On the Pointwise Slant Submanifolds

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Abstract In this survey paper, we consider several kinds of submanifolds in Riemannian manifolds, which are obtained by many authors. (i.e., slant submanifolds, pointwise slant submanifolds, semi-slant submanifolds, pointwise semi-slant submanifolds, pointwise almost h-slant submanifolds, pointwise almost h-semi-slant submanifolds, etc.) And we deal with some results, which are obtained by many authors at this area. Finally, we give some open problems at this area.

1 Introduction

Given a Riemannian manifold $(\overline{M}, g)$ with some additional structures, there are several kinds of submanifolds:

(Almost) complex submanifolds, totally real submanifolds, slant submanifolds, pointwise slant submanifolds, semi-slant submanifolds, pointwise semi-slant submanifolds, etc.

In 1990, Chen [3] defined the notion of slant submanifolds of an almost Hermitian manifold as a generalization of almost complex submanifolds and totally real submanifolds.

In 1994, Papaghiuc [7] introduced a semi-slant submanifold of an almost Hermitian manifold as a generalization of CR-submanifolds and slant submanifolds.

In 1996, Lotta [6] introduced a slant submanifold of an almost contact metric manifold.

In 1998, Etayo [5] defined the notion of pointwise slant submanifolds of an almost Hermitian manifold under the name of quasi-slant submanifolds as a generalization of slant submanifolds.

In 1999, Cabrerizo, Carriazo, Fernandez, Fernandez [2] defined the notion of semi-slant submanifolds of an almost contact metric manifold.

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Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_21

In 2012, Chen and Garay [4] studied deeply pointwise slant submanifolds of an almost Hermitian manifold.

In 2013, Sahin [10] introduced pointwise semi-slant submanifolds of an almost Hermitian manifold.

In 2014, Park [8] defined the notion of pointwise almost h-slant submanifolds and pointwise almost h-semi-slant submanifolds of an almost quaternionic Hermitian manifold.

In 2015, Park [9] introduced pointwise slant and pointwise semi-slant submanifolds of an almost contact metric manifold.

In this paper, we consider some results, which are obtained by many authors at this area. And we give some open problems at this area.

2 Preliminaries

Let $(\overline{M}, g, J)$ be an *almost Hermitian manifold*, where $\overline{M}$ is a C^∞-manifold, g is a Riemannian metric on $\overline{M}$, and J is an almost complex structure on $\overline{M}$ which is compatible with g.

I.e., $J \in \mathrm{End}(T\overline{M})$, $J^2 = -id$, $g(JX, JY) = g(X, Y)$ for $X, Y \in \Gamma(T\overline{M})$.

Let M be a submanifold of $\overline{M} = (\overline{M}, g, J)$. We have the following notions.

We call M an *almost complex submanifold* of $\overline{M}$ if $J(T_x M) \subset T_x M$ for $x \in M$.

The submanifold M is said to be a *totally real submanifold* if $J(T_x M) \subset T_x M^\perp$ for $x \in M$.

The submanifold M is called a *CR-submanifold* if there exists a distribution $\mathscr{D} \subset TM$ on M such that $J(\mathscr{D}_x) = \mathscr{D}_x$ and $J(\mathscr{D}_x^\perp) \subset T_x M^\perp$ for $x \in M$, where $\mathscr{D}^\perp$ is the orthogonal complement of $\mathscr{D}$ in TM.

The almost Hermitian manifold $\overline{M} = (\overline{M}, g, J)$ is said to be *Kähler* if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

Now we recall other notions. Let N be a $(2n + 1)$-dimensional C^∞-manifold with a tensor field ϕ of type $(1, 1)$, a vector field ξ, and a 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where I denotes the identity endomorphism of TN.

Then we have [1]

$$\phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2}$$

And we call (ϕ, ξ, η) an *almost contact structure* and (N, ϕ, ξ, η) an *almost contact manifold*.

