Real Hypersurfaces in Hermitian Symmetric Space of Rank Two with Killing Shape Operator

Ji-Eun Jang, Young Jin Suh and Changhwa Woo

Abstract We have considered a new notion of the shape operator A satisfies Killing tensor type for real hypersurfaces M in complex Grassmannians of rank two. With this notion we prove the non-existence of real hypersurfaces M in complex Grassmannians of rank two.

1 Introduction

A typical example of Hermitian symmetry spaces of rank two is the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Another one is complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$.

Characterizing model spaces of real hypersurfaces under certain geometric conditions is one of our main interests in the classification theory in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ or complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$. In this paper, we use the same geometric condition on real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ as used in $G_2(\mathbb{C}^{m+2})$ to compare the results.

$G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2{\cdot}U_m)$ has a compact transitive group SU_{2+m}, however $SU_{2,m}/S(U_2{\cdot}U_m)$ has a noncompact indefinite transitive group $SU_{2,m}$. This

J.-E. Jang
Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea
e-mail: ji-eun82@daum.net

Y.J. Suh
Department of Mathematics and RIRCM,
Kyungpook National University, Daegu 41566, Republic of Korea
e-mail: yjsuh@knu.ac.kr

C. Woo (✉)
Department of Mathematics Education, Woosuk University, Wanju, Jeonbuk 565-701, Republic of Korea
e-mail: legalgwch@woosuk.ac.kr

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_23

distinction gives various remarkable results. Riemannian symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$ has a remarkable geometrical structure. It is the unique noncompact, Kähler, irreducible, quaternionic Kähler manifold with negative curvature.

Suppose that M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$). Let N be a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$). Since $G_2(\mathbb{C}^{m+2})$(or $SU_{2,m}/S(U_2{\cdot}U_m)$) has the Kähler structure J, we may define the *Reeb vector field* $\xi = -JN$ and a one dimensional distribution $[\xi] = \mathscr{C}^\perp$ where $\mathscr{C}$ denotes the orthogonal complement in T_xM, $x \in M$, of the Reeb vector field ξ. The Reeb vector field ξ is said to be *Hopf* if $\mathscr{C}$ (or $\mathscr{C}^\perp$) is invariant under the shape operator A of M. The one dimensional foliation of M defined by the integral curves of ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* if and only if the Hopf foliation of M is totally geodesic. By the formulas in [5, Sect. 2], it can be checked that ξ is Hopf vector field if and only if M is Hopf hypersurface.

From the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$), there naturally exist *almost contact 3-structure* vector fields $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Put $\mathscr{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$. It is a 3-dimensional distribution in the tangent bundle TM of M. In addition, we denoted by $\mathscr{Q}$ the orthogonal complement of $\mathscr{Q}^\perp$ in TM. It is the quaternionic maximal subbundle of TM. Thus the tangent bundle of M is expressed as a direct sum of $\mathscr{Q}$ and $\mathscr{Q}^\perp$.

For any geodesic γ in M, a (1,1) type tensor field T is said to be Killing if $T\dot\gamma$ is parallel displacement along γ, which gives $0 = \nabla_{\dot\gamma}(T\dot\gamma) = (\nabla_{\dot\gamma}T)\dot\gamma + T(\nabla_{\dot\gamma}\dot\gamma) = (\nabla_{\dot\gamma}T)\dot\gamma$. That is, $(\nabla_X T)X = 0$ for any tangent vector field X on M (see [2]).

$$\begin{aligned} 0 &= (\nabla_{X+Y}T)(X+Y) \\ &= (\nabla_X T)X + (\nabla_X T)Y + (\nabla_Y T)X + (\nabla_Y T)Y \\ &= (\nabla_X T)Y + (\nabla_Y T)X \end{aligned}$$

for any vector fields X and Y on M.

Thus the Killing tensor field T is equivalent to $(\nabla_X T)Y + (\nabla_Y T)X = 0$.

From this notion, in this paper we consider a new condition related to the shape operator A of M defined in such a way that

$$(\nabla_X A)Y + (\nabla_Y A)X = 0 \tag{C-1}$$

for any vector fields X on M.

