The Chern-Moser-Tanaka Invariant on Pseudo-Hermitian Almost CR Manifolds

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Abstract We study on the Chern-Moser-Tanaka invariant (Chern, Acta Math 133:219–271, 1974, [5], Tanaka, Japan J Math 12:131–190, 1976, [14]) of pseudo-conformal transformations on pseudo-Hermitian almost CR manifolds.

1 Introduction

A contact manifold (M, η) admits the fundamental structures which enrich the geometry. One is a Riemannian metric g compatible to the contact form η and we obtain a *contact Riemannian manifold* $(M; \eta, g)$. The other is a *pseudo-Hermitian and strictly pseudo-convex structure* (η, L) (or (η, J)), where L is the *Levi form* associated with an endomorphism J on D (= kernel of η) such that $J^2 = -I$. $(M; \eta, J)$ is called a *strictly pseudo-convex, pseudo-Hermitian manifold (or almost CR manifold)*. Then we have a one-to-one correspondence between the two associated structures by the relation $g = L + \eta \otimes \eta$, where we denote by the same letter L the natural extension ($i_\xi L = 0$) of the Levi form to a (0,2)-tensor field on M. So, we treat contact Riemannian structures together with strictly pseudo-convex almost CR structures. In earlier works [6–8, 10], the present author started the intriguing study of the interactions between them. For complex analytical considerations, it is desirable to have integrability of the almost complex structure J (on D). If this is the case, we speak of an *(integrable) CR structure* and of a *CR manifold.* Indeed, S. Webster [21, 22] introduced the term *pseudo-Hermitian structure* for a CR manifold with a non-degenerate Levi-form. In the present paper, we treat the pseudo-Hermitian structure as an extension to the case of non-integrable $\mathscr{H}$.

There is a canonical affine connection in a non-degenerate CR manifold, the so-called pseudo-Hermitian connection (or the Tanaka-Webster connection). S. Tanno [16] extends the Tanaka-Webster connection for strictly pseudo-convex almost CR manifolds (in which $\mathscr{H}$ is in general non-integrable). We call it the *generalized Tanaka-Webster connection*. Using this we have the *pseudo-Hermitian Ricci*

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curvature tensor. If the pseudo-Hermitian Ricci curvature tensor is a scalar (field) multiple of the Levi form in a strictly pseudo-convex almost CR manifold, then it is said to have the *pseudo-Einstein structure*. A *pseudo-Hermitian CR space form* is a strictly pseudo-convex CR manifold of constant holomorphic sectional curvature (for Tanaka-Webster connection). Then we have that a pseudo-Hermitian CR space form is pseudo-Einstein. In Sect. 4, we study the generalized Chern-Moser-Tanaka curvature tensor C as a pseudo-conformal invariant in a strictly pseudo-convex almost CR manifold. Then we first prove that the Chern-Moser-Tanaka curvature tensor vanishes for a pseudo-Hermitian CR space form. Moreover, we prove that for a strictly pseudo-convex almost CR manifold M^{2n+1} ($n > 1$) with vanishing C, M is pseudo-Einstein if and only if M is of pointwise constant holomorphic sectional curvature.

2 Preliminaries

We start by collecting some fundamental materials about contact Riemannian geometry and strictly pseudo-convex pseudo-Hermitian geometry. All manifolds in the present paper are assumed to be connected, oriented and of class C^∞.

2.1 Contact Riemannian Structures

A *contact manifold* (M, η) is a smooth manifold M^{2n+1} equipped with a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. For a contact form η, there exists a unique vector field ξ, called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there also exist a Riemannian metric g and a $(1, 1)$-tensor field φ such that

$$\eta(X) = g(X, \xi),\ d\eta(X, Y) = g(X, \varphi Y),\ \varphi^2 X = -X + \eta(X)\xi, \tag{1}$$

where X and Y are vector fields on M. From (1), it follows that

$$\varphi\xi = 0,\ \eta \circ \varphi = 0,\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (1) is said to be a *contact Riemannian manifold* or *contact metric manifold* and it is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M, we define a $(1, 1)$-tensor field h by $h = \frac{1}{2}\pounds_\xi \varphi$, where $\pounds_\xi$ denotes Lie differentiation for the characteristic direction ξ. Then we may observe that h is self-adjoint and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h, \tag{3}$$