If there is a Riemannian metric g on N such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3}$$

for $X, Y \in \Gamma(TN)$, then we call (ϕ, ξ, η, g) an *almost contact metric structure* and (N, ϕ, ξ, η, g) an *almost contact metric manifold*.

The metric g is called a *compatible metric*.

Then we obtain

$$\eta(X) = g(X, \xi). \tag{4}$$

Define $\Phi(X, Y) := g(X, \phi Y)$ for $X, Y \in \Gamma(TN)$.

Since ϕ is anti-symmetric with respect to g, the tensor Φ is a 2-form on N and is called the *fundamental 2-form* of the almost contact metric structure (ϕ, ξ, η, g).

An almost contact metric manifold (N, ϕ, ξ, η, g) is said to be a *contact metric manifold* (or *almost Sasakian manifold*) if it satisfies

$$\Phi = d\eta. \tag{5}$$

It is easy to check that given a contact metric manifold (N, ϕ, ξ, η, g), we get

$$(d\eta)^n \wedge \eta \neq 0. \tag{6}$$

The *Nijenhuis tensor* of a tensor field ϕ is defined by

$$N(X, Y) := \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \tag{7}$$

for $X, Y \in \Gamma(TN)$.

We call the almost contact metric structure (ϕ, ξ, η, g) *normal* if

$$N(X, Y) + 2d\eta(X, Y)\xi = 0 \tag{8}$$

for $X, Y \in \Gamma(TN)$.

A contact metric manifold (N, ϕ, ξ, η, g) is said to be a *K-contact manifold* if the characteristic vector field ξ is Killing.

It is well-known that for a contact metric manifold (N, ϕ, ξ, η, g), ξ is Killing if and only if the tensor $\bar{h} := \frac{1}{2} L_\xi \phi$ vanishes, where L denotes the Lie derivative [1].

An almost contact metric manifold (N, ϕ, ξ, η, g) is called a *Sasakian manifold* if it is contact and normal.

Given an almost contact metric manifold (N, ϕ, ξ, η, g), we know that it is Sasakian if and only if

$$(\overline{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{9}$$

for $X, Y \in \Gamma(TN)$ [1].

If an almost contact metric manifold (N, ϕ, ξ, η, g) is Sasakian, then we have

$$\overline{\nabla}_X \xi = -\phi X \tag{10}$$

for $X \in \Gamma(TN)$ [1].

Moreover, a Sasakian manifold is a K-contact manifold [1].

An almost contact metric manifold (N, ϕ, ξ, η, g) is said to be a *Kenmotsu manifold* if it satisfies

$$(\overline{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{11}$$

for $X, Y \in \Gamma(TN)$ [1].

Then we easily obtain

$$\overline{\nabla}_X \xi = X - \eta(X)\xi \tag{12}$$

for $X \in \Gamma(TN)$ [1].

An almost contact metric manifold (N, ϕ, ξ, η, g) is called an *almost cosymplectic manifold* if η and Φ are closed.

An almost cosymplectic manifold (N, ϕ, ξ, η, g) is said to be a *cosymplectic manifold* if it is normal.

Given an almost contact metric manifold (N, ϕ, ξ, η, g), we also know that it is cosymplectic if and only if ϕ is parallel (i.e., $\overline{\nabla}\phi = 0$) [1].

Given a cosymplectic manifold (N, ϕ, ξ, η, g), we easily get

$$\overline{\nabla}\phi = 0, \ \overline{\nabla}\eta = 0, \ \text{and } \overline{\nabla}\xi = 0. \tag{13}$$

Let $\overline{M}$ be a $4m$-dimensional C^∞-manifold and let E be a rank 3 subbundle of $\text{End}(T\overline{M})$ such that for any point $p \in \overline{M}$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call E an *almost quaternionic structure* on $\overline{M}$ and $(\overline{M}, E)$ an *almost quaternionic manifold*.

