The Schwarz Lemma for Super-Conformal Maps

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Abstract A super-conformal map is a conformal map from a two-dimensional Riemannian manifold to the Euclidean four-space such that the ellipse of curvature is a circle. Quaternionic holomorphic geometry connects super-conformal maps with holomorphic maps. We report the Schwarz lemma for super-conformal maps and related results.

1 Introduction

For a smooth manifold M, we denote the tangent bundle by TM and its fiber at $p \in M$ by T_pM. Let Σ be a two-dimensional oriented Riemannian manifold and $f : \Sigma \to \mathbb{R}^4$ be an isometric immersion. We denote the Riemannian metric of Σ by g. For a tangent vector $X \in T_p\Sigma$, we denote the norm with respect to the Riemannian metric by $\|X\|$. We denote the second fundamental form of f by h. Then

$$\mathscr{E}_p = \left\{h(X, X) : X \in T_p\Sigma, \|X\| = 1\right\}$$

is called the *ellipse of curvature* or the *curvature ellipse* of f at $p \in M$ [9]. It is indeed an ellipse in the normal space at p if it does not degenerate to a point or a line segment. If the ellipse of curvature is a circle or a point at any point p, then f is said to be *super-conformal* [2].

The author showed that a super-conformal map is a Bäcklund transform of a minimal surface [6]. Regarding f as an isometric immersion, the inequality

$$|\mathscr{H}|^2 - K - |K^{\perp}| \geq 0 \tag{1}$$

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holds between the mean curvature vector $\mathscr{H}$, the Gaussian curvature K and the normal curvature $K^{\perp}$ [10]. The equality holds if and only if f is super-conformal. From this point of view, a super-conformal map is called a Wintgen ideal surface [8]. The integral of the left-hand side of (1) over Σ is the Willmore energy of f. This implies that a super-conformal map is a Willmore surface with vanishing Willmore energy. Hence the super-conformality is invariant under Möbius transforms of $\mathbb{R}^4$.

We discuss the Möbius geometry of super-conformal immersion by exchanging a two-dimensional oriented Riemannian manifold with a Riemann surface and an isometric immersion with a conformal immersion. Regarding $\mathbb{C}$ as a subspace of $\mathbb{R}^4$ and a holomorphic function on Σ as a map form Σ to $\mathbb{R}^4$, a holomorphic function satisfies (1) and it is super-conformal. We may regard Möbius geometric theory of holomorphic functions on a Riemann surface as a special case of Möbius geometry of super-conformal immersion.

The author [7] discussed super-conformal maps as a higher codimensional analogue of holomorphic functions and meromorphic functions. In this paper, we report a part of the paper which discusses the Schwarz lemma for super-conformal maps.

For the discussion, we use quaternionic holomorphic geometry [3]. Quaternionic holomorphic geometry of surfaces in $\mathbb{R}^4$ connects theory of holomorphic functions with theory of surfaces in $\mathbb{R}^4$.

2 Classical Results

We begin with reviewing the classical results of super-conformal maps by Friedrich [4] and Wong [11].

Throughout this paper, all manifolds and maps are assumed to be smooth. We compute the ellipse of curvature. We denote the inner product of $\mathbb{R}^4$ by $\langle\ ,\ \rangle$. Let e_1, e_2, e_3, e_4 be an adapted orthonormal local frame of the pull-back bundle $f^*T\mathbb{R}^4$ and θ_1, θ_2, θ_3, θ_4 the dual frame. Assume that the second fundamental form is

$$h = \sum_{p=3}^{4} \sum_{i,j=1}^{2} h_{ijp}\, \theta_i \otimes \theta_j \otimes e_p.$$

Then the ellipse of curvature is parametrized by the map

$$\begin{aligned} \varepsilon(u) &= h(e_1 \cos u + e_2 \sin u, e_1 \cos u + e_2 \sin u) \\ &= \mathscr{H} + \left(\frac{h_{113} - h_{223}}{2} e_3 + \frac{h_{114} - h_{224}}{2} e_4 \right) \cos 2u + (h_{123}e_3 + h_{124}e_4) \sin 2u. \end{aligned}$$

The map f is super-conformal map if and only if $\varepsilon(u)$ parametrizes a circle. The map f is minimal if and only if $\varepsilon(u)$ parametrize a curve of the linear transform of the circle centered at the origin. The linear transform is given by

$$P\left(e_3\ e_4\right) = \left(e_3\ e_4\right)\begin{pmatrix} h_{113} & h_{123} \\ h_{114} & h_{124} \end{pmatrix}$$

Hence f is super-conformal and minimal if and only if the ellipse of curvature is a circle centered at the origin.

