Derivatives on Real Hypersurfaces of Non-flat Complex Space Forms

Juan de Dios Pérez

Abstract Let M be a real hypersurface of a nonflat complex space form, that is, either a complex projective space or a complex hyperbolic space. On M we have the Levi-Civita connection and for any nonnull real number k the corresponding generalized Tanaka-Webster connection. Therefore on M we consider their associated covariant derivatives, the Lie derivative and, for any nonnull k, the so called Lie derivative associated to the generalized Tanaka-Webster connection and introduce some classifications of real hypersurfaces in terms of the concidence of some pairs of such derivations when they are applied to the shape operator of the real hypersurface, the structure Jacobi operator, the Ricci operator or the Riemannian curvature tensor of the real hypersurface.

Keywords Real hypersurfaces · Complex space form · g-Tanaka-Webster connection · Covariant derivatives · Lie derivatives

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1 Introduction

A complex space form is an m-dimensional Kaehler manifold of constant holomorphic sectional curvature c and will be denoted by $M_m(c)$. A complete and simply connected complex space form is complex analytically isometric to

1. A complex projective space $\mathbb{C}P^m$, if $c > 0$.
2. A complex Euclidean space $\mathbb{C}^m$, if $c = 0$.
3. A complex hyperbolic space $\mathbb{C}H^m$, if $c < 0$.

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We will deal with non-flat complex space forms and if (J, g) is the Kaehlerian structure of such a manifold, the metric g is going to be considered with its holomorphic sectional curvature equal to either 4 or -4. That is, $c = \pm 4$.

Let M be a real hypersurface of $M_m(c)$. Let N be a locally defined unit normal vector field on M. Writing $\xi = -JN$, this is a tangent vector field to M called the structure vector field on M (it is also known as Reeb vector field or Hopf vector field). Let A be the shape operator of M associated to N, ∇ the Levi-Civita connection on M and $\mathbb{D}$ the maximal holomorphic distribution on M. That is, for any $p \in M$ $\mathbb{D}_p = \{X \in T_pM / g(X, \xi) = 0\}$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX. Clearly $\eta(X) = g(X, \xi)$ and (ϕ, ξ, η, g) is an almost contact metric structure on M. That is, we have

- $\phi^2 X = -X + \eta(X)\xi$
- $\eta(\xi) = 1$
- $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$
- $\phi\xi = 0$
- $\nabla_X \xi = \phi A X$
- $(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$

for any X, Y tangent to M.

Since the ambient space is of constant holomorphic sectional curvature ± 4, the Gauss and Codazzi equations are respectively given by

$$R(X, Y)Z = \varepsilon\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY \tag{1}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \varepsilon\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2}$$

for any X, Y, Z tangent to M, where $\varepsilon = \pm 1$, depending on the ambient space is either complex projective space or complex hyperbolic space.

A real hypersurface in $M_m(c)$ is Hopf if its structure vector field is principal.

The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$, $m \geq 2$ was obtained by Takagi [29, 30] and consists in six distinct types of real hypersurfaces. Kimura, [11], proved that Takagi's real hypersurfaces are the unique Hopf real hypersurfaces with constant principal curvatures. Takagi's list is as follows:

(A_1) Geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$. They have 2 distinct constant principal curvatures $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$ and $cot(r)$ with eigenspace $\mathbb{D}$.

(A_2) Tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$. They have 3 distinct constant principal curvatures $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $cot(r)$ and $-tan(r)$. The corresponding eigenspaces of $cot(r)$ and $-tan(r)$ are complementary and ϕ-invariant distributions in $\mathbb{D}$.

(B) Tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric. They have 3 distinct constant principal curvatures $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $cot(r - \frac{\pi}{4})$ and $-tan(r - \frac{\pi}{4})$ whose corresponding eigenspaces are complementary and equal dimensional distributions in $\mathbb{D}$ such that $\phi T_{cot(r-\frac{\pi}{4})} = T_{-tan(r-\frac{\pi}{4})}$.