Moreover, let g be a Riemannian metric on $\overline{M}$ such that for any point $p \in \overline{M}$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{14}$$

$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \tag{15}$$

for $X, Y \in \Gamma(T\overline{M})$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call $(\overline{M}, E, g)$ an *almost quaternionic Hermitian manifold*.

Conveniently, the above basis $\{J_1, J_2, J_3\}$ satisfying (14) and (15) is said to be a *quaternionic Hermitian basis*.

Let $(\overline{M}, E, g)$ be an almost quaternionic Hermitian manifold.

We call $(\overline{M}, E, g)$ a *quaternionic Kähler manifold* if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$$

for $X \in \Gamma(T\overline{M})$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on $\overline{M}$ (i.e., $\nabla J_\alpha = 0$ for $\alpha \in \{1, 2, 3\}$, where ∇ is the Levi-Civita connection of the metric g), then $(\overline{M}, E, g)$ is said to be a *hyperkähler manifold*.

Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on $\overline{M}$ and g a *hyperkähler metric*.

Let $\overline{M} = (\overline{M}, E, g)$ be an almost quaternionic Hermitian manifold and M a submanifold of $\overline{M}$.

We call M a *QR-submanifold* (quaternionic-real submanifold) of $\overline{M}$ if there exists a vector subbundle $\mathscr{D}$ of $TM^\perp$ on M such that given a local quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of E, we have $J_\alpha \mathscr{D} = \mathscr{D}$ and $J_\alpha(\mathscr{D}^\perp) \subset TM$ for $\alpha \in \{1, 2, 3\}$, where $\mathscr{D}^\perp$ is the orthogonal complement of $\mathscr{D}$ in $TM^\perp$.

The submanifold M is said to be a *quaternion CR-submanifold* if there exists a distribution $\mathscr{D} \subset TM$ on M such that given a local quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of E, we get $J_\alpha \mathscr{D} = \mathscr{D}$ and $J_\alpha(\mathscr{D}^\perp) \subset TM^\perp$ for $\alpha \in \{1, 2, 3\}$, where $\mathscr{D}^\perp$ is the orthogonal complement of $\mathscr{D}$ in TM.

Throughout this paper, we will use the above notations.

3 Some Results

In this section, we consider some results at this area.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and M a submanifold of $\overline{M}$.

We call M a *slant submanifold* [3] of $\overline{M}$ if the angle $\theta = \theta(X)$ between JX and the tangent space $T_x M$ is constant for nonzero $X \in T_x M$ and any $x \in M$.

Given $X \in \Gamma(TM)$, we have

$$JX = PX + FX, \tag{16}$$

where $PX \in \Gamma(TM)$ and $FX \in \Gamma(TM^\perp)$.

Lemma 1 ([3]) *Let M be a submanifold of an almost Hermitian manifold $\overline{M}$.*

Then $\nabla P = 0$ if and only if M is locally the Riemannian product $M_1 \times \cdots \times M_k$, where each M_i is either a Kähler submanifold, a totally real submanifold, or a Kählerian slant submanifold.

Theorem 1 ([3]) *Let M be a surface in $\mathbb{C}^2$ which is neither holomorphic nor totally real.*

Then M is a minimal slant surface if and only if $\nabla F = 0$.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and M a submanifold of $\overline{M}$.

The submanifold M is said to be a *semi-slant submanifold* [7] if there is a distribution $\mathscr{D} \subset TM$ on M such that $J(\mathscr{D}_x) = \mathscr{D}_x$ for $x \in M$ and the angle $\theta = \theta(X)$ between JX and the space $\mathscr{D}_x^\perp$ is constant for nonzero $X \in \mathscr{D}_x^\perp$ and any $x \in M$, where $\mathscr{D}^\perp$ is the orthogonal complement of $\mathscr{D}$ in TM.

Proposition 1 ([7]) *Let M be a semi-slant submanifold of a Kähler manifold $(\overline{M}, g, J)$.*

Then the complex distribution $\mathscr{D}$ is integrable if and only if we have $h(X, JY) = h(JX, Y)$ for $X, Y \in \Gamma(\mathscr{D})$.

Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and M a submanifold of N.

We call M a *slant submanifold* [6] of N if the angle $\theta = \theta(X)$ between ϕX and the tangent space $T_x M$ is constant for nonzero $X \in T_x M$ with X, ξ being linearly independent and any $x \in M$.

Given $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX, \tag{17}$$

where $PX \in \Gamma(TM)$ and $FX \in \Gamma(TM^\perp)$.

Theorem 2 ([6]) *Let M be a m-dimensional slant submanifold of an almost contact metric manifold N and suppose $\theta \neq \frac{\pi}{2}$.*

Then we have

$$m \text{ is even } \Leftrightarrow \xi \text{ is orthogonal to } N$$

$$m \text{ is odd } \Leftrightarrow \xi \text{ is tangent to } N.$$

Theorem 3 ([6]) *Let M be an immersed submanifold of a K-contact manifold N such that the characteristic vector field ξ is tangent to M. Let $\theta \in [0, \frac{\pi}{2}]$. The following statements are equivalent:*

(a) M is slant in N with the slant angle θ.

(b) For any $x \in M$ the sectional curvature of any 2-plane of $T_x M$ containing ξ_x equals $\cos^2 \theta$.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and M a submanifold of $\overline{M}$.

The submanifold M is called a *pointwise slant submanifold* [4, 5] of $\overline{M}$ if at each given point $x \in M$, the angle $\theta = \theta(X)$ between JX and the tangent space $T_x M$ is constant for nonzero $X \in T_x M$.

Proposition 2 ([5]) *Let M be a pointwise slant submanifold of an almost Hermitian manifold $(\overline{M}, g, J)$.*

If M has odd dimension, then M is a totally real submanifold.

Theorem 4 ([5]) *Let M be a submanifold of an almost Hermitian manifold $(\overline{M}, g, J)$.*

Then M is a pointwise slant submanifold if and only if P_x is a homothety for $x \in M$.

Theorem 5 ([5]) *Let M be a pointwise slant complete totally geodesic submanifold of a Kähler manifold $(\overline{M}, g, J)$.*

Then M is a slant submanifold.

Define $\Omega(X, Y) := g(X, PY)$ for $X, Y \in \Gamma(TM)$.

Theorem 6 ([4]) *Let M be a proper pointwise slant submanifold of a Kähler manifold $(\overline{M}, g, J)$.*

Then Ω is a non-degenerate closed 2-form on M.

Consequently, Ω defines a canonical cohomology class of Ω:

$$[\Omega] \in H^2(M; R).$$

Theorem 7 ([4]) *Let M be a compact $2n$-dimensional differentiable manifold with $H^{2i}(M; R) = 0$ for some $i \in \{1, \dots, n\}$.*

Then M cannot be immersed in any Kähler manifold as a pointwise proper slant submanifold.

Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and M a submanifold of N.

We call M a *semi-slant submanifold* [2] of N if there is a distribution $\mathscr{D} \subset TM$ on M such that $\phi(\mathscr{D}_x) = \mathscr{D}_x$ for $x \in M$ and the angle $\theta = \theta(X)$ between ϕX and the space $\mathscr{D}_x^\perp$ is constant for nonzero $X \in \mathscr{D}_x^\perp$ with X, ξ being linearly independent and any $x \in M$, where $\mathscr{D}^\perp$ is the orthogonal complement of $\mathscr{D}$ in TM.

Theorem 8 ([2]) *Let M be a submanifold of an almost contact metric manifold $N = (N, \phi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$.*

Then M is semi-slant if and only if there exists a constant $\lambda \in [0, 1)$ such that (i) $D = \{X \in TM | P^2 X = -\lambda X\}$ is a distribution. (ii) For any $X \in TM$, orthogonal to D, $FX = 0$.

