Dual Orlicz Mixed Quermassintegral

Jia He, Denghui Wu and Jiazu Zhou

Abstract We study the dual Orlicz mixed Quermassintegral. For arbitrary monotone continuous function ϕ, the dual Orlicz radial sum and dual Orlicz mixed Quermassintegral are introduced. Then the dual Orlicz–Minkowski inequality and dual Orlicz–Brunn–Minkowski inequality for dual Orlicz mixed Quermassintegral are obtained. These inequalities are just the special cases of their L_p analogues (including cases $-\infty < p < 0$, $p = 0$, $0 < p < 1$, $p = 1$, and $1 < p < +\infty$). These inequalities for $\phi = \log t$ are related to open problems including log-Minkowski problem and log-Brunn-Minkowski problem. Moreover, the equivalence of the dual Orlicz–Minkowski inequality for dual Orlicz mixed Quermassintegral and dual Orlicz–Brunn–Minkowski inequality for dual Orlicz mixed Quermassintegral is shown.

Keywords Star body · Orlicz radial sum · Dual Orlicz mixed Quermassintegral · Dual Orlicz–Minkowski inequality · Dual Orlicz–Brunn–Minkowski inequality

2010 Mathematics Subject Classification Primary: 52A30 · 52A40

1 Introduction

The classical Brunn–Minkowski theory for convex bodies (compact convex sets with nonempty interior) is known as consequences of the combination of Minkowski addition and volume, which constitutes the core of convex geometry. Significant results in this theory, for instance the Minkowski's first inequality and the

J. He · D. Wu · J. Zhou (✉)
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
e-mail: zhoujz@swu.edu.cn

J. He
e-mail: 13608343695@163.com

D. Wu
e-mail: wudenghui66@163.com

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_11

Brunn–Minkowski inequality, have important applications in analysis, geometry, random matrices, and many other fields (see [28]).

In 1960s, Firey extended Minkowski addition to L_p addition in [2]. Since then, the Brunn–Minkowski theory has gained amazing developments. This extended theory is called L_p Brunn–Minkowski theory, which connects volumes with L_p addition (see e.g. [7, 9, 19–23, 31]). As a development of L_p Brunn–Minkowski theory, Orlicz–Brunn–Minkowski theory is a new blossom in recent years, which is motivated by [8, 15, 16, 24, 25]. For more references, see [3, 11, 14, 34, 35, 38]. Specifically, Xiong and Zou studied Orlicz mixed Quermassintegral in [35].

In [17, 18], Lutwak introduced duality of the Brunn–Minkowski theory, in which the research object substitutes star bodies for convex bodies, obtained dual counterparts of the several wonderful results in the Brunn–Minkowski theory. Intersection body is a useful geometrical object in dual Brunn–Minkowski theory, introduced by Lutwak in [18]. The class of intersection bodies and mixed intersection bodies are valuable in geometry, especially in answering the known Busemann–Petty problem (see [12]). We refer the reader to [5, 6, 13, 26, 32, 33] for the extended intersection bodies and their applications.

In [37], a dual Orlicz–Brunn–Minkowski theory was presented and the dual Orlicz–Brunn–Minkowski inequality for volume was established. An Orlicz radial sum and dual Orlicz mixed volumes were introduced. The dual Orlicz–Minkowski inequality and the dual Orlicz–Brunn–Minkowski inequality were established. The variational formula for the volume with respect to the Orlicz radial sum was proved. The equivalence between the dual Orlicz–Minkowski inequality and the dual Orlicz–Brunn–Minkowski inequality was demonstrated. Orlicz intersection bodies were introduced and the Orlicz–Busemann–Petty problem was posed. It should noted that analog theory was also discussed in [4]. Following ideas of [4, 37], the dual Orlicz–Brunn–Minkowski inequality for dual mixed Quermassintegral was also discussed in [35].

Motivated by works of [4, 37], we consider the dual Orlicz–Brunn–Minkowski inequality for dual mixed Quermassintegral in the n-dimensional Euclidean space $\mathbb{R}^n$. We denote by $\mathscr{C}^{in}$ the set of all increasing continuous functions $\phi : (0, \infty) \to (-\infty, \infty)$ and by $\mathscr{C}^{de}$ the set of all decreasing continuous functions $\phi : (0, \infty) \to (-\infty, \infty)$. Let $\mathscr{C}$ denote the union of $\mathscr{C}^{in}$ and $\mathscr{C}^{de}$. The n dimensional unit ball and the unit sphere are denoted by B and S^{n-1} respectively.

A set K in $\mathbb{R}^n$ is star-shaped set with respect to $z \in K$ if the intersection of every line through z with K is a line segment. The radial function, $\rho_K : S^{n-1} \to [0, \infty)$, of a compact star-shaped set (about the origin) is defined by

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \qquad u \in S^{n-1}. \tag{1}$$

If $\rho(K, \cdot)$ is positive and continuous, K is called a star body. Let $\mathscr{S}^n$ and $\mathscr{S}_0^n$ denote the set of start bodies and the set of start bodies about the origin in $\mathbb{R}^n$, respectively.

Definition 1 Let $K, L \in \mathscr{S}_0^n, a, b > 0$.
If $\phi \in \mathscr{C}^{in}$, then Orlicz radial sum $a \cdot K \tilde{+}_\phi b \cdot L$ is defined by

$$\rho_{a\cdot K \tilde{+}_\phi\, b\cdot L}(u) = \inf\left\{t > 0 : a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) \leq \phi(1)\right\}, \forall u \in S^{n-1}. \tag{2}$$

If $\phi \in \mathscr{C}^{de}$, then Orlicz radial sum $a \cdot K \tilde{+}_\phi\, b \cdot L$ is defined by

$$\rho_{a\cdot K \tilde{+}_\phi\, b\cdot L}(u) = \sup\left\{t > 0 : a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) \leq \phi(1)\right\}, \forall u \in S^{n-1}. \tag{3}$$

The dual mixed Quermassintegral $\tilde{W}_i(K, L)$, defined in [17], is

$$\tilde{W}_i(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-i-1}\rho_L(u)dS(u). \tag{4}$$

Motivated by this, we define the following dual Orlicz mixed Quermassintegral.