We normalize the second fundamental form and the ellipse of curvature. Let $n(u) = e_3 \cos u + e_4 \sin u$. Because

$$\langle h, n(u)\rangle = \sum_{i,j=1}^{2} h_{ij3}\,\theta_i \otimes \theta_j \cos u + \sum_{i,j=1}^{2} h_{ij4}\,\theta_i \otimes \theta_j \sin u,$$
$$\begin{aligned} \mathrm{tr}\langle h, n(u)\rangle &= h_{113}\cos u + h_{114}\sin u + h_{223}\cos u + h_{224}\sin u \\ &= (h_{113} + h_{223})\cos u + (h_{114} + h_{224})\sin u, \end{aligned}$$

we may assume that $h_{114} + h_{224} = 0$. Let A_{e_4} be the shape operator such that $\langle A_{e_4}(X), Y\rangle = \langle h(X, Y), e_4\rangle$ for any $X, Y \in T_p\Sigma$. Taking e_1 and e_2 as the eigenvectors of A_{e_4}, we may assume that $h_{124} = 0$. The ellipse of curvature becomes

$$\varepsilon(u) = \frac{h_{113} + h_{223}}{2}e_3 + \left(\frac{h_{113} - h_{223}}{2}e_3 + h_{114}e_4\right)\cos 2u + (h_{123}e_3)\sin 2u.$$

Then f is super-conformal if and only if

$$(h_{113} - h_{223})h_{123} = 0, \quad \left(\frac{h_{113} - h_{223}}{2}\right)^2 + h_{114}^2 = h_{123}^2$$

This is equivalent to

$$h_{123} = h_{114} = 0,\ \ h_{113} = h_{223} \text{ or } h_{113} = h_{223},\ \ h_{114}^2 = h_{123}^2.$$

Hence the ellipse of curvature of a super-conformal map becomes

$$\varepsilon(u) = 0 \text{ or } \varepsilon(u) = h_{113} + (h_{114}e_4)\cos 2u + (\pm h_{114}e_3)\sin 2u.$$

If f is minimal, then the ellipse of curvature is

$$\varepsilon(u) = (h_{113}e_3 + h_{114}e_4)\cos 2u + (h_{123}e_3)\sin 2u.$$

Hence f is super-conformal and minimal if and only if

$$\varepsilon(u) = (h_{114}e_4)\cos 2u + (\pm h_{114}e_3)\sin 2u.$$

Another notion is defined by the second fundamental form for surfaces in $\mathbb{R}^4$.

Definition 1 ([5, 11]) The set

$$\mathscr{I}_p = \left\{ \langle h, n \rangle : n \in T_p\Sigma^{\perp}, \|n\| = 1 \right\}.$$

is called the *indicatrix* of the normal curvature or the *Kommerell conic* of f.

The indicatrix is parametrized by

$$\iota(u) = \langle h, n(u) \rangle = \sum_{i,j=1}^{2} (h_{ij3} \cos u + h_{ij4} \sin u) \theta_i \otimes \theta_j.$$

By the normalization, we may assume that

$$\begin{aligned} \iota(u) = (h_{113} \cos u + h_{114} \sin t)\theta_1 \otimes \theta_1 + h_{123} \cos u\, \theta_1 \otimes \theta_2 \\ + h_{213} \cos u\, \theta_2 \otimes \theta_1 + (h_{223} \cos u - h_{114} \sin u)\theta_2 \otimes \theta_2. \end{aligned}$$

We regard $\langle h, n(u) \rangle$ as the shape operator which is is a symmetric $(1, 1)$-tensor. With the standard inner product of symmetric $(1, 1)$-tensors, the curve $\langle h, n(u) \rangle$ is isometrically the curve parametrized by

$$\iota(u) = \left(\frac{1}{\sqrt{2}} (h_{113} \cos u + h_{114} \sin u), \sqrt{2} h_{123} \cos u, \frac{1}{\sqrt{2}} (h_{223} \cos u - h_{114} \sin u) \right)$$

in $\mathbb{R}^3$. Hence f is super-conformal if and only if the indicatrix is parametrized by

$$\iota(u) = \left(\frac{1}{\sqrt{2}} (h_{113} \cos u), 0, \frac{1}{\sqrt{2}} (h_{113} \cos u) \right)$$

or

$$\iota(u) = \left(\frac{1}{\sqrt{2}} (h_{113} \cos u + h_{114} \sin u), \pm\sqrt{2} h_{114} \cos u, \frac{1}{\sqrt{2}} (h_{113} \cos u - h_{114} \sin u) \right).$$