Furthermore, in this case, $\lambda = \cos^2\theta$, where θ denotes the slant angle of M.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and M a submanifold of $\overline{M}$.

We call M a *pointwise semi-slant submanifold* [10] of $\overline{M}$ if there is a distribution $\mathscr{D} \subset TM$ on M such that $J(\mathscr{D}_x) = \mathscr{D}_x$ for $x \in M$ and at each given point $x \in M$, the angle $\theta = \theta(X)$ between JX and the space $\mathscr{D}_x^\perp$ is constant for nonzero $X \in \mathscr{D}_x^\perp$, where $\mathscr{D}^\perp$ is the orthogonal complement of $\mathscr{D}$ in TM.

Theorem 9 ([10]) *Let $\overline{M}$ be a Kähler manifold.*

Then there exist no non-trivial warped product submanifolds $M = M_\theta \times_f M_T$ of a Kähler manifold $\overline{M}$ such that M_T is a holomorphic submanifold and M_θ is a proper pointwise slant submanifold of $\overline{M}$.

Theorem 10 ([10]) *Let M be an $m + n$-dimensional non-trivial warped product pointwise semi-slant submanifold of the form $M_T \times_f M_\theta$ in a Kähler manifold $\overline{M}^{m+2n}$, where M_T is a holomorphic submanifold and M_θ is a proper pointwise slant submanifold of $\overline{M}^{m+2n}$.*

Then we have

(i) The squared norm of the second fundamental form of M satisfies

$$||h||^2 \geq 2n(\csc^2\theta + \cot^2\theta)||\nabla(\ln f)||^2, \quad \dim(M_\theta) = n. \tag{18}$$

(ii) If the equality of (18) holds identically, then M_T is a totally geodesic submanifold and M_θ is a totally umbilical submanifold of $\overline{M}^{m+2n}$.

Moreover, M is a minimal submanifold of $\overline{M}^{m+2n}$.

Let $(\overline{M}, E, g)$ be an almost quaternionic Hermitian manifold and M a submanifold of $(\overline{M}, g)$.

The submanifold M is called a *pointwise almost h-slant submanifold* [8] if given a point $p \in M$ with a neighborhood V, there exist an open set $U \subset \overline{M}$ with $U \cap M = V$ and a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, at each given point $q \in V$ the angle $\theta_R = \theta_R(X)$ between RX and the tangent space T_qM is constant for nonzero $X \in T_qM$.

We call such a basis $\{I, J, K\}$ a *pointwise almost h-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-slant functions* as functions on V.

The submanifold M is called a *pointwise almost h-semi-slant submanifold* [8] if given a point $p \in M$ with a neighborhood V, there exist an open set $U \subset \overline{M}$ with $U \cap M = V$ and a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathscr{D}_1^R \subset TM$ on V such that

$$TM = \mathscr{D}_1^R \oplus \mathscr{D}_2^R, \ R(\mathscr{D}_1^R) = \mathscr{D}_1^R,$$

and at each given point $q \in V$ the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathscr{D}_2^R)_q$ is constant for nonzero $X \in (\mathscr{D}_2^R)_q$, where $\mathscr{D}_2^R$ is the orthogonal complement of $\mathscr{D}_1^R$ in TM.

We call such a basis $\{I, J, K\}$ a *pointwise almost h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-semi-slant functions* as functions on V.

Let M be a pointwise almost h-semi-slant submanifold of a hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $\{I, J, K\}$ is a pointwise almost h-semi-slant basis. We call M *proper* if $\theta_R(p) \in [0, \frac{\pi}{2})$ for $p \in M$ and $R \in \{I, J, K\}$.

Let M be a proper pointwise almost h-slant submanifold of a hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $\{I, J, K\}$ is a pointwise almost h-slant basis.

Define

$$\Omega_R(X, Y) := g(\phi_R X, Y) \tag{19}$$

for $X, Y \in \Gamma(TM)$ and $R \in \{I, J, K\}$.

