The Solvable Models of Noncompact Real Two-Plane Grassmannians and Some Applications

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Abstract Every Riemannian symmetric space of noncompact type is isometric to some solvable Lie group equipped with a left-invariant Riemannian metric. The corresponding metric solvable Lie algebra is called the solvable model of the symmetric space. In this paper, we give explicit descriptions of the solvable models of noncompact real two-plane Grassmannians, and mention some applications to submanifold geometry, contact geometry, and geometry of left-invariant metrics.

1 Introduction

In the studies on Riemannian symmetric spaces of noncompact type, the solvable models have played important roles. Let $M = G/K$ be a Riemannian symmetric space of noncompact type, where G is the identity component of the isometry group $\mathrm{Isom}(M)$. Let $G = KAN$ be an Iwasawa decomposition, where K is maximal compact, A is abelian, and N is nilpotent. Then M is isometric to the solvable Lie group $S := AN$, by being equipped with a suitable left-invariant metric $\langle , \rangle$. The

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solvmanifold $(S, \langle , \rangle)$, or the corresponding metric solvable Lie algebra $(\mathfrak{s}, \langle , \rangle)$, is called the *solvable model* of the symmetric space $M = G/K$.

For a real hyperbolic space $\mathbb{R}\mathrm{H}^n$, its solvable model is the so-called Lie algebra of $\mathbb{R}\mathrm{H}^n$, which is of a quite simple form and has several interesting properties (see [13, 16, 18]). For other (complex, quaternion, and octonion) hyperbolic spaces, which are of rank one, their solvable models are given by Damek–Ricci spaces [1, 5]. Particularly in the case of a complex hyperbolic space $\mathbb{C}\mathrm{H}^n$, the solvable model provides a lot of interesting examples of isometric actions and homogeneous submanifolds. We refer to a survey paper [11] and references therein. These studies have still continued, for examples, the third author [14] studied the geometry of polar foliations on $\mathbb{C}\mathrm{H}^n$, and the second author and Kajigaya [10] studied homogeneous Lagrangian submanifolds in $\mathbb{C}\mathrm{H}^n$.

For higher rank cases, the solvable models are theoretically known, and can be described in terms of the root systems. They have played fundamental roles in the studies on symmetric spaces of noncompact type. Among others, successive examples would be the studies on homogeneous codimension one foliations [3] and hyperpolar foliations [4]. However, we sometimes need more explicit descriptions of the solvable models, in order to study more detailed properties, as in the case of complex hyperbolic spaces $\mathbb{C}\mathrm{H}^n$.

In this paper, we concentrate on a noncompact real two-plane Grassmannian $G_2^*(\mathbb{R}^{n+2})$, and explicitly describe its solvable model according to [8]. It is not difficult to determine the structure of the solvable model, but as far as we know, it is hard to find it in the literature. We also give several applications of the solvable model of $G_2^*(\mathbb{R}^{n+2})$. The topics contain cohomogeneity one actions (homogeneous codimension one foliations), geometry of Lie hypersurfaces, particular contact metric manifolds, and left-invariant Einstein and Ricci soliton metrics on Lie groups. We believe that our solvable model would play a fundamental role in further studies on geometry of $G_2^*(\mathbb{R}^{n+2})$.

2 The Solvable Model

In this section, we recall a description of the solvable models of noncompact real two-plane Grassmannians $G_2^*(\mathbb{R}^{n+2}) = \mathrm{SO}^0(2, n)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(n))$, according to the description given in [8].

2.1 A Description of the Solvable Model

In this subsection we give a definition of the solvable model of $G_2^*(\mathbb{R}^{n+2})$. We usually assume $n \geq 3$, since, in the case of $n = 2$, the symmetric space $G_2^*(\mathbb{R}^4)$ is not irreducible and has different features.

