Springer Proceedings in Mathematics & Statistics

Young Jin Suh
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Hyunjin Lee *Editors*

Hermitian–Grassmannian Submanifolds

Daegu, Korea, July 2016

Springer

Springer Proceedings in Mathematics & Statistics

Volume 203

Springer Proceedings in Mathematics & Statistics

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Editors

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Editors
Young Jin Suh
Department of Mathematics and RIRCM
Kyungpook National University
Daegu
Republic of Korea

Byung Hak Kim
Department of Applied Mathematics
Kyung Hee University
Gyeonggi
Republic of Korea

Yoshihiro Ohnita
Department of Mathematics and OCAMI
Osaka City University
Osaka
Japan

Hyunjin Lee
Research Institute of Real and Complex Manifolds (RIRCM)
Kyungpook National University
Daegu
Republic of Korea

Jiazu Zhou
School of Mathematics and Statistics
Southwest University
Chongqing
China

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-981-10-5555-3 ISBN 978-981-10-5556-0 (eBook)
DOI 10.1007/978-981-10-5556-0

Library of Congress Control Number: 2017946621

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

The 20th International Workshop on Hermitian Symmetric Spaces and Submanifolds (IWHSSS 2016) was held at Kyungpook National University (KNU), Daegu, in Korea from July 26, Tuesday, to July 30, Saturday, 2016. The organizing committee was composed of the following members:

> Young Jin Suh (Kyungpook National University and Research Institute of Real and Complex Manifolds (RIRCM), Korea), Yoshihiro Ohnita (Osaka City University and Osaka City University Advanced Mathematical Institute (OCAMI), Japan), Jiazu Zhou (Southwest University, China), and Byung Hak Kim (Kyung Hee University, Korea).

This conference was largely supported by the Korea Institute for Advanced Study (KIAS, Research Station Program) and the National Research Foundation of Korea (NRF, Project No. 2015-R1A2A1A-01002459) (see http://rircm.knu.ac.kr or http://webbuild.knu.ac.kr/~yjsuh).

Related to IWHSSS 2016, we have organized many kinds of mini-international workshops, including intensive lectures, since December 2015. Among them, the editors want to mention the 11th RIRCM-OCAMI Joint Differential Geometry Workshop on Submanifolds and Lie Theory held at Osaka City University from March 20, Sunday, through March 23, Wednesday, 2016. On behalf of the organizing committee, the editors would especially like to express their gratitude to speakers who submitted their articles as manuscripts for Part II. Invited Talks of these proceedings included:

> Prof. Jost-Hinrich Eschenburg (University of Augsburg, Germany), Prof. Hiroshi Tamaru (Hiroshima University, Japan), Prof. Leonardo Biliotti (University of Parma, Italy), and Prof. Young Jin Suh (Kyungpook National University and RIRCM, Korea).

The organizing committee of IWHSSS 2016 invited many famous differential geometers from all over the world. All participants including speakers in this workshop discussed new developments for research subjects. On behalf of the organizing committee, the editors extend their deep gratitude to all participants and speakers. The editors without exception hope that the 20th proceedings published by Springer provide a fine view of recent topics in differential geometry and related fields.

The editorial committee invited the following professors as a member of the program committee, which referred all manuscripts submitted to our proceedings and provided a great contribution to the publication of the proceedings. The editors gratefully acknowledge the program committee, who gave valuable comments on all submitted articles in these proceedings:

> Prof. Jürgen Berndt (King's College London, UK), Prof. Leonardo Biliotti (University of Parma, Italy), Prof. Mitsuhiro Itoh (University of Tsukuba, Japan), Prof. Byung Hak Kim (Kyung Hee University, Korea), Prof. Reiko Miyaoka (Tohoku University, Japan), Prof. Yoshihiro Ohnita (Osaka City University and OCAMI, Japan), Prof. Juan de Dios Pérez (University of Granada, Spain), Prof. Alfonso Romero (University of Granada, Spain), and Prof. Jiazu Zhou (Southwest University, China).

Finally, the editors are eager to express their hearty thanks to all staff members at KIAS and Springer Japan for their support and publication of our valuable manuscripts of this workshop. The editors will be truly happy if this volume of the Springer Proceedings in Mathematics and Statistics is helpful for differential geometers and graduate students in conducting their research more creatively and successfully. Thank you very much.

Daegu, Korea	Young Jin Suh
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Gyeonggi, Korea	Byung Hak Kim
Daegu, Korea	Hyunjin Lee
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Yoshihiro Ohnita, Osaka City University and Osaka City University Advanced Mathematical Institute (OCAMI), Japan
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Alfonso Romero, University of Granada, Spain
Young Jin Suh, Kyungpook National University, Korea
Jiazu Zhou, Southwest University, China

Contents

Constant Mean Curvature Spacelike Hypersurfaces in Spacetimes with Certain Causal Symmetries

Alfonso Romero

Abstract The role of some causal symmetries of spacetime which naturally arise in General Relativity is discussed. The importance of spacelike hypersurfaces of constant mean curvature (CMC) in the study of the Einstein equation is recalled. In certain spacetimes with symmetry defined by a timelike gradient conformal vector field or by a lightlike parallel vector field, uniqueness theorems of complete CMC spacelike hypersurfaces are given. In several cases, results of Calabi–Bernstein type are obtained as an application.

1 Introduction

The concept of symmetry is basic in General Relativity. It is usually based on a one-parameter group of transformations generated by a Killing or, more generally, a conformal vector field [15]. In fact, the main simplification for the search of exact solutions of the Einstein equation, is to assume the existence of such symmetries [16]. We remark that a completely general approach to symmetries in General Relativity was developed in [31]. The causal character of the Killing or conformal vector field K in a spacetime $(\overline{M}, \bar{g})$ is not always prefixed. However, it is natural to assume that this vector field is timelike. This is supported by well-known examples of exact solutions of the Einstein equation. At the same time, under this assumption, the integral curves of the reference frame

$$Q := \frac{1}{\sqrt{-\bar{g}(K, K)}} K \tag{1}$$

provide a privileged family of observers or test particles in spacetime. Less often the symmetry of the spacetime is defined by a certain lightlike vector field. In fact, there are exact solutions whose symmetry is defined by a parallel lightlike vector field K.

A. Romero (✉)
Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain
e-mail: aromero@ugr.es

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_1

In this case the integral curves of K are interpreted as photons moving at the speed of light, and the spacetime models electromagnetic or gravitational radiation. Finally, spacelike vector fields on $\overline{M}$ does not have a special relevance in General Relativity.

This paper surveys several recent classification results of compact CMC spacelike hypersurfaces in spacetimes with a symmetry defined by the existence of certain causal vector field. Its content is organized as follows. Section 2 is devoted to show the causal symmetries in spacetime we consider here and to describe the corresponding families of spacetimes: Gradient Conformally Stationary (GCS) spacetimes, with its subfamily of Generalized Robertson–Walker (GRW) spacetimes and Brinkmann spacetimes. In Sect. 3 we recall some basic facts on the role of CMC spacelike hypersurfaces in General Relativity. In Sect. 4 several uniqueness theorems on compact CMC spacelike hypersurfaces in GCS spacetimes, and in GRW spacetimes are given. In the last case, the corresponding uniqueness Calabi–Bernstein type results are stated in Sect. 5. Finally, compact spacelike hypersurfaces of constant mean curvature are studied in Sect. 6.

2 Some Types of Causal Symmetries

2.1 The Timelike Case

A vector field K on a Lorentzian manifold $(\overline{M}, \bar{g})$ is called conformal if any of its local flows consists in conformal transformations of $\bar{g}$, i.e., the Lie derivative of $\bar{g}$ with respect to K satisfies

$$\mathscr{L}_K \, \bar{g} = 2\rho \, \bar{g}, \tag{2}$$

where ρ is a (smooth) function on $\overline{M}$. When $\rho =$constant, the vector field is called homothetic and if, in particular, $\rho = 0$ then it is called Killing. The assumption of the existence of a (nontrivial) conformal vector field in spacetime give rise to a symmetry for the Lorentzian metric $\bar{g}$ which simplifies the Einstein equation, where $\bar{g}$ has the role of the unknown.

A conformal vector field K does not have a fixed causal character, but for a spacetime, i.e., a time orientable Lorentzian manifold endowed with one of its two possible time orientations, it is natural to assume that K is timelike, i.e., it satisfies $\bar{g}(K, K) < 0$, everywhere on $\overline{M}$. In such a case the integral curves of the reference frame Q given in (1) yield a family of privileged observers.

A spacetime $(\overline{M}, \bar{g})$ admitting a timelike Killing vector field is called stationary. If a spacetime $\overline{M}$ admits a timelike conformal vector field K, then Lorentzian metric $\frac{-1}{\bar{g}(K,K)} \, \bar{g}$ is stationary. For this reason, the spacetime $(\overline{M}, \bar{g})$ is called conformally stationary (CS). Given a timelike conformal vector field K, when its $\bar{g}$-equivalent 1-form is closed, or equivalently, K is locally the gradient of a function, we have

$$\overline{\nabla}_X K = \rho\, X, \tag{3}$$

for any vector field X on $\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{M}$. Directly from (3) we obtain

$$\rho = K\left(\log \sqrt{-\bar{g}(K, K)}\right) = Q\left(\sqrt{-\bar{g}(K, K)}\right), \tag{4}$$

and therefore,

$$(\rho \circ \gamma)(t) = \frac{d}{dt}(h \circ \gamma)(t), \tag{5}$$

for any observer γ in Q, where $h := \sqrt{-\bar{g}(K, K)}$. Moreover, we also get

$$\overline{\nabla}_Q Q = 0, \tag{6}$$

which describes that the observers in Q are free falling in spacetime.

Any CS spacetime admits the distribution $K^{\perp}$ which is integrable if the timelike conformal vector field K is also assumed to be closed. Each leaf of $K^{\perp}$ is a spacelike hypersurface which is interpreted for an observer in Q as its physical space at one instant of its proper time.

Remark 1 There exist CS spacetimes whose distribution $K^{\perp}$ is not integrable. For instance, each odd dimensional sphere $\mathbb{S}^{2n+1}$ admits a natural Lorentzian metric (see for instance [18, Sec. 4]) with a unit timelike Killing vector field K. If $K^{\perp}$ is integrable, then K should be closed. But the 1-connectedness of $\mathbb{S}^{2n+1}$ implies that K must be in fact a gradient vector field, which is not compatible with the compactness of $\mathbb{S}^{2n+1}$.

Now we will focus on an interesting subfamily of CS spacetimes. Let $\overline{M}$ be a spacetime which admits a timelike conformal vector field K such that

$$K = \overline{\nabla}\phi, \tag{7}$$

for some $\phi \in C^{\infty}(\overline{M})$, in particular K is closed. In this case K is called a timelike gradient conformal vector field and ϕ a potential function of K. The spacetime $\overline{M}$ is then called a gradient conformally stationary (GCS) spacetime [10, 14]. The existence of a timelike gradient conformal vector field in spacetime has been used to study certain cosmological models [25] and plays a relevant role for vacuum and perfect fluid spacetimes [14].

Each leaf S of the foliation $K^{\perp}$ is a spacelike hypersurface in $\overline{M}$. The shape operator of S with respect to the unit normal vector field $Q|_S$ is

$$A = -\frac{\rho}{h}\Big|_S I, \tag{8}$$

where ρ and $h = \sqrt{-\bar{g}(K,K)}$ are constants on S and I is the identity transformation. Thus, S is totally umbilical with constant mean curvature

$$H = \frac{\rho}{h}\Big|_S . \tag{9}$$

Let us notice that a GCS spacetime $\overline{M}$ is noncompact, otherwise formula (7) would imply that K has a zero at any critical point of ϕ. Moreover, the potential ϕ is a global time function [6]. Therefore, the spacetime $\overline{M}$ is stably causal [19], i.e., there is a fine C^0 neighbourhood of the original metric $\bar{g}$ of the spacetime such that any of its Lorentzian metrics is causal [6]. Thus, there is no closed nonspacelike curve in $\overline{M}$. Recall that the existence of a closed timelike curve in a spacetime makes it physically unrealistic because such a curve, after a suitable parametrization, would model an observer travelling in time which experiments in its future an event of its past [29].

Remark 2 Previous arguments may be used to show that there exist CS spacetimes with K timelike conformal and closed which are not globally GSC spacetimes. For instance, the Misner cylinder spacetime [22] is a 2-dimensional Lorentzian manifold admitting a timelike parallel vector field which cannot be globally a gradient because the Misner cylinder spacetime is not causal [10].

There is an important subfamily of GCS spacetimes we will recall now. For a Generalized Robertson–Walker (GRW) spacetime we mean the product manifold $I \times F$, of an open interval I of the real line $\mathbb{R}$ and an $n(\geq 2)$-dimensional Riemannian manifold (F, g_F), endowed with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F)\,, \tag{10}$$

where π_I and π_F denote the projections onto I and F, respectively, and f is a positive smooth function on I. This $(n+1)$-dimensional spacetime $\overline{M}$ is a warped product in the sense of [22] with base $(I, -dt^2)$, fiber (F, g_F) and warping function f.

The class of GRW spacetimes widely extends to those that are called Robertson–Walker (RW) spacetimes. Contrary to these spacetimes, the fiber of a GRW spacetimes has not constant sectional curvature and, therefore, it is not necessarily locally spatially-homogeneous. Intuitively, GRW spacetimes could be then good candidates to model proper portions of universe [24]. On the other hand, small deformations of the metric on the fiber of a RW spacetime fit into the class of GRW spacetimes, and then GRW spacetimes may be also considered to study stability properties of RW spacetimes [17]. Let us remark finally that when GRW spacetimes were introduced in [2], the name RW spacetime was most common in the literature, but to be fair, GRW spacetimes should be called Generalized Friedman–Lemaître–Robertson–Walker (GFLRW) spacetimes.

On any GRW spacetime, the vector field $K = f(\pi_I)\partial_t$ is a timelike gradient conformal vector field with

$$\rho = f'(\pi_I) \qquad \text{and} \qquad \phi = -\varphi(\pi_I), \tag{11}$$

where φ is a primitive function of f. However, the class of GCS spacetimes is clearly bigger than the class of GRW spacetimes. For instance, any open subset of a GRW spacetime is a GCS spacetime but it is not globally isometric to a GRW spacetime, in general (see [10] for more examples). But locally, the converse is also true: i.e., for each point of a GCS spacetime (or more generally of a CS spacetime with K closed) there exists an open neighbourhood which is isometric to a GRW spacetime [30]. On the other hand, under several natural assumptions on either the flow of Q or the leaves of $Q^\perp$, one can prove that certain GCS spacetimes admit a global decomposition as GRW spacetimes [10, Th. 3.1, 3.4].

To end this subsection we notice that each leaf of the foliation $\partial_t^\perp$ in a GRW spacetime is a level hypersurface $t = t_0$, called spacelike slice. The shape operator of the spacelike slice $t = t_0$ with respect to the unit normal vector field $\partial_t|_{t=t_0}$ is

$$A = -\frac{f'(t_0)}{f(t_0)}\, I\,. \tag{12}$$

Thus, S is totally umbilical with constant mean curvature

$$H = \frac{f'(t_0)}{f(t_0)}\,. \tag{13}$$

2.2 *The Lightlike Case*

Now we will consider the case when the symmetry of spacetime is defined by a lightlike vector field. Let K be a conformal and closed vector field on a Lorentzian manifold $(\overline{M}, \bar{g})$. Assume K is lightlike, i.e., $\bar{g}(K, K) = 0$ and $K_p \neq 0$ for any $p \in \overline{M}$. From (3) we have

$$0 = \frac{1}{2} X\left(\bar{g}(K, K)\right) = \bar{g}(\overline{\nabla}_X K, K) = \bar{g}(X, \rho K)$$

for all X. Hence, $\rho K = 0$, which gives $\rho = 0$, i.e., K is parallel. Note that the same previous argument proves that given a nowhere zero conformal and closed vector field K on a Lorentzian manifold $(\overline{M}, \bar{g})$, if $\bar{g}(K, K) =$constant, then K is parallel.

A Lorentzian manifold which admits a parallel lightlike vector field K is called a Brinkmann spacetime (see for instance [7]). Let us remark that making use of such a vector field, a time orientation can be defined and, therefore, a Lorentzian manifold which admits a lightlike vector field is time orientable. On the other hand, a Brinkmann spacetime has a natural foliation, namely $K^\perp$. Each leaf S of the foliation $K^\perp$ is a degenerate hypersurface in the sense that the induced metric from $\bar{g}$ on each leaf has less rank that the dimension of the leaf (note that $T_pS \cap T_p^\perp S =$Span$\{K_p\}$).

The leaves of the foliation has no so clear physical interpretation as in the timelike case [15, Sect. 2.7].

The Lorentzian metric $\bar{g}$ of any $n(\geq 3)$-dimensional Brinkmann spacetime can be locally expressed as follows [8],

$$H(u,x)\,du^2 + 2\,du\,dv + 2\sum_i W_i(u,x)\,du\,dx_i + \sum_{ij} \bar{g}_{ij}\,(u,x)dx_i\,dx_j,$$

where $x = (x_1, \ldots, x_{n-2})$, $1 \leq i, j \leq n-2$ and the parallel vector field K coincides with the coordinate vector field ∂_v on the corresponding coordinate neighbourhood.

A relevant subfamily of n-dimensional Brinkmann spacetimes consists in the so-called pp-wave spacetimes, namely, each of them is given by a Lorentzian metric on $\mathbb{R}^n$ of the following type,

$$H(u, x_1, \ldots x_{n-2})du^2 + 2\,du\,dv + dx_1^2 + \cdots + dx_{n-2}^2,$$

where $(u, v, x_1, \ldots, x_{n-2})$ are the usual coordinates and $H = H(u, x_1, \ldots, x_{n-2})$ a smooth function with no required sign.

Remark 3 At this point, it is natural to wonder if the symmetry defined by a spacelike vector field K in spacetime $(\overline{M}, \bar{g})$, i.e., K satisfies $\bar{g}(K_p, K_p) > 0$ of $K_p = 0$ for all $p \in \overline{M}$, may be considered. There is no mathematical objection although the integral curves of a spacelike vector field do not have a physical interpretation in term of realistic particles in spacetime.

3 Mean Curvature of Spacelike Hypersurfaces in General Relativity

A spacelike hypersurface in a spacetime is a hypersurface which inherits a Riemannian metric from the ambient Lorentzian one. Such a hypersurface defines the family of normal observers: each (inextensible) geodesic in the spacetime determined from a point of the spacelike hypersurface and the future pointing unit normal vector at this point. For each of these observers, the spacelike hypersurface is the spatial universe at one instant of proper time. Moreover, these observers can be locally collected as the integral curves of a (local) reference frame Q in spacetime.

If H denotes the mean curvature function of a spacelike hypersurface S with respect to its future pointing unit normal vector field N we have

$$\mathrm{div}(Q) = nH,$$

on the open subset of S where Q is defined. Thus, the function $\mathrm{div}(Q)$, which measures the expansion/contraction for the observers in Q [29], is up to a positive constant the mean curvature function. Therefore, if the inequality $H > 0$ (resp. $H <$

0) holds, it is interpreted as normal observers get away (resp. come together) after passing through S, and near it.

Now, we will recall the important role of the case $H =$constant in the study of the Einstein equation. Let $(\overline{M}, \bar{g})$ be a 4-dimensional spacetime. Denote by $\overline{\text{Ric}}$ and $\overline{R}$ its Ricci tensor and its scalar curvature, respectively. Consider a stress-energy tensor field T on $\overline{M}$, that is, a 2–covariant symmetric tensor which is assumed to satisfy some reasonable conditions from a physical viewpoint [22, 29]. The spacetime $(\overline{M}, \bar{g})$ obeys the Einstein equation (with zero cosmological constant) with source T if we have

$$\overline{\text{Ric}} - \frac{1}{2}\,\overline{R}\,\bar{g} = T. \tag{G}$$

Equation (G) postulates how mass and radiation are described by the metric tensor, i.e., by the geometry of the spacetime. When $T = 0$ this equation is called the Einstein vacuum equation, and then (G) is equivalent to

$$\overline{\text{Ric}} = 0\,. \tag{G*}$$

When equation (G) holds in spacetime a $\overline{M}$, we have on each spacelike hypersurface S in $\overline{M}$, thanks to the Gauss and Codazzi equations, the following constraint equations,

$$R(g) - \text{trace}(A^2) + (\text{trace}(A))^2 = \varphi, \tag{C$_1$}$$

$$\text{div}(A) - \nabla\text{trace}(A) = X, \tag{C$_2$}$$

where g is the Riemannian metric on S induced by $\bar{g}$, $R(g)$ its scalar curvature, A is the shape operator relative to a future pointing unit timelike normal vector field, and $\varphi \in C^\infty(S)$ and $X \in \mathfrak{X}(S)$ depend on the stress energy tensor T, in such a way that $T = 0$ implies $\varphi = 0$ and $X = 0$. These equations can be seen as PDE with unknows g and A. Consider by simplicity the previous case (G*). Following [12, 13] we recall the following definitions.

An initial data set for the Einstein vacuum equation (G*) is a triple (S, g, A) where S is a 3-dimensional manifold, g is a Riemannian metric on S and A is a $(1, 1)$-tensor field self-adjoint with respect to g, which satisfies the constraint equations $(\text{C})_1$ and $(\text{C})_2$ with $\varphi = 0$ and $X = 0$. A solution of the Cauchy problem corresponding to the initial data set (S, g, A) is a spacetime $(\overline{M}, \bar{g})$ such that $\bar{g}$ is Ricci-flat and there exists a spacelike embedding $j : S \longrightarrow \overline{M}$ such that $j^*\bar{g} = g$ and A is the shape operator with respect to a chosen unit timelike normal vector field.

Thus, the Cauchy problem for the Einstein equation requires to solve previously the constraint equations $(\text{C})_1$ and $(\text{C})_2$. Now we recall the conformal method initiated by A. Lichnerowicz [20] to solve the constraint equations for (G*). First of all, we choose an arbitrary Riemannian metric g_0 on S. Next we put

$$g := \phi^4 g_0, \quad \phi \in C^\infty(S), \quad \phi > 0 \quad \text{and} \quad B := \phi^6\left(A - \frac{1}{3}\,\text{trace}(A)\, I\right)$$

and denote $\tau := \text{trace}(A)$. Then, the constraint equations become

$$\Delta^0 \phi - \frac{R(g_0)}{8}\,\phi + \frac{\text{trace}(B^2)}{8}\,\frac{1}{\phi^7} - \frac{1}{12}\,\tau^2 \phi^5 = 0, \tag{14}$$

$$\text{div}^0(B) - \frac{2}{3}\,\phi^6\,\nabla^0 \tau = 0. \tag{15}$$

where Δ^0, div^0 and ∇^0 denote the Laplacian operator, the divergence and the gradient with respect to g_0, respectively. The first equation is elliptic and it is called the Lichnerowicz equation.

Assume B is a solution of the following linear system

$$\text{div}^0(B) = 0, \quad \text{trace}(B) = 0,$$

and $\phi > 0$ is a solution of the Lichnerowicz equation (14). If we consider

$$g = \phi^4\, g_0 \quad \text{and} \quad A = \frac{1}{\phi^6}\, B + \frac{1}{3}\,\tau I, \quad \tau \in \mathbb{R},$$

then (S, g, A) is an initial data set for the vacuum Einstein equation (G*), [21]. Note that S is a spacelike hypersurface in a spacetime $(\overline{M}, \bar{g})$ solution of the corresponding Cauchy problem of constant mean curvature $H = \frac{-1}{3}\,\tau$.

4 CMC Spacelike Hypersurfaces in a GCS Spacetime

Consider a spacelike hypersurface $x : S \longrightarrow \overline{M}$ in a GCS spacetime $(\overline{M}, \bar{g})$ of dimension $n+1$ and gradient timelike conformal vector field K. Denote by g the induced metric on S and by $\phi_S := \phi \circ x$ the restriction of a potential function ϕ of K on S. Note that ϕ_S is constant if and only if K_p is orthogonal to S, for all $p \in S$.

Along the spacelike immersion x we decompose

$$K = K^T + K^N \tag{16}$$

where K^T (resp. K^N) is the tangent (resp. normal) component of K. The timelike character of K gives $\bar{g}(K^N, K^N) < 0$ everywhere on S. Therefore, we have a globally defined unit timelike normal vector field on S given by

$$N := \frac{1}{\sqrt{-\bar{g}(K^N, K^N)}}\, K^N \tag{17}$$

which clearly lies in the same time-orientation of K, in fact, $\bar{g}(K, N) = -h < 0$ holds on all S. If Q is the reference frame constructed from K in (1), the wrong-way

Schwarz inequality [22, Prop. 5.30] gives $\bar{g}(Q, N) \leq -1$ and equality holds at $p \in S$ if and only if $N(p) = Q(p)$. The tangential component of K satisfies

$$\nabla \phi_S = K^T, \tag{18}$$

where ∇ denotes the gradient operator of g. A standard computation from (18) gives

$$\Delta \phi_S = n\,\rho + n\,H \bar{g}(K, N), \tag{19}$$

where H is the mean curvature of S with respect to N and Δ the Laplacian of the induced metric.

Now we will focus on spatially closed GSC spacetimes. Recall that a spacetime is called spatially closed if it admits a compact spacelike hypersurface. Now, we are in a position to state,

Theorem 1 ([10]) *Let S be a compact CMC spacelike hypersurface in a GCS spacetime. Let p_0 and p^0 be two points of S where ϕ_S attains its minimum and maximum values, respectively. The mean curvature H of S satisfies*

$$\frac{\rho(p^0)}{h(p^0)} \leq H \leq \frac{\rho(p_0)}{h(p_0)}.$$

Proof Taking into account (18) we have $K^T(p_0) = K^T(p^0) = 0$ and, as consequence, the equality $\bar{g}(K, N) = -h$ holds at the points p_0 and p^0. On the other hand, from $\Delta\phi_S(p_0) \geq 0$ and $\Delta\phi_S(p^0) \leq 0$ and (19) we have

$$\rho(p_0) - h(p_0)\,H \geq 0 \quad \text{and} \quad \rho(p^0) - h(p^0)\,H \leq 0,$$

respectively, which give the announced inequalities.

Now consider $p \in \overline{M}$ and an observer γ in Q passing through p. Note that we have

$$\frac{d}{dt}\left(\frac{\rho(\gamma(t))}{h(\gamma(t))}\right) = (\log h(\gamma(t)))'',$$

from (5). Therefore, if we suppose that the function $\frac{\rho}{h}$ is decreasing on γ, then

$$(\log h(\gamma(t)))'' \leq 0\,.$$

On the other hand, ϕ is a global time function. Therefore, ϕ is strictly decreasing along any observer of Q. The inequalities in Theorem 1 become equalities. Therefore we have that $\rho = H\,h$ on S which gives that ϕ_S is sub(or super)-harmonic on S compact and hence ϕ_S must be constant. Thus, we arrive to the following result

Theorem 2 ([10]) *Let $(\overline{M}, \bar{g})$ be a GCS spacetime and suppose that*

$$(\log h(\gamma(t)))'' \leq 0\,,$$

for any observer γ in Q. Any compact CMC spacelike hypersurfaces in $\overline{M}$ is a leaf of the foliation $Q^{\perp}$ (which is totally umbilical with mean curvature $H = \frac{\rho}{h}$).

In particular, Theorem 2 may be restricted to the case of GRW spacetimes to get,

Corollary 1 ([1, Th. 3.1]) *In a GRW spacetime $\overline{M}$ whose warping function satisfies*

$$(\log f)'' \leq 0,$$

the only compact CMC spacelike hypersurfaces are the spacelike slices.

Remark 4 The hypothesis of the convexity of the function $-\log f$ in the previous result is related to certain natural assumption on the Ricci tensor $\overline{\text{Ric}}$ of $\overline{M}$, the so called Null Convergence Condition (NCC), namely, $\overline{\text{Ric}}(w, w) \geq 0$ for any lightlike tangent vector w. In fact, if $\overline{M}$ obeys NCC then $-\log f$ is convex. This curvature condition holds naturally if the spacetime obeys the Einstein equation (with zero cosmological constant), and, of course, when the spacetime is Einstein, i.e., its Ricci tensor is proportional to its metric. Under stronger curvature conditions, the so called Timelike Curvature Condition (TCC), $\overline{\text{Ric}}(v, v) \geq 0$ for any timelike tangent vector v, and Ubiquitous Curvature Condition, $\overline{\text{Ric}}(v, v) > 0$ for any timelike tangent vector v, two results contained in Corollary 1 were obtained in [2, Th. 5.1, 5.2], respectively.

To end this section we precisely deal with GRW spacetimes which are Einstein. If $\overline{M}$ has base I, fiber F and warping function f, then $\overline{M}$ is Einstein with $\overline{\text{Ric}} = \bar{c}\,\bar{g}$, if and only if (F, g) has constant Ricci curvature c and f satisfies the following differential equations,

$$\frac{f''}{f} = \frac{\bar{c}}{n} \qquad \text{and} \qquad \frac{\bar{c}(n-1)}{n} = \frac{c + (n-1)(f')^2}{f^2} \tag{20}$$

Moreover, $\overline{M}$ has constant sectional curvature $\overline{C}$ if and only if its fiber F has constant sectional curvature C, i.e., $\overline{M}$ is a Robertson–Walker spacetime and its warping function f satisfies (20) with $c = (n-1)\,C$ and $\bar{c} = n\,\overline{C}$.

All the positive solutions of (20) were obtained in [3], collected in a table (in each case, the interval of definition I of f is the maximal one where f is positive). Note that, from (20), we have

$$(n-1)(\log f)'' = \frac{c}{f^2}. \tag{21}$$

Taking into account previous formula, a direct application of Corollary 1 gives the following wide extension of [3, Cor. 5],

Theorem 3 ([10]) *Every compact CMC spacelike hypersurface in an Einstein GRW spacetime, whose fiber has Ricci curvature $c \leq 0$, must be a spacelike slice.*

5 Calabi–Bernstein Type Results

Let $(\overline{M}, \bar{g})$ be a GRW spacetime with fiber a Riemannian manifold (F, g) and warping function $f : I \longrightarrow \mathbb{R}^+$. Let Ω a domain in F. For each $u \in C^\infty(\Omega)$ such that $u(\Omega) \subset I$, its graph in $\overline{M}$ is the hypersurface

$$\Sigma_u := \{(u(p), p) \ : \ p \in \Omega\}. \tag{22}$$

The metric induced on Σ_u by the Lorentzian metric $\bar{g}$ given in (10) is written on F as follows

$$g_u = -du^2 + f(u)^2 g, \tag{23}$$

where $f(u) := f \circ u$. Therefore, Σ_u is spacelike if and only if $| Du |< f(u)$ everywhere on Ω, where Du denotes the gradient of u with respect to g. We will say that a spacelike graph is entire if $\Omega = F$. In that follows, only entire spacelike graphs will be considered.

The unit timelike normal vector field on Σ_u in the same time orientation of ∂_t is

$$N = \frac{f(u)}{\sqrt{f(u)^2 - |Du|^2}} \left(\partial_t + \frac{1}{f(u)^2} (0, Du) \right). \tag{24}$$

It is not difficult to see that the mean curvature H of Σ_u relative to N satisfies

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 - | Du |^2}} \right) = nH - \frac{f'(u)}{\sqrt{f(u)^2 - | Du |^2}} \left(n + \frac{| Du |^2}{f(u)^2} \right) \tag{E.1}$$

$$| Du |< f(u), \tag{E.2}$$

where div denotes the divergence operator of (F, g) and $f'(u) := f' \circ u$. When H is constant, this equation is called the CMC spacelike hypersurface equation, which we will call it as equation (E).

Equation (E) is the Euler–Lagrange equation of a variational problem [4]. In fact, let (F, g) be a (connected) compact Riemannian manifold, with dimension $n \geq 2$ and let f be a positive smooth function defined on an open interval I of $\mathbb{R}$. Consider the class of smooth real valued functions u on F such that $u(F) \subset I$ and $| Du |< f(u)$.

On this class, consider the n-dimensional functional area

$$\mathscr{A}(u) := \int_F f(u)^{n-1} \sqrt{f(u)^2 - | Du |^2}\, dV_g,$$

where dV_g is the canonical measure defined by g. The Euler–Lagrange equation for critical points of the functional $\mathscr{A}$, under the constraint

$$\int_F \left(\int_{t_0}^u f(t)^n dt \right) dV_g = \text{constant},$$

is precisely (E). Notice that $\mathscr{A}(u)$ is the area of the Riemannian manifold (F, g_u) and the previous formula may be seen as a volume constraint. Therefore, u is a critical point of functional $\mathscr{A}$ under the volume constraint if and only if Σ_u has constant mean curvature.

As an application of Corollary 1 we have the following uniqueness result,

Theorem 4 ([10]) *Let (F, g) be a compact Riemannian manifold and let $f : I \longrightarrow \mathbb{R}^+$ be a smooth function such that*

$$(\log f)'' \leq 0.$$

For each real constant H, the only entire solutions u of equation (E) are the constant functions $u = u_0$ such that $H = \dfrac{f'(u_0)}{f(u_0)}$.

Previous result widely extends [2, Th. 5.9] where a curvature assumption was needed according with the technique used.

Remark 5 The technique used in Sects. 4 and 5 works for compact spacelike hypersurfaces in (necessarily) spatially closed GRW spacetimes. The case of complete (noncompact) CMC spacelike hypersurfaces have been carry out mainly the ideas introduced in the seminal article by Chen and Yau [11]. Recently, techniques of analysis on n-dimensional parabolic Riemannian manifolds have been introduced in [26, 28] for the case of spatially parabolic GRW spacetimes. Let us notice that the family of spatially parabolic GRW spacetimes extend to the one of spatially closed GRW spacetimes from de point of view of the geometric-analysis of the fiber. Moreover, it allows to model open relativistic universes which are not incompatible with certain cosmological principle. On the other hand, the mean curvature function H of a spacelike surface has been studied without assuming its constancy in [27] (and references therein) under the hypothesis of a certain control of H described by a nonlinear inequality involving the restriction of the function f'/f on the spacelike surface. Such complete spacelike surfaces have been classified under certain geometric assumptions and new Calabi–Bernstein type problems have been stated and solved in [27].

6 CMC Spacelike Hypersurfaces in a Brinkmann Spacetime

Consider a spacelike hypersurface $x : S \longrightarrow \overline{M}$ in a Brinkmann spacetime $(\overline{M}, \bar{g})$ of dimension $n + 1$ and parallel lightlike vector field K. Denote by g the induced metric on S and, as the begin of Sect. 4, decompose K along the spacelike immersion x in tangential and normal components K^T and K^N, respectively. The lightlike character of K gives $\bar{g}(K^N, K^N) = -g(K^T, K^T) < 0$ everywhere on S. Therefore, we have a globally defined unit timelike normal vector field on S given by

$$N := \frac{1}{\sqrt{-\bar{g}(K^N, K^N)}} K^N \tag{25}$$

which obviously lies in the same time-orientation defined by K, i.e., such that $\bar{g}(K, N) =< 0$ holds on all S.