Theorem 11 ([8]) *Let M be a proper pointwise almost h-slant submanifold of a hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $\{I, J, K\}$ is a pointwise almost h-slant basis. Then the 2-form Ω_R is closed for each $R \in \{I, J, K\}$.*

Theorem 12 ([8]) *Let M be a $2n$-dimensional compact proper pointwise almost h-slant submanifold of a $4m$-dimensional hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $\{I, J, K\}$ is a pointwise almost h-slant basis.*

Then

$$H^*(M, R) \supseteq \widetilde{H}, \tag{20}$$

where $\widetilde{H}$ is the algebra spanned by $\{[\Omega_I], [\Omega_J], [\Omega_K]\}$.

Theorem 13 ([8]) *Let $(\overline{M}, I, J, K, g)$ be a hyperkähler manifold. Then given $R \in \{I, J, K\}$, there do not exist any non-trivial warped product submanifolds $M = B \times_f F$ of a Kähler manifold $(\overline{M}, R, g)$ such that B is a proper pointwise slant submanifold of $(\overline{M}, R, g)$ and F is a holomorphic submanifold of $(\overline{M}, R, g)$.*

Theorem 14 ([8]) *Let $M = B \times_f F$ be a non-trivial warped product proper pointwise h-semi-slant submanifold of a hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $TB = \mathscr{D}_1$, $TF = \mathscr{D}_2$, $\dim B = 4n_1$, $\dim F = 2n_2$, $\dim \overline{M} = 4m$, $\theta_I(p)\theta_J(p)\theta_K(p) \neq 0$ for any $p \in M$, and $\{I, J, K\}$ is a pointwise h-semi-slant basis.*

Assume that $m = n_1 + n_2$.

Then given $R \in \{I, J, K\}$, we get

$$||h||^2 \geq 4n_2(\csc^2\theta_R + \cot^2\theta_R)||\nabla(\ln f)||^2 \tag{21}$$

with equality holding if and only if $g(h(V, W), Z) = 0$ for $V, W \in \Gamma(TF)$ and $Z \in \Gamma(TM^\perp)$.

Let $N = (N, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact metric manifold and M a submanifold of N.

The submanifold M is called a *pointwise slant submanifold* [9] if at each given point $p \in M$ the angle $\theta = \theta(X)$ between ϕX and the space M_p is constant for nonzero $X \in M_p$, where $M_p := \{X \in T_pM \mid g(X, \xi(p)) = 0\}$.

The submanifold M is called a *pointwise semi-slant submanifold* [9] if there is a distribution $\mathscr{D}_1 \subset TM$ on M such that

$$TM = \mathscr{D}_1 \oplus \mathscr{D}_2, \quad \phi(\mathscr{D}_1) \subset \mathscr{D}_1,$$

and at each given point $p \in M$ the angle $\theta = \theta(X)$ between ϕX and the space $(\mathscr{D}_2)_p$ is constant for nonzero $X \in (\mathscr{D}_2)_p$, where $\mathscr{D}_2$ is the orthogonal complement of $\mathscr{D}_1$ in TM.

Theorem 15 ([9]) *Let M be a pointwise slant connected totally geodesic submanifold of a cosymplectic manifold (N, ϕ, ξ, η, g).*

Then M is a slant submanifold of N.

Theorem 16 ([9]) *Let M be a $2m$-dimensional compact proper pointwise slant submanifold of a $(2n + 1)$-dimensional cosymplectic manifold (N, ϕ, ξ, η, g) such that ξ is normal to M.*

Then $[\Omega] \in H^2(M, R)$ is non-vanishing.

Theorem 17 ([9]) *Let M be a $(2m+1)$-dimensional compact proper pointwise slant submanifold of a $(2n+1)$-dimensional cosymplectic manifold (N, ϕ, ξ, η, g) such that ξ is tangent to M.*

Then both $[\eta] \in H^1(M, R)$ and $[\Omega] \in H^2(M, R)$ are non-vanishing.