In this paper, we give a complete classification for real hypersurfaces in $\bar{M}$ ($G_2(\mathbb{C}^{m+2})$ or $SU_{2,m}/S(U_2{\cdot}U_m)$) with Killing shape operator. In order to do it, we consider a problem related to the following:

Theorem 1 *There does not exist any real hypersurface in $\bar{M}$ complex Grassmannians of rank two, $m \geq 3$, with Killing shape operator.*

Since the notion of Killing tensor field is weaker than the notion of parallel tensor field, by Theorem 1, we naturally have the following:

quotation There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel shape operator (see [11]).

On the other hand, by virtue of Theorem 2 we can assert the following:

Corollary 1 *There does not exist any hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$ with parallel shape operator.*

2 Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$ and $SU_{2,m}/S(U_2{\cdot}U_m)$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [5, 6, 11, 12]. The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B, this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way, $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight.

When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ of K has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\Big\} \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\Big\},\end{aligned} \tag{2.1}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

Now we summarize basic material about complex hyperbolic two-plane Grassmann manifolds $SU_{2,m}/S(U_2{\cdot}U_m)$, for details we refer to [14, 16].

The Riemannian symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$, which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, is a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank 2. Let $G = SU_{2,m}$ and $K = S(U_2{\cdot}U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K, respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{A}ut(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m}AI_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$$

I_2 (resp., I_m) denotes the identity 2×2-matrix (resp., $m \times m$-matrix). Then $< X, Y >= -B(X, \theta Y)$ becomes a positive definite $\mathrm{A}d(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2{\cdot}U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2{\cdot}U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2{\cdot}U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2{\cdot}U_m)$. By construction, J commutes with each almost Hermitian structure J_ν in $\mathfrak{J}$ for $\nu = 1, 2, 3$. Recall that a canonical local basis $\{J_1, J_2, J_3\}$ of a quaternionic Kähler structure $\mathfrak{J}$ consists of three almost

Hermitian structures J_1, J_2, J_3 in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is to be taken modulo 3. The tensor field JJ_ν, which is locally defined on $SU_{2,m}/S(U_2{\cdot}U_m)$, is self-adjoint and satisfies $(JJ_\nu)^2 = I$ and $tr(JJ_\nu) = 0$, where I is the identity transformation. For a nonzero tangent vector X we define $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

We identify the tangent space $T_oSU_{2,m}/S(U_2{\cdot}U_m)$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ at o with $\mathfrak{p}$ in the usual way. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since $SU_{2,m}/S(U_2{\cdot}U_m)$ has rank 2, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_oSU_{2,m}/S(U_2{\cdot}U_m) \cong \mathfrak{p}$ is contained in some maximal abelian subspace of $\mathfrak{p}$. Generically this subspace is uniquely determined by X, in which case X is called regular. If there exists more than one maximal abelian subspaces of $\mathfrak{p}$ containing X, then X is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Up to scaling there exists a unique $SU_{2,m}$-invariant Riemannian metric g on complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$. Equipped with this metric $SU_{2,m}/S(U_2{\cdot}U_m)$ is a Riemannian symmetric space of rank 2 which is both Kähler and quaternionic Kähler. For computational reasons we normalize g such that the minimal sectional curvature of $(SU_{2,m}/S(U_2{\cdot}U_m), g)$ is -4. The sectional curvature K of the noncompact symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$ equipped with the Killing metric g is bounded by $-4 \leq K \leq 0$. The sectional curvature -4 is obtained for all 2-planes $\mathbb{C}X$ when X is a non-zero vector with $JX \in \mathfrak{J}X$.

When $m = 1$, $G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1{\cdot}U_2)$ is isometric to the two-dimensional complex hyperbolic space $\mathbb{C}H^2$ with constant holomorphic sectional curvature -4.

When $m = 2$, we note that the isomorphism $SO(4, 2) \simeq SU_{2,2}$ yields an isometry between $G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2{\cdot}U_2)$ and the indefinite real Grassmann manifold $G_2^*(\mathbb{R}_2^6)$ of oriented two-dimensional linear subspaces of an indefinite Euclidean space $\mathbb{R}_2^6$. For this reason we assume $m \geq 3$ from now on, although many of the subsequent results also hold for $m = 1, 2$.

Hereafter X,Y and Z always stand for any tangent vector fields on M.

The Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ is locally given by

$$\begin{aligned}\bar{R}(X, Y)Z = -\frac{1}{2}\Big[& g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ & - g(JX, Z)JY - 2g(JX, Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y \\ & - 2g(J_\nu X, Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\}\Big],\end{aligned} \tag{2.2}$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathfrak{J}$.

3 Basic Formulas

In this section we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$) (see [3, 5, 7, 10–12, 18]).

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$). The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.1}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$) there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{3.2}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$), together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in Sect. 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0,\\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\
&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned} \tag{3.3}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$ in Sect. 2 and (3.1), the relation between these two almost contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{aligned}
&\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu,\\
&\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi.
\end{aligned} \tag{3.4}$$

On the other hand, from the parallelism of Kähler structure J, that is, $\bar{\nabla} J = 0$ and the quaternionic Kähler structure $\mathfrak{J}$, together with Gauss and Weingarten formulas it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{3.5}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{3.6}$$

$$(\nabla_X \phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \tag{3.7}$$

Combining these formulas, we find the following:

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu)\\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu)\\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX\\ &\quad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{3.8}$$

Using the above expression (2.1) for the curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$), the equations of Codazzi is given by

$$\begin{aligned}k\big\{(\nabla_X A)Y - (\nabla_Y A)X\big\} &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\Big\}\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\Big\}\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\Big\}\xi_\nu,\end{aligned} \tag{3.9}$$

where in the case of $G_2(\mathbb{C}^{m+2})$ (resp., $SU_{2,m}/S(U_2{\cdot}U_m)$), the constant $k = 1$ and $SU_{2,m}/S(U_2{\cdot}U_m)$ (resp., $k = -2$).

4 Proof of Theorems

In this section, we classify real hypersurfaces in $\bar{M}$ ($G_2(\mathbb{C}^{m+2})$ or $SU_{2,m}/S(U_2{\cdot}U_m)$) whose shape operator has Killing tensor field.

From (C-1) and the Codazzi equation (3.9), we have

$$\begin{aligned}-2k(\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\Big\}\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\Big\}\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\Big\}\xi_\nu\end{aligned} \tag{4.1}$$

Putting $Y = \xi$ into (4.1),

$$-2k(\nabla_\xi A)X = -\phi X + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu\big\}. \tag{4.2}$$

Lemma 1 *Let M be a real hypersurface in complex Grassmannians of rank two $\bar{M}$, $m \geq 3$ with Killing shape operator. Then the Reeb vector field ξ belongs to either the distribution $\mathscr{Q}$ or the distribution $\mathscr{Q}^\perp$.*

Proof Without loss of generality, ξ is written as

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1, \tag{$**$}$$

where X_0 (resp., ξ_1) is a unit vector in $\mathscr{Q}$ (resp., $\mathscr{Q}^\perp$).

Taking the inner product of (4.2) with ξ, we have

$$-2kg\big((\nabla_\xi A)X, \xi\big) = -4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi X). \tag{4.3}$$

Since $(\nabla_\xi A)$ is self-adjoint, it follows from (C-1) that $-4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi X) = 0$. By putting $X = \phi X_0$ and using (**), we have $-4\eta_1^2(\xi)\eta(X_0) = 0$.

Thus we have only two cases: $\xi \in \mathscr{Q}^\perp$ or $\xi \in \mathscr{Q}$.

- **Case 1.** $\xi \in \mathscr{Q}^\perp$.

Without loss of generality, we may put $\xi = \xi_1 \in \mathscr{Q}^\perp$. Then (4.2) is reduced into

$$-2k(\nabla_\xi A)X = -\phi X - \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3. \tag{4.4}$$

The symmetric endomorphism of (4.4) with respect to the metric g, we have

$$-2k(\nabla_\xi A)X = \phi X + \phi_1 X - 2\eta_3(X)\xi_2 + 2\eta_2(X)\xi_3. \tag{4.5}$$

Combining (4.4) with (4.5), we have $\phi X + \phi_1 X - 2\eta_3(X)\xi_2 + 2\eta_2(X)\xi_3 = 0$. By putting $X = \xi_3$ into the equation above, we have $2\xi_3 = 0$. This is a contradiction.