On the other hand, K^T is a nowhere zero vector field on S. This fact restricts the topology of S in the case we assume S is compact because it means that the Euler-Poincaré number of S is zero.

Consider the function $u = \bar{g}(K, N) = -\sqrt{g(K^T, K^T)} < 0$, whose gradient satisfies

$$\nabla u = -A\, K^T, \tag{26}$$

where A is the shape operator of S associated to N. Now, using the Codazzi equation we obtain, [23],

$$\Delta u = n\, \bar{g}(\nabla H, K) + \overline{\mathrm{Ric}}(K^T, N) + \mathrm{trace}(A^2)\, u. \tag{27}$$

(Compare with [4, 5] where certain integral formulas were obtained for compact spacelike hypersurfaces in a general CS spacetime). Taking into account that K is parallel, we have

$$\overline{\mathrm{Ric}}(K^T, N) = u\, \overline{\mathrm{Ric}}(N, N),$$

and therefore, for any CMC spacelike hypersurface, (27) reduces to

$$\Delta u = \{\overline{\mathrm{Ric}}(N, N) + \mathrm{trace}(A^2)\}\, u. \tag{28}$$

If we assume the Brinkmann spacetime $\overline{M}$ obeys TCC then $\overline{\mathrm{Ric}}(N, N) \geq 0$ everywhere on S. Under this assumption, formula (28) gives that the function u is superharmonic. Moreover, making use of (26) and (28), u is constant if and only if S is totally geodesic. Therefore, we get,

Theorem 5 ([23, Th. 2]) *Let $\overline{M}$ a Brinkmann spacetime which obeys the Timelike Convergent Condition. Any compact CMC spacelike hypersurface must be totally geodesic.*

Remark 6 Contrary to the situation in Sect. 4, we can ensure here that in a Brinkmann spacetime which obeys the Timelike Convergent Condition there is no compact CMC spacelike hypersurface whose mean curvature is different from zero [23, Th. 1].

Remark 7 In the case of a maximal surface S in a 3-dimensional Brinkmann spacetime, the assumption of compactness of S can be weakened to completeness to get that in a 3-dimensional Brinkmann spacetime which obeys TCC, a complete maximal surface must be totally geodesic [23, Th. 4], extending the classical parametric Calabi–Bernstein theorem for complete maximal surfaces in 3-dimensional Lorentz-Minkowski spacetime [9, 11].

Acknowledgements Research partially supported by Spanish MINECO and ERDF project MTM2013-47828-C2-1-P. The author would like to express his deep and sincere thanks to the organizers of the 20th International Workshop on Hermitian Symmetric Spaces and Submanifolds and the 12th RIRCM-OCAMI Joint Differential Geometry Workshop, held in Kyungpook National University, Daegu (Korea), July 26–30, 2016, specially to Professor Young Jin Suh for the support for his participation in the congress and the warm hospitality during his stay in Korea, and to Dr. Hyunjin Lee and Mr. Changhwa Woo for their kindness and the excellent work done as the congress secretariat.

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Sequences of Maximal Antipodal Sets of Oriented Real Grassmann Manifolds II

Hiroyuki Tasaki

Abstract Chen–Nagano introduced the notion of antipodal sets of compact Riemannian symmetric spaces. The author showed a correspondence between maximal antipodal sets of oriented real Grassmann manifolds and certain families of subsets of finite sets and reduced the classifications of maximal antipodal sets of oriented real Grassmann manifolds to a certain combinatorial problem in a previous paper. In this paper we construct new sequences of maximal antipodal sets from those obtained in previous papers and estimate the cardinalities of antipodal sets.

1 Introduction

The present paper is a sequel to a previous paper [4]. Chen–Nagano [1] introduced the notion of antipodal sets of compact Riemannian symmetric spaces. The author showed a correspondence between maximal antipodal sets of the oriented real Grassmann manifolds $\tilde{G}_k(\mathbb{R}^n)$ consisting of k-dimensional oriented subspaces in $\mathbb{R}^n$ and certain families of subsets of cardinality k in $[n] = \{1, 2, 3, \dots, n\}$, and reduced the classifications of maximal antipodal sets of oriented real Grassmann manifolds to a combinatorial problem in [3]. Using this correspondence he showed the classification of maximal antipodal sets of $\tilde{G}_k(\mathbb{R}^n)$ whose ranks are less than 5 in [3] and constructed some sequences of maximal antipodal sets in [3, 4], which were used in estimates of antipodal sets of $\tilde{G}_5(\mathbb{R}^n)$ obtained in [5]. Certain maximal antipodal sets of $\tilde{G}_5(\mathbb{R}^n)$ obtained in [3, 4] attain the maximum of the cardinalities of antipodal sets of $\tilde{G}_5(\mathbb{R}^n)$ for sufficiently large n. Frankl–Tokushige [2] have obtained some estimates of combinatorial objects which lead some estimates of cardinalities of antipodal sets of $\tilde{G}_k(\mathbb{R}^n)$ for general k and sufficiently large n. For these estimates some sequences of maximal antipodal sets obtained in [3, 4] played fundamental roles. So

H. Tasaki (✉)
Division of Mathematics, Faculty of Pure and Applied Sciences,
University of Tsukuba, 305-8571 Tsukuba, Ibaraki, Japan
e-mail: tasaki@math.tsukuba.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_2

we expect that we can get some estimates of antipodal sets using the sequences of maximal antipodal sets obtained in this paper.

In this paper we construct new sequences of maximal antipodal sets from those obtained in [3, 4]. We review the definition of antipodal sets, fundamental properties of them and the sequences Ev_{2m} of antipodal sets in Sect. 2. We construct some sequences of maximal antipodal sets from Ev_{2m} in Sect. 3. Using these maximal antipodal sets we get estimates of cardinalities of antipodal sets in Sect. 4.

2 Definition and Fundamental Results

We review the definition of antipodal sets and results on some sequences of antipodal sets obtained in [3, 4].

We denote by $\binom{X}{k}$ the set of subsets of cardinality k of a set X. Two elements α, β of $\binom{[n]}{k}$ are *antipodal*, if the cardinality $|\beta\backslash\alpha|$ is even, where $\beta\backslash\alpha = \{i \in \beta \mid i \notin \alpha\}$. A subset A of $\binom{[n]}{k}$ is *antipodal*, if any α, β of A are antipodal. We denote by $\mathrm{Sym}(n)$ the symmetric group on $[n]$. Two subsets A, B of $\binom{[n]}{k}$ are *congruent*, if A is transformed to B by an element of $\mathrm{Sym}(n)$. If a subset A of $\binom{[n]}{k}$ is antipodal, then a subset congruent with A is also antipodal.

When $X = X_1 \cup \cdots \cup X_m$ is a disjoint union, we put

$$A_1 \times \cdots \times A_m = \{\alpha_1 \cup \cdots \cup \alpha_m \mid \alpha_i \in A_i\}$$

for subsets A_i of $\binom{X_i}{k_i}$. We get $A_1 \times \cdots \times A_m \subset \begin{pmatrix} X \\ k_1 + \cdots + k_m \end{pmatrix}$. For a natural number m we denote

$$\binom{[2]}{1}_m = \binom{\{1,2\}}{1} \times \cdots \times \binom{\{2m-1, 2m\}}{1} \subset \binom{[2m]}{m}.$$

For $\alpha = \{\alpha_1, \ldots, \alpha_m\} \in \binom{[2]}{1}_m$ $(\alpha_i \in \{2i-1, 2i\})$ we define

$$\alpha^e = \{i \mid \alpha_i \text{is even}\}, \quad \alpha^o = \{i \mid \alpha_i \text{is odd}\}.$$

Definition 1 ([4]) For a natural number m we define a subset Ev_{2m} of $\binom{[2m]}{m}$ by

$$Ev_{2m} = \left\{ \alpha = \{\alpha_1, \ldots, \alpha_m\} \in \binom{[2]}{1}_m \,\middle|\, |\alpha^e| \text{ is even} \right\}.$$

Lemma 1 ([4]) *For* $\alpha, \beta \in \binom{[2]}{1}_m$ *we have* $|\alpha \cap \beta| = 2|\alpha^e \cap \beta^e| + |\beta^o| - |\alpha^e|$.

Using Lemma 1 we showed the following lemma in [4].

Lemma 2 *For a natural number* m, Ev_{2m} *is an antipodal set of* $\binom{[2m]}{m}$.

The following sets defined in [3, 4] are antipodal sets.

$$A(2k, 2l) = \{\alpha_1 \cup \cdots \cup \alpha_k \in \binom{[2l]}{2k} \mid \alpha_i \in \{\{1,2\}, \ldots, \{2l-1, 2l\}\}\},$$
$$A(2k+1, 2l+1) = A(2k, 2l) \times \{\{2l+1\}\}.$$

Example 1 ([3, 4]) As is showed in Example 1 of [4], Ev_6 is transformed by $\begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,5\,1\,6\,3\,4 \end{pmatrix}$ to $B(3,6) = \{\{1,2,3\},\{1,4,5\},\{2,4,6\},\{3,5,6\}\}$ defined in [3], which is a maximal antipodal set of $\binom{[6]}{3}$. Additionally Ev_6 is included in Ev_6^+ defined in Proposition 1, which is transformed by $\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 2\,5\,1\,6\,3\,4\,7 \end{pmatrix}$ to $B(3,7) = B(3,6) \cup \{\{1,6,7\},\{2,5,7\},\{3,4,7\}\}$ defined in [3]. This is a maximal antipodal set of $\binom{[7]}{3}$. Hence Ev_6 is not a maximal antipodal set of $\binom{[7]}{3}$.

Theorem 1 ([4]) *If* $2m \equiv 2, 4, 6 (\mathrm{mod} 8)$*, then* Ev_{2m} *is a maximal antipodal set of* $\binom{[2m]}{m}$*. On the other hand,*

$$Ev_{8m}^+ = Ev_{8m} \cup A(4m, 8m)$$

is a maxmal antipodal set of $\binom{[8m]}{4m}$.

Ev_8^+ is congruent with $B(4,8)$, which is stated in Remark 7.2 of [3]. We have already proved that $B(4,8)$ is a maximal antipodal set of $\binom{[8]}{4}$ in [3].

3 Some Sequences Obtained from Ev_{2m}

For natural numbers k, m, n satisfying $2m \leq n$ and $k < n$, and $\alpha \in \binom{[n]}{k}$, we put

$$I_j(\alpha, 2m) = \{i \mid |\{2i-1, 2i\} \cap \alpha| = j,\ 1 \leq i \leq m\} \quad (j = 0, 1, 2),$$
$$\tilde{I}_j(\alpha, 2m) = \bigcup_{i \in I_j(\alpha, 2m)} \{2i-1, 2i\}.$$

Then we get disjoint unions

$$\{1, 2, \ldots, m\} = I_0(\alpha, 2m) \cup I_1(\alpha, 2m) \cup I_2(\alpha, 2m),$$
$$\{1, 2, \ldots, 2m\} = \tilde{I}_0(\alpha, 2m) \cup \tilde{I}_1(\alpha, 2m) \cup \tilde{I}_2(\alpha, 2m).$$

Proposition 1 *For a nonnegative integer m we define*

$$Ev_{8m+6}^+ = Ev_{8m+6} \cup A(4m+2, 8m+6) \times \{\{8m+7\}\} \subset \binom{[8m+7]}{4m+3}.$$

Ev^+_{8m+6} *is a maximal antipodal set of* $\binom{[8m+7]}{4m+3}$.

Proof We have already known that Ev_{8m+6} and $A(4m+2, 8m+6) \times \{\{8m+7\}\}$ are antipodal sets of $\binom{[8m+7]}{4m+3}$. For $\alpha \in Ev_{8m+6}$ and $\beta \in A(4m+2, 8m+6) \times \{\{8m+7\}\}$ we have $|\alpha \cap \beta| = 2m+1$, which means that α and β are antipodal in $\binom{[8m+7]}{4m+3}$. Hence Ev^+_{8m+6} is an antipodal set of $\binom{[8m+7]}{4m+3}$.

Next we show that Ev^+_{8m+6} is a maximal antipodal set of $\binom{[8m+7]}{4m+3}$. We take $\alpha \in \binom{[8m+7]}{4m+3}$ which is antipodal with all elements of Ev^+_{8m+6}. If α does not contain $8m+7$, then $\alpha \in \binom{[8m+6]}{4m+3}$ and the maximal property of Ev_{8m+6} in $\binom{[8m+6]}{4m+3}$ implies that $\alpha \in Ev_{8m+6} \subset Ev^+_{8m+6}$. Thus we consider the case where α contains $8m+7$. We get

$$4m+3 = |\alpha| = |I_1(\alpha, 8m+6)| + 2|I_2(\alpha, 8m+6)| + 1,$$

where 1 in the right hand side comes from $8m+7 \in \alpha$. We have $4m+2 = |I_1(\alpha, 8m+6)| + 2|I_2(\alpha, 8m+6)|$ and $|I_1(\alpha, 8m+6)|$ is even. If $I_1(\alpha, 8m+6)$ is empty, then we obtain

$$\alpha \in A(4m+2, 8m+6) \times \{\{8m+7\}\} \subset Ev^+_{8m+6}.$$

Thus we consider the case where $I_1(\alpha, 8m+6)$ is not empty. We write $|I_1(\alpha, 8m+6)| = 2k$. We take a subset $I'_1 \subset I_1(\alpha, 8m+6)$ with $|I'_1| = 2k-1$. Since

$$|\{1, \dots, 4m+3\} \backslash I_1(\alpha, 8m+6)| = 4m+3-2k > 2m-2k+2,$$

we can take a subset $J \subset \{1, \dots, 4m+3\} - I_1(\alpha, 8m+6)$ with $|J| = 2m-2k+2$. We take

$$\beta = \bigcup_{j \in I'_1 \cup J} \{2j-1, 2j\} \cup \{8m+7\}.$$

Since $|I'_1| + |J| = 2k-1+2m-2k+2 = 2m+1$, we have

$$\beta \in A(4m+2, 8m+6) \times \{\{8m+7\}\} \subset Ev^+_{8m+6}.$$

Moreover we get

$$\begin{aligned} |\alpha \cap \beta| &\equiv |I'_1| + 1 \pmod 2 \\ &= 2k, \end{aligned}$$

This means that α, β are not antipodal in $\binom{[8m+7]}{4m+3}$, which is a contradiction. Therefore Ev^+_{8m+6} is a maximal antipodal set of $\binom{[8m+7]}{4m+3}$.

Proposition 2 *For a natural number* m, Ev_{8m+2} *is a maximal antipodal set of* $\binom{[8m+4]}{4m+1}$. *We define*

$$Ev^+_{8m+2} = Ev_{8m+2} \cup A(4m-2, 8m+2) \times \{\{8m+3, 8m+4, 8m+5\}\}.$$

Ev^+_{8m+2} is a maximal antipodal set of $\binom{[8m+5]}{4m+1}$.

Proof We first show that Ev_{8m+2} is a maximal antipodal set of $\binom{[8m+3]}{4m+1}$. We take $\alpha \in \binom{[8m+3]}{4m+1}$ which is antipodal with all elements of Ev_{8m+2}. If α does not contain $8m+3$, then $\alpha \in \binom{[8m+2]}{4m+1}$ and the maximal property of Ev_{8m+2} in $\binom{[8m+2]}{4m+1}$ implies that $\alpha \in Ev_{8m+2}$. Thus we consider the case where α contains $8m+3$. We get

$$4m+1 = |\alpha| = |I_1(\alpha, 8m+2)| + 2|I_2(\alpha, 8m+2)| + 1,$$

thus $|I_1(\alpha, 8m+2)|$ is even. If $|I_1(\alpha, 8m+2)| = 0$, then for any $\beta \in Ev_{8m+2}$

$$|\alpha \cap \beta| = |I_2(\alpha, 8m+2)| = 2m,$$

which means that α, β are not antipodal. This is a contradiction. If $|I_1(\alpha, 8m+2)| \geq 2$, then we can take $\gamma \in Ev_{8m+2}$ satisfying

$$\left|\alpha \cap \tilde{I}_1(\alpha, 8m+2) \cap \gamma\right| \equiv \left|\alpha \cap \tilde{I}_2(\alpha, 8m+2) \cap \gamma\right| \pmod 2.$$

So $|\alpha \cap \gamma|$ is even and α, γ are not antipodal. This is a contradiction. Therefore Ev_{8m+2} is a maximal antipodal set of $\binom{[8m+3]}{4m+1}$.

Next we show that Ev_{8m+2} is a maximal antipodal set of $\binom{[8m+4]}{4m+1}$. We take $\alpha \in \binom{[8m+4]}{4m+1}$ which is antipodal with all elements of Ev_{8m+2}. If $|\alpha \cap \{8m+3, 8m+4\}| = 0$, then $\alpha \in \binom{[8m+2]}{4m+1}$, which implies that $\alpha \in Ev_{8m+2}$. If $|\alpha \cap \{8m+3, 8m+4\}| = 1$, this case is reduced to the case where α contains $8m+3$, hence $\alpha \in Ev_{8m+2}$. So it is sufficient to consider the case where α contains both of $8m+3$ and $8m+4$. We get

$$4m+1 = |\alpha| = |I_1(\alpha, 8m+2)| + 2|I_2(\alpha, 8m+2)| + 2,$$

thus $I_1(\alpha, 8m+2)$ is not empty. We can take $\gamma \in Ev_{8m+2}$ satisfying

$$\left|\alpha \cap \tilde{I}_1(\alpha, 8m+2) \cap \gamma\right| \equiv \left|\alpha \cap \tilde{I}_2(\alpha, 8m+2) \cap \gamma\right| \pmod 2.$$

So $|\alpha \cap \gamma|$ is even and α, γ are not antipodal. This is a contradiction. Therefore Ev_{8m+2} is a maximal antipodal set of $\binom{[8m+4]}{4m+1}$.

We finally show that Ev^+_{8m+2} is a maximal antipodal set of $\binom{[8m+5]}{4m+1}$. We have already known that Ev_{8m+2} and $A(4m-2, 8m+2) \times \{\{8m+3, 8m+4, 8m+5\}\}$ are antipodal sets of $\binom{[8m+5]}{4m+1}$. For any $\alpha \in Ev_{8m+2}$ and $\beta \in A(4m-2, 8m+2) \times \{\{8m+3, 8m+4, 8m+5\}\}$ we have $|\alpha \cap \beta| = 2m-1$, which means that α, β are antipodal. We prove that Ev^+_{8m+2} is a maximal antipodal set of $\binom{[8m+5]}{4m+1}$ in the following. We take $\alpha \in \binom{[8m+5]}{4m+1}$ which is antipodal with all elements of Ev^+_{8m+2}. If $|\alpha \cap \{8m+3, 8m+4, 8m+5\}| \leq 2$, then we get $\alpha \in Ev_{8m+2} \subset Ev^+_{8m+2}$ by the results obtained above. So it is sufficient to consider the case where $\{8m+3, 8m+4, 8m+5\} \subset \alpha$. We get

$$4m+1 = |\alpha| = |I_1(\alpha, 8m+2)| + 2|I_2(\alpha, 8m+2)| + 3,$$

thus $|I_1(\alpha, 8m+2)|$ is even. If $|I_1(\alpha, 8m+2)| = 0$, then

$$\alpha \in A(4m-2, 8m+2) \times \{\{8m+3, 8m+4, 8m+5\}\} \subset Ev^+_{8m+2}.$$

If $|I_1(\alpha, 8m+2)| \geq 2$, then we can take $\gamma \in Ev_{8m+2}$ satisfying

$$\left|\alpha \cap \tilde{I}_1(\alpha, 8m+2) \cap \gamma\right| \equiv \left|\alpha \cap \tilde{I}_2(\alpha, 8m+2) \cap \gamma\right| \pmod 2.$$

So $|\alpha \cap \gamma|$ is even and α, γ are not antipodal. This is a contradiction. Therefore Ev^+_{8m+2} is a maximal antipodal set of $\binom{[8m+5]}{4m+1}$.

Proposition 3 *For a nonnegative integer m, Ev_{8m+4} is a maximal antipodal set of $\binom{[8m+5]}{4m+2}$. We define*

$$Ev^+_{8m+4} = Ev_{8m+4} \cup A(4m, 8m+4) \times \{\{8m+5, 8m+6\}\}.$$

Ev^+_{8m+2} is a maximal antipodal set of $\binom{[8m+6]}{4m+2}$.

Proof We first show that Ev_{8m+4} is a maximal antipodal set of $\binom{[8m+5]}{4m+2}$. We take $\alpha \in \binom{[8m+5]}{4m+2}$ which is antipodal with all elements of Ev_{8m+4}. If α does not contain $8m+5$, then $\alpha \in \binom{[8m+4]}{4m+2}$ and the maximal property of Ev_{8m+4} in $\binom{[8m+4]}{4m+2}$ implies that $\alpha \in Ev_{8m+4}$. Thus we consider the case where α contains $8m+5$. We get

$$4m+2 = |\alpha| = |I_1(\alpha, 8m+4)| + 2|I_2(\alpha, 8m+4)| + 1,$$

thus $I_1(\alpha, 8m+4)$ is not empty. We can take $\gamma \in Ev_{8m+4}$ satisfying

$$\left|\alpha \cap \tilde{I}_1(\alpha, 8m+4) \cap \gamma\right| \not\equiv \left|\alpha \cap \tilde{I}_2(\alpha, 8m+4) \cap \gamma\right| \pmod 2.$$

So $|\alpha \cap \gamma|$ is odd and α, γ are not antipodal. This is a contradiction. Therefore Ev_{8m+4} is a maximal antipodal set of $\binom{[8m+5]}{4m+2}$.

Next we show that Ev^+_{8m+4} is a maximal antipodal set of $\binom{[8m+6]}{4m+2}$. We have already known that Ev_{8m+4} and $A(4m, 8m+4) \times \{\{8m+5, 8m+6\}\}$ are antipodal sets of $\binom{[8m+6]}{4m+2}$. We take $\alpha \in \binom{[8m+6]}{4m+2}$ which is antipodal with all elements of Ev^+_{8m+4}. If $|\alpha \cap \{8m+5, 8m+6\}| \leq 1$, then we get $\alpha \in Ev_{8m+4} \subset Ev^+_{8m+4}$ by the result obtained above. So it is sufficient to consider the case where $\{8m+5, 8m+6\} \subset \alpha$. We get

$$4m+2 = |\alpha| = |I_1(\alpha, 8m+4)| + 2|I_2(\alpha, 8m+4)| + 2,$$

thus $|I_1(\alpha, 8m+4)|$ is even. If $|I_1(\alpha, 8m+4)| = 0$, then

$$\alpha \in A(4m, 8m+4) \times \{\{8m+5, 8m+6\}\} \subset Ev^+_{8m+4}.$$

If $|I_1(\alpha, 8m+4)| \geq 2$, then we can take $\gamma \in Ev_{8m+4}$ satisfying

$$\left|\alpha \cap \tilde{I}_1(\alpha, 8m+4) \cap \gamma\right| \not\equiv \left|\alpha \cap \tilde{I}_2(\alpha, 8m+4) \cap \gamma\right| \pmod{2}.$$

So $|\alpha \cap \gamma|$ is odd and α, γ are not antipodal. This is a contradiction. Therefore Ev^+_{8m+4} is a maximal antipodal set of $\binom{[8m+6]}{4m+2}$.

Proposition 4 *For a natural number m, Ev^+_{8m} is a maximal antipodal set of $\binom{[8m+3]}{4m}$.*

Proof We have already known that Ev^+_{8m} is a maximal antipodal set of $\binom{[8m]}{4m}$. We first prove that Ev^+_{8m} is a maximal antipodal set of $\binom{[8m+1]}{4m}$. We take $\alpha \in \binom{[8m+1]}{4m}$ which is antipodal with all elements of Ev^+_{4m}. If α does not contain $8m+1$, then $\alpha \in \binom{[8m]}{4m}$ and the maximal property of Ev^+_{8m} in $\binom{[8m]}{4m}$ implies that $\alpha \in Ev^+_{8m}$. Thus we consider the case where α contains $8m+1$. We get

$$4m = |\alpha| = |I_1(\alpha, 8m)| + 2|I_2(\alpha, 8m)| + 1,$$

thus $I_1(\alpha, 8m)$ is not empty. We can take $\gamma \in Ev_{8m}$ satisfying

$$\left|\alpha \cap \tilde{I}_1(\alpha, 8m) \cap \gamma\right| \not\equiv \left|\alpha \cap \tilde{I}_2(\alpha, 8m) \cap \gamma\right| \pmod{2}.$$

So $|\alpha \cap \gamma|$ is odd and α, γ are not antipodal. This is a contradiction. Therefore Ev^+_{8m} is a maximal antipodal set of $\binom{[8m+1]}{4m}$.

We second show that Ev^+_{8m} is a maximal antipodal set of $\binom{[8m+2]}{4m}$. We take $\alpha \in \binom{[8m+2]}{4m}$ which is antipodal with all elements of Ev^+_{4m}. If $|\alpha \cap \{8m+1, 8m+2\}| \leq 1$, then we get $\alpha \in Ev^+_{8m}$ by the result obtained above. So it is sufficient to consider the case where $\{8m+1, 8m+2\} \subset \alpha$. We get

$$4m = |\alpha| = |I_1(\alpha, 8m)| + 2|I_2(\alpha, 8m)| + 2,$$

thus $|I_1(\alpha, 8m)|$ is even. If $|I_1(\alpha, 8m)| = 0$, then

$$\alpha \in A(4m-2, 8m) \times \{\{8m+1, 8m+2\}\}.$$

For any $\gamma \in Ev_{8m}$ we have $|\alpha \cap \gamma| = 2m-1$, hence α and γ are not antipodal, which is a contradiction. So $|I_1(\alpha, 8m)|$ is a positive even number. We can take $\gamma \in Ev_{8m}$ satisfying

$$\left|\alpha \cap \tilde{I}_1(\alpha, 8m) \cap \gamma\right| \not\equiv \left|\alpha \cap \tilde{I}_2(\alpha, 8m) \cap \gamma\right| \pmod{2}.$$

So $|\alpha \cap \gamma|$ is odd and α, γ are not antipodal. This is a contradiction. Therefore Ev_{8m}^+ is a maximal antipodal set of $\binom{[8m+2]}{4m}$.

We third show that Ev_{8m}^+ is a maximal antipodal set of $\binom{[8m+3]}{4m}$. We take $\alpha \in \binom{[8m+3]}{4m}$ which is antipodal with all elements of Ev_{4m}^+. If $|\alpha \cap \{8m+1, 8m+2, 8m+3\}| \leq 2$, then we get $\alpha \in Ev_{8m}^+$ by the result obtained above. So it is sufficient to consider the case where $\{8m+1, 8m+2, 8m+3\} \subset \alpha$. We get

$$4m = |\alpha| = |I_1(\alpha, 8m)| + 2|I_2(\alpha, 8m)| + 3,$$

thus $|I_1(\alpha, 8m)|$ is not empty. We can take $\gamma \in Ev_{8m}$ satisfying

$$\left|\alpha \cap \tilde{I}_1(\alpha, 8m) \cap \gamma\right| \not\equiv \left|\alpha \cap \tilde{I}_2(\alpha, 8m) \cap \gamma\right| \pmod 2.$$

So $|\alpha \cap \gamma|$ is odd and α, γ are not antipodal. This is a contradiction. Therefore Ev_{8m}^+ is a maximal antipodal set of $\binom{[8m+3]}{4m}$.

We summarize the results obtained above into the following theorem, which is a refinement of Theorem 1.

Theorem 2 *The followings hold.*

(0) For $m \geq 1$, Ev_{8m} is not a maximal antipodal set of $\binom{[8m]}{4m}$.

$$Ev_{8m}^+ = Ev_{8m} \cup A(4m, 8m)$$

is a maximal antipodal set of $\binom{[8m+i]}{4m}$ for $i = 0, 1, 2, 3$.

(1) For $m \geq 1$, Ev_{8m+2} is a maximal antipodal set of $\binom{[8m+i]}{4m+1}$ for $i = 2, 3, 4$.

$$Ev_{8m+2}^+ = Ev_{8m+2} \cup A(4m-2, 8m+2) \times \{\{8m+3, 8m+4, 8m+5\}\}.$$

is a maximal antipodal set of $\binom{[8m+5]}{4m+1}$.

(2) For $m \geq 0$, Ev_{8m+4} is a maximal antipodal set of $\binom{[8m+i]}{4m+2}$ for $i = 4, 5$.

$$Ev_{8m+4}^+ = Ev_{8m+4} \cup A(4m, 8m+4) \times \{\{8m+5, 8m+6\}\}.$$

is a maximal antipodal set of $\binom{[8m+6]}{4m+2}$.

(3) For $m \geq 0$, Ev_{8m+6} is a maximal antipodal set of $\binom{[8m+6]}{4m+3}$.

$$Ev_{8m+6}^+ = Ev_{8m+6} \cup A(4m+2, 8m+6) \times \{\{8m+7\}\}$$

is a maximal antipodal set of $\binom{[8m+7]}{4m+3}$.

The following table shows maximal antipodal sets of $\binom{[n]}{k}$.

$k \backslash n$	$8m$	$8m+1$	$8m+2$	$8m+3$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
$4m$	Ev_{8m}^{+}	Ev_{8m}^{+}	Ev_{8m}^{+}	Ev_{8m}^{+}				
$4m+1$			Ev_{8m+2}	Ev_{8m+2}	Ev_{8m+2}	Ev_{8m+2}^{+}		
$4m+2$					Ev_{8m+4}	Ev_{8m+4}	Ev_{8m+4}^{+}	
$4m+3$							Ev_{8m+6}	Ev_{8m+6}^{+}

4 Estimates of Antipodal Sets

We consider estimates of the cardinalities of antipodal sets of $\binom{[n]}{k}$. For this purpose we define

$$a(k,n) = \max\left\{ |A| \;\middle|\; A \text{ is antipodal in } \binom{[n]}{k} \right\}.$$

If n is sufficiently large with respect to k, there exists estimates of $a(k,n)$ from above where maximal antipodal sets which attain $a(k,n)$ are determined by the results of [2, 5]. We have $a(k,n) = a(n-k,n)$, thus it is enough to consider the case where $2k \le n$. We treat the cases where n are small with respect to k.

Corollary 1 *The followings hold.*

(0) For $m \ge 1$ and $i = 0, 1, 2, 3$

$$2^{4m-1} + \binom{4m}{2m} \le a(4m, 8m+i).$$

(1) For $m \ge 1$ and $i = 2, 3, 4$

$$2^{4m} \le a(4m+1, 8m+i),$$
$$2^{4m} + \binom{4m+1}{2m-1} \le a(4m+1, 8m+5).$$

(2) For $m \ge 0$ and $i = 4, 5$

$$2^{4m+1} \le a(4m+2, 8m+i),$$
$$2^{4m+1} + \binom{4m+2}{2m} \le a(4m+2.8m+6).$$

(3) For $m \ge 0$

$$2^{4m+2} \le a(4m+3, 8m+6), \quad 2^{4m+2} + \binom{4m+3}{2m+1} \le a(4m+3, 8m+7).$$

Acknowledgements The author was partly supported by the Grant-in-Aid for Science Research (C) (No. 15K04835), Japan Society for the Promotion of Science.

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Derivatives on Real Hypersurfaces of Non-flat Complex Space Forms

Juan de Dios Pérez

Abstract Let M be a real hypersurface of a nonflat complex space form, that is, either a complex projective space or a complex hyperbolic space. On M we have the Levi-Civita connection and for any nonnull real number k the corresponding generalized Tanaka-Webster connection. Therefore on M we consider their associated covariant derivatives, the Lie derivative and, for any nonnull k, the so called Lie derivative associated to the generalized Tanaka-Webster connection and introduce some classifications of real hypersurfaces in terms of the concidence of some pairs of such derivations when they are applied to the shape operator of the real hypersurface, the structure Jacobi operator, the Ricci operator or the Riemannian curvature tensor of the real hypersurface.

Keywords Real hypersurfaces · Complex space form · g-Tanaka-Webster connection · Covariant derivatives · Lie derivatives

2010 Mathematics Subject Classification Primary 53C40 · Secondary 53C15

1 Introduction

A complex space form is an m-dimensional Kaehler manifold of constant holomorphic sectional curvature c and will be denoted by $M_m(c)$. A complete and simply connected complex space form is complex analytically isometric to

1. A complex projective space $\mathbb{C}P^m$, if $c > 0$.
2. A complex Euclidean space $\mathbb{C}^m$, if $c = 0$.
3. A complex hyperbolic space $\mathbb{C}H^m$, if $c < 0$.

J. de Dios Pérez (✉)
Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain
e-mail: jdperez@ugr.es

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_3

We will deal with non-flat complex space forms and if (J, g) is the Kaehlerian structure of such a manifold, the metric g is going to be considered with its holomorphic sectional curvature equal to either 4 or -4. That is, $c = \pm 4$.

Let M be a real hypersurface of $M_m(c)$. Let N be a locally defined unit normal vector field on M. Writing $\xi = -JN$, this is a tangent vector field to M called the structure vector field on M (it is also known as Reeb vector field or Hopf vector field). Let A be the shape operator of M associated to N, ∇ the Levi-Civita connection on M and $\mathbb{D}$ the maximal holomorphic distribution on M. That is, for any $p \in M$ $\mathbb{D}_p = \{X \in T_pM / g(X, \xi) = 0\}$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX. Clearly $\eta(X) = g(X, \xi)$ and (ϕ, ξ, η, g) is an almost contact metric structure on M. That is, we have

- $\phi^2 X = -X + \eta(X)\xi$
- $\eta(\xi) = 1$
- $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$
- $\phi\xi = 0$
- $\nabla_X \xi = \phi AX$
- $(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$

for any X, Y tangent to M.

Since the ambient space is of constant holomorphic sectional curvature ± 4, the Gauss and Codazzi equations are respectively given by

$$R(X, Y)Z = \varepsilon\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY \tag{1}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \varepsilon\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2}$$

for any X, Y, Z tangent to M, where $\varepsilon = \pm 1$, depending on the ambient space is either complex projective space or complex hyperbolic space.

A real hypersurface in $M_m(c)$ is Hopf if its structure vector field is principal.

The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$, $m \geq 2$ was obtained by Takagi [29, 30] and consists in six distinct types of real hypersurfaces. Kimura, [11], proved that Takagi's real hypersurfaces are the unique Hopf real hypersurfaces with constant principal curvatures. Takagi's list is as follows:

(A_1) Geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$. They have 2 distinct constant principal curvatures $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$ and $cot(r)$ with eigenspace $\mathbb{D}$.