We see that f is minimal if and only if the indicatrix is parametrized by

$$\iota(u) = \left(\frac{1}{\sqrt{2}} (h_{113} \cos u + h_{114} \sin u), \sqrt{2} h_{123} \cos u, \frac{1}{\sqrt{2}} (-h_{113} \cos u - h_{114} \sin u) \right).$$

Moreover, f is super-conformal and minimal if and only if

$$\iota(u) = 0 \text{ or } \iota(u) = \left(\frac{1}{\sqrt{2}} (h_{114} \sin u), \pm\sqrt{2} h_{114} \cos u, \frac{1}{\sqrt{2}} (-h_{114} \sin u) \right).$$

Set

$$\mathscr{I}_p(X) = \left\{\langle h(X,\ ), n\rangle : n \in T_p\Sigma^{\perp}, \|n\| = 1\right\} \subset (T_p\Sigma)^*.$$

Definition 2 ([4]) An immersion f is called *superminimal* if $\mathscr{I}_p(X)$ is a circle centered at 0 in $(T_p\Sigma)^*$.

The following lemma explains the relation among the super-conformal maps, minimal surfaces and superminimal surfaces.

Lemma 1 *A map f is superminimal if and only if f is super-conformal and minimal.*

Proof For $X = X_1e_1 + X_2e_2$ and the normalization, we have

$$\begin{aligned}\langle h(X,\ ), n(u)\rangle &= \sum_{i,j=1}^{2} X_i h_{ij3}\theta_j \cos u + \sum_{i,j=1}^{2} X_i h_{ij4}\theta_j \sin u \\ &= ((X_1h_{113} + X_2h_{123})\theta_1 + (X_1h_{123} + X_2h_{223})\theta_2)\cos u \\ &\quad + (X_1h_{114}\theta_1 - X_2h_{114}\theta_2)\sin u.\end{aligned}$$

Hence f is superminimal if and only if

$$\begin{aligned}(X_1^2h_{113} - X_2^2h_{223})h_{114} &= 0, \\ (X_1h_{113} + X_2h_{213})^2 + (X_1h_{123} + X_2h_{223})^2 &= (X_1^2 + X_2^2)h_{114}^2\end{aligned}$$

Hence

$$h_{114} = X_1h_{113} + X_2h_{213} = X_1h_{123} + X_2h_{223} = 0$$

or

$$\begin{aligned}X_1^2h_{113} - X_2^2h_{223} &= 0, \\ (X_1h_{113} + X_2h_{123})^2 + (X_1h_{123} + X_2h_{223})^2 &= (X_1^2 + X_2^2)h_{114}^2\end{aligned}$$

Because X_1 and X_2 is arbitrary under $X_1^2 + X_2^2 \neq 0$, we have $h = 0$, or $h_{113} = h_{223} = 0$ and $h_{123}^2 = h_{114}^2$. Hence the lemme holds. □

For a holomorphic function $g(z)$ on $\mathbb{C}$, define a map $\tilde{g}\colon \mathbb{C} \to \mathbb{C}^2 \cong \mathbb{R}^4$ by $\tilde{g}(z) = (z, g(z))$. Then $\tilde{g}$ is called an *R-surface* [5]. Kommerell showed that an R-surface is superminimal.

3 Quaternionic Holomorphic Geometry

We review super-conformal maps by quaternionic holomorphic geometry of surfaces in $\mathbb{R}^4$ [2]. We identify $\mathbb{R}^4$ with the set of all quaternions $\mathbb{H}$. The inner product of $\mathbb{R}^4$ becomes

$$\langle a, b\rangle = \mathrm{Re}(\bar{a}b) = \mathrm{Re}(\bar{b}a) = \frac{1}{2}(\bar{a}b + \bar{b}a).$$

We identify $\mathbb{R}^3$ with the set of all imaginary parts of quaternions $\mathrm{Im}\,\mathbb{H}$. Then two-dimensional sphere with radius one centered at the origin in $\mathbb{R}^3$ is $S^2 = \{a \in \mathrm{Im}\,\mathbb{H} : a^2 = -1\}$.