(C) Tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^n$, where $2n + 1 = m$ and $m \geq 5$. They have 5 distinct principal curvatures, $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $cot(r - \frac{\pi}{4})$ with multiplicity 2, $cot(r - \frac{\pi}{2}) = -tan(r)$, with multiplicity m-3, $cot(r - \frac{3\pi}{4})$, with multiplicity 2 and $cot(r - \pi) = cot(r)$ with multiplicity m-3. Moreover $\phi T_{cot(r-\frac{\pi}{4})} = T_{cot(r-\frac{3\pi}{4})}$ and $T_{-tan(r)}$ and $T_{cot(r)}$ are ϕ-invariant.

(D) Tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Plucker embedding of the complex Grassmannian manifold $G(2, 5)$ in $\mathbb{C}P^9$. They have the same principal curvatures as type C real hypersurfaces, $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, and the other four principal curvatures have the same multiplicity 4 and their eigenspaces have the same behaviour with respect to ϕ as in type C.

(E) Tubes of radius r, $0 < r < \frac{\pi}{4}$, over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^{15}$. They have the same principal curvatures as type C real hypersurfaces, $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $cot(r - \frac{\pi}{4})$ and $cot(r - \frac{3\pi}{4})$ have multiplicities equal to 6 and $-tan(r)$ and $cot(r)$ have multiplicities equal to 8. Their corresponding eigenspaces have the same behaviour with respect to ϕ as in type C real hypersurfaces.

In the case of $\mathbb{C}H^m$, $m \geq 2$, Berndt, [1], classified Hopf real hypersurfaces with constant principal curvatures into three types:

(A_1) Tubes of radius $r > 0$ over $\mathbb{C}H^k$, $0 \leq k \leq m - 1$ with 3 (respectively, 2) distinct constant principal curvatures if $0 < k < m - 1$ (respectively $k = 0$ or $k = m - 1$), $2coth(2r)$ with eigenspace $\mathbb{R}[\xi]$, $tanh(r)$ and $coth(r)$ whose eigenspaces are complementary and ϕ-invariant distributions in $\mathbb{D}$.

(A_2) Horospheres in $\mathbb{C}H^m$ with 2 distinct constant principal curvatures, 2 with eigenspace $\mathbb{R}[\xi]$ and 1 with eigenspace $\mathbb{D}$.

(B) Tubes of radius $r > 0$ over $\mathbb{R}H^m$, with 3 (respectively 2) distinct constant principal curvatures if $r \neq ln(2 + \sqrt{3})$, (respectively, $r = ln(2 + \sqrt{3})$), $2coth(2r)$ with eigenspace $\mathbb{R}[\xi]$, $tanh(r)$ and $coth(r)$, both with multiplicities equal to m-1 and such that $\phi T_{tanh(r)} = T_{coth(r)}$.

Ruled real hypersurfaces can be described as follows: Take a regular curve γ in $M_m(c)$ with tangent vector field X. At each point of γ there is a unique $M_{m-1}(c)$ cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a ruled real hypersurface. It will be an embedded hypersurface locally, although globally it will in general have self-intersections and singularities. Equivalently, a ruled real hypersurface satisfies that $g(A\mathbb{D}, \mathbb{D}) = 0$. For examples see [12] or [15].

In 2007 Berndt and Tamaru, [3], gave a complete classification of homogeneous real hypersurfaces in $\mathbb{C}H^m$, $m \geq 2$, obtaining 6 types of real hypersurfaces including types (A_1), (A_2) and (B). The principal curvatures and eigenspaces of the other 3 types were described by Berndt and Díaz-Ramos, see [2]. Among them, what the authors call type S real hypersurfaces are either the ruled minimal real hypersurfaces W^{2m-1} introduced in 1988 by Lohnherr, [14], for $r = 0$ or an equidistant hypersurface to W^{2m-1} at a distance $r > 0$.

