Maximal Antipodal Subgroups of the Automorphism Groups of Compact Lie Algebras

Makiko Sumi Tanaka and Hiroyuki Tasaki

Abstract We classify maximal antipodal subgroups of the group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of a compact classical Lie algebra $\mathfrak{g}$. A maximal antipodal subgroup of $\mathrm{Aut}(\mathfrak{g})$ gives us as many mutually commutative involutions of $\mathfrak{g}$ as possible. For the classification we use our former results of the classification of maximal antipodal subgroups of quotient groups of compact classical Lie groups. We also use canonical forms of elements in a compact Lie group which is not connected.

1 Introduction

The group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of a compact semisimple Lie algebra $\mathfrak{g}$ is a compact semisimple Lie group which is not necessarily connected. The identity component of $\mathrm{Aut}(\mathfrak{g})$ is the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms. A subgroup of a compact Lie group is an antipodal subgroup if it consists of mutually commutative involutive elements. In this article we give a classification of maximal antipodal subgroups of $\mathrm{Aut}(\mathfrak{g})$ when $\mathfrak{g}$ is a compact classical semisimple Lie algebra $\mathfrak{su}(n)$ $(n \geq 2)$, $\mathfrak{o}(n)$ $(n \geq 5)$ or $\mathfrak{sp}(n)$ $(n \geq 1)$ (Theorem 4). A maximal antipodal subgroup $\mathrm{Aut}(\mathfrak{g})$ gives us as many mutually commutative involutions of $\mathfrak{g}$ as possible.

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Then G is a compact connected semisimple Lie group whose center Z is discrete. The quotient G/Z is isomorphic to $\mathrm{Int}(\mathfrak{g})$ via the adjoint representation. Therefore our results [5] of the classification of maximal antipodal subgroups of G/Z gives the classification of maximal antipodal subgroups of $\mathrm{Int}(\mathfrak{g})$. In order to consider the case where $\mathrm{Aut}(\mathfrak{g})$ is

M.S. Tanaka (✉)
Faculty of Science and Technology, Department of Mathematics,
Tokyo University of Science, Noda, Chiba 278-8510, Japan
e-mail: tanaka_makiko@ma.noda.tus.ac.jp

H. Tasaki
Faculty of Pure and Applied Sciences, Division of Mathematics,
University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan
e-mail: tasaki@math.tsukuba.ac.jp

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not connected, we give a canonical form of an element of a disconnected Lie group (Proposition 3).

After we submitted the manuscript, we found Yu studied elementary abelian 2-subgroups of the automorphism group of compact classical simple Lie algebras in [6]. Elementary abelian 2-subgroups are nothing but antipodal subgroups.

2 Maximal Antipodal Subgroups of Quotient Lie Groups

In this section we refer to our former results in [5].

Although the notion of an antipodal set is originally defined as a subset of a Riemannian symmetric space M in [1], we give an alternative definition when M is a compact Lie group with a bi-invariant Riemannian metric.

Definition 1 Let G be a compact Lie group and we denote by e the identity element of G. A subset A of G satisfying $e \in A$ is called an *antipodal set* if A satisfies the following two conditions.

(i) Every element $x \in A$ satisfies $x^2 = e$.
(ii) Any elements $x, y \in A$ satisfy $xy = yx$.

Proposition 1 ([5]) *If a subset A of G satisfying $e \in A$ is a maximal antipodal set, then A is an abelian subgroup of G which is isomorphic to a product $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ of some copies of $\mathbb{Z}_2$. Here $\mathbb{Z}_2$ denotes the cyclic group of order* 2.

Let

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n).$$

For a subset $X \subset O(n)$ we define $X^{\pm} := \{x \in X \mid \det(x) = \pm 1\}$.

Proposition 2 (cf. [1]) *A maximal antipodal subgroup of $U(n)$, $O(n)$, $Sp(n)$ is conjugate to Δ_n. A maximal antipodal subgroup of $SU(n)$, $SO(n)$ is conjugate to Δ_n^+.*

Let

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \subset O(2),$$

which is a dihedral group. Let

$$Q[8] := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\},$$

which is the quaternion group, where $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are elements of the standard basis of the quaternions $\mathbb{H}$. We decompose a natural number n as $n = 2^k \cdot l$ into the product of the k-th power 2^k of 2 and an odd number l. For s with $0 \leq s \leq k$ we define

$$D(s,n) := D[4] \otimes \cdots \otimes D[4] \otimes \Delta_{n/2^s}$$
$$= \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_1, \ldots, d_s \in D[4], d_0 \in \Delta_{n/2^s}\} \subset O(n).$$

We always use k and l in the above meaning when we write $n = 2^k \cdot l$.