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{4}$$

where ∇ is Levi-Civita connection. From (3) and (4) we see that ξ generates a geodesic flow. Furthermore, we know that $\nabla_\xi \varphi = 0$ in general (cf. p. 67 in [1]). From the second equation of (3) it follows also that

$$(\nabla_\xi h)\varphi = -\varphi(\nabla_\xi h). \tag{5}$$

A contact Riemannian manifold for which ξ is Killing is called a *K-contact manifold.* It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For further details on contact Riemannian geometry, we refer to [1].

2.2 Pseudo-Hermitian Almost CR Structures

For a contact manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM | \eta(v) = 0\}$. Then the $2n$-dimensional distribution (or subbundle) $D : p \to D_p$ is called the *contact distribution (or contact subbundle)*. Its associated almost CR structure is given by the holomorphic subbundle

$$\mathscr{H} = \{X - iJX : X \in \Gamma(D)\}$$

of the complexification $\mathbb{C}TM$ of the tangent bundle TM, where $J = \varphi|D$, the restriction of φ to D. Then we see that each fiber $\mathscr{H}_p$ $(p \in M)$ is of complex dimension n and $\mathscr{H} \cap \bar{\mathscr{H}} = \{0\}$. Furthermore, we have $\mathbb{C}D = \mathscr{H} \oplus \bar{\mathscr{H}}$. For the real representation $\{D, J\}$ of $\mathscr{H}$ we define the Levi form by

$$L : \Gamma(D) \times \Gamma(D) \to \mathscr{F}(M), \quad L(X, Y) = -d\eta(X, JY)$$

where $\mathscr{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is Hermitian and positive definite. We call the pair (η, L) (or (η, J)) a *strictly pseudo-convex, pseudo-Hermitian structure* on M. We say that *the almost CR structure is integrable* if $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$. Since $d\eta(JX, JY) = d\eta(X, Y)$, we see that $[JX, JY] - [X, Y] \in \Gamma(D)$ and $[JX, Y] + [X, JY] \in \Gamma(D)$ for $X, Y \in \Gamma(D)$, further if M satisfies the condition $[J, J](X, Y) = 0$ for $X, Y \in \Gamma(D)$, then the pair (η, J) is called a *strictly pseudo-convex (integrable) CR structure* and $(M; \eta, J)$ is called a *strictly pseudo-convex CR manifold* or a *strictly pseudo-convex integrable pseudo-Hermitian manifold*. A *pseudo-Hermitian torsion* is defined by $\tau = \varphi h$ (cf. [2]).

For a given strictly pseudo-convex pseudo-Hermitian manifold M, the almost CR structure is integrable if and only if M satisfies the integrability condition $\Omega = 0$, where Ω is a (1,2)-tensor field on M defined by

$$\Omega(X, Y) = (\nabla_X \varphi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX) \tag{6}$$

for all vector fields X, Y on M (see [16], Proposition 2.1]). It is well known that for 3-dimensional contact Riemannian manifolds their associated CR structures are always integrable (cf. [16]).

A *Sasakian manifold* is a strictly pseudo-convex CR manifold whose characteristic flow is isometric (or equivalently, vanishing the pseudo-Hermitian torsion). From (6) it follows at once that a Sasakian manifold is also determined by the condition

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{7}$$

for all vector fields X and Y on the manifold.