Thus, there does not exist any hypersurface in $\bar{M}$, $m \geq 3$, with Killing shape operator and $\xi \in \mathscr{Q}^\perp$ everywhere.

- **Case 2.** $\xi \in \mathscr{Q}$.

Equation (4.2) becomes

$$-2k(\nabla_\xi A)X = -\phi X + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu\xi - 3\eta_\nu(\phi X)\xi_\nu\big\}. \tag{4.6}$$

The symmetric endomorphism of (4.6) with respect to the metric g, we have

$$-2k(\nabla_\xi A)X = \phi X + \sum_{\nu=1}^{3}\big\{ -\eta_\nu(\phi X)\xi_\nu + 3\eta_\nu(X)\phi\xi_\nu \big\}. \tag{4.7}$$

Combining (4.6) with (4.7), we have $2\phi X + 2\sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu\big\} = 0$. By putting $X = \xi_1$ into above equation, we have $4\phi\xi_1 = 0$. This is a contradiction, too. Thus, there does not exist any hypersurface in $\bar{M}$, $m \geq 3$, with Killing shape operator and $\xi \in \mathscr{Q}$ everywhere.

Accordingly, we complete the proof of Theorem 1 in the introduction.

Usually, the notion of parallel is stronger than the notion of Killing, we also have a non-existence of parallel hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. Then Corollary 1 in the introduction is naturally proved.

References

1. Alekseevskii, D.V.: Compact quaternion spaces. Func. Anal. Appl. **2**, 106–114 (1966)
2. Blair, D.E.: Riemannian geometry of contact and symplectic manifolds. Springer Science and Business Media (2010)
3. Jeong, I., Machado, C., Pérez, J.D., Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with $\mathscr{D}^{\perp}$-parallel structure Jacobi operator. Int. J. Math. **22**(5), 655–673 (2011)
4. Lee, H., Choi, Y.S., Woo, C.: Hopf hypersurfaces in complex two-plane Grassmannians with Reeb parallel shape operator. Bull. Malaysian Math. Soc. **38**(3), 617–634 (2015)
5. Lee, H., Suh, Y.J.: Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector. Bull. Korean Math. Soc. **47**(3), 551–561 (2010)
6. Lee, H., Suh, Y.J., Woo, C.: Real hypersurfaces with commuting Jacobi operators in complex two-plane Grassmannians. Houston J. Math. **40**(3), 751–766 (2014)
7. Pérez, J.D., Suh, Y.J.: The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians. J. Korean Math. Soc. **44**, 211–235 (2007)
8. Pérez, J.D., Suh, Y.J., Woo, C.: Real hypersurfaces in complex two-plane Grassmannians with GTW Harmonic curvature. Canad. Math. Bull. **58**(4), 835–845 (2015)
9. Pérez, J.D., Suh, Y.J., Woo, C.: Real hypersurfaces in complex two-plane Grassmannians whose shape operator is recurrent for the generalized Tanaka-Webster connection. Turkish J. Math. **39**(3), 313–321 (2015)
10. Pérez, J.D., Suh, Y.J., Woo, C.: Real Hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting shape operator. Open Math. **13**(1), 493–501 (2015)
11. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator. Bull. Austral. Math. Soc. **68**, 493–502 (2003)
12. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel shape operator II. J. Korean Math. Soc. **41**, 535–565 (2004)
13. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor. Proc. Royal Soc. Edinb. A. **142**, 1309–1324 (2012)
14. Suh, Y.J.: Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians. Adv. Appl. Math. **50**, 645–659 (2013)
15. Suh, Y.J.: Real hypersurfaces in complex two-plane Grassmannians with harmonic curvature. J. Math. Pures Appl. **100**, 16–33 (2013)
16. Suh, Y.J.: Real hypersurfaces in complex hyperbolic two-plane Grassmannians with Reeb vector field. Adv. Appl. Math. **55**, 131–145 (2014)

17. Suh, Y.J.: Real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting Ricci tensor. Int. J. Math., World Sci. Publ. **26**, 1550008 (26 pages) (2015)
18. Suh, Y.J., Woo, C.: Real Hypersurfaces in complex hyperbolic two-plane Grassmannians with parallel Ricci tensor. Math. Nachr. **287**, 1524–1529 (2014)