The center of $U(n)$ is $\{z1_n \mid z \in U(1)\}$ and we identify it with $U(1)$. Let $\mathbb{Z}_\mu$ be the cyclic group of degree μ which lies in the center of $U(n)$. Let $\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$ be the natural projection.

Theorem 1 ([5]) *Let $n = 2^k \cdot l$. Let θ be a primitive 2μ-th root of 1. Then a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ is conjugate to one of the following.*

(1) *In the case where n or μ is odd, $\pi_n(\{1,\theta\}D(0,n)) = \pi_n(\{1,\theta\}\Delta_n)$.*
(2) *In the case where both n and μ are even, $\pi_n(\{1,\theta\}D(s,n))$ $(0 \le s \le k)$, where the case $(s,n) = (k-1, 2^k)$ is excluded.*

Remark 1 Since we have an inclusion $\Delta_2 \subsetneq D[4]$ which implies $D(k-1, 2^k) \subsetneq D(k, 2^k)$, the case $(s,n) = (k-1, 2^k)$ is excluded.

Theorem 2 ([5]) *Let n and μ be natural numbers where μ divides n. Let $n = 2^k \cdot l$. Let $\mathbb{Z}_\mu$ be the cyclic group of degree μ which lies in the center of $SU(n)$. Let θ be a primitive 2μ-th root of 1. Then a maximal antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$ is conjugate to one of the following.*

(1) *In the case where n or μ is odd, $\pi_n(\Delta_n^+)$.*
(2) *In the case where both n and μ are even,*
   (a) *when $k = 1$, $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$ or $\pi_n((D^+[4] \cup \theta D^-[4]) \otimes \Delta_l)$, where $\pi_2(\Delta_2^+ \cup \theta\Delta_2^-)$ is excluded when $n = \mu = 2$.*
   (b) *When $k \ge 2$, under the expression $\mu = 2^{k'} \cdot l'$ where $1 \le k' \le k$ and l' divides l,*
      (b1) *if $k' = k$, $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$ or $\pi_n(D(s,n))$ $(1 \le s \le k)$, where the case $(s,n) = (k-1, 2^k)$ is excluded.*
      (b2) *If $1 \le k' < k$, $\pi_n(\{1,\theta\}\Delta_n^+)$ or $\pi_n(\{1,\theta\}D(s,n))$ $(1 \le s \le k)$, where the case $(s,n) = (k-1, 2^k)$ is excluded and, moreover, $\pi_4(\{1,\theta\}\Delta_4^+)$ is excluded when $n = 4$.*

Remark 2 Since $\Delta_4^+ = \Delta_2 \otimes \Delta_2 \subsetneq D[4] \otimes D[4] = D(2,4)$, $\pi_4(\{1,\theta\}\Delta_4^+)$ is excluded.

Theorem 3 ([5]) *Let π_n be one of the natural projections $O(n) \to O(n)/\{\pm 1_n\}$, $SO(n) \to SO(n)/\{\pm 1_n\}$, $Sp(n) \to Sp(n)/\{\pm 1_n\}$. Let $n = 2^k \cdot l$.*

(I) *A maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ is conjugate to one of $\pi_n(D(s,n))$ $(0 \le s \le k)$, where the case $(s,n) = (k-1, 2^k)$ is excluded.*
(II) *When n is even, a maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ is conjugate to one of the following.*
   (1) *In the case where $k = 1$, $\pi_n(\Delta_n^+)$ or $\pi_n(D^+[4] \otimes \Delta_l)$, where $\pi_2(\Delta_2^+)$ is excluded when $n = 2$.*

(2) *In the case where* $k \geq 2$, $\pi_n(\Delta_n^+)$ *or* $\pi_n(D(s,n))$ $(1 \leq s \leq k)$*, where the case* $(s,n) = (k-1, 2^k)$ *is excluded and moreover* $\pi_4(\Delta_4^+)$ *is excluded when* $n = 4$.

