Gromov–Witten Invariants on the Products of Almost Contact Metric Manifolds

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Abstract We investigate Gromov–Witten invariants and quantum cohomologies on the products of almost contact metric manifolds. The product of two cosymplectic manifolds has a Kähler structure. We compute some cohomology classes of compact cosymplectic manifolds and show that any compact simply connected Kähler manifold cannot be a product of two cosymplectic manifolds. On the products we get some geometric properties, Gromov–Witten invariants and quantum cohomologies. We have some relations between Gromov–Witten invariants of the products and the ones of two cosymplectic manifolds.

1 Introduction

Let M be a real $(2n+1)$-dimensional smooth manifold and $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M. An almost co-complex structure on M is defined by a smooth $(1, 1)$-tensor field φ, a smooth vector field ξ, and a smooth 1-form η on M such that $\varphi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$, where I denotes the identity transformation of the tangent bundle TM. Manifolds with an almost co-complex structure (φ, ξ, η) are called almost contact manifolds. An almost contact manifold (M, φ, ξ, η) with a Riemannian metric tensor g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(M)$ is called an almost contact metric manifold, and denote it by $(M, g, \varphi, \xi, \eta)$. An almost contact metric manifold has its structure group of the form $U(n) \oplus (1)$, and the fundamental 2-form ϕ defined by

$$\phi(X, Y) = g(X, \varphi Y)$$

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for all $X, Y \in \mathfrak{X}(M)$. An almost cosymplectic structure (η, ϕ) on M is called cosymplectic if $d\eta = 0 = d\phi$, in this case M is called an almost co-Kähler manifold. When $\phi = d\eta$ the associated almost cosymplectic structure is called a contact structure on M and M an almost Sasakian manifold. The Nijenhuis tensor N_φ of φ is the $(1, 2)$-tensor field defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - [X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y]$$

for all $X, Y \in \mathfrak{X}(M)$, where $[X, Y]$ is the Lie bracket of X and Y. An almost cocomplex structure (φ, ξ, η) is called integrable if $N_\varphi = 0$, and normal if $N_\varphi + 2d\eta \otimes \xi = 0$. An integrable almost cocomplex structure is called a cocomplex structure. An integrable almost co-Kähler manifold is called a co-Kähler manifold, while a Sasakian manifold is a normal almost Sasakian manifold. In this paper we follow definitions and notations on almost contact metric manifolds in the references [1–4].

Let (M, g, ω, J) be a symplectic manifold with an almost complex structure J which is compatible with a symplectic structure ω. To study symplectic manifolds many geometers [5–7] have studied the theory of pseudo-holomorphic maps from a Riemann surface to M. Let $A \in H_2(M; \mathbb{Z})$ be an integral homology class, and $\mathfrak{M}_{g,k}(M, A, J)$ be the moduli space of stable J-holomorphic maps which represent A from a Riemann surface with genus g and k marked points to M. The moduli spaces define the Gromov–Witten invariants via evaluation maps. Using the Gromov–Witten invariants we can define quantum product on cohomologies and have the quantum cohomology ring $QH^*(M; \Lambda)$ [6, 7] with coefficients in some Novikov ring Λ. In [8, 9] we have studied Gromov–Witten invariants and quantum cohomologies on symplectic manifolds, in [10] geodesic surface congruences. We have extended the notion of pseudo-holomorphic map in symplectic manifolds to the one of pseudo-co-holomorphic map in almost contact metric manifolds. We have had some results on Gromov–Witten type invariants and quantum type cohomologies on contact manifolds [2], and on the products of cosymplectic manifolds and circle [11].

In this paper we consider the products of almost contact metric manifolds. On the products we investigate some geometric structures, Gromov–Witten invariants, and quantum cohomologies. In Sect. 2, we induce the fundamental 2-form and almost complex structure on the product of two almost contact metric manifolds. In particular, the product of two cosymplectic manifolds is Kähler. In Sect. 3, we have some topological properties of the product of two cosymplectic manifolds. We show that the cosymplectic structure (η, ϕ) of a compact cosymplectic manifold contributes to each Betti numbers. As a consequence we have that any compact simply connected Kähler manifold can not be a product of two cosymplectic manifolds. In Sect. 4, we study Gromow-Witten invariants on the product of two cosymplectic manifolds. We show that the Gromov–Witten invariant of the product is equal to the product of Gromov–Witten type invariants of two cosymplectic manifolds.