Definition 1 Let $c > 0$ and $n \geq 3$. We call $(\mathfrak{s}(c), \langle , \rangle, J)$ the *solvable model* of $G_2^*(\mathbb{R}^{n+2})$ if

(1) $\mathfrak{s}(c) := \mathrm{span}\{A_1, A_2, X_0, Y_1, \ldots, Y_{n-2}, Z_1, \ldots, Z_{n-2}, W_0\}$ is a $2n$-dimensional Lie algebra whose nonzero bracket relations are defined by

- $[A_1, X_0] = cX_0$, $[A_1, Y_i] = -(c/2)Y_i$, $[A_1, Z_i] = (c/2)Z_i$, $[A_1, W_0] = 0$,
- $[A_2, X_0] = 0$, $[A_2, Y_i] = (c/2)Y_i$, $[A_2, Z_i] = (c/2)Z_i$, $[A_2, W_0] = cW_0$,
- $[X_0, Y_i] = cZ_i$, $[Y_i, Z_i] = cW_0$.

(2) $\langle , \rangle$ is an inner product on $\mathfrak{s}(c)$ so that the above basis is orthonormal,
(3) J is a complex structure on $\mathfrak{s}(c)$ given by

$$J(A_1) = -X_0, \quad J(A_2) = W_0, \quad J(Y_i) = Z_i.$$

Let $S(c)$ denote the connected and simply-connected Lie group with Lie algebra $\mathfrak{s}(c)$, equipped with the induced left-invariant metric $\langle , \rangle$ and the induced complex structure J. The triplet $(S(c), \langle , \rangle, J)$ is also called the solvable model.

Theorem 2 ([8]) *The solvable model* $(S(c), \langle , \rangle, J)$ *is isomorphic to* $G_2^*(\mathbb{R}^{n+2})$ *with minimal sectional curvature* $-c^2$.

The proof is given by describing the Iwasawa decomposition of $\mathfrak{so}(2, n)$ explicitly, in terms of matrices. This is long but a straightforward calculation.

We here see the structure of the Lie algebra $\mathfrak{s}(c)$. One can directly see that

$$\mathfrak{n} := [\mathfrak{s}(c), \mathfrak{s}(c)] = \mathrm{span}\{X_0, Y_1, \ldots, Y_{n-2}, Z_1, \ldots, Z_{n-2}, W_0\}.$$

Furthermore, by the given bracket relations, we have

$$[\mathfrak{n}, \mathfrak{n}] = \mathrm{span}\{Z_1, \ldots, Z_{n-2}, W_0\}, \quad [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \mathrm{span}\{W_0\}, \quad [\mathfrak{n}, [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]]] = 0.$$

Therefore, $\mathfrak{s}(c)$ is solvable, whose derived subalgebra is three-step nilpotent. This is compatible with the root space decomposition, mentioned in the next subsection.

2.2 A Description in Terms of Root Spaces

In this subsection, we describe the root space decomposition of the solvable model $(\mathfrak{s}(c), \langle , \rangle, J)$. We need such description in order to translate some general results stated in terms of the root spaces.

Let us put $\mathfrak{a} := \mathrm{span}\{A_1, A_2\} \subset \mathfrak{s}(c)$, which is an abelian subalgebra. Then, for each $\alpha \in \mathfrak{a}^*$, the root space $\mathfrak{s}_\alpha$ of $\mathfrak{s}(c)$ with respect to $\mathfrak{a}$ is defined by

$$\mathfrak{s}_\alpha := \{X \in \mathfrak{s}(c) \mid [H, X] = \alpha(H)X \quad (\forall H \in \mathfrak{a})\}.$$

Proposition 3 *Let us define* $\varepsilon_i \in \mathfrak{a}^*$ *by*

$$\varepsilon_1(A_1) := c/2, \quad \varepsilon_2(A_1) := -c/2, \quad \varepsilon_1(A_2) := c/2, \quad \varepsilon_2(A_2) := c/2.$$

Then the nontrivial root spaces can be described as follows:

$$\mathfrak{s}_{\varepsilon_1-\varepsilon_2} = \mathrm{span}\{X_0\}, \qquad \mathfrak{s}_{\varepsilon_2} = \mathrm{span}\{Y_1, \ldots, Y_{n-2}\},$$
$$\mathfrak{s}_{\varepsilon_1} = \mathrm{span}\{Z_1, \ldots, Z_{n-2}\}, \qquad \mathfrak{s}_{\varepsilon_1+\varepsilon_2} = \mathrm{span}\{W_0\}.$$