(A_2) Tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m - 1$. They have 3 distinct constant principal curvatures $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $cot(r)$ and $-tan(r)$. The corresponding eigenspaces of $cot(r)$ and $-tan(r)$ are complementary and ϕ-invariant distributions in $\mathbb{D}$.

(B) Tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric. They have 3 distinct constant principal curvatures $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $cot(r - \frac{\pi}{4})$ and $-tan(r - \frac{\pi}{4})$ whose corresponding eigenspaces are complementary and equal dimensional distributions in $\mathbb{D}$ such that $\phi T_{cot(r-\frac{\pi}{4})} = T_{-tan(r-\frac{\pi}{4})}$.

(C) Tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^n$, where $2n + 1 = m$ and $m \geq 5$. They have 5 distinct principal curvatures, $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $cot(r - \frac{\pi}{4})$ with multiplicity 2, $cot(r - \frac{\pi}{2}) = -tan(r)$, with multiplicity m-3, $cot(r - \frac{3\pi}{4})$, with multiplicity 2 and $cot(r - \pi) = cot(r)$ with multiplicity m-3. Moreover $\phi T_{cot(r-\frac{\pi}{4})} = T_{cot(r-\frac{3\pi}{4})}$ and $T_{-tan(r)}$ and $T_{cot(r)}$ are ϕ-invariant.

(D) Tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Plucker embedding of the complex Grassmannian manifold $G(2, 5)$ in $\mathbb{C}P^9$. They have the same principal curvatures as type C real hypersurfaces, $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, and the other four principal curvatures have the same multiplicity 4 and their eigenspaces have the same behaviour with respect to ϕ as in type C.

(E) Tubes of radius r, $0 < r < \frac{\pi}{4}$, over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^{15}$. They have the same principal curvatures as type C real hypersurfaces, $2cot(2r)$ with eigenspace $\mathbb{R}[\xi]$, $cot(r - \frac{\pi}{4})$ and $cot(r - \frac{3\pi}{4})$ have multiplicities equal to 6 and $-tan(r)$ and $cot(r)$ have multiplicities equal to 8. Their corresponding eigenspaces have the same behaviour with respect to ϕ as in type C real hypersurfaces.

In the case of $\mathbb{C}H^m$, $m \geq 2$, Berndt, [1], classified Hopf real hypersurfaces with constant principal curvatures into three types:

(A_1) Tubes of radius $r > 0$ over $\mathbb{C}H^k$, $0 \leq k \leq m - 1$ with 3 (respectively, 2) distinct constant principal curvatures if $0 < k < m - 1$ (respectively $k = 0$ or $k = m - 1$), $2coth(2r)$ with eigenspace $\mathbb{R}[\xi]$, $tanh(r)$ and $coth(r)$ whose eigenspaces are complementary and ϕ-invariant distributions in $\mathbb{D}$.

(A_2) Horospheres in $\mathbb{C}H^m$ with 2 distinct constant principal curvatures, 2 with eigenspace $\mathbb{R}[\xi]$ and 1 with eigenspace $\mathbb{D}$.

(B) Tubes of radius $r > 0$ over $\mathbb{R}H^m$, with 3 (respectively 2) distinct constant principal curvatures if $r \neq ln(2 + \sqrt{3})$, (respectively, $r = ln(2 + \sqrt{3})$), $2coth(2r)$ with eigenspace $\mathbb{R}[\xi]$, $tanh(r)$ and $coth(r)$, both with multiplicities equal to m-1 and such that $\phi T_{tanh(r)} = T_{coth(r)}$.

Ruled real hypersurfaces can be described as follows: Take a regular curve γ in $M_m(c)$ with tangent vector field X. At each point of γ there is a unique $M_{m-1}(c)$ cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a ruled real hypersurface. It will be an embedded hypersurface locally, although globally it will in general have self-intersections and singularities. Equivalently, a ruled real hypersurface satisfies that $g(A\mathbb{D}, \mathbb{D}) = 0$. For examples see [12] or [15].

In 2007 Berndt and Tamaru, [3], gave a complete classification of homogeneous real hypersurfaces in $\mathbb{C}H^m$, $m \geq 2$, obtaining 6 types of real hypersurfaces including types (A_1), (A_2) and (B). The principal curvatures and eigenspaces of the other 3 types were described by Berndt and Díaz-Ramos, see [2]. Among them, what the authors call type S real hypersurfaces are either the ruled minimal real hypersurfaces W^{2m-1} introduced in 1988 by Lohnherr, [14], for $r = 0$ or an equidistant hypersurface to W^{2m-1} at a distance $r > 0$.

Real hypersurfaces satisfying $A\phi = \phi A$ were classified by Okumura in 1975, [20], for the case of the complex projective space and by Montiel and Romero in 1986, [18], for the case of the complex hyperbolic space:

Theorem 1 *Let M be a real hypersurface of $M_m(c)$, $m \geq 2$. Then $A\phi = \phi A$ if and only if M is locally congruent to a homogeneous hypersurface of either the types (A_1) or (A_2) in $\mathbb{C}P^m$ or either the types (A_1), (A_2) or (B) in $\mathbb{C}H^m$.*

The Tanaka-Webster connection, [31, 33], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold. As a generalization of this connection Tanno, [32], defined the generalized Tanaka-Webster connection for contact metric manifolds by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y \tag{3}$$

for any tangent X, Y. Let k be a nonzero number. Using the naturally extended affine connection of Tanno's generalized Tanaka-Webster connection, Cho, [7, 8], defined the k-th g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ for a real hypersurface in $M_m(c)$ by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{4}$$

for any X, Y tangent to M. Then $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}\xi = 0$, $\hat{\nabla}^{(k)}g = 0$, $\hat{\nabla}^{(k)}\phi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the k-th g-Tanaka-Webster connection coincides with the Tanaka-Webster connection.

2 Covariant Derivatives

Let M be a real hypersurface in a non-flat complex space form $M_n(c)$. On M we have the Levi-Civita connection and for any nonzero k the k-th g-Tanaka-Webster connection. Consider both covariant derivatives.

We have the tensor field of type (1, 2) on M given by the difference of both connections $F^{(k)}(X, Y) = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any X, Y tangent to M. We will call this tensor the k-th Cho tensor on M. Associated to it, for any X tangent to M and any non null k we can consider the tensor field of type (1, 1) $F^{(k)}_X Y = F^{(k)}(X, Y)$ for any Y tangent to M. This operator will be called the k-th Cho operator corresponding to X. The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $\hat{T}^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$, for any X, Y tangent to M.

Notice that if $X \in \mathbb{D}$, $F^{(k)}_X$ does not depend on k. In this case we will write simply F_X for $F^{(k)}_X$.

Consider any tensor T of type (1, 1) on M. We can study when the covariant derivatives associated to Levi-Civita and g-Tanaka-Webster connections coincide on T, that is, $\nabla T = \hat{\nabla}^{(k)} T$. This is equivalent to the fact that for any X tangent to M, $TF_X^{(k)} = F_X^{(k)} T$, and its geometrical meaning is that every eigenspace of T is preserved by the k-th Cho operator $F_X^{(k)}$.

On the other hand, as $TM = Span\{\xi\} \oplus \mathbb{D}$, we can weak the above condition to the cases $X = \xi$ or $X \in \mathbb{D}$.

For the case $T = A$, in [27, 28], we obtained the following results:

Theorem 2 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $F_X A = AF_X$ for any $X \in \mathbb{D}$, if and only if M is locally congruent to a ruled real hypersurface.*

and

Theorem 3 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $F_\xi^{(k)} A = AF_\xi^{(k)}$ for a nonnull constant k if and only if M is locally congruent to a type (A) real hypersurface.*

And as consequence of both theorems we get

Corollary 1 *There do not exist real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 3$, such that $F_X^{(k)} A = AF_X^{(k)}$, for any X tangent to M and a nonnull constant k.*

The structure Jacobi operator R_ξ of M is an important tool to study the geometry of M. It is defined by $R_\xi X = R(X, \xi)\xi$, for any X tangent to M. Therefore its expression is given by

$$R_\xi X = \varepsilon\{X - \eta(X)\xi\} + \alpha AX - \eta(AX)A\xi \tag{1}$$

If in our study we take $T = R_\xi$, in [21, 22] we have proved the following results

Theorem 4 *Let M be a real hypersurface in $M_m(c)$, $m \geq 2$. Then $F_X R_\xi = R_\xi F_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

and

Theorem 5 *Let M be a real hypersurface in $M_m(c)$, $m \geq 2$. Then $F_\xi^{(k)} R_\xi = R_\xi F_\xi^{(k)}$ for a nonnull k if and only if M is locally congruent either to a real hypersurface of type (A) or to a real hypersurface with $A\xi = 0$.*

As above we get

Corollary 2 *There do not exist real hypersurfaces M in $M_m(c)$, $m \geq 2$, such that $F_X^{(k)} R_\xi = R_\xi F_X^{(k)}$ for some nonnull constant k and any X tangent to M.*

The Ricci tensor of a real hypersurface M in $M_m(c)$ is given by

$$SX = \varepsilon\{(2m+1)X - 3\eta(X)\xi\} + hAX - A^2X \tag{2}$$

for any X tangent to M, where $h = trace(A)$.

It is well known that $M_n(c)$ does not admit real hypersurfaces with parallel Ricci tensor ($\nabla S = 0$). Therefore it is natural to investigate real hypersurfaces satisfying weaker conditions than the parallelism of S. Most important results on the study of the Ricci tensor of real hypersurfaces in non-flat complex space forms are included in Sect. 6 of [6].

We are going to suppose that $F_X S = SF_X$ for any $X \in \mathbb{D}$. This is equivalent to have

$$g(\phi AX, SY)\xi - \eta(SY)\phi AX = g(\phi AX, Y)S\xi - \eta(Y)S\phi AX \tag{3}$$

for any $X \in \mathbb{D}$, Y tangent to M. In [9] we prove the

Theorem 6 *There do not exist Hopf real hypersurfaces in $M_m(c)$, $m \geq 2$, whose Ricci tensor satisfies $F_X S = SF_X$ for any $X \in \mathbb{D}$.*

Therefore we can locally write $A\xi = \alpha\xi + \beta U$ for a unit $U \in \mathbb{D}$, where α and β are functions defined on M and $\beta \neq 0$. We also will call $\mathbb{D}_U = Span\{\xi, U, \phi U\}^{\perp}$. This is a holomorphic distribution in $\mathbb{D}$.

Taking the scalar product of (3) for $Y \in \mathbb{D}$ with Y yields $\eta(SY)g(\phi AX, Y) = 0$ for any $X, Y \in \mathbb{D}$. If $g(\phi AX, Y) = 0$ for any $X, Y \in \mathbb{D}$, M is a ruled real hypersurface. Therefore

Theorem 7 *Let M be a non Hopf real hypersurface in $M_m(c)$, $m \geq 2$, such that $F_X S = SF_X$ for any $X \in \mathbb{D}$. Then either M is ruled of $\eta(SY) = 0$ for any $Y \in \mathbb{D}$.*

Consider that $\eta(SY) = 0$ for any $Y \in \mathbb{D}$. It is easy to see that $AU = \beta\xi + (h - \alpha)U$ and $A\phi U = 0$. Therefore we have two possibilities

1. $h = \alpha$.
2. $h - \alpha \neq 0$. In this case we obtain $\beta^2 = \alpha(h - \alpha) - 3\varepsilon$. In the case of $\mathbb{C}P^m$ this yields $\alpha \neq 0$ and $h = \frac{\beta^2+\alpha^2+3}{\alpha}$ is also nonnull.

In the first case we obtain

Theorem 8 *Let M be a real hypersurface in $M_m(c)$, $m \geq 2$, such that $h = \alpha$. Then $F_X S = SF_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

So let us consider the second case for a real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$. We have seen that $\mathbb{D}_U$ is A-invariant. From (3) taking $Y \in \mathbb{D}_U$ such that $AY = \lambda Y$, we get either $\lambda = 0$ or if $\lambda \neq 0$, either $\lambda = h$ or $A\phi Y = 0$.

Not any eigenvalue in $\mathbb{D}_U$ can be zero, because in that case the type number is 2 and M should be ruled, giving a contradiction. Moreover there must be distinct than 0 and h and then $A\phi Y = 0$ for an eigenvector Y corresponding to such an eigenvalue.

By the Codazzi equation we see that $\phi T_0 \perp T_0$, where T_0 denotes the eigenspace corresponding to the eigenvalue 0 and $\phi T_h \perp T_h$ for the eigenspace corresponding to the eigenvalue h (that maybe does not appear). Thus we can write $\mathbb{D}_U = T_0 \oplus T_h \oplus \bar{\mathbb{D}}_U$. Where $\phi T_0 = T_h \oplus \bar{\mathbb{D}}_U$.

If either h or $\frac{\beta^2+3}{\alpha}$ is an eigenvalue in $\bar{\mathbb{D}}_U$ we can prove

$$grad(\beta) = (\beta^2 + 5)\phi U$$
$$grad(\alpha) = \frac{\alpha\beta(\beta^2 + 7)}{\beta^2 + 3}\phi U \tag{4}$$

and this provides a contradiction. Thus we have

Theorem 9 *Let M be a non Hopf real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, such that $\alpha = g(A\xi, \xi) \neq h$. Then $F_X S = SF_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a real hypersurface such that $A\xi = \alpha\xi + \beta U$, for a unit $U \in \mathbb{D}$, α and β are nonvanishing functions, $AU = \beta\xi + \frac{\beta^2+3}{\alpha}U$, $A\phi U = 0$ and $\mathbb{D}_U = T_0 \oplus \bar{\mathbb{D}}_U$. All eigenvalues in $\bar{\mathbb{D}}_U$ are nonnull and different from h and $\frac{\beta^2+3}{\alpha}$. Moreover the sum of all nonnull eigenvalues in $\bar{\mathbb{D}}_U$ is* 0.

Remark: The real hypersurface appearing in last theorem satisfies that $Ker(A) = Span\{\phi U\} \oplus T_0$ is an integrable distribution whose integral leaves are totally geodesic and totally real in M. Therefore they are $\mathbb{R}P^{m-1}$.

Now consider the Riemannian curvature tensor R of a real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, and suppose that $\nabla_X R = \hat{\nabla}_X^{(k)} R$ for any $X \in \mathbb{D}$. If M is non Hopf and we follow the above notation, we obtain that $A\phi U = 0$, $AX = 0$ for any $X \in \mathbb{D}_U$ and $\alpha g(AU, U)^2 = (\beta^2 + 3)g(AU, U)$. If $g(AU, U) = 0$, M is ruled. If not, $AU = \beta\xi + \frac{\beta^2+3}{\alpha}U$. Then by Codazzi equation applied to X and ϕX, $X \in \mathbb{D}_U$, we get

$$g([\phi X, X], U) = -\frac{2}{\beta} \tag{5}$$

and

$$\frac{\beta^2 + 3}{\alpha} g([\phi X, X], U) = 0. \tag{6}$$

Both equations give a contradiction.

If M is Hopf we obtain $\alpha = 0$. If $X \in \mathbb{D}$ satisfies $AX = \lambda X$ we get $-\lambda^2 A\phi X = 3\lambda\phi X$. If $\lambda = 0$, as $A\phi X = \mu\phi X$ we arrive to a contradiction, because μ does not exist. Therefore $\lambda \neq 0$ and $-3\lambda = \lambda$. This is impossible and we have, [24],

Theorem 10 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $\nabla_X R = \hat{\nabla}^{(k)}_X R$ for any $X \in \mathbb{D}$ and some nonzero constant k if and only if M is locally congruent to a ruled real hypersurface.*

If now $\nabla_\xi R = \hat{\nabla}^{(k)}_\xi R$ and M is non Hopf, we get $A\phi U = \gamma\phi U$ for a certain function γ and $k\alpha g(AX, U) = 0$ for any $X \in \mathbb{D}_U$. Thus either $\alpha = 0$ or $g(AU, X) = 0$ for any $X \in \mathbb{D}_U$.

If $\alpha = 0$ we can prove that $A\xi = \beta U$, $AU = \beta\xi$, $A\phi U = k\phi U$, where $k^2 = \beta + 3$. Moreover, as $\mathbb{D}_U$ is A-invariant, if $Y \in \mathbb{D}_U$ satisfies $AY = \lambda Y$, $A\phi Y = \frac{k\lambda-1}{k}\phi Y$. But we can also obtain $k^2\lambda\phi Y = k^2 A\phi Y$. Both expressions give a contradiction. Thus $\alpha \neq 0$.

After some work we get $grad(\beta) = (2 + \alpha\frac{\beta^2+3}{k} + 2\beta^2)\phi U$. From this we have $(\frac{\beta^2+3}{k})^2 + \beta^2 + 1 = 0$, which is impossible.

Therefore M must be Hopf and we obtain $\alpha(A\phi - \phi A)X = 0$ for any X tangent to M. If $A\phi = \phi A$, M must be of type (A). If $\alpha = 0$ we find that M has, at most, three distinct constant principal curvatures. Then (see [19]) M is locally congruent to a real hypersurface either of type (A) or of type (B). As type (B) real hypersurfaces do not have $\alpha = 0$, we obtain (see [24])

Theorem 11 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $\nabla_\xi R = \hat{\nabla}^{(k)}_\xi R$ for some nonnull constant k if and only if M is locally congruent to a type (A) real hypersurface.*

As a consequence

Corollary 3 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\nabla R = \hat{\nabla}^{(k)} R$ for some nonnull constant k.*

3 Lie Derivatives

Let $\mathscr{L}$ denote the Lie derivative of a real hypersurface M in $\mathbb{C}P^m$. Then $\mathscr{L}_X Y = \nabla_X Y - \nabla_Y X$ for any X, Y tangent to M. Moreover, for any tensor T of type $(1, 1)$ on M $(\mathscr{L}_X T)Y = \mathscr{L}_X TY - T\mathscr{L}_X Y$.

Associated to the k-th g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ on M we can consider the so-called Lie derivative associated to such a connection (introduced by Jeong, Pak and Suh in [10] for real hypersurfaces of complex two-plane Grassmannians) defined by $\hat{\mathscr{L}}^{(k)}_X Y = \hat{\nabla}^{(k)}_X Y - \hat{\nabla}^{(k)}_Y X$ for any X, Y tangent to M.

Suppose that $\mathscr{L}_\xi A = \hat{\mathscr{L}}^{(k)}_\xi A$. If M is non Hopf, this yields $AU = \beta\xi + kU$, $A\phi U = \frac{\alpha+k}{2}\phi U$ and $\mathbb{D}_U$ is A-invariant. But we also obtain $\frac{k-\alpha}{2}AU = \beta(\frac{k-\alpha}{2})\xi + (\frac{k^2-\alpha^2}{4} - \beta^2)U$. If $\alpha = k$ this yields $\beta^2 U = 0$, which is impossible. Therefore $AU = \beta\xi + \frac{2}{k-\alpha}(\frac{k^2-\alpha^2}{4} - \beta^2)U$. Both expressions for AU give $(k - \alpha)^2 = -4\beta^2$, which is impossible and M must be Hopf.

If M is Hopf it is easy to see that M must be of type (A). Therefore we have [23].

Theorem 12 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 2$. Then $\mathscr{L}_\xi A = \hat{\mathscr{L}}_\xi^{(k)} A$ for some nonnull k if and only if M is locally congruent to a real hypersurface of type (A).*

If now we suppose that $\mathscr{L}_X A = \hat{\mathscr{L}}_X^{(k)} A$ for any $X \in \mathbb{D}$ we can prove that M must be Hopf. In this case, if λ is a principal curvature in $\mathbb{D}$ we obtain

$$\lambda^2 + (k - \alpha)\lambda - k\alpha = 0. \tag{1}$$

Thus either $\lambda = \alpha$ or $\lambda = -k$ and M has, at most, two distinct constant principal curvatures. Therefore M must be locally congruent to a geodesic hypersphere, [5]. As M cannot be totally umbilical, there exists $Y \in \mathbb{D}$ such that $AY = -kY$. But then $A\phi Y = \alpha Y$. Therefore $\alpha \neq -k$ and this contradicts the fact that M is a geodesic hypersphere. Then

Theorem 13 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L}_X A = \hat{\mathscr{L}}_X^{(k)} A$ for some nonnull k and any $X \in \mathbb{D}$.*

As above

Corollary 4 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L} A = \hat{\mathscr{L}}^{(k)} A$ for some nonnull k.*

Now consider the structure Jacobi operator R_ξ on M. In [26] we proved the following

Theorem 14 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L}_\xi R_\xi = 0$. Then either M is locally congruent to a tube of radius $\frac{\pi}{4}$ over a complex submanifold in $\mathbb{C}P^m$ or to a real hypersurface of type (A) with radius $r \neq \frac{\pi}{4}$.*

Suppose now that $\mathscr{L}_\xi R_\xi = \hat{\mathscr{L}}_\xi^{(k)} R_\xi$. Then $(\phi A - A\phi)R_\xi = R_\xi(\phi A - A\phi)$. This yields (see [26]).

Theorem 15 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $\mathscr{L}_\xi R_\xi = \hat{\mathscr{L}}_\xi^{(k)} R_\xi$ for some nonnull k if and only if M is locally congruent to either a real hypersuface of type (A) and radius $r \neq \frac{\pi}{4}$ or to a tube of radius $\frac{\pi}{4}$ around a complex submanifold in $\mathbb{C}P^m$.*

If now we suppose $\mathscr{L}_X R_\xi = \hat{\mathscr{L}}_X^{(k)} R_\xi$ for any $X \in \mathbb{D}$ and M is non Hopf we get $\alpha g(A^2\phi U, U) = 0$.

If $\alpha = 0$, $A\xi = \beta U$, $AU = \beta\xi + kU$, $A\phi U = -k\phi U$. We also prove that the unique eigenvalue in $\mathbb{D}_U$ is k. Now the Codazzi equation yields $k = 0$, a contradiction.

Therefore $\alpha \neq 0$, $AU = \beta\xi + \omega U$, $A\phi U = \delta\phi U$, for some functions ω and δ. Then we obtain $\alpha^2 = 1$, $\omega = \frac{\beta^2-1}{\alpha} = k$, $\delta = k$. That is, $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + kU$, $A\phi U = kU$, $AZ = -\frac{1}{\alpha}Z$, for any $Z \in \mathbb{D}_U$. This case yields $4k^2 - \alpha k + 3 = 0$. There does not exist any k satisfying this equation. Therefore M must be Hopf.

Let X be a unit vector field in $\mathbb{D}$ such that $AX = \lambda X$. From [20], $A\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Then we have three possibilities

- $\lambda + \mu = 0$, $\lambda = k$. Then $k^2 = -1$, which is impossible.
- $\lambda + \mu = 0$, $\mu = -\frac{1}{\alpha}$. Then $\alpha^2 = -1$, also impossible.
- $\lambda = \mu = -\frac{1}{\alpha}$. Then $2 = 0$, also impossible.

Therefore we obtain

Theorem 16 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L}_X R_\xi = \hat{\mathscr{L}}_X^{(k)} R_\xi$ for any $X \in \mathbb{D}$ and some nonnull k.*

We also have the following corollaries

Corollary 5 *There do not exist real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, such that $\mathscr{L} R_\xi = \hat{\mathscr{L}}^{(k)} R_\xi$ for some nonnull k.*

and

Corollary 6 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull constant. Then $\hat{\mathscr{L}}_\xi^{(k)} R_\xi = 0$ if and only if M is locally congruent to either a tube of radius $\frac{\pi}{4}$ around a complex submanifold in $\mathbb{C}P^m$ or to a real hypersurface of type (A) and radius $r \neq \frac{\pi}{4}$.*

Acknowledgements Author is partially supported by MINECO-FEDER Grant MTM2013-47828-C2-1-P.

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Maximal Antipodal Subgroups of the Automorphism Groups of Compact Lie Algebras

Makiko Sumi Tanaka and Hiroyuki Tasaki

Abstract We classify maximal antipodal subgroups of the group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of a compact classical Lie algebra $\mathfrak{g}$. A maximal antipodal subgroup of $\mathrm{Aut}(\mathfrak{g})$ gives us as many mutually commutative involutions of $\mathfrak{g}$ as possible. For the classification we use our former results of the classification of maximal antipodal subgroups of quotient groups of compact classical Lie groups. We also use canonical forms of elements in a compact Lie group which is not connected.

1 Introduction

The group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of a compact semisimple Lie algebra $\mathfrak{g}$ is a compact semisimple Lie group which is not necessarily connected. The identity component of $\mathrm{Aut}(\mathfrak{g})$ is the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms. A subgroup of a compact Lie group is an antipodal subgroup if it consists of mutually commutative involutive elements. In this article we give a classification of maximal antipodal subgroups of $\mathrm{Aut}(\mathfrak{g})$ when $\mathfrak{g}$ is a compact classical semisimple Lie algebra $\mathfrak{su}(n)$ $(n \geq 2)$, $\mathfrak{o}(n)$ $(n \geq 5)$ or $\mathfrak{sp}(n)$ $(n \geq 1)$ (Theorem 4). A maximal antipodal subgroup $\mathrm{Aut}(\mathfrak{g})$ gives us as many mutually commutative involutions of $\mathfrak{g}$ as possible.

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Then G is a compact connected semisimple Lie group whose center Z is discrete. The quotient G/Z is isomorphic to $\mathrm{Int}(\mathfrak{g})$ via the adjoint representation. Therefore our results [5] of the classification of maximal antipodal subgroups of G/Z gives the classification of maximal antipodal subgroups of $\mathrm{Int}(\mathfrak{g})$. In order to consider the case where $\mathrm{Aut}(\mathfrak{g})$ is

M.S. Tanaka (✉)
Faculty of Science and Technology, Department of Mathematics,
Tokyo University of Science, Noda, Chiba 278-8510, Japan
e-mail: tanaka_makiko@ma.noda.tus.ac.jp

H. Tasaki
Faculty of Pure and Applied Sciences, Division of Mathematics,
University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan
e-mail: tasaki@math.tsukuba.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_4

not connected, we give a canonical form of an element of a disconnected Lie group (Proposition 3).

After we submitted the manuscript, we found Yu studied elementary abelian 2-subgroups of the automorphism group of compact classical simple Lie algebras in [6]. Elementary abelian 2-subgroups are nothing but antipodal subgroups.

2 Maximal Antipodal Subgroups of Quotient Lie Groups

In this section we refer to our former results in [5].

Although the notion of an antipodal set is originally defined as a subset of a Riemannian symmetric space M in [1], we give an alternative definition when M is a compact Lie group with a bi-invariant Riemannian metric.

Definition 1 Let G be a compact Lie group and we denote by e the identity element of G. A subset A of G satisfying $e \in A$ is called an *antipodal set* if A satisfies the following two conditions.

(i) Every element $x \in A$ satisfies $x^2 = e$.
(ii) Any elements $x, y \in A$ satisfy $xy = yx$.

Proposition 1 ([5]) *If a subset A of G satisfying $e \in A$ is a maximal antipodal set, then A is an abelian subgroup of G which is isomorphic to a product $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ of some copies of $\mathbb{Z}_2$. Here $\mathbb{Z}_2$ denotes the cyclic group of order* 2.

Let

$$\Delta_n := \left\{ \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \right\} \subset O(n).$$

For a subset $X \subset O(n)$ we define $X^{\pm} := \{x \in X \mid \det(x) = \pm 1\}$.

Proposition 2 (cf. [1]) *A maximal antipodal subgroup of $U(n)$, $O(n)$, $Sp(n)$ is conjugate to Δ_n. A maximal antipodal subgroup of $SU(n)$, $SO(n)$ is conjugate to Δ_n^+.*

Let

$$D[4] := \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix} \right\} \subset O(2),$$

which is a dihedral group. Let

$$Q[8] := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\},$$

which is the quaternion group, where $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are elements of the standard basis of the quaternions $\mathbb{H}$. We decompose a natural number n as $n = 2^k \cdot l$ into the product of the k-th power 2^k of 2 and an odd number l. For s with $0 \le s \le k$ we define

$$D(s,n) := D[4] \otimes \cdots \otimes D[4] \otimes \Delta_{n/2^s}$$
$$= \{d_1 \otimes \cdots \otimes d_s \otimes d_0 \mid d_1, \dots, d_s \in D[4], d_0 \in \Delta_{n/2^s}\} \subset O(n).$$

We always use k and l in the above meaning when we write $n = 2^k \cdot l$.

The center of $U(n)$ is $\{z1_n \mid z \in U(1)\}$ and we identify it with $U(1)$. Let $\mathbb{Z}_\mu$ be the cyclic group of degree μ which lies in the center of $U(n)$. Let $\pi_n : U(n) \to U(n)/\mathbb{Z}_\mu$ be the natural projection.

Theorem 1 ([5]) *Let $n = 2^k \cdot l$. Let θ be a primitive 2μ-th root of 1. Then a maximal antipodal subgroup of $U(n)/\mathbb{Z}_\mu$ is conjugate to one of the following.*

(1) *In the case where n or μ is odd, $\pi_n(\{1,\theta\}D(0,n)) = \pi_n(\{1,\theta\}\Delta_n)$.*
(2) *In the case where both n and μ are even, $\pi_n(\{1,\theta\}D(s,n))$ $(0 \le s \le k)$, where the case $(s,n) = (k-1, 2^k)$ is excluded.*

Remark 1 Since we have an inclusion $\Delta_2 \subsetneq D[4]$ which implies $D(k-1, 2^k) \subsetneq D(k, 2^k)$, the case $(s,n) = (k-1, 2^k)$ is excluded.

Theorem 2 ([5]) *Let n and μ be natural numbers where μ divides n. Let $n = 2^k \cdot l$. Let $\mathbb{Z}_\mu$ be the cyclic group of degree μ which lies in the center of $SU(n)$. Let θ be a primitive 2μ-th root of 1. Then a maximal antipodal subgroup of $SU(n)/\mathbb{Z}_\mu$ is conjugate to one of the following.*

(1) *In the case where n or μ is odd, $\pi_n(\Delta_n^+)$.*
(2) *In the case where both n and μ are even,*
  (a) *when $k = 1$, $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$ or $\pi_n((D^+[4] \cup \theta D^-[4]) \otimes \Delta_l)$, where $\pi_2(\Delta_2^+ \cup \theta\Delta_2^-)$ is excluded when $n = \mu = 2$.*
  (b) *When $k \ge 2$, under the expression $\mu = 2^{k'} \cdot l'$ where $1 \le k' \le k$ and l' divides l,*
    (b1) *if $k' = k$, $\pi_n(\Delta_n^+ \cup \theta\Delta_n^-)$ or $\pi_n(D(s,n))$ $(1 \le s \le k)$, where the case $(s,n) = (k-1, 2^k)$ is excluded.*
    (b2) *If $1 \le k' < k$, $\pi_n(\{1,\theta\}\Delta_n^+)$ or $\pi_n(\{1,\theta\}D(s,n))$ $(1 \le s \le k)$, where the case $(s,n) = (k-1, 2^k)$ is excluded and, moreover, $\pi_4(\{1,\theta\}\Delta_4^+)$ is excluded when $n = 4$.*

Remark 2 Since $\Delta_4^+ = \Delta_2 \otimes \Delta_2 \subsetneq D[4] \otimes D[4] = D(2,4)$, $\pi_4(\{1,\theta\}\Delta_4^+)$ is excluded.

Theorem 3 ([5]) *Let π_n be one of the natural projections $O(n) \to O(n)/\{\pm 1_n\}$, $SO(n) \to SO(n)/\{\pm 1_n\}$, $Sp(n) \to Sp(n)/\{\pm 1_n\}$. Let $n = 2^k \cdot l$.*

(I) *A maximal antipodal subgroup of $O(n)/\{\pm 1_n\}$ is conjugate to one of $\pi_n(D(s,n))$ $(0 \le s \le k)$, where the case $(s,n) = (k-1, 2^k)$ is excluded.*
(II) *When n is even, a maximal antipodal subgroup of $SO(n)/\{\pm 1_n\}$ is conjugate to one of the following.*
  (1) *In the case where $k = 1$, $\pi_n(\Delta_n^+)$ or $\pi_n(D^+[4] \otimes \Delta_l)$, where $\pi_2(\Delta_2^+)$ is excluded when $n = 2$.*

(2) *In the case where* $k \geq 2$, $\pi_n(\Delta_n^+)$ *or* $\pi_n(D(s,n))$ $(1 \leq s \leq k)$, *where the case* $(s,n) = (k-1, 2^k)$ *is excluded and moreover* $\pi_4(\Delta_4^+)$ *is excluded when* $n = 4$.

(III) *A maximal antipodal subgroup of* $Sp(n)/\{\pm 1_n\}$ *is conjugate to one of* $\pi_n(Q[8] \cdot D(s,n))$ $(0 \leq s \leq k)$, *where the case* $(s,n) = (k-1, 2^k)$ *is excluded.*

3 Canonical Forms of Elements of a Disconnected Lie Group

Let G be a compact connected Lie group and let T be a maximal torus of G. Then we have

$$G = \bigcup_{g \in G} gTg^{-1},$$

which means that a canonical form of an element of G with respect to conjugation is an element of T. We give a formulation of canonical forms of elements of G in the case where G is not connected. Let G_0 be the identity component of a compact Lie group G. Then G/G_0 is a finite group and we have

$$G = \bigcup_{[\tau] \in G/G_0} G_0 \tau,$$

where $[\tau]$ denotes the coset represented by $\tau \in G$.

Ikawa showed a canonical form of a certain action on a compact connected Lie group in [3, 4]. Using this canonical form we can obtain the following proposition.

Proposition 3 *For* $\tau \in G$ *we define an automorphism* I_τ *of* G_0 *by* $I_\tau(g) = \tau g \tau^{-1}$ $(g \in G_0)$. *Let* T_τ *be a maximal torus of* $F(I_\tau, G_0) := \{g \in G_0 \mid I_\tau(g) = g\}$. *Then we have*

$$G_0 \tau = \bigcup_{g \in G_0} g(T_\tau \tau) g^{-1}.$$

Therefore a canonical form of an element of a connected component $G_0 \tau$ of G with respect to conjugation under G_0 is an element of $T_\tau \tau$.

4 Maximal Antipodal Subgroups of the Automorphism Groups of Compact Lie Algebras

Let $\mathfrak{g}$ be a compact semisimple Lie algebra. Then the group $\mathrm{Aut}(\mathfrak{g})$ of automorphisms of $\mathfrak{g}$ is a compact semisimple Lie group which is not necessarily connected. By the definition of antipodal sets, the set of maximal antipodal subgroups of $\mathrm{Aut}(\mathfrak{g})$ is equal to the set of maximal subsets of $\mathrm{Aut}(\mathfrak{g})$ satisfying (i) each element has order 2 except for the identity element and (ii) all elements are commutative to each other.

Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Then G is a compact connected semisimple Lie group whose center Z is discrete. The quotient group G/Z is isomorphic to $\mathrm{Int}(\mathfrak{g})$ via the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. Hence the classification of maximal antipodal subgroups of G/Z gives the classification of maximal antipodal subgroups of $\mathrm{Int}(\mathfrak{g})$.