Real hypersurfaces satisfying $A\phi = \phi A$ were classified by Okumura in 1975, [20], for the case of the complex projective space and by Montiel and Romero in 1986, [18], for the case of the complex hyperbolic space:

Theorem 1 *Let M be a real hypersurface of $M_m(c)$, $m \geq 2$. Then $A\phi = \phi A$ if and only if M is locally congruent to a homogeneous hypersurface of either the types (A_1) or (A_2) in $\mathbb{C}P^m$ or either the types (A_1), (A_2) or (B) in $\mathbb{C}H^m$.*

The Tanaka-Webster connection, [31, 33], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold. As a generalization of this connection Tanno, [32], defined the generalized Tanaka-Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y \tag{3}$$

for any tangent X, Y. Let k be a nonzero number. Using the naturally extended affine connection of Tanno's generalized Tanaka-Webster connection, Cho, [7, 8], defined the k-th g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface in $M_m(c)$ by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{4}$$

for any X, Y tangent to M. Then $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the k-th g-Tanaka-Webster connection coincides with the Tanaka-Webster connection.

2 Covariant Derivatives

Let M be a real hypersurface in a non-flat complex space form $M_n(c)$. On M we have the Levi-Civita connection and for any nonzero k the k-th g-Tanaka-Webster connection. Consider both covariant derivatives.

We have the tensor field of type (1, 2) on M given by the difference of both connections $F^{(k)}(X, Y) = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any X, Y tangent to M. We will call this tensor the k-th Cho tensor on M. Associated to it, for any X tangent to M and any non null k we can consider the tensor field of type (1, 1) $F^{(k)}_X Y = F^{(k)}(X, Y)$ for any Y tangent to M. This operator will be called the k-th Cho operator corresponding to X. The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $\hat{T}^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$, for any X, Y tangent to M.

Notice that if $X \in \mathbb{D}$, $F^{(k)}_X$ does not depend on k. In this case we will write simply F_X for $F^{(k)}_X$.

Consider any tensor T of type $(1, 1)$ on M. We can study when the covariant derivatives associated to Levi-Civita and g-Tanaka-Webster connections coincide on T, that is, $\nabla T = \hat{\nabla}^{(k)} T$. This is equivalent to the fact that for any X tangent to M, $TF_X^{(k)} = F_X^{(k)} T$, and its geometrical meaning is that every eigenspace of T is preserved by the k-th Cho operator $F_X^{(k)}$.

On the other hand, as $TM = Span\{\xi\} \oplus \mathbb{D}$, we can weak the above condition to the cases $X = \xi$ or $X \in \mathbb{D}$.

For the case $T = A$, in [27, 28], we obtained the following results:

Theorem 2 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $F_X A = AF_X$ for any $X \in \mathbb{D}$, if and only if M is locally congruent to a ruled real hypersurface.*

and

Theorem 3 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $F_\xi^{(k)} A = AF_\xi^{(k)}$ for a nonnull constant k if and only if M is locally congruent to a type (A) real hypersurface.*

And as consequence of both theorems we get

Corollary 1 *There do not exist real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 3$, such that $F_X^{(k)} A = AF_X^{(k)}$, for any X tangent to M and a nonnull constant k.*

The structure Jacobi operator R_ξ of M is an important tool to study the geometry of M. It is defined by $R_\xi X = R(X, \xi)\xi$, for any X tangent to M. Therefore its expression is given by

$$R_\xi X = \varepsilon\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{1}$$

If in our study we take $T = R_\xi$, in [21, 22] we have proved the following results

Theorem 4 *Let M be a real hypersurface in $M_m(c)$, $m \geq 2$. Then $F_X R_\xi = R_\xi F_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

and

Theorem 5 *Let M be a real hypersurface in $M_m(c)$, $m \geq 2$. Then $F_\xi^{(k)} R_\xi = R_\xi F_\xi^{(k)}$ for a nonnull k if and only if M is locally congruent either to a real hypersurface of type (A) or to a real hypersurface with $A\xi = 0$.*

