Characterizations of a Clifford Hypersurface in a Unit Sphere

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Abstract The Clifford hypersurface is one of the simplest compact hypersurfaces in a unit sphere. We give two different characterizations of Clifford hypersurfaces among constant m-th order mean curvature hypersurfaces with two distinct principal curvatures. One is obtained by assuming embeddedness and by comparing two distinct principal curvatures. The proof uses the maximum principle to the two-point function, which was used in the proof of Lawson conjecture by Brendle (Acta Math. 211(2):177–190, 2013, [6]). The other is given by obtaining a sharp curvature integral inequality for hypersurfaces in a unit sphere with constant m-th order mean curvature and with two distinct principal curvatures, which generalizes Simons integral inequality (Simons, Ann. Math. (2) 88:62–105, 1968, [30]). This article is based on joint works (Min and Seo, Math. Res. Lett. 24(2):503–534, 2017, [18], Min and Seo, Monatsh. Math. 181(2):437–450, 2016, [19]) with Sung-Hong Min.

1 Introduction and Results

Recently minimal surface theory in a 3-dimensional unit sphere $\mathbb{S}^3$ has been extensively studied by many geometers. Among compact minimal surfaces in $\mathbb{S}^3$, the simplest one is the equator, which is totally geodesic. In 1966, Almgren [2] obtained the uniqueness theorem, which states that any immersed 2-sphere in $\mathbb{S}^3$ is totally geodesic. Thereafter Lawson [16] constructed compact embedded minimal surfaces in $\mathbb{S}^3$ with any genus. Moreover he conjectured that the only compact embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus. Brendle [6] proved ingeniously this famous conjecture by using the maximum principle for the two-point function.

Theorem 1 ([6]) *The only embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus.*

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In 1989, Pinkall and Sterling [29] proposed the conjecture that any embedded constant mean curvature(CMC) torus is rotationally symmetric, which is a CMC-version of Lawson conjecture. Applying Brendle's argument in [6], Andrews and Li [3] gave an affirmative answer to Pinkall–Sterling's conjecture.

Theorem 2 ([3]) *Every embedded CMC torus in* $\mathbb{S}^3$ *is rotationally symmetric.*

It would be interesting to obtain an analogue in higher-dimensional cases. However, the situation is more complicated in higher-dimensional cases. In the following we give brief historical review in this direction.

Let M be a compact minimal hypersurface in $\mathbb{S}^{n+1}$. Simons [30] obtained the following identity:

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + |A|^2(n - |A|^2),$$

where Δ, ∇, and A denote the Laplacian, the Levi-Civita connection, and the second fundamental form on M, respectively. Integrating this identity over M, Simons was able to prove the following integral inequality:

$$\int_M |A|^2 \left(|A|^2 - n\right) \geq 0. \tag{1}$$

It follows from the above integral inequality that there are three possibilities: Such M is either totally geodesic, or $|A|^2 \equiv n$, or $|A|^2(x) > n$ at some point $x \in M$. Regarding the second case, Chern, do Carmo and Kobayashi [10] in 1968 and Lawson [15] in 1969 independently proved

Theorem 3 ([10, 15]) *For* $n \geq 3$, *if* $|A|^2 \equiv n$ *on* M, *then* M *is isometric to a Clifford minimal hypersurface* $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-1}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{1}{n}}\right)$.

For higher-dimensional cases, Otsuki deeply investigated minimal hypersurfaces with two distinct principal curvatures as follows:

Theorem 4 ([23–25]) *Let* M *be a minimal hypersurface in* $\mathbb{S}^{n+1}$ *with two distinct principal curvatures* λ *and* μ.

- *The distribution of the space of principal vectors corresponding to each principal curvature is completely integrable.*
- *If the multiplicity of a principal curvature is greater than* 1, *then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*
- *If one of* λ *and* μ *is simple, then there are infinitely many immersed minimal hypersurfaces other than Clifford minimal hypersurfaces.*
- *If* M *is embedded, then* M *is locally congruent to a Clifford minimal hypersurface.*

Hence one sees that the only compact embedded minimal hypersurfaces with two distinct principal curvatures in $\mathbb{S}^{n+1}$ is a Clifford minimal hypersurface. However this uniqueness result does not hold in general. For example, Hsiang [14] constructed infinitely many mutually noncongruent embedded minimal hypersurfaces in $\mathbb{S}^{n+1}$ which are homeomorphic to the Clifford hypersurface using equivariant differential geometry. Furthermore it is well-known that a lot of isoparametric hypersurfaces exist in $\mathbb{S}^{n+1}$, which are all embedded. See [1, 7–9, 12, 13, 21, 22, 26] for examples and [20] for more references.