Theorem 4 *Let $n = 2^k \cdot l$ be a natural number.*

(I) *Let τ denote a map $\tau : \mathfrak{su}(n) \to \mathfrak{su}(n)\ ;\ X \mapsto \bar{X}$. A maximal antipodal subgroup of* $\mathrm{Aut}(\mathfrak{su}(n))$ *is conjugate to* $\{e, \tau\}\mathrm{Ad}(D(s, n))$ $(0 \leq s \leq k)$*, where the case $(s, n) = (k-1, 2^k)$ is excluded. Here e denotes the identity element of* $\mathrm{Aut}(\mathfrak{g})$.

(II) *A maximal antipodal subgroup of* $\mathrm{Aut}(\mathfrak{o}(n))$ *is conjugate to* $\mathrm{Ad}(D(s, n))$ $(0 \leq s \leq k)$*, where the case $(s, n) = (k-1, 2^k)$ is excluded.*

(III) *A maximal antipodal subgroup of* $\mathrm{Aut}(\mathfrak{sp}(n))$ *is conjugate to* $\mathrm{Ad}(Q[8] \cdot D(s, n))$ $(0 \leq s \leq k)$*, where the case $(s, n) = (k-1, 2^k)$ is excluded.*

Before we prove Theorem 4, we need some preparations. Let $\tau' : \mathbb{C}^n \to \mathbb{C}^n$ be the complex conjugation $\tau'(v) = \bar{v}$ for $v \in \mathbb{C}^n$. Since $\tau' \in GL(2n, \mathbb{R})$, $\{1_n, \tau'\}U(n)$ is a subset of $GL(2n, \mathbb{R})$. We have $g\tau' = \tau'\bar{g}$ for $g \in U(n)$. This implies $\mathrm{Ad}(\tau') = \tau$, so we identify τ' with τ. We can see that $\{1_n, \tau\}U(n)$ is a subgroup of $GL(2n, \mathbb{R})$ and the center is $\{\pm 1_n\}$. Let $\mathbb{Z}_\mu := \{z1_n \mid z \in U(1),\ z^\mu = 1\} \subset U(n)$. We can see that $\mathbb{Z}_\mu$ is a normal subgroup of $\{1_n, \tau\}U(n)$. Therefore $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$ is a Lie group. We have $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu = U(n)/\mathbb{Z}_\mu \cup \tau U(n)/\mathbb{Z}_\mu$, which is a disjoint union of the connected components.

Theorem 5 *Let $\pi_n : \{1_n, \tau\}U(n) \to \{1_n, \tau\}U(n)/\mathbb{Z}_\mu$ be the natural projection. Let θ be a primitive 2μ-th root of 1. Let $n = 2^k \cdot l$. Then a maximal antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$ is conjugate to one of the following by an element of $\pi_n(U(n))$.*

(1) *In the case where μ is odd, $\pi_n(\{1_n, \tau\}\{1, \theta\}\Delta_n) = \pi_n(\{1_n, \tau\}\Delta_n)$.*

(2) *In the case where μ is even, $\pi_n(\{1_n, \tau\}\{1, \theta\}D(s, n))$ $(0 \leq s \leq k)$, where the case $(s, n) = (k-1, 2^k)$ is excluded.*

Remark 3 Since $\{1_n, \tau\}\{1, \theta\}\Delta_n \subset \{1_n, \tau\}U(n) \subset GL(2n, \mathbb{R})$, we can consider $\pi_n(\{1_n, \tau\}\{1, \theta\}\Delta_n)$.

Lemma 1 *Let A be a maximal antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$. Then we have $A \cap \tau U(n)/\mathbb{Z}_\mu \neq \emptyset$.*

Proof We assume $A \subset U(n)/\mathbb{Z}_\mu$. By taking conjugation by $U(n)/\mathbb{Z}_\mu$ we can assume $A = \pi_n(\{1, \theta\}D(s, n))$ by Theorem 1. Since $\pi_n(\tau)\pi_n(\theta 1_n) = \pi_n(\theta 1_n)\pi_n(\tau)$, $A \cup \pi_n(\tau)A$ is an antipodal, which contradicts the maximality of A.

Lemma 2 *Let A be a maximal antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$. Let θ be a primitive 2μ-th root of 1. Then $\pi_n(\theta 1_n) \in A$.*

Proof Since we showed that $\pi_n(\theta 1_n)$ and $\pi_n(\tau)$ are commutative in the proof of Lemma 1, $\pi_n(\theta 1_n)$ is commutative with every element of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$. Hence $\pi_n(\theta 1_n) \in A$.

Lemma 3 *A maximal antipodal subgroup of* $\{1_n, \tau\}U(n)$ *is conjugate to* $\{1_n, \tau\}\Delta_n$ *by an element of* $U(n)$.

Proof Let A be a maximal antipodal subgroup of $\{1_n, \tau\}U(n)$. Then $A \cap \tau U(n) \neq \emptyset$ by Lemma 1 for $\mu = 1$. We set $R(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ and $r = \lfloor \frac{n}{2} \rfloor$. Then

$$T = \left\{ \begin{bmatrix} R(\phi_1) & & & \\ & \ddots & & \\ & & R(\phi_r) & \\ & & & (1) \end{bmatrix} \middle| \phi_j \in \mathbb{R}\ (1 \leq j \leq r) \right\}$$

is a maximal torus of $O(n) = F(\tau, U(n))$. Here (1) in the above definition of T means 1 when $n = 2r + 1$ and nothing when $n = 2r$. By Proposition 3 we have

$$\tau U(n) = \bigcup_{g \in U(n)} g(\tau T)g^{-1}.$$

Therefore, by retaking A under the conjugation by $U(n)$ if necessary, we may assume that $A \cap \tau U(n)$ has an element $\tau g_0 \in \tau T$. Since $1_n = (\tau g_0)^2 = g_0^2$, we have $g_0 \in \Delta_n$. Applying $\sqrt{-1}\tau\sqrt{-1}^{-1} = -\tau$ to a diagonal element -1 of g_0, we have $\tau g_0 = g_1 \tau 1_n g_1^{-1}$ for a suitable $g_1 \in U(n)$ which is a diagonal matrix whose diagonal elements are $1, \sqrt{-1}$.[1] Therefore if we retake A under the conjugation by $U(n)$ if necessary, we may assume $\tau \in A$. Hence $A \cap \tau U(n) = \tau(A \cap U(n))$. Since $\tau \in A$ and A is commutative, we have $A \cap U(n) \subset O(n)$. We show that $A \cap U(n)$ is a maximal antipodal subgroup of $O(n)$. If there is an antipodal subgroup $\tilde{A}$ which satisfies $A \cap U(n) \subset \tilde{A} \subset O(n)$, then $\{1_n, \tau\}\tilde{A}$ is an antipodal subgroup of $\{1_n, \tau\}U(n)$ and we have $A = (A \cap U(n)) \cup (A \cap \tau U(n)) = \{1_n, \tau\}(A \cap U(n)) \subset \{1_n, \tau\}\tilde{A}$. By the maximality of A we have $A = \{1_n, \tau\}\tilde{A}$, hence $A \cap U(n) = \tilde{A}$. Therefore $A \cap U(n)$ is a maximal antipodal subgroup of $O(n)$. By Proposition 2, $A \cap U(n)$ is conjugate to Δ_n by $O(n)$. Hence $A = \{1_n, \tau\}(A \cap U(n))$ is conjugate to $\{1_n, \tau\}\Delta_n$ by $O(n)$. Therefore any maximal antipodal subgroup of $\{1_n, \tau\}U(n)$ is conjugate to $\{1_n, \tau\}\Delta_n$ by an element of $U(n)$.

We prove Theorem 5.

Proof Since we proved the case of $\mu = 1$ in Lemma 3, we assume $\mu > 1$. We note that $\bar{\theta} \neq \theta$ in this case. Let A be a maximal antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_\mu$

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[1] We have $\{\tau g \mid g \in U(n),\ (\tau g)^2 = 1_n\} = \bigcup_{g \in U(n)} g\tau 1_n g^{-1}$. It is remarkable in contrast to $\{g \in U(n) \mid g^2 = 1_n\} = \bigcup_{g \in U(n)} g\Delta_n g^{-1}$.

and we set $B = \pi_n^{-1}(A)$. Then $\theta \in B$ by Lemma 2. Since $A \cap \tau U(n)/\mathbb{Z}_\mu \neq \emptyset$ by Lemma 1, we have $B \cap \tau U(n) \neq \emptyset$. Therefore, by retaking A under the conjugation by $U(n)/\mathbb{Z}_\mu$ if necessary, we may assume that $B \cap \tau U(n)$ has an element $\tau g_0 \in \tau T$, where T is a maximal torus of $O(n)$ defined in the proof of Lemma 3. By a similar argument as in the proof of Lemma 3 we may assume $g_0 = 1_n$. Thus $\tau \in B$. We note that B is not commutative because $\tau\theta = \bar{\theta}\tau \neq \theta\tau$. Since $\pi_n(\tau) \in A$, we have

$$A = (A \cap \pi_n(U(n))) \cup (A \cap \pi_n(\tau U(n))) = \pi_n(\{1_n, \tau\})(A \cap \pi_n(U(n))).$$

We consider $A \cap \pi_n(U(n))$. Since every element of $A \cap \pi_n(U(n))$ is commutative with $\pi_n(\tau)$, $A \cap \pi_n(U(n)) \subset \{\pi_n(u) \mid u \in U(n),\ \pi_n(\tau u) = \pi_n(u\tau)\}$. Since $u\tau = \tau\bar{u}$, the condition $\pi_n(\tau u) = \pi_n(u\tau)$ is equivalent to $\pi_n(u) = \pi_n(\bar{u})$, which is equivalent to the condition that there exists an integer m such that $\theta^{2m}u = \bar{u}$. Hence we have $\theta^m u = \theta^{-m}\bar{u} = \overline{\theta^m u}$, which means $\theta^m u \in O(n)$. When m is even, we have $\pi_n(\theta^m u) = \pi_n(u)$. Thus $\pi_n(u) \in \pi_n(O(n))$. When m is odd, we have $\pi_n(\theta^m u) = \pi_n(\theta u)$. Hence $\pi_n(u) = \pi_n(\theta 1_n)^{-1}\pi_n(\theta^m u) = \pi_n(\theta 1_n)\pi_n(\theta^m u)$. Thus $\pi_n(u) \in \pi_n(\theta 1_n)\pi_n(O(n))$. Therefore

$$A \cap \pi_n(U(n)) \subset \pi_n(\{1, \theta\}O(n)).$$

We consider the case where μ is odd. We have $\pi_n(\theta 1_n) = \pi_n(\theta^\mu 1_n) = \pi_n(-1_n)$. Hence $\pi_n(\{1, \theta\}O(n)) = \pi_n(O(n))$. Since $-1_n \notin \operatorname{Ker}\pi_n$, we have $O(n) \cap \operatorname{Ker}\pi_n = \{1_n\}$ and the restriction $\pi_n|_{O(n)}$ gives an isomorphism from $O(n)$ onto $\pi_n(O(n))$. Hence we have $\pi_n(\{1, \theta\}O(n)) = \pi_n(O(n)) \cong O(n)$. Therefore $A \cap \pi_n(U(n))$ is conjugate to $\pi_n(\Delta_n)$ by an element of $\pi_n(O(n))$ by Proposition 2. Hence A is conjugate to $\pi_n(\Delta_n) \cup \pi_n(\tau)\pi_n(\Delta_n) = \pi_n(\{1_n, \tau\}\Delta_n)$ by an element of $\pi_n(U(n))$.

We consider the case where μ is even. In this case $\pi_n(\{1, \theta\}O(n)) = \pi_n(O(n)) \cup \pi_n(\theta O(n))$ is a disjoint union. We show that $A \cap \pi_n(O(n))$ is a maximal antipodal subgroup of $\pi_n(O(n))$. Let $\tilde{A}$ be an antipodal subgroup which satisfies $A \cap \pi_n(O(n)) \subset \tilde{A} \subset \pi_n(O(n))$. Since every element of $\tilde{A}$ is commutative with $\pi_n(\tau)$, it turns out that $\pi_n(\{1_n, \tau\}\{1, \theta\})\tilde{A}$ is an antipodal subgroup of $\pi_n(\{1_n, \tau\}U(n)))$. We have $A \cap \pi_n(U(n)) = \pi_n(\{1_n, \theta 1_n\})(A \cap \pi_n(O(n)))$. Therefore

$$A = \pi_n(\{1_n, \tau\}\{1_n, \theta 1_n\})(A \cap \pi_n(O(n))) \subset \pi_n(\{1_n, \tau\}\{1_n, \theta 1_n\})\tilde{A}.$$

By the maximality of A we have $A = \pi_n(\{1_n, \tau\}\{1_n, \theta 1_n\})\tilde{A}$. Moreover, we have $A \cap \pi_n(O(n)) = \tilde{A}$. Thus $A \cap \pi_n(O(n))$ is a maximal antipodal subgroup of $\pi_n(O(n))$. Since μ is even, we have $-1_n \in \operatorname{Ker}\pi_n$. Hence $\pi_n(O(n)) \cong O(n)/\{\pm 1_n\}$. We decompose n as $n = 2^k \cdot l$. By Theorem 3, $A \cap \pi_n(O(n))$ is conjugate to $\pi_n(D(s, n))$ $(0 \leq s \leq k)$ by an element of $\pi_n(O(n))$. Here the case $(s, n) = (k-1, 2^k)$ is excluded. Therefore A is conjugate to $\pi_n(\{1_n, \tau\}\{1_n, \theta 1_n\}D(s, n))$ by $\pi_n(O(n))$.

We prove Theorem 4.

Proof We have Aut($\mathfrak{g}$)= Int($\mathfrak{g}$) when $\mathfrak{g} = \mathfrak{o}(n)$ where n is odd and $\mathfrak{g} = \mathfrak{sp}(n)$. Hence we obtain (II) when n is odd and (III) by Theorem 3. In general we have

$\mathrm{Aut}(\mathfrak{o}(n)) \cong O(n)/\{\pm 1_n\}$ if $n \neq 8$. Hence we obtain (II) when n is even and $n \neq 8$ by Theorem 3. We consider $\mathrm{Aut}(\mathfrak{o}(8))$. It is known that $\mathrm{Aut}(\mathfrak{o}(8))/\mathrm{Int}(\mathfrak{o}(8)) \cong S_3$, where S_3 denotes the symmetric group of degree 3. S_3 has three elements of order 2, denoted by τ_1, τ_2, τ_3, and two elements of order 3. Using these we can see that if A is an antipodal subgroup of $\mathrm{Aut}(\mathfrak{o}(8))$, there is $\tau \in \mathrm{Aut}(\mathfrak{o}(8))$ which satisfies that the coset $\tau\mathrm{Int}(\mathfrak{o}(8))$ corresponds to $\tau_i \in S_3$ for some $i \in \{1, 2, 3\}$ such that $A \subset \mathrm{Int}(\mathfrak{o}(8)) \cup \tau\mathrm{Int}(\mathfrak{o}(8))$. Therefore a maximal antipodal subgroup of $\mathrm{Aut}(\mathfrak{o}(8))$ is conjugate to a maximal antipodal subgroup of $O(8)/\{\pm 1_8\}$. Hence we obtain (II) when $n = 8$.

Finally we prove (I). The adjoint representation $\mathrm{Ad} : \{1_n, \tau\}SU(n) \to \mathrm{Aut}(\mathfrak{su}(n))$ is surjective (cf. [2, Chap. IX, Corollary 5.5, Chap. X, Theorem 3.29]). We have $\mathrm{Ker}\,\mathrm{Ad} = Z_{\{1_n,\tau\}SU(n)}(SU(n)) = \mathbb{Z}_n$, where $Z_{\{1_n,\tau\}SU(n)}(SU(n))$ denotes the centralizer of $SU(n)$ in $\{1_n, \tau\}SU(n)$ and $\mathbb{Z}_n = \{z1_n \mid z \in \mathbb{C},\ z^n = 1\}$. Thus we obtain an isomorphism $\mathrm{Aut}(\mathfrak{su}(n)) \cong \{1_n, \tau\}SU(n)/\mathbb{Z}_n$. Therefore we determine maximal antipodal subgroups of $\{1_n, \tau\}SU(n)/\mathbb{Z}_n$.

Let $\pi_n : \{1_n, \tau\}SU(n) \to \{1_n, \tau\}SU(n)/\mathbb{Z}_n$ denote the natural projection. We decompose n as $n = 2^k \cdot l$. Let θ be a primitive $2n$-th root of 1. Let A be a maximal antipodal subgroup of $\{1_n, \tau\}SU(n)/\mathbb{Z}_n$. Since $\{1_n, \tau\}SU(n)/\mathbb{Z}_n$ is a subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_n$, A is an antipodal subgroup of $\{1_n, \tau\}U(n)/\mathbb{Z}_n$. Hence there is a maximal antipodal subgroup $\tilde{A}$ of $\{1_n, \tau\}U(n)/\mathbb{Z}_n$ such that $A = \tilde{A} \cap \{1_n, \tau\}SU(n)/\mathbb{Z}_n$. By Theorem 5, $\tilde{A}$ is conjugate by an element of $\pi_n(U(n))$ to $\pi_n(\{1_n, \tau\}\{1, \theta\}D(s, n))$, where $s = 0$ when n is odd and $0 \leq s \leq k$ when n is even, moreover, the case $(s, n) = (k - 1, 2^k)$ is excluded. Hence there is $g \in U(n)$ such that

$$\tilde{A} = \pi_n(g)\pi_n(\{1_n, \tau\}\{1, \theta\}D(s, n))\pi_n(g)^{-1} = \pi_n(g\{1_n, \tau\}\{1, \theta\}D(s, n)g^{-1}).$$

We can write $g = g_1 z$ where $g_1 \in SU(n)$ and $z \in U(1)$. Then

$$g\{1_n, \tau\}\{1, \theta\}D(s, n)g^{-1} = g_1\{1_n, \tau z^{-2}\}\{1, \theta\}D(s, n)g_1^{-1}.$$

Hence $\tilde{A}$ is conjugate to $\pi_n(\{1_n, \tau z^{-2}\}\{1, \theta\}D(s, n))$ by an element of $\pi_n(SU(n))$. Since $A = \tilde{A} \cap \pi_n(\{1_n, \tau\}SU(n))$, A is conjugate to

$$\begin{aligned}&\pi_n(\{1_n, \tau z^{-2}\}\{1, \theta\}D(s, n)) \cap \pi_n(\{1_n, \tau\}SU(n))\\ &= \pi_n(\{1, \theta\}D(s, n)) \cap \pi_n(SU(n)) \cup \pi_n(\tau)\left(\pi_n(z^{-2}\{1, \theta\}D(s, n)) \cap \pi_n(SU(n))\right)\end{aligned}$$

by an element of $\pi_n(SU(n))$. In the proof of Theorem 2 ([5]) we showed

$$\pi_n(\{1, \theta\}D(s, n)) \cap \pi_n(SU(n)) = \pi_n(D^+(s, n) \cup \theta D^-(s, n)).$$

We consider $\pi_n(z^{-2}\{1, \theta\}D(s, n)) \cap \pi_n(SU(n))$. We show

$$\pi_n(z^{-2}\{1, \theta\}D(s, n)) \cap \pi_n(SU(n)) = \pi_n(z^{-2}\{1, \theta\}D(s, n) \cap SU(n)).$$

It is clear $\pi_n(z^{-2}D(s,n)) \cap \pi_n(SU(n)) \supset \pi_n(z^{-2}D(s,n) \cap SU(n))$. Conversely, for $d \in D(s,n)$, $\pi_n(z^{-2}d) \in \pi_n(SU(n))$ holds if and only if $\theta^{2m}z^{-2}d \in SU(n)$ for some m. Since $\det(\theta^{2m}z^{-2}d) = \theta^{2mn}z^{-2n}\det(d) = z^{-2n}\det(d)$, $\theta^{2m}z^{-2}d \in SU(n)$ is equivalent to $z^{-2n}\det(d) = 1$. Since $d \in D(s,n)$, $\det(d) = \pm 1$. When $\det(d) = 1$, $z^{-2n}\det(d) = 1$ is equivalent to $z^{-2n} = 1$. Hence $z^{-2} \in \mathrm{Ker}\,\pi_n$. Therefore $\pi_n(z^{-2}d) \in \pi_n(SU(n))$ is equivalent to $\pi_n(d) \in \pi_n(SU(n))$ when $d \in D^+(s,n)$. When $\det(d) = -1$, $z^{-2n}\det(d) = 1$ is equivalent to $z^{-2n} = -1$, that is, $z^{2n} = -1$. Hence $\pi_n(z^2 1_n) = \pi_n(\theta 1_n)$. Therefore $\pi_n(z^{-2}d) \in \pi_n(SU(n))$ is equivalent to $\pi_n(\theta d) \in \pi_n(SU(n))$ when $d \in D^-(s,n)$. Thus we obtain $\pi_n(z^{-2}D(s,n)) \cap \pi_n(SU(n)) \subset \pi_n(z^{-2}D(s,n) \cap SU(n))$. Moreover, we obtain $\pi_n(z^{-2}D(s,n) \cap SU(n)) = \pi_n(D^+(s,n) \cup \theta D^-(s,n))$ by the argument above. As a consequence, A is conjugate to $\pi_n(\{1_n, \tau\}(D^+(s,n) \cup \theta D^-(s,n)))$, where $s = 0$ when n is odd and $0 \leq s \leq k$ when n is even, moreover, the case $(s,n) = (k-1, 2^k)$ is excluded. The isomorphism between $\pi_n(\{1_n, \tau\}SU(n))$ and $\mathrm{Ad}(\mathfrak{su}(n))$ is given by

$$\pi_n(\{1_n, \tau\}SU(n)) \ni \pi_n(x) \mapsto \mathrm{Ad}(x) \in \mathrm{Ad}(\mathfrak{su}(n)) \quad (x \in \{1_n, \tau\}SU(n)).$$

Hence $\pi_n(\{1_n, \tau\}(D^+(s,n) \cup \theta D^-(s,n)))$ corresponds to $\mathrm{Ad}(\{1_n, \tau\}D(s,n)$ under the isomorphism, because $\mathrm{Ad}(\theta 1_n) = e$. Hence we obtain (I).

Acknowledgements The authors would like to thank Osamu Ikawa for his providing information about Proposition 3. The first author was partly supported by the Grant-in-Aid for Science Research (C) (No. 15K04855), Japan Society for the Promotion of Science. The second author was partly supported by the Grant-in-Aid for Science Research (C) (No. 15K04835), Japan Society for the Promotion of Science.

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A Nearly Kähler Submanifold with Vertically Pluri-Harmonic Lift

Kazuyuki Hasegawa

Abstract We consider a certain lift from an almost Hermite submanifold to the bundle of partially complex structures of the ambient manifold. In particular, nearly Kähler submanifolds in Euclidean spaces such that the lifts are vertically pluri-harmonic are studied.

1 Introduction

Kähler submanifolds in Euclidean spaces are studied very well (see [3], [5] and [11]), in particular, with pluri-harmonic Gauss maps. Although compact totally umbilic submanifolds in Euclidean spaces are the standard spheres, they can not admit any Kähler structure except dimension equals two. In [9], it is shown that the six-dimensional sphere can not admit any integrable complex structure compatible with the standard metric. However the six-dimensional sphere has a nearly Kähler structure. From the viewpoint of submanifold geometry, it is interesting to consider nearly Kähler submanifolds which are not necessary Kähler. In this note, we study nearly Kähler submanifolds in Euclidean spaces and obtain local characterization of the six dimensional sphere with nearly Kähler structure in Euclidean space, and we give an extrinsically decomposition of an immersion with vertically pluri-harmonic lift.

2 The Bundle of Partially Complex Structures

Let V be a real vector space of $\dim V = 2n + k$ with inner product $\langle \cdot, \cdot \rangle$. A partially complex structure on V consists of a subspace W of V of $\dim W = 2n$ and

K. Hasegawa (✉)
Faculty of Teacher Education, Institute of Human and Social Sciences, Kanazawa University, Kakuma-machi, Kanazawa, Ishikawa 920-1192, Japan
e-mail: kazuhase@staff.kanazawa-u.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_5

endomorphism $f : W \to W$ satisfying $f^2 = -id$ and $\langle fx, fy\rangle = \langle x, y\rangle$ for all x, $y \in W$. Such f can be extended to an endomorphism of V by defining $f|_{W^\perp} = \{0\}$, where $W^\perp$ is the orthogonal complement of W with respect to $\langle\cdot,\cdot\rangle$. We use the same letter f for this extended endomorphism. Then we have $f^3 + f = 0$, and hence it is so-called f-structure. We set

$$F_{2n}(V) := \{\lambda \in \mathrm{End}(V) \mid \lambda \text{ is } f\text{-structure with} \dim \mathrm{Im}\lambda = 2n\},$$

where $\mathrm{End}(V)$ is the set of all linear maps from V into itself. The tangent space $T_\lambda F_{2n}(V)$ at λ of $F_{2n}(V)$ can be identified with

$$\{\alpha \in \mathrm{End}(V) \mid \alpha\lambda^2 + \lambda\alpha\lambda + \lambda^2\alpha + \alpha = 0\}.$$

Let π_λ be the orthogonal projection from $\mathrm{End}(V)$ onto $T_\lambda F_{2n}(V)$ at $\lambda \in F_{2n}(V)$. We define an almost complex structure $\mathscr{J}$ of $F_{2n}(V)$ by $\mathscr{J}_\lambda(\alpha) = -\alpha\lambda - \lambda^2\alpha\lambda + \lambda\alpha$ for $\alpha \in T_\lambda F_{2n}(V)$ at each $\lambda \in F_{2n}(V)$. We refer to [4] for the detail.

Throughout this paper, all manifolds and maps are assumed to be smooth. Let E be a vector bundle over a manifold M and E_x the fiber of E over $x \in M$. We write TM for the tangent bundle of M and $\mathrm{End}(E)$ for the vector bundle whose fiber $\mathrm{End}(E)_x$ over $x \in M$ is the space of all linear maps from E_x into itself. Let $\varphi : N \to M$ be a smooth map and F a fiber bundle over M. The pull back bundle of F over N by φ is denoted by $\varphi^\# F$. The space of all sections of a fiber bundle F is denoted by $\Gamma(F)$.

Let $(\tilde{M}, \tilde{g})$ be a $(2n + k)$-dimensional Riemannian manifold with a Riemannian metric $\tilde{g}$. We define a fiber bundle $\mathscr{F}_{2n}(\tilde{M})$ over $\tilde{M}$ by

$$\mathscr{F}_{2n}(\tilde{M}) := \bigcup_{x\in\tilde{M}} F_{2n}(T_x\tilde{M}).$$

We refer to [10]. The bundle projection $p : \mathscr{F}_{2n}(\tilde{M}) \to \tilde{M}$ and the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ induce the vertical and horizontal subbundles of $T\mathscr{F}_{2n}(\tilde{M})$. The almost partially complex structure J_ε on $\mathscr{F}_{2n}(\tilde{M})$ $(\varepsilon = \pm 1)$ is defined by $(J_\varepsilon)_J(X) = (J(p_*(X)))^h_J$ for all horizontal vectors X at $J \in \mathscr{F}_{2n}(\tilde{M})$ and $(J_\varepsilon)_J(Y) = \varepsilon\mathscr{J}(Y)$ for all vertical vectors Y at $J \in \mathscr{F}_{2n}(\tilde{M})$, where $(\,\cdot\,)^h$ stands for the horizontal lift and $\mathscr{J}$ is the almost complex structure on each fiber defined above.

Let (M, g, I) be an almost Hermite manifold of $\dim M = 2n$, by definition, where I is an almost complex structure on M and g is a Riemannian metric compatible with I. Let $\mathbf{R}^m$ be the m-dimensional Euclidean spaces. Consider an isometric immersion $f : M \to \tilde{M}$. We omit the symbol of the differential map f_* if there are no confusions. The second fundamental form (resp. the shape operator) of f is denoted by α (resp. S). The mean curvature vector filed of f is denoted by H. Let $T^\perp M$ be the normal bundle of f. We define a lift $\tilde{I} : M \to \mathscr{F}_{2n}(\tilde{M})$ by $\tilde{I}(X) = I(X)$ and $\tilde{I}(\xi) = 0$ for $X \in TM$ and $\xi \in T^\perp M$. The following lemmas can be obtained by straightforward calculations.

Lemma 1 *We have*

$$(\tilde{\nabla}_X \tilde{I})(Y) = (\nabla_X I)(Y) + \alpha(X, IY), \ \ (\tilde{\nabla}_X \tilde{I})(\xi) = I S_\xi X$$

for $X, Y \in TM$, $\xi \in T^\perp M$.

Lemma 2 *Let* $\Phi \in \mathrm{End}(f^\#(T\tilde{M}))$. *We have*

$$(\pi_{\tilde{I}}(\Phi))(X) = \frac{1}{2}(\Phi(X))^\top + (\Phi(X))^\perp + \frac{1}{2}I(\Phi(IX))^\top, \ \ (\pi_{\tilde{I}}(\Phi))(\xi) = (\Phi(\xi))^\top$$

for $X \in TM$, $\xi \in T^\perp M$, *where* $(\ \cdot\ )^\top$ (*resp.* $(\ \cdot\ )^\perp$) *denotes the tangential* (*resp. normal*) *component of* $(\ \cdot\ )$.

For a vector bundle valued $(0, 2)$-tensor θ on M, we define

$$\theta^-(X, Y) = \frac{1}{2}(\theta(X, Y) - \theta(IX, IY)), \ \ \theta^+(X, Y) = \frac{1}{2}(\theta(X, Y) + \theta(IX, IY)),$$

where $X, Y \in TM$. By Lemmas 1 and 2, we have the following.

Proposition 1 *Let* (M, g, I) *be an almost Hermite manifold of* $\dim M = 2n$ *and* $f : (M, g) \to (\tilde{M}, \tilde{g})$ *an isometric immersion. Then the lift* $\tilde{I} : M \to \mathscr{F}_{2n}(\tilde{M})$ *is holomorphic with respect to* J_1 (*resp.* J_{-1}) *if and only if* $(\nabla_{IX} I)(Y) = I(\nabla_X I)(Y)$ (*resp.* $(\nabla_{IX} I)(Y) = -I(\nabla_X I)(Y)$) *for all* $X, Y \in TM$ *and* $\alpha^- = 0$ (*resp.* $\alpha^+ = 0$).

Note that $(\nabla_{IX} I)(Y) = I(\nabla_X I)(Y)$ holds for all $X, Y \in TM$ if and only if I is integrable. An almost Hermite manifold satisfying $(\nabla_{IX} I)(Y) = -I(\nabla_X I)(Y)$ for all $X, Y \in TM$ is called $(1, 2)$- symplectic. In particular, nearly Kähler manifolds, that is, ∇I is skew-symmetric, are $(1, 2)$-symplectic.

In the case of $\dim M = 2$, Proposition 1 corresponds to Theorem 7.1 in [12]. Let $G_k(\mathbf{C}^m)$ be the complex Grassmannian of all complex k-planes in $\mathbf{C}^m$. A k-plane $E \in G_k(\mathbf{C}^m)$ is said to be isotropic if the bilinear form $\langle x, y \rangle = \sum_{i=1}^m x_i y_j$ on $\mathbf{C}^m$ vanishes on $E \times E$. The set of all isotropic k-planes is denoted by $H_k(\mathbf{C}^m)$. Note that $H_k(\mathbf{C}^m) = F_{2k}(\mathbf{R}^m)$ and $H_k(\mathbf{C}^m) \subset G_k(\mathbf{C}^m)$ is a complex submanifold in $G_k(\mathbf{C}^m)$. For an almost Hermite manifold (M, g, I) of $\dim M = 2n$ isometrically immersed in $\mathbf{R}^m$ by f, we can define the complex Gauss map $\tau : M \to G_n(\mathbf{C}^m)$ by

$$\tau(x) := f_*(T_x M^{(0,1)}) = \{f_*(X) - \sqrt{-1} f_*(IX) \mid X \in T_x M\}.$$

It is easy to see $\tau(M) \subset H_n(\mathbf{C}^m)$. Since $\mathscr{F}_{2n}(\mathbf{R}^m) = \mathbf{R}^m \times H_n(\mathbf{C}^m)$, we have $\iota \circ \pi_2 \circ \tilde{I} = \tau$, where $\iota : H_n(\mathbf{C}^m) \to G_n(\mathbf{C}^m)$ is the inclusion and $\pi_2 : \mathbf{R}^m \times H_n(\mathbf{C}^m) \to H_n(\mathbf{C}^m)$ is the projection onto the second factor $H_n(\mathbf{C}^m)$. From Proposition 1, we have the following fact (Theorem 4 in [3]).

Corollary 1 *Let* (M, g, I) *be a Kähler manifold. An isometric immersion* $f : M \to \mathbf{R}^m$ *is pluriminimal, that is,* $\alpha^+ = 0$ *if and only if* $\pi_2 \circ \tilde{I} : M \to H_n(\mathbf{C}^m)$ *is holomorphic.*

Although the six dimensional sphere S^6 does not admit any Kähler structure for topological reasons, the six dimensional sphere S^6 with the standard metric g_0 has a nearly Kähler structure as follows. Let $\times$ be the vector cross product on $\mathbf{R}^7$ induced by the Cayley multiplication. We define an almost complex structure I on the six dimensional sphere $S^6 \subset \mathbf{R}^7$ by $I_x(X) = x \times X$ for $X \in T_x S^6$ at $x \in S^6$. It is well-known that I is a nearly Kähler. For almost complex structures compatible with g_0, the following fact is essentially proved in [9].

Corollary 2 *Let (M, g, I) be an Hermite manifold of* $\dim M = 2n$. *Let $f : M \to \mathbf{R}^m$ is an isometric immersion such that $\pi_2 \circ \tilde{I}$ is an immersion. If $\alpha^- = 0$, then M admits a Kähler structure. In particular, any almost complex structure I on S^6 compatible with g_0 is never integrable.*

Proof Since I is integrable and $\alpha^- = 0$, $\tilde{I}$ is holomorphic with respect to J_1 by Proposition 1, and hence, $\pi_2 \circ \tilde{I} : M \to H_{2n}(\mathbf{C}^m)$ is a holomorphic immersion. This means that M admits a Kähler structure, since $H_{2n}(\mathbf{C}^m)$ is Kähler. Since S^6 is totally umbilic, it holds $\alpha(X, Y) = g_0(X, Y)H$ for all $X, Y \in TM$. Therefore we see that $\alpha^- = 0$ for any almost complex structure I compatible compatible with g_0. □

Therefore, if an Hermite manifold admits an isometric immersion into Euclidean spaces with $\alpha^- = 0$, it gives topological restrictions. In the next section, we consider nearly Kähler submanifolds in Euclidean spaces.