(III) *A maximal antipodal subgroup of* $Sp(n)/\{\pm 1_n\}$ *is conjugate to one of* $\pi_n(Q[8] \cdot D(s,n))$ $(0 \leq s \leq k)$*, where the case* $(s,n) = (k-1, 2^k)$ *is excluded.*

3 Canonical Forms of Elements of a Disconnected Lie Group

Let G be a compact connected Lie group and let T be a maximal torus of G. Then we have

$$G = \bigcup_{g \in G} gTg^{-1},$$

which means that a canonical form of an element of G with respect to conjugation is an element of T. We give a formulation of canonical forms of elements of G in the case where G is not connected. Let G_0 be the identity component of a compact Lie group G. Then G/G_0 is a finite group and we have

$$G = \bigcup_{[\tau] \in G/G_0} G_0 \tau,$$

where $[\tau]$ denotes the coset represented by $\tau \in G$.

Ikawa showed a canonical form of a certain action on a compact connected Lie group in [3, 4]. Using this canonical form we can obtain the following proposition.

Proposition 3 *For* $\tau \in G$ *we define an automorphism* I_τ *of* G_0 *by* $I_\tau(g) = \tau g \tau^{-1}$ $(g \in G_0)$. *Let* T_τ *be a maximal torus of* $F(I_\tau, G_0) := \{g \in G_0 \mid I_\tau(g) = g\}$. *Then we have*

$$G_0 \tau = \bigcup_{g \in G_0} g(T_\tau \tau) g^{-1}.$$

Therefore a canonical form of an element of a connected component $G_0\tau$ of G with respect to conjugation under G_0 is an element of $T_\tau \tau$.

4 Maximal Antipodal Subgroups of the Automorphism Groups of Compact Lie Algebras

Let $\mathfrak{g}$ be a compact semisimple Lie algebra. Then the group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$ is a compact semisimple Lie group which is not necessarily connected. By the definition of antipodal sets, the set of maximal antipodal subgroups of $\mathrm{Aut}(\mathfrak{g})$ is equal to the set of maximal subsets of $\mathrm{Aut}(\mathfrak{g})$ satisfying (i) each element has order 2 except for the identity element and (ii) all elements are commutative to each other.

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Then G is a compact connected semisimple Lie group whose center Z is discrete. The quotient group G/Z is isomorphic to $\mathrm{Int}(\mathfrak{g})$ via the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. Hence the classification of maximal antipodal subgroups of G/Z gives the classification of maximal antipodal subgroups of $\mathrm{Int}(\mathfrak{g})$.

Theorem 4 *Let $n = 2^k \cdot l$ be a natural number.*

(I) *Let τ denote a map $\tau : \mathfrak{su}(n) \to \mathfrak{su}(n)\ ;\ X \mapsto \bar{X}$. A maximal antipodal subgroup of* $\mathrm{Aut}(\mathfrak{su}(n))$ *is conjugate to* $\{e, \tau\}\mathrm{Ad}(D(s, n))$ $(0 \le s \le k)$*, where the case* $(s, n) = (k-1, 2^k)$ *is excluded. Here e denotes the identity element of* $\mathrm{Aut}(\mathfrak{g})$.

(II) *A maximal antipodal subgroup of* $\mathrm{Aut}(\mathfrak{o}(n))$ *is conjugate to* $\mathrm{Ad}(D(s, n))$ $(0 \le s \le k)$*, where the case* $(s, n) = (k-1, 2^k)$ *is excluded.*

(III) *A maximal antipodal subgroup of* $\mathrm{Aut}(\mathfrak{sp}(n))$ *is conjugate to* $\mathrm{Ad}(Q[8] \cdot D(s, n))$ $(0 \le s \le k)$*, where the case* $(s, n) = (k-1, 2^k)$ *is excluded.*

