_{L^{p)'}(\Omega)} \leq \|g\|_{L^{(p'}(\Omega)}.$$

On the other hand, for $g \in L^{(p'}(\Omega)$, we have

$$\|g\|_{L^{(p'}(\Omega)} \leq \|g\|_{L^{p)'}(\Omega)}.$$

Consequently, for $g \in L^{(p'}(\Omega)$, the norms $\|\cdot\|_{L^{(p'}(\Omega)}$ and $\|\cdot\|_{L^{p)'}(\Omega)}$ are equivalent.

The following theorems give various properties of the space $L^{p)'}(\Omega)$.

Theorem 10 ([13]) *The small Lebesgue space $L^{p)'}(\Omega)$ is a Banach function space.*

Theorem 11 ([13]) *For $1 < p < \infty$, the following Hölder's inequality holds:*

$$\fint_{\Omega} fg \leq \|f\|_{L^{p)}(\Omega)} \|g\|_{L^{p)'}(\Omega)}$$

for all $f \in L^{p)}(\Omega),\ g \in L^{p)'}(\Omega)$.

Theorem 12 ([13]) *The space $L^{p)'}(\Omega)$ is rearrangement invariant, and $\left(L^{p)'}(\Omega)\right)' = L^{p)}(\Omega)$.*

Now, in view of Theorem 2, the following completes the search for the associate space of the grand Lebesgue space $L^{p)}(\Omega)$.

Theorem 13 ([13]) *The space $L^{p)}(\Omega)$ is the associate space of $L^{p)'}(\Omega)$ and vice-versa.*

In view of the Remark 3 and Theorem 5, we have the following.

Theorem 14 ([13]) *The spaces $L^{p)}(\Omega)$ and $L^{p)'}(\Omega)$ are non reflexive.*

In view of Theorems 4 and 8, the following is obtained.

Theorem 15 ([13]) *The fundamental function of $L^{p)'}(\Omega)$ is*

$$\phi_{L^{p)'}(\Omega)}(t) \approx t^{\frac{1}{p'}} \left(log\frac{1}{t}\right)^{1/p} \quad as \quad t \to 0^+.$$

In view of Theorems 6, 15 and 8, we get the following easily:

Theorem 16 ([13]) *The dual of $L_b^{p)}(\Omega)$ is isometrically isomorphic to $L^{p)'}(\Omega)$ and the dual of $L_b^{p)'}(\Omega)$ is isometrically isomorphic to $L^{p)}(\Omega)$.*

4 Equivalence of Small and Auxiliary Space

The small Lebesgue space $L^{p)'}(\Omega)$ being a BFS (see Theorem 10), it possesses Fatou property. Also note that the norm of the functions in the space $L^{p)'}(\Omega)$ is defined in terms of the norm in the auxiliary space $L^{(p'}(\Omega)$. This observation leaves a possibility if the later space also possesses the Fatou property and consequently, the norm $\|\cdot\|_{L^{p)'}(\Omega)}$ can be written in the simplified way, i.e., in terms of $\|\cdot\|_{L^{(p'}(\Omega)}$. This was done using the "Levi's theorem of monotone convergence for small Lebesgue spaces" in [16]. Before stating the Levi's theorem, first we state the following lemma:

Lemma 1 *For $g \in \mathcal{M}_0^+$, we have*

$$\|g\|_{L^{(p'}(\Omega)} \approx \inf_{\substack{g=\sum_{k=1}^{\infty} g_k \\ g_k \in \mathcal{M}_0^+}} \sum_{k=1}^{\infty} \inf_{0<\varepsilon\leq\frac{p-1}{2}} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\fint_{\Omega} (g_k(t))^{(p-\varepsilon)'} dt\right)^{1/(p-\varepsilon)'}.$$

Theorem 17 ([16]) **Levi's theorem of monotone convergence for small Lebesgue spaces.**

Let $\{f_m\}$ *be a monotone nondecreasing sequence such that*

$$\sup_m \|f_m\|_{L^{(p'}(\Omega)} < \infty.$$

Then, the function $f = \sup_m f_m$ *is such that*

(i) $f \in L^{(p'}(\Omega)$; (ii) $f_m \uparrow f$ *a.e. and* (iii) $f_m \to f$ *in* $L^{(p'}(\Omega)$.

We have Lemma 1 for a smaller range of ε, i.e., $0 < \varepsilon \le \frac{p-1}{2}$, but we can have it for an arbitrary σ such that $0 < \varepsilon \le \sigma$. More precisely, we have the following (the proof given here should be compared with that one of Lemma 2 in [16]):

Lemma 2 *For* $1 < p < \infty$, $g \in \mathcal{M}_0$, *we have*

$$\|g\|_{L^{(p'}(\Omega)} \approx \inf_{\substack{|g| = \sum_{k=1}^{\infty} g_k \\ g_k \in \mathcal{M}_0^+}} \sum_{k=1}^{\infty} \inf_{0<\varepsilon\le\sigma} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\fint_{\Omega} (g_k(t))^{(p-\varepsilon)'} dt \right)^{1/(p-\varepsilon)'}$$

for $0 < \sigma < p-1$.

Proof Let $0 < \sigma < p-1$. For $\varepsilon \in (\sigma, p-1)$, take $\lambda = \frac{(p-\varepsilon)'}{(p-\sigma)'}$ and choose μ such that $\frac{1}{\lambda} + \frac{1}{\mu} = 1$. Then by Hölder's inequality, we have

$$\|g_k\|_{L^{(p-\sigma)'}(\Omega)} \le \|g_k\|_{L^{(p-\varepsilon)'}(\Omega)} |\Omega|^{\frac{1}{(p-\sigma)'} - \frac{1}{(p-\varepsilon)'}}. \tag{2}$$

Now by using (2), we have

$$\begin{aligned}
\|g\|_{L^{(p'}(\Omega)} = \min\Bigg\{ & \inf_{|g|=\sum_{k=1}^{\infty} g_k} \left(\sum_{k=1}^{\infty} \inf_{0<\varepsilon\le\sigma} \left(\frac{1}{|\Omega|}\right)^{\frac{1}{(p-\varepsilon)'}} \varepsilon^{-\frac{1}{p-\varepsilon}} \|g_k\|_{L^{(p-\varepsilon)'}(\Omega)} \right), \\
& \inf_{|g|=\sum_{k=1}^{\infty} g_k} \left(\sum_{k=1}^{\infty} \inf_{\sigma<\varepsilon<p-1} \left(\frac{1}{|\Omega|}\right)^{\frac{1}{(p-\varepsilon)'}} \varepsilon^{-\frac{1}{p-\varepsilon}} \|g_k\|_{L^{(p-\varepsilon)'}(\Omega)} \right) \Bigg\} \\
\ge \min\Bigg\{ & \inf_{|g|=\sum_{k=1}^{\infty} g_k} \left(\sum_{k=1}^{\infty} \inf_{0<\varepsilon\le\sigma} \left(\frac{1}{|\Omega|}\right)^{\frac{1}{(p-\varepsilon)'}} \varepsilon^{-\frac{1}{p-\varepsilon}} \|g_k\|_{L^{(p-\varepsilon)'}(\Omega)} \right), \\
& \inf_{|g|=\sum_{k=1}^{\infty} g_k} \left(\sum_{k=1}^{\infty} \inf_{\sigma<\varepsilon<p-1} \left(\frac{1}{|\Omega|}\right)^{\frac{1}{(p-\varepsilon)'}} \varepsilon^{-\frac{1}{p-\varepsilon}} \|g_k\|_{L^{(p-\sigma)'}(\Omega)} |\Omega|^{\frac{1}{(p-\varepsilon)'} - \frac{1}{(p-\sigma)'}} \right) \Bigg\}
\end{aligned}$$

$$= \min\left\{ \inf_{|g|=\sum_{k=1}^{\infty} g_k} \left(\sum_{k=1}^{\infty} \inf_{0<\varepsilon\le\sigma} \left(\frac{1}{|\Omega|}\right)^{\frac{1}{(p-\varepsilon)'}} \varepsilon^{-\frac{1}{p-\varepsilon}} \|g_k\|_{L^{(p-\varepsilon)'}(\Omega)} \right), \right.$$

$$\left. \inf_{|g|=\sum_{k=1}^{\infty} g_k} \left(\sum_{k=1}^{\infty} \sigma^{-\frac{1}{p-\sigma}} \left(\frac{1}{|\Omega|}\right)^{\frac{1}{(p-\sigma)'}} \|g_k\|_{L^{(p-\sigma)'}(\Omega)} \right) \times \inf_{\sigma<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \sigma^{\frac{1}{p-\sigma}} \right\}$$

$$= \min\left\{1, \left(\inf_{\sigma<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \right) \left(\sigma^{\frac{1}{p-\sigma}} \right) \right\} \times$$

$$\times \inf_{|g|=\sum_{k=1}^{\infty} g_k} \left(\sum_{k=1}^{\infty} \inf_{0<\varepsilon\le\sigma} \varepsilon^{-\frac{1}{p-\varepsilon}} \|g_k\|_{L^{(p-\varepsilon)'}(\Omega)} \left(\frac{1}{|\Omega|}\right)^{\frac{1}{(p-\varepsilon)'}} \right)$$

$$= C(\sigma, p) \inf_{\substack{|g| = \sum_{k=1}^{\infty} g_k \\ g_k \in \mathcal{M}_0^+}} \left(\sum_{k=1}^{\infty} \inf_{0<\varepsilon\le\sigma} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\fint_{\Omega} (g_k(t))^{(p-\varepsilon)'} dt \right)^{1/(p-\varepsilon)'} \right) \quad (3)$$

where $C(\sigma, p) = \min\left\{1, \frac{\sigma^{\frac{1}{p-\sigma}}}{p}\right\}$. Also, by the definition of infimum we have

$$\|g\|_{L^{(p'}(\Omega)} \le \inf_{\substack{|g| = \sum_{k=1}^{\infty} g_k \\ g_k \in \mathcal{M}_0^+}} \sum_{k=1}^{\infty} \inf_{0<\varepsilon\le\sigma} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\fint_{\Omega} (g_k(t))^{(p-\varepsilon)'} dt \right)^{1/(p-\varepsilon)'}. \quad (4)$$

Now, we are done by (3) and (4). □

Remark 5 Theorem 17 can also be proved on using Lemma 2 by making suitable modifications.

As an application of Theorem 17, it can be proved that the space $L^{(p'}(\Omega)$ has absolutely continuous norm, i.e., $L_a^{(p'}(\Omega) = L^{(p'}(\Omega)$. More precisely, we have the following:

Theorem 18 ([16]) *Let $f \in L^{(p'}(\Omega)$ and let $\{E_m\}_{n=1}^{\infty}$ be a sequence of measurable sets in Ω such that*

(i) $\Omega \supseteq E_1 \supseteq E_2 \ldots \supseteq E_m \ldots,$
(ii) $|E_m| \to 0.$

Then $\|f\chi_m\|_{L^{(p'}(\Omega)} \to 0$.

The following holds:

Theorem 19 ([5]) *The space $L^{(p'}(\Omega)$ is a BFS and $L^{p)'}(\Omega) = L^{(p'}(\Omega)$.*

In view of Theorem 19, we get various interesting properties for the associate space $L^{(p'}(\Omega)$ of $L^{p)}(\Omega)$. To begin with, Theorems 18 and 1 give the following:

Theorem 20 ([5]) *The set of bounded functions is dense in $L^{(p'}(\Omega)$, i.e., $L_b^{(p'}(\Omega) = L^{(p'}(\Omega)$.*

Next, we have the following:

Theorem 21 ([5]) *The dual of* $L^{(p'}(\Omega)$ *is canonically isometrically isomorphic to the associate space of* $L^{(p'}(\Omega)$, *i.e.,*

$$\left(L^{(p'}(\Omega)\right)^* = \left(L^{(p'}(\Omega)\right)' = L^{p)}(\Omega).$$

Proof The first equality is obtained by using Theorems 3 and 18, and the second equality follows from Theorems 19 and 2. □

Following theorem is a consequence of Theorems 19 and 14.

Theorem 22 ([5]) *The spaces* $L^{p)}(\Omega)$ *and* $L^{(p'}(\Omega)$ *are non reflexive.*

Theorems 19 and 12 give the following:

Theorem 23 ([5]) *The spaces* $L^{p)}(\Omega)$ *and* $L^{(p'}(\Omega)$ *are rearrangement invariant spaces.*

Recall the generalized grand Lebesgue space $L^{p),\theta}(\Omega)$ having norm (1). Towards the associate space of $L^{p),\theta}(\Omega)$, Capone and Fiorenza in [5] introduced the so called *generalized auxiliary space* $L^{(p',\theta}(\Omega)$, $\theta > 0$ which consists of all the functions $g \in \mathcal{M}_0^+$ such that

$$\|g\|_{L^{(p',\theta}(\Omega)} := \inf_{\substack{g=\sum_{k=1}^{\infty} g_k \\ g_k \in \mathcal{M}_0^+}} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\fint_{\Omega} (g_k(t))^{(p-\varepsilon)'} dt \right)^{1/(p-\varepsilon)'}.$$

Note that for $\theta = 1$, $\|g\|_{L^{(p',\theta}(\Omega)} = \|g\|_{L^{(p'}(\Omega)}$ and for $\theta = 0$, $\|g\|_{L^{(p',\theta}(\Omega)} = \|g\|_{L^{p'}(\Omega)}$.

All the results mentioned in this section in regard to the auxiliary space $L^{(p'}(\Omega)$ or the small Lebesgue space $L^{p)'}(\Omega)$ were, in fact, proved in [5] for the generalized auxiliary space $L^{(p',\theta}(\Omega)$ and the generalized small space $L^{p)',\theta}(\Omega)$, which can be defined in a way similar to as the space $L^{p)'}(\Omega)$.

In 2004, Fiorenza and Karadzhov [14] found the equivalent, explicit expressions for the norms of the small and grand Lebesgue spaces in terms of nonincreasing rearrangement, namely, for $1 < p < \infty$, $|\Omega| = 1$

$$\|f\|_{L^{(p}(\Omega)} \approx \int_{\Omega} (1 - logt)^{-1/p} \left(\int_0^t (f^*(s))^p ds \right)^{1/p} \frac{dt}{t} \tag{5}$$

and

$$\|f\|_{L^{p)}(\Omega)} \approx \sup_{0<t<1} (1 - logt)^{-1/p} \left(\int_t^1 (f^*(s))^p ds \right)^{1/p}, \tag{6}$$

where f^* is nonincreasing rearrangement of f.

Later in 2009, Fratta and Fiorenza [22] proved the above equivalence for $\|\cdot\|_{L^{(p,\theta}(\Omega)}$ and $\|\cdot\|_{L^{p),\theta}(\Omega)}$, $\theta > 0$, i.e., (5) and (6) for generalized spaces by using elementary methods. More precisely, the method is based entirely on integral estimates and asymptotic properties of Euler's gamma function. It is worthy to mention that by this method explicit estimate for constants can also be obtained.

In 2015, Rafeiro and Vargas [51] studied the compactness of subsets of more general grand Lebesgue spaces $L^{p),\delta}(\Omega)$, defined in next section. Here, we state the result for $L^{p)}(\Omega)$ only. Succinctly, they proved the following.

Theorem 24 *For* $1 < p < \infty$, *a subset* $\mathcal{F}$ *of* $L_0^{p)}(\Omega)$, *is relatively compact if and only if the following conditions are satisfied (i)* $\mathcal{F}$ *is bounded in* $L^{p)}(\Omega)$, *and (ii)* $\lim_{h\to 0}\|\tau_h f - f\|_{L^{p)}(\Omega)} = 0$ *uniformly for* $f \in \mathcal{F}$, *where* τ_h *denotes the translation operator given as* $(\tau_h f)(x) = f(x+h)$ *for* $h \in \mathbb{R}^n$ *and* $L_0^{p)}(\Omega)$ *denotes the closure of* C_0^∞ *in* $L^{p)}(\Omega)$.

In 1931 Kolmogorov [41], proved the necessary part of Theorem 24 in the setting of classical L_p spaces, by a simple and elegant contradiction argument. Instead, in [51], the authors have taken a more of constructive approach.

5 Recent Advancements

As mentioned in Sect. 1, Greco, Iwaniec and Sbordone [24] generalized the grand Lebesgue spaces $L^{p)}(\Omega)$ to $L^{p),\theta}(\Omega)$ by replacing ε with ε^θ, $\theta > 0$. Very recently, in 2013, Capone, Formica and Giova [7] further generalized the grand Lebesgue spaces, by replacing ε^θ, $\theta > 0$ with a general measurable function δ. They defined grand Lebesgue spaces $L^{p),\delta}(\Omega)$ with respect to δ, to be the collection of all functions $f \in \mathcal{M}_0$ for which

$$\|f\|_{L^{p),\delta}(\Omega)} := \sup_{0<\epsilon<p-1} \delta^{\frac{1}{p-\varepsilon}}(\varepsilon)\left(\fint_\Omega |f(t)|^{p-\varepsilon}dt\right)^{1/(p-\varepsilon)} < \infty,$$

where $\delta \in L^\infty(0,\ p-1)$ is left continuous, $\lim\limits_{\varepsilon\to 0^+}\delta(\varepsilon) = 0,\ 0 < \delta \leq 1$ and $\delta^{\frac{1}{p-\varepsilon}}(\cdot)$ is nondecreasing.

Note that for $\delta(\varepsilon) = \varepsilon$, $L^{p),\delta}(\Omega) = L^{p)}(\Omega)$, and for $\delta(\varepsilon) = \varepsilon^\theta$, $L^{p),\delta}(\Omega) = L^{p),\theta}(\Omega)$. It was proved in [7] that the spaces $L^{p),\delta}(\Omega)$ are BFSs, the spaces are rearrangement invariant, include $L^p(\Omega)$ and are included in each one of the spaces $L^{p-\varepsilon}(\Omega)$, $0 < \varepsilon < p-1$. Also, they proved that the equivalence

$$\|f\|_{L^{p),\delta}(\Omega)} \approx \sup_{0<\epsilon<\sigma} \delta^{\frac{1}{p-\varepsilon}}(\varepsilon)\left(\fint_\Omega |f(t)|^{p-\varepsilon}dt\right)^{1/(p-\varepsilon)}$$

holds for some $0 < \sigma < p-1$. Moreover, the classical Hardy inequality in the framework of these spaces was also proved.

The most recent general form of grand Lebesgue spaces, known at present, is due to Anatriello and Fiorenza [2] who called their space a fully measurable grand Lebesgue space, defined below.

For $\Omega = I$, let $p(\cdot) \in \mathcal{M}$ be defined on Ω such that $p(\cdot) \geq 1$ a.e., $\delta \in L^\infty(\Omega)$, $\delta > 0$ a.e. and $0 < \|\delta\|_{L^\infty(\Omega)} \leq 1$. The *fully measurable grand Lebesgue space*, denoted by $L^{p[\cdot],\delta(\cdot)}(\Omega)$, consists of all measurable finite a.e. functions f defined on Ω for which $\|f\|_{L^{p[\cdot],\delta(\cdot)}(\Omega)} := \rho_{p[\cdot],\delta(\cdot)}(|f|) < \infty$, where

$$\rho_{p[\cdot],\delta(\cdot)}(|f|) := \operatorname*{ess\,sup}_{x\in\Omega} \rho_{p(x)}(\delta(x)|f(\cdot)|)$$

and for each $x \in \Omega$,

$$\rho_{p(x)}(\delta(x)|f(\cdot)|) := \begin{cases} \left(\int_\Omega (\delta(x)|f(t)|)^{p(x)} dt\right)^{\frac{1}{p(x)}} & \text{if } 1 \leq p(x) < \infty; \\ \operatorname*{ess\,sup}_{t\in\Omega} (\delta(x)|f(t)|) & \text{if } p(x) = \infty. \end{cases}$$

These spaces are rearrangement invariant BFSs. In [2], some properties of the space $L^{p[\cdot],\delta(\cdot)}(\Omega)$ have been established, and also, classical Hardy inequality has been obtained in the context of these spaces.

Another direction in which the grand Lebesgue spaces have been generalized is their weighted version, where by a weight we mean a positive, measurable finite a.e. function defined on Ω. In [21], Fiorenza, Gupta and Jain introduced the weighted version of grand Lebesgue spaces: for $1 < p < \infty$ and a weight $w \in L^1(\Omega)$, they defined

$$L^{p)}_w(\Omega) := \left\{ f \in \mathcal{M}_0 \; : \; \|f\|_{L^{p)}_w(\Omega)} := \sup_{0<\epsilon<p-1} \left(\varepsilon \fint_\Omega |f(t)|^{p-\varepsilon} w(t) dt\right)^{1/(p-\varepsilon)} < \infty \right\}.$$

These spaces are BFSs, and except for the trivial case when w is constant, are not rearrangement invariant.

Remark 6 In the setting of Lebesgue spaces, for a weight w, we have $f \in L^p_w(\Omega) \Leftrightarrow fw^{1/p} \in L^p(\Omega)$. But this kind of equivalence does not hold in the case of weighted grand Lebesgue spaces. Indeed, for $\Omega = I$, if we take $w(t) = t^\alpha$, $\alpha > 0$ and $f(t) = t^\beta$, $\beta > -\alpha - 1$, $t \in \Omega$, then it can be easily checked that $f \in L^{p)}_w(\Omega)$ but $fw^{1/p} \notin L^{p)}(\Omega)$.

In literature, boundedness of different operators, viz., maximal operator, Hilbert transform, singular integrals, fractional integral operators, Hardy operator and their generalized forms etc. have been studied in the case of weighted L^p spaces. The boundedness of these operators have been characterized with the help of weight classes like, A_p- class, B_p-class or some other likewise classes of weights. One may refer to [42] to have an idea of various such weight classes. In the literature, we find that the efforts have been made by several people to make similar kind of studies on weighted grand Lebesgue spaces or, generalized weighted versions of grand Lebesgue spaces. For example, in [21, 36], the boundedness of maximal operator and Hilbert operator have been characterized by A_p-class of weights in the frame work of weighted grand Lebesgue spaces $L^{p)}_w(\Omega)$. Consequently, leading us to the observation that the boundedness of these operators on $L^{p)}_w(\Omega)$ spaces is equivalent to their boundedness on $L^p_w(\Omega)$ spaces. One may also refer to [10, 29, 32–35, 37, 45, 47] and the references therein.

Besides the above mentioned aspects of development for Lebesgue spaces leading to grand Lebesgue spaces, people have also been working on developing the grand versions of various other spaces and to study their properties. Some of them to mention are: grand Sobolev spaces, grand Orlicz spaces, grand Morrey and grand grand Morrey spaces, grand Lorentz spaces, grand Lebesgue spaces with variable exponents, grand Bochner Lebesgue spaces, bilateral grand Lebesgue spaces, composed grand Lebesgue spaces, iterated grand Lebesgue spaces and many more. For details, one may refer to [1, 4, 6, 12, 17, 19, 20, 28, 31, 38–40, 43, 44, 46, 49, 50, 54] and the references therein.

Althrough Ω has been a bounded open subset of $\mathbb{R}^n$, usually, $\Omega = I$. But a very natural question is that: what happens if $|\Omega| = \infty$? In fact, here it is a situation, when the weighted version of grand Lebesgue spaces come to our rescue. Precisely, in this direction, it is due to Samko and Umarkhadzhiev [52], who introduced grand Lebesgue spaces on open sets $\Omega \subseteq \mathbb{R}^n$ with $|\Omega| = \infty$, by controlling the integrability of $|f(x)|^{p-\varepsilon}$ at infinity by means of weight depending on ε. They showed that the space $L^{p)}(\Omega)$ could be defined on an arbitrary set of infinite measure in $\mathbb{R}^n$ if considered with weight introduced, and being dependent on the entity ε.

6 Extrapolation Results and Applications

The motivation of this section comes from the following celebrity extrapolation result of Rubio De Francia [11]: For some fixed $q > 1$, if a sublinear operator T is bounded in $L^q_w(\Omega)$ for all $w \in A_q$-class, then T is bounded in $L^s_w(\Omega)$ for every $w \in A_s$-class for all $1 < s < \infty$. A similar result was proved in [9] for B_q-class. Before mentioning that result, let us fix some notations:

Throughout this section we take $\Omega = I$. If there is no ambiguity, for simplicity sake, we shall avoid using I at places where it should have occurred, e.g., we shall write L^s_w instead of $L^s_w(I)$ and so on.

The weight class B_q is defined for $0 < q < \infty$ as follows:

$$B_q := \left\{ w \,:\, \int_r^1 \left(\frac{r}{t}\right)^q w(t)\, dt \le C \int_0^r w(t)\, dt, \quad \text{for all} \quad 0 < r < 1 \right\}. \tag{7}$$

Also, we denote

$$\|w\|_{B_q} := \inf \left\{ C > 0 : \int_0^r w(t)\,dt + \int_r^1 \left(\frac{r}{t}\right)^q w(t)\,dt \right.$$
$$\left. \le C \int_0^r w(t)dt, \quad \text{for all} \quad 0 < r < 1 \right\}.$$

Observe that

(i) $\|w\|_{B_q} > 1$;

(ii) If $w \in B_s$, then $w \in B_q$ and $\|w\|_{B_q} \le \|w\|_{B_s}$ for $q \ge s > 0$;

(iii) If $w \in B_q$, then there exists $\sigma > 0$ such that $w \in B_{q-\sigma}$ and

$$\|w\|_{B_{q-\sigma}} \le \frac{C_0 \|w\|_{B_q}}{1 - \sigma\alpha^q \|w\|_{B_q}},$$

where $C_0, \ \alpha\,(0 < \alpha < 1)$ are universal constants and $\sigma < \frac{1}{\alpha^q \|w\|_{B_q}}$.

Theorem 25 ([9]) *Let* ψ *be a nonnegative nondecreasing function defined on* I. *Assume that* (f, g) *is a pair of nonnegative nonincreasing functions on I. Let* $0 < s_0 < \infty$. *Suppose for every* $w \in B_{s_0}$

$$\|f\|_{L_w^{s_0}} \le \psi^{\frac{1}{s_0}}(\|w\|_{B_{s_0}})\|g\|_{L_w^{s_0}},$$

then for every $s > 0$ *and* $w \in B_s$ *the inequality*

$$\|f\|_{L_w^s} \le \widetilde{\psi}(\|w\|_{B_s})\|g\|_{L_w^s}$$

holds, where

$$\widetilde{\psi}(x) = \inf_{0<\eta<\frac{s_0}{sx\alpha^s}} \psi^{1/s_0}\left(\frac{s_0}{\eta}\right)\left(\frac{C_0 x}{1 - \eta x \frac{s\alpha^s}{s_0}}\right)^{\frac{1}{s}}$$

and $C_0 > 0, \ \alpha\,(0 < \alpha < 1)$ *being the universal constants.*

These extrapolation results have been used to characterize the boundedness of various integral operators in the framework of Lebesgue spaces.

In this section, we prove Theorem 25 in the setting of fully measurable weighted grand Lebesgue spaces and investigate the boundedness of Hardy averaging operator and Riemann-Liouville fractional operator.

We define below the weighted version of fully measurable grand Lebesgue space:

Definition. Let $p(\cdot) \in \mathcal{M},\ p(\cdot) \geq 1$ a.e., $\delta \in L^\infty,\ \delta > 0$ a.e., w be a weight such that $w \in L^\infty$. Then the space $L_w^{p[\cdot],\delta(\cdot)}$, called *fully measurable weighted grand Lebesgue space*, is the space of all $f \in \mathcal{M}_0$ for which

$$\|f\|_{L_w^{p[\cdot],\delta(\cdot)}} := \rho_{p[\cdot],\delta(\cdot),w}(|f|) < \infty,$$

where

$$\rho_{p[\cdot],\delta(\cdot),w}(|f|) = \operatorname*{ess\,sup}_{x \in I} \rho_{p(x),w(\cdot)}(\delta(x)|f(\cdot)|)$$

and for each $x \in I$,

$$\rho_{p(x),w(\cdot)}(\delta(x)|f(\cdot)|) = \begin{cases} \left(\int_I (\delta(x)|f(t)|)^{p(x)} w(t)dt\right)^{\frac{1}{p(x)}} & \text{if } 1 \leq p(x) < \infty; \\ \operatorname*{ess\,sup}_{t \in I}(\delta(x)|f(t)|w(t)) & \text{if } p(x) = \infty. \end{cases}$$

The spaces $L_w^{p[\cdot],\delta(\cdot)}$ are non rearrangement invariant BFSs, except for the case when w is constant, see Proposition 2.2 [30]. Also, the following continuous embeddings hold:

$$L_w^{p_+} \subseteq L_w^{p[\cdot],\delta(\cdot)} \subseteq L_w^{p(x)} \quad \text{a.e. for} \quad x \in I,$$

where $p_+ := \operatorname*{ess\,sup}_{x \in I} p(x)$.

Remark 7 If $E \subseteq I,\ |E| > 0$ and $p(x) = p_+$ for $x \in E$, then for $f \in \mathcal{M}_0$

$$\rho_{p[\cdot],\delta(\cdot),w}(|f|) \approx \rho_{p_+,w}(|f|),$$

where $\rho_{p_+,w}(|f|)$ is weighted L^p norm of f with the exponent p_+. Consequently, it suffices to consider the fully measurable weighted grand Lebesgue spaces only for $1 \leq p(x) < p_+$ a.e. on I.

The following lemma was proved in [30].

Lemma 3 *Let $p(\cdot) \in \mathcal{M}$ be such that $p(\cdot) \geq 1$ a.e., $w \in L^\infty$ be a weight, $\delta \in L^\infty$ and $\delta > 0$ a.e. Then for $f \in \mathcal{M}^+$*

$$\rho_{p[\cdot],\delta(\cdot),w}(f) \approx \operatorname*{ess\,sup}_{x \in p^{-1}([\tau,\, p_+])} \rho_{p(x),w(\cdot)}(\delta(x)f(\cdot)),$$

for all $\tau \in [1, p_+)$. I.e.,

$$\operatorname*{ess\,sup}_{x \in p^{-1}([\tau,\, p_+])} \rho_{p(x),w(\cdot)}(\delta(x)f(\cdot)) \leq \rho_{p[\cdot],\delta(\cdot),w}(f)$$
$$\leq C \operatorname*{ess\,sup}_{x \in p^{-1}([\tau,\, p_+])} \rho_{p(x),w(\cdot)}(\delta(x)f(\cdot))$$

for all $\tau \in [1, p_+)$*, where*

$$C = \max\left\{1, \underset{x\in p^{-1}([1,\tau))}{ess\sup}\ \delta(x)\frac{2}{\|\delta\|_{L^\infty(p^{-1}([\tau,p_+]))}}(w_\tau^*)^2\right\}$$

and $w_\tau^* = \left(\int_I w(t)dt + 1\right)^{\frac{\tau-1}{\tau}}$.

The following is the first main result of this section:

Theorem 26 *Let* ψ *be a nonnegative nondecreasing function defined on* I. *Let* $1 < s_0 < \infty$. *Let* (f, g) *be a pair of nonnegative, nonincreasing functions on* I. *Suppose that for every* $w \in B_{s_0}$

$$\int_I f^{s_0}(t)w(t)dt \le \psi(\|w\|_{B_{s_0}})\int_I g^{s_0}(t)w(t)dt.$$

Then for every $p(\cdot) \in \mathcal{M},\ p(\cdot) \ge 1$ *a.e. and* $w \in B_{p_+}$ *where* $p_+ < \infty$, *the inequality*

$$\|f\|_{L_w^{p[\cdot],\delta(\cdot)}} \le C\|g\|_{L_w^{p[\cdot],\delta(\cdot)}}$$

holds, where C *is a positive constant depending on* $p(\cdot),\ s_0,\ \delta$ *and* w.

Proof Let $w \in B_{p_+}$. Then by observation (iii), there exists a positive constant $\sigma > 0$ such that $w \in B_{p_+-\sigma}$. Take

$$X_\sigma = \{x \in I\ :\ p_+ - \sigma \le p(x) \le p_+\}$$

and

$$Y_\sigma = \{x \in I\ :\ 1 \le p(x) < p_+ - \sigma\}.$$

If $p_+ - \sigma = 1$, then $X_\sigma = I$ and by monotonicity of the weight class B_q with respect to the index q, we have that $w \in B_{p(x)}$ for all $x \in I$. Therefore, by Theorem 25, we get

$$\|f\|_{L_w^{p(x)}} \le \widetilde{\psi}(\|w\|_{B_{p(x)}})\|g\|_{L_w^{p(x)}}$$

for $x \in I$ a.e., which implies

$$\underset{x\in I}{\text{ess sup}}\,\delta(x)\|f\|_{L_w^{p(x)}} \le \underset{x\in I}{\text{ess sup}}\,\delta(x)\widetilde{\psi}(\|w\|_{B_{p(x)}})\|g\|_{L_w^{p(x)}},$$

i.e.,

$$\|f\|_{L_w^{p[\cdot],\delta(\cdot)}} \le \underset{x\in I}{\text{ess sup}}\,\widetilde{\psi}(\|w\|_{B_{p(x)}})\|g\|_{L_w^{p[\cdot],\delta(\cdot)}}. \tag{8}$$

Now

$$\widetilde{\psi}(\|w\|_{B_{p(x)}}) = \inf_{0<\eta<\frac{s_0}{p(x)\alpha^{p(x)}\|w\|_{B_{p(x)}}}} \psi^{1/s_0}\left(\frac{s_0}{\eta}\right)\left(\frac{C_0\|w\|_{B_{p(x)}}}{1-\frac{\eta p(x)\alpha^{p(x)}}{s_0}\|w\|_{B_{p(x)}}}\right)^{1/p(x)}, \quad (9)$$

where $C_0 > 0$ and $0 < \alpha < 1$ are universal constants. Since $p(x) > p_+ - \sigma = 1$, we have that $\|w\|_{B_{p(x)}} \leq \|w\|_{B_1}$ for all $x \in I$, so that

$$\begin{aligned}
&\inf_{0<\eta<\frac{s_0}{p(x)\alpha^{p(x)}\|w\|_{B_{p(x)}}}} \left(\psi^{1/s_0}\left(\frac{s_0}{\eta}\right)\left(\frac{C_0\|w\|_{B_{p(x)}}}{1-\frac{\eta p(x)\alpha^{p(x)}}{s_0}\|w\|_{B_{p(x)}}}\right)^{1/p(x)}\right) \\
&\qquad \leq \inf_{0<\eta<\frac{s_0}{p_+\|w\|_{B_1}\cdot\alpha}} \left(\psi^{1/s_0}\left(\frac{s_0}{\eta}\right)\frac{C_0\|w\|_{B_1}}{1-\frac{\eta\alpha p_+}{s_0}\|w\|_{B_1}}\right)
\end{aligned}$$

for $x \in I$ a.e. Thus we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{x\in I} \widetilde{\psi}(\|w\|_{B_{p(x)}}) &\leq \inf_{0<\eta<\frac{s_0}{p_+\|w\|_{B_1}\cdot\alpha}} \left(\psi^{1/s_0}\left(\frac{s_0}{\eta}\right)\frac{C_0\|w\|_{B_1}}{1-\frac{\eta\alpha p_+}{s_0}\|w\|_{B_1}}\right) \\
&= C(s_0,\ p_+,\ w).
\end{aligned}$$

Hence from (8), we get

$$\|f\|_{L_w^{p[\cdot],\delta(\cdot)}} \leq C(s_0,\ p_+,\ w)\|g\|_{L_w^{p[\cdot],\delta(\cdot)}}.$$

If $p_+ - \sigma > 1$ but $|Y_\sigma| = 0$, then again $X_\sigma = I$ a.e. and we get the required inequality by arguing as above.

So, let us now assume that $p_+ - \sigma > 1$ and $|Y_\sigma| \neq 0$. Clearly $|X_\sigma| > 0$. Since $w \in B_{p_+-\sigma}$, by monotonicity of weight class B_q with respect to the index q, we have $w \in B_{p(x)}$ for all $x \in X_\sigma$. On using Lemma 3 and Theorem 25, for all $x \in X_\sigma$, we have

$$\begin{aligned}
\|f\|_{L_w^{p[\cdot],\delta(\cdot)}} &= \operatorname*{ess\,sup}_{x\in I} \rho_{p(x),w(\cdot)}(\delta(x)f(\cdot)) \\
&= \max\left\{\operatorname*{ess\,sup}_{x\in Y_\sigma} \rho_{p(x),w(\cdot)}(\delta(x)f(\cdot)),\quad \operatorname*{ess\,sup}_{x\in X_\sigma} \rho_{p(x),w(\cdot)}(\delta(x)f(\cdot))\right\} \\
&\leq \max\{C(\sigma),\ 1\} \operatorname*{ess\,sup}_{x\in X_\sigma} \rho_{p(x),w(\cdot)}(\delta(x)f(\cdot)) \\
&= \max\{C(\sigma),\ 1\} \operatorname*{ess\,sup}_{x\in X_\sigma} \delta(x)\rho_{p(x),w}(f) \\
&\leq \max\{C(\sigma),\ 1\} \operatorname*{ess\,sup}_{x\in X_\sigma} \left(\delta(x)\rho_{p(x),w}(g)\widetilde{\psi}(\|w\|_{B_{p(x)}})\right)
\end{aligned}$$

$$\leq \max\{C(\sigma),\ 1\} \operatorname*{ess\,sup}_{x\in X_\sigma} \left(\widetilde{\psi}(\|w\|_{B_{p(x)}})\right) \operatorname*{ess\,sup}_{x\in I} \rho_{p(x),w(\cdot)}(\delta(x)g(\cdot))$$
$$\leq C(\sigma,\delta,w,p_+,s_0)\|g\|_{L_w^{p[\cdot],\delta(\cdot)}}$$
$$\leq D(\delta,w,p_+,s_0)\|g\|_{L_w^{p[\cdot],\delta(\cdot)}},$$

where for each fixed $x \in I$, $\rho_{p(x),w}(f)$ denotes weighted L^p norm of f, $\widetilde{\psi}(\|w\|_{B_{p(x)}})$ is the same as in (9) and

$$C(\sigma) = \operatorname*{ess\,sup}_{x\in Y_\sigma} \delta(x) \frac{2}{\|\delta\|_{L^\infty(X_\sigma)}} (w^*_{p_+-\sigma})^2,$$

where $w^*_{p_+-\sigma} = \left(\int_I w(t)dt + 1\right)^{\frac{p_+-\sigma-1}{p_+-\sigma}}$,

$$C(\sigma,\delta,w,p_+,s_0) = \max\{1,\ C(\sigma)\} \times$$
$$\times \inf_{0<\eta<\frac{s_0}{p_+\|w\|_{B_{p_+-\sigma}}\cdot\alpha^{p_+-\sigma}}} \left[1 + \psi^{1/s_0}\left(\frac{s_0}{\eta}\right) \frac{C_0\|w\|_{B_{p_+-\sigma}}}{1 - \frac{\eta\alpha^{p_+-\sigma}p_+}{s_0}\|w\|_{B_{p_+-\sigma}}}\right]^{\frac{1}{p_+-\sigma}}$$

and $D(\delta,w,p_+,s_0) = \inf_\sigma C(\sigma,\delta,w,p_+,s_0)$. □

In the above theorem, $p_+ < \infty$. In order to study the situation when $p_+ = \infty$, let us denote

$$B_\infty := \bigcup_{x\in p^{-1}([1,\,p_+])} B_{p(x)},$$

and

$$\|w\|_{B_\infty} := \inf\left\{\|w\|_{B_{p(x)}} \ :\ w \in B_{p(x)}\right\}.$$

In the framework of Lebesgue spaces, the following result is known:

Theorem 27 ([9]) *Let ψ be a nonnegative nondecreasing function defined on I. Let $0 < s_0 < \infty$. Let (f,g) be a pair of nonnegative, nonincreasing functions on I. Suppose for every $w \in B_\infty$*

$$\int_I f^{s_0}(t)w(t)dt \leq \psi(\|w\|_{B_\infty}) \int_I g^{s_0}(t)w(t)dt, \tag{10}$$

then for every $s > 0$ a.e. and $w \in B_\infty$, the inequality

$$\left(\int_I f^s(t)w(t)dt\right)^{1/s} \leq (\psi(1))^{1/s_0}\|w\|_{B_\infty}^{1/s}\left(\int_I g^s(t)w(t)dt\right)^{1/s}$$

holds.

We prove the following:

Theorem 28 *Let ψ be a nonnegative nondecreasing function defined on I. Let $1 \le s_0 < \infty$. Let (f, g) be a pair of nonnegative, nonincreasing functions on I. Suppose for every $w \in B_{s_0}$*

$$\int_I f^{s_0}(t)w(t)dt \le \psi(\|w\|_{B_{s_0}}) \int_I g^{s_0}(t)w(t)dt \tag{11}$$

then for every $p(\cdot) \in \mathcal{M},\ p(\cdot) \ge 1$ a.e. and $w \in B_{p_+}$ where $p_+ = \infty$, the inequality

$$\|f\|_{L_w^{p[\cdot],\delta(\cdot)}} \le C\|g\|_{L_w^{p[\cdot],\delta(\cdot)}} \tag{12}$$

holds, where C is a positive constant depending on $p(\cdot),\ \delta,\ s_0$ and w.

Proof Let $w \in B_\infty$, then there exists $x_0 \in p^{-1}([1,\ p_+])$ such that $w \in B_{p(x_0)}$, without loss of generality we may assume that $p(x_0) > 1$. Therefore, there exists $\sigma > 0$ such that $w \in B_{p(x_0)-\sigma}$. Define

$$X_\sigma = \{x \in I \mid p(x_0) - \sigma \le p(x) < \infty\}$$

and

$$Y_\sigma = \{x \in I \mid 1 \le p(x) < p(x_0) - \sigma\}.$$

Now, on following the steps of Theorem 26, we get (12) with the constant $C = \inf\limits_{w \in B_{p(x_0)}} D(\delta, w, p(x_0), s_0)$. □

Remark 8 If, in Theorem 28, we replace (11) by (10) for every $w \in B_\infty$ and some $1 \le s_0 < \infty$, then we have Theorem 28 trivially.

Below, we apply Theorem 26 to study the boundedness of some integral operators in the framework of fully measurable grand Lebesgue spaces. To start with, we consider the Hardy averaging operator $(Af)(x) := \frac{1}{x}\int_0^x f(t)dt$. In fact, the boundedness of this operator on fully measurable grand Lebesgue spaces has also been proved in [30] (Theorem 5.2). Here, we give an alternate proof of Theorem 5.2 [30] using Theorem 26.

First we mention the theorem which characterizes the boundedness of the Hardy averaging operator on Lebesgue spaces by B_s-class of weights.

Theorem 29 ([48]) *Let $1 \le s < \infty$, then*

$$\|Af\|_{L_w^s} \le C\|f\|_{L_w^s} \tag{13}$$

holds for nonnegative, nonincreasing measurable functions f if and only if $w \in B_s$.

Remark 9 The constant C is same in both (13) and (7). Consequently, if we identify (Af, f) as a pair of nonnegative nonincreasing functions in Theorem 26, then the function ψ in that theorem is the identity function.

We prove the following:

Theorem 30 *Let* $p(\cdot) \in \mathcal{M}^+$, $p(\cdot) \geq 1$ *a.e. and finite on* I. *Let* $p_+ < \infty$, $\delta \in L^\infty$, $\delta > 0$ *a.e.,* $\lim\limits_{x \to 0^+} \delta(x) = 0$ *and* w *be a weight in* L^∞. *Then the inequality*

$$\|Af\|_{L_w^{p[\cdot],\delta(\cdot)}} \leq C \|f\|_{L_w^{p[\cdot],\delta(\cdot)}} \tag{14}$$

holds for all nonnegative nonincreasing measurable functions f *if and only if* $w \in B_{p_+}$.

Proof For the sufficiency part, let f be any nonnegative nonincreasing measurable function, then (Af, f) is a pair of nonnegative, nonincreasing measurable functions. Hence by using Theorem 29, Remark 9 and Theorem 26, we get the inequality (14) with constant

$$C(\delta, w, p_+, s_0) = \inf_{\sigma} \left[\max\left(1, \ \operatorname*{ess\,sup}_{x \in Y_\sigma} \delta(x) \frac{2}{\|\delta\|_{L^\infty(X_\sigma)}} (w^*_{p_+-\sigma})^2 \right) \times \right.$$
$$\left. \times \inf_{0 < \eta < \frac{s_0}{p_+ \alpha^{p_+-\sigma} \|w\|_{B_{p_+-\sigma}}}} \left(1 + \left(\frac{s_0}{\eta}\right)^{1/s_0} \frac{C_0 \|w\|_{B_{p_+-\sigma}}}{1 - \frac{\eta \alpha^{p_+-\sigma} p_+}{s_0} \|w\|_{B_{p_+-\sigma}}} \right)^{\frac{1}{p_+-\sigma}} \right].$$

We get the necessary part from Theorem 5.3 [30]. □

Remark. One may be interested to compare the constants in (14) and Theorem 5.3 in [30].

Next theorem gives the boundedness of R_β, the Riemann-Liouville fractional operator, in the framework of fully measurable grand Lebesgue spaces, which is defined as

$$(R_\beta f)(x) := x^{-\beta} \int_0^x \frac{f(t)}{(x-t)^{1-\beta}} dt, \quad 0 < \beta < 1.$$

Theorem 31 *Let* $p(\cdot) \in \mathcal{M}^+$, $p(\cdot) \geq 1$ *a.e. and finite on* I. *Let* $p_+ < \infty$, $\delta \in L^\infty$, $\delta > 0$ *a.e.,* $\lim\limits_{x \to 0^+} \delta(x) = 0$ *and* w *be a weight in* L^∞ *such that* $\int_0^r w(t)dt > 0$ *for all* $0 < r < 1$. *Then the inequality*

$$\|R_\beta f\|_{L_w^{p[\cdot],\delta(\cdot)}} \leq C \|f\|_{L_w^{p[\cdot],\delta(\cdot)}} \tag{15}$$

holds for all nonnegative nonincreasing measurable functions f *if and only if* $w \in B_{p_+}$.

Proof The following point-wise estimate of the fractional operator R_β for nonnegative nonincreasing functions f is well known:

$$C_1 R_\beta f(x) \leq Af(x) \leq C_2 R_\beta f(x), \tag{16}$$

where C_1 and C_2 are positive constants independent of x and f.

We get the sufficiency part by using the first estimate in (16) and Theorem 30. The necessary part can be obtained by using the right estimate of (16) and Theorem 30. □

Remark 10 For the sufficiency part of Theorems 30 and 31 there is no need to assume the extra condition on δ, i.e., $\lim_{x \to 0^+} \delta(x) = 0$ and finiteness of $p(\cdot)$ on I.

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On Certain New Method to Construct Weighted Hardy-Type Inequalities and Its Application to the Sharp Hardy-Poincaré Inequalities

Agnieszka Kałamajska and Iwona Skrzypczak

Abstract We apply the recent method of Drábek and the authors in order to construct the Hardy–Poincaré–type inequalities

$$\bar{C}_{\gamma,n,p,r} \int_{\mathbb{R}^n} |\xi|^p \left(1 + r|x|^{\frac{p}{p-1}}\right) \left(1 + |x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)-p} dx \\ \leq \int_{\mathbb{R}^n} |\nabla \xi|^p \left(1 + |x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)} dx.$$

Some of the derived inequalities are proven to hold with the best constants.

Keywords Degenerate PDEs · Nonlinear eigenvalue problems · p–harmonic PDEs · p–Laplacian · Quasilinear PDEs

1 Introduction

In this paper we derive the family of inequalities

$$\bar{C}_{\gamma,n,p,r} \int_{\mathbb{R}^n} |\xi|^p \left(1 + r|x|^{\frac{p}{p-1}}\right) \left(1 + |x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)-p} dx \\ \leq \int_{\mathbb{R}^n} |\nabla \xi|^p \left(1 + |x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)} dx \tag{1}$$

involving parameters $r > 0$ and $\gamma \in \mathbb{R}$, where ξ is an arbitrary Lipschitz compactly supported function defined on $\mathbb{R}^n$. The inequality (1) is proven to hold with the optimal constant for the certain range of parameters.

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Inequalities having a similar form to (1), called the Hardy–Poincaré inequalities, the improved Hardy inequalities, or the improved Hardy–Sobolev inequalities, are of the particular importance in the study on nonlinear partial differential equations of elliptic and parabolic type. The optimal constants in the inequalities indicate critical values for existence or sharp rate of decay of solutions to nonlinear problems having the form

$$u_t(x,t) - \Delta_p u(x,t) = \lambda |u(x,t)|^{p-2} u(x,t) f(x) \ \text{ or } \ -\Delta_p u(x) = \lambda |u(x)|^{p-2} u(x) f(x),$$

for various approaches, see e.g. [1, 2, 6, 7, 12, 23, 24].

One of the well–known variants of the improved Hardy–Sobolev inequality

$$C_1 \int_\Omega \frac{u^2}{|x|^2} dx + C_2 \left(\int_\Omega |u|^q dx \right)^{\frac{2}{q}} \le \int_\Omega |\nabla u|^2 dx$$

has been introduced in [6] in the study of qualitative properties of solutions to $-\Delta u = \lambda f(u)$ with convex and increasing function f.

Inequalities similar to (1), in the case $p = 2$, $\gamma < 0$, are of interest in the theory of nonlinear diffusions. Namely, in several papers, e.g. [3–5], the estimates for the constants in the inequalities were obtained and applied in the study on the rate of convergence of solutions to fast diffusion equations

$$u_t = \Delta u^m.$$

For example, [4, Theorem 1, p. 376] supplies the optimal constant for the inequality

$$C \int_{\mathbb{R}^n} |\xi|^2 \left(D + \frac{1}{2|\gamma+1|} |x|^2 \right)^{\gamma-1} dx \le \int_{\mathbb{R}^n} |\nabla \xi|^2 \left(D + \frac{1}{2|\gamma+1|} |x|^2 \right)^{\gamma} dx,$$

while [5, Theorem 2] provides the optimal constant for

$$\Lambda_{\gamma,n} \int_{\mathbb{R}^n} |\xi|^2 \left(1 + |x|^2\right)^{\gamma-1} dx \le \int_{\mathbb{R}^n} |\nabla \xi|^2 \left(1 + |x|^2\right)^{\gamma} dx.$$

In [23, Theorem 2.1], among the other inequalities, the authors study the following one

$$C_1 \int_\Omega u^2 \left(1 + \frac{C_2}{|x|^2} \right) dx \le \int_\Omega |\nabla u|^2 dx.$$

The inequalities studied in [23] were used to investigate the well–possedness of the parabolic problem with an unbounded potential V, namely $u_t = \Delta u + V(x)u$, as well as to the associated spectral problem $\Delta u + V(x)u + \mu u = 0$.

The optimal constant in the Hardy–type inequalities provides an information in the spectral analysis for nonlinear eigenvalue problems in degenerate setting. Indeed, the nontrivial nonnegative solution of the problem

$$-\mathrm{div}(\omega_2 |\nabla u|^{p-2} \nabla u) = \lambda_1 \omega_1 |u|^{p-2} u,$$

involving certain weight functions $\omega_1, \omega_2 \geq 0$, minimises the Rayleigh quotient

$$\lambda_1 = \inf \left\{ \frac{\int_\Omega \omega_2 |\nabla \phi|^p dx}{\int_\Omega \omega_1 |\phi|^p dx} : \phi \in W_0^{1,p}(\Omega, \omega_2) \right\}.$$

See e.g. [1, 7, 9, 12, 23] for further studies on the related problems.

In the paper [13], the authors study the connection between the Hardy–type inequalities with two radially symmetric weight functions and certain elliptic eigenvalue problems. In particular, [13] provides estimates on constants contributing to the discussion from [3–5, 10, 21]. One of the inequalities supplied in [13, Theorem 2.13, part II] has the form

$$c \int_{\mathbb{R}^n} (a + b|x|^\alpha)^{\beta - \frac{2}{\alpha}} \xi^2 \, dx \leq \int_{\mathbb{R}^n} (a + b|x|^\alpha)^\beta \, |\nabla \xi|^2 \, dx.$$

We apply the recent result from [11, Theorem 4.1] (see also [16, 19, 20] for the related earlier results), providing the new method of construction of the general Hardy–type inequalities

$$\int_\Omega |\xi|^p \mu_1(dx) \leq \int_\Omega |\nabla \xi|^p \mu_2(dx).$$

The measures μ_1 and μ_2 involve the nonnegative solution $u : \Omega \to \mathbb{R}$ to the partial differential inequality (PDI) of the form

$$-\Delta_{p,a} u \geq b(x) \Phi(u),$$

where $\Omega \subseteq \mathbb{R}^n$ is an arbitrary open domain, $p > 1$, the operator $\Delta_{p,a} u = \mathrm{div}(a(x) |\nabla u|^{p-2} \nabla u)$ is the degenerate p–Laplacian involving a weight function $a(\cdot) : \Omega \to [0, \infty)$, $b(\cdot)$ is a measurable function defined on Ω, and $\Phi : [0, \infty) \to [0, \infty)$ is a given continuous function. We apply [11, Theorem 4.1] taking into account that if $u_s(x) := (1 + |x|^{p'})^{-s}$, then we have

$$-\Delta_{p,u_\beta} u_\alpha = C_1 (1 + C_2 |x|^{p'}) (u_\alpha)^\delta,$$

with certain parameters $\alpha, \beta \in \mathbb{R}$. Some of the resulting inequalities are provided with the optimal constants. This application is one another example of inequalities that can be obtained via this recent method with the best constants. For the discussion on the related optimal consequences of the result from [11, Theorem 4.1] see [11, Remark 4.1] referring also to [8, 15, 20, 21].

2 Preliminaries

Basic notation

In the sequel we assume that $p > 1$, $\Omega \subseteq \mathbb{R}^n$ is an open subset. By $a(\cdot) - p$–harmonic problems we understand those which involve degenerated p–Laplace operator

$$\Delta_{p,a} u = \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u), \tag{2}$$

with some nonnegative function $a(\cdot)$. The derivatives which appear in (2) are understood in the distributional sense. By $D'(\Omega)$ we denote the space of distributions defined on Ω. If f is defined on Ω, then by $f\chi_\Omega$ we understand a function defined on $\mathbb{R}^n$ which is equal to f on Ω and which is extended by 0 outside Ω. We set $f^- := \min\{f, 0\}$, $f^+ := \max\{f, 0\}$. Moreover, every time when we deal with infimum, we assume $\inf \emptyset = +\infty$.

Weighted Beppo Levi and Sobolev spaces The general method that we apply requires the following setting discussed in details in [11].

Definition 1 (*Class* $B_p(\Omega)$) Let $\mathcal{M}(\Omega)$ be the set of all Borel measurable real functions defined on Ω, $W(\Omega) := \{\varrho \in \mathcal{M}(\Omega) : \ 0 < \varrho(x) < \infty, \ \text{for a.e. } x \in \Omega\}$, and let $p > 1$. We say that a weight $\varrho \in W(\Omega)$ satisfies $B_p(\Omega)$–condition ($\varrho \in B_p(\Omega)$ for short) if $\varrho^{-1/(p-1)} \in L^1_{loc}(\Omega)$.

We define

$$L^p_\varrho(\Omega) := \left\{ u \in \mathcal{M}(\Omega) : \int_\Omega \varrho |u|^p \, dx < \infty \right\}.$$

The Hölder inequality shows that if $p > 1$ and $\varrho \in B_p(\Omega)$, then $L^p_{\varrho,loc}(\Omega) \subseteq L^1_{loc}(\Omega)$, [17].

Assume that $\varrho(\cdot) \in B_p(\Omega)$. We deal with the weighted Beppo Levi space

$$\mathcal{L}^{1,p}_\varrho(\Omega) := \{u \in D'(\Omega) : \frac{\partial u}{\partial x_i} \in L^p_\varrho(\Omega) \text{ for } i = 1, \ldots, n\}.$$

It can be shown that $\mathcal{L}^{1,p}_\varrho(\Omega) \subseteq W^{1,1}_{loc}(\Omega)$ (see e.g. [18, Theorem 1, Sect. 1.1.2]). We also consider local variants of Beppo Levi spaces

$$\mathcal{L}^{1,p}_{\varrho,loc}(\Omega) := \left\{ u \in D'(\Omega) : \frac{\partial u}{\partial x_i} \in L^p_\varrho(U) \text{ for } i = 1, \ldots, n \text{ and every } U \subset\subset \Omega. \right\}$$

Let us observe that $\mathcal{L}^{1,p}_{\varrho,loc}(\Omega) \subseteq W^{1,1}_{loc}(\Omega)$.

Let $\varrho_1(\cdot) \in W(\Omega)$, $\varrho_2(\cdot) \in B_p(\Omega)$. We consider the two–weighted Sobolev spaces $W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega) = L^p_{\varrho_1}(\Omega) \cap \mathcal{L}^{1,p}_{\varrho_2}(\Omega)$:

$$W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega) := \left\{ f \in L^p_{\varrho_1}(\Omega) \cap D'(\Omega) : \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \in L^p_{\varrho_2}(\Omega) \right\}, \tag{3}$$

with the norm $\|f\|_{W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega)} := \|f\|_{L^p_{\varrho_1}(\Omega)} + \|\nabla f\|_{L^p_{\varrho_2}(\Omega,\mathbb{R}^n)}$.

It is known that when $p > 1$, $\varrho_1(\cdot) \in W(\Omega)$, $\varrho_2(\cdot) \in B_p(\Omega)$, then $W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega)$ defined by (3) and equipped with the norm $\|\cdot\|_{W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega)}$ is a Banach space, [17].

When $\varrho_1 \equiv \varrho_2$, we deal with the usual weighted Sobolev space $W^{1,p}_{\varrho_1}(\Omega)$. By $W^{1,p}_{(\varrho_1,\varrho_2),0}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in the space $W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega)$ and we use the standard notation $W^{1,p}_{(\varrho_1,\varrho_1),0}(\Omega) = W^{1,p}_{\varrho_1,0}(\Omega)$ when $\varrho_1 = \varrho_2$.

Degenerate p–Laplacian and differential inequality Assume that $p > 1$, $a \in B_p(\Omega) \cap L^1_{loc}(\Omega)$ (see Definition 1), and $u \in \mathcal{L}^{1,p}_{a,loc}(\Omega)$. Then $a|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ and so the weak divergence of $a|\nabla u|^{p-2}\nabla u \in L^1_{loc}(\Omega,\mathbb{R}^n)$ denoted by $\Delta_{p,a}u$ is well defined via the formula

$$\langle \Delta_{p,a}u, w\rangle = \langle \mathrm{div}\left(a|\nabla u|^{p-2}\nabla u\right), w\rangle := -\int_\Omega a|\nabla u|^{p-2}\nabla u \cdot \nabla w dx \tag{4}$$

where $w \in C_0^\infty(\Omega)$. Obviously, in the case $a \equiv 1$ the operator $\Delta_{p,a}u$ reduces to the p–Laplacian div $\left(|\nabla u|^{p-2}\nabla u\right)$, which coincides with the Laplace operator in the case $p = 2$. We note that when $u \in \mathcal{L}^{1,p}_a(\Omega)$, formula (4) extends for $w \in W^{1,p}_{(b,a),0}(\Omega)$, whenever $b \in W(\Omega)$. Therefore, in that case $\Delta_{p,a}u$ can be also treated as an element of $(W^{1,p}_{(b,a),0}(\Omega))^*$, the dual to the Banach space $W^{1,p}_{(b,a),0}(\Omega)$. We preserve the same notation $\Delta_{p,a}u$ for this functional extension of formula (4).

Our analysis is based on the following differential inequality.

Definition 2 Let $a(\cdot) \in B_p(\Omega) \cap L^1_{loc}(\Omega)$ be a given weight function, $u(\cdot) \in \mathcal{L}^{1,p}_{a,loc}(\Omega)$ be nonnegative, $\Phi : [0,\infty) \to [0,\infty)$ be a continuous function, $b(\cdot)$ be measurable and the function $\Phi(u(\cdot))b(\cdot) \in L^1_{loc}(\Omega)$. Suppose further that for every nonnegative compactly supported function $w \in \mathcal{L}^{1,p}_a(\Omega)$ one has

$$\int_\Omega \Phi(u)b(x)w\, dx > -\infty.$$

We say that the partial differential inequality (PDI for short)

$$-\Delta_{p,a}u \geq \Phi(u)b(x),$$

holds if for every nonnegative compactly supported function $w \in \mathcal{L}^{1,p}_a(\Omega)$ we have

$$\langle -\Delta_{p,a}u, w\rangle \geq \int_\Omega \Phi(u)b(x)w\, dx,$$

where $\langle -\Delta_{p,a}u, w\rangle$ is given by (4).

Assumption A Let us consider the following three conditions.

(a,b) $a(\cdot) \in L^1_{loc}(\Omega) \cap B_p(\Omega)$, $b(\cdot)$ is measurable;

(Ψ, g) The couple of continuous functions $(\Psi, g) : (0,\infty) \times (0,\infty) \to (0,\infty) \times (0,\infty)$, where Ψ is Lipschitz on every closed interval in $(0,\infty)$, satisfy the following compatibility conditions:

(i) the inequality

$$g(t)\Psi'(t) \leq -C\Psi(t) \quad \text{a.e.} \quad \text{in} \quad (0,\infty) \tag{5}$$

holds with some constant $C \in \mathbb{R}$ independent of t and Ψ is monotone (not necessarily strictly);

(ii) each of the functions

$$t \mapsto \Theta(t) := \Psi(t)g^{p-1}(t) \;\text{ and }\; t \mapsto \Psi(t)/g(t)$$

is nonincreasing or bounded in some neighborhood of 0.

(u) We assume that $u \in \mathcal{L}^{1,p}_{a,loc}(\Omega)$ is nonnegative, (a,b) holds, $\Phi : [0,\infty) \to [0,\infty)$ is a continuous function, such that for every nonnegative compactly supported function $w \in \mathcal{L}^{1,p}_a(\Omega)$ one has $\int_\Omega \Phi(u)b(x)w\,dx > -\infty$ and $\Phi(u)b \in L^1_{loc}(\Omega)$.

We suppose that for the following set

$$\mathcal{A} := \left\{ \sigma \in \mathbb{R} : \; \Phi(u)b(x) + \sigma \frac{a(x)}{g(u)}|\nabla u|^p \geq 0 \quad \text{a.e. in } \Omega \cap \{u > 0\} \right\},$$

we have

$$\sigma_0 := \inf \mathcal{A} \in \mathbb{R}. \tag{6}$$

Since $\inf \emptyset = +\infty$, $\mathcal{A}$ can be neither an empty set nor unbounded from below.

By Assumption A we mean the set of conditions (a)–(d) stated below.

(a) We suppose that (Ψ, g) and (u) hold. Parameter σ satisfies $\sigma_0 \leq \sigma < C$, where C is given by (5) and σ_0 by (6).

(b) We suppose that (Ψ, g) and (u) hold. We assume that for every $R > 0$ we have $b^+(x)(\Phi\Psi)(u)\chi_{\{0<u\leq R\}} \in L^1_{loc}(\Omega)$.

(c) We suppose that (Ψ, g) and (u) hold. When the set $\Omega_0 := \{x : u(x) = 0\}$ has a positive measure, then we assume that at least one of the following conditions are satisfied x) $\Phi(0) = 0$, y) $b(x)\chi_{\Omega_0} \geq 0$, z) $\lim_{s\to 0} \Psi(s) = 0$.