3 Nearly Kähler Submanifolds

In this section, we consider the case that (M, g, I) is a nearly Kähler manifold (see the previous section for the definition of a nearly Kähler manifold). We recall that a nearly Kähler manifold (M, g, I) is said to be strict if $\nabla_X I \neq 0$ for all non-zero tangent vector X of M at each point. Let κ be the scalar curvature of (M, g). We define the $*$-Ricci curvature Ric^* of (M, g, I) by

$$Ric^*(X, Y) = -\frac{1}{2}\sum_{i=1}^{2n} g(R_{e_i, Ie_i}X, IY)$$

for $X, Y \in TM$. Moreover the $*$-scalar curvature κ^* is defined by

$$\kappa^* = \sum_{i=1}^{2n} Ric^*(e_i, e_i).$$

Let $Q^m(c)$ be an m-dimensional space form of constant curvature c.

Lemma 3 *Let (M, g, I) be a nearly Kähler manifold of* $\dim M = 2n$ *and $f : (M, g) \to Q^m(c)$ an isometric immersion. Then we have*

$$2(\|\alpha^+\|^2 - \|\alpha^-\|^2) + 4nc = 4\pi \sum_{i=1}^{2n} c_1(TM)(e_i, Ie_i) + \frac{1}{2}\|\nabla I\|^2,$$

where $(e_1, \ldots, e_{2n})$ *is an orthonormal frame of* g *and* $c_1(TM)$ *is the first Chern form of* (TM, I).

Proof We have

$$\sum_{i,j=1}^{2n} g(R_{e_i, Ie_i} Ie_j, e_j) = 4\pi \sum_i c_1(TM)(e_i, Ie_i) + \frac{1}{2}\|\nabla I\|^2$$

by Lemma 2.1 in [8]. Moreover, it holds that

$$\sum_{i,j=1}^{2n} g(R_{e_i, Ie_i} Ie_j, e_j) = 2 \sum_{i,j=1}^{2n} \tilde{g}(\alpha(e_i, e_j), \alpha(Ie_i, Ie_j)) + 4nc$$

by the Gauss equation. Then we have

$$\begin{aligned} & 2 \sum_{i,j=1}^{2n} \tilde{g}(\alpha(e_i, e_j), \alpha(Ie_i, Ie_j)) + 2\|\alpha\|^2 + 4nc \\ &= 4 \sum \tilde{g}(\alpha^+(e_i, e_j), \alpha^+(e_i, e_j) + \alpha^-(e_i, e_j)) + 4nc \\ &= 4\|\alpha^+\|^2 + 4\tilde{g}(\alpha^+, \alpha^-) + 4nc \end{aligned}$$

and

$$\begin{aligned} & 4\pi \sum_i c_1(TM)(e_i, Ie_i) + \frac{1}{2}\|\nabla I\|^2 + 2\|\alpha\|^2 \\ &= 4\pi \sum_i c_1(TM)(e_i, Ie_i) + \frac{1}{2}\|\nabla I\|^2 + 2\|\alpha^+\|^2 + 4\tilde{g}(\alpha^+, \alpha^-) + 2\|\alpha^-\|^2. \end{aligned}$$

This completes the proof. □

Lemma 4 *Let* (M, g, I) *be a nearly Kähler manifold of* $\dim M = 2n$. *If* $f : (M, g) \to Q^m(c)$ *is minimal immersion, then we have*

$$\|\nabla I\|^2 = -2 \sum_{i,j=1}^{2n} \tilde{g}(\alpha(e_i, e_j), \alpha^+(e_i, e_j)) + 4cn(n-1),$$

where $(e_1, \ldots, e_{2n})$ *is an orthonormal frame of* g. *In particular, if* $\alpha^+ = 0$, *then*

$$\|\nabla I\|^2 = 4cn(n-1).$$

Proof The scalar curvature κ and star-scalar curvature κ^* are given by

$$\kappa = -\sum_{i,j=1}^{2n} \tilde{g}(\alpha(e_i, e_j), \alpha(e_i, e_j)) + 2cn(2n-1)$$

and

$$\kappa^* = \sum_{i,j=1}^{2n} \tilde{g}(\alpha(e_i, e_j), \alpha(Ie_i, Ie_j)) + 2cn.$$

Then we have

$$\begin{aligned} \|\nabla I\|^2 &= \kappa - \kappa^* \\ &= -\sum_{i,j=1}^{2n} \tilde{g}(\alpha(e_i, e_j), \alpha(e_i, e_j)) + 2cn(2n-1) \\ &\quad -\sum_{i,j=1}^{2n} \tilde{g}(\alpha(e_i, e_j), \alpha(Ie_i, Ie_j)) - 2cn \\ &= -2\sum_{i,j=1}^{2n} \tilde{g}(\alpha(e_i, e_j), \alpha^+(e_i, e_j)) + 4cn(n-1). \end{aligned}$$

□

If (M, g, I) is a nearly Kähler manifold which is non-Kähler and $\dim M = 6$, then we have $c_1(TM) = 0$ and $\kappa = 5\kappa^* > 0$ (see [7]). Therefore we obtain

$$2(\|\alpha^+\|^2 - \|\alpha^-\|^2) + 12c = \frac{1}{2}\|\nabla I\|^2 = \frac{1}{2}(\kappa - \kappa^*) = 2\kappa^* > 0.$$

On the study for nearly Kähler submanifolds, it is one of the first step to consider the case that M is non-Kähler with f satisfies $\alpha^+ = 0$ or $\alpha^- = 0$.

Theorem 1 *Let (M, g, I) be a nearly Kähler manifold of* $\dim M = 2n$ *and $f : M \to Q^m(c)$ an isometric immersion with $\alpha^+ = 0$. Then we have* (1) *if $c < 0$, it occurs only the case of $n = 1$,* (2) *if $c = 0$, then (M, g, I) must be Kähler,* (3) *if $c > 0$, then (M, g, I) is Kähler iff $n = 1$.*

Proof By Lemma 4, we have $\|\nabla I\|^2 = 4cn(n-1)$, which means the conclusion. □

Therefore, there exists no isometric immersion from a non-Kähler, nearly Kähler manifold into $\mathbf{R}^m$ with $\alpha^+ = 0$ even locally. On the other hand, a typical example of a nearly Kähler submanifold in the Euclidean space with $\alpha^- = 0$ is $S^6 \subset \mathbf{R}^7$.

Theorem 2 *Let (M, g, I) be a nearly Kähler manifold which is non-Kähler and* $\dim M = 2n = 6$ *and $f : (M, g) \to \mathbf{R}^m$ an isometric immersion. If $\alpha^- = 0$, then f is locally congruent to the standard sphere $S^6 \subset \mathbf{R}^m$.*

Proof Since (M, g, I) be a nearly Kähler manifold which is non-Käler and $\dim M = 6$ and f satisfies $\alpha^- = 0$, we obtain $\|\alpha^+\|^2 = \kappa^* > 0$, which means that f is not totally geodesic. Then it is sufficient to show that f is totally umbilic. From the Gauss equation, we have

$$\|\alpha^+\|^2 + 2\tilde{g}(\alpha^+, \alpha^-) + \|\alpha^-\|^2 = 4n^2\|H\|^2 - \kappa,$$

that is, $\|\alpha^+\|^2 = 4n^2\|H\|^2 - \kappa$. By $\kappa = 5\kappa^*$, it holds $3\kappa^* = 3\|\alpha^+\|^2 = 2n^2\|H\|^2$. We define γ by $\gamma(X, Y) = \alpha(X, Y) - g(X, Y)H$ for $X, Y \in TM$ and calculate

$$\|\gamma\|^2 = \|\alpha^+\|^2 - 2n\|H\|^2 = \tfrac{2}{3}n^2\|H\|^2 - 2n\|H\|^2 = 2n\|H\|^2\left(\tfrac{n}{3} - 1\right).$$

Since $n = 3$, we have $\gamma = 0$, and hence f is totally umbilic. □

Next we study nearly Kähler submanifolds with vertically pluri-harmonic lift $\tilde{I}$. Let $H^{\tilde{\nabla}}$ be the Hessian of the induced connection $\tilde{\nabla}$ (we use the same letter of the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$). The lift $\tilde{I}$ is vertically pluri-harmonic (vph) if

$$\pi_{\tilde{I}}(H^{\tilde{\nabla}}(X, X)\tilde{I} + H^{\tilde{\nabla}}(IX, IX)\tilde{I}) = 0$$

for all $X \in TM$. By Lemma 1, we have the following.

Lemma 5 *We have*

$$\begin{aligned}(H^{\tilde{\nabla}}(X, Y)\tilde{I})(Z) = &(H^{\nabla}(X, Y)I)(Z) + S_{\alpha(Y,IZ)}X + IS_{\alpha(X,Z)}Y\\ &-(\nabla_X\alpha)(Y, IZ) - \alpha(X, (\nabla_Y I)(Z)) - \alpha(Y, (\nabla_X I)(Z)),\\ (H^{\tilde{\nabla}}(X, Y)\tilde{I})(\xi) = &-(\nabla_X I)(S_\xi Y) - (\nabla_Y I)(S_\xi X) - I((\nabla_X S)_\xi Y)\\ &-\alpha(X, IS_\xi Y) - \alpha(Y, IS_\xi X)\end{aligned}$$

for $X, Y, Z \in TM$, $\xi \in T^\perp M$.

By Lemmas 5 and 2, we have

Lemma 6 *We have*

$$\begin{aligned}(\pi_{\tilde{I}}(H^{\tilde{\nabla}}(X, X)\tilde{I}))(Y) &= \frac{1}{2}I[H^{\nabla}(X, X)I, I](Y) - (\nabla_X\alpha)(X, IY) - 2\alpha(X, (\nabla_X I)(Y)),\\ (\pi_{\tilde{I}}(H^{\tilde{\nabla}}(X, X)\tilde{I}))(\xi) &= -2(\nabla_X I)(S_\xi X) - I((\nabla_X S)_\xi X)\end{aligned}$$

for $X, Y \in TM$, $\xi \in T^\perp M$.

By Lemma 6, we have the following theorem.

Theorem 3 *Let* (M, g, I) *be an almost Hermite manifold of* $\dim M = 2n$ *and* $f : (M, g) \to (\tilde{M}, \tilde{g})$ *an isometric immersion. The lift* $\tilde{I}$ *is vertically pluri-harmonic if* *and only if I satisfies $\pi_I(H^{\nabla}(X,X)I + H^{\nabla}(IX,IX)I) = 0$ for all $X \in TM$ as a section of $\mathscr{F}_{2n}(M)$(=the usual twistor space of M) and*

$$-2\alpha(X,(\nabla_X I)(Y)) - 2\alpha(IX,(\nabla_{IX} I)(Y)) - (\nabla_X \alpha)(X,IY) - (\nabla_{IX}\alpha)(IX,IY) = 0$$

for all $X, Y \in TM$.

Proof Using Lemma 6 and the assumption for I, we have

$$(\pi_{\tilde{I}}(H^{\tilde{\nabla}}(X,X)\tilde{I}))(Y) = -(\nabla_X\alpha)(X,IY) - 2\alpha(X,(\nabla_X I)(Y))$$

for all $X, Y \in TM$. Calculating $\pi_{\tilde{I}}(H^{\tilde{\nabla}}(X,X)\tilde{I} + H^{\tilde{\nabla}}(IX,IX)\tilde{I})$ shows the conclusion. □

As a direct consequence, we have the following corollary (see Theorem 5 in [3]).

Corollary 3 *Let (M,g,I) be a Kähler manifold of* $\dim M = 2n$ *and $f:(M,g)\to Q^m(c)$ an isometric immersion. The $\tilde{I}$ is vertically pluri-harmonic if and only if α^+ is parallel, that is, f has the parallel pluri-mean curvature.*

Finally, in addition, we consider the following condition: (pod) there exists a parallel orthogonal decomposition $T^{\perp}M = N^- \oplus N^+$ such that N^-, N^+ contain α^-, α^+, respectively. In the case that M is Kähler and $\tilde{M}$ is the Euclidean space, The condition (pod) is equivalent to the condition that f is isotropic and parallel pluri-mean curvature (see [3] for the detail). Note that if f satisfies a weaker condition $\tilde{g}(\alpha^-(X,Y),\alpha^+(Z,W)) = 0$ for all $X,Y,Z,W \in TM$ is called half isotropic (hi). The conditions (vph) and (hi) holds if and only if the complex Gauss map τ of f is pluri-harmonic (see [5]).

We have the following an extrinsically decomposition of f.

Theorem 4 *Let (M,g,I) be a simply connected complete nearly Kähler manifold and $f:(M,g)\to \mathbf{R}^m$ an isometric immersion. If $\tilde{I}$ is vertically pluri-harmonic (vph) and* (pod) *holds, then we have*

(1) *M is isometric to $M_1\times\cdots\times M_k$, where each M_i is nearly Kähler* $(1\le i\le k)$*,*
(2) *the immersion f decomposes into the product $f = f_1\times\cdots f_k : M_1\times\cdots\times M_k \to \mathbf{R}^{m_1}\times\cdots\times\mathbf{R}^{m_k}$ such that each f_i is pluri-minimal immersion from Kähler manifold M_i into $\mathbf{R}^{m_i}$ or minimal immersion into hypersphere of $\mathbf{R}^{m_i}$.*

Proof Since M is nearly Kähler,then I satisfies $\pi_I(H^{\nabla}(X,X)I + H^{\nabla}(IX,IX)I) = 0$ for all $X\in TM$. By Theorem 3, we see

$$\begin{aligned}&-2\alpha^-(X,(\nabla_X I)(Y)) - \nabla^{\perp}_{IY}\alpha^+(X,X) + 2\alpha^+(\nabla_{IY}X,X)\\ &+\alpha^+(I(\nabla_X I)(Y),IX) + \alpha^-(I(\nabla_X I)(Y),IX) = 0\end{aligned}$$

for all $X, Y \in \Gamma(TM)$. The condition (pod) implies that $\alpha^-(X,(\nabla_X I)(Y)) = 0$ and $-\nabla^{\perp}_{IY}\alpha^+(X,X) + 2\alpha^+(\nabla_{IY}X,X) + \alpha^+((\nabla_X I)(Y),X) = 0$. Since M is nearly Kähler and α^+ is I-invariant, we have

$$\alpha^+((\nabla_X I)(Y), X) + \alpha^+((\nabla_{IX} I)(Y), IX) = 0.$$

Therefore $\nabla^\perp H = 0$ holds. We set $\alpha^H(X, Y) = g(A_H X, Y) = \tilde{g}(\alpha(X, Y), H)$ for $X, Y \in TM$. If $H = 0$, then from Lemma 4, it holds that M is Kähler and f is pluri-minimal, that is, $\alpha^+ = 0$. Hereafter we assume that $H \neq 0$. It is easy to see $(\nabla_X \alpha^H)(Y, Z) = \tilde{g}(H, (\nabla_X \alpha)(Y, Z))$ for $X, Y, Z \in TM$. Therefore, $\nabla \alpha^H$ is totally symmetric by the codazzi equation. Since M is nearly Kähler and $\tilde{I}$ is vertically pluri-harmonic, it holds $(\nabla_X \alpha)(X, X) + (\nabla_X \alpha)(IX, IX) = 0$ for all $X \in TM$ by Theorem 3. Therefore, since M is nearly Kähler and the condition (pod) holds, we can calculate

$$\begin{aligned}
(\nabla_X \alpha^H)(X, X) &= \tilde{g}(H, (\nabla_X \alpha)(X, X)) \\
&= -\tilde{g}(H, (\nabla_X \alpha)(IX, IX)) \\
&= -\tilde{g}(H, \nabla^\perp_X \alpha(IX, IX)) + 2\tilde{g}(H, \alpha(\nabla_X IX, IX)) \\
&= -\tilde{g}(H, \nabla^\perp_X (\alpha(X, X) - 2\alpha^-(X, X))) \\
&\quad + 2\tilde{g}(H, \alpha((\nabla_X I)(X), IX)) + 2\tilde{g}(H, \alpha(I(\nabla_X X), IX) \\
&= -\tilde{g}(H, \nabla^\perp_X \alpha(X, X) - 2\alpha^-(X, X)) \\
&\quad + 2\tilde{g}(H, \alpha(\nabla_X X, X) - 2\alpha^-(\nabla_X X, X)) \\
&= -(\nabla_X \alpha^H)(X, X)
\end{aligned}$$

for all $X \in TM$. Hence we have $\nabla \alpha^H = 0$, which means that A_H is parallel with respect to the Levi-Civita connection of g. By the similar argument as in [5], we have the conclusion. □

Note that any simply connected complete nearly Kähler manifold is decomposed into Kähler and strict nearly Kähler manifolds (intrinsically). See [7].

Corollary 4 *Let (M, g, I) be a simply connected complete nearly Kähler manifold and $f : (M, g) \to \mathbf{R}^m$ an isometric immersion. If $\alpha^- = 0$, then we have the same conclusion as Theorem 4.*

Proof By $\alpha^- = 0$ and the codazzi equation for α, we have

$$\begin{aligned}
2(\nabla_X \alpha)(Y, Z) = \alpha((\bar{\nabla}_Z I)(X), IY) + \alpha(IX, (\bar{\nabla}_Z I)(Y)) \\
+ \alpha((\bar{\nabla}_{IY} I)(IX), IZ) - \alpha(X, (\bar{\nabla}_{IY} I)(Z)) \\
+ \alpha((\bar{\nabla}_X I)(Y), Z) - \alpha(IY, (\bar{\nabla}_X I)(Z))
\end{aligned}$$

for all $X, Y, Z \in TM$. Here we use the same calculation as in [8]. Since M is nearly Kähler, it holds $(\nabla_X \alpha)(X, X) = 0$, and hence α is parallel. Then $\tilde{I}$ is vertically pluri-harmonic. Moreover choosing $N^- = M \times \{0\}$ and $N^+ = T^\perp M$, we can see that (pod) holds. □

Acknowledgements The author would like to thank to organizers for giving him the opportunity to talk at the conference “The 20th International Workshop on Hermitian Symmetric Spaces and Submanifolds". This work is partially supported by JSPS KAKENHI Grant Number 15K04839.

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The Schwarz Lemma for Super-Conformal Maps

Katsuhiro Moriya

Abstract A super-conformal map is a conformal map from a two-dimensional Riemannian manifold to the Euclidean four-space such that the ellipse of curvature is a circle. Quaternionic holomorphic geometry connects super-conformal maps with holomorphic maps. We report the Schwarz lemma for super-conformal maps and related results.

1 Introduction

For a smooth manifold M, we denote the tangent bundle by TM and its fiber at $p \in M$ by T_pM. Let Σ be a two-dimensional oriented Riemannian manifold and $f : \Sigma \to \mathbb{R}^4$ be an isometric immersion. We denote the Riemannian metric of Σ by g. For a tangent vector $X \in T_p\Sigma$, we denote the norm with respect to the Riemannian metric by $\|X\|$. We denote the second fundamental form of f by h. Then

$$\mathscr{E}_p = \left\{h(X, X) : X \in T_p\Sigma, \|X\| = 1\right\}$$

is called the *ellipse of curvature* or the *curvature ellipse* of f at $p \in M$ [9]. It is indeed an ellipse in the normal space at p if it does not degenerate to a point or a line segment. If the ellipse of curvature is a circle or a point at any point p, then f is said to be *super-conformal* [2].

The author showed that a super-conformal map is a Bäcklund transform of a minimal surface [6]. Regarding f as an isometric immersion, the inequality

$$|\mathscr{H}|^2 - K - |K^\perp| \geq 0 \tag{1}$$

K. Moriya (✉)
Division of Mathematics, Faculty of Pure and Applied Sciences,
University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571, Japan
e-mail: moriya@math.tsukuba.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_6

holds between the mean curvature vector $\mathscr{H}$, the Gaussian curvature K and the normal curvature $K^{\perp}$ [10]. The equality holds if and only if f is super-conformal. From this point of view, a super-conformal map is called a Wintgen ideal surface [8]. The integral of the left-hand side of (1) over Σ is the Willmore energy of f. This implies that a super-conformal map is a Willmore surface with vanishing Willmore energy. Hence the super-conformality is invariant under Möbius transforms of $\mathbb{R}^4$.

We discuss the Möbius geometry of super-conformal immersion by exchanging a two-dimensional oriented Riemannian manifold with a Riemann surface and an isometric immersion with a conformal immersion. Regarding $\mathbb{C}$ as a subspace of $\mathbb{R}^4$ and a holomorphic function on Σ as a map form Σ to $\mathbb{R}^4$, a holomorphic function satisfies (1) and it is super-conformal. We may regard Möbius geometric theory of holomorphic functions on a Riemann surface as a special case of Möbius geometry of super-conformal immersion.

The author [7] discussed super-conformal maps as a higher codimensional analogue of holomorphic functions and meromorphic functions. In this paper, we report a part of the paper which discusses the Schwarz lemma for super-conformal maps.

For the discussion, we use quaternionic holomorphic geometry [3]. Quaternionic holomorphic geometry of surfaces in $\mathbb{R}^4$ connects theory of holomorphic functions with theory of surfaces in $\mathbb{R}^4$.

2 Classical Results

We begin with reviewing the classical results of super-conformal maps by Friedrich [4] and Wong [11].

Throughout this paper, all manifolds and maps are assumed to be smooth. We compute the ellipse of curvature. We denote the inner product of $\mathbb{R}^4$ by $\langle\ ,\ \rangle$. Let e_1, e_2, e_3, e_4 be an adapted orthonormal local frame of the pull-back bundle $f^*T\mathbb{R}^4$ and θ_1, θ_2, θ_3, θ_4 the dual frame. Assume that the second fundamental form is

$$h = \sum_{p=3}^{4} \sum_{i,j=1}^{2} h_{ijp}\, \theta_i \otimes \theta_j \otimes e_p.$$

Then the ellipse of curvature is parametrized by the map

$$\varepsilon(u) = h(e_1 \cos u + e_2 \sin u, e_1 \cos u + e_2 \sin u)$$
$$= \mathscr{H} + \left(\frac{h_{113} - h_{223}}{2} e_3 + \frac{h_{114} - h_{224}}{2} e_4\right) \cos 2u + (h_{123} e_3 + h_{124} e_4) \sin 2u.$$

The map f is super-conformal map if and only if $\varepsilon(u)$ parametrizes a circle. The map f is minimal if and only if $\varepsilon(u)$ parametrize a curve of the linear transform of the circle centered at the origin. The linear transform is given by

$$P\left(e_3\ e_4\right) = \left(e_3\ e_4\right)\begin{pmatrix} h_{113} & h_{123} \\ h_{114} & h_{124} \end{pmatrix}$$

Hence f is super-conformal and minimal if and only if the ellipse of curvature is a circle centered at the origin.

We normalize the second fundamental form and the ellipse of curvature. Let $n(u) = e_3 \cos u + e_4 \sin u$. Because

$$\begin{aligned} \langle h, n(u)\rangle &= \sum_{i,j=1}^{2} h_{ij3}\,\theta_i \otimes \theta_j \cos u + \sum_{i,j=1}^{2} h_{ij4}\,\theta_i \otimes \theta_j \sin u, \\ \operatorname{tr}\langle h, n(u)\rangle &= h_{113}\cos u + h_{114}\sin u + h_{223}\cos u + h_{224}\sin u \\ &= (h_{113} + h_{223})\cos u + (h_{114} + h_{224})\sin u, \end{aligned}$$

we may assume that $h_{114} + h_{224} = 0$. Let A_{e_4} be the shape operator such that $\langle A_{e_4}(X), Y\rangle = \langle h(X, Y), e_4\rangle$ for any $X, Y \in T_p\Sigma$. Taking e_1 and e_2 as the eigenvectors of A_{e_4}, we may assume that $h_{124} = 0$. The ellipse of curvature becomes

$$\varepsilon(u) = \frac{h_{113} + h_{223}}{2} e_3 + \left(\frac{h_{113} - h_{223}}{2} e_3 + h_{114} e_4\right)\cos 2u + (h_{123} e_3)\sin 2u.$$

Then f is super-conformal if and only if

$$(h_{113} - h_{223})h_{123} = 0, \quad \left(\frac{h_{113} - h_{223}}{2}\right)^2 + h_{114}^2 = h_{123}^2$$

This is equivalent to

$$h_{123} = h_{114} = 0,\ \ h_{113} = h_{223} \text{ or } h_{113} = h_{223},\ \ h_{114}^2 = h_{123}^2.$$

Hence the ellipse of curvature of a super-conformal map becomes

$$\varepsilon(u) = 0 \text{ or } \varepsilon(u) = h_{113} + (h_{114} e_4)\cos 2u + (\pm h_{114} e_3)\sin 2u.$$

If f is minimal, then the ellipse of curvature is

$$\varepsilon(u) = (h_{113} e_3 + h_{114} e_4)\cos 2u + (h_{123} e_3)\sin 2u.$$

Hence f is super-conformal and minimal if and only if

$$\varepsilon(u) = (h_{114} e_4)\cos 2u + (\pm h_{114} e_3)\sin 2u.$$

Another notion is defined by the second fundamental form for surfaces in $\mathbb{R}^4$.

Definition 1 ([5, 11]) The set

$$\mathscr{I}_p = \left\{ \langle h, n \rangle : n \in T_p \Sigma^{\perp}, \|n\| = 1 \right\}.$$

is called the *indicatrix* of the normal curvature or the *Kommerell conic* of f.

The indicatrix is parametrized by

$$\iota(u) = \langle h, n(u) \rangle = \sum_{i,j=1}^{2} (h_{ij3} \cos u + h_{ij4} \sin u) \theta_i \otimes \theta_j.$$

By the normalization, we may assume that

$$\begin{aligned} \iota(u) = (h_{113} \cos u + h_{114} \sin t) \theta_1 \otimes \theta_1 + h_{123} \cos u \, \theta_1 \otimes \theta_2 \\ + h_{213} \cos u \, \theta_2 \otimes \theta_1 + (h_{223} \cos u - h_{114} \sin u) \theta_2 \otimes \theta_2. \end{aligned}$$

We regard $\langle h, n(u) \rangle$ as the shape operator which is is a symmetric $(1, 1)$-tensor. With the standard inner product of symmetric $(1, 1)$-tensors, the curve $\langle h, n(u) \rangle$ is isometrically the curve parametrized by

$$\iota(u) = \left(\frac{1}{\sqrt{2}} (h_{113} \cos u + h_{114} \sin u), \sqrt{2} h_{123} \cos u, \frac{1}{\sqrt{2}} (h_{223} \cos u - h_{114} \sin u) \right)$$

in $\mathbb{R}^3$. Hence f is super-conformal if and only if the indicatrix is parametrized by

$$\iota(u) = \left(\frac{1}{\sqrt{2}} (h_{113} \cos u), 0, \frac{1}{\sqrt{2}} (h_{113} \cos u) \right)$$

or

$$\iota(u) = \left(\frac{1}{\sqrt{2}} (h_{113} \cos u + h_{114} \sin u), \pm \sqrt{2} h_{114} \cos u, \frac{1}{\sqrt{2}} (h_{113} \cos u - h_{114} \sin u) \right).$$

We see that f is minimal if and only if the indicatrix is parametrized by

$$\iota(u) = \left(\frac{1}{\sqrt{2}} (h_{113} \cos u + h_{114} \sin u), \sqrt{2} h_{123} \cos u, \frac{1}{\sqrt{2}} (-h_{113} \cos u - h_{114} \sin u) \right).$$

Moreover, f is super-conformal and minimal if and only if

$$\iota(u) = 0 \text{ or } \iota(u) = \left(\frac{1}{\sqrt{2}} (h_{114} \sin u), \pm \sqrt{2} h_{114} \cos u, \frac{1}{\sqrt{2}} (-h_{114} \sin u) \right).$$

Set

$$\mathscr{I}_p(X) = \left\{\langle h(X,\ ), n\rangle : n \in T_p\Sigma^{\perp}, \|n\| = 1\right\} \subset (T_p\Sigma)^*.$$

Definition 2 ([4]) An immersion f is called *superminimal* if $\mathscr{I}_p(X)$ is a circle centered at 0 in $(T_p\Sigma)^*$.

The following lemma explains the relation among the super-conformal maps, minimal surfaces and superminimal surfaces.

Lemma 1 *A map f is superminimal if and only if f is super-conformal and minimal.*

Proof For $X = X_1e_1 + X_2e_2$ and the normalization, we have

$$\begin{aligned}\langle h(X,\ ), n(u)\rangle &= \sum_{i,j=1}^{2} X_i h_{ij3}\theta_j \cos u + \sum_{i,j=1}^{2} X_i h_{ij4}\theta_j \sin u \\ &= ((X_1h_{113} + X_2h_{123})\theta_1 + (X_1h_{123} + X_2h_{223})\theta_2)\cos u \\ &\quad + (X_1h_{114}\theta_1 - X_2h_{114}\theta_2)\sin u.\end{aligned}$$

Hence f is superminimal if and only if

$$\begin{aligned}(X_1^2h_{113} - X_2^2h_{223})h_{114} &= 0,\\ (X_1h_{113} + X_2h_{213})^2 + (X_1h_{123} + X_2h_{223})^2 &= (X_1^2 + X_2^2)h_{114}^2\end{aligned}$$

Hence

$$h_{114} = X_1h_{113} + X_2h_{213} = X_1h_{123} + X_2h_{223} = 0$$

or

$$\begin{aligned}X_1^2h_{113} - X_2^2h_{223} &= 0,\\ (X_1h_{113} + X_2h_{123})^2 + (X_1h_{123} + X_2h_{223})^2 &= (X_1^2 + X_2^2)h_{114}^2\end{aligned}$$

Because X_1 and X_2 is arbitrary under $X_1^2 + X_2^2 \neq 0$, we have $h = 0$, or $h_{113} = h_{223} = 0$ and $h_{123}^2 = h_{114}^2$. Hence the lemme holds. □

For a holomorphic function $g(z)$ on $\mathbb{C}$, define a map $\tilde{g} : \mathbb{C} \to \mathbb{C}^2 \cong \mathbb{R}^4$ by $\tilde{g}(z) = (z, g(z))$. Then $\tilde{g}$ is called an *R-surface* [5]. Kommerell showed that an R-surface is superminimal.

3 Quaternionic Holomorphic Geometry

We review super-conformal maps by quaternionic holomorphic geometry of surfaces in $\mathbb{R}^4$ [2]. We identify $\mathbb{R}^4$ with the set of all quaternions $\mathbb{H}$. The inner product of $\mathbb{R}^4$ becomes

$$\langle a, b\rangle = \mathrm{Re}(\overline{a}b) = \mathrm{Re}(\overline{b}a) = \frac{1}{2}(\overline{a}b + \overline{b}a).$$

We identify $\mathbb{R}^3$ with the set of all imaginary parts of quaternions $\mathrm{Im}\,\mathbb{H}$. Then two-dimensional sphere with radius one centered at the origin in $\mathbb{R}^3$ is $S^2 = \{a \in \mathrm{Im}\,\mathbb{H} : a^2 = -1\}$.

Let Σ be a Riemann surface with complex structure J_Σ. For a one-form ω on Σ, we define a one-form $*\omega$ by $*\omega = \omega \circ J_\Sigma$. A map $f\colon \Sigma \to \mathbb{H}$ is called a conformal map if $\langle df \circ J_\Sigma, df\rangle = 0$. This is equivalent to that $*df = N\,df = -df\,R$ with maps $N, R\colon \Sigma \to S^2$. Each point where f fails to be an immersion is isolated. This means that a conformal map is a branched immersion. The second fundamental form of f is

$$h(X, Y) = \frac{1}{2}(*df(X)\,dR(Y) - dN(X) * df(Y)).$$

Let $\mathscr{H}\colon \Sigma \to \mathbb{H}$ be the mean curvature vector of f. Then

$$df\,\overline{\mathscr{H}} = -\frac{1}{2}(*dN + N\,dN),\;\; \overline{\mathscr{H}}\,df = \frac{1}{2}(*dR + R\,dR).$$

The ellipse of curvature is

$$\mathscr{E}_p = \left\{\mathscr{H}|df(e_1)|^2 + \frac{1}{4}\cos 2\theta(a-b)(e_1) + \frac{1}{4}\sin 2\theta N(a+b)(e_1) : \theta \in \mathbb{R}\right\},$$
$$a = df\,(*dR - R\,dR),\;\; b = (*dN - N\,dN)\,df.$$

Then f is super-conformal if and only one of the following equations holds.

$$*dR - R\,dR = 0,\;\; *dN - N\,dN = 0$$

at any point $p \in \Sigma$.

In the following, we restrict ourselves to super-conformal maps with $*dN = N\,dN$.

Lemma 2 *A super-conformal map $f\colon \Sigma \to \mathbb{H}$ with $*df = N\,df$ and $*dN = N\,dN$ is superminimal if and only if f is holomorphic with respect to a right quaternionic linear complex structure of $\mathbb{H}$.*

Proof A super-conformal map $f\colon \Sigma \to \mathbb{H}$ with $*df = N\,df$ and $*dN = N\,dN$ satisfies the equation

$$df\,\overline{\mathscr{H}} = -N\,dN.$$

Hence f is minimal if and only if N is a constant map. Define $J\colon \mathbb{H} \to \mathbb{H}$ by $Jv = Nv$ for any $v \in \mathbb{H}$. Then J is a right quaternionic linear complex structure of $\mathbb{H}$. Because $*df = J\,df$, the map f is holomorphic with respect to J.

By the above lemma, we see that a holomorphic map g from Σ to $\mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}j \cong \mathbb{H}$ is superminimal because $*dg = i\,dg$. A holomorphic function and an R-surface are special cases of this superminimal surface.

4 The Schwarz Lemma

Because a holomorphic function is a super-conformal map, we may expect that a super-conformal map is an analogue of a holomorphic function. A factorization of super-conformal map given in Theorem 4.3 in [7] may support this idea. The following is a variant of the theorem.

Theorem 1 ([7], Theorem 4.3) *Let $\phi\colon \Sigma \to \mathbb{H}$ be a super-conformal map with $*d\phi = N\,d\phi$, $*dN = N\,dN$ and $N\phi = \phi i$ and $h\colon \Sigma \to \mathbb{C}^2 \cong \mathbb{C} \oplus \mathbb{C}j \cong \mathbb{H}$ be a holomorphic map. Then, a map $f = \phi h$ is a super-conformal map with $*df = N\,df$.*

This theorem shows that a holomorphic section of a complex vector bundle of rank two, trivialized by the super-conformal map f is a super-conformal map. We see that $N + i$ is a super-conformal map with $N(N + i) = (N + i)i$. The condition $*dN = N\,dN$ implies N is the inverse of the stereographic projection followed by an anti-holomorphic function ([7], Corollary 3.2). Hence if Σ is an open Riemann surface and $N\colon \Sigma \to S^2$ is the inverse of the stereographic projection of an anti-holomorphic function with $N(\Sigma) \subset S^2 \setminus \{-i\}$, then $N + i$ is a global super-conformal trivializing section. A super-conformal map $f\colon \Sigma \to \mathbb{H}$ with $*df = N\,df$, $*dN = N\,dN$ always factors $f = (N + i)h$ with a holomorphic map $h\colon \Sigma \to \mathbb{C} \oplus \mathbb{C}j$. We don't need to see $-i$ in a special light. If $a \in S^2$ and $a \notin N(\Sigma)$, then we can rotate f so that $-i \notin N(\Sigma)$. The condition that N fails to be surjective is necessary.