Proof It follows directly from the bracket relations of the solvable model. □

As usual, we put $\alpha_1 := \varepsilon_1 - \varepsilon_2$ and $\alpha_2 := \varepsilon_2$. We then have the root space decomposition of $\mathfrak{s}(c)$ with respect to $\mathfrak{a}$,

$$\mathfrak{s}(c) = \mathfrak{a} \oplus \mathfrak{s}_{\alpha_1} \oplus \mathfrak{s}_{\alpha_2} \oplus \mathfrak{s}_{\alpha_1+\alpha_2} \oplus \mathfrak{s}_{\alpha_1+2\alpha_2}.$$

Therefore the set of roots is of type B_2, and $\{\alpha_1, \alpha_2\}$ is the set of simple roots. This agrees with the root system of $G_2^*(\mathbb{R}^{n+2})$.

3 Applications

In this section, we mention several applications of the solvable models $(S(c), \langle, \rangle, J)$ of noncompact real two-plane Grassmannians $G_2^*(\mathbb{R}^{n+2})$.

3.1 Cohomogeneity One Actions

In this subsection, we study cohomogeneity one actions on $G_2^*(\mathbb{R}^{n+2})$ in terms of the solvable model.

Definition 4 For an isometric action on a Riemannian manifold, maximal dimensional orbits are said to be *regular*, and other orbits *singular*. The codimension of a regular orbit is called the *cohomogeneity* of the action.

Therefore, a cohomogeneity one action is an isometric action whose regular orbits are of codimension one. For irreducible symmetric spaces of noncompact type, cohomogeneity one actions without singular orbit (equivalently, homogeneous codimension one foliations) have been classified in [3]. The classification result is described in terms of the root systems, but one can translate it into the solvable models as follows.

Theorem 5 ([3]) *An isometric action of a connected group on* $G_2^*(\mathbb{R}^{n+2})$ *is a cohomogeneity one action without singular orbit if and only if it is orbit equivalent to one of the actions given by*

(N) $\mathfrak{h} = \mathrm{span}\{a_1 A_1 + a_2 A_2\} \oplus \mathfrak{n}$ *with* $a_1^2 + a_2^2 = 1$,
(A_1) $\mathfrak{h} = \mathfrak{s}(c) \ominus \mathrm{span}\{X_0\}$,
(A_2) $\mathfrak{h} = \mathfrak{s}(c) \ominus \mathrm{span}\{Y_1\}$.

We refer these actions as the actions of type (N), (A_1), and (A_2), respectively. Note that there exist continuously many actions of type (N). The orbits of these actions play leading roles throughout this section.

Remark 6 Let H be a Lie subgroup of the solvable model $S(c)$. We identify $G_2^*(\mathbb{R}^{n+2}) \cong S(c)$, and hence H acts on $S(c)$ by the multiplication from the left. In this paper we consider H is acting on $G_2^*(\mathbb{R}^{n+2})$ in this way. On the other hand, one has $G_2^*(\mathbb{R}^{n+2}) = \mathrm{SO}^0(2, n)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(n))$, and H acts on this homogeneous space since $H \subset S(c) \subset \mathrm{SO}^0(2, n)$. We note that these two actions are equivariant, by the identification $F : S(c) \to G_2^*(\mathbb{R}^{n+2}) : g \mapsto g.o$, where o denotes the origin.

3.2 Lie Hypersurfaces

In this subsection, we study extrinsic geometry of orbits of cohomogeneity one actions on $G_2^*(\mathbb{R}^{n+2})$ without singular orbits. These orbits are sometimes called *Lie hypersurfaces*.

Proposition 7 ([3, 15]) *For the cohomogeneity one actions on* $G_2^*(\mathbb{R}^{n+2})$ *described in Theorem 5, we have the following:*

(1) For each action of type (N)*, all orbits are isometrically congruent to each other.*
(2) For each of the actions of type (A_1) *and* (A_2)*, there exists the unique minimal orbit.*

It depends on the choice of $a_1 A_1 + a_2 A_2$ whether a cohomogeneity one action of type (N) has minimal orbits or not. In order to study it, we have only to study the minimality of the orbit $H.e$ through the identity $e \in S(c)$. This is equivalent to the minimality of the Lie subgroup $H \subset S(c)$.