Definition 2 Let $K, L \in \mathscr{S}_0^n$, $i \in \mathbb{R}$, $\phi \in \mathscr{C}$. The dual Orlicz mixed Quermassintegral $\tilde{W}_{\phi,i}(K, L)$ is defined by

$$\tilde{W}_{\phi,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)dS(u). \tag{5}$$

When $\phi(t) = t^p$, with $p \neq 0$, the dual Orlicz mixed volume reduces to L_p dual mixed Quermassintegral (see [20] for the case $p \geq 1$ and $i = 0$)

$$\tilde{W}_{p,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i-p}(u)\rho_L^p(u)dS(u).$$

When $\phi(t) = \log t$, one has

$$\tilde{W}_{\phi,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)dS(u).$$

In Sect. 2, we introduce some basic concepts. In Sect. 3, the Orlicz radial sum and some related properties are discussed. Some important properties of dual Orlicz mixed Quermassintegral are investigated in Sect. 4.

In Sect. 5, dual Orlicz–Minkowski inequality and dual Orlicz–Brunn–Minkowski inequality are established for dual Orlicz mixed Quermassintegral. As special cases, these inequalities are just the L_p counterparts, including the cases $-\infty < p < 0$, $p = 0$, $0 < p < 1$, $p = 1$ and $1 < p < +\infty$. These inequalities for $\phi = \log t$ are related to open problems, such as, the log-Brunn–Minkowski problem and the log-Minkowski problem. Moreover, we prove the equivalence of dual Orlicz–Minkowski inequality and dual Orlicz–Brunn–Minkowski inequality.

2 Preliminaries

Let $K, L \in \mathscr{S}_0^n$. By (1), one has

$$K \subset L \quad \text{if and only if} \quad \rho_K(u) \leq \rho_L(u). \tag{6}$$

Two star bodies K and L are dilates (of each other) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. If $t > 0$, we have

$$\rho(tK, u) = t\rho(K, u), \qquad \text{for all } u \in S^{n-1}.$$

We write A^{-1} for the inverse matrix of A where $A \in GL(n)$. So associated with the definition of the radial function, for $A \in GL(n)$, the radial function of the image $AK = \{Ay : y \in K\}$ of K is shown by

$$\rho(AK, u) = \rho(K, A^{-1}u), \qquad \text{for all } u \in S^{n-1}. \tag{7}$$

The radial Hausdorff metric between the star bodies K and L is

$$\tilde{\delta}(K, L) = \max_{u \in S^{n-1}} \mid \rho_K(u) - \rho_L(u) \mid .$$

A sequence $\{K_i\}$ of star bodies is said to be convergent to K if

$$\tilde{\delta}(K_i, K) \to 0, \qquad \text{as } i \to \infty.$$

Therefore, a sequence of star bodies K_i converges to K if and only if the sequence of the radial function $\rho(K_i, \cdot)$ converges uniformly to $\rho(K, \cdot)$ [27].

Let $K, L \in \mathscr{S}^n$. We have

$$K \tilde{+}_\phi \varepsilon L \to K$$

in the radial Hausdorff metric as $\varepsilon \to 0^+$ [36].

The radial Minkowski linear combination of sets $K_1, \ldots, K_r$ in $\mathbb{R}^n$ is defined by

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r\}, \qquad \text{for all } \lambda_i \in \mathbb{R}, \;\; i = 1, \ldots, r.$$

If $K, L \in \mathscr{S}_0^n$ and $a, b > 0$, $aK \tilde{+} bL$ can be defined as a star body with satisfying that

$$\rho_{aK \tilde{+} bL}(u) = a\rho_K(u) + b\rho_L(u), \qquad \text{for all } u \in S^{n-1}. \tag{8}$$

Write $V(K)$ for the volume of the compact set K in $\mathbb{R}^n$. In fact, the volume of the radial Minkowski linear combination $\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r$ is a homogeneous n-th polynomial in λ_i (see [17, 18]).

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum_{r \leq n} \tilde{V}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficient $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$, it is nonnegative and only depends on the sets $K_{i_1}, \ldots, K_{i_n}$. Or write $\tilde{V}_i(K, L) = \tilde{V}(\underbrace{K, \ldots, K}_{n-i}, \underbrace{L, \ldots, L}_{i})$. If $L = B$, the dual mixed volume $\tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$ which is called the dual Quermassintegral of K.

If $K_1, \ldots, K_n \in S_0^n$, the dual mixed volume $\tilde{V}(K_1, \ldots, K_n)$ is defined [17]

$$\tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}(u) \cdots \rho_{K_n}(u) dS(u),$$

where S is the Lebesgue measure on S^{n-1} (i.e., the (n-1)-dimensional Hausdorff measure). Let $K \in \mathscr{S}_0^n$ and $i \in \mathbb{R}$. A slight extension (see [29]) of the notation $\tilde{W}_i(K)$ is

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} dS(u). \tag{9}$$

In (4), let $i = 0$, we immediately get the following integral representation for the first dual mixed volume proved by Lutwak in [17]: if $K, L \in \mathscr{S}_0^n$, then

$$\tilde{V}_1(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-1} \rho_L(u) dS(u).$$

The integral representation (4), together with the Hölder inequality and (9), immediately lead to the following dual Minkowski inequality about the dual mixed Quermassintegral $\tilde{W}_i(K, L)$.