As above we get

Corollary 2 *There do not exist real hypersurfaces M in $M_m(c)$, $m \geq 2$, such that $F_X^{(k)} R_\xi = R_\xi F_X^{(k)}$ for some nonnull constant k and any X tangent to M.*

The Ricci tensor of a real hypersurface M in $M_m(c)$ is given by

$$SX = \varepsilon\{(2m+1)X - 3\eta(X)\xi\} + hAX - A^2X \tag{2}$$

for any X tangent to M, where $h = trace(A)$.

It is well known that $M_n(c)$ does not admit real hypersurfaces with parallel Ricci tensor ($\nabla S = 0$). Therefore it is natural to investigate real hypersurfaces satisfying weaker conditions than the parallelism of S. Most important results on the study of the Ricci tensor of real hypersurfaces in non-flat complex space forms are included in Sect. 6 of [6].

We are going to suppose that $F_X S = S F_X$ for any $X \in \mathbb{D}$. This is equivalent to have

$$g(\phi AX, SY)\xi - \eta(SY)\phi AX = g(\phi AX, Y)S\xi - \eta(Y)S\phi AX \tag{3}$$

for any $X \in \mathbb{D}$, Y tangent to M. In [9] we prove the

Theorem 6 *There do not exist Hopf real hypersurfaces in $M_m(c)$, $m \geq 2$, whose Ricci tensor satisfies $F_X S = S F_X$ for any $X \in \mathbb{D}$.*

Therefore we can locally write $A\xi = \alpha\xi + \beta U$ for a unit $U \in \mathbb{D}$, where α and β are functions defined on M and $\beta \neq 0$. We also will call $\mathbb{D}_U = Span\{\xi, U, \phi U\}^{\perp}$. This is a holomorphic distribution in $\mathbb{D}$.

Taking the scalar product of (3) for $Y \in \mathbb{D}$ with Y yields $\eta(SY)g(\phi AX, Y) = 0$ for any $X, Y \in \mathbb{D}$. If $g(\phi AX, Y) = 0$ for any $X, Y \in \mathbb{D}$, M is a ruled real hypersurface. Therefore

Theorem 7 *Let M be a non Hopf real hypersurface in $M_m(c)$, $m \geq 2$, such that $F_X S = S F_X$ for any $X \in \mathbb{D}$. Then either M is ruled of $\eta(SY) = 0$ for any $Y \in \mathbb{D}$.*

Consider that $\eta(SY) = 0$ for any $Y \in \mathbb{D}$. It is easy to see that $AU = \beta\xi + (h - \alpha)U$ and $A\phi U = 0$. Therefore we have two possibilities

1. $h = \alpha$.
2. $h - \alpha \neq 0$. In this case we obtain $\beta^2 = \alpha(h - \alpha) - 3\varepsilon$. In the case of $\mathbb{C}P^m$ this yields $\alpha \neq 0$ and $h = \frac{\beta^2+\alpha^2+3}{\alpha}$ is also nonnull.

In the first case we obtain

Theorem 8 *Let M be a real hypersurface in $M_m(c)$, $m \geq 2$, such that $h = \alpha$. Then $F_X S = S F_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

So let us consider the second case for a real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$. We have seen that $\mathbb{D}_U$ is A-invariant. From (3) taking $Y \in \mathbb{D}_U$ such that $AY = \lambda Y$, we get either $\lambda = 0$ or if $\lambda \neq 0$, either $\lambda = h$ or $A\phi Y = 0$.

Not any eigenvalue in $\mathbb{D}_U$ can be zero, because in that case the type number is 2 and M should be ruled, giving a contradiction. Moreover there must be distinct than 0 and h and then $A\phi Y = 0$ for an eigenvector Y corresponding to such an eigenvalue.