2 The Product of Two Almost Contact Metric Manifolds

Let $(M_i^{2n_i+1}, g_i, \varphi_i, \xi_i, \eta_i, \phi_i)$, $i = 1, 2$, be almost contact metric manifolds. Then the product $M := M_1 \times M_2$ is a smooth manifold of dimension $2n$, where $n = n_1 + n_2 + 1$. Let g be a Riemannian metric on M defined by

$$g((X_1, X_2), (Y_1, Y_2)) = g_1(X_1, Y_1) + g_2(X_2, Y_2)$$

for every $(X_1, X_2), (Y_1, Y_2) \in \mathfrak{X}(M)$, and J a $(1, 1)$-tensor field on M defined by

$$J(X_1, X_2) = (\varphi_1 X_1 - \eta_2(X_2)\xi_1, \varphi_2 X_2 + \eta_1(X_1)\xi_2)$$

for every $(X_1, X_2) \in \mathfrak{X}(M)$.

The tangent bundles are decomposed as

$$TM_1 = \mathscr{D}_1 \oplus \langle \xi_1 \rangle, \quad TM_2 = \mathscr{D}_2 \oplus \langle \xi_2 \rangle,$$

where $\mathscr{D}_1 = \{X \in TM_1 \mid \eta_1(X) = 0\}$, $\mathscr{D}_2 = \{X \in TM_2 \mid \eta_2(X) = 0\}$, and $\langle \xi_i \rangle$, $i = 1, 2$ are trivial real line bundles on M_i.

Lemma 1 *Let M be the product of almost contact metric manifolds M_1 and M_2. Then we have*

(1) $TM \simeq \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle$ *is isomorphic to a sum of complex vector bundles.*
(2) $J^2 = -I$.
(3) $J = \varphi_1$ *on* $\mathscr{D}_1$, $J = \varphi_2$ *on* $\mathscr{D}_2$, *and* $J := \varphi_3$ *on* $\langle \xi_1, \xi_2 \rangle$, *where* $\varphi_3(\xi_1) = \xi_2$ *and* $\varphi_3(\xi_2) = -\xi_1$.
(4) $g = J^* g$.

Proof By the definitions of the almost contact metric manifold, the metric g, and the $(1, 1)$-tensor J, we can easily prove Lemma 1. □

By Lemma 1 the product of two almost contact metric manifolds is an almost complex manifold. The fundamental 2-form on the product M is given by

$$\phi((X_1, X_2), (Y_1, Y_2)) = g((X_1, X_2), J(Y_1, Y_2))$$

for every $(X_1, X_2), (Y_1, Y_2) \in \mathfrak{X}(M)$.

Lemma 2 *The fundamental 2-form ϕ on the product M is*

$$\phi = \phi_1 + \phi_2 - \eta_1 \wedge \eta_2.$$

Proof For every $(X_1, X_2), (Y_1, Y_2) \in \mathfrak{X}(M)$,

$$\phi((X_1, X_2), (Y_1, Y_2)) = g((X_1, X_2), J(Y_1, Y_2))$$

$$
\begin{aligned}
&= g((X_1, X_2), (\varphi_1 Y_1 - \eta_2(Y_2)\xi_1, \varphi_2 Y_2 + \eta_1(Y_1)\xi_2)) \\
&= g_1(X_1, \varphi_1 Y_1 - \eta_2(X_2)\xi_1) + g_2(X_2, \varphi_2 Y_2 + \eta_1(Y_1)\xi_2) \\
&= g_1(X_1, \varphi_1 Y_1) - \eta_2(X_2) g_1(X_1, \xi_1) + g_2(X_2, \varphi_2 Y_2) + \eta_1(Y_1) g_2(X_2, \xi_2) \\
&= \phi_1(X_1, Y_1) + \phi_2(X_2, Y_2) - \eta_2(X_2)\eta_1(X_1) + \eta_1(Y_1)\eta_2(X_2) \\
&= (\phi_1 + \phi_2 - \eta_1 \wedge \eta_2)((X_1, X_2), (Y_1, Y_2)).
\end{aligned}
$$

Thus we have $\phi = \phi_1 + \phi_2 - \eta_1 \wedge \eta_2$. □

Recall that an almost contact metric manifold $(M, g, \varphi, \xi, \eta, \phi)$ is almost cosymplectic or almost co-Kähler (cosymplectic or co-Kähler) if and only if $d\eta = 0 = d\phi$ $(d\eta = 0 = d\phi = N_\varphi)$, respectively.