Now, we review the *generalized Tanaka-Webster connection* [16] on a strictly pseudo-convex almost CR manifold $M = (M; \eta, J)$. The generalized Tanaka-Webster connection $\hat{\nabla}$ is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X, Y on M. Together with (4), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + B(X, Y), \tag{8}$$

where we have put

$$B(X, Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Y)\xi. \tag{9}$$

Then, we see that the generalized Tanaka-Webster connection $\hat{\nabla}$ has the torsion $\hat{T}(X, Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY$. In particular, for a K-contact manifold we get

$$B(X, Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi. \tag{10}$$

Furthermore, it was proved that

Proposition 1 ([16]) *The generalized Tanaka-Webster connection $\hat{\nabla}$ on a strictly pseudo-convex almost CR manifold $M = (M; \eta, J)$ is the unique linear connection satisfying the following conditions:*

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$, where g is the associated Riemannian metric;
$(iii-1)$ $\hat{T}(X, Y) = 2L(X, JY)\xi$, $X,\ Y \in \Gamma(D)$;
$(iii-2)$ $\hat{T}(\xi, \varphi Y) = -\varphi\hat{T}(\xi, Y)$, $Y \in \Gamma(D)$;
(iv) $(\hat{\nabla}_X \varphi)Y = \Omega(X, Y)$, $X,\ Y \in \Gamma(TM)$.

The pseudo-Hermitian connection (or The Tanaka-Webster connection) [14, 22] on a non-degenerate (integrable) CR manifold is defined as the unique linear connection satisfying (i), (ii), (iii-1), (iii-2) and $\Omega = 0$. We refer to [2] for more details about pseudo-Hermitian geometry in strictly pseudo-convex almost CR manifolds.

2.3 Pseudo-homothetic Transformations

In this subsection, we first review

Definition 1 Let $(M; \eta, \xi.\varphi, g)$ be a contact Riemannian manifold. Then a diffeomorphism f on M is said to be a *pseudo-homothetic transformation* if there exists a positive constant a such that

$$f^*\eta = a\eta,\ f_*\xi = \xi/a,\ \varphi \circ f_* = f_* \circ \varphi,\ f^*g = ag + a(a-1)\eta \otimes \eta.$$

Due to S. Tanno [15], we have

Theorem 1 *If a diffeomorphism f on a contact Riemannian manifold M is φ-holomorphic, i.e.,*

$$\varphi \circ f_* = f_* \circ \varphi,$$

then f is a pseudo-homothetic transformation.

Here, the new contact Riemannian manifold $(M; \bar{\eta}, \bar{\xi}.\bar{\varphi}, \bar{g})$ defined by

$$\bar{\eta} = a\eta,\ \bar{\xi} = \xi/a,\ \bar{\varphi} = \varphi,\ \bar{g} = ag + a(a-1)\eta \otimes \eta, \tag{11}$$

is called a *pseudo-homothetic deformation* of $(M, \eta, \xi.\varphi, g)$. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + A(X, Y), \tag{12}$$

where A is the $(1, 2)$-type tensor defined by

$$A(X, Y) = -(a-1)[\eta(Y)\varphi X + \eta(X)\varphi Y] - \frac{a-1}{a} g(\varphi h X, Y)\xi.$$

Then we have

Proposition 2 ([9]) *The generalized Tanaka-Webster connection is pseudo-homothetically invariant.*

The so-called (k, μ)-spaces are defined by the condition

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y)$$

for $(k, \mu) \in \mathbb{R}^2$, where I denotes the identity transformation. This class involves the Sasakian case for $k = 1$ $(h = 0)$. For a non-Sasakian contact Riemannian manifold,

h has the only two eigenvalues $\sqrt{1-k}$ and $-\sqrt{1-k}$ on D with their multiplicities n respectively. The (k, μ)-spaces have integrable CR structures and further, this class of spaces is invariant under pseudo-homothetic transformations. Indeed, a pseudo-homothetic transformation with constant $a(>0)$ transforms a (k, μ)-space into a $(\bar{k}, \bar{\mu})$-space where $\bar{k} = \frac{k+a^2-1}{a^2}$ and $\bar{\mu} = \frac{\mu+2a-2}{a}$ (cf. [1] or [3]). In particular, we find that $k = 1$ and $\mu = 2$ are the only two invariants under pseudo-homothetic transformations for all $a \neq 1$.