Lemma 1 *If $K, L \in \mathscr{S}_0^n$ and $i < n - 1$, then*

$$\tilde{W}_i(K, L)^{n-i} \leq \tilde{W}_i(K)^{n-i-1} \tilde{W}_i(L), \tag{10}$$

with equality if and only if K and L are dilates of each other.

If $i > n - 1$ and $i \neq n$, (10) is reversed, with equality if and only if K and L are dilates.

We shall obtain the dual Brunn–Minkowski inequality for the dual Quermassintegral $\tilde{W}_i(aK \tilde{+} bL)$.

Lemma 2 *If $K, L \in \mathscr{S}_0^n$, $i < n - 1$ and $a, b > 0$, then*

$$\tilde{W}_i(aK \tilde{+} bL)^{\frac{1}{n-i}} \leq a \tilde{W}_i(K)^{\frac{1}{n-i}} + b \tilde{W}_i(L)^{\frac{1}{n-i}}, \tag{11}$$

with equality if and only if K and L are dilates of each other.

If $i > n-1$ and $i \neq n$, (11) is reversed, with equality if and only if K and L are dilates.

Upon the definition of the function ϕ, suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality (see [10]) shows that if $\phi : I \to \mathbb{R}$ is a concave function, then

$$\int_X \phi(g(x))d\mu(x) \le \phi\left(\int_X g(x)d\mu(x)\right). \tag{12}$$

If $\phi \in \mathscr{C}_2$, the inequality is reverse, that is

$$\int_X \phi(g(x))d\mu(x) \ge \phi\left(\int_X g(x)d\mu(x)\right). \tag{13}$$

If ϕ is strictly concave or convex, each equality in (12) and (13) holds if and only if $g(x)$ is constant for μ-almost all $x \in X$.

3 Orlicz Radial Sum

From (7) and the definition of the Orlicz radial sum, we have

Proposition 1 *Let $K, L \in \mathscr{S}_0^n$, and $a, b > 0$. If $\phi \in \mathscr{C}$, then for $A \in GL(n)$,*

$$A(a \cdot K \tilde{+}_\phi b \cdot L) = a \cdot AK \tilde{+}_\phi b \cdot AL. \tag{14}$$

Proof For $\phi \in \mathscr{C}^{in}$, $u \in S^{n-1}$, by (7)

$$\begin{aligned}
\rho_{a \cdot AK \tilde{+}_\phi b \cdot AL}(u) &= \inf\left\{t > 0 : a\phi\left(\frac{\rho_{AK}(u)}{t}\right) + b\phi\left(\frac{\rho_{AL}(u)}{t}\right) \le \phi(1)\right\} \\
&= \inf\left\{t > 0 : a\phi\left(\frac{\rho_K(A^{-1}u)}{t}\right) + b\phi\left(\frac{\rho_L(A^{-1}u)}{t}\right) \le \phi(1)\right\} \\
&= \rho_{a \cdot K \tilde{+}_\phi b \cdot L}(A^{-1}u) \\
&= \rho_{A(a \cdot K \tilde{+}_\phi b \cdot L)}(u).
\end{aligned}$$

If $\phi \in \mathscr{C}^{de}$, in the same way, we also have (14).

Since $K, L \in \mathscr{S}_0^n$ and $u \in S^{n-1}$, $0 < \rho_K(u) < \infty$ and $0 < \rho_L(u) < \infty$, hence $\frac{\rho_K(u)}{t} \to 0$ and $\frac{\rho_L(u)}{t} \to 0$ as $t \to \infty$. By the assumption that ϕ is monotone increasing (or decreasing) in $(0, \infty)$, so the function

$$t \mapsto a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right),$$

is monotone decreasing (or increasing) in $(0, \infty)$. Since it is also continuous, we have

Lemma 3 *Let $K, L \in \mathscr{S}_0^n$, $a, b > 0$, and $u \in S^{n-1}$. If $\phi \in \mathscr{C}$, then*

$$a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) = \phi(1),$$

if and only if

$$\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u) = t.$$

Remark 1 We shall provide several special examples of the Orlicz radial sum. Let $K, L \in \mathscr{S}_0^n$, $a, b > 0$.

(1) When $\phi(t) = t^p$, with $p \neq 0$, it is easy to show that the Orlicz radial sum reduces to an analogue form of Lutwak's L_p radial combination ($p \geq 1$, see [20])

$$\rho(a \cdot K\tilde{+}_\phi b \cdot L, u)^p = a\rho(K, u)^p + b\rho(L, u)^p.$$

(2) When $\phi(t) = \log t$, we obtain

$$(a + b)\log\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u) = a\log\rho_K(u) + b\log\rho_L(u).$$

This sum is dual of the logarithm sum which is an important notion (see [1, 30]).

(3) When $\phi(t) = \log(t + 1)$, we have

$$\left(\frac{\rho_K(u)}{\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)} + 1\right)^a \left(\frac{\rho_L(u)}{\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)} + 1\right)^b = 2,$$

and $\phi(0) = 0$.

Lemma 4 *Let $K, L \in \mathscr{S}_0^n$, for $0 < \lambda < 1$,*

(1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, then

$$(1 - \lambda) \cdot K\tilde{+}_\phi\lambda \cdot L \subseteq (1 - \lambda)K\tilde{+}\lambda L. \tag{15}$$

When ϕ is strictly concave or convex, the equality holds if and only if $K = L$.

(2) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$ or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, then

$$(1 - \lambda) \cdot K\tilde{+}_\phi\lambda \cdot L \supseteq (1 - \lambda)K\tilde{+}\lambda L. \tag{16}$$

When ϕ is strictly concave or convex, the equality holds if and only if $K = L$.