Before we prove Theorem 4, we need some preparations. Let $\tau' : \mathbb{C}^n \to \mathbb{C}^n$ be the complex conjugation $\tau'(v) = \bar{v}$ for $v \in \mathbb{C}^n$. Since $\tau' \in GL(2n, \mathbb{R})$, $\{1_n, \tau'\}U(n)$ is a subset of $GL(2n, \mathbb{R})$. We have $g\tau' = \tau'\bar{g}$ for $g \in U(n)$. This implies $\mathrm{Ad}(\tau') = \tau$, so we identify τ' with τ. We can see that $\{1_n, \tau\}U(n)$ is a subgroup of $GL(2n, \mathbb{R})$ and the center is $\{\pm 1_n\}$. Let $\mathbb{Z}_\mu := \{z 1_n \mid z \in U(1),\ z^\mu = 1\} \subset U(n)$. We can see that $\mathbb{Z}_\mu$ is a normal subgroup of $\{1_n, \tau\}U(n)$. Therefore $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$ is a Lie group. We have $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu = U(n)/\mathbb{Z}_\mu \cup \tau U(n)/\mathbb{Z}_\mu$, which is a disjoint union of the connected components.

Theorem 5 *Let $\pi_n : \{1_n, \tau\}U(n) \to \{1_n, \tau\}U(n)/\mathbb{Z}_\mu$ be the natural projection. Let θ be a primitive 2μ-th root of 1. Let $n = 2^k \cdot l$. Then a maximal antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$ is conjugate to one of the following by an element of $\pi_n(U(n))$.*

(1) *In the case where μ is odd, $\pi_n(\{1_n, \tau\}\{1, \theta\}\Delta_n) = \pi_n(\{1_n, \tau\}\Delta_n)$.*

(2) *In the case where μ is even, $\pi_n(\{1_n, \tau\}\{1, \theta\}D(s, n))$ $(0 \le s \le k)$, where the case $(s, n) = (k-1, 2^k)$ is excluded.*

Remark 3 Since $\{1_n, \tau\}\{1, \theta\}\Delta_n \subset \{1_n, \tau\}U(n) \subset GL(2n, \mathbb{R})$, we can consider $\pi_n(\{1_n, \tau\}\{1, \theta\}\Delta_n)$.

Lemma 1 *Let A be a maximal antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$. Then we have $A \cap \tau U(n)/\mathbb{Z}_\mu \ne \emptyset$.*

Proof We assume $A \subset U(n)/\mathbb{Z}_\mu$. By taking conjugation by $U(n)/\mathbb{Z}_\mu$ we can assume $A = \pi_n(\{1, \theta\}D(s, n))$ by Theorem 1. Since $\pi_n(\tau)\pi_n(\theta 1_n) = \pi_n(\theta 1_n)\pi_n(\tau)$, $A \cup \pi_n(\tau)A$ is an antipodal, which contradicts the maximality of A.

Lemma 2 *Let A be a maximal antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$. Let θ be a primitive 2μ-th root of 1. Then $\pi_n(\theta 1_n) \in A$.*

Proof Since we showed that $\pi_n(\theta 1_n)$ and $\pi_n(\tau)$ are commutative in the proof of Lemma 1, $\pi_n(\theta 1_n)$ is commutative with every element of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$. Hence $\pi_n(\theta 1_n) \in A$.

Lemma 3 *A maximal antipodal subgroup of* $\{1_n, \tau\}U(n)$ *is conjugate to* $\{1_n, \tau\}\Delta_n$ *by an element of* $U(n)$.

Proof Let A be a maximal antipodal subgroup of $\{1_n, \tau\}U(n)$. Then $A \cap \tau U(n) \neq \emptyset$ by Lemma 1 for $\mu = 1$. We set $R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ and $r = \lfloor \frac{n}{2} \rfloor$. Then

$$T = \left\{ \begin{bmatrix} R(\phi_1) & & & \\ & \ddots & & \\ & & R(\phi_r) & \\ & & & (1) \end{bmatrix} \middle| \phi_j \in \mathbb{R}\ (1 \leq j \leq r) \right\}$$

is a maximal torus of $O(n) = F(\tau, U(n))$. Here (1) in the above definition of T means 1 when $n = 2r + 1$ and nothing when $n = 2r$. By Proposition 3 we have