This fact suggests that we should distinguish the case where the Riemann surface Σ is parabolic or hyperbolic. In the case where $\Sigma = \mathbb{C}$, we have an analogue of Liouville's theorem.

Theorem 2 ([7], Theorem 4.4) *Let $\phi\colon \mathbb{C} \to \mathbb{H}$ be a super-conformal map with $*d\phi = N\,d\phi$, $*dN = N\,dN$ and $N\phi = \phi i$. Assume that $N(\mathbb{C}) \subset S^2 \setminus \{-i\}$ and $|\phi|^{-1}$ is bounded above. If $f\colon \mathbb{C} \to \mathbb{H}$ is a super-conformal map with $*df = N\,df$ and $|f|$ is bounded above, then $f = \phi C$ for some constant $C \in \mathbb{H}$.*

In the case where $\Sigma = B^2 = \{z \in \mathbb{C} : |z| < 1\}$, we have an analogue of the Schwarz lemma.

Theorem 3 ([7], Theorem 4.5) *Let* $\phi\colon B^2 \to \mathbb{H}$ *be a super-conformal map with* $*d\phi = N\,d\phi$, $*dN = N\,dN$ *and* $N\phi = \phi i$. *Assume that* $N(B^2) \subset S^2 \setminus \{-i\}$ *and* $|\phi| < c$ *and* $|\phi|^{-1} < \tilde{c}$. *If* $f\colon B^2 \to \mathbb{H}$ *is a super-conformal map with* $*df = N\,df$ *and* $f(0) = 0$, *then there exists a constant* C *such that*

$$|f(z)| \leq C|z|.$$

Moreover, if $f = \phi(\lambda_0 + \lambda_1 j)$ *for holomorphic functions* λ_0 *and* λ_1, *then there exist constants* C_0, $C_1 > 0$ *such that*

$$|f(z)| \leq c(C_0^2 + C_1^2)^{1/2}|z|.$$

The equality holds if and only if $\phi = c$ *and there exists* $z_0 \in B^2$ *such that* $|\lambda_n(z)| = C_n|z_0|$ $(n = 0, 1)$. *We also have*

$$\left|f_x(0) - N(0)f_y(0)\right| \leq c(C_0^2 + C_1^2)^{1/2}.$$

The equality holds if and only if $f = c$ *and there exists* $z_0 \in B^2$ *such that* $|\lambda_n(z)| = C_n|z_0|$ $(n = 0, 1)$.

Assume that $f(B^2) \subset B^4 = \{a \in \mathbb{H} : |a| < 1\}$. It is known that

$$T(a) = \frac{(1 - |a_1|^2)(a - a_1) - |a - a_1|^2 a_1}{1 + |a|^2|a_1|^2 - 2\langle a, a_1\rangle}$$

is a Möbius transform of $\mathbb{R}^4$ with $T(a_1) = 0$ [1]. The transform T is

$$\begin{aligned} T(a) &= \frac{a - a_1 - |a_1|^2 a + |a_1|^2 a_1 - |a|^2 a_1 + a|a_1|^2 + a_1\overline{a}a_1 - |a_1|^2 a_1}{|1 - \overline{a}_1 a|^2} \\ &= \frac{a - a_1 - |a|^2 a_1 + a_1\overline{a}a_1}{|1 - \overline{a}_1 a|^2} = \frac{(1 - a_1\overline{a})(a - a_1)}{|1 - \overline{a}_1 a|^2} \\ &= (1 - a\overline{a}_1)^{-1}(a - a_1) \end{aligned}$$

and T preserves B^4. If $f\colon B^2 \to B^4$ is a super-conformal map with $*df = N\,df$ and $*dN = N\,dN$, then

$$\begin{aligned} *d(T \circ f) &= (1 - f\overline{a}_1)^{-1}(*df\,\overline{a})(1 - f\overline{a}_1)^{-1}(1 - f) - (1 - f\overline{a}_1)^{-1} * df \\ &= (1 - f\overline{a}_1)^{-1} N (1 - f\overline{a}_1)\, d(T \circ f). \end{aligned}$$

It is known that a Möbius transform of a super-conformal map is super-conformal. Then we have an analogue of the Schwarz-Pick theorem.

Theorem 4 ([7], Theorem 4.7) *Let $\phi\colon B^2 \to B^4$ be a super-conformal map with $*d\phi = N\,d\phi$, $*dN = N\,dN$ and $N\phi = \phi i$. Assume that $|\phi|$ and $|\phi|^{-1}$ are bounded. Let $f\colon \Sigma \to \mathbb{H}$ be a super-conformal map with $*df = N\,df$. Assume that the map*

$$\tilde{N} := (1 - \tilde{f}(z)\overline{\tilde{f}(z_1)})^{-1} N (1 - \tilde{f}(z)\overline{\tilde{f}(z_1)})\colon \Sigma \to S^2$$

satisfies $\tilde{N}(B^2) \subset S^2 \setminus \{-i\}$. Then there exists a constant $C > 0$ such that

$$\frac{|f(z) - f(z_1)|}{\left|1 - \overline{f(z_1)}f(z)\right|} \leq C \frac{|z - z_1|}{|1 - \overline{z_1}z|}$$

Moreover,

$$\frac{|f_x(z_1)|}{1 - |f(z_1)|^2} = \frac{\left|f_y(z_1)\right|}{1 - |f(z_1)|^2} \leq \frac{C}{1 - |z_1|^2}.$$

We fix Riemannian metrics $ds^2_{B^2}$ on B^2 and $ds^2_{B^4}$ on B^4 as

$$ds^2_{B^2} = \frac{4}{(1 - (x^2 + y^2))^2}(dx \otimes dx + dy \otimes dy),$$

$$ds^2_{B^4} = \frac{4}{(1 - \sum_{n=0}^{3} a_n^2)^2}(\sum_{n=0}^{3} da_n \otimes da_n).$$

Then a geometric version of the Schwarz-Pick theorem becomes as follows.

Theorem 5 *Let $\phi\colon B^2 \to B^4$ be a super-conformal map with $*d\phi = N\,d\phi$, $*dN = N\,dN$ and $N\phi = \phi i$. Assume that $|\phi|$ and $|\phi|^{-1}$ are bounded. Let $f\colon \Sigma \to \mathbb{H}$ be a super-conformal map with $*df = N\,df$. Assume that the map*

$$\tilde{N} := (1 - f(z)\overline{f(z_1)})^{-1} N (1 - f(z)\overline{f(z_1)})\colon \Sigma \to S^2$$

satisfies $\tilde{N}(B^2) \subset S^2 \setminus \{-i\}$. Then there exists a constant $C > 0$ such that $f^* ds^2_{B^4} \leq C ds^2_{B^2}$.

Acknowledgements This work was supported by JSPS KAKENHI Grant Number 25400063.

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Reeb Recurrent Structure Jacobi Operator on Real Hypersurfaces in Complex Two-Plane Grassmannians

Hyunjin Lee and Young Jin Suh

Abstract In (Jeong et al., Acta Math Hungar 122(1–2), 173–186, 2009) [7], Jeong, Pérez, and Suh verified that there does not exist any connected Hopf hypersurface in complex two-plane Grassmannians with parallel structure Jacobi operator. In this paper, we consider more general notions as Reeb recurrent or $\mathscr{Q}^{\perp}$-recurrent structure Jacobi operator. By using these general notions, we give some new characterizations of Hopf hypersurfaces in complex two-plane Grassmannians.

1 Introduction

As examples of Hermitian symmetric spaces of rank 2 we can give Riemannian symmetric spaces $SU_{2,m}/S(U_2U_m)$ and SO_{m+2}/SO_mSO_2, which are said to be complex hyperbolic two-plane Grassmannians and complex quadric, respectively. Recently, the second author have studied hypersurfaces of those spaces (see [19–22]). On the other hand, as another kind of Hermitian symmetric spaces with rank 2 of compact type, we have *complex two-plane Grassmannians* $G_2(\mathbb{C}^{m+2})$ which are the sets of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kaehler structure J and a quaternionic Kaehler structure $\mathfrak{J}$ not containing J, for details we refer to [1, 2, 15–18]. In particular, when $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism Spin(6) $\simeq$ SU(4) yields an isometry between

H. Lee (✉)
Research Institute of Real and Complex Manifolds (RIRCM),
Kyungpook National University, Daegu 41566, Republic of Korea
e-mail: lhjibis@hanmail.net

Y.J. Suh
Department of Mathematics and RIRCM, Kyungpook National
University, Daegu 41566, Republic of Korea
e-mail: yjsuh@knu.ac.kr

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_7

$G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. Hereafter, we will assume $m \geq 3$.

On a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be the *Reeb* vector field, where N denotes a local unit normal vector field to M in $G_2(\mathbb{C}^{m+2})$. And a real hypersurface such that $A[\xi] \subset [\xi]$ is called *Hopf hypersurface*. The *almost contact 3-structure* vector fields ξ_ν for the 3-dimensional distribution $\mathscr{Q}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where $\{J_\nu\}_{\nu=1,2,3}$ denotes a canonical local basis of a quaternionic Kaehler structure $\mathfrak{J}$, such that $T_xM = \mathscr{Q} \oplus \mathscr{Q}^\perp$, $x \in M$. In addition, a real hypersurface of $G_2(\mathbb{C}^{m+2})$ satisfying $g(A\mathscr{Q}, \mathscr{Q}^\perp) = 0$ (i.e. $A\mathscr{Q}^\perp \subset \mathscr{Q}^\perp$ or $A\mathscr{Q} \subset \mathscr{Q}$, resp.) is said to be a $\mathscr{Q}^\perp$*-invariant hypersurface*. We can naturally consider two geometric conditions that the 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$ and the 3-dimensional distribution $\mathscr{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M.

In a paper due to Berndt and Suh [3] we have introduced the following theorem.

Theorem 1 *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathscr{Q}^\perp$ are invariant under the shape operator of M if and only if*

(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or

(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

If the integral curves of the Reeb vector field ξ are geodesics on M in $G_2(\mathbb{C}^{m+2})$, we say that the Reeb flow on M is *geodesic*. By the basic property of ξ, that is, $\nabla_\xi \xi = \phi A \xi$, this condition is equivalent to M being a Hopf hypersurface. Specially, if the principal curvature function $\alpha = g(A\xi, \xi)$ of ξ is not vanishing on M, it is said that M has a *non-vanishing geodesic Reeb flow*. In addition, if the smooth function α satisfies $\xi\alpha = 0$, we say that the geodesic Reeb flow is *constant along the Reeb direction*. Moreover, when ξ is Killing, $\mathscr{L}_\xi g = 0$ for the Lie derivative along the direction of ξ, we say that the Reeb flow on M is *isometric*. This means that the metric tensor g is invariant along the Reeb flow on M. Then this is equivalent to the fact that the shape operator A of M commutes with the structure tensor ϕ of M in $G_2(\mathbb{C}^{m+2})$. Related to this concept we obtained a typical characterization of real hypersurfaces of type (A) given in Theorem 1. On the other hand, Lee and Suh [11] gave a characterization of Hopf hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$ as follows.

Theorem 2 *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathscr{Q}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

In this paper, we consider two generalizations of parallelism for the structure Jacobi operator of M in $G_2(\mathbb{C}^{m+2})$, namely, Reeb or $\mathscr{Q}^\perp$-recurrent structure Jacobi operator, respectively. Actually, if R denotes the Riemannian curvature tensor of M and X any tangent vector field to M, then the *Jacobi operator R_X with respect to*

X at $x \in M$ can be defined in such a way that $(R_X Y)(x) = (R(Y, X)X)(x)$ for any $Y \in T_x M$, $x \in M$. It becomes a self-adjoint endomorphism of the tangent bundle TM. From this definition, we obtain the Jacobi operator R_ξ with respect to the Reeb vector field $\xi \in TM$, which is said the *structure Jacobi operator* of M defined by $R_\xi Y = R(Y, \xi)\xi$ for all tangent vector fields Y to M.

Moreover, due to the definition given by Kobayashi and Nomizu [9] the structure Jacobi operator R_ξ on M is said to be *recurrent* if there exists a 1-form ω such that

$$(\nabla_X R_\xi)Y = \omega(X)R_\xi(Y) \qquad (*)$$

for all tangent vector fields X, Y on M. If the covariant derivative of R_ξ along any curve γ on M with $\dot{\gamma}(p) = X$, $p \in M$, vanishes identically, then we say that the structure Jacobi operator R_ξ of M is *parallel* (see [7]). Accordingly, we see that the concept of recurrency for R_ξ naturally generalizes parallelism. Specifically, we say that R_ξ is *Reeb recurrent* (or *$\mathscr{Q}^\perp$-recurrent*, resp.) if R_ξ satisfies (*) for $X = \xi$ (or $X \in \mathscr{Q}^\perp$, resp.). Then they are weaker conditions than recurrent structure Jacobi operator. Concerning such notions, in this paper we give some classifications of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ as follows.

Theorem 3 *Let M be a real hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with non-vanishing and constant geodesic Reeb flow along the Reeb direction. Then M has Reeb recurrent structure Jacobi operator if and only if M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \in (0, \frac{\pi}{4\sqrt{2}}) \cup (\frac{\pi}{4\sqrt{2}}, \frac{\pi}{2\sqrt{2}})$ and the one-form ω satisfies $\omega(\xi) = 0$.*

Remark 1 There are many results for the parallelism of structure Jacobi operator R_ξ of M in $G_2(\mathbb{C}^{m+2})$ ([4, 6, 7, 13], etc.). In particular, we see that if $\omega(\xi) = 0$, the notion of Reeb recurrent structure Jacobi operator is equivalent to Reeb parallelism, that is, $\nabla_\xi R_\xi = 0$. In [4] Jeong, Kim, and Suh introduced the notion of Reeb parallel structure Jacobi operator.

Theorem 4 *There do not exist real hypersurfaces in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with non-vanishing geodesic Reeb flow and $\mathscr{Q}^\perp$-recurrent structure Jacobi operator if the distribution $\mathscr{Q}$ or $\mathscr{Q}^\perp$-component of the Reeb vector field is invariant under the shape operator.*

2 Preliminaries

We use some references [5, 8, 14, 17] to recall the Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and some fundamental formulas including the Codazzi and Gauss equations for a real hypersurface in $G_2(\mathbb{C}^{m+2})$.

In this section let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Now we want to derive the structure Jacobi operator $R_\xi \in \text{End}(T_x M)$, $x \in M$, of M from the equation of Gauss. Since the structure Jacobi operator R_ξ is defined by $R_\xi Y := R(Y,\xi)\xi$ for any tangent vector field Y on M, we obtain

$$\begin{aligned} R_\xi Y &= Y - \eta(Y)\xi + \alpha AY - \alpha^2\eta(Y)\xi \\ &\quad + \sum_{\nu=1}^{3}\Big\{3g(Y,\phi_\nu\xi)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu\phi Y - \eta_\nu(Y)\xi_\nu + \eta_\nu(\xi)\eta(Y)\xi_\nu\Big\}, \end{aligned} \tag{1}$$

where $\alpha = g(A\xi,\xi)$. Moreover, from a well-known fact that the covariant derivative of R_ξ along any direction of X is defined by $(\nabla_X R_\xi)Y = \nabla_X(R_\xi Y) - R_\xi(\nabla_X Y)$, together with $(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX$ we have

$$\begin{aligned} (\nabla_X R_\xi)Y &= -g(\phi AX, Y)\xi - \eta(Y)\phi AX + (X\alpha)AY + \alpha(\nabla_X A)Y \\ &\quad - 2\alpha(X\alpha)\eta(Y)\xi - \alpha^2 g(Y,\phi AX)\xi - \alpha^2\eta(Y)\phi AX \\ &\quad - \sum_{\nu=1}^{3}\Big[\, g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta(\phi_\nu AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\ &\quad + 3g(\phi_\nu AX, \phi Y)\phi_\nu\xi + 3\eta(Y)\eta_\nu(AX)\phi_\nu\xi + 3\eta_\nu(\phi Y)\phi_\nu\phi AX \\ &\quad - 3\alpha\eta_\nu(\phi Y)\eta(X)\xi_\nu + 4\eta_\nu(\xi)\eta_\nu(\phi Y)AX \\ &\quad - 4\eta_\nu(\xi)g(AX,Y)\phi_\nu\xi + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\Big]. \end{aligned} \tag{2}$$

On the other hand, we can derive some facts from our assumption that M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with geodesic Reeb flow, that is, $A\xi = \alpha\xi$ where $\alpha = g(A\xi,\xi)$. Among them, we introduce a lemma which is induced from the equation of Codazzi [14].

Lemma 1 *If M is a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with geodesic Reeb flow, then*

$$\begin{aligned} &\operatorname{grad}\alpha = (\xi\alpha)\xi + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu \\ &\Big(i.e.\ \ X\alpha = (\xi\alpha)\eta(X) + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)g(\phi\xi_\nu, X)\Big) \end{aligned} \tag{1.3-(i)}$$

and

$$\begin{aligned} 2A\phi AX &= \alpha A\phi X + \alpha\phi AX + 2\phi X \\ &\quad + 2\sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu + \eta_\nu(\xi)\phi_\nu X \\ &\quad - 2\eta(X)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\Big\}, \end{aligned} \tag{1.3-(ii)}$$

for any tangent vector field X *on* M *in* $G_2(\mathbb{C}^{m+2})$.

As mentioned in Theorem 1, the complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with two kinds of A-invariances for the distributions $[\xi] = \text{Span}\{\xi\}$ and $\mathscr{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ was introduced in [17, 18]. Related to this result, we have the following two propositions with respect to the principal curvatures of the model spaces (A) and (B), respectively.

Proposition 1 *Let* M *be a connected real hypersurface of* $G_2(\mathbb{C}^{m+2})$. *Suppose that* $A\mathscr{Q} \subset \mathscr{Q}$, $A\xi = \alpha\xi$, *and* ξ *is tangent to* $\mathscr{Q}^\perp$. *Let* $J_1 \in \mathfrak{J}$ *be the almost Hermitian structure such that* $JN = J_1N$. *Then* M *has the following three (if* $r = \pi/2\sqrt{8}$*) or four (otherwise) distinct constant principal curvatures* α, β, λ *and* μ *with some* $r \in (0, \pi/\sqrt{8})$.

principal curvature	multiplicity	eigenspace
$\alpha = \sqrt{8}\cot(\sqrt{8}r)$	1	$T_\alpha = -\mathbb{R}JN = \text{Span}\{\xi\}$
$\beta = \sqrt{2}\cot(\sqrt{2}r)$	2	$T_\beta = \mathbb{C}^\perp N = \text{Span}\{\xi_2, \xi_3\}$
$\lambda = -\sqrt{2}\tan(\sqrt{2}r)$	$2(m-1)$	$T_\lambda = \{X \mid X \perp \mathbb{H}N,\ JX = J_1X\}$
$\mu = 0$	$2(m-1)$	$T_\mu = \{X \mid X \perp \mathbb{H}N,\ JX = -J_1X\}$

Here $\mathbb{R}N$, $\mathbb{C}N$ *and* $\mathbb{H}N$ *respectively denotes real, complex and quaternionic span of the structure vector field* ξ *and* $\mathbb{C}^\perp N$ *denotes the orthogonal complement of* $\mathbb{C}N$ *in* $\mathbb{H}N$.

Proposition 2 *Let* M *be a connected real hypersurface of* $G_2(\mathbb{C}^{m+2})$. *Suppose that* $A\mathscr{Q} \subset \mathscr{Q}$, $A\xi = \alpha\xi$, *and* ξ *is tangent to* $\mathscr{Q}$. *Then the quaternionic dimension* m *of* $G_2(\mathbb{C}^{m+2})$ *is even, say* $m = 2n$, *and* M *has five distinct constant principal curvatures and their corresponding multiplicities and corresponding eigenspaces are as follows.*

principal curvature	multiplicity	eigenspace
$\alpha = -2\tan(2r)$	1	$T_\alpha = \text{Span}\{\xi\}$
$\beta = 2\cot(2r)$	3	$T_\beta = \text{Span}\{\xi_\nu \mid \nu = 1, 2, 3\}$
$\gamma = 0$	3	$T_\gamma = \text{Span}\{\phi_\nu\xi \mid \nu = 1, 2, 3\}$
$\lambda = \cot(r)$	$4n-4$	T_λ
$\mu = -\tan(r)$	$4n-4$	T_μ

Here, the radius r *belongs to* $(0, \pi/4)$ *and the eigenspaces* T_λ *and* T_μ *satisfy the following properties.*

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}N)^\perp,\ \mathfrak{J}T_\lambda = T_\lambda,\ \mathfrak{J}T_\mu = T_\mu,\ JT_\lambda = T_\mu.$$

Finally, we would like to introduce some results which are very useful tools to study the following problem: *Whether or not the Reeb vector field* ξ *belongs to either the distribution* $\mathscr{Q}$ *or its orthogonal complement* $\mathscr{Q}^\perp$ *under our assumptions*. In [14], Pérez and Suh proved the following lemma by using (1.3-(i)):

Lemma 2 *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M has vanishing (or constant) geodesic Reeb flow, then the Reeb vector field ξ belongs to either the distribution $\mathscr{Q}$ or the distribution $\mathscr{Q}^{\perp}$.*

On the other hand, from the property of the gradient of Reeb function α, that is, $g(\nabla_X \text{grad}\, \alpha, Y) = g(X, \nabla_Y \text{grad}\, \alpha)$ for all $X, Y \in TM$ it is proved (see [6, 10]):

Lemma 3 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. The principal curvature α is constant along the direction of ξ if and only if the $\mathscr{Q}$ and $\mathscr{Q}^{\perp}$-components of the structure vector field ξ are invariant by the shape operator A.*

Furthermore, by using the basic formulas for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ we get:

Lemma 4 *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with non-vanishing geodesic Reeb flow. If the Reeb vector field ξ is given by $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit vector field $X_0 \in \mathscr{Q}$ and $\xi_1 \in \mathscr{Q}^{\perp}$ such that $\eta(X_0)\eta(\xi_1) \neq 0$, then we have:*

(i) $\phi X_0 = -\eta(\xi_1)\phi_1 X_0$, $\phi\xi_1 = \eta(X_0)\phi_1 X_0$,
(ii) $g(\phi X_0, \phi X_0) = \eta^2(\xi_1)$, $g(\phi_1\xi, \phi X_0) = -\eta(X_0)\eta(\xi_1)$, $g(\phi_1 X_0, \phi_1 X_0) = 1$.

Moreover, if the $\mathscr{Q}$ (or $\mathscr{Q}^{\perp}$)-component of ξ is principal, then it gives us:

(iii) *the vector fields ϕX_0, $\phi_1 X_0$ and $\phi_1\xi$ are also principal where their corresponding eigenvalues are given by $\lambda = \big(\alpha^2 + 4\eta^2(X_0)\big)/\alpha$,*
(iv) $q_\nu(\xi) = 0$, $q_\nu(X_0) = 0$, $q_\nu(\xi_1) = 0$ *for* $\nu = 2, 3$,
(v) $\nabla_{X_0} X_0 = \delta\phi_1 X_0 = \sigma\phi X_0$ *where* $\delta = (-2\alpha\eta(\xi_1))/\eta(X_0)$ *and* $\sigma = 2\alpha/\eta(X_0)$,
(vi) $\nabla_{X_0}\xi_1 = \alpha\phi_1 X_0$, $\nabla_{\xi_1}\xi_1 = 0$, $\nabla_{\xi_1} X_0 = \alpha\phi_1 X_0$.

3 Hopf Hypersurfaces in Complex Two-Plane Grassmannians with Reeb Recurrent Structure Jacobi Operator

Throughout this section, let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb recurrent structure Jacobi operator. It means that there exists a one form ω on M such that $(\nabla_\xi R_\xi)Y = \omega(\xi)R_\xi Y$ for all tangent vector fields Y to M. Thus, from the Eqs. (1) and (2), the structure Jacobi operator R_ξ of M satisfies

$$
\begin{aligned}
&(\nabla_\xi R_\xi)Y = \omega(\xi)R_\xi Y \\
&\iff (\xi\alpha)AY + \alpha(\nabla_\xi A)Y - 2\alpha(\xi\alpha)\eta(Y)\xi \\
&\quad - \sum_{\nu=1}^{3} 4\alpha\Big\{g(\phi_\nu\xi, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu\xi - \eta_\nu(\xi)g(\phi_\nu\xi, Y)\xi - \eta_\nu(\xi)\eta(Y)\phi_\nu\xi\Big\} \\
&\quad = \omega(\xi)\Big[Y - \eta(Y)\xi + \alpha AY - \alpha^2\eta(Y)\xi \\
&\quad + \sum_{\nu=1}^{3}\Big\{3g(Y, \phi_\nu\xi)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu\phi Y - \eta_\nu(Y)\xi_\nu + \eta_\nu(\xi)\eta(Y)\xi_\nu\Big\}\Big],
\end{aligned} \tag{3}
$$

where Y belongs to the tangent vector space T_xM at any point x of M. Using this equation, we prove that:

Lemma 5 *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb recurrent structure Jacobi operator. If the Reeb function α is constant along the direction of the Reeb vector field ξ, then ξ is tangent either to $\mathscr{Q}$ or to $\mathscr{Q}^\perp$.*

Proof To prove it, we suppose that the Reeb vector field ξ is given by

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{†}$$

with $\eta(X_0)\eta(\xi_1) \neq 0$ for some unit $X_0 \in \mathscr{Q}$ and $\xi_1 \in \mathscr{Q}^\perp$. The result is trivial when the smooth function $\alpha = g(A\xi, \xi)$ identically vanishes by virtue of Lemma 2. So, we only consider the case that α is non-vanishing. Since we now assume the principal curvature α is constant along the direction of ξ, that is, $\xi\alpha = 0$, we see that X_0 and ξ_1 become principal vector fields with their corresponding principal curvatures α given by Lemma 3. From this, if we put $Y = X_0$ in (3), then we get

$$
\begin{aligned}
&\alpha(\nabla_\xi A)X_0 + 4\alpha\eta(\xi_1)\eta(X_0)\phi_1\xi \\
&\quad = \omega(\xi)\Big[X_0 - \eta(X_0)\xi + \alpha^2 X_0 - \alpha^2\eta(X_0)\xi - \eta(\xi_1)\phi_1\phi X_0 + \eta(\xi_1)\eta(X_0)\xi_1\Big].
\end{aligned}
$$

And taking the inner product with ϕX_0, together with formulas in Lemma 4 it follows

$$\alpha g((\nabla_\xi A)X_0, \phi X_0) - 4\alpha\eta^2(\xi_1)\eta^2(X_0) = 0. \tag{4}$$

Moreover, since $(\nabla_\xi A)X_0 = (X_0\alpha)\xi + \alpha\phi AX_0 - A\phi AX_0 + \phi X_0 + \eta(\xi_1)\phi_1 X_0$ from the equation of Codazzi, we see that $g((\nabla_\xi A)X_0, \phi X_0) = -4\alpha\eta^2(\xi_1)\eta(X_0)$ by virtue of Lemma 4. It follows that the equation (4) implies $\alpha\eta^2(\xi_1)\eta^2(X_0) = 0$. In fact, since we only consider the case of $\alpha \neq 0$ with $\eta(X_0)\eta(\xi_1) \neq 0$, it makes a contradiction. Hence, we can assert that the Reeb vector field ξ belongs to either $\mathscr{Q}$ or $\mathscr{Q}^\perp$ under our assumptions. □

Thus, we shall divide our observation in two cases depending on the Reeb vector field ξ belongs to either $\mathscr{Q}$ or $\mathscr{Q}^\perp$. When $\xi \in \mathscr{Q}$, by virtue of Theorem 2, we see that a real hypersurface M with our assumptions becomes a $\mathcal{Q}^\perp$-invariant hypersurface in $G_2(\mathbb{C}^{m+2})$. Next, we consider the case $\xi \in \mathcal{Q}^\perp$. Without loss of generality, we may put $\xi = \xi_1$.

Lemma 6 *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying the following three conditions:*

(a) M has non-vanishing geodesic Reeb flow whose is constant along the Reeb direction, that is, $\alpha = g(A\xi, \xi) \neq 0$ and $\xi\alpha = 0$,
(b) the structure Jacobi operator R_ξ of M is Reeb recurrent, that is, R_ξ satisfies $(\nabla_X R_\xi)Y = \omega(X)R_\xi Y$, and
(c) the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$.

Then the distribution $\mathcal{Q}^\perp$ (or $\mathcal{Q}$, respectively) is invariant under the shape operator A of M, that is, $g(A\mathcal{Q}, \mathcal{Q}^\perp) = 0$.

Proof We may put $\xi = \xi_1$, because $\xi \in \mathcal{Q}^\perp$. From the assumption of $\xi\alpha = 0$, the Eq. (3) can be written as

$$\begin{aligned} &\alpha(\nabla_\xi A)Y \\ &\quad = \omega(\xi)\Big[Y - \eta(Y)\xi + \alpha AY - \alpha^2\eta(Y)\xi + 2\eta_2(Y)\xi_2 + 2\eta_3(Y)\xi_3 - \phi_1\phi Y\Big] \end{aligned} \tag{5}$$

for any tangent vector field Y on M. On the other hand, by the equation of Codazzi, the left-hand side of (5) becomes

$$\begin{aligned} (\nabla_\xi A)Y &= (\nabla_Y A)\xi + \phi Y + \phi_1 Y + 2\eta_3(Y)\xi_2 - 2\eta_2(Y)\xi_3 \\ &= (Y\alpha)\xi + \alpha\phi AY - A\phi AY + \phi Y + \phi_1 Y + 2\eta_3(Y)\xi_2 - 2\eta_2(Y)\xi_3, \end{aligned}$$

where the second equality holds because M is Hopf. By Lemma 1 and $\xi\alpha = 0$, the Eq. (5) becomes

$$\begin{aligned} &\frac{\alpha^2}{2}\phi AY - \frac{\alpha^2}{2}A\phi Y \\ &= \omega(\xi)\Big[Y - \eta(Y)\xi + \alpha AY - \alpha^2\eta(Y)\xi + 2\eta_2(Y)\xi_2 + 2\eta_3(Y)\xi_3 - \phi_1\phi Y\Big] \end{aligned} \tag{6}$$

for all tangent vector fields Y on M.

Let Ω_1 be subset of M given by $\Omega_1 = \{p \in M \mid \omega(\xi)(p) = \omega(\xi_p) \neq 0\}$. From now on, we will show that $g(A\xi_\mu, X) = 0$, $\mu = 1, 2, 3$, for all $X \in \mathcal{Q}$ on the open set Ω_1. Actually, when $\mu = 1$, it is true, since $\xi = \xi_1 \in \mathcal{Q}$ and M is Hopf. So, we will prove that $g(A\xi_2, X) = g(A\xi_3, X) = 0$ for $X \in \mathcal{Q}$ on Ω_1. Putting $Y = \xi_2$ and $Y = \xi_3$ in (6), we obtain $\alpha^2\phi A\xi_2 + \alpha^2 A\xi_3 = 2\omega(\xi)\big\{\alpha A\xi_2 + 2\xi_2\big\}$ and $\alpha^2\phi A\xi_3 - \alpha^2 A\xi_2 = 2\omega(\xi)\big\{\alpha A\xi_3 + 2\xi_3\big\}$, respectively. Taking the inner product with $X \in \mathcal{Q}$, then these equations give us

$$\begin{cases} -\alpha^2\eta_2(A\phi X) + \alpha^2\eta_3(AX) - 2\alpha\omega(\xi)\eta_2(AX) = 0, \\ -\alpha^2\eta_3(A\phi X) - \alpha^2\eta_2(AX) - 2\alpha\omega(\xi)\eta_3(AX) = 0. \end{cases} \tag{7}$$

On the other hand, applying the structure tensor field ϕ to (6), we obtain

$$\begin{aligned} & -\alpha^2 AY + \alpha^3\eta(Y)\xi - \alpha^2\phi A\phi Y \\ & = 2\omega(\xi)\Big[\phi Y + \alpha\phi AY - 2\eta_2(Y)\xi_3 + 2\eta_3(Y)\xi_2 + \phi_1 Y\Big], \end{aligned} \tag{8}$$

where $\phi\phi_1\phi Y = \phi^2\phi_1 Y = -\phi_1 Y$. Similarly, putting $Y = \xi_2$ and ξ_3 in (8) and taking the inner product with $X \in \mathcal{Q}$, we have

$$\begin{cases} -\alpha^2\eta_2(AX) - \alpha^2\eta_3(A\phi X) + 2\alpha\omega(\xi)\eta_2(A\phi X) = 0, \\ -\alpha^2\eta_3(AX) + \alpha^2\eta_2(A\phi X) + 2\alpha\omega(\xi)\eta_3(A\phi X) = 0. \end{cases} \tag{9}$$

The four equations in (7) with (9) can be expressed as the product of matrices as follows.

$$\begin{pmatrix} -k & m & -m & 0 \\ -m & -k & 0 & -m \\ -m & 0 & k & -m \\ 0 & -m & m & k \end{pmatrix} \begin{pmatrix} \eta_2(AX) \\ \eta_3(AX) \\ \eta_2(A\phi X) \\ \eta_3(A\phi X) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{10}$$

where $m = \alpha^2$ and $k = 2\alpha\omega(\xi)$. In fact, the determinant of the 4×4 matrix W is given by

$$\begin{aligned} \det W &= \det \begin{pmatrix} -k & m & -m & 0 \\ -m & -k & 0 & -m \\ -m & 0 & k & -m \\ 0 & -m & m & k \end{pmatrix} = \begin{vmatrix} -k & m & -m & 0 \\ -m & -k & 0 & -m \\ -m & 0 & k & -m \\ 0 & -m & m & k \end{vmatrix} \\ &= -k \begin{vmatrix} -k & 0 & -m \\ 0 & k & -m \\ -m & m & k \end{vmatrix} - m \begin{vmatrix} -m & 0 & -m \\ -m & k & -m \\ 0 & m & k \end{vmatrix} - m \begin{vmatrix} -m & -k & -m \\ -m & 0 & -m \\ 0 & -m & k \end{vmatrix} \\ &= k^2(k^2 + 4m^2), \end{aligned}$$

and it does not vanish on Ω_1. So, there exists the inverse matrix of W denoted by W^{-1}. From this and (10), we see that $\eta_2(AX) = 0$ and $\eta_3(AX) = 0$ for any $X \in \mathcal{Q}$, that is,

$$\begin{pmatrix} \eta_2(AX) \\ \eta_3(AX) \\ \eta_2(A\phi X) \\ \eta_3(A\phi X) \end{pmatrix} = W^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, we assert that $g(A\mathscr{Q}^{\perp}, \mathscr{Q}) = 0$ on Ω_1 under our hypotheses in Lemma 6.