Proof Let $K_\lambda = (1 - \lambda) \cdot K\tilde{+}_\phi\lambda \cdot L$.

(1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$, by Lemma 3 and concavity of ϕ, we have

$$\phi(1) = (1-\lambda)\phi\left(\frac{\rho_K(u)}{\rho_{K_\lambda}(u)}\right) + \lambda\phi\left(\frac{\rho_L(u)}{\rho_{K_\lambda}(u)}\right)$$
$$\leq \phi\left(\frac{(1-\lambda)\rho_K(u) + \lambda\rho_L(u)}{\rho_{K_\lambda}(u)}\right).$$

Since ϕ is monotone increasing on $(0, \infty)$, hence we have

$$(1-\lambda)\rho_K(u) + \lambda\rho_L(u) \geq \rho_{K_\lambda}(u),$$

that is,

$$\rho_{(1-\lambda)K\tilde{+}\lambda L}(u) \geq \rho_{K_\lambda}(u). \tag{17}$$

If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, by Lemma 3 convexity of ϕ and ϕ is monotone decreasing on $(0, \infty)$, in the same way, we can obtain (17). Then by (6), (17) deduces the helpful conclusion (15).

(2) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, by Lemma 3 and convexity of ϕ, we have

$$\phi(1) = (1-\lambda)\phi\left(\frac{\rho_K(u)}{\rho_{K_\lambda}(u)}\right) + \lambda\phi\left(\frac{\rho_L(u)}{\rho_{K_\lambda}(u)}\right)$$
$$\geq \phi\left(\frac{(1-\lambda)\rho_K(u) + \lambda\rho_L(u)}{\rho_{K_\lambda}(u)}\right).$$

Since ϕ is monotone increasing on $(0, \infty)$, hence we also have

$$(1-\lambda)\rho_K(u) + \lambda\rho_L(u) \leq \rho_{K_\lambda}(u),$$

that is,

$$\rho_{(1-\lambda)K\tilde{+}\lambda L}(u) \leq \rho_{K_\lambda}(u). \tag{18}$$

If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, by Lemma 3, concavity of ϕ and ϕ is monotone decreasing on $(0, \infty)$, in the same way, we can obtain (18). Then by (6), (18) deduce the helpful conclusion (16).

From the equality condition in the concavity (or convexity) of ϕ, then each equality in (15) and (16) holds if and only if $K = L$.

Corollary 1 *Let $K, L \in \mathscr{S}_0^n$, $0 < \lambda < 1$ and $\tilde{W}_i(K) = \tilde{W}_i(L)$.*

(1) If $i < n-1$, $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, then

$$\tilde{W}_i((1-\lambda)\cdot K\tilde{+}_\phi\lambda\cdot L) \leq \tilde{W}_i(K), \tag{19}$$

with equality if and only if $K = L$.

(2) If $i > n-1$ and $i \neq n$, $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$ or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, then

$$\tilde{W}_i((1-\lambda)\cdot K\tilde{+}_\phi\lambda\cdot L) \geq \tilde{W}_i(K), \tag{20}$$

with equality if and only if $K = L$.

Proof (1) By Lemmas 4 and 2, we have

$$\begin{aligned}\tilde{W}_i((1-\lambda)\cdot K\tilde{+}_\phi\lambda\cdot L)^{\frac{1}{n-i}} &\le \tilde{W}_i((1-\lambda)\cdot K\tilde{+}\lambda\cdot L)^{\frac{1}{n-i}}\\ &\le (1-\lambda)\tilde{W}_i(K)^{\frac{1}{n-i}}+\lambda\tilde{W}_i(L)^{\frac{1}{n-i}}\\ &= \tilde{W}_i(K)^{\frac{1}{n-i}}.\end{aligned}$$

The equality condition in (19) can be obtained from the equality condition of (11).
(2) Similarly, from Lemmas 3.4 and 2.2, we can obtain (20).

4 Dual Orlicz Mixed Quermassintegral

We denote the right derivative of a real-valued function f by f'_r. In the following Lemma 5 the function ϕ is different from ϕ in Lemma 4.1 of [37]. However, we can use the similar argument to prove Lemma 5, so we omit the details.

Lemma 5 *Let $\phi \in \mathscr{C}$ and $K, L \in \mathscr{S}_0^n$. Then*

$$\lim_{\varepsilon\to 0^+}\frac{\rho_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho_K(u)}{\varepsilon}=\frac{\rho_K(u)}{\phi'_r(1)}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),$$

uniformly for all $u \in S^{n-1}$.

Theorem 1 *Let $\phi \in \mathscr{C}$, $K, L \in \mathscr{S}_0^n$ and $i \neq n$. Then*

$$\frac{n}{n-i}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}_\phi\varepsilon\cdot L)-\tilde{W}_i(K)}{\varepsilon}=\frac{1}{\phi'_r(1)}\int_{S^{n-1}}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)dS(u).$$

Proof Let $\varepsilon > 0$, $K, L \in \mathscr{S}_0^n$, $i \neq n$ and $u \in S^{n-1}$. By Lemma 5, it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{\rho^{n-i}_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho_K^{n-i}(u)}{\varepsilon} &= (n-i)\rho_K^{n-i-1}(u)\lim_{\varepsilon\to 0^+}\frac{\rho_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho_K(u)}{\varepsilon}\\ &= \frac{(n-i)\rho_K^{n-i}(u)}{\phi'_r(1)}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),\end{aligned}$$

uniformly on S^{n-1}. Hence

$$\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}_\phi\varepsilon\cdot L)-\tilde{W}_i(K)}{\varepsilon} = \lim_{\varepsilon\to 0^+}\left(\frac{1}{n}\int_{S^{n-1}}\frac{\rho^{n-i}_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho^{n-i}_K(u)}{\varepsilon}dS(u)\right)$$
$$= \frac{1}{n}\int_{S^{n-1}}\lim_{\varepsilon\to 0^+}\frac{\rho^{n-i}_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho^{n-i}_K(u)}{\varepsilon}dS(u)$$
$$= \frac{n-i}{n\phi'_r(1)}\int_{S^{n-1}}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho^{n-i}_K(u)dS(u),$$

we complete the proof of Theorem 1.