We here recall some general facts on the minimality of Lie subgroups. Let $(G, \langle , \rangle)$ be a Lie group with a left-invariant Riemannian metric, which we identify with the corresponding metric Lie algebra $(\mathfrak{g}, \langle , \rangle)$. First of all, we define the symmetric bilinear form $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$2\langle U(X, Y), Z\rangle = \langle [Z, X], Y\rangle + \langle X, [Z, Y]\rangle \quad (\forall X, Y, Z \in \mathfrak{g}).$$

Then, the Koszul formula yields that the Levi-Civita connection $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of $(\mathfrak{g}, \langle , \rangle)$ can be written as

$$\nabla_X Y = (1/2)[X, Y] + U(X, Y).$$

Let H be a Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then the second fundamental form $h : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g} \ominus \mathfrak{h}$ of the submanifold $H \subset G$ is defined by

$$h(X, Y) := (\nabla_X Y)^{\perp} := U(X, Y)_{\mathfrak{g}\ominus\mathfrak{h}},$$

which means the $(\mathfrak{g} \ominus \mathfrak{h})$-component of $U(X, Y)$ (and $\ominus$ denotes the orthogonal complement). The trace of h is called the *mean curvature vector* of the submanifold H in G, and H is said to be *minimal* if the mean curvature vector vanishes. In order to study the minimality of some Lie subgroups, the following notion is convenient.

Definition 8 A vector $H_0 \in \mathfrak{g}$ is called the *mean curvature vector* of $(\mathfrak{g}, \langle,\rangle)$ if it satisfies

$$\langle H_0, X\rangle = \mathrm{tr}(\mathrm{ad}_X) \quad (\forall X \in \mathfrak{g}).$$

Note that one has to distinguish the mean curvature vector of $(\mathfrak{g}, \langle,\rangle)$ and the mean curvature vector of a submanifold H in G. These two mean curvature vectors are related in the following particular cases.

Proposition 9 *Let H_0 be the mean curvature vector of $(\mathfrak{g}, \langle,\rangle)$, and H be a Lie subgroup of G whose Lie algebra $\mathfrak{h}$ contains $[\mathfrak{g}, \mathfrak{g}]$. Then the mean curvature vector of the submanifold H in G coincides with $(H_0)_{\mathfrak{g}\ominus\mathfrak{h}}$.*

Proof Since $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$, one has a decomposition $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}] \oplus (\mathfrak{h} \ominus [\mathfrak{g}, \mathfrak{g}])$. Let $\{e_i\}$ and $\{e'_j\}$ be orthonormal bases of $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{h} \ominus [\mathfrak{g}, \mathfrak{g}]$, respectively. Then, the mean curvature vector H'_0 of the submanifold H in G is given by

$$H'_0 = \sum h(e_i, e_i) + \sum h(e'_j, e'_j) = \sum U(e_i, e_i)_{\mathfrak{g}\ominus\mathfrak{h}} + \sum U(e'_j, e'_j)_{\mathfrak{g}\ominus\mathfrak{h}}.$$

Here, since $e'_j \perp [\mathfrak{g}, \mathfrak{g}]$, one has $U(e'_j, e'_j) = 0$. We thus have

$$H'_0 = \sum U(e_i, e_i)_{\mathfrak{g}\ominus\mathfrak{h}}.$$

Our claim is $H'_0 = (H_0)_{\mathfrak{g}\ominus\mathfrak{h}}$. Take any $X \in \mathfrak{g} \ominus \mathfrak{h}$. Then we have

$$\langle H'_0, X\rangle = \langle \sum U(e_i, e_i), X\rangle = \sum \langle [X, e_i], e_i\rangle = \mathrm{tr}(\mathrm{ad}_X|_{[\mathfrak{g},\mathfrak{g}]}).$$

On the other hand, by the definition of H_0, one knows

$$\langle H_0, X\rangle = \mathrm{tr}(\mathrm{ad}_X) = \mathrm{tr}(\mathrm{ad}_X|_{[\mathfrak{g},\mathfrak{g}]}),$$

where the last equality follows from $\mathrm{ad}_X(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$. This completes the proof. □