Otsuki's result was extended to higher-order mean curvature cases for hypersurfaces with two distinct principal curvatures. Wu [34] proved that if the multiplicities of two distinct principal curvatures are at least 2, then a compact hypersurface with constant m-th order mean curvature is congruent to a Clifford hypersurface. Thus we shall consider hypersurfaces with constant m-th order mean curvature satisfying that one of the two distinct principal curvatures is simple. We remark that if M is a hypersurface in a space form with two distinct principal curvatures such that one of two distinct principal curvatures is simple, then M is a part of rotationally symmetric hypersurface, which was proved by do Carmo and Dajzer [11]. Recall that the m-th order mean curvature H_m of an n-dimensional hypersurface $M \subset \mathbb{S}^{n+1}$ is defined by the elementary symmetric polynomial of degree m in the principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$ on M as follows:

$$\binom{n}{m} H_m = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \ldots \lambda_{i_m}.$$

We also recall that if an n-dimensional Clifford hypersurface in $\mathbb{S}^{n+1}$ has two distinct principal curvatures λ and μ of multiplicities $n-k$ and k respectively, then it is given by

$$\mathbb{S}^{n-k}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^{k}\left(\frac{1}{\sqrt{1+\mu^2}}\right)$$

with $\lambda\mu + 1 = 0$.

Assume that M is a compact hypersurface in a unit sphere with constant m-th order mean curvature H_m and with two distinct principal curvatures with multiplicities $n-1, 1$. Without loss of generality, we may assume that $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$. Choose the orthonormal frame tangent to M such that $h_{ij} = \lambda_i \delta_{ij}$, that is,

$$\begin{aligned} Ae_i &= \lambda e_i \quad \text{for } i = 1, \ldots, n-1, \\ Ae_n &= \mu e_n. \end{aligned}$$

Then

$$\binom{n}{m} H_m = \binom{n-1}{m} \lambda^m + \binom{n-1}{m-1} \lambda^{m-1} \mu,$$

which gives

$$H_m = \frac{m}{n}\lambda^{m-1}\left(\frac{n-m}{m}\lambda + \mu\right). \tag{2}$$

For some Weingarten hypersurfaces with two distinct principal curvatures, Andrews, Huang, and Li obtained the following:

Theorem 5 ([4]) *Let Σ be a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ, whose multiplicities are m and $n-m$ respectively for $1 \le m \le n-1$. If $\lambda + \alpha\mu = 0$ for some positive constant α, Σ is congruent to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{1}{\alpha+1}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{\alpha}{1+\alpha}}\right)$.*

Using the identity (2) and Theorem 5, we see that any compact embedded hypersurfaces with vanishing m-th order mean curvature and with two distinct principal curvatures is congruent to a Clifford hypersurface. On the other hand, Perdomo [28] constructed compact embedded CMC hypersurfaces in $\mathbb{S}^{n+1}$, which have two distinct principal curvatures, one of them being simple.

Theorem 6 ([28]) *For any integer $m \ge 2$ and H between $\cot\frac{\pi}{m}$ and $\frac{(m^2-2)\sqrt{n-1}}{n\sqrt{m^2-1}}$, there exists a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature H other than the totally geodesic n-spheres and Clifford hypersurfaces.*

We note that the two distinct principal curvatures λ and μ satisfy $\lambda > \mu$ in Theorem 6, where μ is simple. In case where $\lambda < \mu$, it is natural to ask whether one can obtain the uniqueness of Clifford hypersurface or not. In [18], Sung-Hong Min and the author gave the affirmative answer to this question as follows:

Theorem 7 ([18]) *Let Σ be an $n(\ge 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature $H \ge 0$ and with two distinct principal curvatures λ and μ, μ being simple. If $\mu > \lambda$, then Σ is congruent to a Clifford hypersurface.*

In Sect. 3, we give another characterization of Clifford hypersurfaces using Simons-type integral inequality for a compact hypersurface in a unit sphere with constant higher-order mean curvature and with two distinct principal curvatures.