By the Codazzi equation we see that $\phi T_0 \perp T_0$, where T_0 denotes the eigenspace corresponding to the eigenvalue 0 and $\phi T_h \perp T_h$ for the eigenspace corresponding to the eigenvalue h (that maybe does not appear). Thus we can write $\mathbb{D}_U = T_0 \oplus T_h \oplus \bar{\mathbb{D}}_U$. Where $\phi T_0 = T_h \oplus \bar{\mathbb{D}}_U$.

If either h or $\frac{\beta^2+3}{\alpha}$ is an eigenvalue in $\bar{\mathbb{D}}_U$ we can prove

$$\begin{aligned} grad(\beta) &= (\beta^2+5)\phi U \\ grad(\alpha) &= \frac{\alpha\beta(\beta^2+7)}{\beta^2+3}\phi U \end{aligned} \tag{4}$$

and this provides a contradiction. Thus we have

Theorem 9 *Let M be a non Hopf real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, such that $\alpha = g(A\xi, \xi) \neq h$. Then $F_X S = S F_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a real hypersurface such that $A\xi = \alpha\xi + \beta U$, for a unit $U \in \mathbb{D}$, α and β are nonvanishing functions, $AU = \beta\xi + \frac{\beta^2+3}{\alpha}U$, $A\phi U = 0$ and $\mathbb{D}_U = T_0 \oplus \bar{\mathbb{D}}_U$. All eigenvalues in $\bar{\mathbb{D}}_U$ are nonnull and different from h and $\frac{\beta^2+3}{\alpha}$. Moreover the sum of all nonnull eigenvalues in $\bar{\mathbb{D}}_U$ is* 0.

Remark: The real hypersurface appearing in last theorem satisfies that $Ker(A) = Span\{\phi U\} \oplus T_0$ is an integrable distribution whose integral leaves are totally geodesic and totally real in M. Therefore they are $\mathbb{R}P^{m-1}$.

Now consider the Riemannian curvature tensor R of a real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, and suppose that $\nabla_X R = \hat{\nabla}_X^{(k)} R$ for any $X \in \mathbb{D}$. If M is non Hopf and we follow the above notation, we obtain that $A\phi U = 0$, $AX = 0$ for any $X \in \mathbb{D}_U$ and $\alpha g(AU, U)^2 = (\beta^2+3)g(AU, U)$. If $g(AU, U) = 0$, M is ruled. If not, $AU = \beta\xi + \frac{\beta^2+3}{\alpha}U$. Then by Codazzi equation applied to X and ϕX, $X \in \mathbb{D}_U$, we get

$$g([\phi X, X], U) = -\frac{2}{\beta} \tag{5}$$

and

$$\frac{\beta^2+3}{\alpha} g([\phi X, X], U) = 0. \tag{6}$$

Both equations give a contradiction.

If M is Hopf we obtain $\alpha = 0$. If $X \in \mathbb{D}$ satisfies $AX = \lambda X$ we get $-\lambda^2 A\phi X = 3\lambda\phi X$. If $\lambda = 0$, as $A\phi X = \mu\phi X$ we arrive to a contradiction, because μ does not exist. Therefore $\lambda \neq 0$ and $-3\lambda = \lambda$. This is impossible and we have, [24],

Theorem 10 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $\nabla_X R = \hat{\nabla}^{(k)}_X R$ for any $X \in \mathbb{D}$ and some nonzero constant k if and only if M is locally congruent to a ruled real hypersurface.*

If now $\nabla_\xi R = \hat{\nabla}^{(k)}_\xi R$ and M is non Hopf, we get $A\phi U = \gamma\phi U$ for a certain function γ and $k\alpha g(AX, U) = 0$ for any $X \in \mathbb{D}_U$. Thus either $\alpha = 0$ or $g(AU, X) = 0$ for any $X \in \mathbb{D}_U$.