3 Pseudo-Einstein Structures

We define the pseudo-Hermitian curvature tensor (or the generalized Tanaka-Webster curvature tensor) on a strictly pseudo-convex almost CR manifold $\hat{R}$ of $\hat{\nabla}$ by

$$\hat{R}(X, Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X,Y]}Z$$

for all vector fields X, Y, Z in M. We remark that the generalized Tanaka-Webster connection is not torsion-free, and then the Jacobi- or Bianchi-type identities do not hold, in general. From the definition of $\hat{R}$, we have

$$\hat{R}(X, Y)Z = R(X, Y)Z + H(X, Y)Z, \tag{13}$$

and

$$\begin{aligned} H(X, Y)Z = {} & \eta(Y)\big((\nabla_X\varphi)Z - g(X + hX, Z)\xi\big) - \eta(X)\big((\nabla_Y\varphi)Z - g(Y + hY, Z)\xi\big) \\ & + \eta(Z)\big((\nabla_X\varphi)Y - (\nabla_Y\varphi)X + (\nabla_X\varphi h)Y - (\nabla_Y\varphi h)X \\ & + \eta(Y)(X + hX) - \eta(X)(Y + hY)\big) - 2g(\varphi X, Y)\varphi Z \\ & - g(\varphi X + \varphi hX, Z)(\varphi Y + \varphi hY) + g(\varphi Y + \varphi hY, Z)(\varphi X + \varphi hX) \\ & - g((\nabla_X\varphi)Y - (\nabla_Y\varphi)X + (\nabla_X\varphi h)Y - (\nabla_Y\varphi h)X, Z)\xi \end{aligned} \tag{14}$$

for all vector fields X, Y, Z in M.

Now, we introduce the pseudo-Hermitian Ricci (curvature) tensor:

$$\hat{\rho}(X, Y) = \frac{1}{2}\text{trace of } \{V \mapsto J\hat{R}(X, JY)V\},$$

where X, Y are vector fields orthogonal to ξ. This definition was referred as a 2nd kind in the author's earlier work [9]. Indeed, the pseudo-Hermitian Ricci (curvature) tensor of the 1st kind $\hat{\rho}_1$ is defined by

$$\hat{\rho}_1(X, Y) = \text{trace of } \{V \mapsto \hat{R}(V, X)Y\},$$

where V is any vector field on M and X, Y are vector fields orthogonal to ξ. Then we can find the following useful relation between the two notions in general:

$$\begin{aligned}\hat{\rho}(X, Y) =& \hat{\rho}_1(X, Y) - 2(n-1)g(hX, Y) \\ &+ \sum_{i=1}^{2n} \Big(g((\hat{\nabla}_{e_i}\Omega)(X, Y), \varphi e_i) - g((\hat{\nabla}_X \Omega)(e_i, Y), \varphi e_i) \Big)\end{aligned} \tag{15}$$

for $X, Y \in \Gamma(D)$ (cf. [17]). We define the corresponding pseudo-Hermitian Ricci operator $\hat{Q}$ is defined by $L(\hat{Q}X, Y) = \hat{\rho}(X, Y)$. The Tanaka-Webster (or the pseudo-Hermitian) scalar curvature $\hat{r}$ is given by

$$\hat{r} = \text{trace of } \{V \mapsto \hat{Q}V\}.$$

Then, from Proposition 2, we get

Corollary 1 *The pseudo-Hermitian curvature tensor* (*or The generalized Tanaka-Webster curvature tensor*) $\hat{R}$ *and the pseudo-Hermitian Ricci tensor* $\hat{Q}$ *are pseudo-homothetic invariants.*

Definition 2 Let $(M; \eta, J)$ be a strictly pseudo-convex almost CR manifold. Then the pseudo-Hermitian structure (η, J) is said to be pseudo-Einstein if the pseudo-Hermitian Ricci tensor is proportional to the Levi form, namely,

$$\hat{\rho}(X, Y) = \lambda L(X, Y),$$

where $X, Y \in \Gamma(D)$, where $\lambda = \hat{r}/2n$.