Remark 10 In general, the mean curvature vector H_0 of $(\mathfrak{g}, \langle,\rangle)$ satisfies

$$\langle H_0, [\mathfrak{g}, \mathfrak{g}]\rangle = 0,$$

since $\langle H_0, [X, Y]\rangle = \mathrm{tr}(\mathrm{ad}_{[X,Y]}) = \mathrm{tr}([\mathrm{ad}_X, \mathrm{ad}_Y]) = 0$. Therefore, if we consider the particular case $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$, the mean curvature vector of the submanifold $H = [G, G]$ coincides with H_0. This is a reason why H_0 is called the mean curvature vector.

We apply this general theory to the actions of type (N) on $G_2^*(\mathbb{R}^{n+2}) \cong S(c)$. By the given bracket relations of $\mathfrak{s}(c)$, one can directly calculate H_0.

Proposition 11 $H_0 := cA_1 + c(n-1)A_2$ *is the mean curvature vector of the solvable model* $(\mathfrak{s}(c), \langle , \rangle)$.

For an action of type (N), there exist no minimal orbits in a generic case. However, if $a_1A_1 + a_2A_2$ is a particular one, then the action has minimal orbit (and hence all orbits are minimal). Such phenomenon has been known in [3, Corollary 3.2], but we here point out which action has a minimal orbit.

Proposition 12 *A cohomogeneity one action of type* (N) *on* $G_2^*(\mathbb{R}^{n+2})$ *has a minimal orbit (and hence all orbits are minimal) if and only if it is given by*

$$\mathfrak{h} := \mathrm{span}\{A_1 + (n-1)A_2\} \oplus \mathfrak{n}.$$

Proof Let $\mathfrak{h} := \mathrm{span}\{a_1A_1 + a_2A_2\} \oplus \mathfrak{n}$, and H be the connected Lie subgroup of $S(c)$ with Lie algebra $\mathfrak{h}$. We study the condition for the submanifold H in $S(c)$ to be minimal. Note that $[\mathfrak{s}(c), \mathfrak{s}(c)] = \mathfrak{n} \subset \mathfrak{h}$ holds. Therefore, by Proposition 9, H is minimal in $S(c)$ if and only if $(H_0)_{\mathfrak{s}(c)\ominus\mathfrak{h}} = 0$. This is equivalent to $\mathfrak{h} = \mathrm{span}\{H_0\} \oplus \mathfrak{n}$. We thus complete the proof by Proposition 11. □

3.3 Einstein Solvmanifolds

In this subsection, we study intrinsic geometry of orbits of cohomogeneity one actions on $G_2^*(\mathbb{R}^{n+2})$ without singular orbits. In particular, they provide examples of Einstein solvmanifolds. First of all we recall the following notation.

Definition 13 A metric solvable Lie algebra $(\mathfrak{s}, \langle , \rangle)$ is said to be of *Iwasawa type* if

(i) $\mathfrak{a} := \mathfrak{s} \ominus [\mathfrak{s}, \mathfrak{s}]$ is abelian,
(ii) for every $A \in \mathfrak{a}$, ad_A is symmetric with respect to $\langle , \rangle$, and $\mathrm{ad}_A \neq 0$ if $A \neq 0$,
(iii) there exists $A_0 \in \mathfrak{a}$ such that $\mathrm{ad}_{A_0}|_{[\mathfrak{s},\mathfrak{s}]}$ is positive definite.

One can easily see that the solvable model $(\mathfrak{s}(c), \langle , \rangle)$ of $G_2^*(\mathbb{R}^{n+2})$ is of Iwasawa type. More generally, the solvable parts of Iwasawa decompositions of semisimple Lie algebras are of Iwasawa type.

Proposition 14 ([12], Theorem 4.18) *Let* $(\mathfrak{s}, \langle , \rangle)$ *be an Einstein solvable Lie algebra of Iwasawa type, and* H_0 *be the mean curvature vector of* $(\mathfrak{s}, \langle , \rangle)$. *We put* $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$, $\mathfrak{a} := \mathfrak{s} \ominus \mathfrak{n}$, *and take a nonzero subspace* $\mathfrak{a}' \subset \mathfrak{a}$. *Then* $(\mathfrak{s}' := \mathfrak{a}' \oplus \mathfrak{n}, \langle , \rangle|_{\mathfrak{s}'\times\mathfrak{s}'})$ *is Einstein if and only if* $H_0 \in \mathfrak{a}'$.