If $\alpha = 0$ we can prove that $A\xi = \beta U$, $AU = \beta\xi$, $A\phi U = k\phi U$, where $k^2 = \beta + 3$. Moreover, as $\mathbb{D}_U$ is A-invariant, if $Y \in \mathbb{D}_U$ satisfies $AY = \lambda Y$, $A\phi Y = \frac{k\lambda - 1}{k}\phi Y$. But we can also obtain $k^2\lambda\phi Y = k^2 A\phi Y$. Both expressions give a contradiction. Thus $\alpha \neq 0$.

After some work we get $grad(\beta) = (2 + \alpha\frac{\beta^2+3}{k} + 2\beta^2)\phi U$. From this we have $(\frac{\beta^2+3}{k})^2 + \beta^2 + 1 = 0$, which is impossible.

Therefore M must be Hopf and we obtain $\alpha(A\phi - \phi A)X = 0$ for any X tangent to M. If $A\phi = \phi A$, M must be of type (A). If $\alpha = 0$ we find that M has, at most, three distinct constant principal curvatures. Then (see [19]) M is locally congruent to a real hypersurface either of type (A) or of type (B). As type (B) real hypersurfaces do not have $\alpha = 0$, we obtain (see [24])

Theorem 11 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $\nabla_\xi R = \hat{\nabla}^{(k)}_\xi R$ for some nonnull constant k if and only if M is locally congruent to a type (A) real hypersurface.*

As a consequence

Corollary 3 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\nabla R = \hat{\nabla}^{(k)} R$ for some nonnull constant k.*

3 Lie Derivatives

Let $\mathscr{L}$ denote the Lie derivative of a real hypersurface M in $\mathbb{C}P^m$. Then $\mathscr{L}_X Y = \nabla_X Y - \nabla_Y X$ for any X, Y tangent to M. Moreover, for any tensor T of type $(1, 1)$ on M $(\mathscr{L}_X T)Y = \mathscr{L}_X TY - T\mathscr{L}_X Y$.

Associated to the k-th g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ on M we can consider the so-called Lie derivative associated to such a connection (introduced by Jeong, Pak and Suh in [10] for real hypersurfaces of complex two-plane Grassmannians) defined by $\hat{\mathscr{L}}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X$ for any X, Y tangent to M.

Suppose that $\mathscr{L}_\xi A = \hat{\mathscr{L}}^{(k)}_\xi A$. If M is non Hopf, this yields $AU = \beta\xi + kU$, $A\phi U = \frac{\alpha+k}{2}\phi U$ and $\mathbb{D}_U$ is A-invariant. But we also obtain $\frac{k-\alpha}{2}AU = \beta(\frac{k-\alpha}{2})\xi + (\frac{k^2-\alpha^2}{4} - \beta^2)U$. If $\alpha = k$ this yields $\beta^2 U = 0$, which is impossible. Therefore $AU = \beta\xi + \frac{2}{k-\alpha}(\frac{k^2-\alpha^2}{4} - \beta^2)U$. Both expressions for AU give $(k - \alpha)^2 = -4\beta^2$, which is impossible and M must be Hopf.

If M is Hopf it is easy to see that M must be of type (A). Therefore we have [23].

Theorem 12 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Then $\mathscr{L}_\xi A = \hat{\mathscr{L}}_\xi^{(k)} A$ for some nonnull k if and only if M is locally congruent to a real hypersurface of type (A).*

If now we suppose that $\mathscr{L}_X A = \hat{\mathscr{L}}_X^{(k)} A$ for any $X \in \mathbb{D}$ we can prove that M must be Hopf. In this case, if λ is a principal curvature in $\mathbb{D}$ we obtain

$$\lambda^2 + (k - \alpha)\lambda - k\alpha = 0. \tag{1}$$

Thus either $\lambda = \alpha$ or $\lambda = -k$ and M has, at most, two distinct constant principal curvatures. Therefore M must be locally congruent to a geodesic hypersphere, [5]. As M cannot be totally umbilical, there exists $Y \in \mathbb{D}$ such that $AY = -kY$. But then $A\phi Y = \alpha Y$. Therefore $\alpha \neq -k$ and this contradicts the fact that M is a geodesic hypersphere. Then