$$\tau U(n) = \bigcup_{g \in U(n)} g(\tau T)g^{-1}.$$

Therefore, by retaking A under the conjugation by $U(n)$ if necessary, we may assume that $A \cap \tau U(n)$ has an element $\tau g_0 \in \tau T$. Since $1_n = (\tau g_0)^2 = g_0^2$, we have $g_0 \in \Delta_n$. Applying $\sqrt{-1}\tau\sqrt{-1}^{-1} = -\tau$ to a diagonal element -1 of g_0, we have $\tau g_0 = g_1 \tau 1_n g_1^{-1}$ for a suitable $g_1 \in U(n)$ which is a diagonal matrix whose diagonal elements are $1, \sqrt{-1}$.[1] Therefore if we retake A under the conjugation by $U(n)$ if necessary, we may assume $\tau \in A$. Hence $A \cap \tau U(n) = \tau(A \cap U(n))$. Since $\tau \in A$ and A is commutative, we have $A \cap U(n) \subset O(n)$. We show that $A \cap U(n)$ is a maximal antipodal subgroup of $O(n)$. If there is an antipodal subgroup $\tilde{A}$ which satisfies $A \cap U(n) \subset \tilde{A} \subset O(n)$, then $\{1_n, \tau\}\tilde{A}$ is an antipodal subgroup of $\{1_n, \tau\}U(n)$ and we have $A = (A \cap U(n)) \cup (A \cap \tau U(n)) = \{1_n, \tau\}(A \cap U(n)) \subset \{1_n, \tau\}\tilde{A}$. By the maximality of A we have $A = \{1_n, \tau\}\tilde{A}$, hence $A \cap U(n) = \tilde{A}$. Therefore $A \cap U(n)$ is a maximal antipodal subgroup of $O(n)$. By Proposition 2, $A \cap U(n)$ is conjugate to Δ_n by $O(n)$. Hence $A = \{1_n, \tau\}(A \cap U(n))$ is conjugate to $\{1_n, \tau\}\Delta_n$ by $O(n)$. Therefore any maximal antipodal subgroup of $\{1_n, \tau\}U(n)$ is conjugate to $\{1_n, \tau\}\Delta_n$ by an element of $U(n)$.

We prove Theorem 5.

Proof Since we proved the case of $\mu = 1$ in Lemma 3, we assume $\mu > 1$. We note that $\bar{\theta} \neq \theta$ in this case. Let A be a maximal antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$

[1] We have $\{\tau g \mid g \in U(n),\ (\tau g)^2 = 1_n\} = \bigcup_{g \in U(n)} g\tau 1_n g^{-1}$. It is remarkable in contrast to $\{g \in U(n) \mid g^2 = 1_n\} = \bigcup_{g \in U(n)} g\Delta_n g^{-1}$.

and we set $B = \pi_n^{-1}(A)$. Then $\theta \in B$ by Lemma 2. Since $A \cap \tau U(n)/\mathbb{Z}_\mu \neq \emptyset$ by Lemma 1, we have $B \cap \tau U(n) \neq \emptyset$. Therefore, by retaking A under the conjugation by $U(n)/\mathbb{Z}_\mu$ if necessary, we may assume that $B \cap \tau U(n)$ has an element $\tau g_0 \in \tau T$, where T is a maximal torus of $O(n)$ defined in the proof of Lemma 3. By a similar argument as in the proof of Lemma 3 we may assume $g_0 = 1_n$. Thus $\tau \in B$. We note that B is not commutative because $\tau\theta = \bar{\theta}\tau \neq \theta\tau$. Since $\pi_n(\tau) \in A$, we have

$$A = (A \cap \pi_n(U(n))) \cup (A \cap \pi_n(\tau U(n))) = \pi_n(\{1_n, \tau\})(A \cap \pi_n(U(n))).$$

We consider $A \cap \pi_n(U(n))$. Since every element of $A \cap \pi_n(U(n))$ is commutative with $\pi_n(\tau)$, $A \cap \pi_n(U(n)) \subset \{\pi_n(u) \mid u \in U(n),\ \pi_n(\tau u) = \pi_n(u\tau)\}$. Since $u\tau = \tau\bar{u}$, the condition $\pi_n(\tau u) = \pi_n(u\tau)$ is equivalent to $\pi_n(u) = \pi_n(\bar{u})$, which is equivalent to the condition that there exists an integer m such that $\theta^{2m}u = \bar{u}$. Hence we have $\theta^m u = \theta^{-m}\bar{u} = \overline{\theta^m u}$, which means $\theta^m u \in O(n)$. When m is even, we have $\pi_n(\theta^m u) = \pi_n(u)$. Thus $\pi_n(u) \in \pi_n(O(n))$. When m is odd, we have $\pi_n(\theta^m u) = \pi_n(\theta u)$. Hence $\pi_n(u) = \pi_n(\theta 1_n)^{-1}\pi_n(\theta^m u) = \pi_n(\theta 1_n)\pi_n(\theta^m u)$. Thus $\pi_n(u) \in \pi_n(\theta 1_n)\pi_n(O(n))$. Therefore