Remark 1 N. Tanaka [13] and J.M. Lee [11] defined the pseudo-Hermitian Ricci tensor on a non-degenerate CR manifold in a complex fashion. Further, J.M. Lee defined and intensively studied the pseudo-Einstein structure. Then every 3-dimensional strictly pseudo-convex CR manifold is pseudo-Einstein.

Remark 2 From (15), we at once see that for the Sasakian case or the 3-dimensional case $\hat{\rho} = \hat{\rho}_1$.

Moreover, we have

Proposition 3 ([9]) *A non-Sasakian contact* (k, μ)*-space* $(k < 1)$ *is pseudo-Einstein with constant pseudo-Hermitian scalar curvature* $\hat{r} = 2n^2(2 - \mu)$.

In [3] they proved that unit tangent sphere bundles with standard contact metric structures are (k, μ)-spaces if and only if the base manifold is of constant curvature b with $k = b(2 - b)$ and $\mu = -2b$. Thus, we have

Corollary 2 *The standard contact metric structure of* $T_1M(b)$ *of a space of constant curvature* b *is pseudo-Einstein. Its pseudo-Hermitian scalar curvature* $\hat{r} = 4n^2(1 + b)$.

The class of contact (k, μ)-spaces, whose associated CR structures are integrable as stated at the end of Sect. 2, contains non-unimodular Lie groups with left-invariant contact metric structure other than unit tangent bundles of a space of constant curvature (see [4]).

4 Pseudo-Hermitian CR Space Forms

In this section, we give

Definition 3 ([7]) Let $(M; \eta, J)$ be a strictly pseudo-convex almost CR manifold. Then M is said to be of constant holomorphic sectional curvature c (with respect to the generalized Tanaka-Webster connection) if M satisfies

$$L(\hat{R}(X, \varphi X)\varphi X, X) = c$$

for any unit vector field X orthogonal to ξ. In particular, for the CR integrable case we call M a pseudo-Hermitian (strictly pseudo-convex) CR space form.

Then for a strictly pseudo-convex almost CR manifold M, from (13) and (14) we get

$$g(\hat{R}(X, \varphi X)\varphi X, X) = g(R(X, \varphi X)\varphi X, X) + 3g(X, X)^2 - g(hX, X)^2 - g(\varphi hX, X)^2 \tag{16}$$

for any X orthogonal to ξ. From this, we easily see that s Sasakian space form $M^{2n+1}(c_0)$ of constant φ-holomorphic sectional curvature c_0 (with respect to the Levi-Civita connection) is a strictly pseudo-convex CR space form of constant holomorphic sectional curvature (with respect to the Tanaka-Webster connection) $c = c_0 + 3$. Simply connected and complete Sasakian space forms are the unit sphere S^{2n+1} with the natural Sasakian structure with $c_0 = 1$ $(c = 4)$, the Heisenberg group H^{2n+1} with Sasakian φ-holomorphic sectional curvature $c_0 = -3$ $(c = 0)$, or $B^n \times R$ with Sasakian φ-holomorphic sectional curvature $c_0 = -7$ $(c = -4)$, where B^n is a simply connected bounded domain in C^n with constant holomorphic sectional curvature -4.

For a class of the contact (k, μ)-spaces, we proved the following results.

Theorem 2 ([7]) *Let M be a contact (k, μ)-space. Then M is of constant holomorphic sectional curvature c for Tanaka-Webster connection if and only if* (1) *M is Sasakian space of constant φ-holomorphic sectional curvature $c_0 = c - 3$,* (2) *$\mu = 2$ and $c = 0$, or* (3) *dim M=3 and $\mu = 2 - c$.*

Corollary 3 ([7]) *The standard strictly pseudo-convex CR structure on a unit tangent sphere bundle $T_1M(b)$ of $(n+1)$-dimensional space of constant curvature b has constant holomorphic sectional curvature c if and only if $b = -1$ and $c = 0$, or $n = 1$ and $b = (c-2)/2$.*

Remark 3 (1) The standard contact metric structure of the unit tangent sphere bundle $T_1\mathbb{S}^{n+1}(1)$ is Sasakian [20], but it has not constant holomorphic sectional curvature for both Levi-Civita and Tanaka-Webster connection.