From Definition 2 and Theorem 1, we have

$$\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}_\phi\varepsilon\cdot L)-\tilde{W}_i(K)}{\varepsilon} = \frac{n-i}{\phi'_r(1)}\tilde{W}_{\phi,i}(K,L). \tag{21}$$

An immediate consequence of Proposition 1 and (21) is contained in:

Proposition 2 *If $\phi\in\mathscr{C}$, $K,L\in\mathscr{S}^n_0$ and $i\neq n$, then for $A\in SL(n)$,*

$$\tilde{W}_{\phi,i}(AK,AL)=\tilde{W}_{\phi,i}(K,L).$$

Proof From Proposition 1 and (21), for $A\in SL(n)$, we have

$$\tilde{W}_{\phi,i}(AK,AL) = \frac{\phi'_r(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(AK\tilde{+}_\phi\varepsilon\cdot AL)-\tilde{W}_i(AK)}{\varepsilon}$$
$$= \frac{\phi'_r(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(A(K\tilde{+}_\phi\varepsilon\cdot L))-\tilde{W}_i(K)}{\varepsilon}$$
$$= \frac{\phi'_r(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}_\phi\varepsilon\cdot L)-\tilde{W}_i(K)}{\varepsilon}$$
$$= \tilde{W}_{\phi,i}(K,L).$$

5 Geometric Inequalities

For $K\in\mathscr{S}^n_0$ and $i\in\mathbb{R}$, it will be rather good to use the volume-normalized dual conical measure $\tilde{W}^*_i(K)$ defined by

$$d\tilde{W}^*_i(K)=\frac{1}{n\tilde{W}_i(K)}\rho^{n-i}_K dS, \tag{22}$$

where S is the Lebesgue measure on S^{n-1} and $\tilde{W}^*_i(K)$ is a probability measure on S^{n-1}. When i=0, this is same as the definition in [4].

We now set up the dual Orlicz–Minkowski inequality for the dual Quermassintegral as follows:

Theorem 2 *Suppose $K, L \in \mathscr{S}_0^n$.*

(1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ and $i < n-1$, then

$$\tilde{W}_{\phi,i}(K,L) \leq \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right). \tag{23}$$

(2) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$ and $i < n-1$, then

$$\tilde{W}_{\phi,i}(K,L) \geq \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right). \tag{24}$$

(3) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, $i > n-1$ and $i \neq n$, then

$$\tilde{W}_{\phi,i}(K,L) \geq \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right). \tag{25}$$

(4) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, $i > n-1$ and $i \neq n$, then

$$\tilde{W}_{\phi,i}(K,L) \leq \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right). \tag{26}$$

Each equality in (23)–(26) holds if and only if K and L are dilates of each other.

Proof (1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$, then by dual Orlicz mixed Quermassintegral (5), and $\tilde{W}_i^*(K)$ defined by (22) is a probability measure on S^{n-1}, Jensen's inequality (12), the integral formulas of dual mixed Quermassintegral (4), dual Minkowski inequality (10), and the fact that ϕ is increasing on $(0, \infty)$, we have

$$\begin{aligned}
\frac{\tilde{W}_{\phi,i}(K,L)}{\tilde{W}_i(K)} &= \frac{1}{n\tilde{W}_i(K)}\int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)dS(u) \\
&\leq \phi\left(\frac{1}{n\tilde{W}_i(K)}\int_{S^{n-1}} \frac{\rho_L(u)}{\rho_K(u)}\rho_K^{n-i}(u)dS(u)\right) \\
&= \phi\left(\frac{\tilde{W}_i(K,L)}{\tilde{W}_i(K)}\right) \\
&\leq \phi\left(\frac{\tilde{W}_i(K)^{\frac{n-i-1}{n-i}}\tilde{W}_i(L)^{\frac{1}{n-i}}}{\tilde{W}_i(K)}\right)
\end{aligned}$$

$$= \phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),$$

(2) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, from (5), (22), Jensen's inequality (13), (4), (10), and ϕ is decreasing on $(0, \infty)$, we have

$$\begin{aligned}
\frac{\tilde{W}_{\phi,i}(K,L)}{\tilde{W}_i(K)} &= \frac{1}{n\tilde{W}_i(K)}\int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)dS(u) \\
&\geq \phi\left(\frac{1}{n\tilde{W}_i(K)}\int_{S^{n-1}} \frac{\rho_L(u)}{\rho_K(u)}\rho_K^{n-i}(u)dS(u)\right) \\
&= \phi\left(\frac{\tilde{W}_i(K,L)}{\tilde{W}_i(K)}\right) \\
&\geq \phi\left(\frac{\tilde{W}_i(K)^{\frac{n-i-1}{n-i}}\tilde{W}_i(L)^{\frac{1}{n-i}}}{\tilde{W}_i(K)}\right) \\
&= \phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),
\end{aligned}$$

(3) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, $i > n-1$ and $i \neq n$, proof as similar above, we can immediately obtain (25) which have the same form with (24).

(4) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, $i > n-1$ and $i \neq n$, similarly, we can immediately obtain (26) which have the same form with (23).

Each equality in (23)–(26) holds if and only if K and L are dilates of each other. Thus we get the significant dual Orlicz–Minkowski inequality.