In fact, $\Omega_0 := M - \Omega_1$ is the complementary set of Ω_1 in M, and $\Omega_0 = \text{Int}\Omega_0 \cup \partial\Omega_0$ where $\text{Int}\Omega_0$ and $\partial\Omega_0$ denote interior and boundary of Ω_0, respectively. Thus we consider our lemma on $\text{Int}\Omega_0$. From the definition of $\text{Int}\Omega_0$ the structure Jacobi operator R_ξ becomes Reeb parallel, that is, $(\nabla_\xi R_\xi) = 0$. Related to this notion, Jeong, Kim, and Suh [4] already gave a characterization of a real hypersurface of type (A) in $G_2(\mathbb{C}^{m+2})$ with non-vanishing geodesic Reeb flow. According to their method, we know that on $\text{Int}\Omega_0$ it holds $g(A\mathscr{Q}, \mathscr{Q}^{\perp}) = 0$. Finally, let us assume that $p \in \partial\Omega_0$. Then there exists a subsequence $\{p_n\} \subset \Omega_1$ such that $p_n \to p$. Since $g(A\mathscr{Q}^{\perp}, \mathscr{Q})(p_n) = 0$ on the open subset Ω_1 in M, by the continuity we also get $g(A\mathscr{Q}^{\perp}, \mathscr{Q})(p) = 0$ on $\partial\Omega_0$. Hence, we get a complete proof of our lemma. □

Remark 2 Actually, when M has vanishing geodesic Reeb flow (it is denoted M_0 for our convenience), we can also consider a similar problem such as Lemma 6. For this case, we should observe our problem with the following two cases:

- $\Omega_1^* = \{p \in M_0 \,|\, \omega(\xi)(p) = \omega(\xi_p) \neq 0\}$,
- $\Omega_0^* = M_0 - \Omega_1^*$.

Assume that p belongs to Ω_1^* which is an open set in M_0. Then we see that the structure Jacobi operator identically vanishes on Ω_1^*, that is, $R_\xi Y = 0$ for all tangent vector field Y. From this, we obtain that the structure Jacobi operator R_ξ of Ω_1^* satisfies the commuting condition $R_\xi\phi A = 0 = AR_\xi\phi$. So, by virtue of the proof given in [12], we can assert that when $\xi \in \mathscr{Q}^{\perp}$, it holds $g(A\mathscr{Q}, \mathscr{Q}^{\perp}) = 0$ on Ω_1^*. But, on $\text{Int}\,\Omega_0^*$ we do not have enough information to solve our problem, since the Reeb recurrent structure Jacobi operator becomes a meaningless notion. Hence we do not focus our consideration on it. Therefore, we only consider the case that M has non-vanishing geodesic Reeb flow in Lemma 6.

Summing up these discussions, we assert that real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with all assumptions given in Theorem 3 become one of the model spaces given in Theorem 1. Therefore, let us check the converse problem, that is, whether or not model spaces of type (A) and type (B) satisfy the conditions in Theorem 3. In fact, we know that both model spaces are Hopf and their principal curvature $\alpha = g(A\xi, \xi)$ is constant by Propositions 1 and 2, respectively. From such a point of view, in the following two lemmas we finally show if the structure Jacobi operator R_ξ of model spaces is Reeb recurrent or not.

Lemma 7 *Let M_A be a real hypersurface of type (A) in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the structure Jacobi operator R_ξ of M_A is Reeb recurrent if the one-form ω satisfies $\omega(\xi) = 0$.*

Proof By virtue of Proposition 1, we see that the tangent vector space T_pM_A at $p \in M_A$ has four eigenspaces, T_α, T_β, T_λ, and T_μ. So, let us observe whether or not the equality in (6) holds for each eigenspace of M_A.

Since the shape operator A commutes with the structure tensor ϕ on all eigenspaces in T_pM_A, that is, $\phi A = A\phi$, the left-hand side of (6) is zero for all eigenspaces of M_A, that is, $(\nabla_\xi R_\xi)Y = \frac{\alpha^2}{2}(\phi A - A\phi)Y = 0$ for all $Y \in T_pM_A$.
On the other hand, the left-hand side of (6) is given by

$$\omega(\xi)R_\xi Y = \begin{cases} 0, & \text{if } Y = \xi \in T_\alpha, \\ \omega(\xi)(\alpha\beta + 2)\xi_\mu, & \text{if } Y = \xi_\mu \in T_\beta, \\ \omega(\xi)(\alpha\lambda + 2)Y, & \text{if } Y \in T_\lambda, \\ 0, & \text{if } Y \in T_\mu. \end{cases}$$

Thus it should be $\omega(\xi) = 0$ to hold the equality in (6), since $\alpha = \sqrt{8}\cot(\sqrt{8}r)$, $\beta = \sqrt{2}\cot(\sqrt{2}r)$ and $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ for some $r \in (0, \frac{\pi}{\sqrt{8}})$. □

Lemma 8 *Let M_B be a real hypersurface of type (B) in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the structure Jacobi operator R_ξ of M_B is not Reeb recurrent.*

Proof Suppose that M_B has Reeb recurrent structure Jacobi operator. Then putting $Y = \xi_\kappa \in T_\beta = \text{Span}\{\xi_\kappa \,|\, \kappa = 1, 2, 3\}$ in (3), we obtain

$$(\nabla_\xi R_\xi)\xi_\kappa = \omega(\xi)R_\xi\xi_\kappa \iff \alpha(\nabla_\xi A)\xi_\kappa - 4\alpha\phi_\kappa\xi = \omega(\xi)\alpha\beta\xi_\kappa,$$

since $T_pM_B = T_\alpha \oplus T_\beta \oplus T_\gamma \oplus T_\lambda \oplus T_\mu$ and α is constant from Proposition 2. Moreover it consequently becomes $\alpha(\alpha\beta - 4)\phi_\kappa\xi = \omega(\xi)\alpha\beta\xi_\kappa$, by $(\nabla_\xi A)\xi_\kappa = \beta(\nabla_\xi\xi_\kappa) - A(\nabla_\xi\xi_\kappa)$ and $\nabla_\xi\xi_\kappa = q_{\kappa+2}(\xi)\xi_{\kappa+1} - q_{\kappa+1}(\xi)\xi_{\kappa+2} + \alpha\phi_\kappa\xi$. Hence, taking the inner product of this equation with $\phi_\kappa\xi$, we obtain $\alpha = 0$, because the principal curvatures α and β on M_B are given by $\alpha = -2\tan(2r)$ and $\beta = 2\cot(2r)$ for $r \in (0, \frac{\pi}{4})$. It makes a contradiction. So, we complete the proof of our lemma. □

4 Hopf Hypersurfaces in Complex Two-plane Grassmannians with $\mathscr{Q}^\perp$-Recurrent Structure Jacobi Operator

In this section, we will prove our Theorem 4 given in the introduction. It is said that M is a real hypersurface with $\mathscr{Q}^\perp$-recurrent structure Jacobi operator if the structure Jacobi operator R_ξ of M satisfies $(\nabla_{\xi_\kappa} R_\xi)Y = \omega(\xi_\kappa)R_\xi Y$ for all $\kappa = 1, 2, 3$.

Now we want to prove that the Reeb vector field ξ belongs to either the distribution $\mathscr{Q}$ or its orthogonal complement distribution $\mathscr{Q}^\perp$ under our hypotheses.

Lemma 9 *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathscr{Q}^\perp$-recurrent structure Jacobi operator. If the $\mathscr{Q}$ or $\mathscr{Q}^\perp$-components of the Reeb vector field ξ is invariant under the shape operator A of M, then ξ belongs to either the distribution $\mathscr{Q}$ or the distribution $\mathscr{Q}^\perp$.*

Proof To show this lemma, we may put the Reeb vector field ξ as follows.

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{†}$$

for some unit $X_0 \in \mathscr{Q}$ and $\xi_1 \in \mathscr{Q}^{\perp}$ with $\eta(X_0)\eta(\xi_1) \neq 0$. When the function $\alpha = g(A\xi, \xi)$ identically vanishes, the result can be obtained directly from Lemma 2. So we consider that the smooth function α is non-vanishing. Putting $X = \xi_1 \in \mathscr{Q}^{\perp}$ and $Y = \xi$ in (1) and (2), together with two equations (1.3-(i)), (†) and two Lemmas 3 and 4, the condition of $\mathscr{Q}^{\perp}$-recurrent structure Jacobi operator give us

$$(\nabla_{\xi_1} R_{\xi})\xi = \omega(\xi_1)R_{\xi}\xi \iff -\alpha(\alpha^2 + 8\eta^2(X_0))\phi\xi_1 = 0. \tag{11}$$

Since α and $\eta(X_0)$ are non-vanishing, we obtain $\phi\xi_1 = 0$. It makes a contradiction, because $g(\phi\xi_1, \phi\xi_1) = -g(\phi^2\xi_1, \xi_1) = \eta^2(X_0) \neq 0$. Accordingly, we get a complete proof of our Lemma. □

By this lemma, we first study the case that the Reeb vector field ξ belongs to the distribution $\mathscr{Q}^{\perp}$, that is, $\xi \in \mathscr{Q}^{\perp}$. Without loss of generality, we may put $\xi = \xi_1$. Since $\mathscr{Q}^{\perp} = \text{Span}\{\xi_\kappa \mid \kappa = 1, 2, 3\}$, if we take $\kappa = 1$, then the $\mathscr{Q}^{\perp}$-recurrent structure Jacobi operator property R_{ξ} coincides with the Reeb recurrent property. Hence by virtue of the proof in Lemma 6, we can assert that:

Lemma 10 *Let M be a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying the following four conditions:*

(a) M has non-vanishing geodesic Reeb flow, that is, $\alpha = g(A\xi, \xi) \neq 0$,
(b) the structure Jacobi operator R_{ξ} of M is $\mathscr{Q}^{\perp}$-recurrent, that is, R_{ξ} satisfies $(\nabla_{\xi_\kappa} R_{\xi})Y = \omega(\xi_\kappa)R_{\xi}Y$ for all $\kappa = 1, 2, 3$,
(c) the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ, that is, $\xi\alpha = 0$, and
(d) the Reeb vector field ξ belongs to the distribution $\mathscr{Q}^{\perp}$.

Then the distribution $\mathscr{Q}^{\perp}$ (or $\mathscr{Q}$, respectively) is invariant under the shape operator A of M, that is, $g(A\mathscr{Q}, \mathscr{Q}^{\perp}) = 0$.

But a model space M_A of type (A) does not have $\mathscr{Q}^{\perp}$-recurrent structure Jacobi operator. Actually, we suppose that the structure Jacobi operator R_{ξ} of M_A has the property of $\mathscr{Q}^{\perp}$-recurrency. That is, the structure Jacobi operator R_{ξ} should satisfy $(\nabla_{\xi_\kappa} R_{\xi})Y = \omega(\xi_\kappa)R_{\xi}Y$ for all $\kappa = 1, 2, 3$ and for all tangent vector fields $Y \in T_pM_A = T_\alpha \oplus T_\beta \oplus T_\lambda \oplus T_\mu$, $p \in M_A$. From this fact, we obtain $\beta(\alpha\beta + 2) = 0$ for the restricted case of $\kappa = 2$ and $Y \in T_\alpha$. By the way, we see that it makes a contradiction by virtue of Proposition 1.

On the other hand, if the Reeb vector field ξ belongs to the distribution $\mathscr{Q}$, then a real hypersurface with our assumptions in Lemma 9 is of type (B) owing to Theorem 2. As the next step we should observe the converse problem, that is, whether a model space M_B of type (B) in Theorem 1 has our conditions in Lemma 9 or not. Now, among them let us focus our considerations on the hypothesis of $\mathscr{Q}^{\perp}$-recurrent structure Jacobi operator on M_B. Since a tangent vector space T_pM_B at a point $p \in M_B$ has five eigenspaces T_α, T_β, T_γ, T_λ and T_μ, if we check our condition for the case $Y = \xi \in T_\alpha$, then the eigenvalue β must be zero. But on M_B it does not occur for some radius $r \in (0, \frac{\pi}{4})$.

Summing up these observations in Sect. 4, we complete the proof of our Theorem 4 in the introduction. □

Acknowledgements We thank Professor Juan de Dios Pérez for his useful comments and suggestions on the previous version of this paper. And this work was supported by National Research Foundation of Korea(NRF) grants No. 2015-R1A2A1A01002459.

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Hamiltonian Non-displaceability of the Gauss Images of Isoprametric Hypersurfaces (A Survey)

Reiko Miyaoka

Abstract This is a survey of the joint work [13] (Bull Lond Math Soc 48(5), 802–812, 2016) with Hiroshi Iriyeh (Ibaraki U.), Hui Ma (Tsinghua U.) and Yoshihiro Ohnita (Osaka City U.). The Floer homology of Lagrangian intersections is computed in few cases. Here, we take the image $L = \mathscr{G}(N)$ of the Gauss map of isoparametric hypersurfaces N in S^{n+1}, that are minimal Lagrangian submanifolds of the complex hyperquadric $Q^n(\mathbb{C})$. We call L *Hamiltonian non-displaceable* if $L \cap \varphi(L) \neq \emptyset$ holds for any Hamiltonian deformation φ. Hamiltonian non-displaceability is needed to define the Floer homology $HF(L)$, since $HF(L)$ is generated by points in $L \cap \varphi(L)$. We prove the Hamiltonian non-displaceability of $L = \mathscr{G}(N)$ for any isoparametric hypersurfaces N with principal curvatures having plural multiplicities. The main result is stated in Sect. 4.

1 Introduction of Isoparametric Hypersurfaces

The family of isoparametric hypersurfaces in spheres is rich, containing infinitely many homogeneous and non-homogeneous hypersurfaces. Although the topology and differential geometric properties are well investigated, they are not so familiar. Hence we start from a brief introduction of the subject.

1.1 History

The study of isoparametric hypersurfaces began in geometric optics, and the research is divided into 4 periods:

1. 1918–1924: Laura, Somigliana (geometric optics in $\mathbb{R}^3$)
2. 1937–1940: Levi-Civita, B. Segre, É. Cartan (in $\mathbb{R}^{n+1}$ and in the space forms)

R. Miyaoka (✉)
Tohoku University, Institute of Liberal Arts and Sciences, Sendai 9808576, Japan
e-mail: r-miyaok@m.tohoku.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_8

3. 1973–1985: Münzner, Nomizu, Ozeki, Takeuchi, Ferus, Karcher, Abresch, Dorfmeister, Neher (in S^{n+1})
4. 2007: Cecil, Chi, Jensen, H. Ma, Ohnita, and the author.

A nice reference of the history is [24], and a general reference is [5].

Let (M, g) be an $(n+1)$-dimensional Riemannian manifold, and let N be an embedded hypersurface with a unit normal vector field ξ. Consider a wave (light, heat, sound, etc.) starting from N with the velocity $v(p, X) \in \mathbb{R}$, where $X \in T_pN$ is unit. Put

$$\Phi_t(p) = \{\text{points to which the ray emanating from } p \text{ with } v(p, X) \text{ reaches in time } t\}.$$

Then the "wave front" N_t is given by the envelope of $\Phi_t(p)$ by Huygens' principle.

Definition 1 N_t is *parallel* to each other if the distance between N_t and $N_{t'}$ is constant for each t'.

Lemma 1 *N_t is parallel to each other when $v(p, \xi_p)$ is independent of p.*

Definition 2 When a family of parallel submanifolds $\{N_t\}$ sweeps out M where almost N_t are regular hypersurfaces, and some are regular submanifolds of M, we call N_t an *isoparametric hypersurface*, and a *focal submanifold*, according to the dimension.

Now, consider a mechanical displacement $\Phi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ in $\mathbb{R}^3$ of a wave satisfying the wave equation

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \Delta \Phi \tag{1}$$

where c is a constant and Δ is the Laplacian $\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$. A mechanical displacement means, for instance, a gap of a vibrating spring from the original position. Let a be the biggest gap. We call $N_t = \Phi^{-1}(a) \cap (\mathbb{R}^3 \times \{t\})$ the *wave front*.

Fact 1 (Laura 1918, Somigliana 1918–19) *If Φ satisfies the wave equation (1) and each wave front $N_t = \Phi^{-1}(a) \cap (\mathbb{R}^3 \times \{t\})$ is parallel to each other, then N_t is one of*

$$\mathbb{R}^2, \quad S^2, \quad S^1 \times \mathbb{R}.$$

Fact 2 (Segre 1924) *Let $\{N_t\}$ be a family of parallel surfaces in $\mathbb{R}^3$. If there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that f and Δf are constant on each N_t, then N_t is one of*

$$\mathbb{R}^2, \quad S^2, \quad S^1 \times \mathbb{R}.$$

Fact 3 *Let $\varphi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be the heat conduction*

$$\frac{\partial \varphi}{\partial t} = c^2 \Delta \varphi.$$

If φ is reduced to one-dimensional, then the level surface of φ is one of

$$\mathbb{R}^2, \quad S^2, \quad S^1 \times \mathbb{R}.$$

This is because on each level surface, φ is constant since φ is reduced to one-dimensional, and so is $\frac{\partial \varphi}{\partial t}$ as levels are parallel to each other. Then $\Delta\varphi$ is constant on the level, and the conclusion follows from Fact 2.

1.2 Isoprametric Functions

Recall that on a Riemannian manifold (M, g), the *gradient vector* ∇f and the *Laplacian* Δf of a function $f \in C^\infty(M)$ is given, respectively, by

$$\nabla f = g^{ij}\Big(\frac{\partial f}{\partial x_i}\Big)\frac{\partial}{\partial x_j}, \quad \Delta f = \mathrm{div}\nabla f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}\Big(\sqrt{g}g^{ij}\frac{\partial f}{\partial x_i}\Big).$$

Definition 3 (i) A C^2 function $f : M \to \mathbb{R}$ is called an *isoparametric function* if f satisfies

$$\text{(I) } |\nabla f|^2 = b(f)$$

$$\text{(II) } \Delta f = a(f),$$

where $b(f)$ is a C^2, and $a(f)$ is a C^0 function of f.

(ii) When f is an isoparametric function on M, $N_t = f^{-1}(t)$ is called an *isoparametric hypersurface* if t is a regular value of f, and a *focal submanifold* if t is a critical value.

Remark Definition 2 of isoparametric hypersurfaces coincides with this definition.

Example When $M = \mathbb{R}^{n+1}$, the following gives isoparametric functions and isoparametric hypersurfaces.

1. $f(x) = x^{n+1}, \quad |\nabla f|^2 = 1, \quad \Delta f = 0$ and N_t is an n-plane.
2. $f(x) = |x|^2, |\nabla f|^2 = 4f, \quad \Delta f = 2(n+1)$, and N_t is an n-sphere $S^n(\sqrt{t})$, $(t > 0)$, and N_0 is a point.
3. $f(x) = \sum_{i=1}^{k+1} x_i^2, \quad |\nabla f|^2 = 4f, \quad \Delta f = 2(k+1)$, and N_t is $S^k(\sqrt{t}) \times \mathbb{R}^{n-k}$ $(t > 0)$ and $N_0 = \mathbb{R}^{n-k}$.

Remark (a) The condition (I) implies that $\{N_t\}$ is a parallel family. In fact, since f is constant on N_t, ∇f is a normal vector of N_t, and (I) means that the normal velocity is constant on each level.
(b) (II) implies that N_t has constant mean curvature (CMC). See Appendix.
(c) Isoparametric functions are not unique for an isoparametric family $\{N_t\}$. For

instance, take $f(x) = |x|^2$ in Example 2, and consider $g(x) = |x|^4$, that gives the same level sets as f. Here g is also an isoparametric function, since

$$|\nabla g|^2 = 16g\sqrt{g}, \quad \Delta g = 4(n+3)\sqrt{g}.$$

(d) In the space form, functions satisfying (I) have CMC level set, namely, $\{N_t\}$ is a family of isoparametric hypersurfaces (see Fact 4.1 below). This holds only when M is a space form. Functions satisfying (I) is called *transnormal functions*. A transnormal function is not necessarily an isoparametric function, but the level hypersurfaces become isoparametric hypersurfaces if M is a space form [17].

Fact 4 (É. Cartan 1937–38) *Let $M(c) = \mathbb{R}^{n+1}$, S^{n+1} or H^{n+1} according to $c = 0, 1, -1$. Let $\{N_t\}$ be a family of parallel hypersurfaces. Then the following holds:*

(i) *$\{N_t\}$ is a family of isoparametric hypersurfaces $\Leftrightarrow$ All N_t have CMC $\Leftrightarrow$ Some N_t has constant principal curvatures.*
(ii) *When N_t is an isoparametric hypersurface, let $\kappa_1 > \ldots \kappa_g$ be distinct principal curvatures with multiplicities $m_1, \ldots, m_g$, respectively. Then*

$$\sum_{\kappa_z \neq \kappa_i} \frac{m_i(c + \kappa_a \kappa_i)}{\kappa_a - \kappa_i} = 0 \tag{2}$$

holds, and this is called the Cartan formula.
(iii) *When $c \leq 0$, $g \leq 2$ follows from (2). When $g = 1$, N_t is totally geodesic or totally umbilic. When $g = 2$, N_t is a tube over totally geodesic submanifold. In particular when $M = \mathbb{R}^{n+1}$, these are given by*

$$\mathbb{R}^n, \quad S^n, \quad S^k \times \mathbb{R}^{n-k}.$$

(iv) *When $c > 0$, there exist examples for $g = 3$ and 4. In particular, when $g = 3$, they are given as tubes over the standard embedding of the projective 2-plane $\mathbb{F}P^2$ into S^{3d+1}, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathscr{C}ay$, and $d = 1, 2, 4, 8$, respectively. These are called the* Cartan hypersurfaces.

1.3 Isoparametric Hypersurfaces in S^{n+1}

Now, consider more details in the case $M(1) = S^{n+1}$. Namely, let N^n be an isoparametric hypersurface in S^{n+1}, i.e., a hypersurface with constant principal curvatures by Fact 4 (i). Obviously, all homogeneous hypersurfaces in S^{n+1} are isoparametric, since they have constant principal curvatures. They are given as isotropy orbits of rank two symmetric spaces of compact type, and classified completely [12].

Fact 5 (Münzner 1981) *Let $\kappa_1 > \cdots > \kappa_g$ be distinct principal curvatures of an isoparametric hypersurface N in S^{n+1} with multiplicities $m_1, \ldots, m_g$, respectively. Then $g \in \{1, 2, 3, 4, 6\}$ and $m_i = m_{i+2}$ (i : mod g)*

Classification.

1. When $g = 1$, $N = S^n(r)$, $0 < r < 1$, a hypersphere.
2. When $g = 2$, $N = S^k(r) \times S^{n-k}(\sqrt{1-r^2})$, $0 < r < 1$, $1 < k < n$, the so-called *Clifford hypersurface*.
3. When $g = 3$ N is the *Cartan hypersurface* in Fact 4 (iv).
4. When $g = 4$, there exist infinitely many homogeneous and non-homogeneous examples, the so-called OT-FKM type (constructed by using the representations of Clifford algebras) [22], [10], and two other homogeneous ones. All the cases but $(m_1, m_2) = (7, 8)$ turn out to be of OT-FKM-type or homogeneous ones [4], [6]. Very recently, Q.S. Chi announces that this is true for $(m_1, m_2) = (7, 8)$ [7].
5. When $g = 6$, N is an isotropy orbits of either $G_2/SO(4)$ or $G_2 \times G_2/G_2$ [9], [16].

Therefore, we know

Fact 6 When $g = 1, 2, 3, 6$*, all isoparametric hypersurfaces in S^{N+1} are homogeneous.*

1.4 Gauss Map

The *Gauss map* of a hypersurface N in S^{n+1} is given by:

$$\mathcal{G} : N \ni p \mapsto x(p) + \sqrt{-1}\xi(p) \in Q_n(\mathbb{C}) \cong \mathrm{Gr}^+(2, \mathbb{R}^{n+2}),$$

where $\mathrm{Gr}^+(2, \mathbb{R}^{n+2})$ denotes the oriented 2-plane Grassmannian identified with the complex hyperquadric

$$Q_n(\mathbb{C}) = \{z \in \mathbb{C}P^{n+1} \mid \sum_{i=1}^{n+2} z_i^2 = 0\},$$

via $x(p) \wedge \xi_p = x(p) + \sqrt{-1}\xi_p$.

Note that $Q_n(\mathbb{C})$ is a Hermitian symmetric space (with positive Ricci curvature), and regarded as a symplectic manifold.

The following is the starting point of our argument.

Fact 7 (B. Palmer 1997) [23] *When N is an isoparametric hypersurface, $L = \mathcal{G}(N)$ is a minimal Lagrangian submanifold of $Q_n(\mathbb{C})$.*

2 Review of symplectic geometry

2.1 Symplectic Manifolds and Lagrangian Submanifolds

Definition 4 (1) A $2n$-dimensional smooth manifold M is called a *symplectic manifolds* if M is equipped with a non-degenerate closed 2 form ω. We call ω the *symplectic form*.

(2) An n-dimensional submanifold $\iota : L \to M$ is a *Lagrangian submanifold* of M when $i^*\omega = 0$ holds.

Example (1) Consider $M = T^*\mathbb{R}^n$ with coordinates $(q^1, \ldots, q^n, p_1, \ldots . p_n) \in T^*\mathbb{R}^n$. Then $\omega = \sum dq^i \wedge dp_i$ is a symplectic form, and $(T^*\mathbb{R}^n, \omega)$ is a symplectic manifold.

In this case, typical Lagrangian submanifolds are the base manifold $L = \mathbb{R}^n$, and a fiber $L = \pi^{-1}(q)$ for $q \in \mathbb{R}^n$. In fact, $p = (p_1, \ldots, p_n) = 0$ holds on $\mathbb{R}^n$. Also on $L = \pi^{-1}(q)$, $q = (q^1, \ldots, q^n)$ is constant and so $dq = 0$ follows.

(2) Any Kähler manifold is a symplectic manifold with symplectic form given by the Kähler form. In particular, any surface is a symplectic manifold, and a curve on a surface is a Lagrangian submanifold.

[Darboux] For a symplectic manifold (M^{2n}, ω), there exists a local coordinates (q^i, p_i) of M satisfying $\omega = \sum dq^i \wedge dp_i$

Definition 5 This coordinates is called the *Darboux coordinates*, or, the *canonical coordinates*.

Thus any symplectic manifold is locally symplectomorphic to $T^*\mathbb{R}^n$. This means that *in symplectic geometry, global properties are important.*

Remark For any manifold X, its cotangent bundle T^*X is a symplectic manifold. In fact, in a standard coordinates $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ of T^*X, $\omega^X = \sum dx^i \wedge d\xi_i$ is a coordinates free non-degenerate closed 2-form on T^*X.

Definition 6 Let $L \subset (M, \omega)$ be a Lagrangian submanifold. There exists a tubular neighborhood $(N(L), \omega|_{N(L)})$ of L in M which is symplectomorphic with the tubular neighborhood $(N(0_L), \omega^L|_{N(0_L)})$ of the 0-section $0_L = L$ in T^*L. We call $N(L)$ the *Weinstein neighborhood of L.*

2.2 Hamiltonian Diffeomorphism

Hereafter, let (M, ω) be a compact symplectic manifold.

Definition 7 (1) For a Hamiltonian function $H \in C^\infty(M)$, its *Hamiltonian vector filed* X_H is the one defined by $dH = \omega(\ , X_H)$.

(2) For a time dependent Hamiltonian function $H : [0, 1] \times M \to \mathbb{R}$, let X_{H_t} be the Hamiltonian vector field, and $\{\phi_t^H\}_{t\in[0,1]}$ be its associated flow. Then $\{\phi_t^H\}_{t\in[0,1]}$ is called a *Hamiltonian isotopy of* M.

(3) The time-1 map $\varphi = \phi_1^H$ of a Hamiltonian isotopy of M is called a *Hamiltonian diffeomorphism of* M.

Now we put

$$\begin{aligned}&\mathrm{Ham}(M, \omega) = \{\varphi = \phi_1^H \mid H \in C^\infty([0, 1] \times M)\}\\ &\quad \subset \mathrm{Symp}_0(M, \omega) = \{\text{symplectomorphism isotopic to the identity map}\}\end{aligned}$$

2.3 Lagrangian Intersection

For a function $f \in C^\infty(L)$ on a manifold L^n,

$$L_f = \{(q, df(q))\} \subset T^*L$$

is a Lagrangian submanifold of T^*L. In fact, if we put $f_i = \dfrac{\partial f}{\partial q^i}$, $p = df = \sum_{i=1}^n f_i dq^i$ implies

$$\omega^L|_L = \sum dq^i \wedge dp_i = \sum_{i,j=1}^n dq^i \wedge \frac{df_i}{dq^j} dq^j = 0,$$

since $f_{ij} = f_{ji}$. We call L_f the *Lagrangian graph of* T^*L.

In T^*L, the intersection of the 0-section L with L_f is given by $L \cap L_f = \{(q, 0)\}$, namely, $L \cap L_f$ consists of critical points of f. Thus if L is compact and f is a Morse function on L, we have

$$\#(L \cap L_f) \geq \mathrm{SB}(L, \mathbb{Z}_2),$$

where $\mathrm{SB}(L, \mathbb{Z}_2)$ is the sum of the Betti numbers of L. The coefficient could be $\mathbb{Z}$.

Now, for any $\varphi \in$ Ham (M, ω), $\varphi^*\omega = \omega$ holds and so if L is a Lagrangian submanifold of M, so is $\varphi(L)$. Thus we pose the following question:

Question. (Arnold) Let L be an embedded compact Lagrangian submanifold of M. If the intersection $L \cap \varphi(L)$ is transversal for $\varphi \in$ Ham (M, ω), does it hold $\#(L \cap \varphi(L)) \geq SB(L, \mathbb{Z}_2)$?

Generally, this is not the case. In fact, two small circles on S^2 are separable by an isometry, but we have $SB(S^1, \mathbb{Z}_2) = 2$. On the other hand, when S^1 is a great circle, S^1 and $\varphi(S^1)$ always intersect since a Hamiltonian diffeomorphism preserves the area bisected by a great circle S^1. Thus the above inequality holds for great circles.

Now, we consider under what condition, the inequality holds.

When a Lagrangian submanifold L is contained in the Weinstein neighborhood $N(L)$, we can apply the argument in the case of Lagrangian graphs. However, in general, $\varphi(L)$ is not contained in the Weinstein neighborhood. This is the difficulty of the problem concerning with Hamiltonian diffeomorphisms.

3 Floer Homology of Lagrangian Intersection

3.1 Review of the Morse Theory on Finite Dimensional Manifolds

First, we review the Morse theory in a way fitting to the Floer theory.

Let M be an n-dimensional compact manifold and $f \in C^\infty(M)$ be a Morse function. Put $C_k = \{\text{critical points of index } k\}$, and for any $p, q \in \bigcup_{k=0}^{n} C_k$, define

$$\mathscr{M}(p,q) = \{\gamma(t) : \mathbb{R} \to M \mid -\text{grad} f = \frac{d\gamma}{dt},\ \lim_{t\to-\infty} = p,\ \lim_{t\to\infty} = q\}/\sim$$

where $\sim$ is the parameter shift. Then for $p \in C_k$, the boundary operator $\partial : C_k \to C_{k-1}$ is given by

$$\partial p = \sum_{q \in C_{k-1}} \#\mathscr{M}(p,q) q,$$

where $\#\mathscr{M}(p,q)$ is counted modulo 2. Then $\partial \circ \partial = 0$ holds, and we can define the *Morse homology with* $\mathbb{Z}_2$ *coefficient* by

$$H(M, \mathbb{Z}_2) = \frac{\ker\partial}{\text{Im}\partial}$$

3.2 Introduction of the Floer Homology

Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold, and take $\varphi = \phi_1 \in \text{Ham}(M, \omega)$ of a time dependent flow ϕ_t. Consider the set of paths:

$$\Omega = \{l : [0,1] \to M \mid l(0) \in L, l(1) \in \varphi(L), l \text{ is isotopic to } \phi_t(x_0)\}.$$

Fact 8 (Floer [11]) *Under the assumption* $\int_D v^*\omega = 0$ *for all* $v : (D^2, \partial D^2) \to (M, L)$,

(1) there exists a functional $F : \Omega \ni l \mapsto F(l) \in \mathbb{R}$ *on* Ω*, such that a path* $l \in \Omega$ *is a critical point of* F *if and only if* $\frac{dl}{dt} = 0$ *holds, namely, when* l *is a constant path* $l(t) = p \in L \cap \varphi(L)$.

(2) With respect to a family $J = \{J_t\}_{0 \le t \le 1}$ of time dependent almost complex structures compatible with ω,

$$\operatorname{grad} F = J_t \frac{dl}{dt} \tag{3}$$

holds.

Now, denoting $u(s,t) = l(t)(s)$, we put for $p, q \in L \cap \varphi(L)$,

$$\mathscr{M}(p,q) = \{u : \mathbb{R} \times [0,1] \to M \mid \frac{\partial u}{\partial s} = -\operatorname{grad} F, \lim_{s \to -\infty} u = p, \lim_{s \to \infty} u = q\} / \sim,$$

where $\sim$ is the parameter shift. From (3), $u \in \mathscr{M}(p,q)$ satisfies

$$\frac{\partial u}{\partial s} + J_t \frac{du}{dt} = 0, \tag{4}$$

*and u is called a J-*holomorphic strip.

Fact 9 *(1) When the intersection is transversal at $p \in L \cap \varphi(L)$, we have the so-called* Maslov-Viterbo index *$\mu(p) \in \mathbb{Z}$, and $\mathscr{M}(p,q)$ is a $(\mu(p) - \mu(q) - 1)$-dimensional differentiable manifold.*

(2) When $\mu(p) - \mu(q) = 1$, $\mathscr{M}(p,q)$ is compact.

(3) When $\mu(p) - \mu(q) = 2$, the boundary of the compactification $\overline{\mathscr{M}(p,q)}$ is given by

$$\bigcup_{\mu(r) = \mu(p) - 1} \mathscr{M}(p,r) \times \mathscr{M}(r,q).$$

Next, if we put $CF_k := \{p \in L \cap \varphi(L) \mid \mu(p) = k\}$, the boundary operator $\partial_J : CF_k \to CF_{k-1}$ is given by

$$\partial_J p = \sum_{q \in CF_{\mu(p)-1}} \# \mathscr{M}(p,q) q,$$

where $\#\mathscr{M}(p,q)$ is counted modulo 2.