Theorem 1 *Let* $(M_i^{2n_i+1}, g_i, \varphi_i, \xi_i, \eta_i, \phi_i)$, $i = 1, 2$, *be almost contact metric manifolds and* (M^{2n}, g, J, ϕ) *their product, where* $n = n_1 + n_2 + 1$. *Then we have*

(1) if M_i, $i = 1, 2$, *are almost cosymplectic, then* M *is symplectic.*
(2) if M_i, $i = 1, 2$, *are cosymplectic, then* M *is Kähler.*

Proof By Lemma 1, the product (M, g, J, ϕ) is an almost complex manifold. By Lemma 2 the fundamental 2-form on the product is $\phi = \phi_1 + \phi_2 + \eta_2 \wedge \eta_1$.

The exterior derivative of ϕ is

$$d\phi = d\phi_1 + d\phi_2 + d\eta_2 \wedge \eta_1 - \eta_2 \wedge d\eta_1.$$

(1) Let $M_i, i = 1, 2$, be almost cosymplectic. Then $d\phi_i = 0 = d\eta_i, i = 1, 2$. and so $d\phi = 0$. Thus ϕ is closed. The n times exterior product of ϕ is

$$\phi^n = (\phi_1 + \phi_2 + \eta_2 \wedge \eta_1)^n = \phi_1^{n_1} \wedge \phi_2^{n_2} \wedge \eta_2 \wedge \eta_1$$

which does not vanish on M.
Thus the fundamental 2-form ϕ is a nondegenerate closed 2-form on M.

(2) Let $M_i, i = 1, 2$, be cosymplectic. Then by (1) M is symplectic and J is almost complex structure J is compatible with ϕ.
Since the Nigenhuis tensor on M_i is $N_{\varphi_i} = 0$, $i = 1, 2$, the Nijenhuis tensor N_J on M is zero. Thus (M, g, J, ϕ) is Kähler.

□

Remark 1 The odd dimensional spheres S^{2n_1+1} and S^{2n_2+1}, $n_i > 0$, are contact. The product $S^{2n_1+1} \times S^{2n_2+1}$ is a complex manifold but not symplectic [12].

3 The Product of Two Cosymplectic Manifolds

Let $(M^{2n+1}, g, \varphi, \xi, \eta, \phi)$ be a cosymplectic manifold, and $\bigtriangledown$ the Levi-Civita connection which is compatible with the metric g. Define Two operators L and $\wedge$ on M by the exterior product $L = \varepsilon(\phi)$ and the interior product $\wedge = \iota(\phi)$.

Recall the cohomologies of cosymplectic manifolds.

Lemma 3 ([1]) *For a cosymplectic manifold $(M, g, \varphi, \xi, \eta, \phi)$*

(1) $\bigtriangledown_X \phi = 0$ *for every* $X \in \mathfrak{X}(M)$.
(2) L *commutes with the Laplace-Beltrami operator* $\triangle$.
(3) L *maps the space of harmonic* p*-forms into the space of harmonic* $(p+2)$*-forms.*

Theorem 2 ([1]) *Let $(M^{2n+1}, g, \varphi, \xi, \eta, \phi)$ be a compact cosymplectic manifold. Then the each Betti number $B_i(M)$ of M is nonzero, i.e.,*

$$B_i(M) \neq 0 \quad i = 0, 1, \ldots, 2n+1.$$

Recall the topology of compact cosymplectic manifolds. Since the fundamental 2-form ϕ satisfies $\bigtriangledown_X \phi = 0$ for every $X \in \mathfrak{X}(M)$ we have $d\phi = 0$ and $d^*\phi = 0$. Thus $\triangle\phi = (d^*d + dd^*)\phi = 0$, and ϕ is harmonic.

Suppose ϕ^p is harmonic, then we have

$$\triangle(\phi^{p+1}) = \triangle(L\phi^p) = L(\triangle\phi^p) = L(0) = 0.$$

Thus ϕ^{p+1} is harmonic for every p.

Since $\phi^n \neq 0$ and $\phi^p \neq 0$ for every $1 \leq p \leq n$, the Betti numbers $B_{2p}(M) \neq 0$, $0 \leq p \leq n$. By Poincaré duality the odd dimensional Betti numbers

$$B_{2p+1}(M) \neq 0, \quad 0 \leq p \leq n.$$

Let $\{\xi, e_i, \varphi e_i \mid i = 1, \ldots, n\}$ be a local φ-basis and $\{\eta, \omega_i, \omega_i^* \mid i = 1, \ldots, n\}$ its dual basis in M. Then we have

$$\begin{aligned}
\phi &= \sum_{i=1}^{n} \omega_i \wedge \omega_i^*, \\
\phi^n &= n!\omega_1 \wedge \omega_1^* \wedge \cdots \wedge \omega_n \wedge \omega_n^*, \\
*\,\phi^n &= n! * (\omega_1 \wedge \omega_1^* \wedge \cdots \wedge \omega_n \wedge \omega_n^*) = n!\eta,
\end{aligned}$$

and $\phi^n \wedge \eta$ is a nowhere vanishing $(2n+1)$-form.