Remark 2 It immediately follows a few cases for all $K, L \in \mathscr{S}_0^n$.

(1) Let $\phi(t) = t^p$ with $p < 0$. Equation (24) is just a similar result of Lutwak's L_p dual Minkowski inequality for the L_p dual mixed volume (see [20]): for $i < n-1$,

$$\tilde{W}_{p,i}(K,L)^{n-i} \geq \tilde{W}_i(K)^{n-i-p}\tilde{W}_i(L)^p.$$

(2) Let $\phi(t) = \log t$, we have

$$\tilde{W}_{p,i}(K,L) \leq \frac{\tilde{W}_i(K)}{n-i}\log\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)},$$

it is a very meaningful result, see [1, 30].

(3) Let $\phi(t) = t^p$ with $0 < p < 1$. For $i < n-1$, (23) is just

$$\tilde{W}_{p,i}(K,L)^{n-i} \leq \tilde{W}_i(K)^{n-i-p}\tilde{W}_i(L)^p.$$

(4) Let $\phi(t) = t$. From (23) and (25), we have for $i < n-1$,

$$\tilde{W}_i(K, L)^{n-i} \leq \tilde{W}_i(K)^{n-i-1}\tilde{W}_i(L),$$

and for $i > n-1$, $i \neq n$, the above inequality is reversed.

(5) Let $\phi(t) = t^p$ with $p \geq 1$. It follows from (25) that for $i > n-1$, $i \neq n$,

$$\tilde{W}_i(K, L)^{n-i} \geq \tilde{W}_i(K)^{n-i-p}\tilde{W}_i(L)^p.$$

Corollary 2 *Let $K, L \in \mathscr{S}_0^n$, $i < n-1$, $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ (or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$). If*

$$\tilde{W}_{\phi,i}(M, K) = \tilde{W}_{\phi,i}(M, L), \quad \textit{for all } M \in S_0^n, \tag{27}$$

or

$$\frac{\tilde{W}_{\phi,i}(K, M)}{\tilde{W}_i(K)} = \frac{\tilde{W}_{\phi,i}(L, M)}{\tilde{W}_i(L)}, \quad \textit{for all } M \in S_0^n, \tag{28}$$

then $K = L$.

Proof Whatever $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$, *or* $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, the process of proof is almost identical, so we next just prove the situation that $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$.

Suppose (27) holds, if we take K for M, then from Definition 2 and (9), we have

$$\tilde{W}_{\phi,i}(K, L) = \tilde{W}_{\phi,i}(K, K) = \phi(1)\tilde{W}_i(K).$$

However, from (23), we have

$$\tilde{W}_{\phi,i}(K, K) = \tilde{W}_{\phi,i}(K, L) \leq \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),$$

then

$$\phi(1) \leq \phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is monotone increasing on $(0, \infty)$, we get

$$\tilde{W}_i(L) \geq \tilde{W}_i(K),$$

with equality if and only if K and L are dilates of each other. If we take L for M, similarly we get $\tilde{W}_i(K) \geq \tilde{W}_i(L)$ which shows there is in fact equality in both inequalities and that $\tilde{W}_i(K) = \tilde{W}_i(L)$, hence the equality implies that $K = L$.

Next, assume (28) holds, if we take K for M, then from Definition 2 and (9), we have

$$\frac{\tilde{W}_{\phi,i}(K,K)}{\tilde{W}_i(K)} = \phi(1) = \frac{\tilde{W}_{\phi,i}(L,K)}{\tilde{W}_i(L)}.$$

But from (23), we have

$$\frac{\tilde{W}_{\phi,i}(L,K)}{\tilde{W}_i(L)} \leq \frac{\tilde{W}_i(L)\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(L)}\right)^{\frac{1}{n-i}}\right)}{\tilde{W}_i(L)},$$

then

$$\phi(1) \leq \phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(L)}\right)^{\frac{1}{n-i}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly increasing on $(0, \infty)$, we have

$$\tilde{W}_i(K) \geq \tilde{W}_i(L),$$

with equality if and only if K and L are dilates of each other.

On the other hand, taking L for M, similarly we have $\tilde{W}_i(L) \geq \tilde{W}_i(K)$, which shows that in fact equality holds in both inequalities and $\tilde{W}_i(K) = \tilde{W}_i(L)$. Hence the equality implies $K = L$.

We now establish the following dual Orlicz–Brunn–Minkowski inequality for dual Quermassintegral:

Theorem 3 *Let $K, L \in \mathscr{S}_0^n$ and $a, b > 0$.*

(1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ and $i < n-1$ then

$$\phi(1) \leq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right). \tag{29}$$

(2) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$ and $i < n-1$, then

$$\phi(1) \geq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right). \tag{30}$$

(3) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, $i > n-1$ and $i \neq n$, then

$$\phi(1) \geq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right). \tag{31}$$

(4) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, $i > n-1$ and $i \neq n$, then

$$\phi(1) \leq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right). \tag{32}$$

Each equality in (29)–(32) holds if and only if K and L are dilates of each other.

Proof Note $K_\phi = a \cdot K \tilde{+}_\phi b \cdot L$.