(d) We suppose that (Ψ, g) and (u) hold. We assume that for any compact subset $K \subseteq \Omega$ we have

$$\Psi(R)\int_{K\cap\{u\geq R/2\}}|\nabla u(x)|^{p-1}a(x)\,dx \stackrel{R\to\infty}{\to} 0,$$
$$\Psi(R)\int_{K\cap\{u\geq R/2\}}\Phi(u)b(x)\,dx \stackrel{R\to\infty}{\to} 0.$$

Let us recall [11, Theorem 4.1], which is our main tool.

Theorem 1 (Hardy–type inequality) *Suppose* $a\in L^1_{loc}(\Omega)\cap B_p(\Omega)$, $b\in L^1_{loc}(\Omega)$. *Assume that* $1<p<\infty$ *and* $u\in\mathcal{L}^{1,p}_{a,loc}(\Omega)$ *is a nonnegative solution to the PDI* $-\Delta_{p,a}u\geq\Phi(u)b(x)$ *in the sense of Definition 2. Moreover, let Assumption A hold.*

Then for every Lipschitz function $\xi\in\mathcal{L}^{1,p}_a(\Omega)$ *with compact support in* Ω *we have*

$$\int_\Omega|\xi|^p\mu_1(dx)\leq\int_\Omega|\nabla\xi|^p\mu_2(dx),$$

where

$$\mu_1(dx)=\left(\Phi(u)b(x)+\sigma|\nabla u|^p\frac{a(x)}{g(u)}\chi_{\{u\neq 0\}}\right)\Psi(u)\chi_{\{u>0\}}\,dx,$$
$$\mu_2(dx)=\left(\frac{p-1}{C-\sigma}\right)^{p-1}a(x)\Psi(u)g^{p-1}(u)\chi_{\{u>0,\nabla u\neq 0\}}\,dx.$$

3 Construction of the Hardy–Poincaré Inequalities by Using the Barenblatt–Talenti Profiles

Here we focus on the application of Theorem 1 to derive some variants of the Hardy–Poincaré inequalities. Our main goal is the following theorem.

Theorem 2 *Assume that* $1<p<\infty$, $\gamma>1-\frac{n}{p}$, $0<r<\frac{p}{n}+\gamma\frac{p}{n}$,

$$\varrho_1(x)=\left(1+r|x|^{\frac{p}{p-1}}\right)\left(1+|x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)-p},\qquad \varrho_2(x)=\left(1+|x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)}.$$

Then for every $\xi\in W^{1,p}_{(\varrho_1,\varrho_2)}(\mathbb{R}^n)$, *we have*

$$\begin{aligned}\bar{C}_{\gamma,n,p,r}\int_{\mathbb{R}^n}|\xi|^p\left(1+r|x|^{\frac{p}{p-1}}\right)\left(1+|x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)-p}dx\\ \leq\int_{\mathbb{R}^n}|\nabla\xi|^p\left(1+|x|^{\frac{p}{p-1}}\right)^{\gamma(p-1)}dx.\end{aligned}\tag{7}$$

where $\bar{C}_{\gamma,n,p,r}=n\left(\frac{p}{p-1}\right)^{p-1}\left(\gamma-1+\frac{n}{p}(1-r)\right)^{p-1}$. *Moreover, constant* $\bar{C}_{\gamma,n,p,r}$ *is optimal when* $\gamma>nr+1-\frac{n}{p}$ *and when* $\gamma=1+n(1-\frac{1}{p})$, $r=1$.

Remark 1 By our choice of the class of weight functions, Lipschitz compactly supported functions are dense in $W^{1,p}_{(\varrho_1,\varrho_2)}(\mathbb{R}^n)$. Indeed, let $\xi \in W^{1,p}_{(\varrho_1,\varrho_2)}(\mathbb{R}^n)$ and

$$\phi(x) = \begin{cases} 1, & |x| < 1, \\ -|x| + 2, & 1 \le |x| \le 2, \\ 0, & 2 < |x|, \end{cases} \quad \phi_R(x) = \phi\left(\frac{x}{R}\right), \quad \xi_R(x) = \xi(x)\phi_R(x).$$

One shows that $\xi_R \to \xi$ in $W^{1,p}_{(\varrho_1,\varrho_2)}(\mathbb{R}^n)$ as $R \to \infty$. Moreover, standard convolution argument implies that every compactly supported function $u \in W^{1,p}_{(\varrho_1,\varrho_2)}(\mathbb{R}^n)$ can be approximated in $W^{1,p}_{(\varrho_1,\varrho_2)}(\mathbb{R}^n)$ by Lipschitz compactly supported functions.

The above statement can be compared with the following one obtained in [21], which follows as the special case when one substitutes $r = 1$, and consequently one has to assume that $\gamma > 1$.

Theorem 3 ([21]) Suppose $p > 1$ and $\gamma > 1$. Then, for every function $\xi \in W^{1,p}_{(v_1,v_2)}(\mathbb{R}^n)$, where $v_1(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)(\gamma-1)}$, $v_2(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma}$, we have

$$\bar{C}_{\gamma,n,p} \int_{\mathbb{R}^n} |\xi|^p \left[(1 + |x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma-1} dx \le \int_{\mathbb{R}^n} |\nabla\xi|^p \left[(1 + |x|^{\frac{p}{p-1}})^{p-1}\right]^{\gamma} dx,$$

with $\bar{C}_{\gamma,n,p} = n\left(\frac{p(\gamma-1)}{p-1}\right)^{p-1}$. Moreover, for $\gamma > n + 1 - \frac{n}{p}$, the constant $\bar{C}_{\gamma,n,p}$ is optimal and it is achieved by function $\bar{u}(x) = (1 + |x|^{\frac{p}{p-1}})^{1-\gamma}$.

The main idea of the proof of Theorem 2 is based on the fact that the Barenblatt–Talenti profile (playing the crucial role in [22])

$$u_\eta(x) = (1 + |x|^{p'})^{-\eta}, \quad \eta \in \mathbb{R}, \quad p' = \frac{p}{p-1}, \tag{8}$$

satisfy a certain PDE. Namely, we have the following results.

Lemma 1 *When $p > 1$, $\alpha > 0$, $\beta \in \mathbb{R}$, and u_α, u_β are as in (8) and*

$$C_1 := n(\alpha p')^{p-1}, \; C_2 := 1 - \frac{((\alpha+1)(p-1)+\beta)\, p'}{n}, \; \delta := \frac{(\alpha+1)(p-1)+\beta+1}{\alpha},$$

we have

$$-\Delta_{p,u_\beta} u_\alpha = b(x)\Phi(u_\alpha), \quad \text{where} \quad b(x) = C_1(1 + C_2|x|^{p'}), \; \Phi(s) = s^\delta. \tag{9}$$

Moreover, Assumption A is satisfied when

$$\begin{aligned} \Psi(s) &= s^{-\kappa}, \; \kappa \in \mathbb{R}, \; g(s) = s, \; \sigma_0 \le \sigma < \kappa, \\ \sigma_0 &:= \frac{(\alpha+1)(p-1)+\beta}{\alpha} - \frac{n}{\alpha p'}. \end{aligned}$$

Proof Equation (9) is satisfied due to Lemma 4 in Appendix. The verification of the assumptions (a, b) and (Ψ, g) with $\kappa = C$ is left to the reader. To verify the condition (u) we note that

$$b(x)\Phi(u_\alpha) + \sigma\frac{a(x)}{u_\alpha(x)}|\nabla u_\alpha(x)|^p = C_1(1 + C_2|x|^{p'})u_{\alpha\delta}(x) + \sigma C_3 u_{\alpha\delta}(x)|x|^{p'} \quad (10)$$
$$= u_{\alpha\delta}(x)\left\{C_1 + (C_1C_2 + \sigma C_3)|x|^{p'}\right\},$$

where $C_3 = (\alpha p')^p$, hence (u) is satisfied whenever $C_1C_2 + \sigma C_3 \geq 0$, equivalently

$$\sigma \geq -\frac{C_1C_2}{C_3} = -\frac{n - ((\alpha+1)(p-1)+\beta)\, p'}{\alpha p'} = \sigma_0.$$

To verify the condition (c) we note that $\Omega_0 = \{x : u_\alpha(x) = 0\} = \emptyset$. The rest of the assumptions are obviously satisfied.

Remark 2 Condition (u) cannot be satisifed, when α is negative.

As an important step we obtain certain family of inequalities, which—so far—do not have the form (7) and is obtained as a direct application of Theorem 1, Lemma 1 and (10). The easy proof is left to the reader.

Lemma 2 *Assume that* $1 < p < \infty$, $\alpha > 0$, $\beta \in \mathbb{R}$, $\delta := \frac{(\alpha+1)(p-1)+\beta+1}{\alpha}$, $\sigma_0 := \frac{(\alpha+1)(p-1)+\beta}{\alpha} - \frac{n}{\alpha p'} \leq \sigma < \kappa$, *and*

$$C_1 := n(\alpha p')^{p-1},\ C_2 := 1 - \frac{((\alpha+1)(p-1)+\beta)\, p'}{n} \geq 0,\ C_3 = (\alpha p')^p,$$
$$B := C_2 + \sigma\frac{C_3}{C_1} = 1 - \frac{(\alpha+1)p}{n} + \frac{(\sigma\alpha-\beta)p'}{n}.$$

Then for every compactly supported Lipschitz function $\xi \in \mathcal{L}_a^{1,p}(\Omega)$, *we have*

$$\int_{\mathbb{R}^n} |\xi|^p \mu_1(dx) \leq \int_{\mathbb{R}^n} |\nabla\xi|^p \mu_2(dx),$$

where

$$\mu_1(dx) = C_1\left(1 + B|x|^{p'}\right)\left(1 + |x|^{p'}\right)^{\alpha(\kappa-\delta)} dx,$$
$$\mu_2(dx) = \left(\frac{p-1}{\kappa-\sigma}\right)^{p-1}\left(1 + |x|^{p'}\right)^{\alpha(\kappa-p)+\alpha-\beta} dx.$$

We are now to prove Theorem 2.

Proof According to the density argument from Remark 1, it suffices to prove inequality (7) for every Lipschitz compactly supported function.

We apply Lemma 2. We fix parameters $\alpha > 0, \beta \in \mathbb{R}$, $\sigma > \sigma_0$, where $\sigma_0 = \sigma_0(\alpha, \beta) := \frac{(\alpha+1)(p-1)+\beta}{\alpha} - \frac{n}{\alpha p'}$ and consider linear mapping

$$T_{\alpha,\beta} : (\sigma, \infty) \to \left(\frac{\alpha}{p-1}\sigma - \alpha - \frac{\beta}{p-1}, \infty\right), \quad T_{\alpha,\beta}(\kappa) = \frac{\alpha}{p-1}\kappa - \alpha - \frac{\beta}{p-1}.$$

Easy computation shows that

$$\inf\{T_{\alpha,\beta}(\sigma) : \sigma \geq \sigma_0\} = 1 - \frac{n}{p} =: \gamma_{min},$$

it is achieved at σ_0 and it is independent of α and β. Hence, for every given $\gamma > 1 - \frac{n}{p}$, we find κ and σ, such that $T_{\alpha,\beta}(\kappa) = \gamma$ and $\kappa > \sigma \geq \sigma_0$. Namely, we take

$$\kappa = \kappa(\alpha, \beta, \gamma) = T^{-1}_{\alpha,\beta}(\gamma) = \frac{p-1}{\alpha}\gamma + p - 1 + \frac{\beta}{\alpha},$$

so that $\alpha(\kappa - p) + \alpha - \beta = (p-1)\gamma$ and

$$\sigma \in \mathcal{A}(\alpha, \beta, \gamma) := \left[\frac{(\alpha+1)(p-1)+\beta}{\alpha} - \frac{n}{\alpha p'}, \frac{p-1}{\alpha}\gamma + p - 1 + \frac{\beta}{\alpha}\right).$$

Note that for any $\alpha > 0$ and $\beta \in \mathbb{R}$, the set $\mathcal{A}(\alpha, \beta, \gamma)$ is not empty, whenever $\gamma > 1 - \frac{n}{p}$. According to Lemma 2 we have

$$D(\alpha, \beta, \gamma, \sigma) \int_{\mathbb{R}^n} |\xi|^p \left(1 + B(\alpha, \beta, \sigma)|x|^{p'}\right) \left(1 + |x|^{p'}\right)^{\alpha\left(T^{-1}_{\alpha,\beta}(\gamma) - \delta(\alpha,\beta)\right)} dx \leq$$
$$\leq \int_{\mathbb{R}^n} |\nabla \xi|^p \left(1 + |x|^{p'}\right)^{(p-1)\gamma} dx$$

for Lipschitz compactly supported ξ, where

$$\begin{aligned}
\delta(\alpha, \beta) &:= \frac{(\alpha+1)(p-1)+\beta+1}{\alpha}, \\
B(\alpha, \beta, \sigma) &= 1 - \frac{(\alpha+1)p}{n} - \frac{\beta p'}{n} + \sigma\frac{\alpha p'}{n} > 0, \\
\sigma &\in \mathcal{A}(\alpha, \beta, \gamma), \\
D(\alpha, \beta, \gamma, \sigma) &:= n(\alpha p')^{p-1}\left(\frac{\kappa(\alpha, \beta, \gamma) - \sigma}{p-1}\right)^{p-1}, \\
\alpha\left(T^{-1}_{\alpha,\beta}(\gamma) - \delta(\alpha, \beta)\right) &= (p-1)\gamma - p.
\end{aligned}$$

We compute the range of the function $\sigma \mapsto R_{\alpha,\beta}(\sigma) := B(\alpha, \beta, \sigma)$, when $\sigma \in \mathcal{A}(\alpha, \beta, \gamma)$. Simple computation gives

$$R_{\alpha,\beta}\left(\mathcal{A}(\alpha,\beta,\gamma)\right) = \left[0, \frac{p}{n}(\gamma-1) =: y(\gamma)\right)$$

and it does not depend on α, β. Therefore, for any $r \in [0, y(\gamma))$ and any $\alpha > 0$, $\beta \in \mathbb{R}$, $\gamma > 1 - \frac{n}{p}$, we find $\sigma \in \mathcal{A}(\alpha,\beta,\gamma)$ such that $B(\alpha,\beta,\sigma) = r$. Namely, we choose

$$\sigma =: \sigma(\alpha,\beta,r,\gamma) = \left(r - 1 + \frac{(\alpha+1)p}{n} + \frac{\beta p'}{n}\right)\frac{n}{\alpha p'} = \\ = r\frac{n}{\alpha p'} - \frac{n}{\alpha p'} + \frac{(\alpha+1)(p-1)}{\alpha} + \frac{\beta}{\alpha}.$$

Further computation gives

$$\kappa(\alpha,\beta,\gamma) - \sigma(\alpha,\beta,r,\gamma) = \frac{p-1}{\alpha}\{\gamma - 1 + \frac{n}{p}(1-r)\},$$
$$\bar{C}_{\gamma,n,p,r} = D(\alpha,\beta,\gamma,\sigma(\alpha,\beta,r,\gamma)) = n\left(\frac{p}{p-1}\right)^{p-1}\left(\gamma - 1 + \frac{n}{p}(1-r)\right)^{p-1}.$$

This ends the proof of (7).

Let us concentrate on the proof of the optimality of $\bar{C}_{\gamma,n,p,r}$. We assume that $\gamma > nr + 1 - \frac{n}{p}$. Applying Lemma 1 when $\bar{\alpha} := (1-r)\frac{n}{p} + \gamma - 1$, we get

$$-\Delta_{p,\varrho_2} u_{\bar{\alpha}} = \bar{C}_{\gamma,n,p,r}\,\varrho_1 u_{\bar{\alpha}}^{p-1} \quad \text{in} \quad \mathbb{R}^n.$$

Multiplying the above identity by $u_{\bar{\alpha}}$ and integrating over balls, then applying Gauss—Ostrogradzki Theorem, we obtain

$$B(R) := \bar{C}_{\gamma,n,p,r}\int_{\{|x|<R\}} \varrho_1 u_{\bar{\alpha}}^p\,dx = -\int_{\{|x|<R\}} \Delta_{p,\varrho_2} u_{\bar{\alpha}} \cdot u_{\bar{\alpha}}\,dx \\ = \int_{\{|x|<R\}} \varrho_2 |\nabla u_{\bar{\alpha}}|^p\,dx + \int_{\{|x|=R\}} \varrho_2 |\nabla u_{\bar{\alpha}}|^{p-1} u_{\bar{\alpha}}\,dS =: A(R) + C(R),$$

where dS denotes the surface measure on the sphere $S^{n-1}(R)$. Simple computation shows that both $\int_{\mathbb{R}^n} \varrho_1 u_{\bar{\alpha}}^p\,dx$ and $\int_{\mathbb{R}^n} \varrho_2 |\nabla u_{\bar{\alpha}}|^p\,dx$ are finite. So $B(R)$ and $A(R)$ converge to $\bar{C}_{\gamma,n,p,r}\int_{\mathbb{R}^n} \varrho_1 u_{\bar{\alpha}}^p\,dx$ and to $\int_{\mathbb{R}^n} \varrho_2 |\nabla u_{\bar{\alpha}}|^p\,dx$, respectively, via the Lebesgue Dominated Convergence Theorem. Meanwhile $C(R)$ converges to zero, because

$$C(R) \sim R^{n-1} R^{\gamma p} R^{(r-1)n-\gamma p+1} R^{(r-1)\frac{n}{p-1}+(1-\gamma)p'} =: R^L,$$

where $L = (r-1)np' + n + (1-\gamma)p' < 0$.

Let us focus now on the case of $\gamma = nr + 1 - \frac{n}{p}$. We proceed by the similar argument as in [21, Remark 4.2], considering $\xi_t(y) := \xi(ty)$ and after the change of variables $x := ty$, we let $t \to 0$. Then we substitute $\bar{\gamma} = \gamma p$. This gives the classical Hardy inequality

$$\int_{\mathbb{R}^n} \left(\frac{|\xi(x)|}{|x|}\right)^p |x|^{\bar{\gamma}}\, dx \leq \frac{1}{\bar{C}_{\frac{\bar{\gamma}}{p},n,p,r}} \int_{\mathbb{R}^n} (|\nabla \xi(x)|)^p\, |x|^{\bar{\gamma}}\, dx,$$

where $\frac{1}{\bar{C}_{\frac{\bar{\gamma}}{p},n,p,r}} = \frac{1}{n^p}\frac{1}{r^{p-1}} = \frac{1}{n}\frac{1}{\left(\frac{\bar{\gamma}}{p}-1+\frac{n}{p}\right)^{p-1}}$. The choice of $\bar{\gamma} = p + n(p-1)$, equivalently $\gamma = 1 + n(1 - \frac{1}{p})$ (which implies $r = 1$), gives the inequality with a constant $\frac{1}{\bar{C}_{\frac{\bar{\gamma}}{p},n,p,r}} = \frac{p^p}{(\bar{\gamma}+n-p)^p}$, which is optimal in the classical Hardy inequality [14]. Therefore, in this case the constant $\bar{C}_{\gamma,n,p,r}$ cannot be taken larger, so it is optimal. This ends the proof of the statement.

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Appendix

We have the following two lemmas, which we apply to prove Lemma 1.

Lemma 3 *Let*

$$\Phi_q(\lambda) = |\lambda|^{q-2}\lambda, \quad \lambda \in \mathbb{R}^n, \quad q > 1, \quad s \in \mathbb{R},$$

where the same notation is used also for $n = 1$. *Then we have*

$$\begin{aligned} \Phi_q(s\lambda) &= \Phi_q(s)\Phi_q(\lambda); \\ \Phi_q(\Phi_r(\lambda)) &= \Phi_{(q-1)(r-1)+1}(\lambda); \\ \Phi_q(s) &= s^{q-1}, \quad \text{when } s \geq 0; \\ \nabla |x|^q &= q\Phi_q(x); \end{aligned} \qquad \begin{aligned} \nabla \Phi_q(s) &= (q-1)|s|^{q-2}; \\ \Phi_2(\lambda) &= \lambda; \\ \Phi_q(x)\cdot x &= |x|^q. \end{aligned}$$

Using the above lemma, it is easy to verify the statements presented below.

Lemma 4 *When* u_α *is given by* (8)*, we have*

$$\begin{aligned} \nabla u_\alpha &= (-\alpha p')u_{\alpha+1}(x)\Phi_{p'}(x); \\ \Phi_p(\nabla u_\alpha) &= \Phi_p(-\alpha p')u_{(\alpha+1)(p-1)}\cdot x, \quad \Phi_p(-\alpha p') = -\mathrm{sgn}(|\alpha| p')^{p-1}; \\ u_\beta \Phi_p(\nabla u_\alpha) &= \Phi_p(-\alpha p')u_{(\alpha+1)(p-1)+\beta}\cdot x; \\ \mathrm{div}(u_\gamma \cdot x) &= n\cdot u_{\gamma+1}\left[1 + \left(1 - \frac{\gamma p'}{n}\right)|x|^{p'}\right]; \\ -\Delta_{p,u_\beta} u_\alpha &= n\Phi_p(\alpha p')\cdot u_{(\alpha+1)(p-1)+\beta+1}\left[1 + c_{(\alpha,\beta,p,n)}|x|^{p'}\right], \ \text{where} \\ c_{(\alpha,\beta,p,n)} &= 1 - \frac{((\alpha+1)(p-1)+\beta)\, p'}{n} \end{aligned}$$

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Intrinsic Characterization and the Extension Operator in Variable Exponent Function Spaces on Special Lipschitz Domains

Henning Kempka

Abstract We study 2-microlocal Besov and Triebel-Lizorkin spaces with variable exponents on special Lipschitz domains Ω. These spaces are as usual defined by restriction of the corresponding spaces on $\mathbb{R}^n$. In this paper we give two intrinsic characterizations of these spaces using local means and the Peetre maximal operator. Further, we construct a linear and bounded extension operator following the approach done by Rychkov in (J Lond Math Soc 60(1):237–257, 1999, [14]), which at the end also turns out to be universal.

Keywords 2-microlocal spaces · Besov spaces · Triebel-Lizorkin spaces · Variable integrability · Restriction · Extension

1 Introduction

In this paper we study Besov $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ and Triebel-Lizorkin spaces $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ with variable exponents on special Lipschitz domains $\Omega \subset \mathbb{R}^n$, where

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\}$$

for a Lipschitz continuous function $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$. Here the variable integrability is defined with measurable functions $p(\cdot)$ and $q(\cdot)$ and the variable smoothness is defined in the 2-microlocal sense using admissible weight sequences $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0}$, see Sect. 2 for details.

Spaces of this type on $\mathbb{R}^n$ have first been considered by Diening, Hästö and Roudenko in [4] and by the author in [10]. With also $q(\cdot)$ variable in the B-case they have been studied by Almeida and Hästö in [2] and by the author and Vybíral in [12].

In this paper we obtain intrinsic characterizations of $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ using local means and the Peetre maximal operator. Furthermore, a linear and bounded extension operator from the spaces on Ω to the spaces on $\mathbb{R}^n$ is constructed.

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In the whole work we rely very much on the paper of Rychkov [14] where the same results have been shown for constant exponents, i.e. $p(\cdot) = p, q(\cdot) = q$ as constants and $w_j(x) = 2^{js}$ with $s \in \mathbb{R}$. Surprisingly, all results remain true in the variable setting. We refer again to [14] on an exhaustive history of such results.

For variable exponents there are not so many results on intrinsic characterizations and on the extension operator known. An intrinsic characterization for our spaces has been provided in [9] with the help of non smooth atomic characterizations. This approach also works for more general domains than special Lipschitz domains.

If $p(\cdot) = p$ and $q(\cdot) = q$ are constants, then intrinsic characterizations and an extension operator has been presented by Tyulenev in [21] in the Besov space scale. This work also modified the proofs from Rychkov [14], but the focus in [21] lies on more general domains and on more general weight sequences where also Muckenhoupt weights are allowed as variable smoothness functions.

Further, in [5] Diening and Hästö constructed with mollifiers an extension operator for the Sobolev spaces $W^1_{p(\cdot)} = F^1_{p(\cdot),2}$ from the halfspace $\mathbb{R}^n_+$ to $\mathbb{R}^{n+1}$. Also working on the halfspace $\mathbb{R}^n_+$, Noi showed in [13] the boundedness of the trace and extension operator for the Besov and Triebel-Lizorkin spaces with variable exponents using quarkonial decompositions.

The paper is structured as follows. We introduce in Sect. 2 the necessary notation and the Besov and Triebel-Lizorkin spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with variable exponents on $\mathbb{R}^n$. Further, we present there the important local means characterization for these spaces. In Sect. 3, we introduce special Lip-schitz domains and introduce the spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ as usual by restrictions from the corresponding spaces on $\mathbb{R}^n$. Section 4 contains the main results of this paper. Here we prove an intrinsic characterization using local means and define a linear and bounded extension operator on $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$. This is complemented by Sect. 5, where an universal extension operator $\mathcal{E}_u$ is constructed. Here the operator is not depending on the functions $p(\cdot), q(\cdot)$ and the parameters of the weight sequence α, α_1 and α_2.

2 Preliminaries

First of all, we introduce all necessary notation. As usual, we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{N}$ denotes the set of natural numbers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We write $\eta \approx \xi$ if there exist two constants $c_1, c_2 > 0$ with $c_1\eta \le \xi \le c_2\eta$.

Please be aware that $c > 0$ is an universal constant and can change its value from one line to another but is never depending on any variables used in the estimates, except it is clearly noted. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the set of all infinitely often differentiable functions on $\mathbb{R}^n$ with rapid decay at infinity. Its topology is generated by the seminorms

$$\|\Phi\|_{k,l} = \sup_{x\in\mathbb{R}^n}(1+|x|)^k \sum_{|\beta|\le l} |D^\beta \Phi(x)|.$$

By $\mathcal{S}'(\mathbb{R}^n)$ we denote the dual space of $\mathcal{S}(\mathbb{R}^n)$ containing all tempered distributions on $\mathbb{R}^n$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we denote by $\widehat{f}$ the Fourier transform of f and by $f^\vee$ the inverse Fourier transform of f. For a function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ we denote by $L_\Phi \in \mathbb{N}_0$ the number of moment conditions the function provides, i.e. L_Φ is the highest number with

$$\int_{\mathbb{R}^n} x^\beta \Phi(x)dx = 0 \quad \text{with } |\beta| < L_\Phi, \tag{1}$$

which can equivalently be written as

$$D^\beta \hat{\Phi}(0) = 0 \quad \text{with } |\beta| < L_\Phi.$$

Please note that for $L_\Phi = 0$ the function Φ does not have any moment condition. If not otherwise stated, we define for a function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ the dyadic dilates by $\Phi_j(x) = 2^{jn}\Phi(2^j x)$ for $j \in \mathbb{N}$ and any $x \in \mathbb{R}^n$. We remark that Φ_0 is not covered by the construction above because it is usually realized with a different function Φ_0 which has different properties compared to Φ.

2.1 Besov and Triebel-Lizorkin Spaces with Variable Exponents

Here we introduce the spaces which we are interested in. We study Besov and Triebel-Lizorkin spaces with variable integrability and variable smoothness. We take advantage of the concept of admissible weight sequences to define the variable smoothness.

Definition 1 For fixed real numbers $\alpha \ge 0$ and $\alpha_1 \le \alpha_2$ the class of admissible weights $\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ is the collection of all positive weight sequences $\boldsymbol{w} = (w_j)_{j\in\mathbb{N}_0}$ on $\mathbb{R}^n$ with:

(i) There exists a constant $\mathrm{C} > 0$ such that for fixed $j \in \mathbb{N}_0$ and arbitrary $x, y \in \mathbb{R}^n$

$$0 < w_j(x) \le \mathrm{C} w_j(y)(1+2^j|x-y|)^\alpha;$$

(ii) For any $x \in \mathbb{R}^n$ and any $j \in \mathbb{N}_0$ we have

$$2^{\alpha_1} w_j(x) \le w_{j+1}(x) \le 2^{\alpha_2} w_j(x).$$

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\varphi_0(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\}. \tag{2}$$

Now define $\varphi(x) := \varphi_0(x) - \varphi_0(2x)$ and set $\varphi_j(x) := \varphi(2^{-j}x)$ for all $j \in \mathbb{N}$. Then the sequence $(\varphi_j)_{j\in\mathbb{N}_0}$ forms a smooth dyadic decomposition of unity, which means

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

For an open set $\Omega \subset \mathbb{R}^n$ we denote by $\mathcal{P}(\Omega)$ the class of exponents, which are measurable functions $p : \Omega \to (c, \infty]$ for some $c > 0$. Let $p \in \mathcal{P}(\Omega)$, then $p^+ := \text{ess–sup}_{x\in\Omega}\, p(x)$ and $p^- := \text{ess–inf}_{x\in\Omega}\, p(x)$. The set $L_{p(\cdot)}(\Omega)$ is the variable exponent Lebesgue space, which consists of all measurable functions f such that for some $\lambda > 0$ the modular $\varrho_{p(\cdot)}(f/\lambda)$ is finite. The modular is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\Omega_0} |f(x)|^{p(x)}\, dx + \text{ess-sup}_{x\in\Omega_\infty} |f(x)|.$$

Here Ω_∞ denotes the subset of Ω where $p(x) = \infty$ and $\Omega_0 = \Omega \setminus \Omega_\infty$. The Luxemburg (quasi-)norm of a function $f \in L_{p(\cdot)}(\Omega)$ is given by

$$\| f | L_{p(\cdot)}(\Omega)\| := \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1 \right\}.$$

In order to define the mixed spaces $\ell_{q(\cdot)}(L_{p(\cdot)}(\Omega))$, we need to define another modular. For $p, q \in \mathcal{P}(\Omega)$ and a sequence $(f_\nu)_{\nu\in\mathbb{N}_0}$ of complex-valued Lebesgue measurable functions on Ω, we define

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \inf \left\{ \lambda_\nu > 0 : \varrho_{p(\cdot)}\left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}}\right) \le 1 \right\}. \tag{3}$$

If $q^+ < \infty$, then we can replace (3) by the simpler expression

$$\varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \left\| |f_\nu|^{q(\cdot)} \mid L_{\frac{p(\cdot)}{q(\cdot)}}(\Omega) \right\|. \tag{4}$$

The (quasi-)norm in the $\ell_{q(\cdot)}(L_{p(\cdot)}(\Omega))$ spaces is defined as usual by

$$\| f_\nu \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\Omega))\| = \inf \left\{ \mu > 0 : \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\frac{f_\nu}{\mu}\right) \le 1 \right\}.$$

For the sake of completeness, we state also the definition of the space $L_{p(\cdot)}(\ell_{q(\cdot)}(\Omega))$. At first, one just takes the norm $\ell_{q(\cdot)}$ of $(f_\nu(x))_{\nu\in\mathbb{N}_0}$ for every $x \in \Omega$ and then the $L_{p(\cdot)}$-norm with respect to $x \in \Omega$, i.e.

$$\| f_\nu | L_{p(\cdot)}(\ell_{q(\cdot)}(\Omega))\| = \left\| \left(\sum_{\nu=0}^{\infty} |f_\nu(x)|^{q(x)} \right)^{1/q(x)} \right| L_{p(\cdot)}(\Omega) \Bigg\| .$$

Finally, we also give the definition of smoothness spaces for the exponents. To prove results for the spaces under consideration, like characterizations or the independence of the decomposition of unity, we need this extra regularity conditions for the exponents.

Definition 2 Let $g \in C(\Omega)$ be a continuous function on Ω.

(i) We say that g is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{loc}(\Omega)$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \le \frac{c_{\log}(g)}{\log(e + 1/|x-y|)}$$

holds for all $x, y \in \Omega$.

(ii) We say that g is *globally* log-*Hölder continuous*, abbreviated $g \in C^{\log}(\Omega)$, if g is locally log-Hölder continuous and there exists $g_\infty \in \mathbb{R}$ such that

$$|g(x) - g_\infty| \le \frac{c_{\log}}{\log(e + |x|)}$$

holds for all $x \in \Omega$.

The logarithmic Hölder regularity classes turned out to be sufficient to have the boundedness of the Hardy-Littlewood maximal operator on $L_{p(\cdot)}(\Omega)$ and for further properties we refer to [7] for details. We denote by $p \in \mathcal{P}^{\log}$ any exponent $p \in \mathcal{P}(\Omega)$ with $0 < p^- \le p^+ \le \infty$ and $1/p(\cdot) \in C^{\log}(\Omega)$.

Remark 1 The class $\mathcal{P}^{\log}$ is denoted without underlying class Ω. Having an exponent in $\mathcal{P}(\mathbb{R}^n)$ with $1/p \in C^{\log}(\mathbb{R}^n)$, we can always restrict it to an exponent on Ω. Further by [7, Proposition 4.1.7] we can always extend an exponent $p \in \mathcal{P}(\Omega)$ with $1/p \in C^{\log}(\Omega)$ to an exponent $\widetilde{p} \in \mathcal{P}(\mathbb{R}^n)$ with $1/\widetilde{p} \in C^{\log}(\mathbb{R}^n)$ without changing the numbers p^+, p^-, p_∞ and $c_{\log}(1/p)$.
So, in abuse of notation we always write $p \in \mathcal{P}^{\log}$ and mean either the exponent on $\mathbb{R}^n$ or on Ω, which share in any case the same properties.

Now, we are ready to give the definition of the variable exponent spaces which we are interested in.

Definition 3 Let $p, q \in \mathcal{P}^{\log}$, $(w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $(\varphi_j)_{j\in\mathbb{N}_0}$ a smooth decomposition of unity.

(i) The variable Besov space $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\| f | B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| := \left\| \left(w_j(\cdot) \left(\varphi_j \hat{f}\right)^\vee (\cdot) \right)_{j\in\mathbb{N}_0} \right| \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \Bigg\| < \infty.$$

(ii) For $p^+, q^+ < \infty$ the variable Triebel-Lizorkin space $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$\left\| f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| := \left\| \left(\sum_{j=0}^{\infty} |w_j(\cdot) \left(\varphi_j \hat{f}\right)^{\vee}(\cdot)|q(\cdot) \right)^{1/q(\cdot)} \Big| L_{p(\cdot)}(\mathbb{R}^n) \right\|$$
$$= \left\| \left(w_j(\cdot) \left(\varphi_j \hat{f}\right)^{\vee}(\cdot) \right)_{j \in \mathbb{N}_0} \Big| L_{p(\cdot)}(\ell_q(\mathbb{R}^n)) \right\|.$$

For brevity we write $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ where either $A = B$ or $A = F$.

First definitions of these spaces have been given in [10] and with $q(\cdot)$ also variable in the Besov case in [12]. Furthermore, there already exist a lot of characterizations of these scales of spaces: namely by local means in [10], by atoms, molecules and wavelets in [6, 11], by ball means of differences in [12] and recently by non-smooth atoms in [8]. These characterizations also show the independence of the (quasi)norms above of the chosen start function $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ for the decomposition of unity. If one chooses $0 < p, q \leq \infty$ as constants and sets $w_j(x) = 2^{js}$ with $s \in \mathbb{R}$ then one recovers the usual Besov and Triebel-Lizorkin spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ studied in great detail in [17, 18, 20].

Furthermore, by choosing the weight sequence as $w_j(x) = 2^{js(x)}$ with $s \in C^{\log}_{loc}(\mathbb{R}^n)$ we obtain the scales of Besov and Triebel-Lizorkin spaces with variable smoothness and integrability $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ which have been studied in [2, 4].

2.2 Local Means Characterization

Our approach to obtain intrinsic characterizations and an extension operator for $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ for an special Lipschitz domain $\Omega \subset \mathbb{R}^n$ heavily relies on the characterization by local means. To this end, we repeat this characterization for our spaces under consideration from [10, 12]. The crucial tool will be the Peetre maximal operator which assigns to each system $(\Psi_k)_{k \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$, to each distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ and to each number $a > 0$ the following quantities

$$(\Psi_k^* f)_a(x) := \sup_{y \in \mathbb{R}^n} \frac{|(\Psi_k * f)(y)|}{(1 + |2^k(y - x)|)^a}, \quad x \in \mathbb{R}^n \text{ and } k \in \mathbb{N}_0. \tag{5}$$

We start with two given functions $\Psi_0, \Psi_1 \in \mathcal{S}(\mathbb{R}^n)$. We define

$$\Psi_j(x) = 2^{(j-1)n} \Psi_1(2^{(j-1)} x), \quad \text{for } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}.$$

The local means characterization for $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ from [10, 12] then reads.

Proposition 1 *Let $\boldsymbol{w} = (w_k)_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $p, q \in \mathcal{P}^{\log}$ and let $a > 0$, $R \in \mathbb{N}_0$ with $R > \alpha_2$. Further, let Ψ_0, Ψ_1 belong to $\mathcal{S}(\mathbb{R}^n)$ with*

$$\int_{\mathbb{R}^n} x^{\beta}\Psi_1(x)dx = 0, \quad \textit{for } 0 \le |\beta| < R, \tag{6}$$

and

$$|\hat{\Psi}_0(x)| > 0 \quad \textit{on} \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \tag{7}$$

$$|\hat{\Psi}_1(x)| > 0 \quad \textit{on} \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \tag{8}$$

for some $\varepsilon > 0$.

(i) For $a > \frac{n}{p^-} + c_{log}(1/q) + \alpha$ and all $f \in \mathcal{S}'(\mathbb{R}^n)$ we have

$$\left\| f \,|\, B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\right\| \approx \left\| (\Psi_k * f) w_k \,|\, \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\right\| \approx \left\| (\Psi_k^* f)_a w_k \,\middle|\, \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))\right\|.$$

(ii) For $p^+, q^+ < \infty$ and $a > \frac{n}{\min(p^-,q^-)} + \alpha$ we have for all $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\left\| f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\right\| \approx \left\| w_k(\Psi_k * f) \,|\, L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\right\| \approx \left\| w_k(\Psi_k^* f)_a \,\middle|\, L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))\right\|.$$

The above local means characterization alone only ensures the independence of the chosen decomposition of unity $(\varphi_j)_{j\in\mathbb{N}_0}$ if it is constructed as an, so called, admissible pair, see [1, Sect. 3]. But anyhow, since we also have further characterizations of the spaces $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ by atoms, wavelets and differences, the independence in Definition 3 of the decomposition of unity is justified.

Remark 2 (i) One can rewrite (6) also in $D^{\beta}\hat{\Psi}_1(0) = 0$ for all $|\beta| < R$ or, using our notation, in $L_{\Psi_1} = R$.

(ii) Later assertions are done with only one startfunction $\Phi_0 \in \mathcal{D}(\Omega)$ with $\int_{\mathbb{R}^n} \Phi_0(x)dx \neq 0$. From that function one constructs as usual $\Phi(x) = \Phi_0(x) - 2^{-n}\Phi_0(x/2)$ and sets $\Phi_1(x) = 2^n\Phi(2x)$.

Since $\Phi_0 \in \mathcal{D}(\Omega) \subset \mathcal{S}(\mathbb{R}^n)$ is smooth, we can find an $\varepsilon > 0$ such that $|\hat{\Phi}_0(x)| > 0$ on $\{x \in \mathbb{R}^n : |x| < \varepsilon\}$ is satisfied. Further, also $\Phi_1 \in \mathcal{S}(\mathbb{R}^n)$ fulfills $|\hat{\Phi}_1(x)| > 0$ on $\{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\}$ and therefore (7) and (8) hold with Φ_0 and Φ_1 instead of the Ψ_0 and Ψ_1. This also shows, that we can take the functions $\Phi_j(x) = 2^{jn}\Phi(2^j x) = 2^{(j-1)n}\Phi_1(2^{j-1}x)$ and Φ_0 as basic functions in Proposition 1.

3 Function Spaces on Special Lipschitz Domains

We say that $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is a special Lipschitz domain if it is open and there exists a constant $A > 0$ with

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\}$$

and $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz continuous

$$|\omega(x') - \omega(y')| \leq A|x' - y'|.$$

The function spaces from Sect. 2.1 can be used to define them on domains with the help of Definition 3 by restriction.

As usual $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ stands for the space of infinitely often differentiable functions with compact support in Ω. Let $\mathcal{D}'(\Omega)$ be the dual space of distributions on Ω. For $g \in \mathcal{S}'(\mathbb{R}^n)$ we denote by $g|_\Omega$ its restriction to Ω,

$$g|_\Omega : \quad (g|_\Omega)(\varphi) = g(\varphi) \text{ for all } \varphi \in \mathcal{D}(\Omega).$$

Definition 4 Let $\Omega \subset \mathbb{R}^n$ be a special Lipschitz domain as above. Let $p, q \in \mathcal{P}^{\log}$, $(w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ and $(\varphi_j)_{j\in\mathbb{N}_0}$ be a smooth decomposition of unity.

(i) The variable Besov space $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ on Ω is the collection of all $f \in \mathcal{D}'(\Omega)$ such that there exists a $g \in B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with $g|_\Omega = f$. Furthermore

$$\left\| f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega) \right\| := \inf \left\{ \left\| g \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| : g|_\Omega = f \right\}.$$

(ii) For $p^+, q^+ < \infty$ the variable Triebel-Lizorkin space $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ on Ω is the collection of all $f \in \mathcal{D}'(\Omega)$ such that there exists a $g \in F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with $g|_\Omega = f$. Furthermore

$$\left\| f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega) \right\| := \inf \left\{ \left\| g \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| : g|_\Omega = f \right\}.$$

Remark 3 Usually, one defines function spaces on bounded Lipschitz domains Ω. Then one reduces the proofs and assertions by the localization procedure to special Lipschitz domains. This is done by covering $\partial\Omega$ by finitely many balls B_j and using a decomposition of unity Φ_j which is adapted to the balls B_j. Finally, using pointwise multipliers and rotations (diffeomorphisms) all occurring tasks can be reduced to the case of special Lipschitz domains as described above, see [14, 19] for details.

To the best of the authors knowledge there are no results on diffeomorphisms known if the exponents $p(\cdot), q(\cdot)$ are not constant. So we concentrate our studies only on special Lipschitz domains as above, and leave the case of bounded Lipschitz domains for further research.

4 Intrinsic Characterizations and the Extension Operator

In this section we prove our main results. We give an intrinsic characterization of the spaces from Definition 4 with the help of an adapted Peetre maximal operator

$$(\Phi_k^* f)_a^{\Omega}(x) := \sup_{y\in\Omega} \frac{|(\Phi_k * f)(y)|}{(1+|2^k(y-x)|)^a}, \quad x \in \Omega \text{ and } k \in \mathbb{N}_0. \tag{9}$$

Here $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is a special Lipschitz domain i.e.

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\},$$

where

$$|\omega(x') - \omega(y')| \leq A|x' - y'| \quad \text{for all} x', y' \in \mathbb{R}^{n-1}.$$

By K we denote the cone adapted to the special Lipschitz domain with

$$K = \{(x', x_n) \in \mathbb{R}^n : |x'| < A^{-1}x_n\}. \tag{10}$$

This cone has the property that $x + K \in \Omega$ for all $x \in \Omega$ and we denote by $-K = \{-x : x \in K\}$ the reflected cone. The crucial property is now that for all $\gamma \in \mathcal{D}(-K)$ and all $f \in \mathcal{D}'(\Omega)$ the convolution $(\gamma * f)(x) = \langle \gamma(x - \cdot), f\rangle$ is well defined in Ω, since $\operatorname{supp} \gamma(x - \cdot) \subset \Omega$ for all $x \in \Omega$.

Before coming to the intrinsic characterization and the extension operator we state two useful results which are needed later on. First we need a version of Calderon reproducing formula which was proved in [14, Proposition 2.1].

Lemma 1 *Let* $\Phi_0 \in \mathcal{D}(-K)$ *with* $\int_{\mathbb{R}^n} \Phi_0(x)dx \neq 0$ *be given. Further assume that* $\Phi(x) = \Phi_0(x) - 2^{-n}\Phi_0(x/2)$ *fulfills*

$$\int_{\mathbb{R}^n} x^{\beta}\Phi(x)dx = 0 \qquad \textit{for } |\beta| < L_{\Phi}. \tag{11}$$

Then for any given $L_{\Psi} \in \mathbb{N}_0$ *there exist functions* $\Psi_0, \Psi \in \mathcal{D}(-K)$ *with*

$$\int_{\mathbb{R}^n} x^{\beta}\Psi(x)dx = 0 \qquad \textit{for } |\beta| < L_{\Psi} \tag{12}$$

and for all $f \in \mathcal{D}'(\Omega)$ *we have the identity*

$$f = \sum_{j=0}^{\infty} \Psi_j * \Phi_j * f \quad \textit{in } \mathcal{D}'(\Omega). \tag{13}$$

The second lemma is a Hardy type inequality for the mixed variable spaces. Its proof can be found in [12, Lemma 9] and in [1, Lemma 3.4].

Lemma 2 *Let* $p, q \in \mathcal{P}(\mathbb{R}^n)$ *and* $\delta > 0$. *For a sequence* $(h_j)_{j\in\mathbb{N}_0}$ *of measureable functions we denote*

$$H_l(x) = \sum_{j=0}^{\infty} 2^{-|j-l|\delta} h_j(x).$$

Then there exist constants $C_1, C_2 > 0$ *depending on* $p(\cdot), q(\cdot)$ *and* δ *with*

$$\begin{aligned} \left\| H_l | \ell_{q(\cdot)}(L_{p(\cdot)}) \right\| &\le C_1 \left\| h_l | \ell_{q(\cdot)}(L_{p(\cdot)}) \right\| \\ \left\| H_l | L_{p(\cdot)}(\ell_{q(\cdot)}) \right\| &\le C_2 \left\| h_l | L_{p(\cdot)}(\ell_{q(\cdot)}) \right\| . \end{aligned}$$

Now we are ready to formulate our first main theorem about a linear extension operator.

Theorem 1 *Let* $p, q \in \mathcal{P}^{\log}$ *(with* $p^+, q^+ < \infty$ *in the F-case) and* $(w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$. *Further, let* $\Phi_0 \in \mathcal{D}(-K)$ *with* $\int \Phi_0(x)dx \neq 0$ *be given and we assume that* $\Phi(x) = \Phi_0(x) - 2^{-n}\Phi_0(x/2)$ *satisfies* $L_\Phi > \alpha_2$.
Construct $\Psi_0, \Psi \in \mathcal{D}(-K)$ *with* $L_\Psi > \frac{n}{\min(p^-,q^-)} + c_{\log}(1/q) + \alpha - \alpha_1$ *as in Lemma 1 with*

$$f = \sum_{j=0}^{\infty} \Psi_j * \Phi_j * f \quad \text{in } \mathcal{D}'(\Omega).$$

For any $g : \Omega \to \mathbb{R}$ *denote by* g_Ω *its extension from* Ω *to* $\mathbb{R}^n$ *by zero. Then the map* $\mathcal{E} : \mathcal{D}'(\Omega) \to \mathcal{S}'(\mathbb{R}^n)$ *with*

$$f \mapsto \sum_{j=0}^{\infty} \Psi_j * (\Phi_j * f)_\Omega \tag{14}$$

is a linear and bounded extension operator from $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ *to* $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

In more detail, the theorem claims that the series (14) converges in $\mathcal{S}'(\mathbb{R}^n)$ for any $f \in A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ to an $\mathcal{E}f \in \mathcal{S}'(\mathbb{R}^n)$ with:

- $\mathcal{E}f|_\Omega = f$ in the sense of $\mathcal{D}'(\Omega)$;
- $\left\| \mathcal{E}f | A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| \le c \left\| f | A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega) \right\|$ for any $f \in A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$.

The theorem above is directly connected to the question of an intrinsic characterization of the spaces $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$, which will be solved in the next theorem.

Theorem 2 *Let* $p, q \in \mathcal{P}^{\log}$ *and* $(w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$. *Further, let* $\Phi_0 \in \mathcal{D}(-K)$ *be given with* $\int \Phi_0(x)dx \neq 0$ *and* $L_\Phi > \alpha_2$, *where* $\Phi(x) = \Phi_0(x) - 2^{-n}\Phi_0(x/2)$.

(i) For $a > \frac{n}{p^-} + c_{\log}(1/q) + \alpha$ *and any* $f \in \mathcal{D}'(\Omega)$

$$\left\| f \,|\, B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega) \right\| \approx \left\| \left(w_k (\Phi_k^* f)_a^{\Omega}(\cdot)\right)_{k\in\mathbb{N}_0} \middle| \ell_{q(\cdot)}(L_{p(\cdot)}(\Omega)) \right\|$$

(ii) For $a > \frac{n}{\min(p^-,q^-)} + \alpha$, $p^+, q^+ < \infty$ *and any* $f \in \mathcal{D}'(\Omega)$

$$\begin{aligned} \left\| f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega) \right\| &\approx \left\| \left(w_k (\Phi_k^* f)_a^{\Omega}(\cdot)\right)_{k\in\mathbb{N}_0} \middle| L_{p(\cdot)}(\ell_{q(\cdot)}(\Omega)) \right\| \\ &= \left\| \left(\sum_{k=0}^{\infty} |w_k(\cdot)(\Phi_k^* f)_a^{\Omega}(\cdot)|^{q(\cdot)} \right)^{1/q(\cdot)} \middle| L_{p(\cdot)}(\Omega) \right\|. \end{aligned} \tag{15}$$

Proof Theorems 1 and 2 are so closely connected that they will both be proved in one proof. As usual we restrict to the F-case and outline the necessary modifications for the B-case. By Remark 2 we have the local means characterization from Proposition 1 with the functions Φ_0 and Φ_j constructed from Φ_0.

First step: We show $\left\| f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega) \right\| \geq c \left\| \left(\sum_{k=0}^{\infty} |w_k(\cdot)(\Phi_k^* f)_a^{\Omega}(\cdot)|^{q(\cdot)}\right)^{1/q(\cdot)} \middle| L_{p(\cdot)}(\Omega) \right\|$. This is an easy consequence of the characterization from Proposition 1 using

$$(\Phi_k^* f)_a^{\Omega}(x) \leq (\Phi_k^* g)_a(x) \quad \text{on } \Omega \text{ if } g|_{\Omega} = f.$$

Second step: We denote the right hand side of (15) by $\|f\|$. We show if the $\Psi \in \mathcal{D}(-K)$ from Lemma 1 satisfies $L_{\Psi} > a - \alpha_1$, then for every $f \in \mathcal{D}'(\Omega)$ with $\|f\| < \infty$ the series in (14) converges in $\mathcal{S}'(\mathbb{R}^n)$. Furthermore, the limit $\mathcal{E}f$ satisfies

$$\mathcal{E}f|_{\Omega} = f, \quad \mathcal{E}f \in F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \text{ and } \left\| \mathcal{E}f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| \leq c\|f\|.$$

Having this proven, we see that this step actually proves Theorem 1 and gives us the $\leq$ estimate in (15) and therefore finishes the proof of Theorem 2 as well.

Substep 2.1: We denote by $X = X^{\boldsymbol{w},a}_{p(\cdot),q(\cdot)}$ the space of all sequences $(g^j)_{j\in\mathbb{N}_0}$ of measurable functions $g_j : \mathbb{R}^n \to [0,\infty)$ with

$$\|(g^j)\|_X = \left\| \left(\sum_{j=0}^{\infty} |w_j G^j|^{q(\cdot)} \right)^{1/q(\cdot)} \middle| L_{p(\cdot)}(\mathbb{R}^n) \right\|,$$

where

$$G^j(x) = \sup_{y\in\mathbb{R}^n} \frac{g^j(y)}{(1+2^j|x-y|)^a}.$$

We claim that if $L_\Psi > a - \alpha_1$, then the series $\sum_{j=0}^{\infty} \Psi_j * g^j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ and we can find a constant $c > 0$ such that for any sequence $(g^j) \in X$

$$\left\| \sum_{j=0}^{\infty} \Psi_j * g^j \,\middle|\, F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| \le c \|(g^j)\|_X. \tag{16}$$

To prove (16) we can use the same pointwise estimates as in the proof in [14]. By using the moment conditions on Φ and Ψ we get using Taylors formula and the compact support of Φ and Ψ

$$|\Phi_l * \Psi_j * g^j(x)| \le I^a_{l,j} G^j(x)$$

with

$$I^a_{j,l} = \int_{\mathbb{R}^n} |(\Phi_l * \Psi_j)(z)|(1 + 2^j|z|)^a dz \le c \begin{cases} 2^{(l-j)(L_\Psi - a)}, & \text{for } j \ge l \\ 2^{(j-l)L_\Phi}, & \text{for } j \le l \end{cases}. \tag{17}$$

We use the properties of admissible weight sequences and get

$$w_l(x) \le c w_j(x) \begin{cases} 2^{-\alpha_1(j-l)}, & \text{for } j \ge l \\ 2^{\alpha_2(l-j)}, & \text{for } j \le l \end{cases}$$

and obtain with $\delta = \min(L_\Psi - a + \alpha_1, L_\Phi - \alpha_2) > 0$

$$w_l(x)|\Phi_l * \Psi_j * g^j(x)| \le c w_j(x) 2^{-|j-l|\delta} G^j(x). \tag{18}$$

Now we use the same arguments as in [14] to finish the proof. If $\|(g^j)\|_X < \infty$, then each g^j is a function of at most polynomial growth. Therefore we have $\Psi_j * g^j \in \mathcal{S}'(\mathbb{R}^n)$ and with $\widetilde{w}_l(x) = 2^{-l2\delta} w_l(x)$ we obtain from (18)

$$\begin{aligned} \left\| \Psi_j * g^j \,\middle|\, F^{\widetilde{\boldsymbol{w}}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| &\le c \left\| \left(\sum_{l=0}^{\infty} \left| 2^{-2l\delta} 2^{-|j-l|\delta} w_j(\cdot) G^j(\cdot) \right|^{q(\cdot)} \right)^{1/q(\cdot)} \middle| L_{p(\cdot)}(\mathbb{R}^n) \right\| \\ &\le c \left(\sum_{l=0}^{\infty} \left| 2^{-2l\delta} 2^{-|j-l|\delta} \right|^{q^-} \right)^{1/q^-} \left\| w_j(\cdot) G^j(\cdot) \,\middle|\, L_{p(\cdot)}(\mathbb{R}^n) \right\| \\ &\le c 2^{-j\delta} \left\| w_j(\cdot) G^j(\cdot) \,\middle|\, L_{p(\cdot)}(\mathbb{R}^n) \right\| \le c 2^{-j\delta} \|(g^j)\|_X, \end{aligned}$$

where we used $|l - j| \ge j - l$ and $\ell_{q^-} \hookrightarrow \ell_{q(\cdot)}$. Hence, $\sum_{j=0}^{\infty} \Psi_j * g^j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ due to $F^{\widetilde{\boldsymbol{w}}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and we get from (18) the estimate

$$w_l(x)\left|\Phi_l*\left(\sum_{j=0}^{\infty}\Psi_j*g^j\right)(x)\right|\le c\sum_{j=0}^{\infty}2^{-|j-l|\delta}w_j(x)G^j(x). \tag{19}$$

Now, using Lemma 2 with $h_j(x)=w_j(x)G^j(x)$ we conclude from (19)

$$\left\|\sum_{j=0}^{\infty}\Psi_j*g^j\,\middle|\,F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\right\|\le c\|(g^j)\|_X. \tag{20}$$

Substep 2.2: Finally, we argue as follows to apply our general result (16) to the extension operator from Theorem 1. If $x\in\Omega$, then we have $\sup_{y\in\Omega}\frac{|(\Phi_j*f)(y)|}{(1+2^j|x-y|)^a}=(\Phi_j^*f)_a^{\Omega}(x)$ by definition. If $x\notin\bar{\Omega}$ we can construct a point $\widetilde{x}=(x',2\omega(x')-x_n)\in\Omega$ which is symmetric to $x\notin\bar{\Omega}$ with respect to $\partial\Omega$ in the sense $|\widetilde{x}_n-\omega(x')|=|\omega(x')-x_n|$. Then, by $|\widetilde{x}-y|\le B|x-y|$ for all $y\in\Omega$, with B depending on the Lipschitz constant A, we obtain $\sup_{y\in\Omega}\frac{|(\Phi_j*f)(y)|}{(1+2^j|x-y|)^a}\le c(\Phi_j^*f)_a^{\Omega}(\widetilde{x})$ for $x\notin\bar{\Omega}$. So we have the estimate

$$\|(\Phi_j*f)_\Omega\|_X\le c\left\|\left(\sum_{k=0}^{\infty}|w_k(\cdot)(\Phi_k^*f)_a^{\Omega}(\cdot)|^{q(\cdot)}\right)^{1/q(\cdot)}\middle|L_{p(\cdot)}(\Omega)\right\|\ \text{for all } f\in\mathcal{D}'(\Omega).$$

Combining this with (16), we have for all $f\in\mathcal{D}'(\Omega)$ with $\|f\|<\infty$ that $\mathcal{E}f\in\mathcal{S}'(\mathbb{R}^n)$ and

$$\left\|\mathcal{E}f\,|\,F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\right\|\le c\left\|\left(\sum_{k=0}^{\infty}|w_k(\cdot)(\Phi_k^*f)_a^{\Omega}(\cdot)|^{q(\cdot)}\right)^{1/q(\cdot)}\middle|L_{p(\cdot)}(\Omega)\right\|.$$

Finally, the supports of Ψ_0 and Ψ lie within $-K$ and therefore we obtain using Lemma 1

$$\mathcal{E}f|_\Omega=\sum_{j=0}^{\infty}\Psi_j*\Phi_j*f=f,$$

which completes the proof in the F-case.

Third step: We can use the same reasoning as above for the B-case. The only difference is in the use of Proposition 1, where the condition on $a>0$ is different in the B-case. This also explains now the condition on L_Ψ in Theorem 1, where we have just taken a maximal value for $a>0$. □

It is also possible to get an intrinsic characterization of $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ by using just the convolutions $\Phi_j * f$ instead of the maximal functions $(\Phi^*_j f)^\Omega_a$ as in the local means characterization in Proposition 1. To that end, we introduce the space $\mathcal{S}'(\Omega)$ as subspace of $\mathcal{D}'(\Omega)$ by restriction as

$$\mathcal{S}'(\Omega) := \{f \in \mathcal{D}'(\Omega) : \exists c_f, M_f > 0 \text{ with } |\langle f, \gamma\rangle| \le c_f \|\gamma\|_{M_f},\ \forall \gamma \in \mathcal{D}(\Omega)\}$$

where

$$\|\gamma\|_{M_f} = \sup_{y\in\Omega, |\beta|\le M_f} |D^\beta \gamma(y)|(1+|y|)^{M_f}.$$

From [14, Proposition 3.1] we have the following characterization of this class.

Proposition 2 *We have $f \in \mathcal{S}'(\Omega)$ if and only if there exists a $g \in \mathcal{S}'(\mathbb{R}^n)$ such that $g|_\Omega = f$.*

Remark 4 Since all appearing function spaces $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ are also defined by restriction we have $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega) \subset \mathcal{S}'(\Omega)$. Therefore, the proposition above shows that it is no restriction to use $f \in \mathcal{S}'(\Omega)$ instead of $f \in \mathcal{D}'(\Omega)$.

Furthermore, we also need another lemma which can be seen as the replacement for the boundedness of the Hardy-Littlewood maximal operator which is of no use in our variable exponent spaces. We refer to [2, 4] for the proofs of this lemma.

Lemma 3 *Let $p, q \in \mathcal{P}^{\log}$ and $\eta_{\nu,m}(x) = 2^{n\nu}(1+2^\nu|x|)^{-m}$.*

(i) If $p^- \ge 1$ and $m > n + c_{\log}(1/q)$, then there exists a constant $c > 0$ such that for all sequences $(f_\nu)_{\nu\in\mathbb{N}_0} \in \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))$

$$\left\| f_\nu * \eta_{\nu,m} \right| \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \big\| \le c \left\| f_\nu \right| \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n)) \big\| .$$

(ii) If $1 < p^- \le p^+ < \infty$ and $1 < q^- \le q^+ < \infty$ and $m > n$, then there exists a constant $c > 0$ such that for all sequences $(f_\nu)_{\nu\in\mathbb{N}_0} \in L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n))$

$$\left\| f_\nu * \eta_{\nu,m} \right| L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n)) \big\| \le c \left\| f_\nu \right| L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbb{R}^n)) \big\| .$$

Now, the local means intrinsic characterization for the spaces $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ reads as follows.

Theorem 3 *Let $p, q \in \mathcal{P}^{\log}$ and $(w_j) \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$. Further, let $\Phi_0 \in \mathcal{D}(-K)$ be given with $\int \Phi_0(x)dx \ne 0$ and $L_\Phi > \alpha_2$, where $\Phi(x) = \Phi_0(x) - 2^{-n}\Phi_0(x/2)$.*

(i) For all $f \in \mathcal{S}'(\Omega)$ we have

$$\left\| f \right| B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega) \big\| \approx \left\| (w_k(\Phi_k * f)(\cdot))_{k\in\mathbb{N}_0} \right| \ell_{q(\cdot)}(L_{p(\cdot)}(\Omega)) \big\|$$

(ii) For $p^+, q^+ < \infty$ and all $f \in \mathcal{S}'(\Omega)$ we have

$$\left\| f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega) \right\| \approx \left\| \left(\sum_{k=0}^{\infty} |w_k(\cdot)(\Phi_k * f)(\cdot)|^{q(\cdot)} \right)^{1/q(\cdot)} \middle| L_{p(\cdot)}(\Omega) \right\|.$$

Proof Clearly, we want to take the intrinsic norm given in Theorem 2 as a starting point. To use this characterization we need $L_\Phi > \alpha_2$ and choose suitable functions Ψ_0, Ψ which fulfill (1) with $L_\Psi > a - \alpha_1$. Furthermore, we take the $a > 0$ as large as needed in Theorem 2.
First step: The $\geq$ inequality follows easily by observing $(\Phi_k^* f)_a^\Omega(x) \geq (\Phi_k * f)(x)$.
Second step: One way to prove the $\leq$ inequality would be to consult the proof of [12, Theorem 13] and to modify everything from $\mathbb{R}^n$ to Ω. Instead we use formula (3.4) in [14]

$$|(\Phi_j * f)(x)|^r \leq c \sum_{k=j}^{\infty} 2^{(j-k)L_\Psi r} 2^{kn} \int_\Omega \frac{|(\Phi_k * f)(y)|^r}{(1 + 2^j|x-y|)^{ar}} dy \qquad (21)$$

which was obtained by pointwise manipulations only. Here $r > 0$ and the constant $c > 0$ is independent of $f \in \mathcal{S}'(\Omega)$, $x \in \Omega$ and $j \in \mathbb{N}_0$.