Let Σ be a Riemann surface with complex structure J_Σ. For a one-form ω on Σ, we define a one-form $*\omega$ by $*\omega = \omega \circ J_\Sigma$. A map $f\colon \Sigma \to \mathbb{H}$ is called a conformal map if $\langle df \circ J_\Sigma, df\rangle = 0$. This is equivalent to that $*df = N\,df = -df\,R$ with maps $N, R\colon \Sigma \to S^2$. Each point where f fails to be an immersion is isolated. This means that a conformal map is a branched immersion. The second fundamental form of f is

$$h(X, Y) = \frac{1}{2}(*df(X)\,dR(Y) - dN(X) * df(Y)).$$

Let $\mathscr{H}\colon \Sigma \to \mathbb{H}$ be the mean curvature vector of f. Then

$$df\,\overline{\mathscr{H}} = -\frac{1}{2}(*dN + N\,dN),\ \ \overline{\mathscr{H}}\,df = \frac{1}{2}(*dR + R\,dR).$$

The ellipse of curvature is

$$\mathscr{E}_p = \left\{\mathscr{H}|df(e_1)|^2 + \frac{1}{4}\cos 2\theta(a - b)(e_1) + \frac{1}{4}\sin 2\theta N(a + b)(e_1) : \theta \in \mathbb{R}\right\},$$
$$a = df\,(*dR - R\,dR),\ \ b = (*dN - N\,dN)\,df.$$

Then f is super-conformal if and only one of the following equations holds.

$$*dR - R\,dR = 0,\ \ *dN - N\,dN = 0$$

at any point $p \in \Sigma$.

In the following, we restrict ourselves to super-conformal maps with $*dN = N\,dN$.

Lemma 2 *A super-conformal map $f\colon \Sigma \to \mathbb{H}$ with $*df = N\,df$ and $*dN = N\,dN$ is superminimal if and only if f is holomorphic with respect to a right quaternionic linear complex structure of $\mathbb{H}$.*

Proof A super-conformal map $f\colon \Sigma \to \mathbb{H}$ with $*df = N\,df$ and $*dN = N\,dN$ satisfies the equation

$$df\,\overline{\mathscr{H}} = -N\,dN.$$

Hence f is minimal if and only if N is a constant map. Define $J\colon \mathbb{H} \to \mathbb{H}$ by $Jv = Nv$ for any $v \in \mathbb{H}$. Then J is a right quaternionic linear complex structure of $\mathbb{H}$. Because $*df = J\,df$, the map f is holomorphic with respect to J.

By the above lemma, we see that a holomorphic map g from Σ to $\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}j \cong \mathbb{H}$ is superminimal because $*dg = i\,dg$. A holomorphic function and an R-surface are special cases of this superminimal surface.

4 The Schwarz Lemma

Because a holomorphic function is a super-conformal map, we may expect that a super-conformal map is an analogue of a holomorphic function. A factorization of super-conformal map given in Theorem 4.3 in [7] may support this idea. The following is a variant of the theorem.

Theorem 1 ([7], Theorem 4.3) *Let $\phi\colon \Sigma \to \mathbb{H}$ be a super-conformal map with $*d\phi = N\,d\phi$, $*dN = N\,dN$ and $N\phi = \phi i$ and $h\colon \Sigma \to \mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}j \cong \mathbb{H}$ be a holomorphic map. Then, a map $f = \phi h$ is a super-conformal map with $*df = N\,df$.*

This theorem shows that a holomorphic section of a complex vector bundle of rank two, trivialized by the super-conformal map f is a super-conformal map. We see that $N + i$ is a super-conformal map with $N(N + i) = (N + i)i$. The condition $*dN = N\,dN$ implies N is the inverse of the stereographic projection followed by an anti-holomorphic function ([7], Corollary 3.2). Hence if Σ is an open Riemann surface and $N\colon \Sigma \to S^2$ is the inverse of the stereographic projection of an anti-holomorphic function with $N(\Sigma) \subset S^2 \setminus \{-i\}$, then $N + i$ is a global super-conformal trivializing section. A super-conformal map $f\colon \Sigma \to \mathbb{H}$ with $*df = N\,df$, $*dN = N\,dN$ always factors $f = (N + i)h$ with a holomorphic map $h\colon \Sigma \to \mathbb{C} \oplus \mathbb{C}j$. We don't need to see $-i$ in a special light. If $a \in S^2$ and $a \notin N(\Sigma)$, then we can rotate f so that $-i \notin N(\Sigma)$. The condition that N fails to be surjective is necessary.