(2) The unit tangent sphere bundle $T_1\mathbb{H}^{n+1}(-1)$ of a hyperbolic space $\mathbb{H}^{n+1}(-1)$ is a non-Sasakian example of constant holomorphic sectional curvature for Tanaka-Webster connection but not for Levi-Civita connection.

In [7] we determined the Riemannian curvature tensor explicitly for a strictly pseudo-convex CR space of constant holomorphic sectional curvature c. Then we have

$$g(\hat{R}(X,Y)Z,W) = g(H(X,Y)Z,W) + \frac{c}{4}\Big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W)\Big\} \tag{17}$$

for all vector fields $X, Y, Z, W \perp \xi$, where

$$\begin{aligned} g(H(X,Y)Z,W) = {} & g(Y,Z)g(hX,W) - g(X,Z)g(hY,W) \\ & - g(Y,W)g(hX,Z) + g(X,W)g(hY,Z) \\ & + g(\varphi Y,Z)g(\varphi hX,W) - g(\varphi X,Z)g(\varphi hY,W) \\ & - g(\varphi Y,W)g(\varphi hX,Z) + g(\varphi X,W)g(\varphi hY,Z). \end{aligned} \tag{18}$$

Then from (17) we get

$$\hat{\rho}(X,Y) = c(n+1)/2\, g(X,Y). \tag{19}$$

Proposition 4 ([9]) *A strictly pseudo-convex CR space form of constant holomorphic sectional curvature c is pseudo-Einstein with constant pseudo-Hermitian scalar curvature* $\hat{r} = n(n+1)c$.

5 The Chern-Moser-Tanaka Invariant

Now, we review the pseudo-conformal transformations of a strictly pseudo-convex almost CR structure. Given a contact form η, we consider a 1-form $\bar{\eta} = \sigma\eta$ for a positive smooth function σ. By assuming $\bar{\phi}|D = \phi|D$ ($\bar{J} = J$), the associated Riemannian structure $\bar{g}$ of $\bar{\eta}$ is determined in a natural way. Namely, we have

$$\bar{\xi} = (1/\sigma)(\xi + \zeta),\ \zeta = (1/2\sigma)\phi(\mathrm{grad}\,\sigma),\ \bar{\phi} = \phi + (1/2\sigma)\eta\otimes(\mathrm{grad}\,\sigma - \xi\sigma\cdot\xi),$$
$$\bar{g} = \sigma g - \sigma(\eta\otimes\nu + \nu\otimes\eta) + \sigma(\sigma - 1 + \|\zeta\|^2)\eta\otimes\eta,$$

where ν is dual to ζ with respect to g. We call the transformation $(\eta, J) \to (\bar{\eta}, \bar{J})$ a *pseudo-conformal transformation (or gauge transformation)* of the strictly pseudo-convex almost CR structure. We remark in particular that when σ is a constant, then a gauge transformation reduces to a pseudo-homothetic transformation.