Now, dividing (21) by $(1 + 2^j|x-z|)^{ar}$ and using on the left hand side $1 + 2^j|y - z| \leq (1 + 2^j|x-z|)(1 + 2^j|x-y|)$ gives us by taking the supremum with respect to $x \in \Omega$

$$\left((\Phi_j^* f)_a^\Omega(z)\right)^r \leq c \sum_{k=j}^{\infty} 2^{(j-k)L_\Psi r} 2^{kn} \int_\Omega \frac{|(\Phi_k * f)(y)|^r}{(1 + 2^j|y-z|)^{ar}} dy$$

We multiply with $w_j(z)^r$ and use the estimates $(1 + 2^k|y-z|)^{ar} \leq 2^{(k-j)ar}(1 + 2^j|y-z|)^{ar}$ and $w_j(z) \leq C 2^{(j-k)\alpha_1} w_k(y)(1 + 2^k|y-z|)^\alpha$ and obtain

$$\left(w_j(z)(\Phi_j^* f)_a^\Omega(z)\right)^r \leq c \sum_{k=j}^{\infty} 2^{(j-k)(L_\Psi - a + \alpha_1)r} 2^{kn} \int_\Omega \frac{w_k^r(y)|(\Phi_k * f)(y)|^r}{(1 + 2^k|y-z|)^{(a-\alpha)r}} dy$$

which can be rewritten with $\delta = L_\Psi - a + \alpha_1 > 0$ in

$$\left(\chi_\Omega(z) w_j(z)(\Phi_j^* f)_a^\Omega(z)\right)^r \leq c \sum_{k=j}^{\infty} 2^{(j-k)\delta r} \left[(\chi_\Omega w_k (\Phi_k * f))^r * \eta_{k,(a-\alpha)r} \right](z). \qquad (22)$$

Now, we use the usual procedure to end the proof. In the F-case we choose $r > 0$ with $\frac{n}{a-\alpha} < r < \min(p^-, q^-)$. This is possible due to the conditions of the theorem and we get $p/r, q/r \in \mathcal{P}^{\log}$ with $1 < p^-/r \leq p^+/r < \infty, 1 < q^-/r \leq q^+/r < \infty$. Applying the $L_{p(\cdot)/r}(\ell_{q(\cdot)/r}(\mathbb{R}^n))$ norm on (22) we conclude by using Lemmas 2 and 3

$$\begin{aligned}\left\| w_j(z)(\Phi_j^* f)_a^{\Omega}(z)\right| L_{p(\cdot)}(\ell_{q(\cdot)}(\Omega))\Big\|^r &= \left\| \left(\chi_\Omega(z) w_j(z)(\Phi_j^* f)_a^{\Omega}(z)\right)^r \right| L_{p(\cdot)/r}(\ell_{q(\cdot)/r}(\mathbb{R}^n))\Big\| \\ &\leq c \left\| (\chi_\Omega w_k(\Phi_k * f))^r * \eta_{k,(a-\alpha)r} \right| L_{p(\cdot)/r}(\ell_{q(\cdot)/r}(\mathbb{R}^n))\Big\| \\ &\leq c \left\| (\chi_\Omega w_k(\Phi_k * f))^r \right| L_{p(\cdot)/r}(\ell_{q(\cdot)/r}(\mathbb{R}^n))\Big\| \\ &= c \left\| w_k(\Phi_k * f) \right| L_{p(\cdot)}(\ell_{q(\cdot)}(\Omega))\Big\|^r .\end{aligned}$$

This finishes the proof in the F-case using Theorem 2. In the B-case the same reasoning by taking the $\ell_{q(\cdot)/r}(L_{p(\cdot)/r}(\mathbb{R}^n))$ norm of (22) works; only the parameter $r > 0$ has to be chosen as

$$\frac{n}{a-\alpha-c_{\log}(1/q)} < r < p^- \quad \text{where we used } c_{\log}(r/q) = r c_{\log}(1/q).$$

□

5 A Universal Extension Operator

The extension operator $\mathcal{E}$ from Theorem 1 has the serious drawback that it only works for special values of $p(\cdot), q(\cdot)$ and $\alpha_1, \alpha_2, \alpha$. This is due to the fact that all conditions depend on the number of moments we have for the functions Φ and Ψ. More precisely, we know that for fixed numbers of moments L_Φ, L_Ψ the extension operator works for

$$L_\Phi > \alpha_2 \qquad \text{and} \qquad L_\Psi > \frac{n}{\min(p^-, q^-)} + c_{\log}(1/q) + \alpha - \alpha_1 .$$

A good try to widen this region would be to choose $\Phi, \Psi \in \mathcal{D}(-K)$ with $L_\Psi = L_\Phi = \infty$, but clearly this is impossible. Fortunately, this can be done if $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ which are not compactly supported in $-K$, but have support in $-K$ and rapid decay at infinity.

Theorem 4 *(i) There exist functions $\Phi_0, \Phi, \Psi_0, \Psi \in \mathcal{S}(\mathbb{R}^n)$ with supports in $-K$ with $L_\Psi = L_\Phi = \infty$ and*

$$f = \sum_{k=0}^{\infty} \Psi_k * \Phi_k * f \qquad \textit{holds for all } f \in \mathcal{S}'(\Omega).$$

(ii) The map $\mathcal{E}_u : \mathcal{S}'(\Omega) \to \mathcal{S}'(\mathbb{R}^n)$ defined with the functions from (i) by

$$f \mapsto \sum_{k=0}^{\infty} \Psi_k * (\Phi_k * f)_\Omega$$

yields a linear bounded extension operator from $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\Omega)$ to $A^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ for all admissible exponents $p(\cdot), q(\cdot)$ and $(w_j)_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$.

The proof of this theorem can be copied word by word from the proof of [14, Theorem 4.1]. The crucial part there is to construct the needed functions $\Phi_0, \Phi, \Psi_0, \Psi \in \mathcal{S}(\mathbb{R}^n)$ with supports in $K = \{(x', x_n) \in \mathbb{R}^n : |x'| < A^{-1}x_n\}$ with $L_\Psi = L_\Phi = \infty$ which consists in a modification of Stein's function [16, Sect. 6.3]. Finally, with that functions satisfying Calderon's reproducing formula one has to revisit the proof of Theorem 1. Actually, there is only one difficulty to overcome: we estimated in (17) by using the compact support of the functions $\Phi_0, \Phi, \Psi_0, \Psi \in \mathcal{S}(\mathbb{R}^n)$. Since we do not have any compact support of these functions anymore we have to use [3, Lemma 2.1] and the same estimate (17) can be achieved.

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The Boundedness of Sublinear Operators in Weighted Morrey Spaces Defined on Spaces of Homogeneous Type

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Abstract The boundedness of sublinear integral operators in weighted Morrey spaces defined on spaces of homogeneous type is established under the Muckenhoupt conditions on weights. These operators involve Hardy-Littlewood and fractional maximal operators, Calderón-Zygmund operators, potential operators, etc. The boundedness problem for commutators of sublinear operators is also studied. Applications to estimates for hypoelliptic operators in weighted Morrey spaces defined on nilpotent Lie groups are also given.

Keywords Sublinear operators · Weighted Morrey spaces · Spaces of homogeneous type · Weighted inequality · Singular integrals · Fractional integrals · Homogeneous groups · Hypoelliptic operators

1 Preliminaries

In the paper we establish the boundedness of sublinear integral operators and their commutators in weighted Morrey spaces defined with respect to the Muckenhoupt weights. The function spaces under consideration are defined on quasi–metric measure spaces with doubling measure (spaces of homogeneous type, briefly, SHT). Generally speaking, sublinear operators involve many interesting operators of Harmonic Analysis such as the Hardy-Littlewood maximal and Calderón-Zygmund oper-

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ators, C. Fefferman singular multipliers, R. Fefferman singular integrals, Ricci-Stein oscillatory singular integrals, fractional integrals, the Bochner-Riesz means, etc. Similar problems for Euclidean spaces were studied in [21] (see also [16] for the diagonal case). The boundedness of some operators in Morrey spaces with Muckenhoupt weights via extrapolation techniques was established in [20].

Finally we give applications of some of the derived results to estimates for hypoelliptic operators in weighted Morrey spaces defined on homogeneous groups.

Morrey spaces $L^{p,\lambda}$ defined on Euclidean spaces were introduced in 1938 by C. Morrey [14] in connection with regularity of solutions to partial differential equations, and provided a useful tool in the regularity theory of PDEs. In [18] an overview of various generalizations of Morrey spaces can be found. The boundedness of integral operators of Harmonic Analysis in weighted Morrey spaces defined on Euclidean spaces first was studied in [11].

Let (X, d, μ) be a quasi-metric measure space (briefly, QMMS) with a quasi-metric d and measure μ. A quasi-metric d is a function $d : X \times X \to [0, \infty)$ which satisfies the following conditions:

(a) $d(x, y) = 0$ if and only if $x = y$.

(b) There is a constant $a_0 > 0$ such that $d(x, y) \le a_0 d(y, x)$ for all $x, y \in X$.

(c) There is a constant $a_1 > 0$ such that $d(x, y) \le a_1(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

If μ satisfies the doubling condition, i.e., if there is a positive constant b such that for all $x \in X$ and $r > 0$,

$$\mu B(x, 2r) \le b\mu B(x, r),$$

then QMMS (X, d, μ) is called a space of homogeneous type (SHT).

It is known (see [12]) that for any quasi-metric space (X, d) there is a continuous quasi-metric ρ on X which is equivalent to d such that all balls corresponding to ρ are open in the topology induced by ρ, and there exist constants C and $\theta \in (0, 1)$ such that for all $x, y, z \in X$,

$$|\rho(x, z) - \rho(y, z)| \le C\rho^{\theta}(x, y)\big(\rho(x, z) + \rho(y, z)\big)^{1-\theta}.$$

Without loss of generality we assume that d is continuous and all balls are open with respect to d.

For the definition, examples and some properties of an SHT see, e.g., monographs [4, 5, 22].

Let $\ell := \operatorname{diam}(X) = \sup_{x,y\in X} d(x, y)$. Notice that the condition $\ell < \infty$ implies that $\mu(X) < \infty$.

Definition 1 The triple (X, d, μ) is called an RD-space if it is an SHT and μ satisfies the reverse doubling condition: there exist constants $a, b > 1$ such that for all $x \in X$ and $0 < r < \ell/a$,

$$b\mu(B(x, r)) \le \mu B(x, ar).$$

Remark 1 (i) It is known that (X, d, μ) is an RD-space if and only if it is an SHT and there is a constant $\overline{c}$ such that for all $x \in X$ and $0 < r < \frac{\ell}{\overline{c}}$,

$$B(x, \overline{c}r)\backslash B(x, r) \neq \emptyset, \quad x \in X,$$

(for the proof we refer to see, e.g., [22], p. 11, Lemma 20, [8], Remark 1.2).

Throughout the paper we assume that (X, d, μ) is an RD-space.

Let w_1 and w_2 be weight functions on X, i.e. a. e. positive and locally integrable functions on X. The weighted Morrey space $M^{p,\lambda}_{w_1,w_2}(X)$, $1 \leq p < \infty$, $0 < \lambda < 1$, is defined as follows:

$$M^{p,\lambda}_{w_1,w_2}(X) = \left\{ f : \|f\|_{M^{p,\lambda}_{w_1,w_2}(X)} < \infty \right\},$$

where

$$\|f\|_{M^{p,\lambda}_{w_1,w_2}(X)} := \sup_{B\subset X} \left(\frac{1}{\big(w_2(B)\big)^{\lambda}} \int\limits_B |f(x)|^p w_1(x) d\mu(x) \right)^{\frac{1}{p}}.$$

Here B denotes a ball in X.

If $w_1 \equiv w_2 := w$, then we denote $M^{p,\lambda}_{w_1,w_2}(X)$ by $M^{p,\lambda}_{w}(X)$.

For a weight function w on X we denote by $L^{p,\infty}_w(X)$, $1 \leq p < \infty$, the weak weighted Lebesgue space which is defined with respect to the quasi-norm:

$$\|f\|_{L^{p,\infty}_w(X)} = \sup_{\alpha>0} \alpha \Big(w(\{x \in X : |f(x)| > \alpha\}) \Big)^{1/p} < \infty.$$

We say that a weight function w belongs to the Muckenhoupt class $A_r(X)$, $1 < r < \infty$, if

$$\|w\|_{A_r} := \sup_B \left(\frac{1}{\mu(B)} \int_B w(x) d\mu(x) \right) \left(\frac{1}{\mu(B)} \int_B w^{1-r'}(x) d\mu(x) \right)^{r-1} < \infty,$$

where the supremum is taken over all balls $B \subset X$.

Further, $w \in A_1(X)$ if there is a positive constant C such that for all balls $B \subset X$,

$$\frac{1}{\mu(B)} \int\limits_B w(x) d\mu(x) \leq C \; ess \; inf_B w(x).$$

It is easy to check that the condition $w \in A_r(X)$ implies that the measure $d\nu(x) = w(x) d\mu(x)$ satisfies the doubling condition.

Notation:

ℓ denotes the diameter of the X set.

$B(x, r) := \{y \in X : d(x, y) < r\}$.

By c and C we denote various absolute positive constants, which may have different values even in the same line.
p' stands for the conjugate exponent $1/p + 1/p' = 1$.
By the symbol $D(X)$ is denoted the class of bounded functions on X with compact supports.
We denote $aB := B(x, ar)$ for a ball $B := B(x, r)$ and constant $a, k \in \mathbf{Z}$.
$\overline{a} := a_1 (a_1 (a_0 + 1) + 1)$ with the quasi-metric constants a_0 and a_1.
$B_k(x_0, r) := \{x \in X : d(x_0, x) < \overline{a}^k r\}$.
$A_k(x_0, r) := B_k(x_0, r) \backslash B_{k-1}(x_0, r), \quad k \in \mathbf{Z}$, where x_0 is a point in X and $\overline{a}$ is the constant defined above.
For a ball B with radius r we denote by B_k the ball with the same center and radius $\overline{a}^k r$.
We denote by f_B average of a function f: $f_B := \frac{1}{\mu(B)} \int\limits_B f(x) d\mu(x)$.
Under the symbol $f_{B,w}$ we mean the average of f with respect to a weight function w: $f_{B,w} := \frac{1}{w(B)} \int\limits_B f(x) w(x) d\mu(x)$;
For a weight ρ on X and a μ-measurable set E, we denote $\rho(E) := \int\limits_E \rho(x) d\mu(x)$.
If $\mu(X) < \infty$, we will assume that m_0 is integer depending on $r > 0$ such that the number $d_{x_0} := \sup_{x \in X} d(x_0, x)$ belongs to the interval $[\overline{a}^{m_0} r, \overline{a}^{m_0+1} r)$; if $\mu(X) = \infty$, then we will suppose that $m_0 = \infty$.

2 Diagonal Case

We will assume that a sublinear operator T defined on a class of measurable functions $f : X \longrightarrow \mathbf{R}$ satisfies the condition: there is a positive constant c_0 such that for all $f \in L^1(X)$ with compact support and $x \notin \operatorname{supp} f$,

$$|Tf(x)| \le c_0 \int\limits_X \frac{|f(y)|}{\mu B(x, d(x, y))} d\mu(y). \tag{1}$$

In this case we write that T satisfies condition $S(X)$.

We will also suppose that together with condition (1) the boundedness of T holds in appropriate weighted Lebesgue space.

We say that a sublinear operator T satisfies the condition $\mathcal{B}_r(X)$, $1 < r < \infty$, if there is a positive constant C independent of f such that for every weight function $w \in A_r(X)$,

$$\|Tf\|_{L^r_w(X)} \le c \|f\|_{L^r_w(X)}, \quad f \in D(X). \tag{2}$$

Further, a sublinear operator T satisfies $\mathcal{B}(X)$ if there is a positive constant C independent of f such that for every weight function $w \in A_1(X)$,

$$\|Tf\|_{L^{1,\infty}_w(X)} \le c \|f\|_{L^1_w(X)}, \quad f \in D(X).$$

For example, conditions (1) and (2) ($1 < r < \infty$) are satisfied for the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$

where the supremum is taken over all balls in X containing x, and singular integral operators

$$Kf(x) = p.v. \int_X k(x, y) f(y) d\mu(y),$$

where k is the Calderón-Zygmund kernel (see also e.g., [5] Ch. 8 for the definition of k): $k : X \times X \backslash \{(x, x) : x \in X\} \to \mathbf{R}$ is a measurable function satisfying the conditions:

(i) $|K(x, y)| \leq \frac{C}{\mu B(x, d(x, y))}, \quad x, y \in X, \quad x \neq y;$

(ii)

$$|K(x_1, y) - K(x_2, y)| + |K(y, x_1) - K(y, x_2)| \leq Cw\left(\frac{d(x_2, x_1)}{d(x_2, y)}\right) \frac{1}{\mu B(x_2, d(x_2, y))}$$

for all x_1, x_2 and y with $d(x_2, y) \geq Cd(x_1, x_2)$, where w is a positive non-decreasing function on $(0, \infty)$ which satisfies the Δ_2 condition $w(2t) \leq cw(t)$ $(t > 0)$ and the Dini condition $\int_0^1 w(t)/t \, dt < \infty$. It is also assumed that k is such that Kf exists almost everywhere on X in the principal value sense for all $f \in L^2(X)$ and that K is bounded in $L^2(X)$.

Theorem 1 *Let* $1 < p < \infty$, $0 < \lambda < 1$, $w \in A_p(X)$. *Suppose that sublinear operator* T *satisfies the conditions* $S(X)$ *and* $\mathcal{B}_p(X)$. *Then there is a positive constant* c *such that the following inequality is true:*

$$\|Tf\|_{M_w^{p,\lambda}(X)} \leq c\|f\|_{M_w^{p,\lambda}(X)}, \quad f \in D(X). \tag{3}$$

Further, if $0 < \lambda < 1$, $w \in A_1(X)$, *the conditions* $S(X)$ *and* $\mathcal{B}(X)$ *are satisfied for a sublinear operator* T, *Then there is a positive constant* c *such that for all* $f \in D(X)$, $\alpha > 0$ *and balls* B,

$$w\Big(\{x \in B : |Tf(x)| > \alpha\}\Big) \leq \frac{c}{\alpha} \|f\|_{M_w^{1,\lambda}(X)} w(B)^\lambda. \tag{4}$$

Proof Let $\mu(X) = \infty$. We prove the strong-type inequality (3). The proof of (4) is similar.

Let us take a ball $B := B(x_0, r)$ with sufficiently small r. Represent the function f as $f = f_1 + f_2$ where $f_1 = f\chi_{\overline{a}B}$ and $f_2 = f - f_1$.

Due to condition (2), and the doubling condition for the measure $d\nu(x) = w(x)d\mu(x)$ we have for f_1 and B,

$$\left(\frac{1}{\big(w(B)\big)^{\lambda}}\int_B |Tf_1(x)|^p w(x)d\mu(x)\right)^{\frac{1}{p}} \le C\|f\|_{M_w^{p),\theta,\lambda}(X)}.$$

Further, observe that if $x \in B$ and $y \in A_k(x_0, r)$, $k \ge 2$, then

$$\mu B_k(x_0, r) \le C\mu B(x_0, d(x_0, y)) \le C\mu B(x, d(x, y))$$

with the constant C depending on the quasi-metric constants for d. Consequently, by condition (1) we have that

$$\begin{aligned}
&\frac{1}{w(B)^{\lambda/p}}\left(\int\limits_B |Tf_2(x)|^p w(x)d\mu(x)\right)^{\frac{1}{p}}\\
&= \frac{1}{w(B)^{\lambda}/p}\left(\int\limits_B |T(f\sum_{k=2}^{\infty}\chi_{A_k(x_0,r)})(x)|^p w(x)d\mu(x)\right)^{\frac{1}{p}}\\
&\le \frac{1}{\big(w(B)\big)^{\lambda/p}}\left(\sum_{k=2}^{\infty}\left(\int_B |T(f\chi_{A_k(x_0,r)})(x)|^p w(x)d\mu(x)\right)^{\frac{1}{p}}\right.\\
&\le C\frac{1}{\big(w(B)\big)^{\frac{\lambda}{p}-\frac{1}{p}}}\sum_{k=2}^{\infty}(\mu(B_k(x_0,r)))^{-1}\|f\|_{L^1(A_k(x_0,r))}\\
&\le C\frac{1}{\big(w(B)\big)^{\lambda/p-1/p}}\sum_{k=2}^{\infty}(\mu(B_k(x_0,r)))^{-1}\left(\int_{A_k(x_0,r)}|f|^p w d\mu\right)^{\frac{1}{p}}\\
&\times\left(\int_{A_k(x_0,r)} w^{1-p'}(x)d\mu(x)\right)^{\frac{1}{p'}}\\
&\le C\|w\|_{A_p(X)}^{\frac{1}{p}}\|f\|_{L_w^{p,\lambda}(X)}\sum_{k=2}^{\infty}\left(\frac{w(B(x_0,r))}{w(B_k(x_0,r))}\right)^{\frac{1}{p}-\frac{\lambda}{p}}\\
&\le C\|f\|_{L_w^{p,\lambda}(X)}\|w\|_{A_p(X)}^{\frac{1}{p}}.
\end{aligned}$$

In the latter inequality we use the condition $0 < \lambda < 1$ and the fact that measure $d\nu(x) = w(x)d\mu(x)$ satisfies the reverse doubling condition (it is doubling because $w \in A_p(X)$).

If $\mu(X) < \infty$, then we take a ball $B := B(x_0, r)$ with sufficiently small r and use the same representation $f = f_1 + f_2$. In this case we take sums $\sum_{k=2}^{m_0+1}$ instead of $\sum_{k=2}^{\infty}$ and argue as in the case $\mu(X) = \infty$.

3 Non-diagonal Case

Let $0 < \alpha < 1$. We say that a sublinear operator T_α of fractional type satisfies the condition $S_\alpha(X)$ if there is a positive constant C such that for all functions $f \in L^1(X)$ with compact supports and all $x \notin \operatorname{supp} f$,

$$|T_\alpha f(x)| \leq C \int\limits_X \frac{f(y)}{\mu(B(x, d(x, y))^{1-\alpha}} d\mu(y). \tag{5}$$

Like the diagonal case, we will also suppose that the sublinear operator T_α is bounded between appropriate weighted Lebesgue spaces.

Further, it can be checked easily that condition (5) is satisfied for fractional maximal function

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)| d\mu(y),$$

and fractional integral operator

$$I_\alpha f(x) = \int_X \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\alpha}} d\mu(y). \tag{6}$$

By the well-known one-weight characterization for fractional integrals (see [15] for Euclidean spaces and e.g., [5], Chap. 6 for an SHT), the inequality

$$\|(N_\alpha f)w\|_{L^q(X)} \leq C\|fw\|_{L^p(X)}, \quad 0 < \alpha < 1, \quad 1 < p < 1/\alpha, \ q = \frac{p}{1 - \alpha p},$$

where N_α is I_α or M_α, holds if and only if $w \in A_{p,q}(X)$, i.e.

$$\|w\|_{A_{p,q}} := \sup_B \left(\frac{1}{\mu B} \int\limits_B w^q(x) d\mu(x) \right) \left(\frac{1}{\mu B} \int\limits_B w^{-p'}(x) d\mu(x) \right)^{q/p'} < \infty.$$

It can be reformulated as follows: the inequality

$$\|N_\alpha(f u^\alpha)\|_{L^q_u(X)} \leq C\|f\|_{L^p_u(X)},$$

where N_α is I_α or M_α, holds if and only if $u \in A_{1+q/p'}(X)$.

Further, let $0 < \alpha < 1$ and let $\frac{1}{q} = 1 - \alpha$. Then N_α is bounded from $L^1_w(X)$ to $L^{q,\infty}_{w^q}(X)$ if and only if

$$\|w\|_{A_{1,q}} := \sup_B \left(\frac{1}{\mu B} \int\limits_B w^q(x) d\mu(x) \right) \left(ess\ sup_B \frac{1}{w(x)} \right) < \infty.$$

Definition 2 Let $0 < \alpha < 1$ and $1 < r < 1/\alpha$. We set $s = \frac{r}{1-\alpha r}$. We say that a sublinear operator T_α satisfies the condition $\mathcal{B}_{\alpha,r,s}(X)$ if there exists a positive constant C independent of f such that for every weight $w \in A_{1+s/r'}(X)$ the inequality

$$\|T_\alpha(f w^\alpha)\|_{L^s_w(X)} \le C\|f\|_{L^r_w(X)}, \quad f \in D(X),$$

is fulfilled.

We say that a sublinear operator T_α satisfies the condition $\overline{\mathcal{B}}_{\alpha,r,s}(X)$, $1 < r < s < \infty$, if it is bounded from $L^r_{w^r}(X)$ to $L^s_{w^s}(X)$ for every weight $w \in A_{r,s}(X)$.

Further, let $0 < \alpha < 1$ and let $s = \frac{1}{1-\alpha}$. An operator T_α satisfies the condition $\overline{\mathcal{B}}_{\alpha,s}(X)$ if is bounded from $L^1_w(X)$ to $L^{s,\infty}_{w^s}(X)$.

Now we prove the next statement:

Theorem 2 *Let $0 < \alpha < 1$ and $1 < p < 1/\alpha$. We set $q = \frac{p}{1-\alpha p}$. Let sublinear operator T_α satisfy the conditions $S_\alpha(X)$ and $\mathcal{B}_{\alpha,p,q}(X)$. Suppose that $0 < \lambda < 1$ and that $w \in A_{1+q/p'}(X)$. Then there is a constant $c > 0$ such that for all $f \in M^{p,\lambda p/q}_w(X)$,*

$$\|T_\alpha\big(f w^\alpha\big)\|_{M^{q,\lambda}_w(X)} \le c\|f\|_{M^{p,\lambda p/q}_w(X)}, \quad f \in D(X).$$

Proof We begin as in the proof of Theorem 1. We assume that $\mu(X) = \infty$. The case $\mu(X) < \infty$ follows as in the proof of that theorem.

Using the representation $f = f_1 + f_2$ with the same f_1 and f_2 as in the proof of Theorem 1, we see that

$$\begin{aligned}
&\frac{1}{\big(w(B)\big)^{\lambda/q}}\bigg(\int_B |T_\alpha(f w^\alpha)(x)|^q w(x)\mu(x)\bigg)^{\frac{1}{q-\varepsilon}}\\
&\le \frac{1}{\big(w(B)\big)^{\lambda/q}}\bigg(\int_B |T_\alpha(f_1 w^\alpha)(x)|^q w(x)\mu(x)\bigg)^{\frac{1}{q}}\\
&+ \frac{1}{\big(w(B)\big)^{\lambda/q}}\bigg(\int_B |T_\alpha(f_2 w^\alpha)(x)|^q w(x)\mu(x)\bigg)^{\frac{1}{q}}.
\end{aligned}$$

By the assumption,

$$\|T_\alpha(f w^\alpha)\|_{L^q_w(X)} \le C\|f\|_{L^p_w(X)}.$$

Consequently, by using the doubling condition for the measure $d\nu(x) = w d\mu(x)$ we find that

$$
\begin{aligned}
&\frac{1}{\big(w(B)\big)^{\lambda/q}}\bigg(\int_B |T_\alpha(f_1 w^\alpha)(x)|^q w(x)d\mu(x)\bigg)^{\frac{1}{q}}\\
&\le C\frac{1}{\big(w(\overline{a}B)\big)^{\lambda/q}}\bigg(\int_{\overline{a}B} |f(x)|^p w(x)d\mu(x)\bigg)^{\frac{1}{p}}\\
&\le C\|f\|_{M_w^{p,\lambda p/q}(X)}.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
&\frac{1}{\big(w(B)\big)^{\lambda/q}}\bigg(\int_B |T_\alpha(f_2 w^\alpha)(x)|^q w(x)d\mu(x)\bigg)^{\frac{1}{q}}\\
&= \frac{1}{\big(w(B)\big)^{\lambda/q}}\bigg(\int_B |T_\alpha(f w^\alpha \sum_{k=2}^\infty \chi_{A_k(x_0,r)})(x)|^q w(x)d\mu(x)\bigg)^{\frac{1}{q}}\\
&\le \frac{1}{\big(w(B)\big)^{\lambda/q-1/q}}\sum_{k=2}^\infty\bigg(\int_B |T_\alpha(f w^\alpha \chi_{A_k(x_0,r)})(x)|^q w(x)d\mu(x)\bigg)^{\frac{1}{q}}\\
&\le C\frac{1}{\big(w(B)\big)^{\lambda/q-1/q}}\sum_{k=2}^\infty\Big(\mu(B_k(x_0,r)\Big)^{\alpha-1}\|f w^\alpha\|_{L^1(A_k(x_0,r))}\\
&\le C\frac{1}{\big(w(B)\big)^{\lambda/q-1/q}}\sum_{k=2}^\infty(\mu(B_k(x_0,r)))^{\alpha-1}\bigg(\int_{A_k(x_0,r)}|f|^p w d\mu\bigg)^{\frac{1}{p}}\\
&\times\bigg(\int_{A_k(x_0,r)} w^{-p'/q}(x)d\mu(x)\bigg)^{\frac{1}{p'}}\\
&\le C\|w\|^{\frac{1}{q}}_{A_{1+q/p'}(X)}\sum_{k=2}^\infty\Bigg(\frac{\big(w(B(x_0,r))\big)}{\big(w(B_k(x_0,r))\big)}\Bigg)^{\frac{1}{q}-\frac{\lambda}{q}}\|f\|_{M_w^{p,\lambda p/q}(X)}\\
&\le C\|w\|^{\frac{1}{q}}_{A_{1+q/p'}(X)}\|f\|_{M_w^{p,\lambda p/q}(X)}.
\end{aligned}
$$

In the latter inequality the condition $0<\lambda<1$ and the fact that the measure $d\nu(x)=w(x)d\mu(x)$ is doubling (consequently it satisfies the reverse doubling condition) are used.

Theorem 3 *Let $0<\alpha<1$ and $1<p<1/\alpha$. We set $q=\frac{p}{1-\alpha p}$. Suppose that $0<\lambda<1$ and that $w\in A_{p,q}(X)$. Let a sublinear operator T_α satisfy the conditions $S_\alpha(X)$ and $\mathcal{B}_{\alpha,p,q}(X)$. Then there is a positive constant c such that*

$$
\|T_\alpha f\|_{M^{q,\lambda}_{w^q,w^q}(X)}\le c\|f\|_{M^{p,\lambda p/q}_{w^p,w^q}(X)},\quad f\in D(X). \tag{7}
$$

Further, if $0<\alpha<1$ and $q=\frac{1}{1-\alpha}$, then there is a positive constant C such that for all $f\in D(X)$, balls $B\subset X$ and $\alpha>0$, the inequality

$$w^q\Big(\{x\in B : |T_\alpha f(x)|>\alpha\Big)^{1/q} \le \frac{C}{\alpha}\|f\|_{M^{p,\lambda}_{w,w^q}(X)}\big(w^q(B)\big)^\lambda \tag{8}$$

provided that $w\in A_{1,q}(X)$ and conditions $S_\alpha(X)$ and $\mathcal{B}_{\alpha,q}(X)$ are satisfied for T_α.

Proof Strong-type inequality (7) is a consequence of Theorem 2 and the fact that $w\in A_{1+q/p'}(X)$ if and only if $w^{1/q}\in A_{p,q}(X)$. Weak type inequality (8) can be obtained by repeating the arguments of the proof of Theorems 1 and 2. In this case condition $\mathcal{B}_{\alpha,q}(X)$ is used instead of $\mathcal{B}_{\alpha,p,q}(X)$.

Corollary 1 *Let α, λ, p, q and w satisfy the conditions of Theorem 3. Then estimate (7) (resp. (8)) holds for the operator N_α, where N_α is either fractional integral operator I_α or fractional maximal operator M_α.*

4 Commutators

The space of functions of *bounded mean oscillation*, denoted by $\mathrm{BMO}(X,\mu)$, is the set of all real-valued locally integrable functions such that

$$\|f\|_{\mathrm{BMO}(X,\mu)} = \sup_{x\in X, 0<r<\ell}\frac{1}{\mu B(x,r)}\int_{B(x,r)}|f(y)-f_{B(x,r)}|d\mu(y)<\infty, \tag{9}$$

where $f_{B(x,r)}$ is the integral average over the ball $B(x,r)$. $\mathrm{BMO}(X,\mu)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathrm{BMO}(X,\mu)}$ when we regard the space BMO as the class of equivalent functions modulo additive constants.

Remark 2 In this remark, we give equivalent norms for functions in the space $\mathrm{BMO}(X,\mu)$, namely

(i) we can define an equivalent norm in $\mathrm{BMO}(X,\mu)$ as

$$\|f\|_{\mathrm{BMO}(X,\mu)} \sim \sup_{x\in X,0<r<\ell}\inf_{c\in\mathbf{R}}\frac{1}{\mu B(x,r)}\int_{B(x,r)}|f(y)-c|d\mu(y), \tag{10}$$

(ii) the John-Nirenberg inequality (see e.g., [1]) gives us another equivalent norm for $\mathrm{BMO}(X,\mu)$-functions given by

$$\|f\|_{\mathrm{BMO}(X,\mu)} \sim \sup_{x\in X,0<r<\ell}\left(\frac{1}{\mu B(x,r)}\int_{B(x,r)}|f(y)-f_{B(x,r)}|^p d\mu(y)\right)^{\frac{1}{p}} \tag{11}$$

valid for $1<p<\infty$, where f_B stands for the integral average.

We denote by $A_\infty(X)$ the class of weights w satisfying the condition: there are positive constants C and ε such that for every ball B and measurable subset $E\subset B$ the inequality

$$\frac{w(E)}{w(B)} \le C\left(\frac{\mu(E)}{\mu(B)}\right)^{\varepsilon} \tag{12}$$

holds. The infimum of C for which (12) holds is denoted by $\|w\|_\infty$.

Let U be an operator and b a locally integrable function. We define the *commutator* $U_b f$ as

$$U_b f = bU(f) - U(bf).$$

Commutators are very useful when studying problems related with regularity of solutions of elliptic partial differential equations of the second order (see [3]).

For the sublinear operators T and T_α we will have the following assumptions on their commutators T_b and $T_{\alpha,b}$ respectively:

$$|T_b f(x)| \le C_1 \int\limits_X \frac{|b(x)-b(y)||f(y)|}{\mu B(x, d(x,y))} d\mu(y), \quad x \notin \operatorname{supp} f; \tag{13}$$

$$|T_{\alpha,b} f(x)| \le C_2 \int\limits_X \frac{|b(x)-b(y)||f(y)|}{\big(\mu B(x, d(x,y))\big)^{1-\alpha}} d\mu(y), \quad x \notin \operatorname{supp} f. \tag{14}$$

The following statements are well-known (see [17] and [2] respectively):

Theorem A *Let* $1 < p < \infty$, $w \in A_p(X)$. *If* $b \in \mathrm{BMO}(X,\mu)$*, then the following inequality holds with the positive constant independent of* $f \in D(X)$*:*

$$\|K_b f\|_{L^p_w(X)} \le C\|b\|_{\mathrm{BMO}(X,\mu)(X)}\|f\|_{L^p_w(X)},$$

where K *is the Calderón-Zygmund operator on* X.

Theorem B *Let* $1 < p < \infty$, $0 < \alpha < 1/p$. *We set* $q = \frac{p}{1-\alpha p}$. *Suppose that* $w \in A_{p,q}(X)$. *If* $b \in \mathrm{BMO}(X,\mu)$*, then the following inequality holds with the positive constant independent of* f*:*

$$\|wI_{\alpha,b} f\|_{L^q(X)} \le C\|b\|_{\mathrm{BMO}(X,\mu)}\|wf\|_{L^p(X)},$$

where I_α *is the potential operator on* X.

In fact, the latter results deal with the commutators of the type:

$$(S_b^m f)(x) = \int\limits_{\mathbf{R}^n} f(y)(b(x)-b(y))^m k(x,y) d\mu(y),$$

for appropriate kernel.

Now we formulate the main results of this section.

Theorem 4 *Let* $1 < p < \infty$, $0 < \lambda < 1$, $w \in A_p(X)$, $b \in \mathrm{BMO}(X,\mu)$. *Let a sublinear operator* T_b *satisfy the conditions (13) and* $\mathcal{B}_p(X)$. *Then there is a positive* *constant c independent of f such that*

$$\|T_b f\|_{M_w^{p,\lambda}(X)} \le c\|f\|_{M_w^{p,\lambda}(X)}, \quad f \in D(X).$$

Theorem 5 *Let* $1 < p < \infty$ *and* $0 < \alpha < 1/p$. *We set* $q = \frac{p}{1-\alpha p}$. *Let* $b \in \mathrm{BMO}(X,\mu)$, $0 < \lambda < 1$ *and let* $w \in A_{1+q/p'}(X)$. *Suppose that* $T_{\alpha,b}$ *satisfies conditions (14) and* $\mathcal{B}_{\alpha,p,q}(X)$. *Then there is a positive constant c such that the following inequality is true:*

$$\|T_{\alpha,b}(f w^{\alpha})\|_{M_w^{q,\lambda}(X)} \le c\|f\|_{M_w^{p,\lambda}(X)}, \quad f \in D(X)$$

Theorem 6 *Let* $1 < p < \infty$ *and* $0 < \alpha < 1/p$. *We set* $q = \frac{p}{1-\alpha p}$. *Let* $0 < \lambda < 1$ *and let* $b \in \mathrm{BMO}(X,\mu)$. *Suppose that* $w \in A_{p,q}(X)$. *Let* $T_{\alpha,b}$ *satisfy conditions (14) and* $\overline{\mathcal{B}}_{\alpha,p,q}(X)$. *Then the following inequality holds:*

$$\|T_{\alpha,b} f\|_{M^{q,\lambda}_{w^q,w^q}(X)} \le C\|b\|_{\mathrm{BMO}(X,\mu)(X)}\|f\|_{M^{p,\lambda p/q}_{w^p,w^q}(X)}, \quad f \in D(X)$$

with the positive constant C independent of f.

To prove the main results of this paper we need some auxiliary statements:

Lemma 1 *[1] Let* $1 < p < \infty$. *There exist positive constants* C_1 *and* C_2 *independent of b and* $B \subset X$ *such that*

$$C_1\|b\|_{\mathrm{BMO}(X,\mu)} \le \left(\frac{1}{\mu(B)}\int_B |b(x)-b_B|^p d\mu(x)\right)^{1/p} \le C_2\|b\|_{\mathrm{BMO}(X,\mu)}. \tag{15}$$

Remark 3 If $p < 1$, the second inequality of (15) still holds, because of Hölder's inequality.

Lemma 2 *Let* $w \in A_\infty(X)$. *Suppose that* $b \in \mathrm{BMO}(X,\mu)$. *Then there are positive constants* C_1 *and* C_2 *such that for all balls* $B \subset X$, *the following inequalities hold:*

(i)

$$\frac{1}{w(B)}\int_B |b(x)-b_B|w(x)d\mu(x) \le C_1; \tag{16}$$

(ii)

$$\frac{1}{w(B)}\int_B |b(x)-b_{B,w}|w(x)d\mu(x) \le C_2. \tag{17}$$

Proof For (16) we refer e.g., [7] and [13]. Inequality (17) follows easily from (16) by the following simple observation:

$$\int_B |b(x) - b_{B,w}| w(x) d\mu(x) \leq \int_B |b(x) - b_B| w(x) d\mu(x) + w(B) |b_{B,w} - b_B|$$
$$\leq 2 \int_B |b(x) - b_B| w(x) d\mu(x).$$

The next lemma in Euclidean spaces is given in [10] (see P. 121).

Lemma 3 *The following inequality holds for all* $b \in \mathrm{BMO}(X, \mu)$*:*

$$|b_{B_k} - b_B| \leq kA \|b\|_{\mathrm{BMO}(X,\mu)},$$

where $A := D^{\log_2 \overline{a}+1}$*,* D *is the doubling constant.*

Proof

$$\begin{aligned} |b_{B_1} - b_B| &= \frac{1}{\mu(B)} \left| \int_B \big(b(y) - b_{B_1}\big) d\mu(y) \right| \\ &\leq \frac{A}{\mu(B_1)} \int_{B_1} \big|b(y) - b_{B_1}\big| d\mu(y) \leq A \|b\|_{\mathrm{BMO}(X,\mu)}. \end{aligned}$$

Further, taking this argument into account and by adding and subtracting the terms b_{B_2}, b_{B_3}, etc. $b_{B_{k-1}}$ we get the desired result.

The next statement for Euclidean spaces was proved in [21].

Lemma 4 *Let* $1 < p < \infty$ *and* $0 < \lambda < 1$*. Suppose that* $w \in A_\infty$*. Then there is a positive constant* C *such that for all* $f \in M_w^{p,\lambda}(X)$*, all balls* $B := B(x_0, r)$ *the inequality*

$$\left(\int_{X \setminus \overline{a}B} \frac{|f(y)|}{\mu(B(x_0, d(x_0, y))} |b_{B,w} - b(y)| d\mu(y) \right)^p w(B)^{1-\lambda} \leq C \|f\|^p_{M_w^{p,\lambda}(X)}.$$

Proof Applying Hölder's inequality we have

$$\begin{aligned} &\left(\int_{X \setminus \overline{a}B} \frac{|f(y)|}{\mu(x_0, d(x_0, y))} |b_{B,w} - b(y)| d\mu(y) \right)^p w(B)^{1-\lambda} \\ &\leq \left(\sum_{k=1}^{\infty} \int_{A_{k+1}} \frac{|f(y)|}{\mu(B(x_0, d(x_0, y))} |b_{B,w} - b(y)| d\mu(y) \right)^p w(B)^{1-\lambda} \\ &\leq \left(\sum_{k=1}^{\infty} \frac{1}{\mu(B(x_0, \overline{a}^k r))} \int_{A_{k+1}} |f(y)| |b_{B,w} - b(y)| d\mu(y) \right)^p w(B)^{1-\lambda} \end{aligned}$$

$$\le \Bigg(\sum_{k=1}^{\infty} \frac{1}{\mu(B(x_0, \overline{a}^k r))} \Bigg(\int\limits_{A_{k+1}} |f(y)|^p w(y) d\mu(y) \Bigg)^{1/p}$$

$$\times \Bigg(\int\limits_{B(x_0, \overline{a}^k r))} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'} d\mu(y) \Bigg)^{1/p'} \Bigg)^p w(B)^{1-\lambda}$$

$$\le C \|f\|^p_{M_w^{p,\lambda}(X)} \Bigg(\sum_{k=1}^{\infty} \frac{w(B_{k+1})^{\lambda/p}}{\mu(B(x_0, \overline{a}^k r))}$$

$$\times \Bigg(\int\limits_{B(x_0, \overline{a}^k r)} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'} d\mu(y) \Bigg)^{1/p'} \Bigg)^p w(B)^{1-\lambda}.$$

Further, it is easy to see that by adding and subtracting $b_{B_{k+1}, w^{1-p'}}$ we find that

$$\Bigg(\int\limits_{B(x_0, \overline{a}^k r)} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'} d\mu(y) \Bigg)^{1/p'}$$

$$\le \Bigg(\int\limits_{B_{k+1}} |b_{B_{k+1}, w^{1-p'}} - b(y)|^{p'} w^{1-p'} d\mu(y) \Bigg)^{1/p'}$$

$$+ |b_{B_{k+1}, w^{1-p'}} - b_{B,w}| \big(w^{1-p'}(B_{k+1}) \big)^{1/p'} := I_1 + I_2.$$

By the fact that $w^{1-p'} \in A_\infty(X)$ and Lemmas 1, 2 we find that

$$I_1 \le C \|b\|_{\mathrm{BMO}(X,\mu)(X, w^{1-p'})} \big(w^{1-p'}(B_{k+1}) \big)^{1/p'} \le C \big(w^{1-p'}(B_{k+1}) \big)^{1/p'}.$$

Now Lemma 3 yields that

$$|b_{B_{k+1}, w^{1-p'}} - b_{B,w}| \le |b_{B_{k+1}, w^{1-p'}} - b_{B_{k+1}}| + |b_{B_{k+1}} - b_B| + |b_B - b_{B,w}|$$

$$\le \frac{1}{w^{1-p'}(B_{k+1})} \int\limits_{B_{k+1}} |b(y) - b_{B_{k+1}}| w^{1-p'}(y) d\mu(y) + A(k+1) \|b\|_{\mathrm{BMO}(X,\mu)}$$

$$+ \frac{1}{w(B)} \int\limits_{B} |b(y) - b_B| w(y) d\mu(y) := I_{21} + I_{22} + I_{23}.$$

Observe that Lemma 2 yields that

$$I_{23} \le C$$

and

$$I_{21} \le C.$$

In the latter inequality the fact that $w^{1-p'} \in A_{p'}(X)$ is used. Consequently,

$$I_2 \le C(A(k+1)+2)w^{1-p'}(B_{k+1})^{1/p'}.$$

Summarizing these inequalities we find that

$$\left(\sum_{k=1}^{\infty} \frac{w(B_{k+1})^{\lambda/p}}{\mu(B(x_0, \overline{a}^k r))}\left(\int\limits_{B(x_0,\overline{a}^k r))} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'} d\mu(y)\right)^{1/p'}\right)^p w(B)^{1-\lambda}$$
$$\le C\left(\sum_{k=1}^{\infty} \frac{(k+1)w(B)^{(1-\lambda)/p}}{w(B_{k+1})^{(1-\lambda)/p}}\right)^p \le C.$$

Proof of Theorem 4. Using the representation $f = f_1 + f_2$, where $f_1 = f\chi_{\overline{a}B}$, $f_2 = f - f_1$, and $B := B(x_0, r)$, we have

$$\int\limits_B |T_b f(x)|^p w(x) d\mu(x)$$
$$\le C\left(\int\limits_B |T_b f_1(x)|^p w(x) d\mu(x) + \int\limits_B |T_b f_2(x)|^p w(x) d\mu(x)\right) := I_1 + I_2.$$

By the hypothesis T_b is bounded in $L^p_w(X)$. Therefore,

$$I_1 \le C \int\limits_{\overline{a}B} |f(x)|^p w(x) d\mu(x) \le C\|f\|^p_{M_w^{p,\lambda}(X)} w(B)^{\lambda}.$$

To estimate I_2 first we observe that if $x \in B$ and $y \notin \overline{a}B$, then

$$\mu(B(x_0, d(x_0, y)) \le C\mu(B(x, d(x, y))$$

with a positive constant C independent of x, x_0, y. Consequently, by condition (13) we get

$$|T_b f_2(x)|^p \le C\left(\int\limits_X \frac{|f_2(y)||b(x) - b(y)|}{\mu(B(x, d(x, y))} d\mu(y)\right)^p$$
$$\le C\left(\int\limits_{X \setminus \overline{a}B} \frac{|f(y)|}{\mu(B(x_0, d(x_0, y))} d\mu(y)\right)^p |b(x) - b_{B,w}|^p$$
$$+ C\left(\int\limits_{X \setminus \overline{a}B} \frac{|f(y)||b(y) - b_{B,w}|}{\mu(B(x_0, d(x_0, y))} d\mu(y)\right)^p.$$

Hence,

$$I_2 \le C\bigg(\int\limits_{X \setminus \overline{a}B} \frac{|f(y)|}{\mu(B(x_0, d(x_0, y))} d\mu(y)\bigg)^p \bigg(\int\limits_B |b(x) - b_{B,w}|^p w(x) d\mu(x)\bigg)$$
$$+ C\bigg(\int\limits_{X \setminus \overline{a}B} \frac{|f(y)|}{\mu(B(x_0, d(x_0, y))} |b(y) - b_{B,w}| d\mu(y)\bigg)^p w(B) := I_{21} + I_{22}.$$

Observe that Lemma 4 yields that

$$I_{22} \le C \|f\|^p_{M^{p,\lambda}_w(X)} w(B)^\lambda.$$

Further, observe that the condition $w \in A_p(X)$ and Lemmas 1, 2 imply that

$$\int\limits_B |b(x) - b_{B,w}|^p w(x) d\mu(x) \le C w(B)$$

with the positive constant C independent of B.

Consequently, this estimate together with the reverse doubling condition for the measure $d\nu(x) = w(x) d\mu(x)$ yields that

$$I_{21} \le C\bigg(\sum_{k=1}^\infty \frac{1}{\mu(B(x_0, \overline{a}^k r))} \int\limits_{B_{k+1}} |f(y)| d\mu(y)\bigg)^p \int\limits_B |b(x) - b_{B,w}|^p w(x) d\mu(x)$$
$$\le C\bigg(\sum_{k=1}^\infty \frac{1}{\mu(B_{k+1})} \bigg(\frac{1}{w(B_{k+1})^\lambda} \int\limits_{B_{k+1}} |f(y)|^p w(y) d\mu(y)\bigg)^{1/p} w(B_{k+1})^{\lambda/p}$$
$$\times \bigg(\int\limits_{B_{k+1}} w^{1-p'} d\mu(y)\bigg)^{1/p'}\bigg)^p \int\limits_B |b(x) - b_{B,w}|^p w(x) d\mu(x)$$
$$\le C \|w\|_{A_p(X)} \|f\|^p_{M^{p,\lambda}_w(x)}$$
$$\times \bigg(\sum_{k=1}^\infty \frac{\mu(B_{k+1})^{1/p'}}{\mu(B_k)} \bigg(\frac{1}{\mu(B_{k+1})} \int\limits_{B_{k+1}} w(y) d\mu(y)\bigg)^{-1/p} w(B_{k+1})^{\lambda/p}\bigg)^p$$
$$\times \int\limits_B |b(x) - b_{B,w}|^p w(x) d\mu(x)$$
$$\le C \|f\|^p_{M^{p,\lambda}_w(X)} \bigg(\sum_{k=1}^\infty \frac{w(B)^{(1-\lambda)/p}}{w(B_k)^{(1-\lambda)/p}}\bigg)^p w(B)^\lambda$$
$$\le C \|f\|^p_{M^{p,\lambda}_w(X)} w(B)^\lambda.$$

Summarizing these estimates we get

$$I_2 \leq C\|f\|^p_{M_w^{p,\lambda}(X)} w(B)^\lambda$$

which completes the proof of the theorem.

Proof of Theorem 5 is similar to that of Theorem 4; therefore we omit the proof. Theorem 6 is a consequence of Theorem 5. Details are omitted (see also the comment after Theorem 3).

5 Applications to Estimates for Hypoelliptic Operators

A homogeneous Lie group is a connected Lie group G endowed with a family of automorphisms $\{D_t\}_{t>0}$ such that its Lie algebra g is homogeneous under $\delta_t = (D_t)_*$. Let G be a homogeneous group with homogeneous dimension Q, the quasi-norm $x \to r(x)$ and Haar measure $d\mu = dx$. It is known that G is a space of homogeneous type with the quasi-metric $d(x, y) := r(xy^{-1})$ and doubling measure dx (we refer to e.g., [6] for definitions and properties of homogeneous groups). Heisenberg group is one of the interesting examples of homogeneous group.

For a fixed basis $(e_1, \ldots, e_n)$ in g. Denote by X_j the left-invariant vector field such that $(X_j)_0 = \partial_{e_j}$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, we set $X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n}$. By Poincaré-Birkhoff-Witt theorem (see, e.g., [19], Theorem 4.1) if L is a left-invariant differential operator on G, then L can be written in one and only one way as

$$L = \sum_{|\alpha| \leq m} c_\alpha X^\alpha.$$

Let G be a homogeneous group. We say that a left-invariant differential operator L is homogeneous of order μ if $L(f \circ \delta_t) = t^{-\mu}(Lf) \circ \delta_t$. Poincaré-Birkhoff-Witt theorem implies that a left-invariant differential operator L is homogeneous of order μ if and only if $L = \sum_{d(\alpha)=\mu} c_\alpha X^\alpha$.

Let L be a linear differential operator with smooth coefficients on an open set U. Denote by ${}^tL$, the transpose of L, the operator such that

$$\int_U Lf(x)\varphi(x)dx = \int_U f(x)^t L(x)dx,$$

for every test functions f, φ.

By the definition, L is hypoelliptic operator if U an open set in G and u is a distribution on U such that $Lu \in C^\infty(U)$, then it follows that $u \in C^\infty(U)$ (see [9])

The following statements are known (for the proof see e.g., [19]):

Theorem C *Let L be a left-invariant differential operator on G, homogeneous of order $\mu < Q$. Assume that L and ${}^tL$ are hypoelliptic. Then L has a global funda-*

mental solution, which is smooth away from the origin and homogeneous of order $-Q+\mu$. *This solution is a unique homogeneous fundamental solution.*

Theorem D *Let* $0<\alpha<Q$, K_α *be a distribution homogeneous of degree* $-Q+\alpha$ *and continuous away from the origin. Then* K_α *is a locally integrable function satisfying*

$$|K_\alpha(x)| \le \frac{C}{(r(x))^{Q-\alpha}}$$

for some $C>0$.

Definition 3 We say that a weight function w on G satisfies the condition $A_{p,q}(G)$, $1<p<q<\infty$, if

$$\sup_B \left(\frac{1}{|B|}\int_B w^q(x)dx\right)^{1/q}\left(\int_B w^{-p'}(x)dx\right)^{1/p'} < \infty.$$

The main statements of this section read as follows:

Theorem 7 *Let* L *be a left-invariant differential operator of order* μ, $\mu<Q$. *Suppose that both* L *and* ${}^tL$ *are hypoelliptic operators. Let* $X_1,\cdots,X_n$ *be basis of* g *consisting of homogeneous vector fields. Let* $X^\alpha=X^{\alpha_1}\cdots X^{\alpha_n}$ *be homogeneous of order* $d(\alpha)$, *where* $0\le d(\alpha)<\mu$. *Further, let* $\mu-d(\alpha)<\frac{Q}{p}$ *and let* $q=\frac{Qp}{Q-(\mu-d(\alpha))p}$. *Suppose that* $w\in A_{p,q}$. *If* $f\in D'(G)$, *with compact support, then* $Lf\in M^{p,\lambda p/q}_{w^p,w^q}(G)$ *implies that* $X^\alpha f\in M^{q,\lambda}_{w^q,w^q}(G)$.

Theorem 8 *Let* $L=\sum_j X_j$ *be a homogeneous sub-Laplacian on a stratified group* G. *Let conditions of Theorem 7 be satisfied corresponding to* $\mu=2$ *and* $d(\alpha)=0,1$. *Then the inequality*

$$\|X^\alpha f\|_{M^{q,\lambda}_{w^q,w^q}(G)} \le C\|Lf\|_{L^{p,\lambda p/q}_{w^p,w^q}(G)}$$

holds.

Proof of Theorem 7. Let K_α be a homogeneous fundamental solution of L. By Theorem C, K_α exists, is unique and is homogeneous of order $\mu-Q$. Consequently, $X^\alpha K_\alpha$ is homogeneous of degree $-Q+\mu-d(\alpha)$ and is smooth away from the origin. By Theorem D and Theorem 3 we have that $(Lf)*(X^\alpha K_\alpha)\in M^{q,\lambda}_{w^q,w^q}(G)$ if $Lf\in M^{p,\lambda p/q}_{w^p,w^q}(G)$. Taking $\psi=f-(Lf)*K_\alpha$. Then since $L\psi=0$ we have that $\psi\in C^\infty(G)$. Hence $X^\alpha f=X_\alpha\psi+(Lf)*(X^\alpha K_\alpha)$ belongs to $M^{q,\lambda}_{w^q,w^q}(G)$.

Proof of Theorem 8. Since L is a sub-Laplacian, then (see e.g., [19]) we have that $f=(Lf)*K_\alpha$, where K_α is a fundamental solution of L. Hence, $X^\alpha f=(Lf)*(X^\alpha K_\alpha)$ which implies the desired result by Theorem 3.

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Essentially Algebraic Composition Operators on Lorentz Sequence Spaces with a Weight

Romesh Kumar, Ajay K. Sharma, Sumit Dubey and Shagoon Wasir

Abstract In this paper, we characterize the essentially algebraic composition operators on Lorentz sequence spaces with a weight. The techniques used in the proofs of the results of this paper are essentially the same as in the work of Bӧttcher and Heidler (Integr Eqn Oper Theory 15:389–411, 1992 [2], St. Petersburg Math J 5:1099–1119, 1994 [3]).

Keywords Composition operators · Algebraic operators · Essentially algebraic operators · Lorentz sequence spaces

1 Introduction

Let X = (X, Σ, μ) be a σ-finite complete measure space and let $T : X \to X$ be a measurable transformation, that is, $T^{-1}(A) \in \Sigma$ for any $A \in \Sigma$. If $\mu o T^{-1}(A) = 0$ for each $A \in \Sigma$ with $\mu(A) = 0$, then T is said to be non-singular.

Any non-singular measurable transformation T induces a linear operator C_T from $L^0(X)$ into itself defined by

$$(C_T f)(t) = f o T(t) = f(T(t)), \quad t \in X, \quad f \in L^0(X),$$

where $L^0(X)$ denotes the linear space of all equivalence classes of Σ-measurable functions on X. Here we identify any two functions that are equal μ-almost everywhere on X.

Let M_0 be the class of all functions f in $L^0(X)$ that are finite μ-almost everywhere on X. For $f \in M_0$, we define the distribution function μ_f of f on $(0, \infty)$ by

$$\mu_f(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\})$$

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and the decreasing rearrangement f^* of f on $(0, \infty)$ by

$$
\begin{aligned}
f^*(t) &= \inf\{\lambda > 0 : \mu_f(\lambda) \le t\} \\
&= \sup\{\lambda > 0 : \mu_f(\lambda) > t\}.
\end{aligned}
$$

The Lorentz space $L^{p,q}(\mu)$ is the set of all classes of Σ-measurable functions f on X such that the functional $||f||_{pq} < \infty$, where

$$
||f||_{pq} = \begin{cases} (\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t})^{1/q} \ , & \text{if } 1 < p < \infty, 1 \le q < \infty \\ \sup_{t>0} t^{1/p} f^*(t) \ , & \text{if } 1 < p \le \infty, q = \infty. \end{cases}
$$

Take X = $\mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$ and $\mu(\{n\}) = 1$. Then the Lorentz sequence space $\ell^{p,q}$ is the set of all sequences $a = \{a_n\} \in c_o$ such that the functional $||a||_{pq} < \infty$, where

$$
||a||_{pq} = \begin{cases} (\sum_{n=1}^\infty (n^{1/p} a_n^*)^q n^{-1})^{1/q} \ , & \text{if } 1 < p < \infty, 1 \le q < \infty \\ \sup_{n\ge1} n^{1/p} a_n^* \ , & \text{if } 1 < p \le \infty, q = \infty. \end{cases}
$$

If we take $X = \mathbb{N}$, $\Sigma = 2^{\mathbb{N}}$ and $\mu : \mathbb{N} \to (0, \infty)$ is the weight function, then the corresponding Lorentz sequence space with weight μ is denoted by $\ell_\mu^{p,q}$ and

$$
||a||_{pq,\mu} = \begin{cases} (\sum_{n=1}^\infty (n^{1/p} a_n^* \mu(n))^q)^{1/q} \ , & \text{if } 1 < p < \infty, 1 \le q < \infty \\ \sup_{n\ge1} n^{1/p} a_n^* \mu(n) \ , & \text{if } 1 < p \le \infty, q = \infty. \end{cases}
$$

Note that the Lorentz spaces are quasi-normed linear spaces and the functional $||.||_{pq}$ is a norm if and only if $1 \le q \le p < \infty$ or $p = q = \infty$.

For any Σ-measurable set A of finite measure, we have

$$
||\chi_A||_{pq} = \begin{cases} (p/q)^{1/q} (\mu(A))^{1/p} \ , & \text{if } 1 < p < \infty, 1 \le q < \infty \\ (\mu(A))^{1/p} \ , & \text{if } 1 < p \le \infty, q = \infty. \end{cases}
$$

For details about Lorentz space one can refer to [1, 4] and references therein.

Theorem 1 (Cf [[8], Theorem 2.3]). *Let $T : X \to X$ be a non-singular measurable transformation. Then T induces a bounded composition operator C_T on $L^{pq}(\mu)$, $1 < p < \infty$, $1 \le q \le \infty$ if and only if there exists some constant $M > 0$ such that*

$$
\mu o T^{-1}(A) \le M \mu(A),
$$

for each $A \in \Sigma$. Moreover,

$$||C_T|| = \sup_{A\in\Sigma, 0<\mu(A)<\infty} (\frac{\mu o T^{-1}(A)}{\mu(A)})^{1/p}$$

For Orlicz spaces see [7, 10, 11] and for composition operators on Orlicz spaces one can refer to [5, 6, 11] and references therein. The work of this paper is motivated by the interesting work of Bӧttcher and Heidler [2, 3].

Definition 1 Let $\mathbb{C}[z]$ denote the ring of univariate polynomials with complex coefficients. A polynomial $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$ is said to be monic if $a_n = 1$.

Definition 2 An operator U on a Banach Space B is said to be algebraic if there is a non zero polynomial p(z) such that p(U) = 0 and U will be called essentially algebraic if there is a non zero polynomial q(z) such that q(U) is compact.

Definition 3 The monic polynomial p(z) of the least degree such that p(U) is zero is called the chracterstic polynomial of U.

Definition 4 The monic polynomials q(z) of the least degree such that q(U) is compact is called the essentially chracterstic polynomial of U.

Let p_a and q_a represent the chracterstic and essentially chracterstic polynomials associated with the linear operator U. Also for a polynomial $p(z) \in \mathbb{C}[z]$, $(p(z))$ denotes the two sided ideal $p(z)\mathbb{C}[z]$.

Let $\mathcal{B}(X)$ be the collection of all bounded operators on a Banach space X and $\mathcal{K}(X)$ be the collection of all compact operators on X. $\mathcal{B}(X)$ is a Banach algebra under the operator norm and $\mathcal{K}(X)$ is a two sided ideal in $\mathcal{B}(X)$. Let $\pi : \mathcal{B}(X) \to \mathcal{B}(X)/\mathcal{K}(X)$ be the natural map of $\mathcal{B}(X)$ onto the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$. Let $alg(U) = \{p(U) : p(z) \in \mathbb{C}[z]\}$ and $alg\pi(U) = \{p(\pi(U)) : p(z) \in \mathbb{C}[z]\}$. In case $alg(U)$ is finite dimensional, the $alg(U)$ is isomorphic to $\mathbb{C}[z]/(p_a(z))$ and is closed subalgebra of $\mathcal{B}(X)$. In case $alg\pi(U)$ is finite dimensional, the $alg\pi(U)$ is isomorphic to $\mathbb{C}[z]/(q_a(z))$ and is closed subalgebra of $\mathcal{B}(X)/\mathcal{K}(X)$.The essential characteristic polynomial of U is actually the characteristic polynomial of the Calkin image $\pi(U)$ of U in the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$. See [2, 3] for more details about the above definitions and notations.

Algebraic and essentially algebraic composition operators on ℓ^p, $1 \le p \le \infty$ are studied by Bottcher and Heidler [3] and these results were extended to Orlicz spaces by Kumar and Kumar [9].

2 Essentially Algebraic Composition Operators

In this section we study essentially algebraic composition operators. The set $\{n, T(n), T^2(n), ...\}$ is called the orbit of n. If the orbit of $n \in \mathbb{N}$ is a finite set, then we define the enter length $ent_T(n)$ and the cycle length $cyc_T(n)$ of the point n as

$$ent_T(n) = min\{k \in \mathbb{N} \cup \{0\} : T^k(n) = T^{k+m}(n) \text{ for some } m \geq 1\}$$

and

$$cyc_T(n) = min\{k \in \mathbb{N} \cup \{0\} : T^{k+ent_T(n)}(n) = T^{ent_T(n)}(n)\}.$$

If the orbit is infinite, then we take $ent_T(n) = cyc_T(n) = \infty$.