The above procedure is called the rank reduction of an Einstein solvable Lie algebra $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle , \rangle)$. Note that our solvable model is Einstein, since it is isometric to an irreducible symmetric space. Hence, by applying the above procedure, we immediately have the following.

Proposition 15 *Let* $(\mathfrak{s}(c), \langle , \rangle, J)$ *be the solvable model of* $G_2^*(\mathbb{R}^{n+2})$*, and put* $\mathfrak{h} := \mathrm{span}\{A_1 + (n-1)A_2\} \oplus \mathfrak{n}$. *Then, for the corresponding cohomogeneity one action of type* (N)*, all orbits are Einstein hypersurfaces with respect to the induced metrics.*

In particular, $G_2^*(\mathbb{R}^{n+2})$ admits (homogeneous) real hypersurfaces which are Einstein. This is an easy observation, but would be interesting from the viewpoint of submanifold geometry. In fact, this is in contrast to the case of $\mathbb{C}\mathrm{H}^n$, namely, $\mathbb{C}\mathrm{H}^n$ do not admit any Einstein real hypersurfaces (see [19]).

3.4 Contact Metric Manifolds

In this subsection, we apply the solvable model $(\mathfrak{s}(c), \langle , \rangle, J)$ of $G_2^*(\mathbb{R}^{n+2})$ to study contact metric manifolds. Let M be a smooth manifold and $\mathfrak{X}(M)$ denote the set of all smooth vector field. A contact metric structure is denoted by (η, ξ, φ, g). The following notion has been introduced in [6].

Definition 16 Let $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold $(M, \eta, \xi, \varphi, g)$ is called a (κ, μ)-*space* if the Riemannian curvature tensor R satisfies

$$R(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y) \quad (\forall X, Y \in \mathfrak{X}(M)),$$

where I denotes the identity transformation and $h := (1/2)\mathscr{L}_\xi \varphi$ is the Lie derivative of φ along ξ.

It has been known that $\kappa \leq 1$ always holds. Furthermore, a contact metric manifold is Sasakian if and only if it is a $(1, \mu)$-space [6]. Therefore, the class of (κ, μ)-spaces is a kind of generalization of Sasakian manifolds. Typical examples of non-Sasakian (κ, μ)-spaces are the unit tangent sphere bundles $T_1(M(c))$ over Riemannian manifolds $M(c)$ of constant curvature $c \neq 1$. Non-Sasakian (κ, μ)-spaces have been studied deeply by Boeckx [7], but a geometric understanding seems to be not enough. The following gives a realization of $(0, 4)$-spaces.

Theorem 17 ([8]) *Let* $(\mathfrak{s}(2\sqrt{2}), \langle , \rangle, J)$ *be the solvable model of* $G_2^*(\mathbb{R}^{n+2})$ *with normalization* $c = 2\sqrt{2}$*, where* $n \geq 3$. *Then,* $\mathfrak{h} := \mathfrak{s}(2\sqrt{2}) \ominus \mathrm{span}\{A_1 + A_2\}$ *is a subalgebra, and the corresponding Lie group* H *equipped with the standard almost contact metric structure is a* $(0, 4)$*-space of dimension* $2n - 1$.

Recall that every real hypersurface in a Kähler manifold admits an almost contact metric structure. Note that $G_2^*(\mathbb{R}^{n+2})$ is a Hermitian symmetric space, which is Kähler, of dimension $2n$. Therefore, the above Lie subgroup H is equipped with an almost contact metric structure, and of dimension $2n-1$. The proof is given by showing that $\mathfrak{h}$ is isomorphic to the example constructed by Boeckx [7].

We also note that this result is relevant to the study by Berndt and Suh [2], who classified contact real hypersurfaces in $G_2^*(\mathbb{R}^{n+2})$ with constant principal curvatures. The above $(0, 4)$-space is an example of such hypersurfaces, and hence is contained in their classification list (which is called a horosphere).