$$A \cap \pi_n(U(n)) \subset \pi_n(\{1, \theta\}O(n)).$$

We consider the case where μ is odd. We have $\pi_n(\theta 1_n) = \pi_n(\theta^\mu 1_n) = \pi_n(-1_n)$. Hence $\pi_n(\{1, \theta\}O(n)) = \pi_n(O(n))$. Since $-1_n \notin \operatorname{Ker}\pi_n$, we have $O(n) \cap \operatorname{Ker}\pi_n = \{1_n\}$ and the restriction $\pi_n|_{O(n)}$ gives an isomorphism from $O(n)$ onto $\pi_n(O(n))$. Hence we have $\pi_n(\{1, \theta\}O(n)) = \pi_n(O(n)) \cong O(n)$. Therefore $A \cap \pi_n(U(n))$ is conjugate to $\pi_n(\Delta_n)$ by an element of $\pi_n(O(n))$ by Proposition 2. Hence A is conjugate to $\pi_n(\Delta_n) \cup \pi_n(\tau)\pi_n(\Delta_n) = \pi_n(\{1_n, \tau\}\Delta_n)$ by an element of $\pi_n(U(n))$.

We consider the case where μ is even. In this case $\pi_n(\{1, \theta\}O(n)) = \pi_n(O(n)) \cup \pi_n(\theta O(n))$ is a disjoint union. We show that $A \cap \pi_n(O(n))$ is a maximal antipodal subgroup of $\pi_n(O(n))$. Let $\tilde{A}$ be an antipodal subgroup which satisfies $A \cap \pi_n(O(n)) \subset \tilde{A} \subset \pi_n(O(n))$. Since every element of $\tilde{A}$ is commutative with $\pi_n(\tau)$, it turns out that $\pi_n(\{1_n, \tau\}\{1, \theta\})\tilde{A}$ is an antipodal subgroup of $\pi_n(\{1_n, \tau\}U(n))$. We have $A \cap \pi_n(U(n)) = \pi_n(\{1_n, \theta 1_n\})(A \cap \pi_n(O(n)))$. Therefore

$$A = \pi_n(\{1_n, \tau\}\{1_n, \theta 1_n\})(A \cap \pi_n(O(n))) \subset \pi_n(\{1_n, \tau\}\{1_n, \theta 1_n\})\tilde{A}.$$

By the maximality of A we have $A = \pi_n(\{1_n, \tau\}\{1_n, \theta 1_n\})\tilde{A}$. Moreover, we have $A \cap \pi_n(O(n)) = \tilde{A}$. Thus $A \cap \pi_n(O(n))$ is a maximal antipodal subgroup of $\pi_n(O(n))$. Since μ is even, we have $-1_n \in \operatorname{Ker}\pi_n$. Hence $\pi_n(O(n)) \cong O(n)/\{\pm 1_n\}$. We decompose n as $n = 2^k \cdot l$. By Theorem 3, $A \cap \pi_n(O(n))$ is conjugate to $\pi_n(D(s, n))$ $(0 \leq s \leq k)$ by an element of $\pi_n(O(n))$. Here the case $(s, n) = (k-1, 2^k)$ is excluded. Therefore A is conjugate to $\pi_n(\{1_n, \tau\}\{1_n, \theta 1_n\}D(s, n))$ by $\pi_n(O(n))$.

We prove Theorem 4.