The maximal enter length of the self map $T : \mathbb{N} \to \mathbb{N}$ is denoted by ent(T) and defined by

$$ent(T) = \sup_{n \in \mathbb{N}} ent_T(n)$$

and the set of occuring periods or cycle lengths is

$$Per(T) = \{k \in \mathbb{N} : \mathbb{N}_k(T) \neq \phi\},$$

where

$$\mathbb{N}_k(T) = \{n \in \mathbb{N} : ent_T(n) = 0 \text{ and } cyc_T(n) = k\}, \qquad (k \geq 1).$$

In case $|\mathbb{N}_k(T)| = card(\mathbb{N}_k(T)) = \infty$, we say that k is an essential period of T. The set of all essential periods is denoted by $Wper(T)$. The set of all unessential periods is defined as

$$Uper(T) = Per(T) \backslash Wper(T)$$

and the set of all periodic points with unessential periods is the set

$$\mathbb{N}_U(T) = \{n \in \mathbb{N} : ent_T(n) = 0 \text{ and } cyc_T(n) \in Uper(T)\}.$$

The essential enter length Went(T) for the self-map $T : \mathbb{N} \to \mathbb{N}$ depends on decreasing rearrangement '*' and μ. If the set

$$\mathbb{N}_0(T) = \{n \in \mathbb{N} : ent_T(n) \geq 1\}$$

is finite, then we put Went(T) = 0. If $\mathbb{N}_0(T)$ is infinite, then we define Went(T) as the minimal $m \in \mathbb{N}$ such that the set

$$\mathbb{N}^{*,\mu}_{m,\epsilon} = \{n \in \mathbb{N} : ent_T(n) \geq 1 \text{ and } \mu(T^{-m}(n)) > \epsilon\}$$

is finite, for each $\epsilon > 0$.

In other words,

$$\lim_{n\in\mathbb{N}_0(T)} \frac{\mu(T^{-m}(n))}{\mu(n)} = 0. \qquad ....(3.1)$$

If there is no $m \in \mathbb{N}$ such that the set $\mathbb{N}^{*,\mu}_{m,\epsilon}$ is finite, then we put $Went(T) = \infty$ (Fig. 1).

Example 1 Let $T : \mathbb{N} \mapsto \mathbb{N}$ be defined as:

$$T(x) = \begin{cases} x + 1 \; if \; x \neq 5k; \, k \in \mathbb{N} \\ x - 3 \; if \; x = 5k; \, k \in \mathbb{N} \end{cases}.$$

Then $ent_T(n) = 0$ for all $n \in \mathbb{N}\backslash\{5k - 4 : k \in \mathbb{N}\}$ and $ent_T(5k - 4) = 1$ for all $k \in \mathbb{N}$. Also $cyc_T(n) = 4$ for all $n \in \mathbb{N}$.

Example 2 Let $T : \mathbb{N} \mapsto \mathbb{N}$ be defined as:

$$T(x) = \begin{cases} 2n \text{ if n is odd} \\ \frac{n}{2} \text{ if n is even} \end{cases}.$$

Then $ent_T(2^k(2n - 1)) = k - 1$ and $ent_T(2n - 1) = 0$ for all $n \in \mathbb{N}$. Also $cyc_T(n) = 2$ for all $n \in \mathbb{N}$.

For details about the above discussions and definitions we refer to [2, 3] (Fig. 2).

One can easily prove the following:-

Proposition 1 *If $T : \mathbb{N} \to \mathbb{N}$ is any self-map such that $C_T \in L(\ell^{pq}_\mu)$ and Went(T) = $m < \infty$, then*

$$\lim_{n\in\mathbb{N}_0(T)} \frac{\mu(T^{-k}(n))}{\mu(n)} = 0,$$

for $k > m$.

We shall denote the functional $||.||_{pq,\mu}$ of ℓ^{pq}_μ by $||.||$ in the next part of the paper.

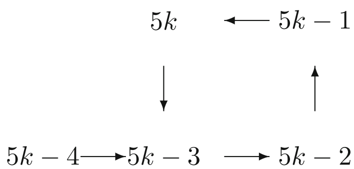

Fig. 1 $(k \in \mathbb{N})$

Fig. 2 For $k \in \mathbb{N}$, $n \geq 1$

Lemma 1 *Suppose that the set*

$$\bigcup_{s\geq t}\mathbb{N}_s(T)=\{n\in\mathbb{N}: ent_T(n)=0 \text{ and } cyc_T(n)\geq t\}$$

is infinite for some $t\in\mathbb{N}$. *Let p(z) be a polynomial such that* $p(C_T)$ *is compact. Then degree* $(p(z))\geq t$.

Proof Consider a polynomial $p(z)=p_o+p_1z+...+p_{m-1}z^{m-1}+z^m$ and let $m<t$. Suppose that the set $\mathbb{N}^{*,\mu}_{m,\epsilon}(T)$ is infinite. Then there are infinite number of orbits of T each having length atleast t. Choose a point n_k from each such orbit such that

$$\mu(T^m(n_k))<\mu(T^{\ell}(n_k))$$

for each $\ell\geq 0$. So we obtain a sequence $\{n_k\}$ such that

$$\{T^r(n_k)\}_{r=0}^{t-1}\cap\{T^r(n_\ell)\}_{r=0}^{t-1}=\phi,\ \text{if } k\neq\ell$$

with $cyc_T(n_k)\geq t$ and $\mu(n_k)\geq\mu(T^m(n_k)$ for each k. For each $i\in\mathbb{N}$, $\chi_{\{i\}}\in\ell^{p,q}_{\mu}$ and let

$$\widetilde{\chi}_{\{T^m(n_k)\}}(n)=\begin{cases}(p/q)^{-1/q}(\mu(T^m(n_k))^{-1/p}\chi_{\{T^m(n_k)\}}(n), \\ \qquad\qquad \text{if } 1<p<\infty,\ 1\leq q<\infty \\ \\ (\mu(T_m(n_k))^{-1/p}\chi_{\{T^m(n_k)\}}(n), \\ \qquad\qquad \text{if } 1<p\leq\infty,\ q=\infty.\end{cases}$$

for each $n\in\mathbb{N}$. Then $||\widetilde{\chi}_{\{T^m(n_k)\}}||=1$ and

$$p(C_T)\widetilde{\chi}_{\{T^m(n_k)\}}(n_\ell)=\begin{cases}(p/q)^{-1/q}(\mu(T^m(n_k))^{-1/p}\sum_{j=0}^{m}p_j\chi_{\{T^m(n_k)\}}(T^j(n_\ell)), \\ \qquad\qquad\qquad\qquad \text{if } 1<p<\infty, 1\leq q<\infty \\ \\ (\mu(T^m(n_k))^{-1/p}\sum_{j=0}^{m}p_j\chi_{\{T^m(n_k)\}}(T^j(n_\ell)), \\ \qquad\qquad \text{if } 1<p<\infty, q=\infty.\end{cases}$$

$$=\begin{cases}0, & \text{if } k\neq\ell \\ (p/q)^{-1/q}(\mu(T^m(n_k))^{-1/p}, & \text{if } k=\ell,\ 1<p<\infty, 1\leq q<\infty \\ (\mu(T^m(n_k))^{-1/p}, & \text{if } k=\ell,\ 1<p\leq\infty, q=\infty.\end{cases}$$

Note that $p_m = 1$. For $1 < p < \infty$, $1 \leq q < \infty$ and $k \neq \ell$, we have

$$\sum_{n=1}^{\infty}\left\{n^{1/p}(p(C_T)(\widetilde{\chi}_{\{T^m(n_k)\}} - \widetilde{\chi}_{\{T^m(n_\ell)\}})(n))^*\mu(n)\right\}^q \frac{1}{n}$$

$$\begin{aligned}
&\geq \sum_{n=1}^{\infty}\left[n^{1/p}|p(C_T)(\widetilde{\chi}_{\{T^m(n_k)\}} - n\widetilde{\chi}_{\{T^m(n_\ell)\}})(n)|\mu(n)\right]^q \frac{1}{n} \\
&\geq (n_k^{1/p}|p(C_T)\widetilde{\chi}_{\{T^m(n_k)\}}\mu(n_k|))^q \frac{1}{n_k} \\
&\qquad + (n_\ell^{1/p}|p(C_T)\widetilde{\chi}_{\{T^m(n_k)\}}\mu(n_\ell)|)^q \frac{1}{n_\ell} \\
&= (\frac{p}{q})^{-1}\{(n_k^{1/p}\frac{\mu(n_k)}{\mu(T^m(n_k))^{1/p}})^q \frac{1}{n_k} + (n_\ell^{1/p}\frac{\mu(n_\ell)}{\mu(T^m(n_\ell))^{1/p}})^q . \frac{1}{n_\ell}\} \\
&\geq \frac{q}{p}(n_k^{q/p^{-1}} + n_\ell^{q/p^{-1}}) > 1 \text{ whenever } q \geq p.
\end{aligned}$$

This implies that $||p(C_T)(\widetilde{\chi}_{\{T^m(n_k)\}} - \widetilde{\chi}_{\{T^m(n_\ell)\}})|| > 1$, whenever $q \geq p$. Thus $p(C_T)$ can not be compact for $q \geq p$.

Similarly, for $1 < p \leq \infty$, $q = \infty$ and $k \neq \ell$, we can see that $p(C_T)$ cannot be compact.

Finally, we consider the case $q < p$. In this case $L^{p,q} \subset L^p$ and $p|q > 1$. For $1 < p < \infty$, $1 \leq q < \infty$ and $k \neq \ell$, we obtain

$$\sum_{n=1}^{\infty}(n^{1/p}(p(C_T)(\widetilde{\chi}_{\{T^m(n_k)\}} - \widetilde{\chi}_{\{T^m(n_\ell)\}})(n))^*\mu(n))^q n^{-1}$$

$$\begin{aligned}
&\geq \sum_{n=1}^{\infty}|(p(C_T)(\widetilde{\chi}_{\{T^m(n_k)\}} - \widetilde{\chi}_{\{T^m(n_\ell)\}})(n)|\mu(n))^p \\
&\geq |(p(C_T)(\widetilde{\chi}_{\{T^m(n_k)\}}(n_k)|\mu(n_k))^p + |(p(C_T)(\widetilde{\chi}_{\{T^m(n_\ell)\}}(n_\ell)|\mu(n_\ell))^p \\
&= (\frac{p}{q})^{-p/q}[\frac{(\mu(n_k))^p}{\mu(T^m(n_k))} + \frac{(\mu(n_\ell))^p}{\mu(T^m(n_\ell))}] \\
&\geq (p/q)^{-p/q} \\
&= (q/p)^{p/q} \\
&> q/p.
\end{aligned}$$

This proves that $p(C_T)$ can not be compact for $q < p$. ∎

Lemma 2 *Let the set*

$$\mathbb{N}_s(T) = \{n \in \mathbb{N} : ent_T(n) = 0 \text{ and } cyc_T(n) = s\}$$

be infinite for some $s \in \mathbb{N}$. Further if p(z) is a polynomial such that $p(C_T)$ is compact, then $z^s - 1$ divides p(z).

Proof Let $p(z) = p_o + p_1 z + ... + p_t z^t$. Then $z^s - 1$ divides p(z) if and only if $\sum_{j+\ell \equiv 0(s)} p_j = 0 \quad \forall \quad \ell \in \{0, 1, ..., s-1\}$. We assume that $z^s - 1$ does not divide p(z). Then there exists an integer ℓ and some $c > 0$ such that

$$|\sum_{j+\ell=0(s)} p_j| > c.$$

Choose a sequence $\{n_k\}_{k=1}^{\infty}$ from the infinite set $\mathbb{N}_s(T)$ such that $cyc_T(n) = s$, $\mu(n_k) \leq \mu(T^{\ell}(n_k))$ and $\{T^r(n_\ell\}_{r=0}^{s-1} \cap \{T^r(n_k)\}_{r=0}^{s-1} = \phi, \quad k \neq \ell$.

Take

$$\widetilde{\chi}_{\{n_k\}}(n) = \begin{cases} (p/q)^{-1/q}(\mu(n_k))^{-1/p}\chi_{\{n_k\}}(n), & \text{if } 1 < p < \infty,\ 1 \leq q < \infty \\ (\mu(n_k))^{-1/p}\chi_{\{n_k\}}(n), & \text{if } 1 < p \leq \infty,\ q = \infty \end{cases}$$

for each $n \in \mathbb{N}$. Then $||\widetilde{\chi}_{\{n_k\}}|| = 1$ for each k.
For $1 < p < \infty$, $1 \leq q < \infty$ and $k \neq \ell$, we obtain

$$\sum_{n=1}^{\infty}(n^{1/p}(p(C_T)(\widetilde{\chi}_{\{n_k\}} - \widetilde{\chi}_{\{n_\ell\}})(n))^* \frac{\mu(n)}{c})^q \frac{1}{n}$$

$$\geq \sum_{n=1}^{\infty}\left(n^{1/p}|p(C_T)(\widetilde{\chi}_{\{n_k\}} - \widetilde{\chi}_{\{n_\ell\}})(n)|\frac{\mu(n)}{c}\right)^q \frac{1}{n}$$

$$\geq (n_k^{1/p}|(\sum_{j=0}^{t} p_j(\widetilde{\chi}_{\{n_k\}}(T^{j+\ell}(n_k)) - \widetilde{\chi}_{\{n_\ell\}}(T^{j+\ell}(n_k)))|\frac{\mu(T^\ell(n_k))}{c})^q \frac{1}{n_k}$$

$$= (n_k^{1/p}(p/q)^{-1/q}(\mu(n_k))^{-1/p}|\frac{\sum_{j=0}^{t} p_j\chi_{\{n_k\}}(T^{j+\ell}(n_k))}{c}|\mu(T^\ell(n_k))^q \frac{1}{n_k}$$

$$> ((n_k^{1/p}(p/q)^{-1/p}\frac{\mu(T^\ell(n_k))}{(\mu(n_k))^{1/p}})^q \frac{1}{n_k}$$

$$\geq \frac{q}{p}(n_k)^{q/p-1}$$

> 1 whenever $q \geq p$.

Thus, for $1 < p < \infty$ and $1 \leq q < \infty$, $p(C_T)$ cannot be compact whenever $q \geq p$. As in the Lemma 1, we can see that $p(C_T)$ can not be compact for $q < p$. Similarly, for $1 < p \leq \infty$ and $q = \infty$, $p(C_T)$ cannot be compact. ■

Lemma 3 *Suppose* $m = Went(T) \geq s$ *and p(z) is a polynomial such that* $p(C_T)$ *is compact. Then* z^s *divides p(z).*

Proof Let $p(z) = p_m z^m + ... + p_t z^t = \sum_{j=m}^{t} p_j z^j$ be a polynomial with $p_m \neq 0$, $m < s$. By Proposition 1, there is some $c > 0$ such that the set

$$\mathbb{N}^{*,\mu}_{m,c}(T) = \{n \in \mathbb{N} : ent_T(n) \geq 1 \text{ and } \frac{\mu(T^{-m}(n))}{\mu(n)} > c\}$$

is infinite. So, we can choose a sequence $\{n_k\}_{k=1}^{\infty}$ with the properties:

$$\frac{\mu(T^{-m}(n_k))}{\mu(n_k)} > c; \quad n_k \neq n_\ell \text{ for } k \neq \ell, \quad ent_T(n_k) \geq 1,$$

that is, $n_k \neq T^j(n_k)$ for $j \geq 1$, $n_k \neq T^j(n_\ell)$ if $k > \ell$ and $j \in \{0, 1, ..., n-m\}$. As in the Lemma 1, take

$$\widetilde{\chi}_{\{n_k\}}(n) = \begin{cases} (p/q)^{-1/q}(\mu(n_k))^{-1/p}\chi_{\{n_k\}}(n), & \text{if } 1 < p < \infty,\ 1 \leq q < \infty \\ (\mu(n_k))^{-1/p}\chi_{\{n_k\}}(n), & \text{if } 1 < p \leq \infty,\ q = \infty \end{cases}$$

and its norm is 1 in ℓ^{pq}_{μ} for each k.
Now,

$$P(C_T)\widetilde{\chi}_{\{n_k\}}(n) = \begin{cases} (p/q)^{-1/q}(\mu(n_k))^{-1/p}\sum_{j=m}^{t} p_j\chi_{\{n_k\}}(T^j(n)), \\ \qquad \text{if } 1 < p < \infty, 1 < q < \infty \\ (\mu(n_k))^{-1/p}\sum_{j=m}^{t} p_j\chi_{\{n_k\}}(T^j(n)), \\ \qquad \text{if } 1 < p \leq \infty, q = \infty \end{cases}$$

$$= \begin{cases} 0, & \text{if } n \in T^{-m}(n_\ell) \text{ and } \ell < k \\ p_m(p/q)^{-1/q}(\mu(n_k))^{-1/p}, & \text{if } n \in T^{-m}(n_k);\ 1 < p < \infty,\ 1 \leq q < \infty \\ p_m(\mu(n_k))^{-1/p}, & \text{if } n \in T^{-m}(n_k);\ 1 < p \leq \infty,\ q = \infty \end{cases}$$

Thus, for $1 < p \leq \infty,\ 1 \leq q < \infty$ and $l < k$, we obtain

$$\sum_{n=1}^{\infty}(n^{1/p}(\left(\frac{p(C_T)(\widetilde{\chi}_{\{n_k\}}-\widetilde{\chi}_{\{n_\ell\}})}{|p_m|}\right)(n))^{*}\mu(n))^{q}\frac{1}{n}$$
$$\geq \sum_{n=1}^{\infty}(n^{1/p}|\left(\frac{p(C_T)(\widetilde{\chi}_{\{n_k\}}-\widetilde{\chi}_{\{n_\ell\}})}{|p_m|}\right)(n)|\mu(n))^{q}\frac{1}{n}$$
$$\geq \sum_{n\in T^{-m}(n_\ell)}(n^{1/p}|\left(\frac{p(C_T)(\widetilde{\chi}_{\{n_k\}}-\widetilde{\chi}_{\{n_\ell\}})}{|p_m|}\right)(n)|\mu(n))^{q}\frac{1}{n}$$
$$= \sum_{n\in T^{-m}(n_\ell)}(n^{1/p}\frac{(p/q)^{-1/q}(\mu(n_\ell))^{-1/p}}{|p_m|}|p_m|\mu(n))^{q}\frac{1}{n}$$
$$\geq (n_\ell^{1/p}(p/q)^{-1/q}\frac{\mu(T^{-m}(n_k))}{(\mu(n_\ell))^{1/p}})^{q}\frac{1}{n_\ell}$$
$$> c(\frac{q}{p})(n_\ell^{q/p-1})$$
$$\geq c \text{ whenever } q \geq p.$$

Thus, the set $\{p(C_T)\widetilde{\chi}_{\{n_k\}} : k \in \mathbb{N}\}$ has no cluster point. So, for $1 < p < \infty$ and $1 \leq q < \infty$, $p(C_T)$ cannot be compact whenever $q \geq p$. As in Lemma 1, one can prove that $p(C_T)$ cannot be compact for $q < p$. Similarly, for $1 < p \leq \infty$ and $q = \infty$, $p(C_T)$ cannot be compact. ■

Theorem 2 *Let $T : \mathbb{N} \to \mathbb{N}$ be a self-map such that $C_T \in L(\ell_\mu^{p,q})$. Then the following conditions are equivalent:*

(i) $dim(alg\pi(C_T)) < \infty$;
(ii) $Went(T) < \infty, \quad |Per(T)| < \infty$;
(iii) $Went(T) < \infty, \quad |Wper(T)| < \infty, |\mathbb{N}_U(T)| < \infty$.

Further suppose that $alg\pi(C_T)$ is finite - dimensional Then

$$alg\pi(C_T) \cong \mathbb{C}[z]/(q(z)) \text{ for } q(z) = z^m \Pi_{\lambda\in H}(z-\lambda)$$

where m = Went(T) and $H = \bigcup_{k\in Wper(T)} G_k, \quad G_k = \{\lambda \in \mathbb{C} : \lambda^k = 1\}$.

Proof Proof follows almost on similar lines as in [3]. ■

Corollary 1 *Let $T : \mathbb{N} \to \mathbb{N}$ be a self map. Then the following conditions are equivalent :-*

(i) $C_T \in K(\ell_\mu^{p,q})$;
(ii) $C_T \in L(\ell_\mu^{p,q}), \quad alg\pi(C_T) \cong \mathbb{C}[z]/(z) \cong \mathbb{C}$;

(iii) $C_T \in L(\ell^{p,q}_\mu)$, $|Per(T)| < \infty, Wper(T) = \phi, Went(T) = 1$;

(iv) $|Per(T)| < \infty, Wper(T) = \phi, \quad \mu(T^{-1}(n)) < \infty$ *and* $\lim_{n\in\mathbb{N}} \dfrac{\mu(T^{-1}(n))}{\mu(n)} = 0.$

Corollary 2 *If μ is the counting measure, then every essentially composition operator on $\ell^{p,q}_\mu$ is algebraic.*

Proof Proof is on the similar lines as in [[3], Corollary 3.6] ■

Example 3 The forward shift operator S on $\ell^{p,q}$ is defined by

$$S(z_1, z_2, ...) = (0, z_1, z_2, ...)$$

Then for every non-zero element 'a' in $\ell^{p,q}$, we have $ent_S(a) = cyc_S(a) = \infty$, $ent(S) = \infty$ and for $a_0 = (0, 0, ...,)$ in $\ell^{p,q}$, we have $ent_S(a_0) = 0, cyc_S(a_0) = 1$.

Thus, $Per(S) = \{1\}$, $Wper(S) = \phi$, $Uper(S) = \{1\}$ and $X_U(S) = \{a_0\}$.

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Higher Dimensional Hardy-Type Inequalities

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Abstract We give necessary and sufficient conditions for certain multidimensional Hardy inequalities over spherical cones. The inequalities involve adjoint Hardy operator. A mixed norm inequality has also been characterised.

Keywords Hardy operator · Adjoint Hardy operator · Mixed norm · Higher dimensional inequalities

1 Introduction

Let $\varSigma_N$ be the surface of the unit ball in R^N, i.e., $\varSigma_N = \{x \in R^N : |x| = 1\}$, where $|x|$ denotes the Euclidean norm of the vector $x \in R^N$. Let B_N be a measurable subset of $\varSigma_N$ and $E \subset R^N$ be a spherical cone, i.e.,

$$E = \{x \in R^N : x = s\sigma, 0 \leq s < \infty, \sigma \in B_N\}.$$

Let S_{N_x}, $x \in R^N$ denote the part of E with radius $\leq |x|$, i.e.,

$$S_{N_x} = \{y \in R^N : y = s\sigma, 0 \leq s \leq |x|, \sigma \in B_N\}.$$

Further, we denote by $\alpha S_N, \alpha > 0$, the part of E with radius $\leq \alpha$. Note that $E = \bigcup_{\alpha>0} \alpha S_N$. For $x \in E \setminus \{0\}$, we denote by $|S_{N_x}|$, the volume of S_{N_x}. The symbols B_M, F, S_{M_y}, $|S_{M_y}|$ are defined similarly for an M-dimensional setting.

Consider a multidimensional Hardy operator H_E defined by

$$(H_E f)(x) = \int_{S_{N_x}} f(y)dy, \quad x \in E.$$

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In [10], Sinnamon studied the Hardy inequality

$$\left(\int_E (H_E f)^q(x)w_0(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_E f^p(x)w(x)dx\right)^{\frac{1}{p}} \tag{1}$$

in terms of the standard one-dimensional inequality

$$\left(\int_0^\infty (Hg)^q(x)W_0(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty g^p(x)W(x)dx\right)^{\frac{1}{p}}, \tag{2}$$

where H is the classical Hardy operator $(Hf)(x) = \int_0^x f(t)dt$, w_0, w are weights defined on E and W_0, W are weights defined suitably on $(0, \infty)$.

Later this technique of studying Higher dimensional inequalities in terms of one-dimensional inequalities was used in [3–5] for different operators, e.g., Hardy-Steklov operator, Geometric mean operator etc. Using this technique, higher dimensional compactness of Hardy operator and Hardy-Steklov operators were also obtained in [4, 5].

In this paper, we shall give the same treatment to the adjoint of H_E given by

$$(H_E^* f)(x) = \int_{E\setminus S_{N_x}} f(y)dy.$$

We shall show that the inequality (1) with H_E replaced by H_E^* holds for all functions $f \geq 0$ if and only if the inequality (2) with H replaced by the adjoint of H, i.e.,

$$(H^* g)(x) = \int_x^\infty g(t)dt$$

holds for all functions $g \geq 0$. This result is proved in Sect. 2.

Next, consider the double sized multidimensional operator

$$(H_{E,F} f)(x, y) = \int_{S_{M_x}} \int_{S_{N_y}} f(s, t)dtds$$

and its adjoint

$$(H_{E,F}^* f)(x, y) = \int_{E\setminus S_{M_x}} \int_{F\setminus S_{N_y}} f(s, t)dtds\,.$$

In [3], the Hardy type inequality involving $H_{E,F}$ has been studied in terms of the inequality involving the two dimensional operator

$$(H_2 g)(x, y) = \int_0^x \int_0^y g(s, t)dtds.$$

In this paper, we study similar result for the operator $H^*_{E,F}$. Finally, we consider a mixed norm type inequality involving the operator $H^*_{E,F}$ and characterise it in terms of another mixed norm inequality involving the adjoint operator H^*_2 given by

$$(H^*_2 g)(x, y) = \int_x^\infty \int_y^\infty g(s, t)dtds.$$

These results are proved in Sect. 3.

2 The Operator H^*_E

We prove the following:

Theorem 1 *Let* $1 < p < \infty$, $0 < q < \infty$, E, S_{N_x}, B_N *be as defined above and* w, w_0 *be weight functions on* E. *Then the inequality*

$$\left(\int_E w_0(x)(H^*_E f)^q(x)dx\right)^{\frac{1}{q}} \le C\left(\int_E w(x) f^p(x)dx\right)^{\frac{1}{p}} \tag{3}$$

holds for all $f \ge 0$ *if and only if*

$$\left(\int_0^\infty W_0(x)(H^* g)^q(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty W(x)g^p(x)dx\right)^{\frac{1}{p}} \tag{4}$$

holds for all $g \ge 0$ *with*

$$W_0(x_0) = \int_{B_N} w_0(x_0x')x_0{}^{N-1}dx', \qquad x_0 > 0 \tag{5}$$

$$W(x_0) = \left(\int_{B_N} w^{1-p'}(x_0x')x_0{}^{N-1}dx'\right)^{1-p}, \qquad x_0 > 0. \tag{6}$$

Proof Suppose (4) holds. Fix a non-negative locally integrable function $f : E \to \mathbb{R}$ and define

$$g(x_0) = \int_{B_N} f(x_0x')x_0{}^{N-1}dx', \quad x_0 > 0. \tag{7}$$

By making variable transformation $x = x_0x'$, where $x \in E$, $x_0 = |x| \in (0, \infty)$, $x' \in B_N$ and similarly $s = s_0s'$, we have

$$\begin{aligned}(H_E^* f)(x) &= \int_{E\setminus S_{N_x}} f(s)ds \\ &= \int_{x_0}^{\infty}\int_{B_N} f(s_0 s')s_0^{N-1}ds'ds_0 \\ &= \int_{x_0}^{\infty} g(s_0)ds_0 \\ &= (H^* g)(x_0). \end{aligned} \tag{8}$$

Now, making variable transformation $x = x_0 x'$, using (5), (8) and (4), we have

$$\begin{aligned}\left(\int_E w_0(x)(H_E^* f)^q(x)dx\right)^{\frac{1}{q}} &= \left(\int_0^{\infty}\int_{B_N} w_0(x_0x')(H^*g)^q(x_0)x_0^{N-1}dx'dx_0\right)^{\frac{1}{q}} \\ &= \left(\int_0^{\infty} W_0(x_0)(H^*g)^q(x_0)dx_0\right)^{\frac{1}{q}} \\ &\le C\left(\int_0^{\infty} W(x_0)g^p(x_0)dx_0\right)^{\frac{1}{p}}.\end{aligned}$$

Next, by using (7), applying Hölder's inequality for the inner integral and using (6), we get

$$\begin{aligned}\left(\int_E w_0(x)(H_E^* f)^q(x)dx\right)^{\frac{1}{q}} &\le C\left(\int_0^{\infty} W(x_0)\left(\int_{B_N} f(x_0x')x_0^{N-1}dx'\right)^p dx_0\right)^{\frac{1}{p}} \\ &\le C\left(\int_0^{\infty} W(x_0)\left\{\int_{B_N} f^p(x_0x')w(x_0x')x_0^{N-1}dx'\right\}\right. \\ &\qquad \times \left.\left(\int_{B_N} w^{1-p'}(x_0x')x_0^{N-1}dx'\right)^{\frac{p}{p'}} dx_0\right)^{\frac{1}{p}} \\ &= C\left(\int_0^{\infty}\int_{B_N} w(x_0x')f^p(x_0x')x_0^{N-1}dx'dx_0\right)^{\frac{1}{p}} \\ &= C\left(\int_E w(x)f^p(x)dx\right)^{\frac{1}{p}}\end{aligned}$$

and therefore (3) holds.

For the converse, assume that (3) holds. Fix a non-negative locally integrable function $g : (0, \infty) \to \mathbb{R}$ and define $f : E \to \mathbb{R}$ by

$$f(x_0x') = g(x_0)W^{p'-1}(x_0)w^{1-p'}(x_0x'), \quad x_0 > 0,\ x' \in B_N.$$

Note that

$$\int_{B_N} f(x_0x')x_0^{N-1}dx' = g(x_0).$$

Therefore, as in the first part of the proof, we have

$$\left(\int_0^\infty W_0(x_0)(H^*g)^q(x_0)dx_0\right)^{\frac{1}{q}} = \left(\int_E w_0(x)(H_E^*f)^q(x)dx\right)^{\frac{1}{q}}.$$

Now, using (3) and then making use of (6) and (9), we have

$$\begin{aligned}\left(\int_0^\infty W_0(x_0)(H^*g)^q(x_0)dx_0\right)^{\frac{1}{q}} &\le C\left(\int_E w(x)f^p(x)dx\right)^{\frac{1}{p}}\\ &= C\left(\int_0^\infty\int_{B_N} f^p(x_0x')w(x_0x')x_0^{N-1}dx'dx_0\right)^{\frac{1}{p}}\\ &= C\left(\int_0^\infty g^p(x_0)W^{p'}(x_0)\right.\\ &\quad\left.\times\left(\int_{B_N} w^{1-p'}(x_0x')x_0^{N-1}dx'\right)dx_0\right)^{\frac{1}{p}}\\ &= C\left(\int_0^\infty W(x_0)g^p(x_0)dx_0\right)^{\frac{1}{p}},\end{aligned}$$

i.e., (4) holds and the assertion is obtained.

The boundedness of the Hardy operator between the weighted Lebesgue spaces is also well known. The following is the corresponding result.

Theorem A ([7, 8]) *Let W_0, W be weight functions on* $(0,\infty)$.

(i) For $1 < p \le q < \infty$, the inequality

$$\left(\int_0^\infty\left(\int_x^\infty g(t)dt\right)^q W_0(x)dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty g^p(x)W(x)dx\right)^{\frac{1}{p}}$$

holds for $g \ge 0$ if and only if

$$B^* := \sup_{x>0}\left(\int_0^x W_0(y)dy\right)^{\frac{1}{q}}\left(\int_x^\infty W^{1-p'}(y)dy\right)^{\frac{1}{p'}} < \infty. \tag{9}$$

Moreover, if C is the best possible constant then

$$B^* \le C \le \left(1+\frac{q}{p'}\right)^{\frac{1}{q}}\left(1+\frac{p'}{q}\right)^{\frac{1}{p'}} B^*.$$

(ii) For $0 < q < p < \infty$, $p > 1$, *the inequality holds for* $g \ge 0$ *if and only if*

$$A^* := \left(\int_0^\infty \left(\int_0^x W_0(y)dy\right)^{\frac{r}{q}} \left(\int_x^\infty W^{1-p'}(y)dy\right)^{\frac{r}{q'}} W^{1-p'}(x)dx\right)^{\frac{1}{r}} < \infty. \tag{10}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. *Moreover, if C is the best possible constant then*

$$q^{\frac{1}{q}}\left(\frac{p'q}{r}\right)^{\frac{1}{q'}} A^* \le C \le q^{\frac{1}{q}}(p')^{\frac{1}{q'}} A^*.$$

In view of Theorems 1 and A, the precise weight conditions for the inequality (3) can be written. We do it in the following theorem:

Theorem 2 *Let the assumptions of Theorem 1 be satisfied.*

(a) For $p \le q$, *the inequality (3) holds if and only if (9) holds with* W_0 *and* W *given, respectively, by (5) and (6).*
(b) For $q < p$, *the inequality (3) holds if and only if (10) holds with* W_0 *and* W *given, respectively, by (5) and (6).*

3 The Operator $H^*_{E,F}$ with Usual Norm and Mixed Norm

Here, we shall be considering cones in R^N as well as in R^M. Consider a double sized multidimensional operator

$$(H_{E,F}f)(x,y) = \int_{S_{M_x}} \int_{S_{N_y}} f(s,t)dtds$$

and its adjoint

$$(H^*_{E,F}f)(x,y) = \int_{E\setminus S_{M_x}} \int_{F\setminus S_{N_y}} f(s,t)dtds\,.$$

Now, we prove the following result which characterises the boundedness of the operator $H^*_{E,F}$. In fact, the characterisation is obtained in terms of the boundedness of the operator H^*_2.

Theorem 3 *Let* $1 < p < \infty$, $0 < q < \infty$ *and* E, F, S_{M_x}, S_{N_y}, B_M, B_N *be as defined above and* w, w_0 *be weight functions on* $E \times F$. *Then the inequality*

$$\left(\int_E \int_F w_0(x, y)(H^*_{E,F} f)^q(x, y)dydx \right)^{\frac{1}{q}} \le C \left(\int_E \int_F w(x, y) f^p(x, y)dydx \right)^{\frac{1}{p}} \tag{11}$$

holds for all $f \ge 0$ *if and only if*

$$\left(\int_0^\infty \int_0^\infty W_0(x, y)(H^*_2 g)^q(x, y)dydx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \int_0^\infty W(x, y)g^p(x, y)dydx \right)^{\frac{1}{p}} \tag{12}$$

holds for all $g \ge 0$ *with*

$$W_0(x_0, y_0) = \int_{B_M} \int_{B_N} w_0(x_0x', y_0y') {x_0}^{M-1} {y_0}^{N-1} dy'dx', \qquad x_0 > 0, y_0 > 0 \tag{13}$$

$$W(x_0, y_0) = \left(\int_{B_M} \int_{B_N} w^{1-p'}(x_0x', y_0y') x_0^{M-1} y_0^{N-1} dy'dx' \right)^{1-p}, \quad x_0 > 0, y_0 > 0. \tag{14}$$

Proof Suppose (12) holds. Fix a non-negative locally integrable function $f : E \times F \to \mathbb{R}$. Define

$$g(x_0, y_0) = \int_{B_M} \int_{B_N} f(x_0x', y_0y') {x_0}^{M-1} {y_0}^{N-1} dy'dx', \quad x_0, y_0 > 0 \tag{15}$$

so that on making the changes of variables $x = x_0x'$ where $x \in E$, $x_0 = |x| \in (0, \infty)$, $x' \in B_M$ and similarly $y = y_0y'$, $s = s_0s'$, $t = t_0t'$, we have

$$\begin{aligned}
(H^*_{E,F} f)(x, y) &= \int_{E \setminus S_{M_x}} \int_{F \setminus S_{N_y}} f(s, t)dtds \\
&= \int_{x_0}^\infty \int_{y_0}^\infty \int_{B_M} \int_{B_N} f(s_0s', t_0t') s_0^{M-1} t_0^{N-1} dt'ds'dt_0ds_0 \\
&= \int_{x_0}^\infty \int_{y_0}^\infty g(s_0, t_0)dt_0ds_0 \\
&= (H^*_2 g)(x_0, y_0).
\end{aligned} \tag{16}$$

Now, making variable transformations $x = x_0x'$, $y = y_0y'$, using (16), (13) and then (12), we have

$$\begin{aligned}
&\left(\int_E\int_F w_0(x,y)(H^*_{E,F}f)^q(x,y)dydx\right)^{\frac{1}{q}}\\
&=\left(\int_0^\infty\int_0^\infty\int_{B_M}\int_{B_N} w_0(x_0x',y_0y')(H_2^*g)^q(x_0,y_0)x_0^{M-1}y_0^{N-1}dy'dx'dy_0dx_0\right)^{\frac{1}{q}}\\
&=\left(\int_0^\infty\int_0^\infty W_0(x_0,y_0)(H_2^*g)^q(x_0,y_0)dy_0dx_0\right)^{\frac{1}{q}}\\
&\le C\left(\int_0^\infty\int_0^\infty W(x_0,y_0)g^p(x_0,y_0)dy_0dx_0\right)^{\frac{1}{p}}.
\end{aligned}$$

Next, we use (15), apply Hölder's inequality for the inner integral and then use (14) to get

$$\begin{aligned}
&\left(\int_E\int_F w_0(x,y)(H^*_{E,F}f)^q(x,y)dydx\right)^{\frac{1}{q}}\\
&\le C\left(\int_0^\infty\int_0^\infty W(x_0,y_0)\left(\int_{B_M}\int_{B_N} f(x_0x',y_0y')x_0^{M-1}y_0^{N-1}dy'dx'\right)^p dy_0dx_0\right)^{\frac{1}{p}}\\
&\le C\left(\int_0^\infty\int_0^\infty W(x_0,y_0)\left(\int_{B_M}\int_{B_N} f^p(x_0x',y_0y')w(x_0x',y_0y')x_0^{M-1}y_0^{N-1}dy'dx'\right)\right.\\
&\quad\left.\times\left\{\int_{B_M}\int_{B_N} w^{1-p'}(x_0x',y_0y')x_0^{M-1}y_0^{N-1}dy'dx'\right\}^{\frac{p}{p'}}dy_0dx_0\right)^{\frac{1}{p}}\\
&\le C\left(\int_E\int_F w(x,y)f^p(x,y)dydx\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus (11) holds.

To prove the converse, suppose (11) holds and fix a non-negative locally integrable function $g : (0,\infty)\times(0,\infty)\to\mathbb{R}$. Define $f : E\times F\to\mathbb{R}$ by

$$\begin{aligned}
&f(x_0x',y_0y') = g(x_0,y_0)W^{p'-1}(x_0,y_0)w^{1-p'}(x_0x',y_0y'),\\
&x_0,y_0>0,\quad x'\in B_M,\quad y'\in B_N.
\end{aligned}\tag{17}$$

Then, using (17)

$$\left(\int_{B_M}\int_{B_N} f(x_0x',y_0y')x_0^{M-1}y_0^{N-1}dy'dx'\right) = g(x_0,y_0).$$

Therefore, as in the first part of the proof, we have

$$\left(\int_0^\infty\int_0^\infty W_0(x_0,y_0)(H_2^*g)^q(x_0,y_0)dy_0dx_0\right)^{\frac{1}{q}} = \left(\int_E\int_F w_0(x,y)(H_{E,F}^*f)^q(x,y)dydx\right)^{\frac{1}{q}}.$$

Now, using (11) and then making use of (14) and (17), we have

$$\begin{aligned}
&\left(\int_0^\infty\int_0^\infty W_0(x_0,y_0)(H_2^*g)^q(x_0,y_0)dy_0dx_0\right)^{\frac{1}{q}}\\
&\le C\left(\int_E\int_F w(x,y)f^p(x,y)dydx\right)^{\frac{1}{p}}\\
&= C\left(\int_0^\infty\int_0^\infty\int_{B_M}\int_{B_N} f^p(x_0x',y_0y')w(x_0x',y_0y')x_0^{M-1}y_0^{N-1}dy'dx'dy_0dx_0\right)^{\frac{1}{p}}\\
&= C\left(\int_0^\infty\int_0^\infty g^p(x_0,y_0)W^{p'}(x_0,y_0)\right.\\
&\quad\left.\times\int_{B_M}\int_{B_N} w^{1-p'}(x_0x',y_0y')x_0^{M-1}y_0^{N-1}dy'dx'dy_0dx_0\right)^{\frac{1}{p}}\\
&= C\left(\int_0^\infty\int_0^\infty W(x_0,y_0)g^p(x_0,y_0)dy_0dx_0\right)^{\frac{1}{p}}
\end{aligned}$$

Thus (12) holds.

In order to obtain the precise weight conditions for the inequality (11), let us mention that in [9], Sawyer obtained the following results in connection with the operator H_2:

Theorem B *Let* $1 \le p \le q < \infty$ *and* W_0, W *be weight functions defined on* $(0,\infty)\times(0,\infty)$. *Then the inequality*

$$\left(\int_0^\infty\int_0^\infty (H_2g)^q(x,y)W_0(x,y)dxdy\right)^{\frac{1}{q}} \le C\left(\int_0^\infty\int_0^\infty W(x,y)g^p(x,y)dxdy\right)^{\frac{1}{p}} \quad (18)$$

holds for $g \ge 0$ *if and only if the following three conditions are satisfied:*

$$C_0 = \sup_{x>0, y>0} \left(\int_x^\infty \int_y^\infty W_0(s,t) dt ds \right)^{\frac{1}{q}} \left(\int_0^x \int_0^y W^{1-p'}(s,t) dt ds \right)^{\frac{1}{p'}} < \infty,$$

$$\int_0^x \int_0^y \left(\int_0^s \int_0^t W^{1-p'}(\sigma,\tau) d\tau d\sigma \right)^q W_0(s,t) dt ds \le C_0^q \left(\int_0^x \int_0^y W^{1-p'}(s,t) dt ds \right)^{\frac{q}{p}},$$

$$\int_x^\infty \int_y^\infty \left(\int_s^\infty \int_t^\infty W_0(\sigma,\tau) d\tau d\sigma \right)^{p'} W^{1-p'}(s,t) dt ds \le C_0^{p'} \left(\int_x^\infty \int_y^\infty W_0(s,t) dt ds \right)^{\frac{p'}{q'}}.$$

Note by using the duality arguments that the inequality (18) holds if and only if the inequality

$$\left(\int_0^\infty \int_0^\infty (H_2^* g)^{p'}(x,y) W^{1-p'}(x,y) dx dy \right)^{\frac{1}{p'}} \le C \left(\int_0^\infty \int_0^\infty W_0^{1-q'}(x,y) g^{q'}(x,y) dx dy \right)^{\frac{1}{q'}} \tag{19}$$

holds for $g \ge 0$. Consequently, using the variable transformation $q = p'$, $p = q'$, $W^{1-p'} = W_0$ and $W_0^{1-q'} = W$, we immediately have the following:

Theorem 4 *Let* $1 < p \le q < \infty$ *and* W_o, W *be weight functions on* $(0,\infty) \times (0,\infty)$. *Then, the inequality*

$$\left(\int_0^\infty \int_0^\infty W_0(x,y) (H_2^* g)^q(x,y) dy dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty \int_0^\infty W(x,y) g^p(x,y) dy dx \right)^{\frac{1}{p}}$$

holds for all non-negative functions g *if and only if the following three conditions are satisfied:*

$$C_0 = \sup_{x>0, y>0} \left(\int_x^\infty \int_y^\infty W^{1-q}(s,t) dt ds \right)^{\frac{1}{p'}} \left(\int_0^x \int_0^y W_0(s,t) dt ds \right)^{\frac{1}{q}} < \infty, \tag{20}$$

$$\int_0^x \int_0^y \left(\int_0^s \int_0^t W_0(\sigma,\tau) d\tau d\sigma \right)^{p'} W^{1-q}(s,t) dt ds \le C_0^{p'} \left(\int_0^x \int_0^y W_0(s,t) dt ds \right)^{\frac{p'}{q'}}, \tag{21}$$

$$\int_x^\infty \int_y^\infty \left(\int_s^\infty \int_t^\infty W^{1-q}(\sigma,\tau) d\tau d\sigma \right)^q W_0(s,t) dt ds \le C_0^q \left(\int_x^\infty \int_y^\infty W^{1-q}(s,t) dt ds \right)^{\frac{q}{p}}. \tag{22}$$

Now, Theorems 3 and 4 immediately yield the following:

Theorem 5 *Let the assumptions of Theorem 3 be satisfied. Further, assume that* $1 \le p \le q < \infty$. *Then the inequality (11) holds if and only if (20), (21) and (22) are satisfied with* W_0 *and* W *given, respectively, by (13) and (14).*

Remark 1 It is of interest to obtain Theorem 5 for the case $q < p$. Unfortunately, the Sawyer's result (Theorem B) is not available for this case.

Next, we prove a mixed norm type inequality for the operator $H^*_{E,F}$ the motivation for which is derived from the papers [1, 2].

Theorem 6 *Let* $0 < q_i < \infty$, $1 < p_i < \infty$ $(i = 1, 2)$ *and let* w_1, v_1 *be weight functions on* E *and* w_2, v_2 *be weight functions on* F. *Then the inequality*

$$\left(\int_E w_1(x)\left(\int_F (H^*_{E,F} f)^{q_2}(x, y)w_2(y)dy\right)^{\frac{q_1}{q_2}} dx\right)^{\frac{1}{q_1}} \leq C\left(\int_E v_1(x)\left(\int_F f^{p_2}(x, y)v_2(y)dy\right)^{\frac{p_1}{p_2}} dx\right)^{\frac{1}{p_1}} \quad (23)$$

holds for all $f \geq 0$ *defined on* $E \times F$ *if and only if the inequality*

$$\left(\int_0^\infty W_1(x_0)\left(\int_0^\infty (H^*_2 g)^{q_2}(x_0, y_0)W_2(y_0)dy_0\right)^{\frac{q_1}{q_2}} dx_0\right)^{\frac{1}{q_1}} \leq C\left(\int_0^\infty V_1(x_0)\left(\int_0^\infty g^{p_2}(x_0, y_0)V_2(y_0)dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}} \quad (24)$$

holds for all $g \geq 0$ *with*

$$W_1(x_0) = \int_{B_M} w_1(x_0x')x_0{}^{M-1}dx', \qquad x_0 > 0 \quad (25)$$

$$W_2(y_0) = \int_{B_N} w_2(y_0y')y_0^{N-1}dy', \qquad y_0 > 0 \quad (26)$$

$$V_1(x_0) = \left(\int_{B_M} v_1^{1-p_1'}(x_0x')x_0{}^{M-1}dx'\right)^{1-p_1}, \qquad x_0 > 0 \quad (27)$$

$$V_2(y_0) = \left(\int_{B_N} v_2^{1-p_2'}(y_0y')y_0{}^{N-1}dy'\right)^{1-p_2}, \qquad y_0 > 0 \quad (28)$$

Proof Suppose (24) holds. Let $x' \in B_M$ and $y' \in B_N$. Fix a non-negative locally integrable function $f : E \times F \to \mathbb{R}$. Define

$$g(x_0, y_0) = \int_{B_M}\int_{B_N} f(x_0x', y_0y')x_0{}^{M-1}y_0{}^{N-1}dy'dx', x_0, y_0 > 0 \quad (29)$$

For $x \in E$, we use polar coordinates $x = x_0x'$, $x_0 = |x| \in (0, \infty)$, $x' \in B_M$ and similarly $y = y_0y'$, $s = s_0s'$, $t = t_0t'$. Thus, we have

$$
\begin{aligned}
(H^*_{E,F} f)(x, y) =& \int_{E\setminus S_{M_x}} \int_{F\setminus S_{N_y}} f(s,t)dtds \\
=& \int_{x_0}^{\infty} \int_{y_0}^{\infty} \int_{B_M} \int_{B_N} f(s_0 s', t_0 t') s_0^{M-1} t_0^{N-1} dt' ds' dt_0 ds_0 \\
=& \int_{x_0}^{\infty} \int_{y_0}^{\infty} g(s_0, t_0) dt_0 ds_0 \\
=& (H_2^* g)(x_0, y_0).
\end{aligned} \tag{30}
$$

Therefore, using (27), Hölder's inequality, Minkowski's integral inequality, (28), Hölder's inequality to the inner integral and using (29), we get

$$
\begin{aligned}
&\left(\int_E v_1(x)\left(\int_F v_2(y) f^{p_2}(x,y)dy\right)^{\frac{p_1}{p_2}} dx\right)^{\frac{1}{p}} \\
&= \left(\int_0^{\infty} \int_{B_M} \left(\int_F f^{p_2}(x_0 x', y) v_2(y) dy\right)^{\frac{p_1}{p_2}} v_1(x_0 x') x_0^{M-1} dx' dx_0\right)^{\frac{1}{p_1}} \\
&= \left(\int_0^{\infty} V_1(x_0) \int_{B_M} \left(\int_F f^{p_2}(x_0 x', y) v_2(y) dy\right)^{\frac{p_1}{p_2}} v_1(x_0 x') x_0^{M-1} dx' \right. \\
&\quad \times \left. \left(v_1^{1-p_1'}(x_0 x') x_0^{M-1} dx'\right)^{p_1 - 1} dx_0\right)^{\frac{1}{p_1}} \\
&\geq \left(\int_0^{\infty} V_1(x_0) \left(\int_{B_M} \left(\int_F f^{p_2}(x_0 x', y) v_2(y) dy\right)^{\frac{1}{p_2}} x_0^{M-1} dx'\right)^{p_1} dx_0\right)^{\frac{1}{p_1}} \\
&= \left(\int_0^{\infty} V_1(x_0) \left(\int_{B_M} \left(\int_0^{\infty} \int_{B_N} f^{p_2}(x_0 x', y_0 y') v_2(y_0 y') \right.\right.\right. \\
&\quad \times \left.\left.\left. y_0^{N-1} dy' (x_0^{M-1})^{p_2} dy_0\right)^{\frac{1}{p_2}} dx'\right)^{p_1} dx_0\right)^{\frac{1}{p_1}} \\
&\geq \left(\int_0^{\infty} V_1(x_0) \left(\int_0^{\infty} \left(\int_{B_M} \left(\int_{B_N} f^{p_2}(x_0 x', y_0 y') v_2(y_0 y') y_0^{N-1} dy'\right)^{\frac{1}{p_2}} \right.\right.\right. \\
&\quad \times \left.\left.\left. x_0^{M-1} dx'\right)^{p_2} dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}} \\
&= \left(\int_0^{\infty} V_1(x_0) \left(\int_0^{\infty} V_2(y_0) \left(\int_{B_M} \left(\int_{B_N} f^{p_2}(x_0 x', y_0 y') v_2(y_0 y') y_0^{N-1} dy'\right)^{\frac{1}{p_2}} \right.\right.\right. \\
&\quad \times \left.\left.\left. \left(\int_{B_N} v_2(y_0 y') y_0^{N-1} dy'\right)^{\frac{1}{p_2'}} x_0^{M-1} dx'\right)^{p_2} dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}} \\
&\geq \left(\int_0^{\infty} V_1(x_0) \left(\int_0^{\infty} V_2(y_0) \left(\int_{B_M} \int_{B_N} f(x_0 x', y_0 y') \times y_0^{N-1} dy' x_0^{M-1} dx'\right)^{p_2} dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}} \\
&= \left(\int_0^{\infty} V_1(x_0) \left(\int_0^{\infty} V_2(y_0) g^{p_2}(x_0, y_0) dy_0\right)^{\frac{p_1}{p_2}} dx_0\right)^{\frac{1}{p_1}}.
\end{aligned}
$$

Next, we use (24), (25), (26) and (30) to get

$$\left(\int_E w_1(x)\left(\int_F (H^*_{E,F}f)^{q_2}(x,y)w_2(y)dy\right)^{\frac{q_1}{q_2}}dx\right)^{\frac{1}{q_1}}$$
$$=\left(\int_0^\infty\int_{B_M} w_1(x_0x')\left(\int_0^\infty\int_{B_N}(H^*_2g)^{q_2}(x_0,y_0)w_2(y_0y')y_0^{N-1}dy'dy_0\right)^{\frac{q_1}{q_2}}\right.$$
$$\left.\times\, x_0^{M-1}dx'dx_0\right)^{\frac{1}{q_1}}$$
$$=\left(\int_0^\infty W_1(x_0)\left(\int_0^\infty (H^*_2g)^{q_2}(x_0,y_0)W_2(y_0)dy_0\right)^{\frac{q_1}{q_2}}dx_0\right)^{\frac{1}{q_1}}$$
$$\leq C\left(\int_0^\infty V_1(x_0)\left(\int_0^\infty V_2(y_0)g^{p_2}(x_0,y_0)dy_0\right)^{\frac{p_1}{p_2}}dx_0\right)^{\frac{1}{p_1}}$$
$$\leq C\left(\int_E v_1(x)\left(\int_F v_2(y)f^{p_2}(x,y)dy\right)^{\frac{p_1}{p_2}}dx\right)^{\frac{1}{p}}$$

Conversely, assume that (23) holds. Fix a locally integrable function $g:(0,\infty)\times(0,\infty)\to\mathbb{R}$ and define $f:E\times F\to\mathbb{R}$ by

$$f(x_0x',y_0y')=g(x_0,y_0)V_2^{p_2'-1}(y_0)v_2^{1-p_2'}(y_0y')V_1^{p_1'-1}(x_0)v_1^{1-p_1'}(x_0x'),$$

where $x_0,y_0>0,\quad x'\in B_M,\quad y'\in B_N$. Then (27) and (28) give

$$\int_{B_M}\int_{B_N} f(x_0x',y_0y')x_0{}^{M-1}y_0{}^{N-1}dy'dx'=g(x_0,y_0)$$

and consequently, we get

$$\left(\int_0^\infty W_1(x_0)\left(\int_0^\infty (H^*_2g)^{q_2}(x_0,y_0)W_2(y_0)dy_0\right)^{\frac{q_1}{q_2}}dx\right)^{\frac{1}{q_1}}$$
$$=\left(\int_0^\infty w_1(x)\left(\int_0^\infty (H^*_{E,F}f)^{q_2}(x,y)w_2(y)dy\right)^{\frac{q_1}{q_2}}dx\right)^{\frac{1}{q_1}}$$
$$\leq C\left(\int_E v_1(x)\left(\int_F v_2(y)f^{p_2}(x,y)dy\right)^{\frac{p_1}{p_2}}dx\right)^{\frac{1}{p_1}}$$
$$=C\left(\int_0^\infty\int_{B_M} v_1(x_0x')\left\{\int_0^\infty\int_{B_N} v_2(y_0y')f^{p_2}(x_0x',y_0y')y_0^{N-1}dy'dy_0\right\}^{\frac{p_1}{p_2}}\right.$$
$$\left.\times\, x_0^{M-1}dx'dx_0\right)^{\frac{1}{p_1}}$$

$$= C\Bigg(\int_0^\infty \int_{B_M} v_1(x_0x')\int_0^\infty g^{p_2}(x_0, y_0)V_2^{p_2'}(y_0)$$

$$\times \Bigg(\int_{B_N} v_2^{1-p_2'}(y_0y')y_0^{N-1}dy'dy_0\Bigg)^{\frac{p_1}{p_2}} V_1^{p_1'}(x_0)v_1^{-p_1'}(x_0x')x_0^{M-1}dx'dx_0\Bigg)^{\frac{1}{p_1}}$$

$$= C\Bigg(\int_0^\infty V_1{}^{p_1'}(x_0)\Bigg\{\int_{B_M} v_1^{1-p_1'}(x_0x')x_0^{M-1}dx'\Bigg\}$$

$$\times (\int_0^\infty g^{p_2}(x_0, y_0)V_2(y_0)dy_0\Bigg)^{\frac{p_1}{p_2}} dx_0^{\frac{1}{p_1}}$$

$$= C\Bigg(\int_0^\infty V_1(x_0)\Bigg(\int_0^\infty V_2(y_0)g^{p_2}(x_0, y_0)dy_0\Bigg)^{\frac{p_1}{p_2}} dx_0\Bigg)^{\frac{1}{p_1}}$$

and we are done.

Recently, in [6], the following result was proved:

Theorem C *Let* $1 < p_i < \infty$, $0 < q_i < \infty$, $q_i \neq 1$ $(i = 1, 2)$ *and let* W_1, W_2, V_1, V_2 *be weight functions on* $(0, \infty)$. *Assume, in addition, that either* $p_1 \le p_2 \le q_1$ *or* $p_1 \le q_2 \le q_1$. *Then the inequality*

$$\Bigg(\int_0^\infty W_1(x)\Bigg(\int_0^\infty (H_2^* f)^{q_2}(x, y)W_2(y)dy\Bigg)^{\frac{q_1}{q_2}} dx\Bigg)^{\frac{1}{q_1}} \le C\Bigg(\int_0^\infty V_1(x)\Bigg(\int_0^\infty f^{p_2}(x, y)V_2(y)dy\Bigg)^{\frac{p_1}{p_2}} dx\Bigg)^{\frac{1}{p_1}}$$

holds for all measurable non-negative functions f *defined on* $(0, \infty) \times (0, \infty)$ *if and only if the inequalities*

$$\Bigg(\int_0^\infty (H^*g)^{q_1}(x)W_1(x)dx\Bigg)^{\frac{1}{q_1}} \le C_1\Bigg(\int_0^\infty g^{p_1}(x)V_1(x)dx\Bigg)^{\frac{1}{p_1}}$$

and

$$\Bigg(\int_0^\infty (H^*h)^{q_2}(y)W_2(y)dy\Bigg)^{\frac{1}{q_2}} \le C_2\Bigg(\int_0^\infty h^{p_2}(y)V_2(y)dy\Bigg)^{\frac{1}{p_2}}$$

hold for all measurable non-negative functions g *and* h *defined on* $(0, \infty)$.

The above theorem suggests that the two dimensional mixed norm inequality can be studied in terms of two one-dimensional inequalities. Consequently, in view of Theorems 1, 4 and C, we obtain the following:

Theorem 7 *Let* $1 < p_i < \infty$, $0 < q_i < \infty$, $q_i \neq 1$ $(i = 1, 2)$ *and let* w_1, w_2, v_1, v_2 *be weight functions on* E. *Assume in addition that either* $p_1 \leq p_2 \leq q_1$ *or* $p_1 \leq q_2 \leq q_1$. *Then the inequality*

$$\left(\int_E w_1(x)\left(\int_F (H^*_{E,F} f)^{q_2}(x, y) w_2(y) dy\right)^{\frac{q_1}{q_2}} dx\right)^{\frac{1}{q_1}} \leq C\left(\int_E v_1(x)\left(\int_F f^{p_2}(x, y) v_2(y) dy\right)^{\frac{p_1}{p_2}} dx\right)^{\frac{1}{p_1}}$$

holds for all measurable non-negative functions f *defined on* $E \times F$ *if and only if the inequalities*

$$\left(\int_E (H^*_E g)^{q_1}(x) w_1(x) dx\right)^{\frac{1}{q_1}} \leq C_1\left(\int_E g^{p_1}(x) v_1(x) dx\right)^{\frac{1}{p_1}}$$

and

$$\left(\int_F (H^*_F h)^{q_2}(y) w_2(y) dy\right)^{\frac{1}{q_2}} \leq C_2\left(\int_F h^{p_2}(y) v_2(y) dy\right)^{\frac{1}{p_2}}$$

hold for all measurable non-negative functions g *and* h *defined, respectively, on* E *and* F.

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Recent Advances on Generalized Trigonometric Systems in Higher Dimensions

Jan Lang and Osvaldo Méndez

Abstract We present a survey of current research on the basis properties of several trigonometric systems in higher dimensions.

Keywords p-Laplacian · Generalized trigonometric functions · Riesz theorem · Schauder basis · Multi-dimensional Fourier series

1 Introduction

In this work we address recent results (in fact, some of them haven't been yet published) regarding the construction of trigonometric bases in higher dimensions. It is well known that the classical trigonometric functions emerge from the consideration of, though intimately related, quite different mathematical situations, such as extremal functions of the Sobolev embedding theorem, as eigenfunctions of the classical Laplacian and as inverse functions of integrals of irrational expressions.

Though it is true that the above L^2-based framework enjoys the enormous advantages of linearity and of an underlying Hilbert-space structure, there is in principle no impediment for analyzing the same situations in the context of an L^p theory for $p \in (1, \infty)$. The authors were able to track the first attempts in this direction to Lundberg (see [14]).

There are reasons for this undertaking beyond the mere interest in these generalized functions per se. Such generalized functions, it turns out, have recently proved to play an undeniably important role in the spectral theory of non-linear operators and in approximation theory, as they naturally appear in the study of s-numbers

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of Sobolev embeddings and of Hardy operators ([9]). They also appear in classical studies of exact constants for integral operators (see: [11, 15]).

A variety of interesting properties of these trigonometric systems, such as the generalized Pythagorean identities, have recently opened the door toward Fourier-type analysis based on them. It is noteworthy that the Gibbs phenomenon does not seem to occur in the context of this generalized L^p-Fourier analysis in such form as it does in the standard Fourier series, which could, in principle, make these functions look amenable to the study of discontinuous signals.

2 Generalized Trigonometric Functions

We set about to survey very recent results involving basis properties of several systems of generalized trigonometric functions in higher dimensions, specifically the $\sin_{pq}$, $\cos_{pq}$ functions, the Lindqvist-Peetre functions and the Lindqvist-Peetre p-exponential functions. We refer the reader to the body of the paper for the definitions of these functions. We start by recalling same basic terms. For a real number $p \in (1, \infty)$ and a bounded domain $\Omega \subset \mathbb{R}^n$, the eigenvalue problem for the p-Laplacian

$$\Delta_p(u) := \mathbf{div}\left(|\nabla u|^{p-2}\,\nabla u\right) \tag{1}$$

is given by:

$$\Delta_p(u) = -\lambda|u|^{p-2}u. \tag{2}$$

In particular, the eigenvalue problem for the Dirichlet p-Laplacian is obtained by adjoining the subsidiary boundary condition

$$u|_{\partial\Omega} = 0.$$

The latter, in turn is a particular case of the eigenvalue problem for the Dirichlet pq-Laplacian operator $\Delta_{p,q}$, $p, q \in (1, \infty)$, which in one dimension takes up the form:

$$\begin{aligned} -\Delta_{p,q}u = -\left(|u'|^{p-2}u'\right)' &= \lambda|u|^{q-2}u \quad \mathbf{in}\,(0,1),\\ u(0) = u(1) &= 0, \end{aligned} \tag{3}$$

whose connection to the present work will become apparent shortly.