Since the Hodge star $*$ operator commutes to $\triangle$, i.e., $*\triangle = \triangle *$,

$$n!\triangle\eta = \triangle n!\eta = \triangle * \phi^n = *\triangle\phi^n = *0 = 0.$$

Thus the η is a nonzero harmonic 1-form in M.

For every $1 \leq p \leq n$, since $\triangle(\phi^p \wedge \eta) = (\triangle\phi^p) \wedge \eta + \phi^p \wedge (\triangle\eta) = 0$, the $\phi^p \wedge \eta$ are nonzero harmonic $(2p+1)$-forms.

Theorem 3 *Let $(M^{2n+1}, g, \varphi, \xi, \eta, \phi)$ be a compact cosymplectic manifold. Then we have*

(1) the cohomology classes, $1, \eta, \phi, \phi \wedge \eta, \phi^2, \ldots, \phi^n, \phi^n \wedge \eta$ contribute the Betti numbers $B_i(M)$, $i = 0, \ldots, 2n+1$, respectively.
(2) every Morse function $f : M \to \mathbb{R}$ has critical points more than $n+2$ points such that there are critical points $x_k \in M$ of f satisfying $ind_f(x_k) = k$ for $k = 0, 1, \ldots, 2n+1$.

Let $(M^{2n_i}, g_i, \varphi_i, \xi_i, \eta_i, \phi_i)$ be compact cosymplectic manifolds, $i = 1, 2$ and $(M = M_1 \times M_2, g, J, \phi)$ the product of M_1 and M_2. By Theorem 1 M is a Kähler manifold. By the Künneth Theorem the cohomology of M is

$$H^*(M, \mathbb{Q}) = H^*(M_1, \mathbb{Q}) \otimes H^*(M_2, \mathbb{Q}).$$

The k-dimensional cohomology of M is

$$H^k(M, \mathbb{Q}) = \sum_{k_1+k_2=k} H^{k_1}(M_1, \mathbb{Q}) \otimes H^{k_2}(M_2, \mathbb{Q}).$$

and the kth Betti number of M,

$$B_k(M) = \sum_{k_1+k_2=k} B_{k_1}(M_1) B_{k_2}(M_2).$$

By Theorem 3 the first Betti number of M is $B_1(M) = B_1(M_1) + B_1(M_2) \geq 2$.

Theorem 4 *Let M be a product of two compact cosymplectic manifolds. Then the $B_1(M)$ is even and greater than or equal to 2.*

Theorem 5 *A compact simply connected Kähler manifold cannot be the product of two cosymplectic manifolds.*

4 Gromov–Witten Invariants on the Products

Let $(M^{2n_i+1}, g_i, \varphi_i, \xi_i, \eta_i, \phi_i)$, $i = 1, 2$, compact cosymplectic manifolds and $\mathscr{D}_i = \{X \in TM_i \mid \eta_i(X) = 0\}$, $i = 1, 2$, the distribution bundles associated with η_i on M_i, respectively. As in Sect. 2 we denote (M, g, J, ϕ) the product of M_1 and M_2. We decompose the tangent bundle TM into three complex subbundles as follows:

$$\begin{tikzcd} TM = \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle \arrow[rr, "J"] \arrow[dr] & & TM = \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle \arrow[dl] \\ & M, & \end{tikzcd}$$

for every $(X_1, X_2, r_1\xi_1, r_2\xi_2) \in \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle$.

In the decomposition $(\mathscr{D}_1, \varphi_1)$, $(\mathscr{D}_2, \varphi_2)$, $(\langle \xi_1, \xi_2 \rangle, \varphi_3)$ are Hermitian vector bundles of rank n_1, n_2, and 1, respectively. By the Künneth formula the 2-dimensional homology of M is

$$H_2(M) = H_2(M_1) \oplus H_2(M_2) \oplus (H_1(M_1) \otimes H_1(M_2)).$$

The first Chern class of M is

$$\begin{aligned} c_1(TM) &= c_1(\mathscr{D}_1) + c_1(\mathscr{D}_2) + c_1(\langle \xi_1, \xi_2 \rangle) \\ &= c_1(\mathscr{D}_1) + c_1(\mathscr{D}_2), \end{aligned}$$

where $\langle \xi_1, \xi_2 \rangle$ is a trivial complex line bundle.