Let ω be a nowhere vanishing $(2n+1)$-form on M and fix it. Let $dM(g) = ((-1)^n/2^n n!)\eta \wedge (d\eta)^n$ denote the volume element of (M, η, g). We define β by $dM(g) = \pm e^{\beta}\omega$ and $\theta \in \Gamma(D^*)$ by $\theta(X) = X\beta$ for $X \in \Gamma(D)$. For a strictly pseudo-convex almost CR manifold, the generalized Chern-Moser-Tanaka curvature tensor $C \in \Gamma(D \otimes D^{*3})$ is defined by S. Tanno in [18] (see also, [8]).

$$\begin{aligned}
&(2n+4)g(C(X,Y)Z,W)\\
&\quad= (2n+4)g(\hat{R}(X,Y)Z,W)\\
&\qquad - \hat{\rho}(Y,Z)g(X,W) + \hat{\rho}(X,Z)g(Y,W) - g(Y,Z)\hat{\rho}(X,W) + g(X,Z)\hat{\rho}(Y,W)\\
&\qquad + \hat{\rho}(Y,\varphi Z)g(\varphi X,W) - \hat{\rho}(X,\varphi Z)g(\varphi Y,W) - [\hat{\rho}(X,\varphi Y) - \hat{\rho}(\varphi X,Y)]g(\varphi Z,W)\\
&\qquad + \hat{\rho}(X,\varphi W)g(\varphi Y,Z) - \hat{\rho}(Y,\varphi W)g(\varphi X,Z) - [\hat{\rho}(Z,\varphi W) - \hat{\rho}(\varphi Z,W)]g(\varphi X,Y)\\
&\qquad + [\hat{r}/(2n+2)][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\\
&\qquad + g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W)]\\
&\qquad - (2n+4)[g(hY,Z)g(X,W) - g(hX,Z)g(Y,W) + g(Y,Z)g(hX,W)\\
&\qquad - g(X,Z)g(hY,W) + g(\varphi hY,Z)g(\varphi X,W) - g(\varphi hX,Z)g(\varphi Y,W)\\
&\qquad + g(\varphi hX,W)g(\varphi Y,Z) - g(\varphi hY,W)g(\varphi X,Z)]\\
&\qquad - (n+2)/(n+1)g(U(X,Y,Z;\theta),W)).
\end{aligned} \tag{20}$$

Here $U \in \Gamma(D^2 \otimes D^{*3})$ and $U(X,Y,Z;\theta) = (\theta_j U^{ji}_{lhk} X^h Y^k Z^l)$ in terms of an adapted frame $\{e_\alpha\} = \{e_j,\ e_0 = \xi;\ 1 \le j \le 2n\}$. For a full understanding, we may describe it by using the components of U in terms of $\{e_j, e_0\}$ (cf. [18]). That is,

$$\begin{aligned}
U^{ji}_{lhk} = 2\Big[&1/(n+2)\{-\delta^i_h(\Omega^j_{km} + \Omega^j_{mk})\phi^m_l - \phi^i_h(\Omega^j_{lk} + \Omega^j_{kl}) + g_{hl}(\Omega^j_{km} + \Omega^j_{mk})\phi^{mi}\\
&- \phi_{hl}(\Omega^j_{km} + \Omega^j_{mk})g^{mi}\} + \Omega^j_{lk}\phi^i_h + \phi_{hl}\Omega^j_{mk}g^{im} + \Omega^j_{hk}\phi^i_l\\
&+ (1/2)(\Omega^j_{ml} - \Omega^j_{lm})g^{mi}\phi_{hk} + \phi^j_l\Omega^i_{hk} + \phi^j_h\Omega^i_{lk} - (1/2)\phi^{ij}\Omega^m_{kl}g_{hm}\Big]_{hk},
\end{aligned}$$

where $[\cdots]_{hk}$ denotes the skew-symmetric part of $[\cdots]$ with respect to h, k.

Remark 4 (1) If $n = 1$ (dim M=3), then we always have $C = 0$ (see Remark in [18]).

(2) When $(M; \eta, g)$ is Sasakian, then ($h = 0$ and) C reduces to the C-Bochner curvature tensor, which is the corresponding (through the Boothby-Wang fibration) to the Bochner curvature tensor in a Kähler manifold [12].