2.1 The Functions $\sin_{pq}$, $\cos_{pq}$.

For $1 < p, q < \infty$, a system of generalized trigonometric functions can be defined by setting:

$$\sin^{-1}_{p,q} x := \int_0^x (1 - t^q)^{-1/p}\, dt \quad , \quad x \in (0, 1) \tag{4}$$

and

$$\pi_{p,q} := 2 \int_0^1 (1 - x^q)^{-\frac{1}{p}}\, dx. \tag{5}$$

The function $\sin_{p,q} : [0, \pi_{p,q}/2] \to [0, 1]$ can be extended symmetrically about the line $x = \frac{\pi_{p,q}}{2}$ into $[\frac{\pi_{p,q}}{2}, \pi_{p,q}]$, as an odd function to the interval $[-\pi_{p,q}, 0]$ and finally periodically from the interval $[-\pi_{p,q}, \pi_{p,q}]$ to $(-\infty, \infty)$. In the sequel we set $\pi_p := \pi_{p,p}$ and $\sin_p(x) := \sin_{p,p}(x)$ for $1 < p < \infty$. We remark that the eigenvalues of Δ_{pq} are given by (see [7])

$$\lambda_k = \frac{q(p-1)}{pq^q}(2k\pi_{pq})^q$$

and their associated eigenfunctions are

$$u_k(x) = \frac{\sin_{pq}(k\pi_{pq}x)}{k\pi_{pq}},$$

with $k \in \mathbb{N}$ (). In particular, the functions $\sin_p(n\pi_p x)$, $n \in \mathbb{N}$ are eigenfunctions of (2) on the interval (0, 1).

We observe in passing that natural extensions of the preceding definitions exist for the full range $(p, q) \in [1, \infty] \times [1, \infty]$. Since the end-point cases fall beyond the scope of this work, we omit every mention to the case when either of the subindexes p or q is 1 or ∞. The reader is referred to [9] for more details in connection with this remark.

2.2 The Generalized Trigonometric Functions of Lindqvist-Peetre Type

In [12, 13] Lindqvist and Peetre introduced generalized sine and cosine functions ($\mathbf{S}_{\frac{1}{p}}$ and $\mathbf{C}_{\frac{1}{p}}$, respectively) which, as it turns out, can be expressed in the following way:

$$\mathbf{S}_{\frac{1}{p}}(x) = \sin_{p,p'}(x), \quad \mathbf{C}_{\frac{1}{p}}(x) = \left(\cos_{p,p'}(x)\right)^{p-1}, \quad x \in \mathbb{R}, \tag{6}$$

where, for the sake of simplicity, we have used the notation

$$\left(\cos_{p,p'}(x)\right)^{p-1} = \left|\cos_{p,p'}(x)\right|^{p-2}\cos_{p,p'}(x). \tag{7}$$

Next, we summarize basic properties of these functions. For all $x \in \mathbb{R}$:

$$\begin{aligned} \frac{d}{dx}\mathbf{C}_{\frac{1}{p'}}(x) &= -\left(\mathbf{S}_{\frac{1}{p'}}(x)\right)^{p-1} \\ \frac{d}{dx}\mathbf{S}_{\frac{1}{p'}}(x) &= \left(\mathbf{C}_{\frac{1}{p'}}(x)\right)^{p-1} \\ \left(\mathbf{S}_{p'}(x)\right)^{p} + \left(\mathbf{C}_{p'}(x)\right)^{p} &= 1. \end{aligned}$$

Observe that $\pi_{p,p'}$, $1 < p < \infty$, is equal to the area of the set $\mathcal{S}_{p'}$ enclosed by the p'-circle, that is,

$$\mathcal{S}_{p'} = \{(x, y) \in \mathbb{R}^2; |x|^{p'} + |y|^{p'} \leq 1\}. \tag{8}$$

Set

$$\mathcal{S}_{p'} = \mathcal{S}_{\infty} = \{(x, y) \in \mathbb{R}^2; \max(|x|, |y|) \leq 1\} \;\; if \;\; p = 1. \tag{9}$$

For $1 < p_1 < p_2 < \infty$ the obvious inequalities

$$\max(|x|, |y|) \leq (|x|^{p_1'} + |y|^{p_1'})^{1/p_1'} \leq (|x|^{p_2'} + |y|^{p_2'})^{1/p_2'} \leq |x| + |y|$$

imply that

$$\mathcal{S}_1 \subset \mathcal{S}_{p_1'} \subset \mathcal{S}_{p_2'} \subset \mathcal{S}_{\infty}.$$

Thus,

$$2 \leq \pi_{p,p'} \leq 4 \tag{10}$$

(cf. [8, Lemma 2.4]). Moreover, the function

$$p \mapsto \pi_{p,p'} \quad \textbf{is decreasing on} \quad (1, \infty) \tag{11}$$

and the following estimate holds, for whose proof we refer the reader to [8, Proposition 3.3]:

$$\frac{2}{\pi_{p,p'}} \leq \frac{\mathbf{S}_{\frac{1}{p}}(x)}{x} \leq 1, \quad x \in \left(0, \frac{\pi_{p,p'}}{2}\right). \tag{12}$$

2.3 The p-Exponential of Lindqvist and Peetre

We introduce the p-exponential function $\mathbf{E}_{\frac{1}{p}}$ by

$$\mathbf{E}_{\frac{1}{p}}(i\,y) = \mathbf{C}_{\frac{1}{p}}(y) + i\,\mathbf{S}_{\frac{1}{p}}(y), \quad y \in \mathbb{R}. \tag{13}$$

As usual, i stands for the imaginary unit and in the sequel we will use the standard notation $\overline{z}$ for the complex conjugate of any complex number z.

For the record, we recall from standard functional analysis that a sequence (u_j) in a Banach space X is said to be a Schauder basis (or simply a basis, if there is no room for confusion, as it will be the case in the sequel) for X if for any $x \in X$ there exists a unique sequence of scalars (x_j) with $x = \sum_1^\infty x_j u_j$.

In what follows we denote a multi-index by $\mathbf{k} := (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$ and by $\mathbf{k} \leq \mathbf{l}$ we mean that $k_i \leq l_i$ for each $1 < i < n$. It is well known that for $f \in L^r((-1, 1)^n)$

$$\left\| f - \sum_{\mathbf{k} \leq \mathbf{l}} \hat{f}(k_1, \ldots, k_n) e^{-2\pi i k_j x_j} \right\|_{L^r((-1,1)^n)} \longrightarrow 0 \tag{14}$$

as $\min\{l_1, l_2, \ldots, l_n\} \to \infty$ (this type of convergence is known as convergence in the Pringsheim sense), where as is customary the ordinary Fourier coefficients of f will be written as

$$\hat{f}(k_1, k_2, \ldots, k_n) := \int_{(-1,1)^n} f(x_1, \ldots, x_n)\, \Pi_{j=1}^n e^{-2\pi i k_j x_j} dx_1 \ldots dx_n. \tag{15}$$

Since any $f \in L^r((0, 1)^n)$ can be uniquely extended to $(-1, 1)^n$ as an odd function, it is readily concluded that

Theorem 1 *For $r \in (1, \infty)$ and $f \in L^r((0, 1)^n)$ the sine Fourier partial sums $S_\mathbf{l}$ converge in L^r-norm to f in the Pringsheim sense, i.e.:*

$$\|f - S_\mathbf{l}\|_{L^r((0,1)^n)} \longrightarrow 0 \quad as \quad \min\{l_1, l_2, \ldots, l_n\} \to \infty, \tag{16}$$

where $\mathbf{l} := (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n$,

$$S_\mathbf{l} := \sum_{\mathbf{k} \leq \mathbf{l}} \hat{f}(k_1, \ldots, k_n) \Pi_{j=1}^n \sin(\pi k_j x_j). \tag{17}$$

and

$$\hat{f}(k_1, k_2, \ldots, k_n) := 2^n \int_{(0,1)^n} f(x_1, \ldots, x_n) \Pi_{j=1}^n \sin(\pi k_j x_j)\, dx_1 \ldots dx_n. \tag{18}$$

Recently, in [10] the authors exhibited sufficient conditions on the subindexes p and q such that for each $r \in (1, \infty)$, the systems

$$\left\{\sin_{p,q}(n\pi_{p,q}x)\, \sin_{p,q}(m\pi_{p,q}y)\right\}_{(n,m)\in\mathbb{N}^2} \tag{19}$$

and

$$\left\{\sin_{p,q}(n\pi_{p,q}x)\, \sin_{p,q}(m\pi_{p,q}y)\, \sin_{p,q}(k\pi_{p,q}z)\right\}_{(n,m,k)\in\mathbb{N}^3} \tag{20}$$

constitute a basis for $L^r((0,1)^2)$ and $L^r((0,1)^3)$, respectively. The one dimensional case has been extensively treated for example in [4, 8] among others.

In what follows we briefly describe the methods and ideas involved in the results of [2, 3, 10].

The following theorems were obtained in [10].

Theorem 2 *There exist real numbers $p_0 > 1$, $p_1 > 2$ such that for any $r \in (1,\infty)$, the system (19) is a basis for $L^r((0,1)^2)$ if $(p,q) \in (p_0,2) \times (p_0,2) \cup (p_1,\infty) \times (p_1,\infty)$.*

Theorem 3 *There exists $p_2 > 6$ such that for any $r \in (1,\infty)$ the system (20) is a basis for $L^r((0,1)^3)$ when $(p,q) \in (p_2,\infty) \times (p_2,\infty)$.*

(See Corollary 4 for the precise definition of p_0 and p_1). In order to proceed, it is necessary to introduce some terminology. We set $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $x_j \in \mathbb{R}$ for $j = 1, 2, \ldots, n$; for $\mathbf{k} \in \mathbb{N}^n$ we define the function $g_{\mathbf{k},p,q} \in L^r((0,1)^n)$ by

$$g_{\mathbf{k},p,q}(\mathbf{x}) = \Pi_{j=1}^n \sin_{p,q}(\pi_{p,q} k_j x_j); \tag{21}$$

the corresponding Fourier coefficients are given by

$$\begin{aligned} \hat{g}_{\mathbf{k},p,q}(l_1,\ldots,l_n) &= 2^n \int_{[0,1]^n} \Pi_{j=1}^n \sin_{p,q}(\pi_{p,q} k_j x_j)\, \Pi_{i=1}^n \sin(\pi l_i x_i) d\mathbf{x} \\ &= 2^n \Pi_{j=1}^n \int_0^1 \sin_{p,q}(\pi_{p,q} k_j x_j) \sin(\pi l_j x_j)\, dx_j. \end{aligned} \tag{22}$$

It is easy to see that because of the symmetry of $\sin_{p,q} x$ about the vertical line $x = \pi_{p,q}/2$, one has

$$\hat{g}_{\mathbf{1},p,q}(\mathbf{k}) = 0$$

when $\mathbf{k}$ has at least an even component. The next lemma is a direct consequence of this observation.

Lemma 1

$$\begin{aligned} \hat{g}_{\mathbf{k},p,q}(l_1,\ldots l_n) &= \Pi_{j=1}^n \widehat{\sin_{p,q}(k_j \pi_{p,q} x_j)}(l_j) \\ &= \Pi_{j=1}^n \widehat{\sin_{p,q}(\pi_{p,q} x_j)}\left(l_j/k_j\right) \end{aligned} \tag{23}$$

if l_j/k_j is odd for all $j \in \mathbb{N}$ and 0 otherwise.

For the sake of completeness we state the following lemma which follows immediately from Proposition 4.1 in [8]:

Lemma 2 *Let $1 < p, q < \infty$ and m odd:*

$$\left|\hat{f}_{1,p,q}(m)\right| \le 4\pi_{p,q}/(\pi m)^2. \tag{24}$$

The next lemma is a direct consequence of Proposition 4.2 in [8].

Lemma 3 *For $1 < p, q < \infty$, one has*

$$\widehat{\sin}_{p,q}\pi_{p,q}\,(\cdot)(1) := \tau_{p,q}(1) \geq 8/\pi^2. \tag{25}$$

Definition 1 For a function $f : [0,1]^n \to \mathbb{R}$ we define its extension as the function $\tilde{f} : [0,\infty)^n \to \mathbb{R}$ as follows:

$$\tilde{f}(\mathbf{x}) = -\tilde{f}(2\mathbf{k} - \mathbf{x}) \ \text{ for } \ \mathbf{x} \in \Pi_{j-1}^{n}[k_j, k_j + 1)\ ,\ \ k_j \in \mathbb{N}^n,$$

$$\text{and } \ \tilde{f} \equiv f \ \text{ on } [0,1)^n.$$

It is a matter of routine to verify that given $r \in (1,\infty)$, for each $\mathbf{k} \in \mathbb{N}^n$, the map

$$M_{\mathbf{k}} : L^r((0,1)^n) \to L^r((0,1)^n) \tag{26}$$

which is defined as $M_{\mathbf{k}}(g)(\mathbf{x}) := \tilde{g}(\mathbf{x}\mathbf{k})$ is well defined, linear and in fact, an isometry (i.e. we have $\|M_{\mathbf{k}}\| = 1$). Note that here $\mathbf{x}\mathbf{k} = (x_1, x_2, ...x_n)(k_1, k_2, ...k_n) = (x_1k_1, x_2k_2, ...x_nk_n)$. Let us set

$$\begin{aligned} \tau_{p,q}(\mathbf{k}) &:= 2^n \int_{(0,1)^n} \Pi_{j=1}^{n} \sin_{p,q}(\pi_{p,q}k_jx_j)\sin(\pi k_jx_j)\,d\mathbf{x} \\ &= 2^n \int_0^1 \Pi_{j=1}^{n} \sin_{p,q}(\pi_{p,q}k_jx_j)\sin(k_j\pi x_j)\,dx_j \\ &= \Pi_{j=1}^{n}\tau_{p,q}^{\circ}(k_j), \end{aligned} \tag{27}$$

where $\tau_{p,q}^{\circ}(k_j) := 2\int_0^1 \sin_{p,q}(\pi_{p,q}x_j)\sin(k_j\pi x_j)\,dx_j$. Then the (linear) operator $T : L^r((0,1)^n) \longrightarrow L^r((0,1)^n)$ which is defined by:

$$T(g) := \sum_{\mathbf{k}\in\mathbb{N}^n} \tau_{p,q}(\mathbf{k})M_{\mathbf{k}}(g) \tag{28}$$

is well defined and bounded (just observe that $\sum_{\mathbf{k}\in\mathbb{N}^n} |\tau_{p,q}(\mathbf{k})| < \infty$).

Next we point out that

$$\begin{aligned} \|T - id\cdot\tau_{p,q}(\mathbf{1})\| \quad & \leq n[\tau_{p,q}^{\circ}(1)]^1 \textstyle\sum_{k_2>1,...k_n>1} \Pi_2^n \tau_{p,q}^{\circ}(k_j) + \cdots \\ & \textstyle\binom{n}{s}[\tau_{p,q}^{\circ}(1)]^s \sum_{k_s>1,...k_n>1} \Pi_{j+1}^n \tau_{p,q}^{\circ}(k_j) + \cdots + \sum_{k_1>1,k_2>1,...k_n>1} \Pi_{j=1}^n \tau_{p,q}^{\circ}(k_j) \\ & = \textstyle\sum_{k=1}^{n-1} \binom{n}{k}[\tau_{p,q}^{\circ}(1)]^k \left(\sum_{j=3}^{\infty} \tau_{p,q}^{\circ}(j)\right)^{n-k} + \left(\sum_{j=3}^{\infty} \tau_{p,q}^{\circ}(j)\right)^n \\ & \leq \textstyle\sum_{k=1}^{n-1} \binom{n}{k}[\tau_{p,q}^{\circ}(1)]^k \left(\frac{4\pi p,q}{\pi^2}\left(\frac{\pi^2}{8} - 1\right)\right)^{n-k} + \left(\frac{4\pi p,q}{\pi^2}\left(\frac{\pi^2}{8} - 1\right)\right)^n \\ & \leq \left(\tfrac{4\pi p,q}{\pi^2}\right)^n \left[\textstyle\sum_{k=1}^{n-1} \binom{n}{k}\left(\frac{\pi^2}{8} - 1\right)^{n-k} + \left(\frac{\pi^2}{8} - 1\right)^n\right] \\ & = \left(\tfrac{4\pi p,q}{\pi^2}\right)^n \left(\left(\tfrac{\pi^2}{8}\right)^n - 1\right). \end{aligned} \tag{29}$$

Lemma 4 *For* $1 < q < q' < \infty$, $1 < p < p' < \infty$, *the function*

$$w(x) = \frac{\sin_{p,q}^{-1} x}{\sin_{p'q'}^{-1} x} \tag{30}$$

is strictly increasing on (0, 1).

Proof See [10]

Corollary 1 *If* $1 < p < p' < \infty$, $1 < q < q' < \infty$ *one has*

(i)

$$\frac{\sin_{p,q}^{-1} x}{\pi_{p,q}} < \frac{\sin_{p'q'}^{-1} x}{\pi_{p',q'}} \tag{31}$$

for $x \in (0, 1)$.

(ii) If $x \in (0, 1/2)$, *then*

$$\sin_{p',q'}(\pi_{p',q'}x) < \sin_{p,q}(\pi_{p,q}x). \tag{32}$$

(iii) Uniformly on (0, 1)*:*

$$1 < \frac{\frac{\sin_{p',q'}^{-1} x}{\pi_{p',q'}}}{\frac{\sin_{p,q}^{-1} x}{\pi_{p,q}}} < \frac{\pi_{p,q}}{\pi_{p',q'}} \tag{33}$$

Proof Claim (i) follows immediately since

$$\sin_{p',q'}^{-1}(1) = \frac{\pi_{p',q'}}{2}$$

and

$$\sin_{p,q}^{-1}(1) = \frac{\pi_{p,q}}{2}.$$

With regard to (ii) it is sufficient to compare the inverse functions

$$\left(\frac{\sin_{p',q'}^{-1}(\cdot)}{\pi_{p',q'}}\right)^{-1}$$

and

$$\left(\frac{\sin_{p,q}^{-1}(\cdot)}{\pi_{p,q}}\right)^{-1}$$

on the interval $(0, \frac{1}{2})$ using the information provided by (i). Claim (iii) follows from (i).

Corollary 2 *For* $1 < p < 2$, $1 < q < 2$,

$$\tau_{p,q}(\mathbf{1}) > 1.$$

Proof By virtue of (27) and Corollary 1 (ii), one has

$$\begin{aligned} \tau_{p,q}(\mathbf{1}\ ) &= \Pi_{j=1}^{n} \tau_{p,q}(1)\tau_{p,q}^{\circ}(1) = \left(\tau_{p,q}^{\circ}(1)\right)^n \\ &= \left(2\int_0^1 \sin_{p,q}(\pi_{p,q}t)\sin(\pi t)\,dt\right)^n \qquad (34) \\ &> \left(2\int_0^1 \sin^2(\pi t)\,dt\right)^n = 1. \qquad (35) \end{aligned}$$

Corollary 3 *The system*

$$\left\{\sin_{pq}(k_1\pi_{pq}x_1)\sin_{pq}(k_2\pi_{pq}x_2)\ldots\sin_{pq}(k_n\pi_{pq}x_n)\right\}_{(k_1,k_2,\ldots k_n)\in\mathbb{N}^n}$$

is a basis in $L^r((0,1)^n)$ *if* $1 < p < 2$, $1 < q < 2$ *and*

$$\pi_{p,q} < \frac{2\pi^2}{(\pi^{2n} - 8^n)^{1/n}}$$

or if either $p \geq 2$ *or* $q \geq 2$, *and*

$$\pi_{p,q} < \frac{16}{(\pi^{2n} - 8^n)^{1/n}}.$$

Proof Both claims follow, respectively, from Lemma 3, Corollary 2, formula (29) and the standard functional-analytic argument that if K is an operator with norm strictly less than one on a Banach space X, then $I + K$ is invertible on X.

Corollary 4 *In particular, if* p_0 *and* p_1 *are defined by the equalities*

$$\pi_{p_0} = \frac{2\pi^2}{(\pi^4 - 8^2)^{1/2}} \quad (p_0 \approx 1.85) \tag{36}$$

$$\pi_{p_1} = \frac{16}{(\pi^4 - 8^2)^{1/2}} \quad (p_1 \approx 2.33) \tag{37}$$

then

(i) *for* $p = q \in (p_0, 2) \cup (p_1, \infty)$, *the system*

$$\{\sin_p(m\pi_p x)\sin_p(n\pi_p y)\}_{(m,n)\in\mathbb{N}^2}$$

constitutes a basis for $L^r((0,1)^2)$, $r \in (1,\infty)$.

(ii) *For* $p = q$, $r \in (1,\infty)$ *the system*

$$\{\sin_p(m\pi_p x)\sin_p(n\pi_p y)\sin_p(k\pi_p x)\}_{(m,n,k)\in\mathbb{N}^3}$$

is a basis in $L^r((0,1)^3)$ *for* $p > p_2$*, where* $p_2 \approx 6.5$ *is given by*

$$\pi_{p_2} = \frac{16}{(\pi^6 - 8^3)^{1/3}}. \tag{38}$$

Remark 1 Notice that

$$\lim_{n\to\infty} \frac{2\pi^2}{(\pi^{2n} - 8^n)^{1/n}} = 2$$

and that

$$\frac{16}{(\pi^{2n} - 8^n)^{1/n}} < 2 \ \textbf{if} \ n > 3 \tag{39}$$

Therefore the highest dimension n for which a conclusion can be reached using Corollary 3 is $n = 3$.

We will next establish that the basis property holds for any $p \in (p_0, \infty)$, thus improving Corollary 4. The following simple observation follows from the right-hand-side inequality in Corollary 1 *(iii)*: For $2 < p$ one has:

$$\sin_p^{-1}(x) < \sin^{-1}(x) \tag{40}$$

on $(0, 1)$. Since $\pi_p < \pi$, (40) forces the following relation between the inverse functions, on the interval $(0, \frac{\pi_p}{2})$:

$$\sin x < \sin_p x, \tag{41}$$

which in turn implies the following estimate on $(0, 1/2)$:

$$\sin_p(\pi_p x) > \sin(\pi_p x). \tag{42}$$

In conclusion,

$$\begin{aligned} \tau(p) = 4\int_0^{1/2} \sin_p(\pi_p x)\sin(\pi x)\,dx &> 4\int_0^{1/2} \sin(\pi_p x)\sin(\pi x)\,dx \\ &= 2\left(\tfrac{1}{\pi-\pi_p}\sin\left(\tfrac{\pi-\pi_p}{2}\right) - \tfrac{1}{\pi+\pi_p}\sin\left(\tfrac{\pi+\pi_p}{2}\right)\right) \\ &= \tfrac{4\pi_p\cos(\pi_p/2)}{(\pi+\pi_p)(\pi-\pi_p)} = \gamma(\pi_p). \end{aligned} \tag{43}$$

Since

$$\gamma(x) = \frac{4x\cos\left(\frac{x}{2}\right)}{(\pi + x)(\pi - x)}$$

is increasing in $(\pi/2, \pi)$, π_p increases to π as p decreases to 2 and

$$\lim_{p\to 2^+} \gamma(\pi_p) = 1,$$

it is immediate that if for any $\delta > 0$ and $\pi_{p^*} > \pi/2$ it held that

$$\gamma(\pi_{p^*}) > 1 - \delta$$

it would follow that

$$\gamma(\pi_p) > 1 - \delta$$

for any p with $2 \le p < p^*$. On the other hand,

$$\gamma(\pi_{2.33}) = \gamma\left(\frac{2\pi}{2.33 \sin \frac{\pi}{2.33}}\right) > \frac{93}{100} > \frac{8}{\pi^2}.$$

Thus,

$$\gamma(\pi_p) = \tau(p) > \frac{93}{100}$$

for $p \in [2, 2.33]$. Since the inequality

$$\left(\frac{4\pi_p}{\pi^2}\right)^2 \left(\left(\frac{\pi^2}{8}\right)^2 - 1\right) < .93^2 \tag{44}$$

is satisfied whenever $\pi_p < 3.17$, i.e., whenever $p < 2.33$, one has the following result:

Lemma 5 *The system (19) is a basis for $L^r((0,1)^2)$ for $p = q \in (p_0, \infty)$.*

We next move on to the basis properties of Lindqvist-Peetre functions as presented in [2]. The authors prove the following Theorem along the same lines as the proof of Corollary 3.

Theorem 4 *Let $p \in (1, \infty)$ and $n \in \mathbb{N}$ satisfy the inequality:*

$$\left(\frac{\pi^2}{8}\right)^n - 1 < \left(\frac{2}{\pi_{p,p'}}\right)^n. \tag{45}$$

Then the sequence $\left\{\Pi_{i=1}^n \mathbf{S}_{\frac{1}{p}}(\pi_{p,p'} k_i \cdot)\right\}_{\mathbf{k}} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ is a basis in $L^r((0,1)^n)$ for any $r \in (1, \infty)$.

Theorem 5 *Let $p \in (1, \infty)$ and $n \in \mathbb{N}$.*

(i) The sequence $\left\{\mathbf{S}_{\frac{1}{p}}(\pi_{p,p'}\, k\, x)\right\}_{k\in\mathbb{N}}$ is a basis in $L^r((0,1))$ for any $r \in (1, \infty)$.
(ii) There exists $p_2 > 1$ such that, for every $p \in (p_2, \infty)$, the sequence $\left\{\mathbf{S}_{\frac{1}{p}}(\pi_{p,p'}\, k_1\, x_1)\, \mathbf{S}_{\frac{1}{p}}(\pi_{p,p'}\, k_2\, x_2)\right\}_{(k_1,k_2)\in\mathbb{N}^2}$ is a basis in $L^r((0,1)^2)$ for any $r \in (1, \infty)$.
(iii) There exists $p_3 > 1$ such that, for every $p \in (p_3, \infty)$, the sequence $\left\{\mathbf{S}_{\frac{1}{p}}(\pi_{p,p'}\, k_1\, x_1)\, \mathbf{S}_{\frac{1}{p}}(\pi_{p,p'}\, k_2\, x_2)\, \mathbf{S}_{\frac{1}{p}}(\pi_{p,p'}\, k_3\, x_3)\right\}_{(k_1,k_2,k_3)\in\mathbb{N}^3}$ is a basis in $L^r((0,1)^3)$ for any $r \in (1, \infty)$.

Remark 2 Numerical computations yield the values

$$p_2 = 2.89026,$$

exact up to the fifth decimal place and

$$p_3 = 22.8508,$$

exact up to te fourth decimal place.

Proof (i): For any $p \in (1, \infty)$ (by 10), $\pi_{p,p'} \in [2, 4]$, which implies the validity of condition (45). Thus, the assertion holds by Theorem 8.

(ii),(iii): Put

$$g(p) = \left(\frac{2}{\pi_{p,p'}}\right)^n. \tag{46}$$

By (11), g is increasing and, using (10), we obtain

$$\sup\{g(p) : p \in (1, \infty)\} = 1. \tag{47}$$

On the other hand, one has (see [2])

$$p_n = \inf\left\{g(p) - (\pi^2/8)^n + 1 > 0 : p \in (1, \infty)\right\} > 1. \tag{48}$$

Thus, for any $p > p_n$, due to (11), condition (45) is satisfied. Notice that for $n > 3$, $p_n = \infty$.

Theorem 6 *Let $p \in (1, \infty)$ and $n \in \mathbb{N}$ be so that the condition (45) is satisfied. Then the sequence $\left\{(\mathbf{C}_{\frac{1}{p}}(\pi_{p,p'}\cdot))_{\mathbf{k}}\right\}_{\mathbf{k}} \in \mathbb{N}^n$ is a basis in $L^r((0, 1)^n)$ for any $r \in (1, \infty)$.*

In the same spirit, the following Theorem is proved. We refer the interested reader to [2] for the details:

Theorem 7 *Let p_1, p_2 be the numbers from Theorem 5.*

(i) The sequence $\left\{\Pi_{i=1}^n \mathbf{C}_{\frac{1}{p}}(\pi_{p,p'}\, k_i\cdot)\right\}_{\mathbf{k}=(k_1,\ldots k_n)\in\mathbb{N}}$, $p \in (1, \infty)$, is a basis in $L^r((0, 1))$ for any $r \in (1, \infty)$.

(ii) The sequence $\left\{\mathbf{C}_{\frac{1}{p}}(\pi_{p,p'}\, k_1\, x_1)\, \mathbf{C}_{\frac{1}{p}}(\pi_{p,p'}\, k_2\, x_2)\right\}_{(k_1,k_2)\in\mathbb{N}^2}$, $p \in (p_2, \infty)$, is a basis in $L^r((0, 1)^2)$ for any $r \in (1, \infty)$.

(iii) The sequence $\left\{\mathbf{C}_{\frac{1}{p}}(\pi_{p,p'}\, k_1\, x_1)\, \mathbf{C}_{\frac{1}{p}}(\pi_{p,p'}\, k_2\, x_2)\, \mathbf{C}_{\frac{1}{p}}(\pi_{p,p'}\, k_3\, x_3)\right\}_{(k_1,k_2,k_3)\in\mathbb{N}^3}$, $p \in (p_3, \infty)$, is a basis in $L^r((0, 1)^3)$ for any $r \in (1, \infty)$.

In the same line of thought, an analog of the classical complex exponential function is introduced in [3] and utilized to generate a basis for $L^r((-1, 1))$. Specifically (as usual, i denotes the imaginary unit and for $z = a + i\, b$, where $a, b \in \mathbb{R}$ we write $\overline{z} = a - i\, b$), we define the p-exponential function $\mathbf{E}_{\frac{1}{p}}$ by the equality

$$\mathbf{E}_{\frac{1}{p}}(i\,y) = \mathbf{C}_{\frac{1}{p}}(y) + i\,\mathbf{S}_{\frac{1}{p}}(y), \quad y \in \mathbb{R}. \tag{49}$$

Let $e(t) = \frac{1}{\sqrt{2}} \exp(i\,\pi t)$, $t \in \mathbb{R}$. We denote by

$$e_k(t) = e(kt) = \frac{1}{\sqrt{2}} \exp(i\,\pi k t), t \in \mathbb{R},\ k \in \mathbb{Z}; \tag{50}$$

and recall the standard fact that the system (e_k) constitutes an orthonormal basis in the complex Lebesgue space $L^2((-1, 1))$. It follows that the family of functions

$$\begin{aligned} e_{\mathbf{m}}(x) &= e_{m_1}(x_1)...e_{m_n}(x_n) \\ &= 2^{-n/2} \exp(i\,\pi m_1 x_1) \cdots \exp(i\,\pi m_n x_n), x \in \mathbb{R}^n,\ \mathbf{m} \in \mathbb{Z}^n. \end{aligned}$$

is an orthonormal basis in the complex Lebesgue space $L^2((-1, 1)^n)$. For the proof of the following result see Weisz [16]:

Lemma 6 *Let $f \in L^r((-1, 1)^n)$, where $r \in (1, \infty)$. Denote*

$$\widehat{f}(\mathbf{k}) = \int_{(-1,1)^n} f(x)\, \overline{e_{\mathbf{k}}(x)}\, dx, \quad \mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n. \tag{51}$$

Then

$$f = \sum_{\mathbf{m} \in \mathbb{Z}^n} \widehat{f}(\mathbf{m})\, e_{\mathbf{m}} \tag{52}$$

in the Pringsheim sense.

Throughout this section assume that $1 < p < \infty$ and put

$$\varphi(x) = \mathbf{E}_{\frac{1}{p}}(i\,\pi_{p,p'}\,x), \quad x \in \mathbb{R}. \tag{53}$$

Since each φ_n, $\mathbf{n} \in \mathbb{Z}^n$, is continuous, it has a Fourier expansion (52) with coefficients (51). That is,

$$\varphi_n(x) = \sum_{\mathbf{k} \in \mathbb{Z}^n} \widehat{\varphi_n}(\mathbf{k})\, e_k(x), \quad \textbf{where} \quad \widehat{\varphi_n}(\mathbf{k}) = \int_{(-1,1)^n} \varphi_n(x)\, \overline{e_k(x)}\, dx. \tag{54}$$

Due to the symmetry of $\varphi = \mathbf{S}_{\frac{1}{p}}(\pi_{p,p'} \cdot) = \sin_{p,p'}(\pi_{p,p'} \cdot)$ about $t = 1/2$, for every $\mathbf{k} = (k_1, \ldots, k_n)$ with some even k_i, $i \in \{1, \ldots, n\}$, we have $\widehat{\varphi_{\mathbf{1}}}(\mathbf{k}) = 0$ and

$$\begin{aligned}\widehat{\varphi_n}(\mathbf{k}) &= \int_{(-1,1)^n} \varphi_n(x)\,\overline{e_k(x)}\,dx \\ &= \sum_{\mathbf{m}\in ZZ^n} \widehat{\varphi_1}(\mathbf{m}) \int_{(-1,1)^n} e_{mn}(x)\,\overline{e_k(x)}\,dx \\ &= \begin{cases} \widehat{\varphi_1}(\mathbf{m}) \ \textbf{if}\ k_i = m_i n_i \ \textbf{for all}\ i = 1, \ldots, n; \\ 0 \quad \textbf{otherwise}. \end{cases}\end{aligned}$$

Let us put, for $\mathbf{m} = (m_1, \ldots, m_n) \in ZZ^n$,

$$\tau_{\mathbf{m}} = \prod_{j=1}^{n} \tau_{m_j} = \widehat{\varphi_{\mathbf{1}}}(\mathbf{m}), \tag{55}$$

where

$$\tau_{m_j} = \int_0^1 \varphi(x_j)\,\overline{e(m_j x_j)}\,dx_j = \frac{1}{\sqrt{2}} \int_0^1 \varphi(x_j)\, \exp(-i\,\pi m_j x_j)\,dx_j, \quad j = 1, \ldots, n. \tag{56}$$

Any function f on $[-1, 1)^n$, can be extended to $\widetilde{f}$ on $\mathbb{R}^n$ by setting $\widetilde{f}(x) = \widetilde{f}(2\mathbf{k} + x)$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\mathbf{k} = (k_1, \ldots, k_n) \in ZZ^n$, such that $x_j \in [2k_j - 1, 2k_j + 1)$, $j = 1, \ldots, n$. Define the mapping $\mathcal{E}_{\mathbf{m}} : L^r((-1, 1)^n) \to L^r((-1, 1))^n$, $\mathbf{m} \in ZZ^n$, $r \in (1, \infty)$, by

$$\mathcal{E}_{\mathbf{m}} f(x) = \widetilde{f}(\mathbf{m}x) \tag{57}$$

and note that $\mathcal{E}_{\mathbf{m}}(e_{\mathbf{n}}) = e_{\mathbf{mn}}$. Just as in the preceding sections, one can show that $\mathcal{E}_{\mathbf{m}}$ is a linear isometry, $\|\mathcal{E}_{\mathbf{m}}\| = 1$, and that the map T,

$$Tf(x) = \sum_{\mathbf{m}\in\mathbb{N}^n} \tau_{\mathbf{m}}\,\mathcal{E}_{\mathbf{m}} f(x), \tag{58}$$

is linear and bounded on $L^r((-1, 1))^n$, with the property that, for all $\mathbf{n} \in ZZ^n$,

$$T(e_{\mathbf{n}}) = \varphi_n. \tag{59}$$

It is sufficient to show that T is a homeomorphism, then it will follow that the φ_n inherits from the e_n the property of forming a basis in $L^r((-1, 1)^n)$ for every $r \in (1, \infty)$. In the following lemma we state a criteria for this operator T to be a homeomorphism on $L^r((-1, 1)^n)$.

To this effect, the following statement is proved in [3]:

$$Tf(x) = \sum_{\mathbf{k}\in\mathbb{N}^n} \tau_{\mathbf{2k-1}}\,\mathcal{E}_{\mathbf{2k-1}} f(x), \tag{60}$$

Lemma 7 *The following estimate holds:*

$$|\tau_{2k-1}| \leq \frac{8\,\pi_{p,p'}}{\sqrt{2}\,\pi^2}\frac{1}{(2k-1)^2}, \quad k \in \mathbb{N}. \tag{61}$$

Theorem 8 *Let $p \in (1,\infty)$ and $n \in \mathbb{N}$ be so that the condition(45) is satisfied. Then the sequence $\left\{\Pi_{i=1}^{n}\mathbf{E}_{\frac{1}{p}}(i\,\pi_{p,p'}k_i\cdot))\right\}_{\mathbf{k}=(k_1,\ldots k_n)\in ZZ^n}$ is a basis in $L^r((-1,1)^n)$ for any $r \in (1,\infty)$.*

Proof Inequality (45) coupled with identity (60) in conjunction with standard functional-analytic results imply that T is an isomorphism.

The following corollaries follow in the same spirit (the reader is referred to [3] for the details):

Corollary 5 (case $n = 1$) *Let $p \in (1,\infty)$. The sequence $\left\{(\mathbf{E}_{\frac{1}{p}}(i\,\pi_{p,p'}kx))\right\}_{k=-\infty}^{\infty}$ is a basis in $L^r((-1,1))$ for any $r \in (1,\infty)$.*

Proof For any $p \in (1,\infty)$ (by (10)), $\pi_{p,p'} \in [2,4]$, which implies the validity of condition (45). Thus, the assertion holds by Theorem 4.

Corollary 6 (case $n = 2$) *For p_2 as in Theorem 5, for every $p \in (p_2,\infty)$, the sequence $\left\{\mathbf{E}_{\frac{1}{p}}(i\pi_{p,p'}\,k_1\,x_1)\,\mathbf{E}_{\frac{1}{p}}(i\pi_{p,p'}\,k_2\,x_2)\right\}_{(k_1,k_2)\in\mathbb{N}^2}$ is a basis in $L^r((-1,1)^2)$ for any $r \in (1,\infty)$.*

Corollary 7 (case $n = 3$) *If p_3 is as defined in Theorem 5, then for every $p \in (p_3,\infty)$, the sequence $\left\{\mathbf{E}_{\frac{1}{p}}(\pi_{p,p'}\,k_1\,x_1)\,\mathbf{E}_{\frac{1}{p}}(\pi_{p,p'}\,k_2\,x_2)\,\mathbf{E}_{\frac{1}{p}}(\pi_{p,p'}\,k_3\,x_3)\right\}_{(k_1,k_2,k_3)\in\mathbb{N}^3}$ is a basis in $L^r((-1,1)^3)$ for any $r \in (1,\infty)$.*

3 Concluding Remarks

The two-dimensional generalized Fourier system opens the way for the use of non-orthogonal systems in the treatment of signal processing, which conceivably could be a valuable tool in studying image processing in the case of discontinuous gradient (see [1, 5, 6]), due to the fact that generalized trigonometric functions have a lesser degree of smoothness than the usual trigonometric functions ($p = q = 2$). In fact, the smoothness of generalized trigonometric function can in principle, be controlled by a suitable variation of the parameters p and q.

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Pointwise Multipliers on Musielak-Orlicz-Morrey Spaces

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Abstract In this paper we characterize pointwise multipliers from a Musielak-Orlicz-Morrey space to another Musielak-Orlicz-Morrey space. The set of all pointwise multipliers is also a Musielak-Orlicz-Morrey space.

Keywords Musielak-Orlicz space · Morrey space · Variable exponent · Pointwise multiplier · Pointwise multiplication

1 Introduction

Let (Ω, μ) be a complete σ-finite measure space. We denote by $L^0(\Omega)$ the set of all measurable functions from Ω to $\mathbb{R}$ or $\mathbb{C}$. Let E_1 and E_2 be subspaces of $L^0(\Omega)$. We say that a function $g \in L^0(\Omega)$ is a pointwise multiplier from E_1 to E_2, if the pointwise multiplication fg is in E_2 for any $f \in E_1$. We denote by $\mathrm{PWM}(E_1, E_2)$ the set of all pointwise multipliers from E_1 to E_2. We abbreviate $\mathrm{PWM}(E, E)$ to $\mathrm{PWM}(E)$.

For $p \in (0, \infty]$, we denote by $L^p(\Omega)$ the usual Lebesgue spaces. It is well known as Hölder's inequality that

$$\|fg\|_{L^{p_2}(\Omega)} \le \|f\|_{L^{p_1}(\Omega)} \|g\|_{L^{p_3}(\Omega)},$$

for $1/p_2 = 1/p_1 + 1/p_3$ with $p_i \in (0, \infty]$, $i = 1, 2, 3$. This shows that

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) \supset L^{p_3}(\Omega).$$

Dedicated to Professor Kôzô Yabuta on his 77th birthday

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Conversely, we can show the reverse inclusion by using the closed graph theorem or the uniform boundedness theorem. That is,

$$\mathrm{PWM}(L^{p_1}(\Omega), L^{p_2}(\Omega)) = L^{p_3}(\Omega).$$

This equality was extended to Orlicz spaces by [7, 8], to Musielak-Orlicz spaces by [18] and to Morrey spaces by [13, 14]. In this paper we further extend the equality to Musielak-Orlicz-Morrey spaces. We treat wide class of Musielak-Orlicz-Morrey spaces with generalized Young functions and growth functions, which include generalized Morrey spaces with variable exponent and variable growth condition. We consider function spaces which are defined on a complete σ-finite measure space with a metric or a quasi-metric. For example, spaces of homogeneous type in the sense of Coifman and Weiss [1, 2] or metric measure spaces with non-doubling measure. In this paper we don't always assume the doubling condition on the measure.

Recall that, for a normed or quasi-normed space $E \subset L^0(\Omega)$, we say that E has the lattice (ideal) property if the following holds:

$$f \in E,\ h \in L^0(\Omega),\ |h(x)| \le |f(x)| \text{a.e. } x \quad \Rightarrow \quad h \in E,\ \|h\|_E \le \|f\|_E.$$

It is known that, if E has the lattice property and is complete, then

$$\mathrm{PWM}(E) = L^\infty(\Omega) \quad \text{and} \quad \|g\|_{\mathrm{Op}} = \|g\|_{L^\infty(\Omega)},$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(E)$ (see [8, 12, 19] for example). In this paper we characterize pointwise multipliers from a Musielak-Orlicz-Morrey space to another Musielak-Orlicz-Morrey space.

For Young functions and Orlicz and Musielak-Orlicz spaces, see [6, 10, 21, 23, 24, 26, 30], etc. For Morrey and Orlicz-Morrey spaces, see [9, 11, 15–17, 25, 28], etc.

In the next section we give notion of Young functions and their generalization. We state definitions and properties of Musielak-Orlicz and Musielak-Orlicz-Morrey spaces in Sects. 3 and 4, respectively. Then we state main results in Sect. 5 and prove them in Sect. 6.

2 Young Functions and Their Generalization

Let $\bar{\boldsymbol{\Phi}}$ be the set of all functions $\Phi : [0, \infty] \to [0, \infty]$ such that

$$\lim_{t\to+0} \Phi(t) = \Phi(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} \Phi(t) = \Phi(\infty) = \infty.$$

Let

$$a(\Phi) = \sup\{t \ge 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \ge 0 : \Phi(t) = \infty\}.$$

Definition 1 A function $\Phi \in \bar{\boldsymbol{\Phi}}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is nondecreasing on $[0, \infty)$ and convex on $[0, b(\Phi))$, and

$$\lim_{t \to b(\Phi)-0} \Phi(t) = \Phi(b(\Phi)) \ (\le \infty).$$

We denote by $\boldsymbol{\Phi}_Y$ the set of all Young functions. Any Young function is neither identically zero nor identically infinity on $(0, \infty)$. We define three subsets $\mathcal{Y}^{(i)}$ $(i = 1, 2, 3)$ of Young functions as

$$\mathcal{Y}^{(1)} = \{\Phi \in \boldsymbol{\Phi}_Y : b(\Phi) = \infty\},$$
$$\mathcal{Y}^{(2)} = \{\Phi \in \boldsymbol{\Phi}_Y : b(\Phi) < \infty,\ \Phi(b(\Phi)) = \infty\},$$
$$\mathcal{Y}^{(3)} = \{\Phi \in \boldsymbol{\Phi}_Y : b(\Phi) < \infty,\ \Phi(b(\Phi)) < \infty\}.$$

Then we have the following properties of $\Phi \in \boldsymbol{\Phi}_Y$:

(i) If $\Phi \in \mathcal{Y}^{(1)}$, then Φ is absolutely continuous on any closed interval in $[0, \infty)$ by the convexity and nondecreasingness, and Φ is bijective from $[a(\Phi), \infty)$ to $[0, \infty)$.
(ii) If $\Phi \in \mathcal{Y}^{(2)}$, then Φ is absolutely continuous on any closed interval in $[0, b(\Phi))$, and Φ is bijective from $[a(\Phi), b(\Phi))$ to $[0, \infty)$.
(iii) If $\Phi \in \mathcal{Y}^{(3)}$, then Φ is absolutely continuous on $[0, b(\Phi)]$ and Φ is bijective from $[a(\Phi), b(\Phi)]$ to $[0, \Phi(b(\Phi))]$.

Next we recall the generalized inverse of Young function Φ in the sense of O'Neil [22, Definition 1.2]. See also [29]. For a Young function Φ and $u \in [0, \infty]$, let

$$\Phi^{-1}(u) = \inf\{t \ge 0 : \Phi(t) > u\}, \tag{1}$$

where $\inf \emptyset = \infty$. Then $\Phi^{-1}(u)$ is finite for all $u \in [0, \infty)$. If Φ is bijective from $[0, \infty)$ to itself, then Φ^{-1} is the usual inverse function of Φ.

We have the following properties of $\Phi \in \boldsymbol{\Phi}_Y$ and its inverse:

(P1) $\Phi(\Phi^{-1}(u)) \le u$ for all $u \in [0, \infty)$ and $t \le \Phi^{-1}(\Phi(t))$ if $\Phi(t) \in [0, \infty)$ (Property 1.3 in [22]).
(P2) $\Phi^{-1}(\Phi(t)) = t$ if $\Phi(t) \in (0, \infty)$.
(P3) If $\Phi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$, then $\Phi(\Phi^{-1}(u)) = u$ for all $u \in [0, \infty)$.
(P4) If $\Phi \in \mathcal{Y}^{(3)}$ and $0 < \delta < 1$, then there exists a Young function $\Psi \in \mathcal{Y}^{(2)}$ such that $b(\Phi) = b(\Psi)$ and

$$\Psi(\delta t) \le \Phi(t) \le \Psi(t) \quad \text{for all } t \in [0, \infty).$$

To see (P4) we only set $\Psi = \Phi + \Theta$, where we choose $\Theta \in \mathcal{Y}^{(2)}$ such that $a(\Theta) = \delta\, b(\Phi)$ and $b(\Theta) = b(\Phi)$.

Definition 2 Let $\boldsymbol{\Phi}_Y^v$ be the set of all $\Phi : \Omega \times [0, \infty] \to [0, \infty]$ such that $\Phi(x, \cdot)$ is a Young function for every $x \in \Omega$, and that $\Phi(\cdot, t)$ is measurable on Ω for every

$t \in [0, \infty]$. Assume also that, for any subset $A \subset \Omega$ with finite measure, there exists $t \in (0, \infty)$ such that $\Phi(\cdot, t)\chi_A$ is integrable, where χ_A is the characteristic function of A.

Definition 3 (i) Let $\boldsymbol{\Phi}_{GY}$ be the set of all $\Phi \in \bar{\boldsymbol{\Phi}}$ such that $\Phi((\cdot)^{1/\ell})$ is in $\boldsymbol{\Phi}_Y$ for some $\ell \in (0, 1]$.

(ii) Let $\boldsymbol{\Phi}^v_{GY}$ be the set of all $\Phi : \Omega \times [0, \infty] \to [0, \infty]$ such that $\Phi(\cdot, (\cdot)^{1/\ell})$ is in $\boldsymbol{\Phi}^v_Y$ for some $\ell \in (0, 1]$.

If $\Phi \in \boldsymbol{\Phi}_Y$, then it follows from the convexity and $\Phi(0) = 0$ that

$$\Phi(ct) \le c\Phi(t) \quad \text{for all } c \in [0, 1] \text{ and } t \in [0, \infty). \tag{2}$$

Hence, if $\Phi \in \boldsymbol{\Phi}_{GY}$ and $\Phi((\cdot)^{1/\ell}) \in \boldsymbol{\Phi}_Y$ with $\ell \in (0, 1]$, then

$$\Phi(ct) \le c^\ell \Phi(t) \quad \text{for all } c \in [0, 1] \text{ and } t \in [0, \infty). \tag{3}$$

For $\Phi, \Psi \in \bar{\boldsymbol{\Phi}}$, we write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(C^{-1}t) \le \Psi(t) \le \Phi(Ct) \quad \text{for all } t \in (0, \infty).$$

For $\Phi, \Psi : \Omega \times [0, \infty] \to [0, \infty]$, we also write $\Phi \approx \Psi$ if there exists a positive constant C such that

$$\Phi(x, C^{-1}t) \le \Psi(x, t) \le \Phi(x, Ct) \quad \text{for all } (x, t) \in \Omega \times (0, \infty).$$

Definition 4 Let $\bar{\boldsymbol{\Phi}}_Y$, $\bar{\boldsymbol{\Phi}}^v_Y$, $\bar{\boldsymbol{\Phi}}_{GY}$ and $\bar{\boldsymbol{\Phi}}^v_{GY}$ be the sets of all $\Phi \in \bar{\boldsymbol{\Phi}}$ such that $\Phi \approx \Psi$ for some Ψ in $\boldsymbol{\Phi}_Y$, $\boldsymbol{\Phi}^v_Y$, $\boldsymbol{\Phi}_{GY}$ and $\boldsymbol{\Phi}^v_{GY}$, respectively.

For $\Phi \in \bar{\boldsymbol{\Phi}}^v_{GY}$, we define also its generalized inverse with respect to t by (1) for each x and denote it by Φ^{-1}. That is,

$$\Phi^{-1}(x, u) = \inf\{t \ge 0 : \Phi(x, t) > u\}, \quad (x, u) \in \Omega \times [0, \infty]. \tag{4}$$

Then we have the following:

$$\Psi(x, t) = \Phi(x, t^{1/\ell}) \Rightarrow \Psi^{-1}(x, u) = (\Phi^{-1}(x, u))^\ell, \tag{5}$$

$$\Psi(x, t/C) \le \Phi(x, t) \le \Psi(x, Ct) \Rightarrow \Psi^{-1}(x, u)/C \le \Phi^{-1}(x, u) \le C\Psi^{-1}(x, u). \tag{6}$$

From (5) it follows that

$$\begin{cases} \Phi(x, \Phi^{-1}(x, u)) = \Psi(x, \Psi^{-1}(x, u)), \\ \Phi^{-1}(x, \Phi(x, t)) = (\Psi^{-1}(x, \Psi(x, t^\ell)))^{1/\ell}. \end{cases} \tag{7}$$

Therefore, if $\Phi \in \boldsymbol{\Phi}^v_{GY}$ and $\Phi(\cdot, (\cdot)^{1/\ell}) \in \boldsymbol{\Phi}^v_Y$ with $\ell \in (0, 1]$, then, from the properties (P1)–(P3) we have the following:

(P1') $\Phi(x, \Phi^{-1}(x, u)) \le u$ for all $u \in [0, \infty)$ and $x \in X$, and, $t \le \Phi^{-1}(x, \Phi(x, t))$ if $\Phi(x, t) \in [0, \infty)$.
(P2') $\Phi^{-1}(x, \Phi(x, t)) = t$ if $\Phi(x, t) \in (0, \infty)$.
(P3') If $\Phi(x, (\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$, then $\Phi(x, \Phi^{-1}(x, u)) = u$ for all $u \in [0, \infty)$.

For $\Phi \in \bar{\boldsymbol{\Phi}}^v_{GY}$ and $x \in \Omega$, let

$$a(\Phi; x) = \sup\{t \ge 0 : \Phi(x, t) = 0\}, \quad b(\Phi; x) = \inf\{t \ge 0 : \Phi(x, t) = \infty\}.$$

From the property (P4) we have the following:

(P4') For any $\Phi \in \boldsymbol{\Phi}^v_{GY}$ and $0 < \delta < 1$, there exists $\Psi \in \boldsymbol{\Phi}^v_{GY}$ such that $\Psi(x, (\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ for all $x \in \Omega$ and for some $\ell \in (0, 1]$, and

$$\Psi(x, \delta t) \le \Phi(x, t) \le \Psi(x, t) \quad \text{for all } (x, t) \in \Omega \times [0, \infty).$$

To see (P4') we only set $\Psi = \Phi + \Theta$, where we choose $\Theta(x, t)$ by the following way: If $\Phi(x, (\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$, then $\Theta(x, \cdot) \equiv 0$. If $\Phi(x, (\cdot)^{1/\ell}) \in \mathcal{Y}^{(3)}$, then $\Theta(x, \cdot) \in \mathcal{Y}^{(2)}$ such that $a(\Theta; x) = \delta\, b(\Phi; x)$ and $b(\Theta; x) = b(\Phi; x)$.

At the end of this section we state a lemma which is in [18].

Lemma 1 ([18]) *Let $\Phi \in \boldsymbol{\Phi}^v_{GY}$. For a subset $A \subset \Omega$ with $0 < \mu(A) < \infty$, let $\Phi^A(t) = \int_A \Phi(x, t)\, d\mu(x)$. Then $\Phi^A \in \boldsymbol{\Phi}_{GY}$.*

3 Musielak-Orlicz Spaces

Definition 5 (Musielak-Orlicz space) For a function $\Phi \in \bar{\boldsymbol{\Phi}}^v_{GY}$, let

$$L^\Phi(\Omega) = \left\{ f \in L^0(\Omega) : \int_\Omega \Phi(x, c|f(x)|)\, d\mu(x) < \infty \text{ for some } c > 0 \right\},$$
$$\|f\|_{L^\Phi(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1 \right\}.$$

Then $|f(x)| < \infty$ a.e. $x \in \Omega$ for all $f \in L^\Phi(\Omega)$. By the assumption in Definition 2 any characteristic function of a subset of Ω with finite measure is in $L^\Phi(\Omega)$. Moreover, $\|\cdot\|_{L^\Phi}$ is a quasi-norm, that is, there exists $k \in [1, \infty)$ such that, for all $f, g \in L^\Phi(\Omega)$ and a scalar c,

(i) $\|f\|_{L^\Phi(\Omega)} \ge 0$, $\|f\|_{L^\Phi(\Omega)} = 0 \Leftrightarrow f = 0$,
(ii) $\|cf\|_{L^\Phi(\Omega)} = |c|\|f\|_{L^\Phi(\Omega)}$,
(iii) $\|f + g\|_{L^\Phi(\Omega)} \le k(\|f\|_{L^\Phi(\Omega)} + \|g\|_{L^\Phi(\Omega)})$.

By the definition, if $\Phi \approx \Psi$, then $L^{\Phi}(\Omega) = L^{\Psi}(\Omega)$ with equivalent quasi-norms. If $\Phi \in \boldsymbol{\Phi}_{GY}^{v}$ and $\Phi(\cdot, (\cdot)^{1/\ell}) \in \boldsymbol{\Phi}_{Y}^{v}$ with $\ell \in (0, 1]$, then

(iv) $\|f + g\|_{L^{\Phi}(\Omega)}^{\ell} \le \|f\|_{L^{\Phi}(\Omega)}^{\ell} + \|g\|_{L^{\Phi}(\Omega)}^{\ell}$.

If $\Phi \in \boldsymbol{\Phi}_{Y}^{v}$, then $\|\cdot\|_{L^{\Phi}(\Omega)}$ is a norm.

Musielak-Orlicz spaces satisfy the lattice property (ideal property):

(v) If $g \in L^{\Phi}(\Omega)$ and $|f| \le |g|$ a.e. Ω, then $f \in L^{\Phi}(\Omega)$ and $\|f\|_{L^{\Phi}(\Omega)} \le \|g\|_{L^{\Phi}(\Omega)}$.

Let $\Phi \in \boldsymbol{\Phi}_{GY}^{v}$. Then by the left-continuity of $\Phi(x, t)$ with respect to t and the theory of the Lebesgue integral we have the following:

(vi) If $\sup_j \|f_j\|_{L^{\Phi}(\Omega)} < \infty$, $0 \le f_1 \le f_2 \le \cdots \to f$ a.e. Ω, then $f \in L^{\Phi}(\Omega)$ and $\lim_{j\to\infty} \|f_j\|_{L^{\Phi}(\Omega)} = \|f\|_{L^{\Phi}(\Omega)}$.

The property (vi) is called the Fatou property.

The properties of normed spaces of measurable functions, see [5, pp. 94–99]. By using the method in [3, pp. 38–40] or [10, pp. 35–36], we can prove the following proposition and lemma:

Proposition 1 *Let $\Phi \in \bar{\boldsymbol{\Phi}}_{GY}^{v}$. Then $L^{\Phi}(\Omega)$ is complete.*

Lemma 2 *Let $\Phi \in \bar{\boldsymbol{\Phi}}_{GY}^{v}$. If a sequence $\{f_j\}$ converges in $L^{\Phi}(\Omega)$ to f, then there exists a subsequence $\{f_{j(k)}\}$ which converges μ-almost everywhere to f.*

The following theorem is known.

Theorem 1 ([18]) *Let $\Phi_i \in \bar{\boldsymbol{\Phi}}_{GY}^{v}$, $i = 1, 2, 3$. Assume that there exists a positive constant C such that*

$$C^{-1}\Phi_2^{-1}(x, t) \le \Phi_1^{-1}(x, t)\Phi_3^{-1}(x, t) \le C\Phi_2^{-1}(x, t) \quad \text{for all } (x, t) \in \Omega \times (0, \infty). \tag{8}$$

Assume also that there exists $\Psi_3 \in \boldsymbol{\Phi}_{GY}^{v}$ such that

$$\Phi_3 \approx \Psi_3 \quad \text{and} \quad \Psi_3^A((\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}, \tag{9}$$

for some $\ell \in (0, 1]$ and for any $A \subset \Omega$ with $0 < \mu(A) < \infty$, where $\Psi_3^A(t) = \int_A \Psi_3(x, t)\, d\mu(x)$. Then

$$\mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega)) = L^{\Phi_3}(\Omega).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(L^{\Phi_1}(\Omega), L^{\Phi_2}(\Omega))$ is comparable to $\|g\|_{L^{\Phi_3}(\Omega)}$.

For a measurable set $A \subset \Omega$, we denote the characteristic function of A by χ_A. A finitely simple function has the form

$$\sum_{k=1}^{N} c_k \chi_{A_k},$$

where $N \in \mathbb{N}, c_k \in \mathbb{C}$, and A_k are pairwise disjoint measurable sets with $\mu(A_k) < \infty$. To prove Theorem 1 the following lemma was used.

Lemma 3 ([18]) *Let* $\Phi \in \boldsymbol{\Phi}_{GY}^{v}$ *and* $\Phi(\cdot, (\cdot)^{1/\ell}) \in \boldsymbol{\Phi}_{Y}^{v}$ *for some* $\ell \in (0, 1]$. *Assume that* $\Phi^A((\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ *for any* $A \subset \Omega$ *with* $0 < \mu(A) < \infty$, *where* $\Phi^A(t) = \int_A \Phi(x, t)\, d\mu(x)$. *Then, for all finitely simple functions* g *such that* $g \neq 0$,

$$\int_\Omega \Phi\left(x, \frac{|g(x)|}{\|g\|_{L^\Phi(\Omega)}}\right) d\mu(x) = 1.$$

We give the proof for convenience.

Proof We may assume that $g \geq 0$. Let

$$g = \sum_{k=1}^{N} c_k \chi_{A_k}, \quad 0 < c_1 < c_2 < \cdots < c_N,$$
$$0 < \mu(A_k) < \infty \ (k = 1, 2, \ldots, N), \text{ and } A_j \cap A_k = \emptyset \text{ if } j \neq k,$$

and let

$$\Phi^g(t) = \int_\Omega \Phi(x, |g(x)|t)\, d\mu(x), \quad \Phi^{A_k}(t) = \int_{A_k} \Phi(x, t)\, d\mu(x).$$

Then

$$\Phi^g(t) = \sum_{k=1}^{N} \Phi^{A_k}(c_k t),$$

and

$$a(\Phi^g) = \min_k a(\Phi^{A_k})/c_k, \quad b(\Phi^g) = \min_k b(\Phi^{A_k})/c_k.$$

In this case $\Phi^g((\cdot)^{1/\ell})$ is continuous and convex on $[0, b(\Phi^g))$ and bijective from $(a(\Phi^g), b(\Phi^g))$ to $(0, \infty)$. Since

$$\|g\|_{L^\Phi(\Omega)} = \inf\{\lambda > 0 : \Phi^g(1/\lambda) \leq 1\},$$

we have

$$\Phi^g(1/\|g\|_{L^\Phi(\Omega)}) = 1.$$

This shows the conclusion. □

Remark 1 There exists $\Phi \in \boldsymbol{\Phi}_{Y}^{v}$ such that $\Phi(x, \cdot) \in \mathcal{Y}^{(1)}$ for all $x \in \Omega$ and $\Phi^\Omega \in \mathcal{Y}^{(3)}$. Actually, let $\Omega = (0, 1)$ be the open interval in the real line with the Lebesgue measure and take Young functions $\Phi(x, t)$ for all $x \in \Omega$ such that $\Phi(x, 1) = 1$ and $\Phi(x, 1 + x) = 2/x$. This example is in [18].

4 Musielak-Orlicz-Morrey Spaces

In this section we define Musielak-Orlicz-Morrey spaces on (X, d, μ) which is a complete σ-finite measure space with a metric or a quasi-metric. More precisely, X is a topological space endowed with a metric (or quasi-metric) d and a nonnegative measure μ such that

$$\begin{aligned} d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y, \\ d(x, y) = d(y, x), \\ d(x, y) \leq K\,(d(x, z) + d(z, y)), \end{aligned} \tag{10}$$

the balls $B(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0 < \mu(B(x, r)) < \infty,$$

where $K \geq 1$ is a constant independent of $x, y, z \in X$ and $r > 0$. If $K = 1$, then d is a metric.

If μ satisfies the doubling condition, that is, there exists a positive constant C such that

$$\mu(B(x, 2r)) \leq C\, \mu(B(x, r)) \quad \text{for all } x \in X \text{ and } r > 0, \tag{11}$$

then (X, d, μ) is a space of homogeneous type introduced by Coifman and Weiss [1, 2]. In this case, if $\mu(X) < \infty$, then there exists a positive constant R_0 such that

$$X = B(x, R_0) \quad \text{for all } x \in X, \tag{12}$$

see [20, Lemma 5.1]. A space of homogeneous type (X, d, μ) is called Q-homogeneous ($Q > 0$), if there exists a positive constant C such that

$$C^{-1} r^Q \leq \mu(B(x, r)) \leq C r^Q \quad \text{for} \quad \begin{cases} 0 < r < \infty & \text{when} \mu(X) = \infty, \\ 0 < r < R_0 & \text{when} \mu(X) < \infty, \end{cases} \tag{13}$$

where R_0 is the constant in (12). For example, the Euclidean space $\mathbb{R}^n$ with the Lebesgue measure is n-homogeneous. In this paper we don't always assume (11).

For $\phi : X \times (0, \infty) \to (0, \infty)$ and $B = B(x, r)$, we write $\phi(B) = \phi(x, r)$. For a ball $B = B(x, r)$ and $k > 0$, we shall denote $B(x, kr)$ by kB.

Definition 6 (Musielak-Orlicz-Morrey space) For $\Phi \in \bar{\boldsymbol{\Phi}}^v_{GY}$, $\phi : X \times (0, \infty) \to (0, \infty)$, $\kappa \in [1, \infty)$ and a ball B, let

$$\|f\|_{\Phi,\phi,\kappa,B} = \inf \left\{ \lambda > 0 : \frac{1}{\phi(\kappa B)\mu(\kappa B)} \int_B \Phi\left(x, \frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1 \right\},$$

and let

$$L^{(\Phi,\phi,\kappa)}(X) = \left\{ f \in L^0(X) : \|f\|_{L^{(\Phi,\phi,\kappa)}(X)} < \infty \right\},$$
$$\|f\|_{L^{(\Phi,\phi,\kappa)}(X)} = \sup_B \|f\|_{\Phi,\phi,\kappa,B},$$

where the supremum is taken over all balls B. For $\kappa = 1$, denote $L^{(\Phi,\phi,1)}(X)$ by $L^{(\Phi,\phi)}(X)$ simply.