(1) When $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ and $i < n-1$, by (9), Lemma 3, Definition 2 and (23), then

$$\begin{aligned}
\phi(1) &= \frac{1}{n\tilde{W}_i(K_\phi)} \int_{S^{n-1}} \phi(1)\rho_{K_\phi}^{n-i}(u)dS(u) \\
&= \frac{1}{n\tilde{W}_i(K_\phi)} \int_{S^{n-1}} \left[a\phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right) + b\phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right)\right] \rho_{K_\phi}^{n-i}(u)dS(u) \\
&= \frac{a}{n\tilde{W}_i(K_\phi)} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right) \rho_{K_\phi}^{n-i}(u)dS(u) \\
&\quad + \frac{b}{n\tilde{W}_i(K_\phi)} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right) \rho_{K_\phi}^{n-i}(u)dS(u) \\
&= \frac{a}{\tilde{W}_i(K_\phi)} \tilde{W}_{\phi,i}(K_\phi, K) + \frac{b}{\tilde{W}_i(K_\phi)} \tilde{W}_{\phi,i}(K_\phi, L) \\
&\leq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\phi)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\phi)}\right)^{\frac{1}{n-i}}\right).
\end{aligned}$$

(2) When $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$ and $i < n-1$, by (9), Lemma 3, Definition 2 and (24), we obtain (30).

(3) When $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, $i > n-1$ and $i \neq n$, by (9), Lemma 3, Definition 2 and (25), we obtain (31).

(4) When $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, $i > n-1$ and $i \neq n$, by (9), Lemma 3, Definition 2 and (26), we obtain (32).

Each equality in (29)–(32) holds as an equality if and only if K and L are dilates of each other. We obtain the desired dual Orlicz–Brunn–Minkowski inequality (29)–(32).

Remark 3 For $K, L \in \mathscr{S}_0^n$, $a, b > 0$, some particular cases are as follows: each equality holds if and only if K and L are dilates of each other.

(1) Let $\phi(t) = t^p$ with $p < 0$. From (30) we can deduces to the analogous form of Lutwak's L_p dual Brunn–Minkowski inequality (see [20]): for $i < n-1$,

$$\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)^{\frac{p}{n-i}} \geq a\tilde{W}_i(K)^{\frac{p}{n-i}} + b\tilde{W}_i(L)^{\frac{p}{n-i}}.$$

(2) Let $\phi(t) = \log t$, from (29), we obtain

$$\frac{a}{n-i}\log\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)}\right)+\frac{b}{n-i}\log\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)}\right)\geq 0.$$

(3) Let $\phi(t) = t^p$ with $0 < p < 1$. For $i < n-1$, (29) is just

$$\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)^{\frac{p}{n-i}} \leq a\tilde{W}_i(K)^{\frac{p}{n-i}} + b\tilde{W}_i(L)^{\frac{p}{n-i}}.$$

(4) Let $\phi(t) = t$. From (29) and (31), we have for $i < n-1$,

$$\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)^{\frac{1}{n-i}} \leq a\tilde{W}_i(K)^{\frac{1}{n-i}} + b\tilde{W}_i(L)^{\frac{1}{n-i}},$$

and for $i > n-1$, $i \neq n$, the above inequality reversed.

(5) Let $\phi(t) = t^p$ with $p > 1$. From (31), it follows that for $i > n-1$, $i \neq n$,

$$\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)^{\frac{p}{n-i}} \geq a\tilde{W}_i(K)^{\frac{p}{n-i}} + b\tilde{W}_i(L)^{\frac{p}{n-i}}.$$

We derive the equivalence between the dual Orlicz–Minkowski inequalities (23)–(26) and the dual Orlicz–Brunn–Minkowski inequalities (29)–(32), respectively. Since we proved that (23)–(26) implies (29)–(32), respectively, so now we just need to prove that (29)–(32) can deduce (23)–(26), respectively. Since all the process are similar, so we just prove (23) by (29).

Proof of the implication (29) *to* (23). For $\varepsilon \geq 0$, let $K_\varepsilon = K\tilde{+}_\phi \varepsilon\cdot L$. By (29), the following function

$$G(\varepsilon) = \phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) + \varepsilon\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) - \phi(1),$$

is non-negative and it easily get $G(0) = 0$. Then,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) + \varepsilon\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) - \phi(1)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) - \phi(1)}{\varepsilon} + \phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)\\
&= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) - \phi(1)}{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}} - 1}\cdot\lim_{\varepsilon\to 0^+}\frac{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}} - 1}{\varepsilon}\end{aligned}$$

$$+\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right). \tag{33}$$

Let $t=\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}$ and note that $t\to 1^+$ as $\varepsilon\to 0^+$, consequently,

$$\lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)-\phi(1)}{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}-1}=\lim_{t\to 1^+}\frac{\phi(t)-\phi(1)}{t-1}=\phi_r'(1). \tag{34}$$

By (21), we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}-1}{\varepsilon}&=-\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K_\varepsilon)^{\frac{1}{n-i}}-\tilde{W}_i(K)^{\frac{1}{n-i}}}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\tilde{W}_i(K_\varepsilon)^{-\frac{1}{n-i}}\\&=-\frac{1}{n-i}\tilde{W}_i(K)^{\frac{1-n+i}{n-i}}\cdot\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K_\varepsilon)-\tilde{W}_i(K)}{\varepsilon}\cdot\tilde{W}_i(K)^{-\frac{1}{n-i}}\\&=-\frac{\tilde{W}_{\phi,i}(K,L)}{\phi_r'(1)\tilde{W}_i(K)}.\end{aligned} \tag{35}$$

From (33), (34), (35) and since $G(\varepsilon)$ is non-negative, thus

$$\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon}=-\frac{\tilde{W}_{\phi,i}(K,L)}{\tilde{W}_i(K)}+\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)\geq 0. \tag{36}$$

Therefore, we have the formula (23). The equality holds as an equality in (36) if and only if $G(\varepsilon)=G(0)=0$, and this means that the equality case in (23) can be obtained from the equality condition of (29). □

Acknowledgements This research is supported in part by National Natural Science Foundation of China (Grant No. 11271302) and Fundamental Research Funds for the Central Universities (No. XDJK2016D026). We like to thank referees for helpful suggestions.