3.5 Ricci Soliton Solvmanifolds

In this subsection, we see that the orbits of cohomogeneity one actions of type (N) provide examples of Ricci soliton solvmanifolds. Recall that a Riemannian manifold (M, g) is called a *Ricci soliton* if there exist $c \in \mathbb{R}$ and $X \in \mathfrak{X}(M)$ such that the Ricci tensor Ric_g satisfies

$$\mathrm{Ric}_g = cg + \mathscr{L}_X g,$$

where $\mathscr{L}_X g$ denotes the Lie derivative of g along X.

Definition 18 A metric Lie algebra $(\mathfrak{g}, \langle , \rangle)$ is called an *algebraic Ricci soliton with constant* $c \in \mathbb{R}$ *if there exists* a derivation $D \in \mathrm{Der}(\mathfrak{g})$ such that

$$\mathrm{Ric} = c \cdot \mathrm{id} + D.$$

An algebraic Ricci soliton is called a *solvsoliton* if $\mathfrak{g}$ is solvable, and a *nilsoliton* if $\mathfrak{g}$ is nilpotent. Note that any algebraic Ricci soliton gives rise to a Ricci soliton metric on the corresponding simply-connected Lie group (see [17]).

Proposition 19 ([8, 17]) *Let* $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle , \rangle)$ *be a solvsoliton with constant* $c < 0$. *Take any subspace* $\mathfrak{a}'$ *of* $\mathfrak{a}$, *and put* $\mathfrak{s}' := \mathfrak{a}' \oplus \mathfrak{n}$. *Then,* $\mathfrak{s}'$ *is a subalgebra, and* $(\mathfrak{s}', \langle , \rangle|_{\mathfrak{s}' \times \mathfrak{s}'})$ *is also a solvsoliton with constant* c.

Recall that our solvable model $\mathfrak{s}(c)$ is Einstein with negative scalar curvature and solvable, which is a special case of solvsolitons with constant $c < 0$. Therefore, the above proposition yields the following.

Proposition 20 *All orbits of cohomogeneity one actions of type* (N) *on* $G_2^*(\mathbb{R}^{n+2})$ *are Ricci soliton solvmanifolds.*

Recall that a particular choice of $\mathfrak{a}'$, that is $\mathfrak{h} := \mathrm{span}\{A_1 + (n-1)A_2\} \oplus \mathfrak{n}$, gives rise to an Einstein solvmanifold (see Proposition 15). Other choices of $\mathfrak{a}'$ provide nontrivial (not Einstein) Ricci soliton solvmanifolds.

Corollary 21 *The connected, simply-connected and complete* $(0, 4)$*-space with dimension* ≥ 5 *is a nontrivial Ricci soliton.*

Proof It has been known in [7] that non-Sasakian (κ, μ)-spaces are locally determined by its dimension and the values $(\kappa, \mu) \in \mathbb{R}^2$. Therefore, a connected, simply-connected and complete $(0, 4)$-space is isometric to the one given in Theorem 17 by

$$\mathfrak{h} = \mathfrak{s}(2\sqrt{2}) \ominus \mathrm{span}\{A_1 + A_2\} = \mathrm{span}\{A_1 - A_2\} \oplus \mathfrak{n}.$$

In particular, it is an orbit of a cohomogeneity one action of type (N). By Proposition 20, it must be Ricci soliton. Furthermore, it is not Einstein, since $A_1 - A_2$ is not proportional to H_0. □

Note that Ghosh–Sharma [9] have studied non-Sasakian (κ, μ)-spaces which are Ricci soliton. In fact, they have proved the following classification result.

Theorem 22 ([9]) *Let M be a non-Sasakian (κ, μ)-space whose metric is a Ricci soliton. Then M is locally isometric to either $(0, 0)$-space or $(0, 4)$-space as a contact metric manifold.*

For $(0, 4)$-spaces with dimension ≥ 5, the converse statement would not be explicitly examined (they have used the software MATLAB). Our argument above complements the theorem of Ghosh–Sharma, by giving a Lie-theoretic proof of the converse direction, which can be checked by hand.

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