2 Proof of Theorem 7

Here we give the brief sketch of the proof of Theorem 7. If $H = 0$, then it is already known that Σ is congruent to a Clifford minimal hypersurfaces by the work due to Otsuki. We now assume that $H > 0$. Since Σ is a compact embedded hypersurface, Σ divides $\mathbb{S}^{n+1}$ into two connected components. We may assume that $H > 0$ by the suitable choice of the orientation of Σ. Let R be the region satisfying that ν points out of R. The *mean curvature vector* $\mathbf{H}$ satisfies that $\mathbf{H} = -nH\nu(x)$. For a positive function Ψ on Σ, we denote by $B_T(x, \frac{1}{\Psi(x)})$ a geodesic ball with radius $\frac{1}{\Psi(x)}$ which touches Σ at $F(x)$ inside the region R in $\mathbb{S}^{n+1}$. Then $B_T(x, \frac{1}{\Psi(x)})$ is given by the intersection of $\mathbb{S}^{n+1}$ and a ball of radius $\frac{1}{\Psi(x)}$ centered at $p(x) = F(x) - \frac{1}{\Psi(x)}\nu(x)$ in $\mathbb{R}^{n+2}$. Define the two-point function $Z : \Sigma \times \Sigma \to \mathbb{R}$ by

$$Z(x, y) := \Psi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle. \tag{3}$$

We introduce the notion of the interior ball curvature at $x \in \Sigma$, which was originally given by Andrews-Langford-McCoy [5] (see also [3]).

Definition 1 The *interior ball curvature* k is a positive function on Σ defined by

$$k(x) := \inf\left\{\frac{1}{r} : B_T(x, r) \cap \Sigma = \{x\},\ r > 0\right\}.$$

Since Σ is compact and embedded in $\mathbb{S}^{n+1}$, the function k is a well-defined positive function on Σ. From the definition of $k(x)$ for every point $x \in \Sigma$, we have

$$k(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geq 0$$

for all $y \in \Sigma$.

Let $\Phi(x) := \max\{\lambda(x), \mu(x)\}$ be the maximum value of the principal curvatures of Σ in $\mathbb{S}^{n+1}$ at $F(x)$. It is easy to see that $\Phi(x) - H > 0$. Motivated by the works of Brendle [6] and Andrews-Li [3], we introduce the constant κ as follows:

$$\kappa := \sup_{x\in\Sigma} \frac{k(x) - H}{\Phi(x) - H}.$$

For convenience, we will write $\varphi(x) := \Phi(x) - H$. It follows that there exists a constant $K > 0$ satisfying

$$1 \leq \kappa < K.$$

By definition, we see that $\Phi(x) \leq k(x)$ for every $x \in \Sigma$ in general. Indeed, we have the equality case under our setting.

Proposition 1 *Let Σ be an $n(\geq 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature H with two distinct principal curvatures, one of them being simple. If $H > 0$. Then*

$$k(x) = \Phi(x)$$

for all $x \in \Sigma$.

Proof See [18] for the proof. □

From this observation, it follows that $k(x) = \Phi(x)$ and hence

$$\Phi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geq 0,$$

for all $x, y \in \Sigma$. Fix $x \in \Sigma$ and choose an orthonormal frame $\{e_1, \dots, e_n\}$ in a neighborhood of x such that $h(e_n, e_n) = \Phi$. Let $\gamma(t)$ be a geodesic on Σ such that $\gamma(0) = F(x)$ and $\gamma'(0) = e_n$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) := Z(F(x), \gamma(t)) = \Phi(x)(1 - \langle F(x), \gamma(t)\rangle) + \langle \nu(x), \gamma(t)\rangle.$$

Then one sees that $f(t) \geq 0$ and $f(0) = 0$. Moreover

$$f'(t) = -\langle \Phi(x)F(x) - \nu(x), \gamma'(t)\rangle,$$

$$f''(t) = \langle \Phi(x)F(x) - \nu(x), \gamma(t) + h(\gamma'(t), \gamma'(t))\nu(\gamma(t))\rangle,$$

$$\begin{aligned} f'''(t) = &\ \langle \Phi(x)F(x) - \nu(x), \gamma'(t) + (\nabla^{\Sigma}_{\gamma'(t)} h)(\gamma'(t), \gamma'(t))\nu(\gamma(t))\rangle \\ &+ \langle \Phi(x)F(x) - \nu(x), h(\gamma'(t), \gamma'(t))\nabla_{\gamma'(t)}\nu(\gamma(t))\rangle, \end{aligned}$$

where ∇ is the covariant derivative of $\mathbb{R}^{n+2}$. In particular, at $t = 0$,

$$\begin{aligned} f(0) &= f'(0) = 0, \\ f''(0) &= \langle \Phi(x)F(x) - \nu(x), F(x) + \Phi(x)\nu(x)\rangle = 0. \end{aligned}$$

Because f is nonnegative, we get $f'''(0) = 0$. Hence

$$0 = f'''(0) = \langle \Phi(x)F(x) - \nu(x), e_n + h_{nnn}(x)\nu(x)\rangle = -h_{nnn}(x).$$

Therefore we see that $e_n\lambda = h_{11n} = -\frac{1}{n-1}h_{nnn} = 0$, which implies that λ and μ are constant on Σ by Ostuki. It follows that Σ is an isoparametric hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. From the classification of isoparametric hypersurfaces with two principal curvatures due to Cartan [7], Σ is congruent to the Clifford hypersurface.