Using (17) and (19), from the Eq. (9) we find

Proposition 5 *On a pseudo-Hermitian CR space form, the Chern-Moser-Tanaka invariant C vanishes.*

Moreover we have

Theorem 3 *Let $(M^{2n+1}; \eta, J)$ $(n > 1)$ be a strictly pseudo-convex almost CR manifold with vanishing C. Then M is pseudo-Einstein if and only if M is of pointwise constant holomorphic sectional curvature for the Tanaka-Webster connection.*

The argument and computation of present paper gives a simpler proof of [9, Theorem 22].

Remark 5 The unit tangent sphere bundle $T_1\mathbb{H}^{n+1}(-1)$ of a hyperbolic space $\mathbb{H}^{n+1}(-1)$ is a non-Sasakian example which supports Theorem 3 well. It was proved that the Chern-Moser-Tanaka curvature tensor C on $T_1\mathbb{H}^{n+1}(-1)$ vanishes [19] and within the class of (k, μ)-spaces, it is the only such an example [8].

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References

1. Blair, D.E.: Riemannian geometry of contact and symplectic manifolds, Second edn. In: Progress in Mathematics 203, Birkhäuser (2010)
2. Blair, D.E., Dragomir, S.: Pseudohermitian geometry on contact Riemannian manifolds. Rend. Mat. Appl. **22**(7), 275–341 (2002)
3. Blair, D.E., Koufogiorgos, T., Papantoniou, B.J.: Contact metric manifolds satisfying a nullity condition. Israel J. Math. **91**, 189–214 (1995)
4. Boeckx, E.: A full classification of contact metric (k, μ)-spacesIllinois. J. Math. **44**, 212–219 (2000)
5. Chern, S.S., Moser, J.K.: Real hypersurfaces in complex manifolds. Acta Math. **133**, 219–271 (1974)
6. Cho, J.T.: A new class of contact Riemannian manifolds. Israel J. Math. **109**, 299–318 (1999)
7. Cho, J.T.: Geometry of contact strongly pseudo-convex CR manifolds. J. Korean Math. Soc. **43**(5), 1019–1045 (2006)
8. Cho, J.T.: Contact Riemannian manifolds with vanishing gauge invariant. Toyama Math. J. **31**, 1–16 (2008)
9. Cho, J.T.: Pseudo-Einstein manifolds. Topology Appl. **196**, 398–415 (2015)
10. Cho, J.T., Chun, S.H.: On the classification of contact Riemannian manifolds satisfying the condition (C). Glasgow Math. J. **45**, 99–113 (2003)
11. Lee, J.M.: Pseudo-Einstein structures on CR manifolds. Amer. J. Math. **110**, 157–178 (1988)
12. Matsumoto, M., Chūman, G.: On the C-Bochner curvature tensor. TRU Math. **5**, 21–30 (1969)
13. Tanaka, N.: A differential geometric study on strongly pseudo-convex manifolds. Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9, Kinokuniya Book-Store Co., Ltd., Tokyo (1975)
14. Tanaka, N.: On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections. Japan J. Math. **12**, 131–190 (1976)
15. Tanno, S.: Some transformations on manifolds with almost contact and contact metric structures. Tôhoku Math. J. **15**, 140–147 (1963)
16. Tanno, S.: Variational problems on contact Riemannian manifolds. Trans. Amer. Math. Soc. **314**, 349–379 (1989)
17. Tanno, S.: The Bochner type curvature tensor of contact Riemannian structure. Hokkaido Math. J. **19**, 55–66 (1990)
18. Tanno, S.: Pseudo-conformal invariants of type (1, 3) of CR manifolds. Hokkaido Math. J. **28**, 195–204 (1991)
19. Tanno, S.: The standard CR structure on the unit tangent bundle. Tôhoku Math. J. **44**, 535–543 (1992)

20. Tashiro, Y.: On contact structures of unit tangent sphere bundles. Tôhoku Math. J. **21**, 117–143 (1969)
21. Webster, S.M.: Real hypersurfaces in complex space, thesis, University of California, Berkeley (1975)
22. Webster, S.M.: Pseudohermitian structures on a real hypersurface. J. Diff. Geometry **13**, 25–41 (1978)