Proof We have Aut($\mathfrak{g}$)= Int($\mathfrak{g}$) when $\mathfrak{g} = \mathfrak{o}(n)$ where n is odd and $\mathfrak{g} = \mathfrak{sp}(n)$. Hence we obtain (II) when n is odd and (III) by Theorem 3. In general we have

$\mathrm{Aut}(\mathfrak{o}(n)) \cong O(n)/\{\pm 1_n\}$ if $n \neq 8$. Hence we obtain (II) when n is even and $n \neq 8$ by Theorem 3. We consider $\mathrm{Aut}(\mathfrak{o}(8))$. It is known that $\mathrm{Aut}(\mathfrak{o}(8))/\mathrm{Int}(\mathfrak{o}(8)) \cong S_3$, where S_3 denotes the symmetric group of degree 3. S_3 has three elements of order 2, denoted by τ_1, τ_2, τ_3, and two elements of order 3. Using these we can see that if A is an antipodal subgroup of $\mathrm{Aut}(\mathfrak{o}(8))$, there is $\tau \in \mathrm{Aut}(\mathfrak{o}(8))$ which satisfies that the coset $\tau\mathrm{Int}(\mathfrak{o}(8))$ corresponds to $\tau_i \in S_3$ for some $i \in \{1, 2, 3\}$ such that $A \subset \mathrm{Int}(\mathfrak{o}(8)) \cup \tau\mathrm{Int}(\mathfrak{o}(8))$. Therefore a maximal antipodal subgroup of $\mathrm{Aut}(\mathfrak{o}(8))$ is conjugate to a maximal antipodal subgroup of $O(8)/\{\pm 1_8\}$. Hence we obtain (II) when $n = 8$.

Finally we prove (I). The adjoint representation $\mathrm{Ad} : \{1_n, \tau\}SU(n) \to \mathrm{Aut}(\mathfrak{su}(n))$ is surjective (cf. [2, Chap. IX, Corollary 5.5, Chap. X, Theorem 3.29]). We have $\mathrm{Ker}\,\mathrm{Ad} = Z_{\{1_n,\tau\}SU(n)}(SU(n)) = \mathbb{Z}_n$, where $Z_{\{1_n,\tau\}SU(n)}(SU(n))$ denotes the centralizer of $SU(n)$ in $\{1_n, \tau\}SU(n)$ and $\mathbb{Z}_n = \{z1_n \mid z \in \mathbb{C},\ z^n = 1\}$. Thus we obtain an isomorphism $\mathrm{Aut}(\mathfrak{su}(n)) \cong \{1_n, \tau\}SU(n)/\mathbb{Z}_n$. Therefore we determine maximal antipodal subgroups of $\{1_n, \tau\}SU(n)/\mathbb{Z}_n$.

Let $\pi_n : \{1_n, \tau\}SU(n) \to \{1_n, \tau\}SU(n)/\mathbb{Z}_n$ denote the natural projection. We decompose n as $n = 2^k \cdot l$. Let θ be a primitive $2n$-th root of 1. Let A be a maximal antipodal subgroup of $\{1_n, \tau\}SU(n)/\mathbb{Z}_n$. Since $\{1_n, \tau\}SU(n)/\mathbb{Z}_n$ is a subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_n$, A is an antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_n$. Hence there is a maximal antipodal subgroup $\tilde{A}$ of $\{1_n, \tau\}U(n)/\mathbb{Z}_n$ such that $A = \tilde{A} \cap \{1_n, \tau\}SU(n)/\mathbb{Z}_n$. By Theorem 5, $\tilde{A}$ is conjugate by an element of $\pi_n(U(n))$ to $\pi_n(\{1_n, \tau\}\{1, \theta\}D(s, n))$, where $s = 0$ when n is odd and $0 \leq s \leq k$ when n is even, moreover, the case $(s, n) = (k - 1, 2^k)$ is excluded. Hence there is $g \in U(n)$ such that

$$\tilde{A} = \pi_n(g)\pi_n(\{1_n, \tau\}\{1, \theta\}D(s, n))\pi_n(g)^{-1} = \pi_n(g\{1_n, \tau\}\{1, \theta\}D(s, n)g^{-1}).$$

We can write $g = g_1 z$ where $g_1 \in SU(n)$ and $z \in U(1)$. Then

$$g\{1_n, \tau\}\{1, \theta\}D(s, n)g^{-1} = g_1\{1_n, \tau z^{-2}\}\{1, \theta\}D(s, n)g_1^{-1}.$$