Then $\|\cdot\|_{L^{(\Phi,\phi,\kappa)}(X)}$ is a quasi-norm and $L^{(\Phi,\phi,\kappa)}(X)$ is a quasi-Banach space (complete quasi-normed space), since

$$\|f\|_{\Phi,\phi,\kappa,B} = \|f\|_{L^{\Phi}(B,\mu/(\phi(\kappa B)\mu(\kappa B)))},$$

which is a quasi-norm on the Musielak-Orlicz space $L^{\Phi}(B)$ with the measure $\mu/(\phi(\kappa B)\mu(\kappa B))$. If $\Phi \in \boldsymbol{\Phi}_Y^v$, then $\|\cdot\|_{L^{(\Phi,\phi,\kappa)}(X)}$ is a norm and $L^{(\Phi,\phi,\kappa)}(X)$ is a Banach space.

By Lemma 2 we also have the following

Lemma 4 *Let $\Phi \in \bar{\boldsymbol{\Phi}}_{GY}^v$, $\phi : X \times (0,\infty) \to (0,\infty)$ and $\kappa \in [1,\infty)$. If a sequence $\{f_j\}$ converges in $L^{(\Phi,\phi,\kappa)}(X)$ to f, then there exists a subsequence $\{f_{j(k)}\}$ which converges μ-almost everywhere to f.*

Definition 7 Let p be a variable exponent, that is, it is a measurable function defined on X valued in $(0,\infty]$, and let

$$p_- = \operatorname*{ess\,inf}_{x\in X} p(x), \quad p_+ = \operatorname*{ess\,sup}_{x\in X} p(x). \tag{14}$$

If $\Phi(x,t) = t^{p(x)}$ and $p_- > 0$, then $\Phi \in \boldsymbol{\Phi}_{GY}^v$ and $\Phi(x,(\cdot)^{\max(1,1/p_-)}) \in \boldsymbol{\Phi}_Y^v$. Here, use the following interpretation:

$$t^\infty = \begin{cases} 0, & t \in [0,1], \\ \infty, & t \in (1,\infty]. \end{cases}$$

In this case, denote $L^{(\Phi,\phi,\kappa)}(X)$ by $L^{(p,\phi,k)}(X)$.

Definition 8 Let w be a weight function, that is, it is a measurable function defined on X valued in $(0,\infty)$ a.e., and $\int_A w(x)\,d\mu(x) < \infty$ for any $A \subset X$ with finite measure. If $\Phi(x,t) = t^{p(x)}w(x)$ and $p_- > 0$, then $\Phi \in \boldsymbol{\Phi}_{GY}^v$. In this case, denote $L^{(\Phi,\phi,\kappa)}(X)$ by $L_w^{(p,\phi,\kappa)}(X)$.

Remark 2 If $X = \mathbb{R}^n$ is the Euclidean space with the usual metric and a non-doubling measure μ, $1 \le q \le p < \infty$, $\Phi(x,t) = t^q$ and $\phi(B) = \mu(B)^{-q/p}$, then $L^{(\Phi,\phi,\kappa)}(\mathbb{R}^n)$ coincides with the Morrey space $\mathcal{M}_q^p(\kappa,\mu)$ introduced by Sawano and Tanaka [27]. To put it more precisely, they used cubes Q instead of balls B in [27].

A function $\theta : X \times (0, \infty) \to (0, \infty)$ is said to satisfy the doubling condition if there exists a positive constant C such that

$$\frac{1}{C} \le \frac{\theta(x, r)}{\theta(x, s)} \le C \quad \text{for all } x \in X \text{ and } r, s \in (0, \infty) \text{ with } \frac{1}{2} \le \frac{r}{s} \le 2. \tag{15}$$

For functions $\theta, \kappa : X \times (0, \infty) \to (0, \infty)$, we write $\theta \sim \kappa$ if there exists a positive constant C such that

$$\frac{1}{C} \le \frac{\theta(x, r)}{\kappa(x, r)} \le C \quad \text{for all } (x, r) \in X \times (0, \infty).$$

By the definition we have the following.

Proposition 2 *(i) If $\phi(B) = 1/\mu(B)$, then $L^{(\Phi,\phi,\kappa)}(X)$ coincides with the Musielak-Orlicz space $L^{\Phi}(X)$ for all $\kappa \in [1, \infty)$.*

(ii) If $\Phi \approx \Psi$ and $\phi \sim \psi$, then $L^{(\Phi,\phi,\kappa)}(X) = L^{(\Psi,\psi,\kappa)}(X)$ with equivalent quasi-norms.

(iii) If μ satisfies the doubling condition (11) and ϕ satisfies the doubling condition (15), then, for all $\kappa \in [1, \infty)$, $L^{(\Phi,\phi,\kappa)}(X) = L^{(\Phi,\phi)}(X)$ with equivalent quasi-norms.

(iv) Assume that the Lebesgue differentiation theorem holds on (X, d, μ), that is, for any locally integrable function f, for μ-almost all $x \in X$ one can find a sequence of balls $\{B(x, r_k)\}_k$ with $r_k \to 0$ such that

$$\lim_{k\to\infty} \frac{1}{\mu(B(x, r_k))} \int_{B(x,r_k)} f(y)\, d\mu(y) = f(x).$$

If $\phi(x, r_k) \to 0$ as $k \to \infty$ for each $x \in X$, then $L^{(\Phi,\phi,\kappa)}(X) = \{0\}$.

(v) If there exists $x \in X$ such that $\phi(x, r)\mu(B(x, r)) \to 0$ as $r \to \infty$, then $L^{(\Phi,\phi,\kappa)}(X) = \{0\}$.

We can avoid the cases (iv) and (v) by use of $\phi \in \mathcal{G}^v$ in Definition 10 bellow.

Definition 9 A function $\theta : X \times (0, \infty) \to (0, \infty)$ is *almost increasing* (*almost decreasing*) *with respect to the order by ball inclusion* if there exists a positive constant C such that

$$\theta(B_1) \le C\theta(B_2) \quad (\theta(B_1) \ge C\theta(B_2)) \quad \text{for all balls } B_1 \text{ and } B_2 \text{ with } B_1 \subset B_2.$$

Definition 10 Let $\mathcal{G}^v$ be the set of all $\phi : X \times (0, \infty) \to (0, \infty)$ such that ϕ is almost decreasing with respect to the order by ball inclusion and $\phi(B)\mu(B)$ is almost increasing with respect to the order by ball inclusion.

Example 1 (i) Let $\phi(x, r) = \mu(B(x, r))^{\lambda}$ and $-1 \le \lambda \le 0$. Then ϕ is almost decreasing with respect to the order by ball inclusion and $\phi(B)\mu(B)$ is almost increasing with respect to the order by ball inclusion.

(ii) Let $\lambda(\cdot)$ be a variable exponent and λ^* is a constant, and let

$$\phi(x,r) = \begin{cases} r^{\lambda(x)}, & r \le 1/e, \\ (\min(r, r_*))^{\lambda^*}, & r > 1/e, \end{cases} \quad \lambda_+ \le 0,\ \lambda_* \le 0,$$

where $r_* = \sup\{d(x,y) : x, y \in X\} \in (0, \infty]$. If $\lambda(\cdot)$ satisfies the local log-Hölder condition, that is, there exists a positive constant C such that

$$|\lambda(x) - \lambda(y)| \le \frac{C}{\log(1/d(x,y))} \quad \text{for} \quad d(x,y) \le 1/e,$$

then ϕ satisfies the following two conditions:

$$\phi(x,r) \ge \phi(x,s) \quad \text{for} \quad x \in X,\ r < s,$$

and

$$\phi(x,r) \sim \phi(y,r) \quad \text{for} \quad d(x,y) \le r.$$

These show that ϕ is almost decreasing with respect to the order by ball inclusion. Moreover, if (X, d, μ) is a space of homogeneous type and Q-homogeneous, and if $\lambda_- \ge -Q$ and $\lambda^* \ge -Q$, then $\phi(B)\mu(B)$ is almost increasing with respect to the order by ball inclusion.

Remark 3 Let (X, d, μ) be a space of homogeneous type and $\phi \in \mathcal{G}^v$. Then ϕ satisfies the doubling condition.

By the above remark we have the following.

Proposition 3 *Let (X, d, μ) be a space of homogeneous type. Assume that $\Phi \in \bar{\boldsymbol{\Phi}}^v_{GY}$ and $\phi \in \mathcal{G}^v$. Then, for all $\kappa \in [1, \infty)$, $L^{(\Phi,\phi,\kappa)}(X) = L^{(\Phi,\phi)}(X)$ with equivalent quasi-norms.*

Definition 11 (Hytönen [4]) A metric space (X, d) is called geometrically doubling if there exists $N \in \mathbb{N}$ such that any ball $B(x, r) \subset X$ can be covered by at most N balls $B(x_i, r/2)$.

The following proposition is a generalization of the result in [28].

Proposition 4 *Let (X, d, μ) be a metric measure space and (X, d) be geometrically doubling. Assume that $\Phi \in \bar{\boldsymbol{\Phi}}^v_{GY}$ and $\phi \in \mathcal{G}^v$. Then, for $\kappa_1, \kappa_2 \in (1, \infty)$,*

$$L^{(\Phi,\phi,\kappa_1)}(X) = L^{(\Phi,\phi,\kappa_2)}(X)$$

with equivalent quasi-norms.

Proof We may assume that $\Phi \in \boldsymbol{\Phi}^v_{GY}$ and $\Phi(\cdot, (\cdot)^{1/\ell}) \in \boldsymbol{\Phi}^v_Y$ with $\ell \in (0, 1]$ by Proposition 2 (ii). Let $\kappa_1, \kappa_2 \in (1, \infty)$ and $\kappa_1 < \kappa_2$. Then $\mu(\kappa_1 B)\phi(\kappa_1 B) \le C_\phi\, \mu(\kappa_2 B)\phi(\kappa_2 B)$ for some constant C_ϕ, since $\phi \in \mathcal{G}^v$. Hence

$$\|f\|_{L^{(\Phi,\phi,\kappa_2)}(X)} \le C_\phi \|f\|_{L^{(\Phi,\phi,\kappa_1)}(X)}.$$

Let $\delta = (k_1 - 1)/(k_2 + 1) \in (0, 1)$. Then, any ball $B(x, r)$ can be covered by at most $N\delta^{-\log_2 N}$ balls $B(x_i, \delta r)$, see [4, Lemma 2.3]. Here, N is the integer in Definition 11. We may assume that $B(x, r) \cap B(x_i, \delta r) \ne \emptyset$, that is, $d(x, x_i) < r + \delta r$. Then $B(x_i, k_2\delta r) \subset B(x, k_1 r)$. Using the properties of $\phi \in \mathcal{G}^v$ and letting $C = \left(C_\phi N\delta^{-\log_2 N}\right)^{1/\ell}$, we have

$$\begin{aligned}
&\frac{1}{\mu(B(x,\kappa_1 r))\phi(x,\kappa_1 r)} \int_{B(x,r)} \Phi\left(x, \frac{|f(x)|}{C\|f\|_{L^{(\Phi,\phi,\kappa_2)}(X)}}\right) d\mu(x)\\
&\le \frac{1}{C^\ell} \sum_i \frac{1}{\mu(B(x,\kappa_1 r))\phi(x,\kappa_1 r)} \int_{B(x_i,\delta r)} \Phi\left(x, \frac{|f(x)|}{\|f\|_{L^{(\Phi,\phi,\kappa_2)}(X)}}\right) d\mu(x)\\
&\le \frac{1}{C^\ell} \sum_i \frac{C_\phi}{\mu(B(x_i,\kappa_2\delta r))\phi(x_i,\kappa_2\delta r)} \int_{B(x_i,\delta r)} \Phi\left(x, \frac{|f(x)|}{\|f\|_{L^{(\Phi,\phi,\kappa_2)}(X)}}\right) d\mu(x)\\
&\le \frac{C_\phi N\delta^{-\log_2 N}}{C^\ell} = 1.
\end{aligned}$$

That is,

$$\|f\|_{L^{(\Phi,\phi,\kappa_1)}(X)} \le C\|f\|_{L^{(\Phi,\phi,\kappa_2)}(X)}.$$

This shows that $L^{(\Phi,\phi,\kappa_1)}(X) = L^{(\Phi,\phi,\kappa_2)}(X)$ with equivalent quasi-norms. □

5 Main Results

In this section we characterize pointwise multipliers from a Musielak-Orlicz-Morrey space to another Musielak-Orlicz-Morrey space.

5.1 Generalized Hölder's Inequality

In this subsection, we prove a generalized Hölder's inequality for Musielak-Orlicz-Morrey spaces.

Theorem 2 *Let* $\Phi_i \in \bar{\boldsymbol{\Phi}}_{GY}^{v}$, $\phi_i : X \times (0,\infty) \to (0,\infty)$ *and* $\kappa_i \in [1,\infty)$, $i = 1, 2, 3$. *Assume that there exists a positive constant* C *such that*

$$\Phi_1^{-1}(x, t\phi_1(x,\kappa_1 r))\Phi_3^{-1}(x, t\phi_3(x,\kappa_3 r)) \le C\,\Phi_2^{-1}(x, t\phi_2(x,\kappa_2 r)) \quad \text{for all } x \in X \text{ and } r, t \in (0,\infty).$$

If $\max(\kappa_1, \kappa_3) \le \kappa_2$, *then there exists a positive constant* C' *such that, for all* $f \in L^{(\Phi_1,\phi_1,\kappa_1)}(X)$ *and* $g \in L^{(\Phi_3,\phi_3,\kappa_3)}(X)$,

$$\|fg\|_{L^{(\Phi_2,\phi_2,\kappa_2)}(X)} \le C'\|f\|_{L^{(\Phi_1,\phi_1,\kappa_1)}(X)}\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}.$$

Remark 4 If $\Phi_2 \in \boldsymbol{\Phi}_{GY}^{v}$ and $\Phi_2(\cdot,(\cdot)^{1/\ell_2}) \in \boldsymbol{\Phi}_{Y}^{v}$ with $\ell_2 \in (0,1]$, then we can take $C' = 2^{1/\ell_2}C$ in Theorem 2.

From Theorem 2 we have the following inclusion:

$$\mathrm{PWM}(L^{(\Phi_1,\phi_1,\kappa_1)}(X), L^{(\Phi_2,\phi_2,\kappa_2)}(X)) \supset L^{(\Phi_3,\phi_3,\kappa_3)}(X), \tag{16}$$

and

$$\|g\|_{\mathrm{Op}} \le C'\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)},$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(L^{(\Phi_1,\phi_1,\kappa_1)}(X), L^{(\Phi_2,\phi_2,\kappa_2)}(X))$.

From Theorem 2 we have the following corollaries immediately.

Corollary 1 *Let* $\Phi_i \in \bar{\boldsymbol{\Phi}}_{GY}^{v}$ *and* $\phi_i : X \times (0,\infty) \to (0,\infty)$, $i = 1, 2, 3$, *and let* $\kappa \in [1,\infty)$. *Assume that there exists a positive constant* C *such that*

$$\Phi_1^{-1}(x, t\phi_1(x,r))\Phi_3^{-1}(x, t\phi_3(x,r)) \le C\,\Phi_2^{-1}(x, t\phi_2(x,r)) \quad \text{for all } x \in X \text{ and } r, t \in (0,\infty).$$

Then there exists a positive constant C' *such that, for all* $f \in L^{(\Phi_1,\phi_1,\kappa)}(X)$ *and* $g \in L^{(\Phi_3,\phi_3,\kappa)}(X)$,

$$\|fg\|_{L^{(\Phi_2,\phi_2,\kappa)}(X)} \le C'\|f\|_{L^{(\Phi_1,\phi_1,\kappa)}(X)}\|g\|_{L^{(\Phi_3,\phi_3,\kappa)}(X)}.$$

Corollary 2 *Let* $\Phi_i \in \boldsymbol{\Phi}_{GY}$ *and* $\Phi((\cdot)^{1/\ell_i}) \in \boldsymbol{\Phi}_{Y}$ *with* $\ell_i \in (0,1]$, *and* $\phi_i : (0,\infty) \to (0,\infty)$, $i = 1, 2, 3$, *and let* $\kappa \in [1,\infty)$. *Assume that there exists a positive constant* C *such that*

$$\Phi_1^{-1}(t\phi_1(r))\Phi_3^{-1}(t\phi_3(r)) \le C\,\Phi_2^{-1}(t\phi_2(r)) \quad \text{for all } r, t \in (0,\infty).$$

If $f \in L^{(\Phi_1,\phi_1,\kappa)}(X)$ *and* $g \in L^{(\Phi_3,\phi_3,\kappa)}(X)$, *then* $fg \in L^{(\Phi_2,\phi_2,\kappa)}(X)$ *and*

$$\|fg\|_{L^{(\Phi_2,\phi_2,\kappa)}(X)} \le 2^{1/\ell_2}C\|f\|_{L^{(\Phi_1,\phi_1,\kappa)}(X)}\|g\|_{L^{(\Phi_3,\phi_3,\kappa)}(X)}.$$

Corollary 3 *Let* $\Phi_i \in \boldsymbol{\Phi}_{GY}$ *and* $\Phi((\cdot)^{1/\ell_i}) \in \boldsymbol{\Phi}_Y$ *with* $\ell_i \in (0, 1]$, $i = 1, 2, 3$, *and let* $\phi : (0, \infty) \to (0, \infty)$ *and* $\kappa \in [1, \infty)$. *Assume that there exists a positive constant* C *such that*

$$\Phi_1^{-1}(t)\Phi_3^{-1}(t) \le C\,\Phi_2^{-1}(t) \quad \text{for all } t \in (0, \infty).$$

If $f \in L^{(\Phi_1,\phi,\kappa)}(X)$ *and* $g \in L^{(\Phi_3,\phi,\kappa)}(X)$, *then* $fg \in L^{(\Phi_2,\phi,\kappa)}(X)$ *and*

$$\|fg\|_{L^{(\Phi_2,\phi,\kappa)}(X)} \le 2^{1/\ell_2} C \|f\|_{L^{(\Phi_1,\phi,\kappa)}(X)} \|g\|_{L^{(\Phi_3,\phi,\kappa)}(X)}.$$

Corollary 4 *Let* $p_i(\cdot)$ *be variable exponents with* $0 < (p_i)_- \le (p_i)_+ \le \infty$, w_i *be weights, and* $\phi_i : X \times (0, \infty) \to (0, \infty)$, $i = 1, 2, 3$, *and let* $\kappa \in [1, \infty)$. *Assume that* $1/p_1(x) + 1/p_3(x) = 1/p_2(x)$ *and that there exists a positive constant* C *such that*

$$(\phi_1(x,r)/w_1(x))^{1/p_1(x)} (\phi_3(x,r)/w_3(x))^{1/p_3(x)} \le C\,(\phi_2(x,r)/w_2(x))^{1/p_2(x)} \quad \text{for all } x \in X \text{ and } r \in (0, \infty).$$

If $f \in L_{w_1}^{(p_1,\phi_1,\kappa)}(X)$ *and* $g \in L_{w_3}^{(p_3,\phi_3,\kappa)}(X)$, *then* $fg \in L_{w_2}^{(p_2,\phi_2,\kappa)}(X)$ *and*

$$\|fg\|_{L_{w_2}^{(p_2,\phi_2,\kappa)}(X)} \le 2^{1/\min(1,(p_2)_-)} C \|f\|_{L_{w_1}^{(p_1,\phi_1,\kappa)}(X)} \|g\|_{L_{w_3}^{(p_3,\phi_3,\kappa)}(X)}.$$

Remark 5 If (X, d, μ) is the Euclidean space $\mathbb{R}^n$ with the Lebesgue measure, then Corollary 2 is a generalization of [16, Theorem 4.1]. If p_i are constants, $w_i \equiv 1$ and $\phi_i : (0, \infty) \to (0, \infty)$, then Corollary 4 is a generalization of the results in [13, 14].

5.2 Characterization of the Pointwise Multipliers

In this subsection we state the reverse inclusion to (16). As corollaries we characterize the pointwise multipliers on Musielak-Orlicz-Morrey spaces.

Theorem 3 *Let* $\Phi_i \in \bar{\boldsymbol{\Phi}}_{GY}^{v}$, $\phi_i \in \mathcal{G}^v$ *and* $\kappa_i \in [1, \infty)$, $i = 1, 2, 3$. *Assume that there exists a positive constant* C *such that*

$$\Phi_2^{-1}(x, t\phi_2(x, \kappa_2 r)) \le C\Phi_1^{-1}(x, t\phi_1(x, \kappa_1 r))\Phi_3^{-1}(x, t\phi_3(x, \kappa_3 r)) \quad \text{for all } x \in X \text{ and } r, t \in (0, \infty),$$

and that ϕ_3/ϕ_1 *is almost increasing with respect to the order by ball inclusion. Assume also one of the following:*

(i) Φ_3 *satisfies the* Δ_2 *condition, that is, there exists a positive constant* C_{Φ_3} *such that*

$$\Phi_3(x, 2t) \le C_{\Phi_3}\Phi_3(x, t) \quad \text{for all } x \in X \text{ and } t \in (0, \infty).$$

(ii) $\lim_{r\to\infty} \inf_{x\in X} \phi_3(x,r)\mu(B(x,r)) = \infty$, *$\phi_3(x,r)$ and $\mu(B(x,r))$ are continuous with respect to x and r, and, for all balls B,*

(a) any countable subset in B has an accumulation point in X,
(b) there exists $\Psi_B \in \boldsymbol{\Phi}_{GY}$ satisfying $\Psi_B((\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)}$ for some $\ell \in (0,1]$ such that $\sup_{x\in B} \Phi_3(x,t) \le \Psi_B(t)$ for all t, and,
(c) $\lim_{r\to+0} \inf_{x\in B} \phi_3(x,r) = \infty$.

If $\kappa_2 \le \kappa_3$ and $3K^2\kappa_3 \le \kappa_1$, then

$$\mathrm{PWM}(L^{(\Phi_1,\phi_1,\kappa_1)}(X), L^{(\Phi_2,\phi_2,\kappa_2)}(X)) \subset L^{(\Phi_3,\phi_3,\kappa_3)}(X),$$

and there exists a positive constant C' such that

$$\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)} \le C'\|g\|_{\mathrm{Op}},$$

where $\|g\|_{\mathrm{Op}}$ is the operator norm of $g \in \mathrm{PWM}(L^{(\Phi_1,\phi_1,\kappa_1)}(X), L^{(\Phi_2,\phi_2,\kappa_2)}(X))$.

Remark 6 In Theorem 3 the assumption that ϕ_3/ϕ_1 is almost increasing with respect to the order by ball inclusion is necessary, see [14].

If $\phi \in \mathcal{G}^v$ and ϕ satisfies the doubling condition, then $\Phi^{-1}(x, t\phi(x,r)) \sim \Phi^{-1}(x, t\phi(x, 3K^2r))$. By Theorems 2 and 3 we have the following corollary.

Corollary 5 *Let $\Phi_i \in \bar{\boldsymbol{\Phi}}^v_{GY}$ and $\phi_i \in \mathcal{G}^v$, $i = 1,2,3$, and let $\kappa \in [1,\infty)$. Assume that there exists a positive constant C such that*

$$\begin{aligned} C^{-1}\Phi_2^{-1}(x, t\phi_2(x,r)) &\le \Phi_1^{-1}(x, t\phi_1(x,r))\Phi_3^{-1}(x, t\phi_3(x,r)) \\ &\le C\Phi_2^{-1}(x, t\phi_2(x,r)) \quad \text{for all } x \in X \text{ and } r,t \in (0,\infty), \end{aligned}$$

and that ϕ_3/ϕ_1 is almost increasing with respect to the order by ball inclusion. Assume also one of the conditions (i) and (ii) in Theorem 3. If ϕ_1 or ϕ_2 satisfies the doubling condition, then

$$\mathrm{PWM}(L^{(\Phi_1,\phi_1,\kappa)}(X), L^{(\Phi_2,\phi_2,\kappa)}(X)) = L^{(\Phi_3,\phi_3,\kappa)}(X),$$

and the operator norm of $g \in \mathrm{PWM}(L^{(\Phi_1,\phi_1,\kappa)}(X), L^{(\Phi_2,\phi_2,\kappa)}(X))$ is comparable to $\|g\|_{L^{(\Phi_3,\phi_3,\kappa)}(X)}$.

If $\Phi(x,r) = r^{p(x)}w(x)$ and $0 < p_- \le p_+ < \infty$, then Φ satisfies the Δ_2 condition.

Corollary 6 *Let $p_i(\cdot)$ be variable exponents with $0 < (p_i)_- \le (p_i)_+ \le \infty$, w_i be weights and $\phi_i \in \mathcal{G}^v$, $i = 1, 2, 3$, and let $\kappa \in [1, \infty)$. Assume that $1/p_1(x) + 1/p_3(x) = 1/p_2(x)$, that there exists a positive constant C such that*

$$C^{-1}(\phi_2(x,r)/w_2(x))^{1/p_2(x)} \le (\phi_1(x,r)/w_1(x))^{1/p_1(x)}(\phi_3(x,r)/w_3(x))^{1/p_3(x)} \\ \le C\,(\phi_2(x,r)/w_2(x))^{1/p_2(x)} \quad \text{for all } x \in X \text{ and } r \in (0,\infty),$$

and that ϕ_3/ϕ_1 is almost increasing with respect to the order by ball inclusion. If $(p_3)_+ < \infty$ and if ϕ_1 or ϕ_2 satisfies the doubling condition, then

$$\mathrm{PWM}(L_{w_1}^{(p_1,\phi_1,\kappa)}(X), L_{w_2}^{(p_2,\phi_2,\kappa)}(X)) = L_{w_3}^{(p_3,\phi_3,\kappa)}(X),$$

and the operator norm of $g \in \mathrm{PWM}(L_{w_1}^{(p_1,\phi_1,\kappa)}(X), L_{w_2}^{(p_2,\phi_2,\kappa)}(X))$ is comparable to $\|g\|_{L_{w_3}^{(p_3,\phi_3,\kappa)}(X)}$.

By Remark 3 we have the following corollary.

Corollary 7 *Let (X, d, μ) be a space of homogeneous type, and let $\Phi_i \in \bar{\boldsymbol{\Phi}}_{GY}^{v}$ and $\phi_i \in \mathcal{G}^v$, $i = 1, 2, 3$. Assume that there exists a positive constant C such that*

$$C^{-1}\Phi_2^{-1}(x, t\phi_2(x,r)) \le \Phi_1^{-1}(x, t\phi_1(x,r))\Phi_3^{-1}(x, t\phi_3(x,r)) \\ \le C\,\Phi_2^{-1}(x, t\phi_2(x,r)) \quad \text{for all } x \in X \text{ and } r, t \in (0,\infty),$$

and that ϕ_3/ϕ_1 is almost increasing with respect to the order by ball inclusion. Assume also one of the conditions (i) and (ii) in Theorem 3. Then

$$\mathrm{PWM}(L^{(\Phi_1,\phi_1)}(X), L^{(\Phi_2,\phi_2)}(X)) = L^{(\Phi_3,\phi_3)}(X),$$

and the operator norm of $g \in \mathrm{PWM}(L^{(\Phi_1,\phi_1)}(X), L^{(\Phi_2,\phi_2)}(X))$ is comparable to $\|g\|_{L^{(\Phi_3,\phi_3)}(X)}$.

By Example 1 (ii) we have the following corollary.

Corollary 8 *Let (X, d, μ) be a space of homogeneous type and Q-homogeneous. Let $p_i(\cdot)$ and $\lambda_i(\cdot)$ be variable exponents with $0 < (p_i)_- \le (p_i)_+ \le \infty$ and $-Q \le (\lambda_i)_- \le (\lambda_i)_+ < 0$, w_i be weights, $i = 1, 2, 3$. Let λ^* be a constant with $-Q \le \lambda^* < 0$, and let*

$$\phi_i(x,r) = \begin{cases} r^{\lambda_i(x)}, & r \le 1/e, \\ (\min(r, r_*))^{\lambda^*}, & r > 1/e, \end{cases}$$

where $r_* = \sup\{d(x,y) : x, y \in X\} \in (0, \infty]$. Assume that $(p_3)_+ < \infty$, that $\lambda_i(\cdot)$, $i = 1, 2, 3$, are log-Hölder continuous, and that

$$\begin{cases} \frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)}, \\ \frac{\lambda_1(x)}{p_1(x)} + \frac{\lambda_3(x)}{p_3(x)} = \frac{\lambda_2(x)}{p_2(x)}, \\ w_1(x)^{1/p_1(x)} w_3(x)^{1/p_3(x)} = w_2(x)^{1/p_2(x)}, \\ \lambda_3(x) \geq \lambda_1(x), \end{cases} \quad \text{for all } x \in X.$$

Then

$$\mathrm{PWM}(L_{w_1}^{(p_1,\phi_1)}(X), L_{w_2}^{(p_2,\phi_2)}(X)) = L_{w_3}^{(p_3,\phi_3)}(X),$$

and the operator norm of $g \in \mathrm{PWM}(L_{w_1}^{(p_1,\phi_1)}(X), L_{w_2}^{(p_2,\phi_2)}(X))$ *is comparable to* $\|g\|_{L_{w_3}^{(p_3,\phi_3)}(X)}$.

Let $p(\cdot)$ be a variable exponent, and let

$$\Phi(x,t) = \begin{cases} 1/\exp(1/t^{p(x)}), & t \in [0,1], \\ \exp(t^{p(x)}), & t \in (1,\infty]. \end{cases}$$

Here we use the following interpretation:

$$\begin{cases} 1/\exp(1/t^{\infty}) = 0, & t \in [0,1], \\ \exp(t^{\infty}) = \infty, & t \in (1,\infty]. \end{cases}$$

If $p_- > 0$, then $\Phi \in \bar{\boldsymbol{\Phi}}_Y^v$, see [18, Examples 3.2 and 3.5].

Corollary 9 *Let* $X = \mathbb{R}^n$, $d(x,y) = |x-y|$ *and* $\mu = w dx$, *where* w *is a weight function and* dx *is the Lebesgue measure. Assume that there exist positive constants* c_1 *and* c_2 *such that* $c_1 \leq w(x) \leq c_2$ *a.e* $x \in \mathbb{R}^n$. *Let* $p_i(\cdot)$ *be variable exponents with* $0 < (p_i)_- \leq (p_i)_+ \leq \infty$, *and let*

$$\Phi_i(x,t) = \begin{cases} 1/\exp(1/t^{p_i(x)}), & t \in [0,1], \\ \exp(t^{p_i(x)}), & t \in (1,\infty], \end{cases} \quad i = 1,2,3.$$

Let λ *be a constant with* $-1 < \lambda < 0$, *and let* $\phi(B) = \mu(B)^{\lambda}$. *Assume that* $(p_3)_+ < \infty$ *and that* $1/p_1(x) + 1/p_3(x) = 1/p_2(x)$. *Then*

$$\mathrm{PWM}(L^{(\Phi_1,\phi)}(X), L^{(\Phi_2,\phi)}(X)) = L^{(\Phi_3,\phi)}(X),$$

and the operator norm of $g \in \mathrm{PWM}(L^{(\Phi_1,\phi)}(X), L^{(\Phi_2,\phi)}(X))$ *is comparable to* $\|g\|_{L^{(\Phi_3,\phi)}(X)}$.

For generalized Lebesgue spaces $L^{p(\cdot)}(X)$ with variable exponent, we have the following, which is a corollary of Theorem 1.

Corollary 10 ([18]) *Let $p_i(\cdot)$ be variable exponents, $i = 1, 2, 3$, and*

$$X_\infty = \{x \in X : p_3(x) = \infty\}.$$

Assume that $(p_i)_- > 0$, $i = 1, 2, 3$, $\sup_{x \in X \setminus X_\infty} p_3(x) < \infty$ and

$$\frac{1}{p_1(x)} + \frac{1}{p_3(x)} = \frac{1}{p_2(x)} \quad \text{for all } x \in X. \tag{17}$$

Then

$$\mathrm{PWM}(L^{p_1(\cdot)}(X), L^{p_2(\cdot)}(X)) = L^{p_3(\cdot)}(X).$$

Moreover, the operator norm of $g \in \mathrm{PWM}(L^{p_1(\cdot)}(X), L^{p_2(\cdot)}(X))$ is comparable to $\|g\|_{L^{p_3(\cdot)}(X)}$.

6 Proof of Main Results

In this section we prove Theorems 2 and 3. In this section, we write $f \lesssim g$ or $g \gtrsim f$ if $f \le Cg$ for some positive constant C.

Proof (Proof of Theorem 2) We follow the proof method of [22, Theorem 2.3]. We may assume that $\Phi_i \in \boldsymbol{\Phi}^v_{GY}$ by Proposition 2 (ii) and (6). Let $\Phi_2(\cdot, (\cdot)^{1/\ell_2}) \in \boldsymbol{\Phi}^v_Y$ with $\ell_2 \in (0, 1]$. We may also assume that $\|f\|_{L^{(\Phi_1,\phi_1,\kappa_1)}} = \|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}} = 1$. For any ball B and $x \in B$, let

$$t = \max\left(\frac{\Phi_1(x, |f(x)|)}{\phi_1(\kappa_1 B)}, \frac{\Phi_3(x, |g(x)|)}{\phi_3(\kappa_3 B)}\right).$$

We note that $t < \infty$ a.e. $x \in B$, since $\int_B \Phi_1(x, |f(x)|)d\mu(x) \le \mu(\kappa_1 B)\phi_1(\kappa_1 B)$ and $\int_B \Phi_3(x, |g(x)|)d\mu(x) \le \mu(\kappa_3 B)\phi_3(\kappa_3 B)$. From (P1') and $\Phi_1(x, |f(x)|) \le t\phi_1(\kappa_1 B)$ it follows that

$$|f(x)| \le \Phi_1^{-1}(x, \Phi_1(x, |f(x)|)) \le \Phi_1^{-1}(x, t\phi_1(\kappa_1 B)).$$

In the same way we have

$$|g(x)| \le \Phi_3^{-1}(x, \Phi_3(x, |g(x)|)) \le \Phi_3^{-1}(x, t\phi_3(\kappa_3 B)).$$

Hence, for a.e. $x \in B$,

$$|f(x)g(x)| \le \Phi_1^{-1}(x, t\phi_1(\kappa_1 B))\, \Phi_3^{-1}(x, t\phi_3(\kappa_3 B)) \le C\, \Phi_2^{-1}(x, t\phi_2(\kappa_2 B)).$$

Since $\Phi_2(x, (\cdot)^{1/\ell_2})$ is a Young function, by (3) and (P1') we have

$$\Phi_2\left(x, \frac{|f(x)g(x)|}{2^{1/\ell_2}C}\right) \le \frac{1}{2}\Phi_2\left(x, \frac{|f(x)g(x)|}{C}\right)$$
$$\le \frac{1}{2}\Phi_2(x, \Phi_2^{-1}(x, t\phi_2(\kappa_2 B))) \le \frac{1}{2}t\phi_2(\kappa_2 B)$$
$$\le \frac{1}{2}\left(\frac{\Phi_1(x, |f(x)|)}{\phi_1(\kappa_1 B)} + \frac{\Phi_3(x, |g(x)|)}{\phi_3(\kappa_3 B)}\right)\phi_2(\kappa_2 B).$$

Therefore

$$\int_B \Phi_2\left(x, \frac{|f(x)g(x)|}{2^{1/\ell_2}C}\right) d\mu(x)$$
$$\le \frac{1}{2}\left(\int_B \frac{\Phi_1(x, |f(x)|)}{\phi_1(\kappa_1 B)}\, d\mu(x) + \int_B \frac{\Phi_3(x, |g(x)|)}{\phi_3(\kappa_3 B)}\, d\mu(x)\right)\phi_2(\kappa_2 B)$$
$$\le \frac{1}{2}(\mu(\kappa_1 B) + \mu(\kappa_3 B))\phi_2(\kappa_2 B)$$
$$\le \mu(\kappa_2 B)\phi_2(\kappa_2 B),$$

since $\max(\kappa_1, \kappa_3) \le \kappa_2$. This shows

$$\|fg\|_{\Phi_2,\phi_2,\kappa_2,B} \le 2^{1/\ell_2}C.$$

Then we have the conclusion. □

To preove Theorem 3 we show the following two lemmas.

Lemma 5 *Let* $\Phi \in \boldsymbol{\Phi}_{GY}^v$ *and* $\Phi(\cdot, (\cdot)^{1/\ell}) \in \boldsymbol{\Phi}_Y^v$ *with* $\ell \in (0, 1]$, $\phi \in \mathcal{G}^v$ *and* $3K^2\kappa \le \kappa'$. *Let* $\phi(B_1)\mu(B_1) \le C_\phi\, \phi(B_2)\mu(B_2)$ *for all balls* B_1 *and* B_2 *with* $B_1 \subset B_2$. *If* $f \in L^{(\Phi,\phi,\kappa)}(X)$, $f = 0$ *outside of some ball* B_0, *and*

$$\sup_{B \subset 3K^2 B_0} \|f\|_{\Phi,\phi,\kappa,B} = M,$$

then $f \in L^{(\Phi,\phi,\kappa')}(X)$ *and*

$$\|f\|_{L^{(\Phi,\phi,\kappa')}(X)} \le C_\phi^{1/\ell} M.$$

Proof Let $B_0 = B(x_0, r_0)$. For any ball $B = B(x, r)$ with $B \cap B_0 \ne \emptyset$, if $r \le r_0$, then $B \subset 3K^2 B_0$, and then $\|f\|_{\Phi,\phi,\kappa',B} \le \|f\|_{\Phi,\phi,\kappa,B} \le M$. If $r > r_0$, then $B_0 \subset 3K^2 B$. In this case we have $\kappa B_0 \subset \kappa' B$ and, by (3),

$$\frac{1}{\phi(\kappa' B)\mu(\kappa' B)} \int_B \Phi\left(x, \frac{|f(x)|}{C_\phi^{1/\ell} M}\right) d\mu(x)$$
$$\leq \frac{\phi(\kappa B_0)\mu(\kappa B_0)}{C_\phi\, \phi(\kappa' B)\mu(\kappa' B)} \frac{1}{\mu(\kappa B_0)\phi(\kappa B_0)} \int_{B_0} \Phi\left(x, \frac{|f(x)|}{M}\right) d\mu(x) \leq 1.$$

Then we have the conclusion. □

Lemma 6 *Assume the same condition as Theorem 3. Let g be a simple function and $g = 0$ outside of some ball $B_* = (x_*, r_*)$. Then there exists a function $f \in L^{(\Phi_1,\phi_1,\kappa_1)}(X)$ such that*

$$\|f\|_{L^{(\Phi_1,\phi_1,\kappa_1)}(X)} \|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)} \leq C \|fg\|_{L^{(\Phi_2,\phi_2,\kappa_2)}(X)}, \tag{18}$$

where C is a positive constant independent of g, f and B_*.

Proof First note that g is in $L^{(\Phi_3,\phi_3,\kappa_3)}(X)$ by the assumption. By Proposition 2 (ii) and (6) we may assume that $\Phi_i \in \boldsymbol{\Phi}_{GY}^v$ and $\Phi_i(\cdot, (\cdot)^{1/\ell_i}) \in \boldsymbol{\Phi}_Y^v$ with $\ell_i \in (0, 1]$, $i = 1, 2, 3$.

Case (i): Assume that Φ_3 satisfies the Δ_2 condition. Then $\Phi_3(x, (\cdot)^{1/\ell_3}) \in \mathcal{Y}^{(1)}$ for each $x \in X$ and $\Phi_3^A((\cdot)^{1/\ell_3}) \in \mathcal{Y}^{(1)}$ for any $A \subset X$ with $0 < \mu(A) < \infty$, where $\Phi_3^A(t) = \int_A \Phi_3(x, t)\, d\mu(x)$.

Take a ball $B_0 = B(x_0, r_0)$ such that

$$\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)} \leq 2\|g\|_{\Phi_3,\phi_3,\kappa_3,B_0}.$$

Then $g|_{B_0}$ is a simple function in the Musielak-Orlicz space $L^{\Phi_3}(B_0)$ with the measure $\mu/(\mu(\kappa_3 B_0)\phi(\kappa_3 B_0))$ and

$$\|g|_{B_0}\|_{L^{\Phi_3}(B_0,\mu/(\phi_3(\kappa_3 B_0)\mu(\kappa_3 B_0)))} = \|g\|_{\Phi_3,\phi_3,\kappa_3,B_0}.$$

By Lemma 3 we have

$$\frac{1}{\phi_3(\kappa_3 B_0)\mu(\kappa_3 B_0)} \int_{B_0} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{\Phi_3,\phi_3,\kappa_3,B_0}}\right) d\mu(x) = 1.$$

Let

$$h(x) = \frac{1}{\phi_3(\kappa_3 B_0)} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) \chi_{B_0}(x). \tag{19}$$

Then $h|_{B_0}$ is in $L^1(B_0)$ and $h(x) < \infty$ a.e. Let

$$f(x) = \begin{cases} \Phi_1^{-1}(x, \phi_1(\kappa_1 B_0)h(x)), & 0 < h(x) < \infty, \\ 0, & h(x) = 0. \end{cases} \tag{20}$$

Then, by (P1'),

$$\Phi_1(x, f(x)) \le \frac{\phi_1(\kappa_1 B_0)}{\phi_3(\kappa_3 B_0)} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) \chi_{B_0}(x), \quad \text{a.e. } x \in X.$$

We will show that $\|f\|_{L^{(\Phi_1,\phi_1,\kappa_1)}(X)} \lesssim 1$ and that $\|fg\|_{L^{(\Phi_2,\phi_2,\kappa_2)}(X)} \gtrsim \|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}$. Then we have (18)

Now, for any ball $B = B(x, r)$ with $B \subset 3K^2 B_0$, since $3K^2\kappa_3 \le \kappa_1$, we have $\kappa_3 B \subset \kappa_1 B_0$ and

$$\begin{aligned}
&\frac{1}{\phi_1(\kappa_3 B)\mu(\kappa_3 B)} \int_B \Phi_1(x, f(x))\, d\mu(x) \\
&\le \frac{1}{\phi_1(\kappa_3 B)\mu(\kappa_3 B)} \frac{\phi_1(\kappa_1 B_0)}{\phi_3(\kappa_3 B_0)} \int_B \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) \chi_{B_0}(x)\, d\mu(x) \\
&\le \frac{\phi_3(\kappa_3 B)}{\phi_1(\kappa_3 B)} \frac{\phi_1(\kappa_1 B_0)}{\phi_3(\kappa_3 B_0)} \frac{1}{\phi_3(\kappa_3 B)\mu(\kappa_3 B)} \int_B \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) d\mu(x) \\
&\le \frac{\phi_3(\kappa_3 B)}{\phi_1(\kappa_3 B)} \frac{\phi_1(\kappa_1 B_0)}{\phi_3(\kappa_3 B_0)} \lesssim \frac{\phi_3(\kappa_3 B)}{\phi_1(\kappa_3 B)} \frac{\phi_1(\kappa_1 B_0)}{\phi_3(\kappa_1 B_0)} \lesssim 1,
\end{aligned}$$

where we use the almost increasingness of ϕ_3/ϕ_1 in the last inequality. That is,

$$\sup_{B \subset 3K^2 B_0} \|f\|_{\Phi_1,\phi_1,\kappa_3,B} \lesssim 1.$$

By Lemma 5 we have $\|f\|_{L^{(\Phi_1,\phi_1,\kappa_1)}(X)} \lesssim 1$.

Next we show $\|fg\|_{L^{(\Phi_2,\phi_2,\kappa_2)}(X)} \gtrsim \|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}$. We may assume that $\Phi_2(x, (\cdot)^{1/\ell_2}) \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ for all $x \in X$ by (P4'). If $h(x) = 0$, then $f(x) = 0$ and $\Phi_2\left(x, \frac{|f(x)g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) = 0$. If $0 < h(x) < \infty$, then $0 < \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) < \infty$. Then by (P2') we have

$$\begin{aligned}
\frac{|f(x)g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}} &= \Phi_1^{-1}(x, \phi_1(\kappa_1 B_0)h(x))\Phi_3^{-1}\left(x, \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right)\right) \\
&= \Phi_1^{-1}(x, \phi_1(\kappa_1 B_0)h(x))\Phi_3^{-1}(x, \phi_3(\kappa_3 B_0)h(x)) \\
&\ge C^{-1}\, \Phi_2^{-1}(x, \phi_2(\kappa_2 B_0)h(x)),
\end{aligned}$$

and then, by (P3'),

$$\Phi_2\left(x, \frac{C|f(x)g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) \ge \Phi_2(x, \Phi_2^{-1}(x, \phi_2(\kappa_2 B_0)h(x))) = \phi_2(\kappa_2 B_0)h(x).$$

Hence, by $\kappa_2 \le \kappa_3$,

$$\begin{aligned}
&\frac{1}{\mu(\kappa_2 B_0)\phi_2(\kappa_2 B_0)} \int_{B_0} \Phi_2\left(x, \frac{C|f(x)g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) d\mu(x) \\
&\qquad \ge \frac{1}{\mu(\kappa_2 B_0)} \int_{B_0} h(x)\, d\mu(x) \ge \frac{1}{\mu(\kappa_3 B_0)} \int_{B_0} h(x)\, d\mu(x) \\
&\qquad\qquad = \frac{1}{\mu(\kappa_3 B_0)\phi_3(\kappa_3 B_0)} \int_{B_0} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) d\mu(x). \quad (21)
\end{aligned}$$

By the Δ_2 condition of Φ_3 we have

$$\begin{aligned}
&\frac{1}{\mu(\kappa_2 B_0)\phi_2(\kappa_2 B_0)} \int_{B_0} \Phi_2\left(x, \frac{C|f(x)g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) d\mu(x) \\
&\ge \frac{1}{\mu(\kappa_3 B_0)\phi_3(\kappa_3 B_0)} \int_{B_0} \Phi_3\left(x, \frac{|g(x)|}{2\|g\|_{\Phi_3,\phi_3,\kappa_3,B_0}}\right) d\mu(x) \\
&\gtrsim \frac{1}{\mu(\kappa_3 B_0)\phi_3(\kappa_3 B_0)} \int_{B_0} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{\Phi_3,\phi_3,\kappa_3,B_0}}\right) d\mu(x) = 1.
\end{aligned}$$

Therefore, $\|fg\|_{L^{(\Phi_2,\phi_2,\kappa_2)}} \gtrsim \|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}}$ and we have (18).

Case (ii): Assume (ii) in Theorem 3. Take a sequence of balls $B_n = B(x_n, r_n)$ such that $B_* \cap B_n \ne \emptyset$ and that

$$\frac{1}{2}\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)} \le \|g\|_{\Phi_3,\phi_3,\kappa_3,B_1} \le \|g\|_{\Phi_3,\phi_3,\kappa_3,B_2} \le \cdots \to \|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}.$$

By Lemma 3 we have

$$\frac{1}{\mu(\kappa_3 B_n)\phi_3(\kappa_3 B_n)} \int_{B_n} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{\Phi_3,\phi_3,\kappa_3,B_n}}\right) d\mu(x) = 1. \quad (22)$$

By the assumption (ii) (b) there exists $\Psi = \Psi_{B_*} \in \boldsymbol{\Phi}_{GY}$ satisfying $\Psi((\cdot)^{1/\ell}) \in \mathcal{Y}^{(1)}$ such that $\sup_{x\in B_*} \Phi_3(x,t) \le \Psi(t)$ for all t. Then

$$\int_{B_n} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{\Phi_3,\phi_3,\kappa_3,B_n}}\right) d\mu(x) \le \int_{B_*} \Psi\left(\frac{2\|g\|_{L^\infty(X)}}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) d\mu(x) < \infty.$$

By the assumption $\lim_{r\to\infty} \inf_{x\in X} \phi_3(x,r)\mu(B(x,r)) = \infty$, we see that $\sup_n r_n < \infty$ and then all balls B_n are subsets of some ball $\tilde{B}_*$. Since $\lim_{r\to+0} \inf_{x\in\tilde{B}_*} \phi_3(x,r) = \infty$ and

$$\begin{aligned}\phi_3(\kappa_3 B_n) &= \frac{1}{\mu(\kappa_3 B_n)} \int_{B_n} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{\Phi_3,\phi_3,\kappa_3,B_n}}\right) d\mu(x) \\ &\le \frac{1}{\mu(\kappa_3 B_n)} \int_{B_n \cap B_*} \Psi\left(\frac{2\|g\|_{L^\infty(X)}}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) d\mu(x) \\ &\le \Psi\left(\frac{2\|g\|_{L^\infty(X)}}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) < \infty,\end{aligned}$$

we have $\inf_n r_n > 0$. Then, by the assumption (ii) (a) we can take a subsequence if we need, (x_n, r_n) converges to some point $(\tilde{x}, \tilde{r})$ with $\tilde{x} \in X$ and $\tilde{r} > 0$. Let $\tilde{B} = B(\tilde{x}, \tilde{r})$. Then $\mu(\kappa_3 B_n)\phi_3(\kappa_3 B_n) \to \mu(\kappa_3 \tilde{B})\phi_3(\kappa_3 \tilde{B})$ by the assumption (ii), and

$$\int_{B_n} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{\Phi_3,\phi_3,\kappa_3,B_n}}\right) d\mu(x) \to \int_{\tilde{B}} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) d\mu(x).$$

From (22) it follows that

$$\frac{1}{\mu(\kappa_3 \tilde{B})\phi_3(\kappa_3 \tilde{B})} \int_{\tilde{B}} \Phi_3\left(x, \frac{|g(x)|}{\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}}\right) d\mu(x) = 1.$$

Using $\tilde{B}$ instead of B_0, we define h and f by (19) and (20), respectively. Then by the same way as Case (i) we have $\|f\|_{L^{(\Phi_1,\phi_1,\kappa_1)}(X)} \lesssim 1$. By (21) we have $\|fg\|_{L^{(\Phi_2,\phi_2,\kappa_2)}(X)} \gtrsim \|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)}$. This shows (18). □

Proof (Proof of Theorem 3) First note that, by Lemma 4, each element of $\mathrm{PWM}(L^{(\Phi_1,\phi_1,\kappa_1)}(X), L^{(\Phi_2,\phi_2,\kappa_2)}(X))$ is a closed operator. Then it is a bounded operator by the closed graph theorem.

Let $g \in \mathrm{PWM}(L^{(\Phi_1,\phi_1,\kappa_1)}(X), L^{(\Phi_2,\phi_2,\kappa_2)}(X))$. Let $\{B_j\}$ be an increasing sequence of balls such that

$$X = \bigcup_{j=1}^{\infty} B_j.$$

Take a sequence $\{g_j\}$ of simple functions such that $g_j = 0$ outside of B_j and

$$0 \le g_1 \le g_2 \le \cdots \to |g| \text{ a.e.}$$

Then $g_j \in \mathrm{PWM}(L^{(\Phi_1,\phi_1,\kappa_1)}(X), L^{(\Phi_2,\phi_2,\kappa_2)}(X)) \cap L^{(\Phi_3,\phi_3,\kappa_3)}(X)$ and

$$\|g_1\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)} \le \|g_2\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)} \cdots, \quad \|g_j\|_{\mathrm{Op}} \le \|g\|_{\mathrm{Op}}.$$

By Lemma 6 there exist functions $f_j \in L^{(\Phi_1,\phi_1,\kappa_1)}(X)$ such that

$$\|f_j\|_{L^{(\Phi_1,\phi_1,\kappa_1)}(X)} \|g_j\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)} \lesssim \|f_j g_j\|_{L^{(\Phi_2,\phi_2,\kappa_2)}(X)} \le \|g_j\|_{\mathrm{Op}} \|f_j\|_{L^{(\Phi_1,\phi_1,\kappa_1)}(X)}.$$

Then

$$\|g_j\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)} \lesssim \|g_j\|_{\mathrm{Op}} \le \|g\|_{\mathrm{Op}}.$$

By the Fatou property we get $g \in L^{(\Phi_3,\phi_3,\kappa_3)}(X)$ and

$$\|g\|_{L^{(\Phi_3,\phi_3,\kappa_3)}(X)} \lesssim \|g\|_{\mathrm{Op}}.$$

This is the desired conclusion. □

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The Fatou Property of Function Spaces, Heat Kernels, Admissible Norms and Mapping Properties

Hans Triebel

Abstract This paper deals with the Fatou property of some distinguished spaces in the context of tempered distributions. We discuss the close connection with so-called admissible norms, some of them are defined in terms of heat kernels. We illustrate how the Fatou property can be used to prove mapping properties of some operators in limiting situations.

Keywords Fatou property · Spaces of tempered distributions · Morrey spaces · Lorentz spaces · Singular integral operators

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1 Aims and Motivations

Let $S(R^n)$ be the usual Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on the Euclidean n-space R^n. Let $S'(R^n)$ be the space of all tempered distributions in R^n, the dual of $S(R^n)$. Let $D(R^n) = C_0^\infty(R^n)$ be the collection of all $f \in S(R^n)$ with compact support in R^n. Let $A(R^n)$ be a quasi-normed space in $S'(R^n)$ with $A(R^n) \hookrightarrow S'(R^n)$ (continuous embedding). Then $A(R^n)$ is said to have the *Fatou property* if there is a positive constant c such that

$$\sup_{j \in N} \|g_j \,|A(R^n)\| < \infty \qquad \text{and} \qquad g_j \to g \quad \text{in } S'(R^n) \tag{1}$$

implies $g \in A(R^n)$ and

$$\|g \,|A(R^n)\| \le c \sup_{j \in N} \|g_j \,|A(R^n)\|. \tag{2}$$

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We are interested in the nowadays well-known inhomogeneous spaces $A^s_{p,q}(R^n)$, $A \in \{B, F\}$, $s \in R$ and $0 < p, q \le \infty$, their tempered homogeneous counterparts $\overset{*}{A}{}^s_{p,q}(R^n)$, related Lorentz spaces $L_{r,u}(R^n)$ and Morrey spaces $L^r_p(R^n)$, $H^\varrho L_p(R^n)$ as long as they fit in the framework of the dual pairing $\big(S(R^n), S'(R^n)\big)$ asking, among other questions, of whether these spaces have the Fatou property. This paper is a comment on some assertions in our recent publications [17–22] and [16]. We assume that the reader is familiar with these spaces. Standard definitions will not be repeated. Even worse we refer the reader to the (historical) references in the just quoted publications restricting ourselves here to the bare minimum. We are more interested in the interplay of what is already known complemented by a few new aspects. The topics we have in mind are characterized by the key words in the title. Let again $A(R^n) \hookrightarrow S'(R^n)$ be a quasi-Banach space. The related quasi-norm $\| \cdot \, | A(R^n)\|$ is said to be *admissible* if any $f \in S'(R^n)$ can be tested of whether it belongs to $A(R^n)$ or not (which means whether the related quasi-norm is finite or infinite). Within a given fixed quasi-Banach space $A(R^n)$ further equivalent quasi-norms are called *domestic*. A simple but nevertheless illuminating example is the Lebesgue space

$$L_p(R^n) = F^0_{p,2}(R^n), \qquad 1 < p < \infty. \tag{3}$$

Then

$$\|f \, | F^0_{p,2}(R^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^\infty \big| (\varphi_j \widehat{f})^\vee(\cdot) \big|^2 \Big)^{1/2} \, | L_p(R^n) \Big\| \tag{4}$$

according to (15) below is an admissible norm whereas

$$\|f \, | L_p(R^n)\| = \Big(\int_{R^n} |f(x)|^p \, \mathrm{d}x \Big)^{1/p} \tag{5}$$

is a domestic norm. We discussed this point in greater detail in [19, Sect. 1.3, pp. 5, 6]. In the case of the inhomogeneous spaces $A^s_{p,q}(R^n)$ the distinction between admissible, domestic and other types of norms as discussed in [19] might be somewhat pedantic. But the situation is different for the *tempered homogeneous spaces* $\overset{*}{A}{}^s_{p,q}(R^n)$ as considered in [19, 21, 22] in the framework of the dual pairing $\big(S(R^n), S'(R^n)\big)$. Then *admissible quasi-norms* in terms of *heat kernels* in Gauss-Weierstrass semi-groups $W_t f$, $f \in S'(R^n)$, will play a central role. This has a long history going back to [12, 13]. But the recent interest comes from (linear and nonlinear) parabolic equations including Navier–Stokes equations and PDE models of chemotaxis. We discussed these points in [19–21] based on [17, 18]. This will not be repeated here. But these applications suggest to ask for spaces satisfying the continuous embeddings

$$S(R^n) \hookrightarrow A^s_{p,q}(R^n) \hookrightarrow S'(R^n), \qquad S(R^n) \hookrightarrow \overset{*}{A}{}^s_{p,q}(R^n) \hookrightarrow S'(R^n). \tag{6}$$

This is obvious for the inhomogeneous spaces $A^s_{p,q}(R^n)$ but a severe restriction for the tempered homogeneous spaces $\overset{*}{A}{}^s_{p,q}(R^n)$ as treated in [19, 21]. If a space $A(R^n)$ is defined in terms of admissible quasi-norms then the Fatou property in the context of the dual pairing $\big(S(R^n), S'(R^n)\big)$ is quite often a more or less immediate consequence. One can take (4) or characterizations based on heat kernels as examples. This is the case for all inhomogeneous spaces $A^s_{p,q}(R^n)$ and all considered tempered homogeneous spaces $\overset{*}{A}{}^s_{p,q}(R^n)$ as will be detailed in Sect. 2 below. But it applies also to Lorentz spaces $L_{r,u}(R^n)$ and the Morrey spaces $L^r_p(R^n)$, $H^{\varrho}L_p(R^n)$, after the standard norms in terms of Lebesgue-measurable norms have been replaced by suitable admissible norms partly based on duality as will be described in the Sects. 3 and 4 below.

The Fatou property can be used to study mapping properties of linear and also non-linear operators. We describe a typical situation. Let $A(R^n)$ and $B(R^n)$ be two quasi-Banach spaces such that

$$S(R^n) \hookrightarrow A(R^n) \hookrightarrow B(R^n) \hookrightarrow S'(R^n). \tag{7}$$

Let T be a linear operator acting continuously in $B(R^n)$,

$$T: \quad B(R^n) \hookrightarrow B(R^n). \tag{8}$$

One may ask whether the restriction of T to $A(R^n)$ acts continuously in $A(R^n)$. Let $A(R^n)$ be a space with the Fatou property assuming in addition that for any $f \in A(R^n)$ there are functions $f_j \in S(R^n)$ such that

$$\sup_{j \in N} \|f_j \,|A(R^n)\| \le c\, \|f \,|A(R^n)\|, \qquad f_j \to f \quad \text{in} \quad B(R^n). \tag{9}$$

This is a rather typical situation. One may think about atomic or wavelet decompositions where the related building blocks can be mollified. Then one may even assume $f_j \in D(R^n) = C_0^\infty(R^n)$ in (9). Let

$$\|T f_j \,|A(R^n)\| \le c\, \|f_j \,|A(R^n)\|, \qquad j \in N. \tag{10}$$

If $S(R^n)$ is dense in $A(R^n)$ then one has

$$T: \quad A(R^n) \hookrightarrow A(R^n) \tag{11}$$

by completion. If $S(R^n)$ is not necessarily dense in $A(R^n)$ then the Fatou property of $A(R^n)$ and the above assumptions ensure again (11). In other words, the Fatou property may serve as a substitute if completion or other direct arguments do not work. One may think about $A(R^n) = A^s_{p,q}(R^n)$ with $\max(p,q) = \infty$ and suitable spaces $B(R^n) = A^\sigma_{u,v}(R^n)$ with $\max(u,v) < \infty$ (based on appropriate embeddings) or $A(R^n) = L_{r,\infty}(R^n)$ (Marcinkiewicz spaces) or $A(R^n) = L^r_p(R^n)$ (Morrey spaces) where the related spaces $B(R^n)$ might be (weighted) L_p-spaces. We do not deal with mapping properties of operators in detail, however we illustrate in Sect. 5 the above comments by a few examples. But otherwise we are mainly interested in illuminating the interplay of the *Fatou property* of some spaces and related *admissible quasi-norms* in the context of the dual pairing $\big(S(R^n), S'(R^n)\big)$.

2 Tempered Spaces

If $\varphi \in S(R^n)$ then $\widehat{\varphi} = F\varphi$ stands for the Fourier transform and $\varphi^\vee = F^{-1}\varphi$ for the inverse Fourier transform. Both are extended in the standard way to $S'(R^n)$. Let $\varphi_0 \in S(R^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(x) = 0 \text{ if } |x| \ge 3/2, \tag{12}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in R^n, \quad k \in N, \tag{13}$$

where N is the collection of all natural numbers. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$.

Definition 1 Let $n \in N$.
(i) Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in R$. Then $B^s_{p,q}(R^n)$ is the collection of all $f \in S'(R^n)$ such that

$$\|f\,|B^s_{p,q}(R^n)\|_\varphi = \Big(\sum_{j=0}^\infty 2^{jsq}\big\|(\varphi_j\widehat{f})^\vee\,|L_p(R^n)\big\|^q\Big)^{1/q} < \infty \tag{14}$$

(with the usual modification if $q = \infty$).
(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $s \in R$. Then $F^s_{p,q}(R^n)$ is the collection of all $f \in S'(R^n)$ such that

$$\|f\,|F^s_{p,q}(R^n)\|_\varphi = \Big\|\Big(\sum_{j=0}^\infty 2^{jsq}\big|(\varphi_j\widehat{f})^\vee(\cdot)\big|^q\Big)^{1/q}\big|L_p(R^n)\Big\| < \infty \tag{15}$$

(with the usual modification if $q = \infty$).

Remark 1 This is the standard definition of the *tempered inhomogeneous spaces* $A^s_{p,q}(R^n)$ with $A \in \{B, F\}$. The theory of these spaces and related (historical) references may be found in [14–16]. All quasi-norms are *admissible*: Any $f \in S'(R^n)$ can be tested if $\|f\,|A^s_{p,q}(R^n)\|_\varphi$ is finite or not. Recall that $A^s_{p,q}(R^n)$ is independent of φ (equivalent quasi-norms). All spaces have the *Fatou property*: If $\psi \in S(R^n)$ then

$$(\psi\widehat{f})^\vee(x) = c\,\big(f, \psi^\vee(x-\cdot)\big), \qquad f \in S'(R^n), \quad x \in R^n. \tag{16}$$

This reduces (1), (2) with $A(R^n) = A^s_{p,q}(R^n)$ to the classical measure-theoretical Fatou property for L_p-spaces. The first detailed discussion had been given in [5] where also the notation *Fatou property* in the context of distributions had been coined. But this remarkable property had also been used before, at least implicitly, for example in [6, 7].