Assume that an integral curve of ξ_i in M_i is a circle S_i^1, $i = 1, 2$. Then the torus $T := S_1^1 \times S_2^1 \subset M_1 \times M_2$ is an integral surface of $\{\xi_1, \xi_2\}$ whose tangent bundle is $T = \langle \xi_1, \xi_2 \rangle$. For example, $M_i = N_i \times S_i^1$ are the products of Kähler manifolds N_i and circles S_i^1, $i = 1, 2$ [11].

Let $A \in H_2(M)$ be decomposed as $A = A_1 + A_2 + A_3$, where $A_1 \in H_2(M_1)$, $A_2 \in H_2(M_2)$, $A_3 \in H_1(M_1) \otimes H_1(M_2)$ and let $\pi_i : M \to M_1, M_2, T$, $i = 1, 2, 3$ be the projections, respectively. Recall that a smooth map $u : (\Sigma, j) \to (M, J)$ from a Riemann surface (Σ, j) to (M, J) is J-holomorphic if $du \circ j = J \circ du$. For each $i = 1, 2, 3$ the map $u_i := \pi_i \circ u$ is φ_i-holomorphic if $du_i \circ j = \varphi_i \circ du_i$.

Lemma 4 *A smooth map $u : (\Sigma, j) \to (M, J)$ is J-holomorphic if and only if $u_i : (\Sigma, j) \to (M_i, J_i)$ is φ_i-holomorphic $i = 1, 2, 3$, where $(M_3, J_3) = (T, \varphi_3)$ and $J = \varphi_1 \oplus \varphi_2 \oplus \varphi_3$ on $TM = \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle$.*

Let $\mathfrak{M}_{0,3}(M; A, J) := \{u : (\Sigma, j) \to (M, J) \mid u$ is J-holomorphic, $u_*([\Sigma]) = A\}$ be the moduli space of stable J-holomorphic maps from a sphere with 3 marked points to M which represent the 2-dimensional homology class A.

Note that there is no nontrivial rational map to a torus [5, 9].

Lemma 5 *The moduli space of T is*

$$\mathfrak{M}_{0,3}(T; A, \varphi_3) = \begin{cases} T & \text{if } A = 0 \\ \phi & \text{otherwise.} \end{cases}$$

Theorem 6 *(1) The moduli space $\mathfrak{M}_{0,3}(M; A, J)$ is a compact stratified space of dimension* $2[c_1(\mathscr{D}_1)A_1 + c_1(\mathscr{D}_2)A_2 + n]$.

(2) If $A_3 = 0$*, then there is a canonical homeomorphism*

$$\mathfrak{M}_{0,3}(M; A, J) \to \mathfrak{M}_{0,3}(M; A_1, \varphi_1) \times \mathfrak{M}_{0,3}(M_2; A_2, \varphi_2) \times T.$$

Proof (1) By the stability of J-holomorphic maps the moduli space $\mathfrak{M}_{0,3}(M; A, J)$ is compact. The dimension of $\mathfrak{M}_{0,3}(M; A, J)$ is

$$\begin{aligned}
\dim \mathfrak{M}_{0,3}(M; A, J) &= 2c_1(TM)A + 2n \\
&= 2(c_1(\mathfrak{D}_1) + c_1(\mathfrak{D}_2) + c_1(\langle \xi_1, \xi_2 \rangle))(A_1 + A_2 + A_3) + 2(n_1 + n_2 + 1) \\
&= (2c_1(\mathfrak{D}_1)A_1 + 2n_1) + (2c_1(\mathfrak{D}_2)A_2 + 2n_2) + 2 \\
&= \dim \mathfrak{M}_{0,3}(M_2; A_2, \varphi_2) + \dim \mathfrak{M}_{0,3}(M_1; A_1, \varphi_1) + \dim T.
\end{aligned}$$