Hence $\tilde{A}$ is conjugate to $\pi_n(\{1_n, \tau z^{-2}\}\{1, \theta\}D(s, n))$ by an element of $\pi_n(SU(n))$. Since $A = \tilde{A} \cap \pi_n(\{1_n, \tau\}SU(n))$, A is conjugate to

$$\begin{aligned}
&\pi_n(\{1_n, \tau z^{-2}\}\{1, \theta\}D(s, n)) \cap \pi_n(\{1_n, \tau\}SU(n)) \\
&= \pi_n(\{1, \theta\}D(s, n)) \cap \pi_n(SU(n)) \cup \pi_n(\tau)\left(\pi_n(z^{-2}\{1, \theta\}D(s, n)) \cap \pi_n(SU(n))\right)
\end{aligned}$$

by an element of $\pi_n(SU(n))$. In the proof of Theorem 2 ([5]) we showed

$$\pi_n(\{1, \theta\}D(s, n)) \cap \pi_n(SU(n)) = \pi_n(D^+(s, n) \cup \theta D^-(s, n)).$$

We consider $\pi_n(z^{-2}\{1, \theta\}D(s, n)) \cap \pi_n(SU(n))$. We show

$$\pi_n(z^{-2}\{1, \theta\}D(s, n)) \cap \pi_n(SU(n)) = \pi_n(z^{-2}\{1, \theta\}D(s, n) \cap SU(n)).$$

It is clear $\pi_n(z^{-2}D(s,n)) \cap \pi_n(SU(n)) \supset \pi_n(z^{-2}D(s,n) \cap SU(n))$. Conversely, for $d \in D(s,n)$, $\pi_n(z^{-2}d) \in \pi_n(SU(n))$ holds if and only if $\theta^{2m}z^{-2}d \in SU(n)$ for some m. Since $\det(\theta^{2m}z^{-2}d) = \theta^{2mn}z^{-2n}\det(d) = z^{-2n}\det(d)$, $\theta^{2m}z^{-2}d \in SU(n)$ is equivalent to $z^{-2n}\det(d) = 1$. Since $d \in D(s,n)$, $\det(d) = \pm 1$. When $\det(d) = 1$, $z^{-2n}\det(d) = 1$ is equivalent to $z^{-2n} = 1$. Hence $z^{-2} \in \operatorname{Ker} \pi_n$. Therefore $\pi_n(z^{-2}d) \in \pi_n(SU(n))$ is equivalent to $\pi_n(d) \in \pi_n(SU(n))$ when $d \in D^+(s,n)$. When $\det(d) = -1$, $z^{-2n}\det(d) = 1$ is equivalent to $z^{-2n} = -1$, that is, $z^{2n} = -1$. Hence $\pi_n(z^2 1_n) = \pi_n(\theta 1_n)$. Therefore $\pi_n(z^{-2}d) \in \pi_n(SU(n))$ is equivalent to $\pi_n(\theta d) \in \pi_n(SU(n))$ when $d \in D^-(s,n)$. Thus we obtain $\pi_n(z^{-2}D(s,n)) \cap \pi_n(SU(n)) \subset \pi_n(z^{-2}D(s,n) \cap SU(n))$. Moreover, we obtain $\pi_n(z^{-2}D(s,n) \cap SU(n)) = \pi_n(D^+(s,n) \cup \theta D^-(s,n))$ by the argument above. As a consequence, A is conjugate to $\pi_n(\{1_n, \tau\}(D^+(s,n) \cup \theta D^-(s,n)))$, where $s = 0$ when n is odd and $0 \leq s \leq k$ when n is even, moreover, the case $(s,n) = (k-1, 2^k)$ is excluded. The isomorphism between $\pi_n(\{1_n, \tau\}SU(n))$ and $\operatorname{Ad}(\mathfrak{su}(n))$ is given by

$$\pi_n(\{1_n, \tau\}SU(n)) \ni \pi_n(x) \mapsto \operatorname{Ad}(x) \in \operatorname{Ad}(\mathfrak{su}(n)) \quad (x \in \{1_n, \tau\}SU(n)).$$

Hence $\pi_n(\{1_n, \tau\}(D^+(s,n) \cup \theta D^-(s,n)))$ corresponds to $\operatorname{Ad}(\{1_n, \tau\}D(s,n)$ under the isomorphism, because $\operatorname{Ad}(\theta 1_n) = e$. Hence we obtain (I).

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