The above spaces $A^s_{p,q}(R^n)$ can also be introduced in terms of heat kernels what will be of interest for our later considerations. Let $w \in S'(R^n)$. Then

$$W_t w(x) = \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{-\frac{|x-y|^2}{4t}} w(y)\, \mathrm{d}y = \frac{1}{(4\pi t)^{n/2}} \Big(w, e^{-\frac{|x-\cdot|^2}{4t}} \Big), \quad t > 0, \tag{17}$$

$x \in R^n$, is the well-known Gauss-Weierstrass semi-group which can be written on the Fourier side as

$$\widehat{W_t w}(\xi) = e^{-t|\xi|^2} \widehat{w}(\xi), \qquad \xi \in R^n, \quad t > 0. \tag{18}$$

The Fourier transform is taken with respect to the space variables $x \in R^n$. Of course, both (17), (18) must be interpreted in the context of $S'(R^n)$. But we recall that (17) makes sense pointwise: It is the convolution of $w \in S'(R^n)$ and $g_t(y) = (4\pi t)^{-n/2} e^{-\frac{|y|^2}{4t}} \in S(R^n)$. In particular,

$$w * g_t \in C^\infty(R^n), \qquad |(w * g_t)(x)| \le c_t \big(1 + |x|^2\big)^{N/2}, \quad x \in R^n, \tag{19}$$

for some $c_t > 0$ and some $N \in N$. Let

$$s < 0 \quad \text{and} \quad 0 < p, q \le \infty \qquad (\text{with } p < \infty \text{for } F\text{-spaces}). \tag{20}$$

Then

$$\| f \,|B^s_{p,q}(R^n)\| = \Big(\int_0^1 t^{-\frac{sq}{2}} \| W_t f \,| L_p(R^n)\|^q \, \frac{\mathrm{d}t}{t} \Big)^{1/q} \tag{21}$$

and

$$\| f \,|F^s_{p,q}(R^n)\| = \Big\| \Big(\int_0^1 t^{-\frac{sq}{2}} \big| W_t f(\cdot) \big|^q \, \frac{\mathrm{d}t}{t} \Big)^{1/q} | L_p(R^n) \Big\| \tag{22}$$

(usual modification if $q = \infty$) are *admissible* (characterizing) equivalent quasi-norms in the respective spaces in the same understanding as in Remark 1 above. This is essentially covered by [15, Theorem 2.6.4, p. 152]. With (17), (19) in place of (16) it follows again that the above spaces $A^s_{p,q}(R^n)$ have the *Fatou property* reducing this question to the classical measure-theoretical Fatou property for vector-valued L_p-spaces. Of interest for us are the Hölder-Zygmund spaces

$$\mathcal{C}^s(R^n) = B^s_{\infty,\infty}(R^n), \qquad s < 0, \tag{23}$$

which can be admissibly normed by

$$\| f \,|\mathcal{C}^s(R^n)\| = \sup_{x\in R^n, 0<t<1} t^{-s/2} |W_t f(x)|. \tag{24}$$

We extend (21), (22) with (20) to

$$s \in R, \qquad \text{and} \quad 0 < p, q \le \infty \quad (\, p < \infty \text{ for } F\text{-spaces}). \tag{25}$$

Let as usual $\partial_t^m g = \partial^m g/\partial t^m$, $m \in N_0 = N \cup \{0\}$ with $\partial_t^0 g = g$. Let $s/2 < m \in N_0$. Then $B^s_{p,q}(R^n)$ is the collection of all $f \in S'(R^n)$ such that

$$\| f \,|B^s_{p,q}(R^n)\|_m = \|W_1 f | L_p(R^n)\| + \Big(\int_0^1 t^{(m-\frac{s}{2})q} \big\| \partial_t^m W_t f \,|L_p(R^n)\big\|^q \,\frac{\mathrm{d}t}{t}\Big)^{1/q} \tag{26}$$

is finite and $F^s_{p,q}(R^n)$ is the collection of all $f \in S'(R^n)$ such that

$$\| f \,|F^s_{p,q}(R^n)\|_m = \|W_1 f | L_p(R^n)\| + \Big\| \Big(\int_0^1 t^{(m-\frac{s}{2})q} \big| \partial_t^m W_t f(\cdot)\big|^q \,\frac{\mathrm{d}t}{t}\Big)^{1/q} |L_p(R^n)\Big\| \tag{27}$$

is finite (usual modification if $q = \infty$). This is covered by [15, Theorem 2.6.4, p. 152]. According to [15, Remark 2.6.4, p. 155] one can replace $\int_0^1$ in (26), (27) by $\int_0^\infty$ ($\sup_{0<t<1}$ by $\sup_{t>0}$) if $s > \max\big(0, n(\frac{1}{p} - 1)\big)$. In other words, (26) and (27) are equivalent *admissible quasi-norms* in $A^s_{p,q}(R^n)$. Using $\partial_t^m W_t w = \Delta^m W_t w = W_t \Delta^m w$ then one has by (17)

$$\partial_t^m W_t w(x) = \frac{1}{(4\pi t)^{n/2}} \Big(\Delta^m w, e^{-\frac{|x-\cdot|^2}{4t}} \Big), \qquad t > 0, \quad x \in R^n. \tag{28}$$

This shows again that all spaces $A^s_{p,q}(R^n)$ as introduced in Definition 1 have the *Fatou property*.

The above *tempered inhomogeneous spaces* $A^s_{p,q}(R^n)$ can be introduced as in Definition 1 or as the collection of all $f \in S'(R^n)$ such (26), (27) are finite. All quasi-norms are admissible and the Fatou property according to (1), (2) can be reduced to the classical Fatou property for (vector-valued) L_p-spaces for measurable functions. Homogeneous spaces $\dot{A}^s_{p,q}(R^n)$ within the dual pairing $\big(\dot{S}(R^n), \dot{S}'(R^n)\big)$ had been discussed all the time, but the recent interest comes from applications to some (nonlinear) PDEs, including Navier–Stokes equations. In [19, Chap. 2] we have given a description of these homogeneous spaces $\dot{A}^s_{p,q}(R^n)$ extending previous considerations in [14, Chap. 5]. But these spaces and, even more, their applications suffer from the ambiguity modulo polynomials. We introduced in [19, Chap. 3] and [21] tempered homogeneous spaces $\overset{*}{A}{}^s_{p,q}(R^n)$ within the dual pairing $\big(S(R^n), S'(R^n)\big)$ (as the tempered inhomogeneous spaces $A^s_{p,q}(R^n)$) avoiding this shortcoming. In contrast to the inhomogeneous spaces the Fourier-analytical counterpart of Definition 1 with $\varphi^j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$, $j \in Z$ (integers) is of no use for this purpose. But the situation is different if one looks for *admissible* quasi-norms for the tempered homogeneous spaces $\overset{*}{A}{}^s_{p,q}(R^n)$ based on heat kernels. We dealt with these spaces in [19, Chap. 3] and [21] repeating now only a few aspects related to *admissible quasi-norms* and the *Fatou property*. The homogeneous counterpart of the Hölder-Zygmund spaces according to (23), (24) are the Banach spaces $\overset{*}{\mathcal{C}}{}^s(R^n)$, $s < 0$, normed by

$$\|f \mid \overset{*}{\mathcal{C}}{}^s(R^n)\| = \sup_{x\in R^n, t>0} t^{-s/2}|W_t f(x)|. \tag{29}$$

It is again an *admissible norm*. Furthermore $\overset{*}{\mathcal{C}}{}^s(R^n) \hookrightarrow S'(R^n)$ and $\overset{*}{\mathcal{C}}{}^s(R^n)$ has the Fatou property by the same arguments as above. The left-hand side of (6) with $\overset{*}{A}{}^s_{p,q}(R^n) = \overset{*}{\mathcal{C}}{}^s(R^n)$ requires $-n \le s < 0$, [19] and, in particular, [21, Theorem 2.6]. Excluding limiting cases one needs now the restriction of s to the distinguished strip

$$-n < s - \frac{n}{p} = -\frac{n}{r} < 0, \qquad 0 < p \le \infty. \tag{30}$$

In particular $1 < r < \infty$. Under these restrictions one can now define the *tempered homogeneous spaces* $\overset{*}{A}{}^s_{p,q}(R^n)$ as follows. Recall that $N_0 = N \cup \{0\}$, where N stands for the natural numbers.

Definition 2 Let $n \in N$.
(i) Let $0 < p \le \infty$, $0 < q \le \infty$ and

$$n\Big(\frac{1}{p} - 1\Big) < s < \frac{n}{p}, \qquad -\frac{n}{r} = s - \frac{n}{p}. \tag{31}$$

Let $s/2 < m \in N_0$. Then $\overset{*}{B}{}^s_{p,q}(R^n)$ collects all $f \in S'(R^n)$ such that

$$\|f \mid \overset{*}{B}{}^s_{p,q}(R^n)\|_m = \Big(\int_0^\infty t^{(m-\frac{s}{2})q} \big\|\partial_t^m W_t f \,|L_p(R^n)\big\|^q \frac{\mathrm{d}t}{t}\Big)^{1/q} + \|f \mid \overset{*}{\mathcal{C}}{}^{-n/r}(R^n)\| \tag{32}$$

is finite (usual modification if $q = \infty$).

(ii) Let $0 < p < \infty$, $0 < q \le \infty$ and s, r as in (31). Let $s/2 < m < N_0$. Then $\overset{*}{F}{}^s_{p,q}(R^n)$ collects all $f \in S'(R^n)$ such that

$$\|f \mid \overset{*}{F}{}^s_{p,q}(R^n)\|_m = \Big\|\Big(\int_0^\infty t^{(m-\frac{s}{2})q} \big|\partial_t^m W_t f(\cdot)\big|^q \frac{\mathrm{d}t}{t}\Big)^{1/q} |L_p(R^n)\Big\| + \|f \mid \overset{*}{\mathcal{C}}{}^{-n/r}(R^n)\| \tag{33}$$

is finite (usual modification if $q = \infty$).

Remark 2 We refer the reader again to [19] and [21, Definition 2.8]. One has always (6) with $\overset{*}{A} \in \{\overset{*}{B}, \overset{*}{F}\}$. Furthermore (32) are equivalent *admissible quasi-norms* in $\overset{*}{B}{}^s_{p,q}(R^n)$ and (33) are equivalent *admissible quasi-norms* in $\overset{*}{F}{}^s_{p,q}(R^n)$. Using again (28) it follows by the same arguments as above that all spaces $\overset{*}{A}{}^s_{p,q}(R^n)$ have the *Fatou property*.

It seems to be reasonable to summarize the above comments. Recall that we explained in (1), (2) what is meant by the *Fatou property* in the framework of $\big(S(R^n), S'(R^n)\big)$. As above a quasi-norm $\| \cdot \,|A(R^n)\|$ is called *admissible* if any $f \in S'(R^n)$ can be tested of whether it belongs to $A(R^n)$ or not.

Theorem 1 *Let $n \in N$.*
(i) The tempered inhomogeneous spaces $A^s_{p,q}(R^n)$ according to Definition 1 have the Fatou property. The quasi-norms (14) and (26) are admissible in $B^s_{p,q}(R^n)$. The quasi-norms (15), (27) are admissible in $F^s_{p,q}(R^n)$.
(ii) The tempered homogeneous spaces $\overset{*}{A}{}^s_{p,q}(R^n)$ according to Definition 2 have the Fatou property. The quasi-norms (32) are admissible in $\overset{*}{B}{}^s_{p,q}(R^n)$. The quasi-norms (33) are admissible in $\overset{*}{F}{}^s_{p,q}(R^n)$.

Remark 3 All is covered by the above considerations and references. There is essentially nothing new compared with what is already known. But we wanted to collect some aspects which are otherwise somewhat scattered in [16, 19, 21]. In particular we tried to shed some light on the close connection between admissible quasi-norms and the Fatou property as long as these quasi-norms are built on (vector-valued) L_p-spaces.

3 Lorentz Spaces

We follow closely [19, Section 3.6] where we discussed some aspects of the Lorentz spaces $L_{r,u}(R^n)$ in close connection with tempered homogeneous spaces as introduced in Definition 2. One may consult [19, Definition 3.18, p. 73].

Let f be a complex a.e. finite Lebesgue-measurable function in R^n. Then the distribution function $\mu_f(\varrho)$ and the decreasing (which means non-increasing) rearrangement f^* of f are given by

$$\mu_f(\varrho) = \big|\{x \in R^n : \, |f(x)| > \varrho\}\big|, \qquad \varrho \geq 0, \tag{34}$$

and

$$f^*(t) = \inf\{\varrho : \, \mu_f(\varrho) \leq t\}, \qquad t \geq 0. \tag{35}$$

Let $0 < r < \infty$ and $0 < u \leq \infty$. Then $L_{r,u}(R^n)$ collects all Lebesgue-measurable functions such that

$$\| f \,|L_{r,u}(R^n)\| = \Big(\int_0^\infty \big(t^{1/r} f^*(t)\big)^u \,\frac{\mathrm{d}t}{t}\Big)^{1/u} < \infty \tag{36}$$

(with the usual modification if $u = \infty$). Recall that $L_{r,r}(R^n) = L_r(R^n)$, $0 < r < \infty$, are the Lebesgue spaces. The standard references for the theory of Lorentz spaces (and diverse generalizations) in the larger context of measure spaces are [2–4]. Our interest in Lorentz spaces comes from their use in connection with both tempered inhomogeneous spaces $A^s_{p,q}(R^n)$ and tempered homogeneous spaces $\overset{*}{A}{}^s_{p,q}(R^n)$. We refer the reader again to [19, Sects. 3.6, 3.7] and the literature mentioned there. If $1 < r < \infty$ then the spaces $L_{r,u}(R^n)$ fit in the scheme of $\big(S(R^n), S'(R^n)\big)$ as a refinement of related Lebesgue spaces $L_r(R^n)$. Then it makes sense to ask for the *Fatou property* of these spaces and also for *admissible* (equivalent) quasi-norms. Afterwards (36) becomes a domestic quasi-norm again in generalization of (4), (5). The *Fatou property* for $L_{r,u}(R^n)$ with $1 < r < \infty$, $0 < u \le \infty$, means in specification of (1), (2) that there is a positive constant c such that

$$\sup_{k\in N} \|g_k \,|L_{r,u}(R^n)\| < \infty \quad \text{and} \quad g_k \to g \quad \text{in} \quad S'(R^n) \tag{37}$$

imply $g \in L_{r,u}(R^n)$ and

$$\|g \,|L_{r,u}(R^n)\| \le c \sup_{k\in N} \|g_k \,|L_{r,u}(R^n)\|. \tag{38}$$

One should be aware that this is the Fatou property within the framework of $\big(S(R^n), S'(R^n)\big)$ and not in the context of Lebesgue-measurable functions in R^n. Although quite obvious we mention that the spaces $L_{r,u}(R^n)$ fit in the above scheme. They can be obtained by real interpolation

$$S(R^n) \hookrightarrow L_{r,u}(R^n) = \big(L_{r_0}(R^n), L_{r_1}(R^n)\big)_{\theta,u} \hookrightarrow S'(R^n), \tag{39}$$

$1 < r_0 < r < r_1 < \infty$, where $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be again the inhomogeneous dyadic resolution of unity according to (12), (13).

Theorem 2 *The spaces* $L_{r,u}(R^n)$ *with* $1 < r < \infty$ *and* $0 < u \le \infty$ *have the* Fatou property *and*

$$\|f \,|L_{r,u}(R^n)\|_\varphi \sim \Big\| \Big(\sum_{j=0}^\infty \big| \big(\varphi_j \widehat{f}\big)^\vee(\cdot)\big|^2\Big)^{1/2} |L_{r,u}(R^n)\Big\| \tag{40}$$

are admissible *equivalent quasi-norms.*

Remark 4 This theorem is essentially covered by [19, Theorem 3.15, p. 69] complemented by the obvious assertion that the quasi-norms in (40) are admissible. There one finds a detailed proof and also further properties of these spaces which are of interest in connection with some tempered homogeneous spaces $\overset{*}{A}{}^{s}_{p,q}(R^n)$. The above assertions extend (3)–(5) from Lebesgue spaces to Lorentz spaces.

4 Morrey Spaces

We complement [18, Chap. 2]. There one finds also detailed (historical) references which will not be repeated here. In addition one may also consult the recent book [1]. First we take over some material from [18].

Let $L_p(M)$, $0 < p \le \infty$, where M is a Lebesgue measurable subset of R^n, be the usual Lebesgue spaces quasi-normed by

$$\|f\,|L_p(M)\| = \Big(\int_M |f(x)|^p \,\mathrm{d}x \Big)^{1/p} \tag{41}$$

(obvious modification if $p = \infty$). Let $L_p(R^n, w_\gamma)$ with $0 < p \le \infty$ and $w_\gamma(x) = (1+|x|^2)^{\gamma/2}$, $\gamma \in R$, be the weighted Lebesgue spaces, quasi-normed by

$$\|f\,|L_p(R^n, w_\gamma)\| = \|w_\gamma f\,|L_p(R^n)\|. \tag{42}$$

As above Z is the collection of all integers; and Z^n where $n \in N$ (natural numbers) denotes the lattice of all points $m = (m_1, \dots, m_n) \in R^n$ with $m_k \in Z$. Let $Q_{j,m} = 2^{-j}m + 2^{-j}(0,1)^n$ with $j \in Z$ and $m \in Z^n$ be the usual dyadic cubes in R^n with sides of length 2^{-j} parallel to the axes of coordinates and $2^{-j}m$ as the lower left corner. As usual $L_p^{\mathrm{loc}}(R^n)$ collects all locally p-integrable functions, that is $f \in L_p(M)$ for any bounded Lebesgue measurable set M in R^n.

Definition 3 Let $n \in N$ and $1 < p < \infty$.
(i) Let $-n/p \le r < 0$. Then $L^r_p(R^n)$ collects all $f \in L^{\mathrm{loc}}_p(R^n)$ such that

$$\|f\,|L^r_p(R^n)\| = \sup_{J \in Z, M \in Z^n} 2^{J(\frac{n}{p}+r)} \|f\,|L_p(Q_{J,M})\| < \infty. \tag{43}$$

Furthermore $\overset{\circ}{L}{}^r_p(R^n)$ is the completion of $D(R^n) = C_0^\infty(R^n)$ in $L^r_p(R^n)$.
(ii) Let $-n < \varrho \le -n/p$. Then $H^\varrho L_p(R^n)$ collects all $h \in S'(R^n)$ which can be represented as

$$h = \sum_{J \in Z, M \in Z^n} h_{J,M}, \qquad \operatorname{supp} h_{J,M} \subset \overline{Q_{J,M}}, \tag{44}$$

such that

$$\sum_{J \in Z, M \in Z^n} 2^{J(\frac{n}{p}+\varrho)} \|h_{J,M}\,|L_p(Q_{J,M})\| < \infty. \tag{45}$$

Furthermore,

$$\|h \,|H^{\varrho}L_p(R^n)\| = \inf \sum_{J\in Z, M\in Z^n} 2^{J(\frac{n}{p}+\varrho)} \|h_{J,M} \,|L_p(Q_{J,M})\| \tag{46}$$

where the infimum is taken over all representations (44), (45).

Remark 5 Part (i) coincides with [18, Definition 2.1(ii), p. 7] and part (ii) with [18, Definition 2.3(ii), p. 9]. Usually $L^r_p(R^n) = L^r L_p(R^n)$ are called (global) Morrey spaces and $H^{\varrho}L_p(R^n)$ are related dual Morrey spaces. In [18] we dealt also with local Morrey spaces and related local dual Morrey spaces. This will not be done here. According to [18, (2.8), (2.42), pp. 8, 13] one has

$$L_p(R^n) = L_p^{-n/p}(R^n) = \mathring{L}_p^{-n/p}(R^n) = H^{-n/p}L_p(R^n). \tag{47}$$

We wish to deal with the above spaces in the framework of the dual pairing $\big(S(R^n), S'(R^n)\big)$. This has been done in detail in [18]. In particular (44) with (45) converges in some $L_u(R^n)$, specified below, with $1 < u \le p < \infty$, and hence in $S'(R^n)$. We repeat some related assertions. As usual $\frac{1}{p} + \frac{1}{p'} = 1$ where $1 < p < \infty$. Duality must always be interpreted in the framework of the dual pairing $\big(S(R^n), S'(R^n)\big)$.

Proposition 1 *(i) Let*

$$1 < p < \infty, \quad -\frac{n}{p} < r < 0, \quad ru = -n \quad \text{and} \quad \gamma < -\frac{n}{p} - r. \tag{48}$$

Then $L^r_p(R^n)$ are Banach spaces, $1 < p < u < \infty$ and

$$S(R^n) \hookrightarrow L_u(R^n) \hookrightarrow L^r_p(R^n) \hookrightarrow L_p(R^n, w_\gamma) \hookrightarrow S'(R^n). \tag{49}$$

Let $r + \varrho = -n$. Then $-n < \varrho < -\frac{n}{p'}$ and

$$\mathring{L}^r_p(R^n)' = H^{\varrho}L_{p'}(R^n). \tag{50}$$

(ii) Let

$$1 < p < \infty, \quad -n < \varrho < -\frac{n}{p}, \quad \varrho u = -n \quad \text{and} \quad \gamma > \frac{n}{p'}. \tag{51}$$

Then $H^{\varrho}L_p(R^n)$ are Banach spaces, $1 < u < p$ and

$$S(R^n) \hookrightarrow L_p(R^n, w_\gamma) \hookrightarrow H^{\varrho}L_p(R^n) \hookrightarrow L_u(R^n) \hookrightarrow S'(R^n). \tag{52}$$

Furthermore $D(R^n)$, $S(R^n)$ and $L_p(R^n, w_\gamma)$ are dense in $H^{\varrho}L_p(R^n)$. Let $r + \varrho = -n$. Then $-\frac{n}{p'} < r < 0$ and

$$H^{\varrho}L_p(R^n)' = L^r_{p'}(R^n). \tag{53}$$

Remark 6 These assertions are covered by [18, Theorems 2.8, 2.19, Proposition 2.10, pp. 16, 19, 25]. There one finds detailed proofs and related references including [8, 9]. In particular (53) makes sense in $\big(S(R^n), S'(R^n)\big)$ because $S(R^n)$ is dense in $H^\varrho L_p(R^n)$. This justifies also (50).

We wish to complement the above assertions asking for admissible norms and the Fatou property. Recall that a norm is called *admissible* if any $f \in S'(R^n)$ can be tested of whether it belongs to the corresponding space or not. The *Fatou property* has been described by (1), (2) where $A(R^n)$ is now one of the above spaces.

Theorem 3 *Let $n \in N$.*
(i) The spaces $L^r_p(R^n)$ with $1 < p < \infty$, $-n/p < r < 0$ have the Fatou property. Furthermore

$$\sup \big\{ |(f, \varphi)| : \ \varphi \in S(R^n), \ \|\varphi \,|H^\varrho L_{p'}(R^n)\| \le 1 \big\} \tag{54}$$

with $r + \varrho = -n$ and $\frac{1}{p} + \frac{1}{p'} = 1$ is an equivalent admissible norm in $L^r_p(R^n)$.
(ii) The spaces $\overset{\circ}{L}{}^r_p(R^n)$ with $1 < p < \infty$, $-n/p < r < 0$ do not have the Fatou property.
(iii) The spaces $H^\varrho L_p(R^n)$ with $1 < p < \infty$, $-n < \varrho < -n/p$ have the Fatou property. Furthermore

$$\sup \big\{ |(f, \varphi)| : \ \varphi \in S(R^n), \ \|\varphi \,|L^r_{p'}(R^n)\| \le 1 \big\} \tag{55}$$

with $r + \varrho = -n$ is an equivalent admissible norm in $H^\varrho L_p(R^n)$.

Proof Step 1. Proposition 1(i) and Definition 3 imply that $S(R^n)$ is dense in $\overset{\circ}{L}{}^r_{p'}(R^n)$. Then part (iii) follows from the duality (50). Similarly one obtains part (i) from the duality (53) using that $S(R^n)$ is dense in $H^\varrho L_{p'}(R^n)$.
Step 2. We prove part (ii). Let

$$Q_l = Q_{J_l, M^l} = 2^{-J_l} M^l + 2^{-J_l}(0,1)^n, \qquad l \in N, \tag{56}$$

be disjoint cubes $Q_{J_l, M^l} \subset Q = (0,1)^n$ where $J_l \in N$, with $J_1 < J_2 < \cdots$ and suitably chosen $M^l \in Z^n$. Let

$$f^\lambda_L = \sum_{l=1}^{L} \lambda_l \, 2^{-J_l r} \varphi_l, \qquad L \in N, \tag{57}$$

with

$$\varphi_l = \varphi\big(2^{J_l}(x - x^l)\big), \qquad l \in N, \tag{58}$$

where x^l is the center of Q_l, $0 \le \varphi \in S(R^n)$,

$$\varphi(0) = 1, \qquad \varphi(x) = 0 \text{ if } |x| \ge 1/2. \tag{59}$$

Furthermore let $\lambda = \{\lambda_l\}_{l=1}^{\infty}$ where either $\lambda_l = 1$ or $\lambda_l = -1$. This is a modification of a corresponding construction in [18, p. 23]. By the same arguments as there one obtains

$$f_L^\lambda \in \overset{\circ}{L}{}^r_p(R^n), \qquad \sup_{L \in N} \| f_L^\lambda \, | L^r_p(R^n) \| < \infty \tag{60}$$

uniformly in λ and

$$f^\lambda = \sum_{l=1}^{\infty} \lambda_l \, 2^{-J_l r} \, \varphi_l \in L^r_p(R^n). \tag{61}$$

With $K > L$ one has

$$\| f_K^\lambda - f_L^\lambda \, | L_p(R^n) \|^p \le c \sum_{l=L+1}^{K} 2^{-J_l r p} 2^{-J_l n} = c \sum_{l=L+1}^{K} 2^{-J_l(n+rp)} \le c \, 2^{-L(n+rp)} \tag{62}$$

where we used $rp + n > 0$. In particular

$$f_L^\lambda \to f^\lambda \quad \text{in} \quad L_p(R^n) \hookrightarrow S'(R^n) \quad \text{if} \quad L \to \infty. \tag{63}$$

The set of all f^λ is non-countable, having the cardinality of R. Furthermore by the same arguments as in [18, (2.101), p. 24] there is a number $c > 0$ such that

$$\| f^{\lambda^1} - f^{\lambda^2} \, | L^r_p(R^n) \| \ge c \tag{64}$$

for all sequences λ^1, λ^2 with $\lambda^1 \ne \lambda^2$. If one assumes that $\overset{\circ}{L}{}^r_p(R^n)$ has the Fatou property then it follows from (60), (63), (64) that $\{f^\lambda\}$ is a non-separable subset of $\overset{\circ}{L}{}^r_p(R^n)$. But (49) shows that $\overset{\circ}{L}{}^r_p(R^n)$ is a separable Banach space. This contradiction proves that $\overset{\circ}{L}{}^r_p(R^n)$ has not the Fatou property.

Remark 7 The idea to employ duality to find admissible norms in some function spaces as in (54), (55) is not new. It had already been used in [5] to show that $L_\infty(R^n)$ as the dual of $L_1(R^n)$ has the Fatou property using in addition that $S(R^n)$ is dense in $L_1(R^n)$. We hinted in [19, p. 113] on this possibility in connection with some homogeneous spaces $\dot{A}^s_{p,q}(R^n)$. It is also helpful in some limiting situations for tempered homogeneous spaces $\overset{*}{A}{}^s_{p,q}(R^n)$ as introduced in [21, Definition 2.8]. The Fatou property as considered in this paper must always be understood in the framework of the dual pairing $\big(S(R^n), S'(R^n)\big)$ as described in (1), (2). But for spaces consisting entirely of regular distributions one may ask for Fatou properties in the context of Lebesgue-measurable functions. This applies in particular to the Morrey spaces $L^r_p(R^n)$ and $H^\varrho L_p(R^n)$ as introduced in Definition 3. Then one has the embeddings according to Proposition 1. Whereas such a measure-theoretical version of the Fatou property does not cause any problems for the spaces $L^r_p(R^n)$ the situation for the spaces $H^\varrho L_p(R^n)$ seems to be different. But a direct detailed study and an affirmative answer of this question may be found in [10], extended in [11] to Morrey spaces based on more general measures. In our case one has the inclusions (49), (52) which show that these two types of Fatou properties are closely related to each other.

5 Mapping Properties, Revisited

We described at the end of Sect. 1 how the Fatou property can be used to ensure some mapping properties if completion by smooth functions does not work. We illuminate these somewhat cryptical comments having a closer look at Calderón-Zygmund operators T,

$$(Tf)(x) = \lim_{\varepsilon \downarrow 0} \int_{y \in R^n, |y| \geq \varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,\mathrm{d}y, \qquad x \in R^n \quad a.e. \tag{65}$$

where $2 \leq n \in N$,

$$\Omega \in C^1\big(\{y: \ |y| = 1\}\big), \qquad \int_{|\sigma|=1} \Omega(\sigma)\,\mathrm{d}\sigma = 0. \tag{66}$$

If $1 < p < \infty$ and $-\frac{n}{p} < \gamma < n(1 - \frac{1}{p})$ then T is a linear and bounded map in $L_p(R^n, w_\gamma)$, normed by (42). Details and references may be found in [18, Sect. 2.5.1, (2.75), Remark 2.11, pp. 19, 20, 30, 31]. Let again $1 < p < \infty$ and $-n/p < r < 0$. Then $T\varphi \in \overset{\circ}{L}{}^r_p(R^n)$ if $\varphi \in S(R^n)$ and

$$\|T\varphi \,|L^r_p(R^n)\| \leq c\, \|\varphi \,|L^r_p(R^n)\|, \qquad \varphi \in S(R^n). \tag{67}$$

This follows from [18, Theorem 2.22, p. 32] and (49). Let now $A(R^n) = L^r_p(R^n)$ and $B = L_p(R^n, w_\gamma)$ with $-\frac{n}{p} < \gamma < -\frac{n}{p} - r$. Then (7) follows from (49). By the above comments one has (8). Recall that $S(R^n)$ is dense in $B(R^n) = L_p(R^n, w_\gamma)$. For any $f \in A(R^n) = L^r_p(R^n)$ one finds $f_j \in S(R^n)$ with (9), whereas (10) is covered by (67). The Fatou property of $A(R^n) = L^r_p(R^n)$ according to Theorem 3(i) ensures now (11), that is

$$T: \quad L^r_p(R^n) \hookrightarrow L^r_p(R^n), \qquad 1 < p < \infty, \quad -n/p < r < 0, \tag{68}$$

and the a.e. pointwise representation (65). The assertion itself is already known and covered by [18, Theorem 2.22, Proposition 2.25, pp. 32, 37] where we proved first

$$T: \ \overset{\circ}{L}{}^r_p(R^n) \hookrightarrow \overset{\circ}{L}{}^r_p(R^n), \qquad 1 < p < \infty, \quad -n/p < r < 0, \tag{69}$$

and afterwards by the duality assertions (50), (53)

$$T: \quad H^{\varrho}L_p(R^n) \hookrightarrow H^{\varrho}L_p(R^n), \qquad 1 < p < \infty, \quad -n < \varrho < -n/p \qquad (70)$$

and (68). The above arguments may be considered as an alternative way to justify (68) using the Fatou property of the underlying spaces according to Theorem 3. But we inserted these discussions mainly to illustrate the possible use of the Fatou property if completion via smooth functions or other direct arguments do not work.

Another typical example might be pointwise multiplications, in particular pointwise multipliers $m(\cdot) \in L_\infty(R^n)$ in, say, $A(R^n) = A^s_{p,q}(R^n)$ with $\max(p, q) = \infty$, that is

$$Tf = m(\cdot) f, \qquad f \in A(R^n). \qquad (71)$$

Since $S(R^n)$ is not dense in $A(R^n)$ one cannot argue by completion. By embedding there are spaces $B(R^n) = A^{\sigma}_{u,v}(R^n, w_\gamma)$ with $\max(u, v) < \infty$ and suitable weights $w_\gamma(x) = (1 + |x|^2)^{\gamma/2}$, $\gamma \in R$, ensuring (7). In addition $S(R^n)$ is dense in $B(R^n)$ such that (8) with T as in (71) can be defined by completion. As for (9) one may think about smoothed wavelets or atoms. If it is possible to justify (10) then the Fatou property for $A^s_{p,q}(R^n)$ ensures (11).

The above examples show that the Fatou property of underlying spaces is useful if direct definitions of the the operators T considered are not possible (or unclear) and completion arguments are not available.

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A Survey on Some Variable Function Spaces

Dachun Yang, Wen Yuan and Ciqiang Zhuo

Abstract This article is devoted to presenting a recapitulative introduction of some recent progresses, obtained by the authors and their collaborators, on the theory of variable function spaces including the variable Hardy spaces (associated with operators), the variable weak Hardy spaces and the variable Besov-type and Triebel-Lizorkin-type spaces.

Keywords Hardy space · Besov space · Triebel-Lizorkin space · Variable exponent
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1 Introduction

In recent decades, there was a rapidly increasing number of articles dealing with function spaces with variable exponents as well as their wide applications to harmonic analysis (see, for example, [20, 57, 67, 76, 89, 92] for variable Hardy-type spaces and their applications; [3, 4, 23, 25–27, 86, 87] for variable Besov-type and Triebel-Lizorkin-type spaces as well as their applications and [19, 46–48] for some other variable function spaces, including variable Lebesgue spaces, and their applications, especially, in the study on the boundedness of operators), partial differential equations (see, for example, [8, 9, 17, 19, 22, 30, 31]) and potential theory (see, for example, [6, 22, 39]). Apart from theoretical considerations, the function

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spaces with variable exponents also have interesting applications in fluid dynamics [1, 65], image processing [14] and variational calculus [2, 32, 40, 41, 66]. The study of variable exponent function spaces, especially the variable Lebesgue space, can be traced back to Birnbaum-Orlicz [11] and Orlicz [63] (see also Luxemburg [55] and Nakano [58, 59]), but the modern development started with the articles [49] of Kováčik and Rákosník and [32] of Fan and Zhao as well as [18] of Cruz-Uribe and [21] of Diening.

Particularly, Nakai and Sawano [57] introduced the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and established its atomic characterization which was further applied to consider the dual space of such a Hardy space. The theory of variable Hardy spaces extends that of variable Lebesgue spaces and that of the classical Hardy spaces. Later, Sawano [67] improved the atomic characterizations of the space $H^{p(\cdot)}(\mathbb{R}^n)$ from [57] and gave more applications including the boundedness of the fractional integral operator and some commutators. After that, Zhuo et al. [92] established some intrinsic square function characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$ including characterizations via the intrinsic Lusin area function, the intrinsic g-function and the intrinsic g^*_λ-function. Independent of [57], Cruz-Uribe and Wang [20] also investigated the variable Hardy space with some slightly weaker assumptions on $p(\cdot)$ than those used in [57], and obtained an atomic decomposition of such a variable Hardy space, which, in spirit, is more close to the atomic decomposition for weighted Hardy spaces due to Strömberg and Torchinsky [70] than the classical atomic decomposition. Also, the Riesz characterizations of the variable Hardy space in [20] were established in [85]. Very recently, in [89], the theory of the variable Hardy space on Euclidean spaces was further generalized into the setting of RD-spaces introduced in [38], namely, metric measure spaces whose measures satisfy both doubling and inverse doubling conditions. For more information on RD-spaces, we refer the reader to [38, 83].

As another generalization of variable Lebesgue spaces and Hardy spaces, the variable weak Hardy space was introduced very recently by Yan et al. in [76]. Recall that the weak Hardy spaces with constant exponents naturally appear when studying the boundedness of some operators on the classical Hardy spaces $H^p(\mathbb{R}^n)$ in the critical case, and also serve as intermediate spaces when studying the real interpolation between the Hardy space $H^p(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$. In [76], several equivalent characterizations of variable weak Hardy spaces, via maximal functions and Littlewood-Paley square functions, and the boundedness of Calderón-Zygmund-type operators including the critical case on those variable weak Hardy spaces were established.

On the other hand, via the Lusin area function, the variable Hardy spaces $H_L^{p(\cdot)}(\mathbb{R}^n)$ associated with linear operators L on $L^2(\mathbb{R}^n)$ whose heat kernels satisfying certain pointwise upper bound were introduced and studied in [84] and their molecular characterizations and dual spaces were also obtained in [84]. Very recently, under the assumption that L is a non-negative self-adjoint operator and satisfies the Gaussian upper bound estimates, Zhuo and Yang in [90] further established several maximal function characterizations of $H_L^{p(\cdot)}(\mathbb{R}^n)$ by first obtaining their atomic characterizations. Moreover, variable Hardy spaces associated with operators satisfying Davies-Gaffney estimates were also introduced and investigated in [82]. Recall that, when

$p(\cdot)$ is a constant, these Hardy spaces associated with operators were originally introduced in [7, 28, 29, 75] and further studied in many other articles (see, for example, [42–44, 68, 78]).

Along a different line of study on variable function spaces, Xu [72, 73] studied Besov spaces $B^s_{p(\cdot),q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p(\cdot),q}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ but fixed q and s. As was well known from the trace theorem (see, for example, [37, Theorem 11.1]) and Sobolev-type theorem (see, for example, [71, Theorem 2.7.1]) of classical function spaces, the smoothness and the integrability often interact with each other. However, the unification of both trace theorems and Sobolev-type embeddings does not hold true on function spaces with only one variable index; for example, the trace space of the Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^n)$ is no longer a space of the same type (see [22]). To overcome this problem, Diening et al. [23] introduced the variable Triebel-Lizorkin space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ via first mixing up the concepts of function spaces with variable smoothness and variable integrability. Later, Almeida and Hästö [4] introduced the Besov space with variable smoothness and integrability $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. It turns out that these spaces behave nicely with respect to the trace operator (see [23, Theorem 3.13], [5, Theorem 5.2] and [60, Theorem 5.1]).

Based on the Besov-type and the Triebel-Lizorkin-type spaces (see [79, 80, 88]) and the variable Besov and Triebel-Lizorkin spaces (see [4, 23]), the authors in [86, 87] introduced the Besov-type and the Triebel-Lizorkin-type spaces with variable smoothness and integrability, which makes a further step in completing the unification process of function spaces with variable smoothness and integrability. The authors also established their equivalent characterizations in terms of the Peetre maximal function and atoms in [86, 87], and further obtained trace theorems corresponding to these spaces.

The main purpose of this survey is to make a recapitulative introduction of some recent progresses, obtained by the authors and their collaborators, on the theory of variable function spaces including the variable Hardy spaces (associated with operators), the variable weak Hardy spaces and the variable Besov-type and Triebel-Lizorkin-type spaces.

On the other hand, in order to solve some endpoint or sharp problems of analysis, some more general Musielak-Orlicz-type function spaces were introduced (see [50, 56, 77]). These spaces are defined via growth functions which may vary in both the spatial variables and the growth variable. Therefore, by selecting some special growth functions, Musielak-Orlicz-type spaces may contain the corresponding variable function spaces as special cases. In this survey, we also further clarify their relationships.

Due to the rapid development of the variable function spaces and their applications, we surely ignore some important progresses by choosing to focus on our own works; please see, for example, the monographs [19, 22, 46, 47] and their references for more progresses.

The layout of this article is as follows.

In Sect. 2, we make conventions on notions and notation.

In Sect. 3, we recall some equivalent characterizations of variable Hardy spaces by means of Riesz transforms and (intrinsic) square functions, including the (intrinsic)

Littlewood-Paley g-function, the (intrinsic) Lusin area function and the (intrinsic) g_λ^*-function.

In Sect. 4, we give a brief introduction of the variable weak Hardy space including their various equivalent characterizations via radial or non-tangential maximal functions, atoms, molecules and Littlewood-Paley square functions, the boundedness of some Calderón-Zygmund operators and the real interpolation space between the variable Hardy space and the space $L^\infty(\mathbb{R}^n)$.

Section 5 is devoted to presenting some properties of Triebel-Lizorkin-type and Besov-type spaces with variable exponents, such as some basic embeddings, characterizations via atoms and Peetre maximal functions, and trace theorems.

In Sect. 6, we recall some results of variable Hardy spaces associated with operators, which mainly include the characterizations via atoms, molecules and maximal functions, their dual spaces and the boundedness of fractional integrals.

Finally, in Sect. 7, as the end of this article, we make further notes about variable function spaces including some open questions.

2 Notions and Notation

In what follows, let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Let $\mathbf{0}_n$ denote the origin of $\mathbb{R}^n$ and $\lfloor a \rfloor$ the maximal integer not bigger than $a \in \mathbb{R}$. For any $\beta := (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$ and

$$\partial^\beta := \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n}.$$

For any subset $E \subset \mathbb{R}^n$, we use χ_E to denote its *characteristic function*. The *symbol* $f \lesssim g$ means that there exists a positive constant C, independent of the main parameters, such that $f \le Cg$. If $f \lesssim g \lesssim f$, then we write $f \sim g$.

For any $r \in (0, \infty)$, denote by $L^r_{\rm loc}(\mathbb{R}^n)$ the *set of all locally r-integrable functions* on $\mathbb{R}^n$ and, for any measurable set $E \subset \mathbb{R}^n$, by $L^r(E)$ the *set of all r-integrable functions* f on E. For any $s \in \mathbb{Z}_+$, $C^s(\mathbb{R}^n)$ denotes the set of all functions having continuous classical derivatives up to order s. Let $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$ and, for any $x \in \mathbb{R}^n$,

$$\Gamma(x) := \{(y, t) \in \mathbb{R}^{n+1}_+ :\ |y - x| < t\}.$$

Let $\mathscr{P}(\mathbb{R}^n)$ be the set of all polynomials on $\mathbb{R}^n$ and, for any $r \in \mathbb{Z}_+$, $\mathscr{P}_r(\mathbb{R}^n)$ the *set of all polynomials on $\mathbb{R}^n$ with order not bigger than r*. Let $\mathcal{S}(\mathbb{R}^n)$ be the *space of all Schwartz functions* on $\mathbb{R}^n$ equipped with the well-known classical topology and $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual space* equipped with the weak-$*$ topology. The space $\mathcal{S}_\infty(\mathbb{R}^n)$ is defined to be the set of all Schwartz functions φ satisfying that $\int_{\mathbb{R}^n} \varphi(x) x^\gamma\, dx = 0$ for all multi-indices $\gamma \in \mathbb{Z}_+^n$, equipped with the same topology as $\mathcal{S}(\mathbb{R}^n)$, and $\mathcal{S}'_\infty(\mathbb{R}^n)$ its *topological dual space* equipped with the weak-$*$

topology. For any $f \in \mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$, we use $\widehat{f}$ to denote its *Fourier transform*. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{Z}$, let

$$\varphi_j(\cdot) := 2^{jn}\varphi(2^j\cdot). \tag{1}$$

A measurable function $p(\cdot):\ \mathbb{R}^n \to (0,\infty]$ is called a *variable exponent*. For any variable exponent $p(\cdot)$, let

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) \tag{2}$$

and

$$\underline{p} := \min\{p_-, 1\}. \tag{3}$$

Denote by $\mathcal{P}(\mathbb{R}^n)$ the *collection of all variable exponents* $p(\cdot)$ *satisfying* $0 < p_- \le p_+ \le \infty$. For any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the function $\varphi_{p(x)}$ by setting, for any $t \in [0,\infty)$,

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)} & \text{if } p(x) \in (0,\infty), \\ 0 & \text{if } p(x) = \infty \text{ and } t \in [0,1], \\ \infty & \text{if } p(x) = \infty \text{ and } t \in (1,\infty). \end{cases}$$

Then the *variable exponent modular* associated to $p(\cdot)$ on $\mathbb{R}^n$ is defined by setting, for any measurable function f, $\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \varphi_{p(x)}(|f(x)|)\,dx$ and the corresponding *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda \in (0,\infty)$, equipped with the *Luxemburg* (also called *Luxemburg-Nakano*) *quasi-norm*

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty):\ \varrho_{p(\cdot)}\,(f/\lambda) - \le 1\right\}.$$

In particular, if $0 < p_- \le p_+ < \infty$, then $\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx$.

Recall that a measurable function $p(\cdot)$ is said to satisfy the *local* log-*Hölder continuity condition*, denoted by $p(\cdot) \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, if there exists a positive constant $C_{\log}(p)$ such that, for any $x,\ y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e + 1/|x-y|)}, \tag{4}$$

and $p(\cdot)$ is said to satisfy the *global* log-*Hölder continuity condition*, denoted by $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, if $p(\cdot) \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ and there exist a positive constant C_∞ and a constant $p_\infty \in \mathbb{R}$ such that, for any $x \in \mathbb{R}^n$,

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}.$$

3 The Variable Hardy Space

In this section, we first recall the definition of variable Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ introduced in [20, 57] and then present their equivalent characterizations via Riesz transforms and (intrinsic) square functions established recently in [85, 92]. For more characterizations and properties of $H^{p(\cdot)}(\mathbb{R}^n)$, we refer the reader to [20, 57, 67].

3.1 *Definition of the Variable Hardy Space*

In what follows, for any $N \in \mathbb{N}$, let

$$\mathcal{F}_N(\mathbb{R}^n) := \left\{ \psi \in \mathcal{S}(\mathbb{R}^n) : \sup_{\alpha,\, \beta \in \mathbb{Z}_+^n,\, |\alpha|,\, |\beta| \le N} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \psi(x)| \le 1 \right\}.$$

For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $t \in (0, \infty)$ and $\xi \in \mathbb{R}^n$, let

$$\phi_t(\xi) := t^{-n} \phi(\xi/t). \tag{5}$$

Then, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, its *radial grand maximal function* $f^*_{N,+}$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$f^*_{N,+}(x) := \sup_{\psi \in \mathcal{F}_N(\mathbb{R}^n)} \sup_{t \in (0,\infty)} |f * \psi_t(x)|. \tag{6}$$

Recall that the *Hardy-Littlewood maximal operator* $\mathcal{M}$ is defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy, \tag{7}$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

Definition 1 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy $0 < p_- \le p_+ < \infty$ with p_- and p_+ as in (2). Assume that there exists $p_0 \in (0, p_-)$ such that the maximal operator $\mathcal{M}$ in (7) is bounded on $L^{p(\cdot)/p_0}(\mathbb{R}^n)$. Let $N \in (\frac{n}{p_0} + n + 1, \infty)$. Then the *variable Hardy space* $H^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^*_{N,+} \in L^{p(\cdot)}(\mathbb{R}^n)$, equipped with the quasi-norm

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} := \|f^*_{N,+}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Remark 1 (i) If $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \le p_+ < \infty$, then the operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

(ii) It was proved in [20, Theorem 3.1] that the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ is independent of the choice of $N \in (\frac{n}{p_0} + n + 1, \infty)$.

(iii) The variable Hardy space was first studied by Nakai and Sawano in [57] and, independently, by Cruz-Uribe and Wang in [20]. It should be pointed out that, in [57], instead of the assumption that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)/p_0}(\mathbb{R}^n)$, the variable exponent $p(\cdot)$ is required to belong to $C^{\log}(\mathbb{R}^n)$, while, in [20], $p(\cdot)$ just satisfies the assumption as in Definition 1, which is a little weaker than that in [57] by (i) of this remark, together with the observation that, if $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, then $p(\cdot)/p_0 \in C^{\log}(\mathbb{R}^n)$.

(iv) Let $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ be a Musielak-Orlicz growth function as in [50]. Then Ky [50] introduced the Musielak-Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$; see also the monograph [77] for a complete survey on the real-variable theory of Musielak-Orlicz Hardy spaces. Recall that the *Musielak-Orlicz space* $L^{\varphi}(\mathbb{R}^n)$ is defined to be the set of all measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\varphi}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \varphi(x, |f(x)|/\lambda) \, dx \leq 1 \right\} < \infty.$$

Observe that, if

$$\varphi(x, t) := t^{p(x)} \quad \text{for all} \quad (x, t) \in \mathbb{R}^n \times [0, \infty), \tag{8}$$

then $H^{\varphi}(\mathbb{R}^n) = H^{p(\cdot)}(\mathbb{R}^n)$. However, a general Musielak-Orlicz function φ satisfying all assumptions in [50] may not have the form as in (8). On the other hand, it was proved in [81, Remark 2.23(iii)] that there exists a variable exponent function $p(\cdot)$ belonging to $C^{\log}(\mathbb{R}^n)$ and hence satisfying the assumptions of Definition 1, but $t^{p(\cdot)}$ is not a uniform Muckenhoupt weight which was required in [50]. Thus, in general, the Musielak-Orlicz Hardy $H^{\varphi}(\mathbb{R}^n)$ in [50] and the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ in [20, 57] can not cover each other. Moreover, if the Musielak-Orlicz function in [50] is of the form as in (8), then the space $H^{\varphi}(\mathbb{R}^n)$ in this case from [50] is covered by the space $H^{p(\cdot)}(\mathbb{R}^n)$ from [20], since, in this case, there exists $p_0 \in (0, p_-)$ such that the maximal operator $\mathcal{M}$ in (7) is bounded on $L^{p(\cdot)/p_0}(\mathbb{R}^n)$.

3.2 Characterizations via Riesz Transforms and (Intrinsic) Square Functions

Recall that, for any $j \in \{1, \ldots, n\}$, the *j-th Riesz transform* is usually defined by setting, for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$R_j(f)(x) := \lim_{\delta \to 0^+} C_{(n)} \int_{\{y \in \mathbb{R}^n : \, |y| > \delta\}} \frac{y_j}{|y|^{n+1}} f(x - y) \, dy,$$

here and hereafter, $\delta \to 0^+$ means that $\delta \in (0, \infty)$ and $\delta \to 0$, $C_{(n)} := \frac{\Gamma([n+1]/2)}{\pi^{(n+1)/2}}$ and Γ denotes the Gamma function.

Recall that a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is called a *distribution restricted at infinity* if there exists a positive number $r \in (1, \infty)$ large enough such that, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi \in L^r(\mathbb{R}^n)$. Moreover, if f is a distribution restricted at infinity, then, for any $j \in \{1, \dots, n\}$, $R_j(f)$ is well defined as a distribution (see [69, p. 123]).

The following conclusion is just [85, Theorem 1.5].

Theorem 1 *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be as in Definition 1 with $p_- \in (\frac{n-1}{n}, \infty)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$, where p_- is as in (2). Then the following items are equivalent:*

(i) *$f \in H^{p(\cdot)}(\mathbb{R}^n)$;*
(ii) *f is a distribution restricted at infinity and there exists a positive constant A_1 such that, for any $\epsilon \in (0, \infty)$,*

$$\|f * \phi_\epsilon\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j(f) * \phi_\epsilon\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le A_1, \tag{9}$$

where ϕ_ϵ is as in (5) with t replaced by ϵ.

Moreover, $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \inf\{A_1 :\ A_1$ satisfies (9)$\}$ with the equivalent positive constants independent of f and ϵ. Furthermore, if $p_- \in [1, \infty)$, then (9) can be replaced by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le A_1.$$

We also have the following conclusion, which was established in [85, Theorem 1.6].

Theorem 2 *Let*

$$m \in \mathbb{N} \cap [2, \infty)$$

and $p(\cdot)$ be as in Definition 1 with $p_- \in (\frac{n-1}{n+m-1}, \infty)$, where p_- is as in (2), and let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \phi(x)\, dx = 1$. Then the following items are equivalent:

(i) *$f \in H^{p(\cdot)}(\mathbb{R}^n)$;*
(ii) *f is a distribution restricted at infinity and there exists a positive constant A_2 such that, for any $\varepsilon \in (0, \infty)$,*

$$\|f * \phi_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \sum_{k=1}^{m} \sum_{j_1, \dots, j_k=1}^{n} \|R_{j_1} \cdots R_{j_k}(f) * \phi_\varepsilon\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le A_2. \tag{10}$$

Moreover, $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \inf\{A_2 :\ A_2$ satisfies (10)$\}$ with the equivalent positive constants independent of f and ε.

To state the (intrinsic) square function characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$, we first recall some notions. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then, for any distribution f, the *Littlewood-Paley g-function* $g_\phi(f)$, the *Lusin area function* $S_\phi(f)$ and the *Littlewood-Paley* g_λ^**-function* $g_{\lambda,\phi}^*(f)$, with $\lambda \in (0,\infty)$, of f are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$g_\phi(f)(x) := \left\{\int_0^\infty |f * \phi_t(x)|^2 \frac{dt}{t}\right\}^{1/2},$$

$$S_\phi(f)(x) := \left\{\int_0^\infty \int_{\{y\in\mathbb{R}^n:\ |y-x|<t\}} |\phi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$

and

$$g_{\lambda,\phi}^*(f)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} |\phi_t * f(y)|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{\frac{1}{2}}.$$

The following conclusions come from [92, Theorem 1.4 and Corollary 1.5].

Theorem 3 *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *satisfy* $0 < p_- \le p_+ < \infty$ *with* p_- *and* p_+ *as in* (2). *Let* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *be a radial real-valued function satisfying*

$$\operatorname{supp} \widehat{\phi} \subset \{\xi \in \mathbb{R}^n :\ 1/2 \le |\xi| \le 2\}$$

and, for any $3/5 \le |\xi| \le 5/3$, $|\widehat{\phi}(\xi)| \ge constant > 0$. *Then* $f \in H^{p(\cdot)}(\mathbb{R}^n)$ *if and only if* $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ *and* $S_\phi(f) \in L^{p(\cdot)}(\mathbb{R}^n)$*; moreover, there exists a positive constant* C*, independent of* f*, such that*

$$C^{-1}\|S_\phi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le C\|S_\phi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

The above conclusion remains true if $S_\phi(f)$ *is replaced, respectively, by* $g_\phi(f)$ *and* $g_{\lambda,\phi}^*(f)$ *with* $\lambda \in (1+2/\min\{2,p_-\},\infty)$.

Remark 2 We point out that the conclusion of Theorem 3 is understood in the following sense: if $f \in H^{p(\cdot)}(\mathbb{R}^n)$, then $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and there exists a positive constant C such that, for any $f \in H^{p(\cdot)}(\mathbb{R}^n)$, $\|S_\phi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}$; conversely, if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $S_\phi(f) \in L^{p(\cdot)}(\mathbb{R}^n)$, then there exists a unique extension $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ such that, for any $h \in \mathcal{S}_\infty(\mathbb{R}^n)$, $\langle \widetilde{f}, h\rangle = \langle f, h\rangle$ and $\|\widetilde{f}\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le C\|S_\phi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ with C being a positive constant independent of f. In this sense, we identify f with $\widetilde{f}$.

For any $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$, let $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ be the *family of all functions* $\phi \in C^s(\mathbb{R}^n)$ such that $\operatorname{supp}\phi \subset \{x \in \mathbb{R}^n :\ |x| \le 1\}$, $\int_{\mathbb{R}^n} \phi(x)x^\gamma\,dx = 0$ for any $\gamma \in \mathbb{Z}_+^n$ and $|\gamma| \le s$ and, for any $\nu \in \mathbb{Z}_+^n$, with $|\nu| = s$, and any $x_1,\ x_2 \in \mathbb{R}^n$, $|\partial^\nu\phi(x_1) - \partial^\nu\phi(x_2)| \le |x_1 - x_2|^\alpha$.

For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+$, let

$$A_{\alpha,s}(f)(y,t) := \sup_{\phi\in\mathcal{C}_{\alpha,s}(\mathbb{R}^n)} |f * \phi_t(y)| \text{ with } \phi_t \text{ as in (5).}$$

Then the *intrinsic g-function* $g_{\alpha,s}(f)$, the *intrinsic Lusin area function* $S_{\alpha,s}(f)$ and the *intrinsic* g_λ^**-function* $g^*_{\lambda,\alpha,s}(f)$, with $\lambda \in (0,\infty)$, of f are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$g_{\alpha,s}(f)(x) := \left\{\int_0^\infty \left[A_{\alpha,s}(f)(x,t)\right]^2 \frac{dt}{t}\right\}^{1/2},$$

$$S_{\alpha,s}(f)(x) := \left\{\int_0^\infty \int_{\{y\in\mathbb{R}^n:\ |y-x|<t\}} \left[A_{\alpha,s}(f)(y,t)\right]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$

and

$$g^*_{\lambda,\alpha,s}(f)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} [A_{\alpha,s}(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}.$$

These intrinsic square functions, when $s = 0$, were originally introduced by Wilson [74], which were further generalized to $s \in \mathbb{Z}_+$ by Liang and Yang [51].

Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$, where ϕ_t is as in (5); see, for example, [35, p. 50].

We now recall the notion of the Campanato space with variable exponent, which was introduced by Nakai and Sawano in [57].

Definition 2 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, s be a nonnegative integer and $q \in [1,\infty)$. Then the *variable Campanato space* $\mathcal{L}_{q,p(\cdot),s}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{L}_{q,p(\cdot),s}(\mathbb{R}^n)} := \sup_{Q\subset\mathbb{R}^n} \frac{|Q|}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[\frac{1}{|Q|}\int_Q |f(x) - P^s_Q f(x)|^q\,dx\right]^{\frac{1}{q}} < \infty,$$

where the supremum is taken over all cubes Q of $\mathbb{R}^n$ and $P^s_Q g$ denotes the *unique polynomial* $P \in \mathscr{P}_s(\mathbb{R}^n)$ such that, for any $h \in \mathscr{P}_s(\mathbb{R}^n)$,

$$\int_Q [f(x) - P(x)]h(x)\,dx = 0.$$

The following intrinsic square function characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$ were obtained in [92, Theorems 1.8 and 1.10 and Corollary 1.9].

Theorem 4 *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *and* $0 < p_- \le p_+ \le 1$ *with* p_- *and* p_+ *as in* (2). *Assume that* $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ *and* $p_- \in (n/(n+\alpha+s), 1]$. *Then* $f \in H^{p(\cdot)}(\mathbb{R}^n)$ *if and only if* $f \in (\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$, *the dual space of* $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$, f *vanishes weakly at infinity and* $g_{\alpha,s}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$*; moreover, it holds true that* $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \|g_{\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ *with the equivalent positive constants independent of* f.

The above conclusion remains true if $g_{\alpha,s}(f)$ *is replaced, respectively, by* $S_{\alpha,s}(f)$ *and* $g^*_{\lambda,\alpha,s}(f)$ *with* $\lambda \in (3+2(\alpha+s)/n, \infty)$.

We end this section by the following remark.

Remark 3 In [89], Zhuo et al. introduced Hardy spaces with variable exponents on RD-spaces with infinite measure via the grand maximal function, which is a generalization of variable Hardy spaces on Euclidean spaces, and then characterized these spaces by means of the non-tangential maximal function or the dyadic maximal function. The characterizations of these spaces in terms of atoms or Littlewood-Paley functions were also established in this article. As applications, in [89], an Olsen's inequality related to the fractional integral operator and the boundedness of singular integral operators and quasi-Banach valued sublinear operators on these spaces were presented. Finally, a duality theory of these spaces was also developed.

4 The Variable Weak Hardy Space

In this section, we recall some properties of the variable weak Hardy space introduced and studied in [76, 91] and begin with the following definition.

Definition 3 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy $0 < p_- \le p_+ < \infty$ with p_- and p_+ as in (2).

(i) The *variable weak Lebesgue space* $WL^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{WL^{p(\cdot)}(\mathbb{R}^n)} := \sup_{\alpha\in(0,\infty)} \alpha \left\|\chi_{\{x\in\mathbb{R}^n:\ |f(x)|>\alpha\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

(ii) Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $N \in (\frac{n}{\underline{p}} + n + 1, \infty)$ be a positive integer, where $\underline{p}$ is as in (3). Then the *variable weak Hardy space*, denoted by $WH^{p(\cdot)}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^*_{N,+} \in WL^{p(\cdot)}(\mathbb{R}^n)$, equipped with the *quasi-norm*

$$\|f\|_{WH^{p(\cdot)}(\mathbb{R}^n)} := \|f^*_{N,+}\|_{WL^{p(\cdot)}(\mathbb{R}^n)},$$

where $f^*_{N,+}$ is as in (6).

Remark 4 (i) It was proved in [76, Theorem 3.7] that the variable weak Hardy space $WH^{p(\cdot)}(\mathbb{R}^n)$ is independent of the choice of $N \in (\frac{n}{\underline{p}} + n + 1, \infty)$. If $p(\cdot) \equiv p \in (0, 1]$, then the space $WH^{p(\cdot)}(\mathbb{R}^n)$ is just the classical weak Hardy space $WH^p(\mathbb{R}^n)$ studied in [33, 34, 53] (see also [52]).

(ii) Recall that Liang et al. [52] introduced the weak Musielak-Orlicz Hardy space $WH^{\varphi}(\mathbb{R}^n)$, with a Musielak-Orlicz growth function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$, via the *weak Musielak-Orlicz space* $WL^{\varphi}(\mathbb{R}^n)$ which is defined as the set of all measurable functions f such that

$$\|f\|_{WL^{\varphi}(\mathbb{R}^n)} := \inf\left\{\lambda\in(0,\infty):\ \sup_{\alpha\in(0,\infty)}\int_{\{x\in\mathbb{R}^n:\ |f(x)|>\alpha\}}\varphi\left(x,\frac{\alpha}{\lambda}\right)dx\le 1\right\}<\infty.$$

Observe that, when φ is as in (8), then $WL^{\varphi}(\mathbb{R}^n)=WL^{p(\cdot)}(\mathbb{R}^n)$ (see [76, Remark 2.8]) and hence $WH^{\varphi}(\mathbb{R}^n)=WH^{p(\cdot)}(\mathbb{R}^n)$. However, in general, the weak Musielak-Orlicz Hardy space $WH^{\varphi}(\mathbb{R}^n)$ in [52] and the variable weak Hardy space $WH^{p(\cdot)}(\mathbb{R}^n)$ in [76] do not cover each other. For the reason, we refer the reader to [76, Remark 2.14(iii)] (see also Remark 1(iv)).

4.1 Equivalent Characterizations

This subsection presents several equivalent characterizations of $WH^{p(\cdot)}(\mathbb{R}^n)$ via maximal functions, atoms, molecules and Littlewood-Paley square functions.

Let $\psi\in\mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}\psi(x)\,dx\neq 0$. Then the *radial maximal function* $\psi_+^*(f)$ of $f\in\mathcal{S}'(\mathbb{R}^n)$ associated to ψ is defined by setting, for any $x\in\mathbb{R}^n$,

$$\psi_+^*(f)(x):=\sup_{t\in(0,\infty)}|f*\psi_t(x)|,$$

where ψ_t is as in (5) with ϕ replaced by ψ.