References

1. Böröczky, K.J., Lutwak, E., Yang, D., Zhang, G.: The log-Brunn–Minkowski inequality. Adv. Math. **231**, 1974–1997 (2012)
2. Firey, W.: p-means of convex bodies. Math. Scand. **10**, 17–24 (1962)
3. Gardner, R.J., Hug, D., Weil, W.: The Orlicz–Brunn–Minkowski theory: a general framework, additions, and inequalities. J. Differ. Geom. **97**, 427–476 (2014)

4. Gardner, R.J., Hug, D., Weil, W., Ye, D.: The dual Orlicz–Brunn–Minkowski theory. J. Math. Anal. Appl. **430**, 810–829 (2015)
5. Haberl, C.: L_p intersection bodies. Adv. Math. **217**(6), 2599–2624 (2008)
6. Haberl, C., Ludwig, M.: A characterization of L_p intersection bodies. Int. Math. Res. Not. Art ID 10548 (2006)
7. Haberl, C., Schuster, F.E.: Asymmetric affine L_p Sobolev inequalities. J. Funct. Anal. **257**(3), 641–658 (2009)
8. Haberl, C., Schuster, F.E.: General L_p affine isoperimetric inequalities. J. Differ. Geom. **83**(1), 1–26 (2009)
9. Haberl, C., Lutwak, E., Yang, D., Zhang, G.: The even Orlicz Minkowski problem. Adv. Math. **224**(6), 2485–2510 (2010)
10. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, London (1934)
11. Jin, H., Yuan, S., Leng, G.: On the dual Orlicz mixed volumes. Chin. Ann. Math. Ser. B **36**, 1019–1026 (2015)
12. Koldobsky, A.: Fourier Analysis in Convex Geometry, Mathematical Surveys and Monographs, vol. 116. American Mathematical Society, Providence (2005)
13. Koldobsky, A., Paouris, G., Zymonopoulou, M.: Complex intersection bodies. J. Lond. Math. Soc. **88**(2), 538–562 (2013)
14. Li, D., Zou, D., Xiong, G.: Orlicz mixed affine quermassintegrals. Sci. China Math. **58**, 1715–1722 (2015)
15. Ludwig, M.: General affine surface areas. Adv. Math. **224**(6), 2346–2360 (2010)
16. Ludwig, M., Reitzner, M.: A classification of $SL(n)$ invariant valuations. Ann. Math. (2) **172**(2), 1219–1267 (2010)
17. Lutwak, E.: Dual mixed volumes. Pac. J. Math. **58**, 531–538 (1975)
18. Lutwak, E.: Intersection bodies and dual mixed volumes. Adv. Math. **71**, 232–261 (1988)
19. Lutwak, E.: The Brunn–Minkowski–Firey theory I: mixed volumes and the Minkowski problem. J. Differ. Geom. **38**, 131–150 (1993)
20. Lutwak, E.: The Brunn–Minkowski–Firey theory II: affine and geominimal surface areas. Adv. Math. **118**, 244–294 (1996)
21. Lutwak, E., Yang, D., Zhang, G.: L_p affine isoperimetric inequalities. J. Differ. Geom. **56**(1), 111–132 (2000)
22. Lutwak, E., Yang, D., Zhang, G.: Sharp affine L_p Sobolev inequalities. J. Differ. Geom. **62**(1), 17–38 (2002)
23. Lutwak, E., Yang, D., Zhang, G.: L_p John ellipsoids. Proc. Lond. Math. Soc. **90**(3), 497–520 (2005)
24. Lutwak, E., Yang, D., Zhang, G.: Orlicz projection bodies. Adv. Math. **223**, 220–242 (2010)
25. Lutwak, E., Yang, D., Zhang, G.: Orlicz centroid bodies. J. Diffier. Geom. **84**, 365–387 (2010)
26. Rubin, B.: Intersection bodies and generalized cosine transform. Adv. Math. **218**(3), 696–727 (2008)
27. Rudin, W.: Principles of Mathematical Analysis. McGraw-Hill Book Company, New York (1976)
28. Schneider, R.: Convex Bodies: The Brunn–Minkowski Theory. Encyclopedia of Mathematics and Its Applications, vol. 151, 2nd edn. Cambridge University Press, Cambridge (2014)
29. Schuster, F.E.: Volume inequalities and additive maps of convex bodies. Mathematika **53**, 211–234 (2006)
30. Stancu, A.: The logarithmic Minkowski inequality for non-symmetric convex bodies. Adv. Appl. Math. **73**, 43–58 (2016)
31. Werner, E., Ye, D.: New L_p affine isoperimetric inequalities. Adv. Math. **218**(3), 762–780 (2008)
32. Wu, D., Bu, Z., Ma, T.: Two complex combinations and complex intersection bodies. Taiwan. J. Math. **18**(5), 1459–1480 (2014)
33. Wu, D., Ma, T., Zhang, L.: The λ -intersection bodies and an analytic generalized Busemann–Petty problem. Math. Inequal. Appl. **17**(3), 1047–1060 (2014)

34. Xi, D., Jin, H., Leng, G.: The Orlicz Brunn–Minkowski inequality. Adv. Math. **260**, 350–374 (2014)
35. Xiong, G., Zou, D.: Orlicz mixed quermassintegrals. Sci. China Math. **57**, 2549–2562 (2014)
36. Zhao, C.: Orlicz dual mixed volumes. J. Results Math. **68**, 93–104 (2015)
37. Zhu, B., Zhou, J., Xu, W.: Dual Orlicz–Brunn–Minkowski theory. Adv. Math. **264**, 700–725 (2014)
38. Zhu, G.: The Orlicz centroid inequality for star bodies. Adv. Appl. Math. **48**, 432–445 (2012)