Let M be a submanifold of a Riemannian manifold (N, g). We call M a *totally umbilic submanifold* of (N, g) if

$$h(X, Y) = g(X, Y)H \quad \text{for } X, Y \in \Gamma(TM), \tag{22}$$

where H is the mean curvature vector field of M in N.

Lemma 2 ([9]) *Let M be a pointwise semi-slant totally umbilic submanifold of an almost contact metric manifold (N, ϕ, ξ, η, g).*

Assume that ξ is tangent to M and N is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.

Then

$$H \in \Gamma(F\mathscr{D}_2). \tag{23}$$

Theorem 18 ([9]) *Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $M = B \times_f \overline{F}$ a nontrivial warped product submanifold of N. Assume that ξ is normal to M and N is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

Then there does not exist a proper pointwise semi-slant submanifold M of N such that $\mathscr{D}_1 = T\overline{F}$ and $\mathscr{D}_2 = TB$.

Theorem 19 ([9]) *Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $M = B \times_f \overline{F}$ a nontrivial warped product submanifold of N. Assume that ξ is tangent to M and N is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

Then there does not exist a proper pointwise semi-slant submanifold M of N such that $\mathscr{D}_1 = T\overline{F}$ and $\mathscr{D}_2 = TB$.

Theorem 20 ([9]) *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional Sasakian manifold (N, ϕ, ξ, η, g) with the semi-slant function θ such that $\mathscr{D}_1 = TB$, $\mathscr{D}_2 = T\overline{F}$, and ξ is tangent to M.*

Assume that $n = m_1 + 2m_2$.

Then we have

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\phi\nabla(\ln f)||^2 + 4m_2\sin^2\theta \tag{24}$$

with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.

Theorem 21 ([9]) *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional cosymplectic manifold (N, ϕ, ξ, η, g) with the semi-slant function θ such that $\mathscr{D}_1 = TB$, $\mathscr{D}_2 = T\overline{F}$, and ξ is tangent to M.*

Assume that $n = m_1 + 2m_2$.

Then we have

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\phi\nabla(\ln f)||^2 \tag{25}$$

with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.

Theorem 22 ([9]) *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold (N, ϕ, ξ, η, g) with the semi-slant function θ such that $\mathscr{D}_1 = TB$, $\mathscr{D}_2 = T\overline{F}$, and ξ is normal to M with $\xi \in \Gamma(\mu)$.*

Assume that $n = m_1 + 2m_2$.

Then we have

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\nabla(\ln f)||^2 + 2m_1 \tag{26}$$

with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.

4 Open Questions

Question 1. Let M be a (pointwise) slant (or (pointwise) semi-slant) submanifold of a Riemannian manifold $(\overline{M}, g)$ with some geometric structures.

Then

1. Give some examples of the manifold M when $\dim M \geq 3$.
2. What kind of rigidity problems can we do on $M \subset \overline{M}$?

Question 2. Let M be a pointwise almost h-semi-slant submanifold of an almost quaternionic Hermitian manifold $(\overline{M}, E, g)$ with the almost h-semi-slant functions $\{\theta_I, \theta_J, \theta_K\}$.

Then

1. Can we give a characterization of the almost h-semi-slant functions $\{\theta_I, \theta_J, \theta_K\}$?
2. What kind of rigidity problems can we do on $M \subset \overline{M}$?
3. Since the quaternionic Kähler manifolds have applications in physics, what is the relation between this notion and physics?
4. Using this notion, what are the advantages for studying quaternionic geometry?

Question 3. Let M be a pointwise slant (or pointwise semi-slant) submanifold of an almost contact metric manifold (N, ϕ, ξ, η, g) with the slant (or semi-slant) function θ.

Then

1. Can we give a characterization of the slant (or semi-slant) function θ?
2. What kind of rigidity problems can we do on $M \subset N$?
3. Using these notions, what are the advantages for studying contact geometry?

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