A distribution $f\in\mathcal{S}'(\mathbb{R}^n)$ is called a *bounded distribution* if, for any $\phi\in\mathcal{S}(\mathbb{R}^n)$, $f*\phi\in L^{\infty}(\mathbb{R}^n)$. For a bounded distribution f, its *non-tangential maximal function*, with respect to Poisson kernels $\{P_t\}_{t\in(0,\infty)}$, is defined by setting, for any $x\in\mathbb{R}^n$,

$$\mathcal{N}(f)(x):=\sup_{t\in(0,\infty),\,|y-x|<t}|f*P_t(y)|,$$

where, for any $t\in(0,\infty)$ and $x\in\mathbb{R}^n$,

$$P_t(x):=\frac{\Gamma([n+1]/2)}{\pi^{(n+1)/2}}\frac{t}{(t^2+|x|^2)^{(n+1)/2}}.$$

Theorem 5 *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$. Suppose that $N\in(\frac{n}{\underline{p}}+n+1,\infty)$ is a positive integer, where $\underline{p}$ is as in* (3)*. Then the following items are mutually equivalent:*

(i) *$f\in WH^{p(\cdot)}(\mathbb{R}^n)$, namely, $f\in\mathcal{S}'(\mathbb{R}^n)$ and $f_{N,+}^*\in WL^{p(\cdot)}(\mathbb{R}^n)$;*
(ii) *f is a bounded distribution and $\mathcal{N}(f)\in WL^{p(\cdot)}(\mathbb{R}^n)$;*
(iii) *$f\in\mathcal{S}'(\mathbb{R}^n)$ and there exists a $\psi\in\mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n}\psi(x)\,dx=1$ such that $\psi_+^*(f)\in WL^{p(\cdot)}(\mathbb{R}^n)$.*

Moreover, for any $f\in WH^{p(\cdot)}(\mathbb{R}^n)$, it holds true that

$$\|f_{N,+}^*\|_{WL^{p(\cdot)}(\mathbb{R}^n)} \sim \|\mathcal{N}(f)\|_{WL^{p(\cdot)}(\mathbb{R}^n)} \sim \|\psi_+^*(f)\|_{WL^{p(\cdot)}(\mathbb{R}^n)},$$

where the equivalent positive constants are independent of f.

We next recall the definitions of the variable weak atomic Hardy space and the variable weak molecular Hardy space.

Definition 4 Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \le p_+ < \infty$,

$$s \in \left(\frac{n}{p_-} - n - 1, \infty\right) \cap \mathbb{Z}_+$$

and $q \in (1, \infty]$, where p_- and p_+ are as in (2).

(i) A measurable function a on $\mathbb{R}^n$ is called a $(p(\cdot), q, s)$-*atom* if there exists a ball B such that supp $a \subset B$, $\|a\|_{L^q(\mathbb{R}^n)} \le |B|^{1/q}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$ and, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le s$, $\int_{\mathbb{R}^n} a(x)x^\alpha\, dx = 0$.

(ii) The *variable weak atomic Hardy space* $WH_{\text{atom}}^{p(\cdot),q,s}(\mathbb{R}^n)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ which can be decomposed as $f = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j}$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{a_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of $(p(\cdot), q, s)$-atoms, associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z},\ j\in\mathbb{N}}$, satisfying that there exists a positive constant $c \in (0, 1]$ such that, for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{cB_{i,j}}(x) \le A$ with A being a positive constant independent of x and i and, for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{i,j} := \widetilde{A}2^i\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ with $\widetilde{A}$ being a positive constant independent of i and j. Moreover, for any $f \in WH_{\text{atom}}^{p(\cdot),q,s}(\mathbb{R}^n)$, define

$$\|f\|_{WH_{\text{atom}}^{p(\cdot),q,s}(\mathbb{R}^n)} := \inf\left[\sup_{i\in\mathbb{Z}}\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right],$$

where the infimum is taken over all decompositions of f as above and $\underline{p}$ is as in (3).

Definition 5 Let $q \in (1, \infty]$, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \le p_+ < \infty$,

$$s \in (\frac{n}{p_-} - n - 1, \infty) \cap \mathbb{Z}_+ with p_- and p_+ as in 2.1x,$$

and $\epsilon \in (0, \infty)$.

(i) A measurable function m is called a $(p(\cdot), q, s, \epsilon)$-*molecule* associated with some ball $B \subset \mathbb{R}^n$ if, for each $j \in \mathbb{N}$,

$$\|m\|_{L^q(U_j(B))} \le 2^{-j\epsilon}|U_j(B)|^{\frac{1}{q}}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1},$$

where $U_0(B) := B$ and, for any $j \in \mathbb{N}$, $U_j(B) := (2^jB)\backslash(2^{j-1}B)$ and, for any $\beta \in \mathbb{Z}_+^n$ with $|\beta| \le s$, $\int_{\mathbb{R}^n} m(x)x^\beta dx = 0$.

(ii) The *variable weak molecular Hardy space* $WH_{\mathrm{mol}}^{p(\cdot),q,s,\epsilon}(\mathbb{R}^n)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ which can be decomposed as $f = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{i,j} m_{i,j}$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{m_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$ is a sequence of $(p(\cdot), q, s, \epsilon)$-molecules associated with balls $\{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}$, $\lambda_{i,j} := \widetilde{A}2^i \|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ with $\widetilde{A}$ being a positive constant independent of i, j, and there exist positive constants A and C such that, for any $i \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, $\sum_{j\in\mathbb{N}} \chi_{CB_{i,j}}(x) \le A$. Moreover, for any $f \in WH_{\mathrm{mol}}^{p(\cdot),q,s,\epsilon}(\mathbb{R}^n)$, define

$$\|f\|_{WH_{\mathrm{mol}}^{p(\cdot),q,s,\epsilon}(\mathbb{R}^n)} := \inf\left[\sup_{i\in\mathbb{Z}}\left\|\left\{\sum_{j\in\mathbb{N}}\left[\frac{\lambda_{i,j}\chi_{B_{i,j}}}{\|\chi_{B_{i,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right],$$

where the infimum is taken over all decompositions of f as above and $\underline{p}$ is as in (3).

The following conclusion was established in [76, Theorems 4.4 and 5.3].

Theorem 6 *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *with* $0 < p_- \le p_+ < \infty$*,* $q \in (\max\{p_+, 1\}, \infty]$*,* $s \in (\frac{n}{p_-} - n - 1, \infty) \cap \mathbb{Z}_+$ *and* $\epsilon \in (n + s + 1, \infty)$*, where* p_- *and* p_+ *are as in* (2)*. Then*

$$WH_{\mathrm{atom}}^{p(\cdot),q,s}(\mathbb{R}^n) = WH^{p(\cdot)}(\mathbb{R}^n) = WH_{\mathrm{mol}}^{p(\cdot),q,s,\epsilon}(\mathbb{R}^n)$$

with equivalent quasi-norms.

In [76, Theorems 6.1, 6.2 and 6.3], the authors established the following equivalent characterizations of $WH^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 7 *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *satisfy* $0 < p_- \le p_+ < \infty$*, where* p_- *and* p_+ *are as in* (2)*. Let* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *be a radial function satisfying*

$$\operatorname{supp} \phi \subset \{x \in \mathbb{R}^n : |x| \le 1\},$$

$\int_{\mathbb{R}^n} \phi(x)x^\gamma\,dx = 0$ *for any* $\gamma \in \mathbb{Z}_+^n$ *with* $|\gamma| \le \max\{\lfloor \frac{n}{p_-} - n - 1\rfloor, 0\}$ *and, for any* $\xi \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$*,* $\int_0^\infty |\widehat{\phi}(\xi t)|^2 \frac{dt}{t} = 1$*. Then* $f \in WH^{p(\cdot)}(\mathbb{R}^n)$ *if and only if* $f \in \mathcal{S}'(\mathbb{R}^n)$*,* f *vanishes weakly at infinity and* $S_\phi(f) \in WL^{p(\cdot)}(\mathbb{R}^n)$*. Moreover, for any* $f \in WH^{p(\cdot)}(\mathbb{R}^n)$*,*

$$C^{-1}\|S_\phi(f)\|_{WL^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{WH^{p(\cdot)}(\mathbb{R}^n)} \le C\|S_\phi(f)\|_{WL^{p(\cdot)}(\mathbb{R}^n)},$$

where C *is a positive constant independent of* f*.*

The above conclusion remains true if $S_\phi(f)$ *is replaced, respectively, by* $g_\phi(f)$ *and* $g^*_{\lambda,\phi}(f)$ *with* $\lambda \in (1 + \frac{2}{\min\{p_-,2\}}, \infty)$*.*

4.2 Applications to Calderón-Zygmund Operators

In this subsection, we give some applications of $WH^{p(\cdot)}(\mathbb{R}^n)$ to the boundedness of some Calderón-Zygmund operators.

Recall that, for any given $\delta \in (0, 1]$, a *convolutional δ-type Calderón-Zygmund operator T* means that: T is a linear bounded operator on $L^2(\mathbb{R}^n)$ with kernel $k \in \mathcal{S}'(\mathbb{R}^n)$ coinciding with a locally integrable function on $\mathbb{R}^n \setminus \{\mathbf{0}_n\}$ and satisfying that, for any $x, y \in \mathbb{R}^n$ with $|x| > 2|y|$,

$$|k(x-y) - k(x)| \le C \frac{|y|^\delta}{|x|^{n+\delta}}$$

and, for any $f \in L^2(\mathbb{R}^n)$, $Tf(x) = k * f(x)$.

The following conclusion is just [76, Theorem 7.3].

Theorem 8 *Let $p(\cdot) : \mathbb{R}^n \to (0, 1]$ belong to $C^{\log}(\mathbb{R}^n)$ and $\delta \in (0, 1]$. Let T be a convolutional δ-type Calderón-Zygmund operator. If $p_- \in [\frac{n}{n+\delta}, 1]$ with p_- as in (2), then T has a unique extension on $H^{p(\cdot)}(\mathbb{R}^n)$ and, moreover, for any $f \in H^{p(\cdot)}(\mathbb{R}^n)$,*

$$\|Tf\|_{WH^{p(\cdot)}(\mathbb{R}^n)} \le C \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)},$$

where C is a positive constant independent of f.

Remark 5 (i) If $p(\cdot) \equiv p \in (0, 1]$, then $WH^{p(\cdot)}(\mathbb{R}^n) = WH^p(\mathbb{R}^n)$. In this case, Theorem 8 indicates that, if $\delta \in (0, 1]$, $p = \frac{n}{n+\delta}$ and T is a convolutional δ-type Calderón-Zygmund operator, then T is bounded from $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ to $WH^{\frac{n}{n+\delta}}(\mathbb{R}^n)$, which is just [53, Theorem 1] (see also [52, Theorem 5.2]). Here $\frac{n}{n+\delta}$ is called the *critical index*. Thus, the boundedness of the Calderón-Zygmund operator from $H^{p(\cdot)}(\mathbb{R}^n)$ to $WH^{p(\cdot)}(\mathbb{R}^n)$ obtained in Theorem 8 includes the critical case.

(ii) Here we point out that, although $\delta \in (0, 1)$ is required in [76, Theorem 7.3], by an argument similar to that used in the proof of [52, Theorem 5.2], we conclude that the conclusion of [76, Theorem 7.3] also holds true for $\delta = 1$.

Recall that, for any given $\gamma \in (0, \infty)$, a linear operator T is called a *γ-order Calderón-Zygmund operator* if T is bounded on $L^2(\mathbb{R}^n)$ and its kernel

$$k : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x, x) : x \in \mathbb{R}^n\} \to \mathbb{C}$$

satisfies that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le \lceil \gamma \rceil$ and $x, y, z \in \mathbb{R}^n$

$$with |x-y| > 2|y-z|,$$

$$|\partial_x^\alpha k(x, y) - \partial_x^\alpha k(x, z)| \le C \frac{|y-z|^{\gamma - \lceil \gamma \rceil}}{|x-y|^{n+\gamma}},$$

here and hereafter, $\lceil\gamma\rceil$ denotes the maximal integer smaller than γ and, for any $f \in L^2(\mathbb{R}^n)$ having compact support and $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\operatorname{supp} f} k(x, y) f(y)\, dy.$$

For any given $m \in \mathbb{N}$, an operator T is said to satisfy the *vanishing moment condition up to order m* if, for any $a \in L^2(\mathbb{R}^n)$ with compact support satisfying that, for any $\beta \in \mathbb{Z}_+^n$ with $|\beta| \le m$, $\int_{\mathbb{R}^n} a(x)x^\beta\, dx = 0$, it holds true that $\int_{\mathbb{R}^n} x^\beta Ta(x)\, dx = 0$.

We also state the following conclusion, which, when $\gamma \in (0,\infty)\backslash\mathbb{N}$, is just [76, Theorem 7.5] and, when $\gamma \in \mathbb{N}$, can be proved by an argument similar to that used in the proof of [76, Theorem 7.5], the details being omitted.

Theorem 9 *Let $p(\cdot) : \mathbb{R}^n \to (0, 1]$ belong to $C^{\log}(\mathbb{R}^n)$ and $\gamma \in (0,\infty)$. Let T be a γ-order Calderón-Zygmund operator and have the vanishing moment condition up to order $\lceil\gamma\rceil$. If $\lceil\gamma\rceil \le n(\frac{1}{p_-} - 1) \le \gamma$ with p_- as in (2), then T has a unique extension on $H^{p(\cdot)}(\mathbb{R}^n)$ and, moreover, for any $f \in H^{p(\cdot)}(\mathbb{R}^n)$, $\|Tf\|_{WH^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}$, where C is a positive constant independent of f.*

Comparing with Remark 5(i), we know that $\frac{n}{n+\gamma}$ is the critical index of the γ-order Calderón-Zygmund operator.

4.3 Real Interpolation Between $H^{p(\cdot)}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$

In this subsection, we give another application of $WH^{p(\cdot)}(\mathbb{R}^n)$ to the real interpolation between the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and the space $L^\infty(\mathbb{R}^n)$ (see [91]).

We first recall some basic notions about the theory of real interpolation (see [10]). Let (X_0, X_1) be a compatible couple of quasi-normed spaces, namely, X_0 and X_1 are two quasi-normed linear spaces which are continuously embedded into some large topological vector space. Let

$$X_0 + X_1 := \{f_0 + f_1 :\ f_0 \in X_0 \text{ and } f_1 \in X_1\}.$$

For any $t \in (0,\infty)$, the *Peetre K-functional* $K(t, f; X_0, X_1)$ on $X_0 + X_1$ is defined by setting, for any $f \in X_0 + X_1$,

$$K(t, f; X_0, X_1) := \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} :\ f = f_0 + f_1,\ f_0 \in X_0 \text{ and } f_1 \in X_1\}.$$

Then, for any $\theta \in (0, 1)$ and $q \in (0,\infty]$, the *real interpolation space* $(X_0, X_1)_{\theta,q}$ between X_0 and X_1 is defined as

$$(X_0, X_1)_{\theta,q} := \left\{f \in X_0 + X_1 :\ \|f\|_{\theta,q} < \infty\right\},$$

where, for any $f \in X_0 + X_1$,

$$\|f\|_{\theta,q} := \begin{cases} \left[\int_0^\infty \left\{t^{-\theta}K(t, f; X_0, X_1)\right\}^q \frac{dt}{t}\right]^{1/q} & \text{if } q \in (0, \infty), \\ \sup_{t\in(0,\infty)} t^{-\theta}K(t, f; X_0, X_1) & \text{if } q = \infty. \end{cases}$$

Theorem 10 *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $\theta \in (0, 1)$. Then it holds true that*

$$(H^{p(\cdot)}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta,\infty} = WH^{\widetilde{p}(\cdot)}(\mathbb{R}^n), \tag{11}$$

where $\frac{1}{\widetilde{p}(\cdot)} = \frac{1-\theta}{p(\cdot)}$.

As a consequence of Theorem 10 and [57, Lemma 3.1], we immediately obtain the following conclusion.

Corollary 1 *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. If $p_- \in (1, \infty)$ with p_- as in (2), then*

$$WH^{p(\cdot)}(\mathbb{R}^n) = WL^{p(\cdot)}(\mathbb{R}^n)$$

with equivalent quasi-norms.

Remark 6 (i) When $p(\cdot) \equiv p \in (0, 1)$, Theorem 10 goes back to [33, Theorem 1], which states that

$$(H^p(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta,\infty} = WH^{p/(1-\theta)}(\mathbb{R}^n), \quad \theta \in (0, 1).$$

(ii) When $p(\cdot) \equiv 1$, (11) becomes

$$(H^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta,\infty} = WH^{1/(1-\theta)}(\mathbb{R}^n) = WL^{1/(1-\theta)}(\mathbb{R}^n), \quad \theta \in (0, 1),$$

which was presented in [64, (2)].

(iii) When $p(\cdot) \equiv p \in (1, \infty)$, (11) is a special case of [64, Theorem 7], namely,

$$(L^p(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta,\infty} = WL^{p/(1-\theta)}(\mathbb{R}^n), \quad \theta \in (0, 1).$$

5 Besov-Type and Triebel-Lizorkin-Type Spaces with Variable Exponents

In this section, we make an introduction of Besov-type and Triebel-Lizorkin-type spaces with variable exponents, which were introduced and studied in [86, 87]. These two kinds of spaces are generalizations of Besov and Triebel-Lizorkin spaces with variable exponents, which were, respectively, introduced in [4, 23] and further studied in [24, 60–62].

5.1 Definitions and Some Basic Embeddings

We begin with the following notation and notions.

For any $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, denote by Q_{jk} the *dyadic cube* $2^{-j}([0,1)^n + k)$ and $\ell(Q_{jk})$ its *side length*. Let $\mathcal{Q} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$,

$$\mathcal{Q}^* := \{Q \in \mathcal{Q} : \ell(Q) \le 1\}$$

and, for any $Q \in \mathcal{Q}$, $j_Q := -\log_2 \ell(Q)$.

Let $\mathcal{G}(\mathbb{R}^{n+1}_+)$ be the *set* of all measurable functions $\phi : \mathbb{R}^{n+1}_+ \to (0,\infty)$ having the following properties: there exist positive constants c_1 and c_2 such that, for any $x \in \mathbb{R}^n$ and $r \in (0,\infty)$,

$$c_1^{-1}\phi(x,2r) \le \phi(x,r) \le c_1\phi(x,2r) \tag{12}$$

and, for any $x, y \in \mathbb{R}^n$ and $r \in (0,\infty)$ with $|x-y| \le r$,

$$c_2^{-1}\phi(y,r) \le \phi(x,r) \le c_2\phi(y,r).$$

In the following, for any cube $Q := Q(x,r)$ with $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, let

$$\phi(Q) := \phi(Q(x,r)) := \phi(x,r).$$

Recall that a pair (φ, Φ) of functions on $\mathbb{R}^n$ is said to be *admissible* if $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}\widehat{\varphi} \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\right\} \text{ and } |\widehat{\varphi}(\xi)| \ge \text{constant} > 0 \text{ when } \frac{3}{5} \le |\xi| \le \frac{5}{3},$$

and

$$\operatorname{supp}\widehat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\} \text{ and } |\widehat{\Phi}(\xi)| \ge \text{constant} > 0 \text{ when } |\xi| \le \frac{5}{3}.$$

We first recall the following mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$, which was introduced by Almeida and Hästö [4].

Definition 6 Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ and E be a measurable subset of $\mathbb{R}^n$. Then the *mixed Lebesgue-sequence space* $\ell^{q(\cdot)}(L^{p(\cdot)}(E))$ is defined to be the set of all sequences $\{f_v\}_{v\in\mathbb{N}}$ of functions in $L^{p(\cdot)}(E)$ such that

$$\begin{aligned}&\|\{f_v\}_{v\in\mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(E))}\\ &\quad := \inf\left\{\lambda \in (0,\infty) : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_v\chi_E/\lambda\}_{v\in\mathbb{N}}) \le 1\right\} < \infty,\end{aligned}$$

where, for any sequence $\{g_v\}_{v\in\mathbb{N}}$ of measurable functions,

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{g_v\}_{v\in\mathbb{N}}) := \sum_{v\in\mathbb{N}} \inf\left\{\mu_v \in (0,\infty) :\ \varrho_{p(\cdot)}\left(g_v/\mu_v^{1/q(\cdot)}\right) \le 1\right\}$$

with the convention $\lambda^{1/\infty} = 1$ for any $\lambda \in (0,\infty)$.

We now recall the definitions of Besov-type and Triebel-Lizorkin-type spaces with variable exponents as follows (see [87, Definition 2.12] and [86, Definition 1.4]).

Definition 7 Let (φ, Φ) be a pair of admissible functions on $\mathbb{R}^n$. Let

$$s \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),$$

$\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$ and $p,\ q \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$.

(i) Then the *Besov-type space with variable exponents*, $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\|\left\{2^{js(\cdot)}|\varphi_j * f|\right\}_{j\ge\max\{j_P,0\}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} < \infty,$$

where, when $j = 0$, φ_0 is replaced by Φ, and the supremum is taken over all dyadic cubes P in $\mathbb{R}^n$.

(ii) If $0 < p_- \le p_+ < \infty$ and $0 < q_- \le q_+ < \infty$ with p_- and p_+ as in (2) and q_- and q_+ as in (2) via replaced p by q, then the *Triebel–Lizorkin-type space with variable exponents*, $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\|\left\{\sum_{j=\max\{j_P,0\}}^{\infty} \left[2^{js(\cdot)}|\varphi_j * f(\cdot)|\right]^{q(\cdot)}\right\}^{\frac{1}{q(\cdot)}}\right\|_{L^{p(\cdot)}(P)} < \infty,$$

where, when $j = 0$, φ_0 is replaced by Φ, and the supremum is taken over all dyadic cubes P in $\mathbb{R}^n$.

In what follows, we use the symbol $A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to denote either $F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ or $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Notice that, when $A = F$, we always require $0 < p_- \le p_+ < \infty$ and $0 < q_- \le q_+ < \infty$.

Remark 7 (i) The spaces $A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ are independent of the choice of the admissible function pairs (φ, Φ); see [86, Corollary 2.4] and [87, Corollary 3.5].

(ii) If $\phi(Q) := 1$ for any cube Q of $\mathbb{R}^n$, then

$$B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \quad\text{and}\quad F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n),$$

where $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ denote the *variable Besov space* (see [4]), respectively, the *variable Triebel-Lizorkin space* (see [23]).

(iii) Here, it should be pointed out that, when considered the variable Besov space in [4], the authors assumed that $\frac{1}{p}$, $\frac{1}{q} \in C^{\log}(\mathbb{R}^n)$, which seems to be weaker than that of Definition 7(i). Indeed, let $p \in \mathcal{P}(\mathbb{R}^n)$. If $p_+ \in (0,\infty)$, then $p \in C^{\log}(\mathbb{R}^n)$ if and only if $\frac{1}{p} \in C^{\log}(\mathbb{R}^n)$; if $p_+ = \infty$, then $p \in C^{\log}(\mathbb{R}^n)$ implies $p(x) \equiv \infty$ for any $x \in \mathbb{R}^n$ and hence $\frac{1}{p} \in C^{\log}(\mathbb{R}^n)$.

(iv) When p, q, s are constant exponents and $\phi(Q) := |Q|^\tau$ with $\tau \in [0,\infty)$ for any cube Q, then

$$B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = B^{s,\tau}_{p,q}(\mathbb{R}^n) \quad \text{and} \quad F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s,\tau}_{p,q}(\mathbb{R}^n),$$

where the symbols $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ denote, respectively, the inhomogeneous Besov-type and Triebel-Lizorkin-type spaces introduced in [88]. We also recall that the homogeneous Besov-type and Triebel-Lizorkin-type spaces were introduced in [79, 80] and, moreover, when $\tau = 0$, they are the classical Besov and Triebel-Lizorkin spaces (see [36, 37, 71]).

(v) Recall that Yang et al. [81] introduced the Musielak-Orlicz Besov-type and Triebel-Lizorkin-type spaces via Musielak-Orlicz functions. By some arguments similar to those used in [81, Remark 2.23], we conclude that the Musielak-Orlicz Besov-type and Triebel-Lizorkin-type spaces in [81] and the Besov-type and the Triebel-Lizorkin-type spaces with variable exponents in [86, 87] do not cover each other.

The following conclusions come from [87, Propositions 4.1 and 5.6] and [86, Proposition 3.20].

Proposition 1 *Let* $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$, $s,\ s_0,\ s_1 \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$,

$$p,\ q,\ q_0,\ q_1 \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n).$$

(i) *If* $q_0 \le q_1$, *then* $A^{s(\cdot),\phi}_{p(\cdot),q_0(\cdot)}(\mathbb{R}^n) \hookrightarrow A^{s(\cdot),\phi}_{p(\cdot),q_1(\cdot)}(\mathbb{R}^n)$.

(ii) *If* $p_+,\ q_+ \in (0,\infty)$ *with* p_+ *and* q_+ *as in Definition 7, then*

$$B^{s(\cdot),\phi}_{p(\cdot),\min\{p(\cdot),\,q(\cdot)\}}(\mathbb{R}^n) \hookrightarrow F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot),\phi}_{p(\cdot),\max\{p(\cdot),\,q(\cdot)\}}(\mathbb{R}^n).$$

In particular, if $p_+ \in (0,\infty)$, *then* $B^{s(\cdot),\phi}_{p(\cdot),p(\cdot)}(\mathbb{R}^n) = F^{s(\cdot),\phi}_{p(\cdot),p(\cdot)}(\mathbb{R}^n)$.

(iii) $\mathcal{S}(\mathbb{R}^n) \hookrightarrow A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

Proposition 2 *Let* $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$, $p_0,\ p_1,\ q \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ *and* $s_0,\ s_1 \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Let, for any* $x \in \mathbb{R}^n$, $s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}$ *and* $s_1(x) \le s_0(x)$. *Then* $A^{s_0(\cdot),\phi}_{p_0(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow A^{s_1(\cdot),\phi}_{p_1(\cdot),q(\cdot)}(\mathbb{R}^n)$.

5.2 Several Equivalent Characterizations

In this subsection, we present several equivalent characterizations of the spaces $A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, via atoms and Peetre maximal functions, and begin with the definitions of atoms and sequence spaces corresponding to $A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Definition 8 Let $K \in \mathbb{Z}_+$, $L \in \mathbb{Z}$ and $R \in \mathbb{N}$. A measurable function a_Q on $\mathbb{R}^n$ is called a (K, L)-*smooth atom* supported near $Q := Q_{jk} \in \mathcal{Q}$, where $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, if supp $a_Q \subset 3Q$, $\int_{\mathbb{R}^n} x^\gamma a_Q(x)\,dx = 0$ for any $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| < L$ when $j \in \mathbb{N}$ and, for any multi-index $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \le K$, $|\partial^\alpha a_Q(x)| \le 2^{(|\alpha|+n/2)j}$ for any $x \in \mathbb{R}^n$.

Definition 9 Let ϕ, $p(\cdot)$, $q(\cdot)$ and $s(\cdot)$ be as in Definition 7. Then the *sequence spaces* $f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ are, respectively, defined to be the set of all sequences $t := \{t_Q\}_{Q\in\mathcal{Q}^*} \subset \mathbb{C}$ such that

$$\|t\|_{f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{Q\subset P,\, Q\in\mathcal{Q}^*} \left[|Q|^{-[\frac{s(\cdot)}{n}+\frac{1}{2}]}|t_Q|\chi_Q\right]^{q(\cdot)} \right\}^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}(P)}$$

and

$$\|t\|_{b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{\substack{Q\subset P,\, Q\in\mathcal{Q}^* \\ \ell(Q)=2^{-j}}} |Q|^{-[\frac{s(\cdot)}{n}+\frac{1}{2}]}|t_Q|\chi_Q \right\}_{j\ge\max\{j_P,0\}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}$$

are finite, where the suprema are taken over all dyadic cubes P in $\mathbb{R}^n$.

For the presentation simplicity, we also use $a^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ to denote

$$either\ f^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\ or\ b^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

The following atomic characterizations of the spaces $A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ were established in [86, Theorem 3.8] and [87, Theorem 5.9], respectively.

Theorem 11 *Let p, q, s and ϕ be as in Definition 7. Let s_- and s_+ be as in (2) with p replaced by s.*

(i) *Let $K \in (s_+ + \max\{0, \log_2 c_1\}, \infty)$ and*

$$L \in (n/\min\{1, p_-, q_-\} - n - s_-, \infty) \quad for\ F{-}space$$

or

$$L \in (n/\min\{1, p_-\} - n - s_-, \infty) \quad for\ B-space.$$

Suppose that $\{m_Q\}_{Q\in\mathcal{Q}^*}$ *are* (K, L)*-smooth atoms and* $t := \{t_Q\}_{Q\in\mathcal{Q}^*} \in a^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*. Then* $f := \sum_{Q\in\mathcal{Q}^*} t_Q m_Q$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *and*

$$\|f\|_{A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \le C\|t\|_{a^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$$

with C *being a positive constant independent of* t.

(ii) *Conversely, if* $f \in A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*, then, for any given* $K,\ L \in \mathbb{Z}_+$*, there exist a sequence* $t := \{t_Q\}_{Q\in\mathcal{Q}^*} \subset \mathbb{C}$ *and a sequence* $\{a_Q\}_{Q\in\mathcal{Q}^*}$ *of* (K, L)*-smooth atoms such that* $f = \sum_{Q\in\mathcal{Q}^*} t_Q a_Q$ *in* $\mathcal{S}'(\mathbb{R}^n)$ *and*

$$\|t\|_{a^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$$

with C *being a positive constant independent of* f.

Let (φ, Φ) be a pair of admissible functions. Recall that the *Peetre maximal function* of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by setting, for any $j \in \mathbb{Z}_+$, $a \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$(\varphi_j^* f)_a(x) := \sup_{y\in\mathbb{R}^n} \frac{|\varphi_j * f(x+y)|}{(1+2^j|y|)^a},$$

where φ_0 is replaced by Φ and φ_j with $j \in \mathbb{N}$ is as in (1).

Then we have the following conclusion, which was established in [86, Theorem 3.11].

Theorem 12 *Let* p*,* q*,* s *and* ϕ *be as in Definition 7. Let*

$$a \in \left(\frac{n}{\min\{p_-, q_-\}} + \log_2 c_1 + C_{\log}(s), \infty\right),$$

where p_- *and* q_- *are as in Definition 7,* c_1 *is as in (12) and* $C_{\log}(s)$ *as in (4) with* p *replaced by* s*. Then* $f \in F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *if and only if* $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\|f\|^*_{F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} < \infty$,
where

$$\|f\|^*_{F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{j=\max\{j_P,0\}}^{\infty} \left[2^{js(\cdot)}(\varphi_j^* f)_a\right]^{q(\cdot)} \right\}^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}(P)}.$$

Moreover, for any $f \in F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*,* $\|f\|_{F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \sim \|f\|^*_{F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ *with the equivalent positive constants independent of* f.

For the Besov-type spaces with variable exponents, their Peetre maximal function characterizations were also obtained in [87, Theorem 5.1]. Here, for any $f \in \mathcal{S}'(\mathbb{R}^n)$,

$j \in \mathbb{Z}_+$, $a \in (0,\infty)$ and $x \in \mathbb{R}^n$, define

$$\varphi_j^{*,a}(2^{js(\cdot)}f)(x) := \sup_{y\in\mathbb{R}^n} \frac{2^{js(y)}|\varphi_j * f(y)|}{(1+2^j|x-y|)^a},$$

where φ_0 is replaced by Φ and φ_j with $j \in \mathbb{N}$ is as in (1), which is also called the *Peetre maximal function* of f.

Theorem 13 *Let p, q, s, ϕ be as in Definition 7 and $a \in ([n+\log_2 c_1]/p_-, \infty)$, where c_1 is as in (12). Then $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f\|^*_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is finite, where*

$$\|f\|^*_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{\varphi_j^{*,a}(2^{js(\cdot)}f)\right\}_{j\geq \max\{j_P,0\}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}.$$

Moreover, for any $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \sim \|f\|^*_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ with the equivalent positive constants independent of f.

As applications of Theorems 12 and 13, we have two equivalent quasi-norms of the spaces $A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ (see [86, Theorem 3.12], respectively, [87, Theorem 5.5]). To state them, let (φ, Φ) be a pair of admissible functions. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, let

$$\left\| f \left| F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right.\right\|_1 := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{j=0}^{\infty} \left[2^{js(\cdot)}|\varphi_j * f|\right]^{q(\cdot)} \right\}^{\frac{1}{q(\cdot)}} \right\|_{L^{p(\cdot)}(P)},$$

$$\left\| f | B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|_1 := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{2^{js(\cdot)}|\varphi_j * f|\right\}_{j\geq 0} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}$$

and

$$\begin{aligned}\left\| f \left| F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right.\right\|_2 &:= \sup_{Q\in\mathcal{Q}} \sup_{x\in Q} |Q|^{-\frac{s(x)}{n}} [\phi(Q)]^{-1} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} |\varphi_{j_Q} * f(x)| \\ &=: \left\| f \left| B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right.\right\|_2,\end{aligned}$$

where φ_0 is replaced by Φ, φ_j with $j \in \mathbb{N}$ is as in (1) and φ_{j_Q} as in (1) with j replaced by j_Q.

Theorem 14 *Let p, q, s, ϕ be as in Definition 7 with $p_+ \in (0,\infty)$, where p_+ is as in (2). Let p_- be as in (2) and c_1 as in (12).*

(i) *Assume that $c_1 \in (0, 2^{n/p_+})$. Then $f \in A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and the quasi-norm $\|f|A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_1 < \infty$; moreover, there exists a positive constant C, independent of f, such that*

$$C^{-1}\|f\|_{A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \le \left\| f \left| A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|_1 \le C\|f\|_{A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.\right.$$

(ii) *Assume that* $c_1 \in (0, 2^{-n/p_-})$. *Then* $f \in A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *if and only if* $f \in \mathcal{S}'(\mathbb{R}^n)$ *and the quasi-norm* $\|f|A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\|_2 < \infty$; *moreover, there exists a positive constant* C, *independent of* f, *such that*

$$C^{-1}\|f\|_{A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \le \left\| f \left| A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|_2 \le C\|f\|_{A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.\right.$$

5.3 A Trace Theorem

In this subsection, we present the properties of the trace operator on spaces $A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ obtained in [86, 87] and begin with some notation.

For measurable functions p, q, s and a set function ϕ being as in Definition 7, let $A^{s(\widetilde{\cdot},0),\widetilde{\phi}}_{p(\widetilde{\cdot},0),q(\widetilde{\cdot},0)}(\mathbb{R}^{n-1})$ denote either the Besov-type or the Triebel-Lizorkin-type spaces with variable exponents $p(\widetilde{\cdot},0)$, $q(\widetilde{\cdot},0)$ and $s(\widetilde{\cdot},0)$ on $\mathbb{R}^{n-1}\times\{0\}$, where $\widetilde{\phi}$ is defined by setting, for any cube $\widetilde{Q}$ of $\mathbb{R}^{n-1}$, $\widetilde{\phi}(\widetilde{Q}) := \phi(\widetilde{Q}\times[0,\ell(\widetilde{Q})))$. In what follows, let $\mathbb{R}^n_+ := \mathbb{R}^{n-1}\times[0,\infty)$ and $\mathbb{R}^n_- := \mathbb{R}^{n-1}\times(-\infty,0]$.

Let $f \in A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Then, by Theorem 11, we have $f = \sum_{Q\in\mathcal{Q}^*} t_Q a_Q$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\left\|\{t_Q\}_{Q\in\mathcal{Q}^*}\right\|_{a^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)},$$

where C is a positive constant independent of f and, for each $Q\in\mathcal{Q}^*$, $t_Q\in\mathbb{C}$ and a_Q is a smooth atom of $A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Define the *trace* of f by setting, for any $\widetilde{x}\in\mathbb{R}^{n-1}$,

$$\mathrm{Tr}(f)(\widetilde{x}) := \sum_{Q\in\mathcal{Q}^*} t_Q a_Q(\widetilde{x},0). \tag{13}$$

This definition of $\mathrm{Tr}(f)$ is determined canonically for any $f \in A^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, since the actual construction of a_Q in the proof of Theorem 11 implies that $t_Q a_Q$ is obtained canonically. Moreover, the summation in (13) converges in $\mathcal{S}'(\mathbb{R}^{n-1})$ (see [86, Lemma 4.3] and [87, Lemma 6.3]) and the trace operator is well defined.

The trace theorem is stated as follows; see [86, Theorem 4.1] and [87, Theorem 6.1].

Theorem 15 *Let* $n\ge 2$, $p,\ q\in\mathcal{P}(\mathbb{R}^n)\cap C^{\log}(\mathbb{R}^n)$, $s\in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ *and* $\phi\in\mathcal{G}(\mathbb{R}^{n+1}_+)$. *Let* p_- *be as in* (2) *and* s_- *as in* (2) *with* p *replaced by* s. *If*

$$s_- - \frac{1}{p_-} - (n-1)\left[\frac{1}{\min\{1,p_-\}} - 1\right] > 0,$$

then

$$\mathrm{Tr}\, F^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\widetilde{\cdot},0)-\frac{1}{p(\widetilde{\cdot},0)},\widetilde{\phi}}_{p(\widetilde{\cdot},0),\,p(\widetilde{\cdot},0)}(\mathbb{R}^{n-1})$$

and

$$\mathrm{Tr}\, B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = B^{s(\widetilde{\cdot},0)-\frac{1}{p(\widetilde{\cdot},0)},\widetilde{\phi}}_{p(\widetilde{\cdot},0),q(\widetilde{\cdot},0)}(\mathbb{R}^{n-1}).$$

6 The Variable Hardy Spaces Associated with Operators

The purpose of this section is to make an introduction of the variable Hardy spaces associated with operators introduced and investigated in [84, 90]. We first make the following two assumptions on the operator L considered in this section.

Assumption 1 Assume that the operator L is one-to-one, has dense range in $L^2(\mathbb{R}^n)$ and a bounded H^∞ functional calculus on $L^2(\mathbb{R}^n)$.

Assumption 2 The kernels $\{K_t\}_{t\in(0,\infty)}$ of $\{e^{-tL}\}_{t\in(0,\infty)}$ are bounded measurable functions on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfy that, for any $x,\ y \in \mathbb{R}^n$,

$$|K_t(x,y)| \le t^{-\frac{n}{m}} g\left(\frac{|x-y|}{t^{\frac{1}{m}}}\right), \tag{14}$$

where m is a positive constant and g is a positive, bounded and decreasing function satisfying that, for some $\varepsilon \in (0,\infty)$,

$$\lim_{r\to\infty} r^{n+\varepsilon} g(r) = 0. \tag{15}$$

For any $\beta \in (0,\infty)$, let $\mathcal{M}_\beta(\mathbb{R}^n)$ be the *set* of all functions $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathcal{M}_\beta(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} \frac{|f(x)|^2}{1+|x|^{n+\beta}}\,dx\right\}^{1/2} < \infty.$$

We point out that the space $\mathcal{M}_\beta(\mathbb{R}^n)$ was introduced by Duong and Yan in [28] and it is a Banach space under the norm $\|\cdot\|_{\mathcal{M}_\beta(\mathbb{R}^n)}$. Let

$$\theta(L) := \sup\{\varepsilon \in (0,\infty) :\ (14) \text{ and } (15) \text{ hold true}\} \tag{16}$$

and

$$\mathcal{M}(\mathbb{R}^n) := \begin{cases} \mathcal{M}_{\theta(L)}(\mathbb{R}^n) & \text{if } \theta(L) < \infty, \\ \bigcup_{\beta\in(0,\infty)} \mathcal{M}_\beta(\mathbb{R}^n) & \text{if } \theta(L) = \infty. \end{cases}$$

Let $s \in \mathbb{Z}_+$. For any $f \in \mathcal{M}(\mathbb{R}^n)$ and $(x,t) \in \mathbb{R}^{n+1}_+$, let

$$P_{s,t}f(x) := f(x) - (I - e^{-tL})^{s+1}f(x) \quad \text{and} \quad Q_{s,t}f(x) := (tL)^{s+1}e^{-tL}f(x).$$

For any function $f \in L^2(\mathbb{R}^n)$, define the *Lusin area function* $S_L(f)$ by setting, for any $x \in \mathbb{R}^n$,

$$S_L(f)(x) := \left\{ \int_{\Gamma(x)} \left| Q_{0,t^m} f(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}.$$

Definition 10 Let L be an operator satisfying Assumptions 1 and 2, and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $0 < p_- \le p_+ \le 1$ with p_- and p_+ as in (2). A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\widetilde{H}_L^{p(\cdot)}(\mathbb{R}^n)$ if $S_L(f) \in L^{p(\cdot)}(\mathbb{R}^n)$; moreover, define $\|f\|_{H_L^{p(\cdot)}(\mathbb{R}^n)} := \|S_L(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Then the *variable Hardy space associated with the operator* L, denoted by $H_L^{p(\cdot)}(\mathbb{R}^n)$, is defined to be the completion of $\widetilde{H}_L^{p(\cdot)}(\mathbb{R}^n)$ in the quasi-norm $\|\cdot\|_{H_L^{p(\cdot)}(\mathbb{R}^n)}$.

Remark 8 Let $\varphi: \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ be a growth function as in [50] and L an operator satisfying the reinforced off-diagonal estimates in [12]. Then Bui et al. [12] introduced the Musielak-Orlicz-Hardy space associated with the operator L via the Lusin area function. By the same reason as that used in Remark 1(iv), we find that Musielak-Orlicz-Hardy spaces associated with operators in [12] and variable exponent Hardy spaces associated with operators in [84] do not cover each other (see also [84, Remark 2.8]).

6.1 The Molecular Characterization and the Duality

In what follows, for any $q \in (0,\infty)$, let $L^q(\mathbb{R}_+^{n+1})$ be the *set of all* q*-integrable functions on* $\mathbb{R}_+^{n+1}$ and $L^q_{\mathrm{loc}}(\mathbb{R}_+^{n+1})$ the *set of all locally* q*-integrable functions on* $\mathbb{R}_+^{n+1}$. For any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, let

$$s_0 := \lfloor (n/m)(1/p_- - 1) \rfloor, \tag{17}$$

namely, s_0 denotes the largest integer not bigger than $\frac{n}{m}(\frac{1}{p_-} - 1)$, where p_- is as in (2).

Let m be as in (14) and $s \in [s_0, \infty)$. Let $C_{(m,s)}$ be a positive constant, depending on m and s, such that

$$C_{(m,s)} \int_0^\infty t^{m(s+2)} e^{-2t^m} (1 - e^{-t^m})^{s_0+1} \frac{dt}{t} = 1.$$

Let $q \in (0,\infty)$. Recall that the *tent space* $T_2^q(\mathbb{R}_+^{n+1})$ is defined to be the set of all measurable functions g on $\mathbb{R}_+^{n+1}$ such that

$$\|g\|_{T_2^q(\mathbb{R}_+^{n+1})} := \left\| \left\{ \int_{\Gamma(\cdot)} |g(y,t)|^2 \frac{dydt}{t^{n+1}} \right\}^{1/2} \right\|_{L^q(\mathbb{R}^n)} < \infty.$$

Let $T_{2,c}^q(\mathbb{R}_+^{n+1})$ be the set of all functions in $T_2^q(\mathbb{R}_+^{n+1})$ with compact supports and define the *operator* π_L by setting, for any $f \in T_{2,c}^q(\mathbb{R}_+^{n+1})$ and $x \in \mathbb{R}^n$,

$$\pi_L(f)(x) := C_{(m,s)} \int_0^\infty Q_{s,t^m}(I - P_{s_0,t^m})(f(\cdot,t))(x) \frac{dt}{t}.$$

We now recall the notion of the molecule introduced in [84, Definition 3.11]. In what follows, for any $(y,t) \in \mathbb{R}_+^{n+1}$, let $B(y,t) := \{x \in \mathbb{R}^n : |x-y| < t\}$.

Definition 11 Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \le p_+ \le 1$ and $s \in [s_0, \infty)$ with s_0 as in (17), where p_- and p_+ are as in (2). A measurable function α on $\mathbb{R}^n$ is called a $(p(\cdot), s, L)$-*molecule* if $\alpha(x) = \pi_L(a)(x)$ for any $x \in \mathbb{R}^n$, where a is a measurable function on $\mathbb{R}_+^{n+1}$ such that supp $a \subset \widehat{Q} := \{(y,t) \in \mathbb{R}_+^{n+1} : B(y,t) \subset Q\}$ for some cube $Q \subset \mathbb{R}^n$ and, for any $q \in (1,\infty)$,

$$\left\| \left\{ \int_{\Gamma(\cdot)} |a(y,t)|^2 \frac{dydt}{t^{n+1}} \right\}^{1/2} \right\|_{L^q(\mathbb{R}^n)} \le |Q|^{1/q} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}.$$

When it is necessary to specify the cube Q, then a is called a $(p(\cdot), s, L)$-*molecule associated with* Q.

The molecular characterization of $H_L^{p(\cdot)}(\mathbb{R}^n)$ is stated as follows (see [84, Theorem 3.13]).

Theorem 16 *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy*

$$0 < p_- \le p_+ \le 1 \text{ and } p_- \in \left(\frac{n}{n+\theta(L)}, 1 \right],$$

and $s \in [s_0, \infty)$ with p_+, p_-, $\theta(L)$ and s_0, respectively, as in (2), (16) and (17).

(i) *If $f \in H_L^{p(\cdot)}(\mathbb{R}^n)$, then there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_{j\in\mathbb{N}}$ of $(p(\cdot), s, L)$-molecules, respectively, associated with cubes $\{Q_j\}_{j\in\mathbb{N}}$ such that $f = \sum_{j\in\mathbb{N}} \lambda_j \alpha_j$ in $H_L^{p(\cdot)}(\mathbb{R}^n)$ and*

$$\mathcal{B}(\{\lambda_j\alpha_j\}_{j\in\mathbb{N}}) := \left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{|\lambda_j|\chi_{Q_j}}{\|Q_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \|f\|_{H_L^{p(\cdot)}(\mathbb{R}^n)} \tag{18}$$

with C being a positive constant independent of f and $\underline{p}$ as in (3).

(ii) *Suppose that* $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ *and* $\{\alpha_k\}_{k\in\mathbb{N}}$ *is a family of* $(p(\cdot), s, L)$*-molecules satisfying* $\mathcal{B}(\{\lambda_k\alpha_k\}_{k\in\mathbb{N}}) < \infty$. *Then* $\sum_{k\in\mathbb{N}} \lambda_k\alpha_k$ *converges in* $H_L^{p(\cdot)}(\mathbb{R}^n)$ *and*

$$\left\|\sum_{k\in\mathbb{N}} \lambda_k\alpha_k\right\|_{H_L^{p(\cdot)}(\mathbb{R}^n)} \le C\mathcal{B}(\{\lambda_k\alpha_k\}_{k\in\mathbb{N}})$$

with C *being a positive constant independent of* $\{\lambda_k\alpha_k\}_{k\in\mathbb{N}}$.

To present the duality of $H_L^{p(\cdot)}(\mathbb{R}^n)$, we need to recall the following BMO-type space associated with the operator L and the variable exponent $p(\cdot)$ (see [84, Definition 4.1]).

Definition 12 Let L satisfy Assumptions 1 and 2, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $0 < p_- \le p_+ \le 1$ and $s \in [s_0, \infty)$, where p_-, p_+ and s_0 are, respectively, as in (2) and (17). Then the BMO-*type space* $\mathrm{BMO}^s_{p(\cdot),L}(\mathbb{R}^n)$ is defined to be the set of all functions $f \in \mathcal{M}(\mathbb{R}^n)$ such that $\|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbb{R}^n)} := \sup_{Q\subset\mathbb{R}^n} \frac{|Q|^{1/2}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\{\int_Q |f(x) - P_{s,(r_Q)^m} f(x)|^2\,dx\right\}^{\frac{1}{2}}$$

and the supremum is taken over all cubes Q of $\mathbb{R}^n$.

In what follows, for any $s \in [s_0, \infty)$ with s_0 as in (17) and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, denote by $H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbb{R}^n)$ the *set of all finite linear combinations of* $(p(\cdot), s, L)$*-molecules*. For any $f \in H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbb{R}^n)$, its quasi-norm is defined by

$$\|f\|_{H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbb{R}^n)} := \inf\left\{\mathcal{B}(\{\lambda_j\alpha_j\}_{j=1}^N) : \ N \in \mathbb{N},\ f = \sum_{j=1}^N \lambda_j\alpha_j\right\},$$

where $\mathcal{B}(\{\lambda_j\alpha_j\}_{j=1}^N)$ is as in (18) and the infimum is taken over all finite molecular decompositions of f.

Now we present the following dual theorem established in [84, Theorem 4.3].

Theorem 17 *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *satisfy* $p_+ \in (0, 1]$ *and* $p_- \in (\frac{n}{n+\theta(L)}, 1]$ *with* p_+, p_- *and* $\theta(L)$, *respectively, as in* (2) *and* (16). *Let* s_0 *be as in* (17) *and* L^* *denote the adjoint operator of* L *on* $L^2(\mathbb{R}^n)$. *Then* $(H_L^{p(\cdot)}(\mathbb{R}^n))^*$ *coincides with* $\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbb{R}^n)$ *in the following sense:*

(i) *If* $g \in \mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbb{R}^n)$, *then the linear mapping* ℓ, *which is initially defined on* $H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbb{R}^n)$ *by*

$$\ell_g(f) := \int_{\mathbb{R}^n} f(x)g(x)\,dx, \tag{19}$$

extends to a bounded linear functional on $H_L^{p(\cdot)}(\mathbb{R}^n)$ *and*

$$\|\ell_g\|_{(H_L^{p(\cdot)}(\mathbb{R}^n))^*} \le C\|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbb{R}^n)},$$

where C is a positive constant independent of g.

(ii) *Conversely, let ℓ be a bounded linear functional on $H_L^{p(\cdot)}(\mathbb{R}^n)$. Then ℓ has the form as in (19) with a unique $g \in \mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbb{R}^n)$ for any $f \in H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbb{R}^n)$ and*

$$\|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbb{R}^n)} \le \widetilde{C}\|\ell\|_{(H_L^{p(\cdot)}(\mathbb{R}^n))^*},$$

where $\widetilde{C}$ is a positive constant independent of ℓ.

We also present the boundedness of the fractional integrals on these Hardy spaces, which was obtained in [84, Theorem 5.9]. Recall that, for any $\gamma \in (0, \frac{n}{m})$ with m as in Assumption 1, the generalized fractional integral $L^{-\gamma}$ associated with L is defined by setting, for any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$L^{-\gamma}(f)(x) := \frac{1}{\Gamma(\gamma)}\int_0^\infty t^{\gamma-1}e^{-tL}(f)(x)\,dt.$$

Theorem 18 *Let L satisfy Assumptions 1 and 2, $\gamma \in (0, \frac{n}{m})$ with m as in Assumption 1, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $\frac{n}{n+\theta(L)} < p_- \le p_+ \le 1$ with p_-, p_+ and $\theta(L)$, respectively, as in (2) and (16). Assume that $q(\cdot)$ is defined by setting $\frac{1}{q(x)} := \frac{1}{p(x)} - \frac{m\gamma}{n}$ for any $x \in \mathbb{R}^n$. Then the fractional integral $L^{-\gamma}$ maps $H_L^{p(\cdot)}(\mathbb{R}^n)$ continuously into $H_L^{q(\cdot)}(\mathbb{R}^n)$.*

6.2 Atomic and Maximal Function Characterizations

In this subsection, we recall the equivalent characterizations of $H_L^{p(\cdot)}(\mathbb{R}^n)$ via atoms and maximal functions established in [90]. In the following, we assume that L is a densely defined linear operator on $L^2(\mathbb{R}^n)$ and satisfies the following assumptions:

Assumption 3 L is non-negative and self-adjoint;

Assumption 4 The kernels $\{K_t\}_{t>0}$ of the semigroup $\{e^{-tL}\}_{t>0}$ satisfy the Gaussian upper bound estimates, namely, there exist positive constants C and c such that, for any $t \in (0,\infty)$ and $x,\ y \in \mathbb{R}^n$,

$$|K_t(x,y)| \le \frac{C}{t^{n/2}}\exp\left\{-\frac{|x-y|^2}{ct}\right\}.$$

Definition 13 Let $q \in (1,\infty]$ and $M \in \mathbb{N}$, L satisfy Assumptions 3 and 4, $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $0 < p_- \le p_+ \le 1$ with p_- and p_+ as in (2).

(I) Let $\mathcal{D}(L^M)$ be the domain of L^M and $Q \subset \mathbb{R}^n$ a cube. A function $\alpha \in L^q(\mathbb{R}^n)$ is called a $(p(\cdot), q, M)_L$*-atom* associated with the cube Q if there exists a function $b \in \mathcal{D}(L^M)$ such that

(i) $\alpha = L^M b$ and, for any $j \in \{0, 1, \ldots, M\}$, supp $(L^j b) \subset Q$;

(ii) for any $j \in \{0, 1, \ldots, M\}$,

$$\|([\ell(Q)]^2 L)^j b\|_{L^q(\mathbb{R}^n)} \le [\ell(Q)]^{2M} |Q|^{1/q} \|\chi_Q\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\ell(Q)$ denotes the *side length* of Q.

(II) Let $f \in L^2(\mathbb{R}^n)$. Then

$$f = \sum_{j\in\mathbb{N}} \lambda_j \alpha_j \tag{20}$$

is called an *atomic* $(p(\cdot), q, M)_L$*-representation of* f if $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and $\{\alpha_j\}_{j\in\mathbb{N}}$ are $(p(\cdot), q, M)_L$-atoms, respectively, associated with cubes $\{Q_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^n$ such that (20) converges in $L^2(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \left\{ \sum_{j\in\mathbb{N}} \left[\frac{|\lambda_j| \chi_{Q_j}(x)}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_-} \right\}^{p(x)/p_-} dx < \infty.$$

(III) Let

$$\mathbb{H}^{p(\cdot),q}_{L,\mathrm{at},M}(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) : \ f \text{ has an atomic } (p(\cdot), q, M)_L\text{-representation} \right\}$$

be equipped with the quasi-norm $\|f\|_{H^{p(\cdot),q}_{L,\mathrm{at},M}(\mathbb{R}^n)}$ which is defined by

$$\inf \left\{ \mathcal{B}(\{\lambda_j \alpha_j\}_{j\in\mathbb{N}}) : \sum_{j\in\mathbb{N}} \lambda_j \alpha_j \text{ is an atomic } (p(\cdot), q, M)_L\text{-representation of } f \right\},$$

where $\mathcal{B}(\{\lambda_j \alpha_j\}_{j\in\mathbb{N}})$ is as in (18) and the infimum is taken over all the atomic $(p(\cdot), q, M)_L$-representations of f as above. The *atomic variable exponent Hardy space* $H^{p(\cdot),q}_{L,\mathrm{at},M}(\mathbb{R}^n)$ is then defined to be the completion of the set $\mathbb{H}^{p(\cdot),q}_{L,\mathrm{at},M}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{p(\cdot),q}_{L,\mathrm{at},M}(\mathbb{R}^n)}$.

The following conclusion is just [90, Theorem 1.8].

Theorem 19 *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *with* $0 < p_- \le p_+ \le 1$, $q \in (1, \infty]$, $M \in (\frac{n}{2}[\frac{1}{p_-} - 1], \infty) \cap \mathbb{N}$ *and* L *be a linear operator on* $L^2(\mathbb{R}^n)$ *satisfying Assumptions*

3 and 4, where p_- and p_+ are as in (2). Then $H^{p(\cdot),q}_{L,\mathrm{at},M}(\mathbb{R}^n)$ and $H^{p(\cdot)}_L(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

Recall that, if L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$, then, for any bounded Borel measurable function $F: [0,\infty)\to\mathbb{C}$, the operator $F(L)$, defined by the formula

$$F(L) := \int_0^\infty F(\lambda)\,dE_L(\lambda),$$

where $E_L(\lambda)$ denotes the spectral decomposition associated with L, is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ (see, for example, [42]). Particularly, if $\phi\in\mathcal{S}(\mathbb{R})$ is an even function, then, for any $t\in(0,\infty)$, the operator $\phi(t\sqrt{L})$ is bounded on $L^2(\mathbb{R}^n)$.

Definition 14 (i) Let $\phi\in\mathcal{S}(\mathbb{R})$ be an even function with $\phi(0)=1$ and L an operator satisfying Assumptions 3 and 4. For any $a\in(0,\infty)$ and $f\in L^2(\mathbb{R}^n)$, the *non-tangential maximal function* of f is defined by setting, for any $x\in\mathbb{R}^n$,

$$\phi^*_{L,\nabla,a}(f)(x) := \sup_{t\in(0,\infty),\,|y-x|<at}\left|\phi(t\sqrt{L})(f)(y)\right|.$$

A function $f\in L^2(\mathbb{R}^n)$ is said to be in the set $\mathbb{H}^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)$ if $\phi^*_{L,\nabla,a}(f)\in L^{p(\cdot)}(\mathbb{R}^n)$; moreover, define $\|f\|_{H^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)} := \|\phi^*_{L,\nabla,a}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Then the *variable exponent Hardy space* $H^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)$ is defined to be the completion of $\mathbb{H}^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)}$.
Particularly, when $\phi(x):=e^{-x^2}$ for any $x\in\mathbb{R}$, use $f^*_{L,\nabla}$ to denote $\phi^*_{L,\nabla,1}(f)$ and, in this case, denote $H^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)$ simply by $H^{p(\cdot)}_{L,\max}(\mathbb{R}^n)$.

(ii) For any $f\in L^2(\mathbb{R}^n)$, define the *grand non-tangential maximal function* of f by setting, for any $x\in\mathbb{R}^n$, $\mathcal{G}^*_{L,\nabla}(f)(x) := \sup_{\phi\in\mathcal{F}(\mathbb{R})}\phi^*_{L,\nabla,1}(f)(x)$, where $\mathcal{F}(\mathbb{R})$ denotes the set of all even functions $\phi\in\mathcal{S}(\mathbb{R})$ satisfying $\phi(0)\neq 0$ and

$$\sum_{k=0}^{N}\int_{\mathbb{R}}(1+|x|)^N\left|\frac{d^k\phi(x)}{dx^k}\right|^2\,dx\le 1$$

with N being a large enough positive integer depending on p_- and n. Then the *variable exponent Hardy space* $H^{p(\cdot),\mathcal{F}}_{L,\max}(\mathbb{R}^n)$ is defined in the same way as $H^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)$ but with $\phi^*_{L,\nabla,a}(f)$ replaced by $\mathcal{G}^*_{L,\nabla}(f)$.

The following conclusion was proved in [90, Theorem 1.11], which, when $p(\cdot)\equiv p\in(0,1)$, coincides with [68, Theorem 1.4 and Corollary 3.2].

Theorem 20 *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $p_+\in(0,1]$, $q\in(1,\infty]$,*

$$M\in\left(\frac{n}{2}\left[\frac{1}{p_-}-1\right],\infty\right)$$

and L be a linear operator on $L^2(\mathbb{R}^n)$ satisfying Assumptions 3 and 4, where p_- and p_+ are as in (2). Then, for any $a \in (0, \infty)$ and ϕ as in Definition 14, the spaces $H^{p(\cdot),q}_{L,\mathrm{at},M}(\mathbb{R}^n)$, $H^{p(\cdot),\mathcal{F}}_{L,\max}(\mathbb{R}^n)$ and $H^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

Definition 15 (i) Let $\phi \in \mathcal{S}(\mathbb{R})$ be an even function with $\phi(0) = 1$. For any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $\phi^*_{L,+}(f)(x) := \sup_{t\in(0,\infty)} |\phi(t\sqrt{L})(f)(x)|$. Particularly, when $\phi(x) := e^{-x^2}$ for any $x \in \mathbb{R}$, use $f^*_{L,+}$ to denote $\phi^*_{L,+}(f)$. The *variable exponent radial Hardy space* $H^{p(\cdot)}_{L,\mathrm{rad}}(\mathbb{R}^n)$ is defined in the same way as $H^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)$ but with $\phi^*_{L,\nabla,a}(f)$ replaced by $f^*_{L,+}$.

(ii) For any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $\mathcal{G}^*_{L,+}(f)(x) := \sup_{\phi\in\mathcal{F}(\mathbb{R})} \phi^*_{L,+}(f)(x)$. The *variable exponent radial Hardy space* $H^{p(\cdot),\mathcal{F}}_{L,\mathrm{rad}}(\mathbb{R}^n)$ is defined in the same way as $H^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)$ but with $\phi^*_{L,\nabla,a}(f)$ replaced by $\mathcal{G}^*_{L,+}(f)$.

Theorem 21 *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $0 < p_- \le p_+ \le 1$, where p_- and p_+ are as in (2), and let L be a linear operator on $L^2(\mathbb{R}^n)$ satisfying Assumptions 3 and 4 and assume that there exist positive constants C and $\mu \in (0, 1]$ such that, for any $t \in (0, \infty)$ and $x,\ y_1,\ y_2 \in \mathbb{R}^n$,*

$$|K_t(y_1, x) - K_t(y_2, x)| \le \frac{C}{t^{n/2}} \frac{|y_1 - y_2|^\mu}{t^{\mu/2}}.$$

If $q \in (1, \infty]$ and

$$M \in (\frac{n}{2}[\frac{1}{p_-} - 1], \infty) \cap \mathbb{N},$$

then the spaces $H^{p(\cdot),q}_{L,\mathrm{at},M}(\mathbb{R}^n)$, $H^{p(\cdot)}_{L,\max}(\mathbb{R}^n)$ and $H^{p(\cdot)}_{L,\mathrm{rad}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

Theorem 21 was proved in [90, Theorem 1.17]. Combining Theorems 19, 20 and 21, we immediately obtain the following conclusion.

Corollary 2 *Let $p(\cdot)$, L, q and M be as in Theorem 21. Then, for any $a \in (0, \infty)$ and ϕ being as in Definition 14, the spaces $H^{p(\cdot)}_L(\mathbb{R}^n)$, $H^{p(\cdot),q}_{L,\mathrm{at},M}(\mathbb{R}^n)$, $H^{p(\cdot),\phi,a}_{L,\max}(\mathbb{R}^n)$, $H^{p(\cdot),\mathcal{F}}_{L,\max}(\mathbb{R}^n)$, $H^{p(\cdot)}_{L,\mathrm{rad}}(\mathbb{R}^n)$ and $H^{p(\cdot),\mathcal{F}}_{L,\mathrm{rad}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

At the end of this section, we give the following remark.

Remark 9 Let L be a one-to-one operator of type ω on $L^2(\mathbb{R}^n)$, with $\omega \in [0, \pi/2)$, which has a bounded holomorphic functional calculus and satisfies Davies-Gaffney estimates. Let $p(\cdot) :\ \mathbb{R}^n \to (0,\ 1]$ belong to $C^{\log}(\mathbb{R}^n)$. Then Yang et al. [82] introduced and investigated the variable Hardy space $H^{p(\cdot)}_L(\mathbb{R}^n)$ associated with L, which is a generalization of the variable Hardy space associated with operator whose heat kernel satisfies certain pointwise upper bounded in [84, 90].

7 Further Notes

Although the theory of variable function spaces has achieved great progress since 1990s, there still exist a lot of unsolved interesting questions related to this subject. We finish this survey with some of such open questions as follows:

(i) In [38], Han, Müller and Yang systematically developed a theory of Besov and Triebel-Lizorkin spaces on RD-spaces. On the other hand, as was mentioned in the introduction and Remark 3, a theory of variable Hardy spaces on RD-space was recently established in [89]. Inspired by these, it is natural to ask whether or not one can develop variable Besov and Triebel-Lizorkin spaces and, more general, variable Besov-type and Triebel-Lizorkin-type spaces on RD-spaces or even on spaces of homogeneous type in the sense of Coifman and Weiss ([15, 16])?

(ii) In recent years, Besov and Triebel-Lizorkin spaces associated with operators have attracted certain attentions; see, for example, [13, 45, 54]. It is interesting to establish the corresponding variable theory.

(iii) Notice that the condition $p(\cdot) \in C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ implies that $p(\cdot)$ is a continuous function. It would be very interesting to know whether or not one can develop variable Besov and Triebel-Lizorkin spaces for some discontinuous variable exponent functions.

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