3 Sharp Curvature Integral Inequality

In this section, we give another uniqueness result of Clifford hypersurfaces in terms of curvature integral inequality. Perdomo [27] and Wang [31] independently obtained a curvature integral inequality for minimal hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, which characterizes a Clifford minimal hypersurface. Later, Wei [32] showed that the similar curvature integral inequality holds for hypersurfaces with the vanishing m-th order mean curvature (i.e., $H_m \equiv 0$).

Theorem 8 ([27, 31, 32]) *Let M be an $n(\geq 3)$-dimensional compact hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple. Then*

$$\int_M |A|^2 \leq \frac{n(m^2 - 2m + n)}{m(n-m)} \mathrm{Vol}(M),$$

where equality holds if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

In [19], Sung-Hong Min and the author obtained a sharp curvature integral inequality for compact hypersurfaces in $\mathbb{S}^{n+1}$ with $H_m \equiv constant$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple.

Theorem 9 ([19]) *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple (i.e., multiplicity 1). For the unit principal direction vector e_n corresponding to μ, we have*

$$\int_M \mathrm{Ric}(e_n, e_n) \geq 0,$$

where Ric *denotes the Ricci curvature. Moreover, equality holds if and only if M is isometric to a Clifford hypersurface.*

We remark that if $H_m \equiv 0$ for $1 \leq m < n$, then

$$\mathrm{Ric}(e_n, e_n) = (n-1)\left(1 - \frac{m(n-m)}{n(m^2 - 2m + n)}|A|^2\right).$$

Theorem 9 can be regarded as an extension of [27, 31, 32]. From this theorem, one sees that if $\mathrm{Ric}(e_n, e_n) \leq 0$ on such a hypersurface M, then M is congruent to a Clifford hypersurface.

Proof of Theorem 9 Here we give a brief idea of the proof of Theorem 9 (See [19] for more details). Note that for $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$, the function $w = |\lambda^m - H_m|^{-\frac{1}{n}}$ is well-defined. From this notion, the Laplacian of $f = f(w)$ on M is given by

$$\Delta f = -\frac{1}{n-1} f'(w) w \,\mathrm{Ric}(e_n, e_n) + \left[f''(w) + (n-1)\frac{f'(w)}{w}\right](e_n w)^2. \quad (4)$$

If we let a function $f(w) = \log w$ in (4), then

$$\Delta f = -\frac{\mathrm{Ric}(e_n, e_n)}{n-1} + \frac{n-2}{w^2}(e_n w)^2.$$

Integrating Δf over M gives

$$\int_M \mathrm{Ric}(e_n, e_n) = (n-1)(n-2)\int_M \frac{(e_n w)^2}{w^2} \geq 0.$$

Equality holds if and only if $e_n\lambda \equiv 0$. Thus both λ and μ are constant, which shows that M is congruent to a Clifford hypersurface by Cartan [7]. □

In the following, we generalize Simons' integral inequality into closed hypersurfaces with two distinct principal curvatures.

Theorem 10 ([19]) *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple. Then we have*

$$\begin{cases} \int_M |A|^p \left(|A|^2 - \dfrac{n(m^2-2m+n)}{m(n-m)}\right) \leq 0 \; if \; p < \frac{n-2}{n}m, \\ \int_M |A|^p \left(|A|^2 - \dfrac{n(m^2-2m+n)}{m(n-m)}\right) \geq 0 \; if \; p > \frac{n-2}{n}m. \end{cases}$$

Equalities in the above hold if and only if M is congruent to a Clifford hypersurface.

Proof See [19] for the proof. □

We remark that if $m = 1$ and $p = 2$, then Theorem 10 is exactly the same as Simons' integral inequality (1), which was mentioned in the introduction. We also remark that when $m = 2$ and $p = 2$, Li [17] obtained some pointwise estimates on $|A|^2$, which gives the above theorem. For $p = 2$ and $3 < m < n$, Wei [33] proved the above theorem for compact and rotational hypersurfaces in a unit sphere with $H_m \equiv 0$.

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