This fact suggests that we should distinguish the case where the Riemann surface Σ is parabolic or hyperbolic. In the case where $\Sigma = \mathbb{C}$, we have an analogue of Liouville's theorem.

Theorem 2 ([7], Theorem 4.4) *Let $\phi\colon \mathbb{C} \to \mathbb{H}$ be a super-conformal map with $*d\phi = N\,d\phi$, $*dN = N\,dN$ and $N\phi = \phi i$. Assume that $N(\mathbb{C}) \subset S^2 \setminus \{-i\}$ and $|\phi|^{-1}$ is bounded above. If $f\colon \mathbb{C} \to \mathbb{H}$ is a super-conformal map with $*df = N\,df$ and $|f|$ is bounded above, then $f = \phi C$ for some constant $C \in \mathbb{H}$.*

In the case where $\Sigma = B^2 = \{z \in \mathbb{C} : |z| < 1\}$, we have an analogue of the Schwarz lemma.

Theorem 3 ([7], Theorem 4.5) *Let $\phi\colon B^2 \to \mathbb{H}$ be a super-conformal map with $* d\phi = N\, d\phi$, $* dN = N\, dN$ and $N\phi = \phi i$. Assume that $N(B^2) \subset S^2 \setminus \{-i\}$ and $|\phi| < c$ and $|\phi|^{-1} < \tilde{c}$. If $f\colon B^2 \to \mathbb{H}$ is a super-conformal map with $* df = N\, df$ and $f(0) = 0$, then there exists a constant C such that*

$$|f(z)| \leq C|z|.$$

Moreover, if $f = \phi(\lambda_0 + \lambda_1 j)$ for holomorphic functions λ_0 and λ_1, then there exist constants C_0, $C_1 > 0$ such that

$$|f(z)| \leq c(C_0^2 + C_1^2)^{1/2}|z|.$$

The equality holds if and only if $\phi = c$ and there exists $z_0 \in B^2$ such that $|\lambda_n(z)| = C_n|z_0|$ $(n = 0, 1)$. We also have

$$\left| f_x(0) - N(0) f_y(0) \right| \leq c(C_0^2 + C_1^2)^{1/2}.$$

The equality holds if and only if $f = c$ and there exists $z_0 \in B^2$ such that $|\lambda_n(z)| = C_n|z_0|$ $(n = 0, 1)$.

Assume that $f(B^2) \subset B^4 = \{a \in \mathbb{H} : |a| < 1\}$. It is known that

$$T(a) = \frac{(1 - |a_1|^2)(a - a_1) - |a - a_1|^2 a_1}{1 + |a|^2|a_1|^2 - 2\langle a, a_1 \rangle}$$

is a Möbius transform of $\mathbb{R}^4$ with $T(a_1) = 0$ [1]. The transform T is

$$\begin{aligned} T(a) &= \frac{a - a_1 - |a_1|^2 a + |a_1|^2 a_1 - |a|^2 a_1 + a|a_1|^2 + a_1\overline{a}a_1 - |a_1|^2 a_1}{|1 - \overline{a}_1 a|^2} \\ &= \frac{a - a_1 - |a|^2 a_1 + a_1\overline{a}a_1}{|1 - \overline{a}_1 a|^2} = \frac{(1 - a_1\overline{a})(a - a_1)}{|1 - \overline{a}_1 a|^2} \\ &= (1 - a\overline{a}_1)^{-1}(a - a_1) \end{aligned}$$

and T preserves B^4. If $f\colon B^2 \to B^4$ is a super-conformal map with $* df = N\, df$ and $* dN = N\, dN$, then

$$\begin{aligned} * d(T \circ f) &= (1 - f\overline{a}_1)^{-1}(* df\, \overline{a})(1 - f\overline{a}_1)^{-1}(1 - f) - (1 - f\overline{a}_1)^{-1} * df \\ &= (1 - f\overline{a}_1)^{-1} N (1 - f\overline{a}_1)\, d(T \circ f). \end{aligned}$$

It is known that a Möbius transform of a super-conformal map is super-conformal. Then we have an analogue of the Schwarz-Pick theorem.