Theorem 13 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L}_X A = \hat{\mathscr{L}}_X^{(k)} A$ for some nonnull k and any $X \in \mathbb{D}$.*

As above

Corollary 4 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L} A = \hat{\mathscr{L}}^{(k)} A$ for some nonnull k.*

Now consider the structure Jacobi operator R_ξ on M. In [26] we proved the following

Theorem 14 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L}_\xi R_\xi = 0$. Then either M is locally congruent to a tube of radius $\frac{\pi}{4}$ over a complex submanifold in $\mathbb{C}P^m$ or to a real hypersurface of type (A) with radius $r \neq \frac{\pi}{4}$.*

Suppose now that $\mathscr{L}_\xi R_\xi = \hat{\mathscr{L}}_\xi^{(k)} R_\xi$. Then $(\phi A - A\phi)R_\xi = R_\xi(\phi A - A\phi)$. This yields (see [26]).

Theorem 15 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $\mathscr{L}_\xi R_\xi = \hat{\mathscr{L}}_\xi^{(k)} R_\xi$ for some nonnull k if and only if M is locally congruent to either a real hypersuface of type (A) and radius $r \neq \frac{\pi}{4}$ or to a tube of radius $\frac{\pi}{4}$ around a complex submanifold in $\mathbb{C}P^m$.*

If now we suppose $\mathscr{L}_X R_\xi = \hat{\mathscr{L}}_X^{(k)} R_\xi$ for any $X \in \mathbb{D}$ and M is non Hopf we get $\alpha g(A^2\phi U, U) = 0$.

If $\alpha = 0$, $A\xi = \beta U$, $AU = \beta\xi + kU$, $A\phi U = -k\phi U$. We also prove that the unique eigenvalue in $\mathbb{D}_U$ is k. Now the Codazzi equation yields $k = 0$, a contradiction.

Therefore $\alpha \neq 0$, $AU = \beta\xi + \omega U$, $A\phi U = \delta\phi U$, for some functions ω and δ. Then we obtain $\alpha^2 = 1, \omega = \frac{\beta^2 - 1}{\alpha} = k, \delta = k$. That is, $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + kU$, $A\phi U = kU$, $AZ = -\frac{1}{\alpha}Z$, for any $Z \in \mathbb{D}_U$. This case yields $4k^2 - \alpha k + 3 = 0$. There does not exist any k satisfying this equation. Therefore M must be Hopf.

Let X be a unit vector field in $\mathbb{D}$ such that $AX = \lambda X$. From [20], $A\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}$. Then we have three possibilities

- $\lambda + \mu = 0$, $\lambda = k$. Then $k^2 = -1$, which is impossible.
- $\lambda + \mu = 0$, $\mu = -\frac{1}{\alpha}$. Then $\alpha^2 = -1$, also impossible.
- $\lambda = \mu = -\frac{1}{\alpha}$. Then $2 = 0$, also impossible.

Therefore we obtain

Theorem 16 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L}_X R_\xi = \hat{\mathscr{L}}_X^{(k)} R_\xi$ for any $X \in \mathbb{D}$ and some nonnull k.*

We also have the following corollaries

Corollary 5 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L} R_\xi = \hat{\mathscr{L}}^{(k)} R_\xi$ for some nonnull k.*

and

Corollary 6 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull constant. Then $\hat{\mathscr{L}}_\xi^{(k)} R_\xi = 0$ if and only if M is locally congruent to either a tube of radius $\frac{\pi}{4}$ around a complex submanifold in $\mathbb{C}P^m$ or to a real hypersurface of type (A) and radius $r \neq \frac{\pi}{4}$.*

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