(2) By Lemmas 4 and 5, (2) is clear. □

There are the canonical evaluation maps given by as follows:

$$\begin{aligned}
&ev : \mathfrak{M}_{0,3}(M; A, J) \to M^3, && ev([u; z_1, z_2, z_3]) = (u(z_1), u(z_2), u(z_3)), \\
&ev_1 : \mathfrak{M}_{0,3}(M_1; A_1, \varphi_1) \to M_1^3, && ev([u_1; z_1, z_2, z_3]) = (u_1(z_1), u_1(z_2), u_1(z_3)), \\
&ev_2 : \mathfrak{M}_{0,3}(M_2; A_2, \varphi_2) \to M_2^3, && ev([u_2; z_1, z_2, z_3]) = (u_2(z_1), u_2(z_2), u_2(z_3)), \\
&ev_3 : \mathfrak{M}_{0,3}(T; A_3, \varphi_3) \to T^3, && ev_3([u_3; z_1, z_2, z_3]) = (u_3(z_1), u_3(z_2), u_3(z_3)).
\end{aligned}$$

The Gromov–Witten invariants are defined by

$$\begin{aligned}
&\Phi_{0,3}^{M,A,J} : H^*(M^3) \to \mathbb{Q}, && \Phi_{0,3}^{M,A,J}(\alpha) = \int_{\mathfrak{M}_{0,3}(M;A,J)} ev^*(\alpha), \\
&\Phi_{0,3}^{M_1,A_1,\varphi_1} : H^*(M_1^3) \to \mathbb{Q}, && \Phi_{0,3}^{M_1,A_1,\varphi_1}(\alpha_1) = \int_{\mathfrak{M}_{0,3}(M_1;A_1,\varphi_1)} ev_1^*(\alpha_1), \\
&\Phi_{0,3}^{M_2,A_2,\varphi_2} : H^*(M_2^3) \to \mathbb{Q}, && \Phi_{0,3}^{M_2,A_2,\varphi_2}(\alpha_2) = \int_{\mathfrak{M}_{0,3}(M_2;A_2,\varphi_2)} ev_2^*(\alpha_2), \\
&\Phi_{0,3}^{T,A_3,\varphi_3} : H^*(T^3) \to \mathbb{Q}, && \Phi_{0,3}^{T,A_3,\varphi_3}(\alpha_3) = \int_T ev_3^*(\alpha_3).
\end{aligned}$$

By Lemma 5 we have

Lemma 6 *If* $A_3 = 0$*, then the Gromov–Witten invariants of* T *are*

$$\Phi_{0,3}^{T,A_3,\varphi_3} : H^*(T^3) \to \mathbb{Q}, \quad \Phi_{0,3}^{T,A_3,\varphi_3}(\alpha_{31} \otimes \alpha_{32} \otimes \alpha_{33}) = \int_T (\alpha_{31} \cup \alpha_{32} \cup \alpha_{33}),$$

where $\alpha_{3i} \in H^*(T)$*,* $i = 1, 2, 3$.

Theorem 7 *Under the above assumptions we have*

$$\Phi_{0,3}^{M,A,J}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = \Phi_{0,3}^{M_1,A_1,\varphi_1}(\alpha_1) \cdot \Phi_{0,3}^{M_2,A_2,\varphi_2}(\alpha_2) \cdot \int_T ev_3^*(\alpha_3),$$

where $\alpha_1 \in H^*(M_1^3)$, $\alpha_2 \in H^*(M_2^3)$, $\alpha_3 \in H^*(T^3)$, *and* $A_3 = 0$.

Proof Let $\alpha_1 \in H^{d_1}(M_1^3)$, $\alpha_2 \in H^{d_2}(M_2^3)$, and $\alpha_3 \in H^2(T^3)$, where $d_i = \dim \mathfrak{M}_{0,3}(M_i; A_i, \varphi_i) = 2c_i(\mathscr{D}_i) + 2n_i$, $i = 1, 2$. Then we have

$$\Phi_{0,3}^{M;A,J}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = \int_{\mathfrak{M}_{0,3}(M,A,J)} ev^*(\alpha_1 \otimes \alpha_2 \otimes \alpha_3)$$
$$= \int_{\mathfrak{M}_{0,3}(M_1,A_1,\varphi_1)} ev_1^*(\alpha_1) \cdot \int_{\mathfrak{M}_{0,3}(M_2,A_2,\varphi_2)} ev_2^*(\alpha_2) \cdot \int_{\mathfrak{M}_{0,3}(T,0,\varphi_3)} ev_3^*(\alpha_3)$$
$$= \Phi_{0,3}^{M_1,A_1,\varphi_1}(\alpha_1) \cdot \Phi_{0,3}^{M_2,A_2,\varphi_2}(\alpha_2) \cdot \int_T ev_3^*(\alpha_3).$$

□

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