Theorem 4 ([7], Theorem 4.7) *Let* $\phi\colon B^2 \to B^4$ *be a super-conformal map with* $*d\phi = N\,d\phi$, $*dN = N\,dN$ *and* $N\phi = \phi i$. *Assume that* $|\phi|$ *and* $|\phi|^{-1}$ *are bounded. Let* $f\colon \Sigma \to \mathbb{H}$ *be a super-conformal map with* $*df = N\,df$. *Assume that the map*

$$\tilde{N} := (1 - \tilde{f}(z)\overline{\tilde{f}(z_1)})^{-1} N (1 - \tilde{f}(z)\overline{\tilde{f}(z_1)})\colon \Sigma \to S^2$$

satisfies $\tilde{N}(B^2) \subset S^2 \setminus \{-i\}$. *Then there exists a constant* $C > 0$ *such that*

$$\frac{|f(z) - f(z_1)|}{\left|1 - \overline{f(z_1)}f(z)\right|} \leq C \frac{|z - z_1|}{|1 - \overline{z_1}z|}$$

Moreover,

$$\frac{|f_x(z_1)|}{1 - |f(z_1)|^2} = \frac{\left|f_y(z_1)\right|}{1 - |f(z_1)|^2} \leq \frac{C}{1 - |z_1|^2}.$$

We fix Riemannian metrics $ds^2_{B^2}$ on B^2 and $ds^2_{B^4}$ on B^4 as

$$ds^2_{B^2} = \frac{4}{(1 - (x^2 + y^2))^2}(dx \otimes dx + dy \otimes dy),$$

$$ds^2_{B^4} = \frac{4}{(1 - \sum_{n=0}^{3} a_n^2)^2}(\sum_{n=0}^{3} da_n \otimes da_n).$$

Then a geometric version of the Schwarz-Pick theorem becomes as follows.

Theorem 5 *Let* $\phi\colon B^2 \to B^4$ *be a super-conformal map with* $*d\phi = N\,d\phi$, $*dN = N\,dN$ *and* $N\phi = \phi i$. *Assume that* $|\phi|$ *and* $|\phi|^{-1}$ *are bounded. Let* $f\colon \Sigma \to \mathbb{H}$ *be a super-conformal map with* $*df = N\,df$. *Assume that the map*

$$\tilde{N} := (1 - f(z)\overline{f(z_1)})^{-1} N (1 - f(z)\overline{f(z_1)})\colon \Sigma \to S^2$$

satisfies $\tilde{N}(B^2) \subset S^2 \setminus \{-i\}$. *Then there exists a constant* $C > 0$ *such that* $f^* ds^2_{B^4} \leq C ds^2_{B^2}$.

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References

1. Ahlfors, L.V.: Möbius transformations in several dimensions. Ordway Professorship Lectures in Mathematics. University of Minnesota, School of Mathematics, Minneapolis, Minn (1981)
2. Burstall, F.E., Ferus, D., Leschke, K., Pedit, F., Pinkall, U.: Conformal Geometry of Surfaces in S^4 and Quaternions. Lecture Notes in Mathematics, vol. 1772. Springer, Berlin (2002)

3. Ferus, D., Leschke, K., Pedit, F., Pinkall, U.: Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2-tori. Invent. Math. **146**(3), 507–593 (2001)
4. Friedrich, T.: On superminimal surfaces. Arch. Math. (Brno) **33**(1–2), 41–56 (1997)
5. Kommerell, K.: Riemannsche Flächen im ebenen Raum von vier dimensionen. Math. Ann. **60**(4), 548–596 (1905)
6. Moriya, K.: Super-conformal surfaces associated with null complex holomorphic curves. Bull. Lond. Math. Soc. **41**(2), 327–331 (2009)
7. Moriya, K.: A factorization of a super-conformal map. Israel J. Math. **207**(1), 331–359 (2015)
8. Petrović-Torgašev, M., Verstraelen, L.: On Deszcz symmetries of Wintgen ideal submanifolds. Arch. Math. (Brno) **44**(1), 57–67 (2008)
9. Schouten, J.A., Struik, D.J.: Einführung in die neueren Methoden der Differentialgeometrie, vol. II. P. Noordhoff n.v, Groningen-Batavia (1938)
10. Wintgen, P.: Sur l'inégalité de Chen-Willmore, C. R. Acad. Sci. Paris Sér. A-B **288**(21), A993–A995 (1979)
11. Wong, Y.-C.: A new curvature theory for surfaces in a Euclidean 4-space. Comment. Math. Helv. **26**, 152–170 (1952)