Combining this with (3), we obtain $\partial_J \circ \partial_J = 0$, and so the Floer homology *is defined by*

$$HF(L) = \frac{ker \partial_J}{Im \partial_J}.$$

Note that the above argument holds under the assumption $\int_D v^* \omega = 0$.

Fact 10 (Floer [11]) *(1) $HF(L)$ does not depend on the choice of H_t and J_t.*
(2) If $\pi_2(M, L) = 0$, $HF(L) \cong H_*(L, \mathbb{Z}_2)$ holds.

Definition 8 A Lagrangian submanifold $L \subset (M, \omega)$ *is Hamiltonian displaceable* if $L \cap \varphi(L) = \emptyset$ holds for some $\varphi \in$ Ham (M, ω).

Since $HF(L)$ is generated by $\mathscr{C} = L \cap \varphi(L)$, $HF(L) = 0$ follows if L is Hamiltonian displaceable. In other words,

$$\boxed{HF(L) \neq 0 \Rightarrow \forall \varphi \in \mathrm{Ham}(M, \omega), L \cap \varphi(L) \neq \emptyset}$$

Definition 9 *L is Hamiltonian non-displaceable* if $L \cap \varphi(L) \neq \emptyset$ holds for any $\varphi \in \mathrm{Ham}(M, \omega)$.

Remark Note that $L \cap \varphi(L) \neq \emptyset$ does *not necessarily imply* $HF(L) \neq 0$.

3.3 Generalization of the Floer Homology by Y.G. Oh

The assumption $\int_D v^*\omega = 0$ put by Floer is too strong and does not work well. Y.G. Oh weakened the condition and make the Floer homology more useful [19, 21]

Define $I_\omega : \pi_2(M, L) \to \mathbb{R}$ for $u : (D, \partial D) \to (M, L)$, $[u] = A \in \pi_2(M, L)$ by

$$I_\omega(A) = \int_D u^*\omega.$$

Let $\Lambda(\mathbb{C}^n)$ be the set of Lagrangian subspaces of $\mathbb{C}^n$, and put $\tilde{u} = u|_{\partial D} : S^1 \to \Lambda(\mathbb{C}^n)$. Let $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z})$ be the Maslov class, and define $I_{\mu,L} : \pi_2(M, L) \to \mathbb{Z}$ by

$$I_{\mu,L}(A) = \mu(\tilde{u}).$$

Definition 10 (1) L is *monotone* if $I_{\mu,L} = \lambda I_\omega$ holds for some $\lambda > 0$.

(2) The positive generator N_L of the image of $I_{\mu,L}$ is called the *minimal Maslov number*.

Fact 11 (Y.G. Oh [19, 21]) When *L is monotone and the minimal Maslov number satisfies $N_L \geq 2$, we can define $HF(L) := H_*(CF(L), \partial_J)$ for (H, J). This is called the* Floer homology of L with $\mathbb{Z}_2$-coefficient. *$HF(L)$ is Hamiltonian isotopy invariant of L.*

In this case, the boundary operator is given by

$$\partial_J = \partial_0 + \partial_1 + \cdots + \partial\nu, \quad \partial_l : CF_*(L) \to CF_{*-1+lN_L}(L),$$

where $\nu = \left[\frac{\dim L + 1}{N_L}\right]$. *Here,* ∂_0 *is the boundary operator of Morse, while the others are operators concerned with the* J*-holomorphic strip, and the indices jump. Thus the computation of* $HF(L)$ *is difficult when* ν *is large.*

Fact 12 $l > \nu \Rightarrow \partial_l = 0$.
$(\because l > \frac{\dim L + 1}{N_L} \Rightarrow -1 + lN_L > \dim L \Rightarrow CF_{*-1+lN_L} = 0.)$
In this way, when N_L *is large,* ν *is small, and* $HF(L)$ *may be computable.*

Remark When $N_L = 2$, the classification of J-holomorphic disks is necessary to obtain $HF(L)$, that causes the difficulty.

The Floer homology has been computed only for toric fibers and a few other cases. In the next section, we focus on a rich family of Lagrangian submanifolds given by the Gauss image of isoparametric hypersurfaces in spheres, and investigate the Hamiltonian non-displaceability.

4 Main Result

4.1 Gauss Image of Isoparametric Hypersurfaces

From the argument in Sect. 1.4, we know that the Gauss images of isoparametric hypersurfaces in S^{n+1} are minimal Lagrangian submanifolds in the complex hyperquadric $Q_n(\mathbb{C})$.

Fact 13 $Q_n(\mathbb{C})$ *is a positive Kähler Einstein manifold, and so* L is monotone.

Remark The following are known:
(1) When $g = 1$, $N = S^n$, and $L = S^n \subset Q_n(\mathbb{C})$ is a real form.
(2) When $g = 2$, $N = S^k \times S^{n-k}$, and $L = S^k \times S^{n-k}/\mathbb{Z}_2$ is a real form. In this case, $HF(L) \cong H_*(L, \mathbb{Z}_2)$ [14, 20].

Fact 14 (Ma-Ohnita [15]) *When* $L = \mathcal{G}(N)$,

$$N_L = \frac{2n}{g} = \begin{cases} m_1 + m_2 & g : \ even \\ 2m & g : \ odd. \end{cases}$$

Now we state our main theorem:

Main Theorem. [13]

(1) When $g = 3$, $L = \mathscr{G}(N)$ is a $\mathbb{Z}_2$-homology sphere

In particular, if $m = m_i \geq 2$, then $HF(L) \cong H_*(L, \mathbb{Z}_2) \otimes \Lambda$

where Λ is a certain algebra of the Laurant polynomials

If the intersection is transversal, then we have

$$\#L \cap \varphi(L) \geq SB(L, \mathbb{Z}_2)$$

(2) When $g = 4$ and $2 \leq m_1 \leq m_2$, L is Hamiltonian non-displaceable

(3) When $g = 6$ and $m = m_i = 2$, L is Hamiltonian non-displaceable.

Remark The condition $m_i \geq 2$ is necessary for $N_L \geq 3$ (see Fac 14).

Sketch of the proof: (see also Ohnita's article in this volume.)

(1) First we show that L is a $\mathbb{Z}_2$-homology sphere. Then using the Biran-Cornea's argument [2], we obtain $HF(L) \cong H_*(L, \mathbb{Z}_2) \otimes \Lambda$.

To show (2) and (3), we use Damian's spectral sequences (see the next subsection.) We take a covering $\bar{L} = N \to L = N/\mathbb{Z}_g$, and lift the Floer complex to $\bar{L}$. Taking the lifted Floer homology $HF^{\bar{L}}(L)$ given by Damian, we suppose $HF^{\bar{L}}(L) = 0$, then the spectral sequences leads us to a contradiction. Thus we obtain $HF^{\bar{L}}(L) \neq 0$, which means that $L \cap \varphi(L) \neq \emptyset$ holds for any $\varphi \in \mathrm{Ham}\,(Q_n(\mathbb{C}), \omega_{\mathrm{std}})$.

4.2 Damian's Lifted Floer Homology of Monotone Lagrangian Submanifolds

Let (M, ω) be a compact symplectic manifold, and let $L \subset M$ be an embedded compact monotone Lagrangian submanifold satisfying $N_L \geq 3$.

For p, $q \in \mathscr{C} = L \cap \varphi(L)$, consider an isolated J-holomorphic strip $u : \mathbb{R} \times [0, 1] \to M$ joining p, q, and put

$$\Gamma = \bigcup_{p,q \in \mathscr{C}} \{\gamma \mid \gamma(s) := u(s, 0),\ \lim_{s \to -\infty} \gamma(s) = p,\ \lim_{s \to \infty} \gamma(s) = q\}.$$

We obtain the set $(\mathscr{C}, \Gamma)$ of points and paths, which reconstructs the Floer complex $(\bigcup_k CF_k(L), \partial_J)$.

Starting from $(\mathscr{C}, \Gamma)$, fixing a covering $\pi : \bar{L} \to L$, let $\bar{\Gamma}$ be the set of the lift of all paths in Γ to $\bar{L}$.

Put $\pi^{-1}(p) = \{p_i\}_{i\in I}$, and $\pi^{-1}(q) = \{q_i\}_{i\in I}$. Then for p_i, q_j $(i, j \in I)$, we know $\#\,\{\text{elements in } \bar{\Gamma} \text{ joining } p_i \text{ and } q_j\} < \infty$. Put its parity $n(p_i, q_j)$.

Now let $CF^{\bar{L}}(L)$ be the free $\mathbb{Z}_2$-module generated by $\bigcup_{p\in\mathscr{C}} \pi^{-1}(p)$, and define the boundary operator $\partial^{\bar{L}}$ of $CF^{\bar{L}}(L)$ by

$$\partial^{\bar{L}}(p_i) = \sum_{\pi(q_j)=q\in\mathscr{C}} n(p_i, q_j)q_j.$$

Fact 15 (Damian [8]) *Let L be a compact monotone Lagrangian submanifold of M, satisfying $N_L \geq 3$. Let $\bar{L} \to L$ be a covering and $(CF^{\bar{L}}(L), \partial^{\bar{L}})$ be the lifted complex. We call its homology $HF^{\bar{L}}(L) := H_*(CF^{\bar{L}}(L), \partial^{\bar{L}})$ the* lifted Floer homology of L. *This is invariant under Hamiltonian isotopies of L.*

Damian proves it by applying Biran's spectral sequence [1] to the lift of L.

An isoparametric hypersurface covers its Gauss image in a finite order, and the homology of isoparametric hypersurfaces is well-known. A use of Damian's lifted Floer homology is an idea of the first author of [13].

[Damian] When L is a monotone closed Lagrangian submanifold of M with $N_L \geq 3$ and $\bar{L} \to L$ is any covering, $(CF^{\bar{L}}(L), \partial^{\bar{L}})$ is an elliptic complex, and the homology $HF^{\bar{L}}(L) := H_*(CF^{\bar{L}}(L), \partial^{\bar{L}})$ is well-defined as the *lifted Floer homology* of L

$HF^{\bar{L}}(L)$ is invariant under the Hamiltonian isotopies of L.

4.3 Damian's Spectral Sequence

Let $\Lambda = \mathbb{Z}_2[T, T^{-1}]$ be the algebra of Laurent polynomials over $\mathbb{Z}_2$, and $\Lambda^i \subset \Lambda$ be the subspace of homogeneous elements of degree i. Then there exists a spectral sequence $\{E_r^{p,q}, d_r\}$ satisfying the properties:

1. $E_0^{p,q} = CF^{\bar{L}}_{p+q-pN_L} \otimes \Lambda^{pN_L}$, $d_0 = [\partial_0^{\bar{L}}] \otimes 1$.
2. $E_1^{p,q} = H_{p+q-pN_L}(\bar{L}, \mathbb{Z}_2) \otimes \Lambda^{pN_L}$, $d_1 = [\partial_1^{\bar{L}}] \otimes T^{-N_L}$, where

$$[\partial_1^{\bar{L}}] : H_{p+q-pN_L}(\bar{L}; \mathbb{Z}_2) \to H_{p+q-1-(p-1)N_L}(\bar{L}; \mathbb{Z}_2)$$

is induced by $\partial_1^{\bar{L}}$.

3. For any $r \geq 1$, $E_r^{p,q} = V_r^{p,q} \otimes \Lambda^{pN_L}$ with $d_r = \delta_r \otimes T^{-rN_L}$, where $V_r^{p,q}$ is a vector space over $\mathbb{Z}_2$ and $\delta_r : V_r^{p,q} \to V_r^{p-r,q+r-1}$ is a homomorphism defined for every p, q and satisfies $\delta_r \circ \delta_r = 0$. More precisely,

$$V_{r+1}^{p,q} = \frac{\mathrm{Ker}(\delta_r : V_r^{p,q} \to V_r^{p-r,q+r-1})}{\mathrm{Im}(\delta_r : V_r^{p+r,q-r+1} \to V_r^{p,q})}$$

$$V_0^{p,q} = CF_{p+q-pN_L}^{\bar{L}}, \quad V_1^{p,q} = H_{p+q-pN_L}(\bar{L}; \mathbb{Z}_2)$$

$$\delta_1 = [\partial_1^{\bar{L}}]$$

4. $E_r^{p,q}$ collapses at $(\nu + 1)$-step and for any $p \in \mathbb{Z}$, $\oplus_{q \in \mathbb{Z}} E_\infty^{p,q} \cong HF^{\bar{L}}(L)$, where $\nu = [\frac{\dim L + 1}{N_L}]$.

Back to the Gauss image, $\nu = [\frac{\dim L + 1}{N_L}] = [\frac{(n+1)g}{2n}]$ implies that for any $p, q \in \mathbb{Z}$, we have

(1) $E_2^{p,q} = E_\infty^{p,q}$ if and only if $g = 3$ and $(m_1, m_2) = (2, 2), (4, 4), (8, 8)$.
(2) $E_3^{p,q} = E_\infty^{p,q}$ if and only if $g = 3$, $(m_1, m_2) = (1, 1)$ or $g = 4$.
(3) $E_4^{p,q} = E_\infty^{p,q}$ if and only if $g = 6$, $(m_1, m_2) = (1, 1)$ or $(2, 2)$.

4.4 Detail of the Proof of the Main Theorem When $g = 4$

We give a detailed proof of the main theorem in the case $g = 4$. When $g = 6$, the argument becomes longer, but the principle is the same.

Suppose $HF^{\bar{L}}(L) = 0$, then from (2) and 4 above, $0 = E_3^{0,q}$ follows, and so from 3 where $r = 2$ and $p = 0$,

$$V_2^{2,q-1} \to V_2^{0,q} \to V_2^{-2,q+1}$$

is exact. Since

$$V_2^{2,q-1} = \frac{\mathrm{Ker}([\partial_1^{\bar{L}}] : H_{q+1-2N_L}(\bar{L}; \mathbb{Z}_2) \to H_{q-N_L}(\bar{L}; \mathbb{Z}_2))}{\mathrm{Im}([\partial_1^{\bar{L}}] : H_{q+2-3N_L}(\bar{L}; \mathbb{Z}_2) \to H_{q+1-2N_L}(\bar{L}; \mathbb{Z}_2))},$$
$$V_2^{-2,q+1} = \frac{\mathrm{Ker}([\partial_1^{\bar{L}}] : H_{q-1+2N_L}(\bar{L}; \mathbb{Z}_2) \to H_{q-2+3N_L}(\bar{L}; \mathbb{Z}_2))}{\mathrm{Im}([\partial_1^{\bar{L}}] : H_{q+N_L}(\bar{L}; \mathbb{Z}_2) \to H_{q-1+2N_L}(\bar{L}; \mathbb{Z}_2))},$$

$V_2^{2,q-1} = V_2^{-2,q+1} = 0$ for $2 \leq q \leq n-2$, and so

$$0 = V_2^{0,q} = \frac{\mathrm{Ker}([\partial_1^{\bar{L}}] : H_q(\bar{L};\mathbb{Z}_2) \to H_{q-1+N_L}(\bar{L};\mathbb{Z}_2))}{\mathrm{Im}([\partial_1^{\bar{L}}] : H_{q+1-N_L}(\bar{L};\mathbb{Z}_2) \to H_q(\bar{L};\mathbb{Z}_2))}$$

holds. Putting $q = N_L = m_1 + m_2$, we know

$$H_1(\bar{L};\mathbb{Z}_2) \to H_{m_1+m_2}(\bar{L};\mathbb{Z}_2) \to H_{2(m_1+m_2)-1}(\bar{L};\mathbb{Z}_2)$$

is exact, but this contradicts Münzner's result:

$$H_k(N;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & \text{for } k = 0, m_1, m_2, 2m_1 + m_2, m_1 + 2m_2, n, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{for } k = m_1 + m_2, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $HF^{\bar{L}}(L) \neq 0$ implies $HF(L) \neq 0$. In particular, L is Hamiltonian non-displaceable.

5 Open Problems and a Conjecture

Problems.

1. Solve the case $g = 3$ and $m = 1$.
2. Solve the case $g = 4$ and $(m_1, m_2) = (1, k)$.
3. Solve the case $g = 6$ and $m = 1$.
4. Compute $HF(L)$ in case 1 and in all other cases for $g \geq 4$.

A conjecture of H. Ono and IMMO.

In an irreducible Hermitian symmetric space of compact type, any compact minimal Lagrangian submanifold is Hamiltonian non-displaceable.

Acknowledgements The author sincerely would like to thank Professor Young Jin Suh (Director of RIRCM) for his organization of twenty times successful workshops. The author also would like to thank Dr. Hyunjin Lee for her excellent supports to those workshops. The author is partly supported by JSPS Grant-in-Aid for Scientific Research (B) 615H036160.

Appendix

When f is a C^2 function on a Riemannian manifold M, a level set $N_t = f^{-1}(t)$ of a regular value t has the mean curvature $H(t)$

$$nH(t) = \frac{\nabla f(|\nabla f|) - |\nabla f|\triangle f}{|\nabla f|^2}, \tag{5}$$

which we will prove later.

When f is an isoparametric function, the condition (II) implies that $\triangle f$ is constant on N_t, and so is

$$\nabla f(|\nabla f|) = \nabla f(\sqrt{b(f)}) = \frac{b'(f)}{2\sqrt{b(f)}}\nabla f(f) = \frac{b'(f)\sqrt{b(f)}}{2},$$

where $b'(f)$ means the differential w.r.t. the variable of b. Thus we obtain

$$nH(t) = \frac{b'(f) - 2a(f)}{2\sqrt{b(f)}}, \tag{6}$$

and N_t has constant mean curvature.
Proof of (5): Take an orthonormal frame of M along N_t

$$X_i, \quad 1 \le i \le n, \quad X_{n+1} = \xi = \frac{\nabla f}{|\nabla f|},$$

where for $1 \le i \le n$, X_i is tangent to N_t and $X_i(f) = 0$ as $f = t$ on N_t. Hence from

$$\begin{aligned}\triangle f &= \mathrm{div}(\nabla f)\\ &= \textstyle\sum_{i=1}^n \langle \nabla_{X_i}(\nabla f), X_i\rangle + \langle \nabla_\xi(\nabla f), \xi\rangle,\end{aligned}$$

and $\nabla f = |\nabla f|\xi$, we obtain

$$\sum_{i=1}^n \langle \nabla_{X_i}(\nabla f), X_i\rangle = \triangle f - \xi(|\nabla f|) = \triangle f - \frac{\nabla f(|\nabla f|)}{|\nabla f|}. \tag{7}$$

Since the shape operator is given by $AX_i = -\nabla_{X_i}\xi$, $1 \le i \le n$, using (7), we obtain

$$\begin{aligned}nH &= \mathrm{Tr}A = -\textstyle\sum_{i=1}^n \langle \nabla_{X_i}(\xi), X_i\rangle\\ &= \textstyle\sum_{i=1}^n \langle \nabla_{X_i}(\frac{\nabla f}{|\nabla f|}), X_i\rangle = -\frac{1}{|\nabla f|}\sum_{i=1}^n \langle \nabla_{X_i}(\nabla f), X_i\rangle\\ &= \frac{\nabla f(|\nabla f|) - |\nabla f|\triangle f}{|\nabla f|^2}.\end{aligned}$$

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Counterexamples to Goldberg Conjecture with Reversed Orientation on Walker 8-Manifolds of Neutral Signature

Yasuo Matsushita and Peter R. Law

Abstract The famous Goldberg conjecture (Goldberg, Proc Am Math Soc 21, 96–100, 1969) [8] states that the almost complex structure of a compact almost-Kähler Einstein Riemannian manifold is Kähler. It is true if the scalar curvature of the manifold is nonnegative (Sekigawa, Math Ann 271, 333–337, 1985) [20], (Sekigawa, J Math Soc Jpn 36, 677–684, 1987) [21]. If we turn our attention to indefinite metric spaces, several counterexamples to the conjecture have been reported (cf. (Matsushita, J Geom Phys 55, 385–398, 2005) [17], (Matsushita et al., Monatsh Math 150, 41–48, 2007) [18], (Matsushita, et al., Proceedings of The 19th International Workshop on Hermitian-Grassmannian Submanifolds and Its Applications and the 10th RIRCM-OCAMI Joint Differential Geometry Workshop, Institute for Mathematical Sciences (NIMS), vol 19, pp 1–14. Daejeon, South Korea, 2015) [19]). It is important to recognize that all known counterexamples to date are constructed on Walker manifolds, equipped with an almost complex structure of normal orientation. In the present paper, we focus our attention on Walker manifolds with an opposite almost complex structure, and consider if counterexamples to the Goldberg conjecture can be constructed. We succeeded in finding such a counterexample on an 8-dimensional compact Walker manifold of neutral signature, but failed in the case of 6-dimensional compact Walker manifold of signature (4, 2) with a canonically defined opposite almost complex structure.

Y. Matsushita (✉)
Osaka City University Advanced Mathematical Institute, Osaka 558-8585, Japan
e-mail: yma24tak@gmail.com

P.R. Law
4 Mack Place, Monroe, NY 10950, USA
e-mail: prld@member.ams.org

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_9

1 Introduction

In his famous book, Steenrod [23] states that *a compact smooth manifold of dimension n admits an indefinite metric of signature $(p, n-p)$ $(p \geq 1)$ if and only if the manifold admits a nonsingular field of tangent p-planes.* This shows that the existence of an indefinite metric on a manifold is closely related to the manifold topology. In fact, a compact orientable manifold admits a Lorentz metric $(1, n-1)$ if and only if it admits a nonsingular vector field, which is the case if and only if the Euler characteristic vanishes. On a 4-dimensional compact orientable manifold, a neutral metric $(2, 2)$ exists on it if and only if it admits an orientable field of 2-planes, for which the existence condition has been settled (see [9, 14–16]).

It is a note worthy observation that for indefinite signature, the existence of a neutral metric $(2, 2)$ on a 4-dimensional compact orientable manifold is equivalent to the existence of a pair (J, J') of an almost complex structure J and an opposite almost complex structure J' with orientation reversed to the preferred one [15, Fact 7].

Around 1950, Walker studied canonical forms of metrics for an n-dimensional manifold which admits a parallel field of null r-planes $(r/2 \leq n)$ [24, 25]. Such manifolds are called Walker manifolds. The canonical forms of such metrics are expressed in local coordinates, i.e., various expressions are written locally. As we shall illustrate below, on a Walker manifold of even dimension and even signature $(2p, 2q)$, one can, at least locally, always construct an almost complex structure J, and associated with J also an opposite almost complex structure J', which commute with each other.

A famous conjecture by Goldberg [8] is well known. It states that the almost complex structure of a compact almost-Kähler Einstein Riemannian manifold is integrable. It is known that the Goldberg conjecture is true if the scalar curvature is non-negative (see Sekigawa [20, 21]), but that otherwise. However, there are many variant conjectures with various arranged conditions and various affirmative and negative results. See [1] for an excellent survey of the Goldberg conjecture by Apostolov and Drăghici.

With these considerations in mind, we have reported three kinds of counterexamples of the indefinite version to the conjecture. The first counterexample by Haze [17] is constructed on a 4-dimensional noncompact neutral Walker manifold, the second one is an 8-dimensional compact neutral Walker manifold [18]. The last one is constructed on a 6-dimensional compact Walker manifold of signature (4, 2) (not neutral) [19]. These known counterexamples are all constructed on Walker manifolds, and only almost complex structures are considered. On the basis of the second counterexample in [18], Sekigawa et al. [22] studied indefinite Einstein manifolds, together with isotropic Kähler structures (see [6] for the definition and examples).

In the present paper, we shall exhibit a counterexample to the Goldberg conjecture consisting of 8-dimensional compact neutral Walker manifold with an opposite almost complex structure. In fact, a variant of the Goldberg conjecture for reversed orientation is treated.

2 Walker Manifolds

A.G. Walker [24] determined a canonical form of metrics on a pseudo-Riemannian n-manifold M^n admitting a parallel field D of null r-planes. We call such a manifold a Walker manifold. The canonical form of metrics with respect to a suitable choice of coordinates $(x^1, \ldots, x^n)$ is given by

$$g = \left[\, g_{ij} \,\right] = \begin{bmatrix} 0 & 0 & I_r \\ 0 & P & H \\ I_r & {}^tH & Q \end{bmatrix}, \tag{1}$$

where I_r is the unit matrix of order r, and P, Q, H are matrix functions of coordinates, satisfying the following conditions:

1. P is a symmetric nonsingular $(n-2r) \times (n-2r)$ matrix, independent of $x^1, \ldots, x^r$,
2. Q is a symmetric $r \times r$ matrix,
3. H is an $(n-2r) \times r$ matrix, independent of $x^1, \ldots, x^r$.

With respect to the local coordinates, the parallel field of null r-planes is spanned by

$$D = \operatorname{span}\{\, \partial_1, \ldots, \partial_r \,\}, \quad g\left(\partial_i, \partial_j\right) = 0 \quad (i, j = 1, \ldots, r), \tag{2}$$

where ∂_i stand for $\partial/\partial x^i$. See also [25].

Remark Four-dimensional Walker manifolds have been intensively studied (see e.g., [2–5, 7, 11–13, 17], and also references therein).

2.1 8-Dimensional Walker Manifolds of Neutral Signature (4, 4)

We now concentrate our attention on an 8-dimensional Walker manifold (M, g, D), where g is a metric of neutral signature (4, 4) and D a parallel field of 4-dimensional null planes. Then, from Walker's theorem, there is, locally, a system of coordinates $(x^1, \ldots, x^8)$ so that g takes the canonical form

$$g = \left[\, g_{ij} \,\right] = \begin{bmatrix} 0 & I_4 \\ I_4 & Q \end{bmatrix}, \tag{3}$$

where I_4 is the unit 4×4 matrix and Q is a 4×4 symmetric matrix whose entries are functions of the coordinates $(x^1, \ldots, x^8)$. Note that $D = \operatorname{span}\{\partial_1, \ldots, \partial_4\}$. With respect to the metric, any vector in D is null: $g(\partial_i, \partial_j) = 0$ for $1 \le i, j \le 4$.

We first considered a simplified Walker metric, with Q of diagonal form $Q = \operatorname{diag}\,[p\ q\ r\ s]$, where p, q, r and s are arbitrary functions of the coordinates $(x^1,$

…, x^8). After some calculations with such a diagonal Q, we further restricted our attention to a simpler metric of the form

$$g = [\, g_{ij} \,] = \begin{bmatrix} 0 & I_4 \\ I_4 & Q \end{bmatrix}, \qquad \text{with} \quad Q = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4}$$

where $p = p(x^1, \ldots, x^8)$ and $r = r(x^1, \ldots, x^8)$. We shall hereafter refer to such an 8-dimensional Walker manifold with the metric above, as simply a *Walker 8-manifold.*

3 An Orthonormal Frame

It is elementary to find an orthonormal frame $\{\, e_1, \ldots, e_8 \,\}$ with respect to the metric (4), as follows:

$$\begin{gathered} g(e_i, e_j) = \varepsilon_i \, \delta_{ij} \\ \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = -\varepsilon_5 = -\varepsilon_6 = -\varepsilon_7 = -\varepsilon_8 = 1 \end{gathered} \tag{5}$$

Among various possibilities, we choose the following orthonormal frame:

$$\begin{aligned} e_1 &= \frac{1-p}{2}\partial_1 + \partial_5, \; e_2 = \frac{1-r}{2}\partial_3 + \partial_7, \; e_3 = \frac{1}{\sqrt{2}}(\partial_2 + \partial_6), \; e_4 = \frac{1}{\sqrt{2}}(\partial_4 + \partial_8) \\ e_5 &= -\frac{1+p}{2}\partial_1 + \partial_5, \; e_6 = -\frac{1+r}{2}\partial_3 + \partial_7, \; e_7 = \frac{1}{\sqrt{2}}(\partial_2 - \partial_6), \; e_8 = \frac{1}{\sqrt{2}}(\partial_4 - \partial_8). \end{aligned} \tag{6}$$

With respect to the orthonormal frame, we define the normal orientation of the Walker 8-manifold as the orientation determined by

$$e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7 \wedge e_8. \tag{7}$$

4 Two Kinds of g-Orthogonal Almost Complex Structures

With respect to any orthonormal frame $\{\, e_1, \ldots, e_8 \,\}$, one can define a canonical **almost complex structure** J with normal orientation, as follows:

$$J\,e_1 = e_2, \quad J\,e_3 = e_4, \quad J\,e_5 = e_6, \quad J\,e_7 = e_8,$$
$$J\,e_2 = -e_1, \quad J\,e_4 = -e_3, \quad J\,e_6 = -e_5, \quad J\,e_8 = -e_7. \tag{8}$$

For the Walker 8-manifold, consisting of a single coordinate chart, the orthonormal frame vectors are globally defined so one can also define an almost complex structure J' with reversed (opposite) orientation as the preferred one:

$$J'\,e_1 = e_2, \quad J'\,e_3 = e_4, \quad J'\,e_5 = e_6, \quad J'\,e_7 = -e_8,$$
$$J'\,e_2 = -e_1, \quad J'\,e_4 = -e_3, \quad J'\,e_6 = -e_5, \quad J'\,e_8 = e_7 \tag{9}$$

Note that the only difference between J and J' lies in the sign for the operations on the two vectors e_7 and e_8. We call such J' an **opposite almost complex structure**.

4.1 The Explicit Form of J (Treated in [18])

The action of J on the coordinate basis of vectors is obtained from (6) and (8) as follows:

$$J\,\partial_1 = \partial_3, \quad J\,\partial_2 = \partial_8, \quad J\,\partial_3 = -\partial_1, \quad J\,\partial_4 = -\partial_6, \quad J\,\partial_6 = \partial_4, \quad J\,\partial_8 = -\partial_2,$$
$$J\,\partial_5 = \frac{p-r}{2}\partial_3 + \partial_7, \qquad J\,\partial_7 = \frac{p-r}{2}\partial_1 - \partial_5. \tag{10}$$

Also in a matrix form:

$$J = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & (p-r)/2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & (p-r)/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{11}$$

with nonzero components

$$J_1^3 = -J_3^1 = J_2^4 = -J_4^2 = J_5^7 = -J_7^5 = J_6^8 = -J_8^6 = 1, \quad J_5^3 = J_7^1 = \frac{p-r}{2}. \tag{12}$$

4.2 The Explicit Form of J'

Similarly, from (6) and (9), we see the action of J' on the coordinate basis of vectors is as follows:

$$J'\,\partial_1 = \partial_3,\quad J'\,\partial_2 = \partial_8,\quad J'\,\partial_3 = -\partial_1,\quad J'\,\partial_4 = -\partial_6,\quad J'\,\partial_6 = \partial_4,\quad J'\,\partial_8 = -\partial_2,$$
$$J'\,\partial_5 = \frac{p-r}{2}\partial_3 + \partial_7,\qquad J'\,\partial_7 = \frac{p-r}{2}\partial_1 - \partial_5, \tag{13}$$

In matrix form,

$$J' = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & (p-r)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & (p-r)/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

with nonzero components:

$$J'^{3}_{\ 1} = J'^{8}_{\ 2} = -J'^{1}_{\ 3} = -J'^{6}_{\ 4} = J'^{7}_{\ 5} = J'^{4}_{\ 6} = -J'^{5}_{\ 7} = -J'^{2}_{\ 8} = 1,\quad J'^{3}_{\ 5} = J'^{1}_{\ 7} = \frac{p-r}{2}. \tag{15}$$

It is this opposite almost complex structure that is the focus of our concern in this paper.

5 Opposite Kähler Form

Similarly to the standard Kähler form Ω, treated in [18], we can define a kind of Kähler form Ω', called an opposite Kähler form, in terms of J' and g, as follows:

$$\Omega'(X, Y) = g(J'X, Y). \tag{16}$$

Setting $\Omega'(\partial_i, \partial_j) = g(J'\partial_i, \partial_j)$, we have the nonzero components:

$$\Omega'(\partial_1, \partial_7) = \Omega'(\partial_2, \partial_4) = -\Omega'(\partial_3, \partial_5) = \Omega'(\partial_6, \partial_8) = 1,\quad \Omega'(\partial_5, \partial_7) = \frac{p+r}{2}. \tag{17}$$

Then, we have

$$\begin{aligned} \Omega' &= \sum_{i<j} \Omega'(\partial_i, \partial_j)\, dx^i \wedge dx^j \\ &= dx^1 \wedge dx^7 + dx^2 \wedge dx^4 - dx^3 \wedge dx^5 + dx^6 \wedge dx^8 + \frac{1}{2}(p+r)\, dx^5 \wedge dx^7. \end{aligned} \tag{18}$$

It is easy to see that Ω' is nondegenerate as follows:

$$\Omega' \wedge \Omega' \wedge \Omega' \wedge \Omega' = -24 dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8, \tag{19}$$

and that its preferred orientation is the reversed one of that of Ω (cf. [18, (7)]).

6 Opposite Almost-Kähler Structure (Symplectic Structure)

At this stage, we have constructed an opposite almost-Hermitian structure (g, J', Ω') on the Walker 8-manifold M. We must now consider if Ω' is symplectic or not, i.e., if the triple (g, J', Ω') is almost-Kähler or not. The differential of Ω' is easily computed:

$$\begin{aligned} d\Omega' = &\frac{1}{2}(p_1 + r_1)\, dx^1 \wedge dx^5 \wedge dx^7 + \frac{1}{2}(p_2 + r_2)\, dx^2 \wedge dx^5 \wedge dx^7 \\ &+ \frac{1}{2}(p_3 + r_3)\, dx^3 \wedge dx^5 \wedge dx^7 + \frac{1}{2}(p_4 + r_4)\, dx^4 \wedge dx^5 \wedge dx^7 \\ &- \frac{1}{2}(p_6 + r_6)\, dx^5 \wedge dx^6 \wedge dx^7 + \frac{1}{2}(p_8 + r_8)\, dx^5 \wedge dx^7 \wedge dx^8. \end{aligned} \tag{20}$$

From this expression, we have the following

Proposition 1 *Ω' is symplectic if and only if the following PDEs hold.*

$$p_1 + r_1 = p_2 + r_2 = p_3 + r_3 = p_4 + r_4 = p_6 + r_6 = p_8 + r_8 = 0. \tag{21}$$

where $p_i = \partial p/\partial x^i$, $r_i = \partial r/\partial x^i$ $(i \neq 5, 7)$.

This gives the following

Corollary 1 *Ω is symplectic, i.e., (g, J', Ω') is opposite almost Kähler if and only if*

$$p + r = p(x^1, \ldots, x^8) + r(x^1, \ldots, x^8) = \xi(x^5, x^7), \tag{22}$$

where ξ is an arbitrary function of x^5 and x^7 only.

7 Opposite Hermitian Structure (J'-Integrability)

The opposite almost complex structure J' is integrable if and only if the torsion of J' (Nijenhuis tensor) vanishes, i.e., the components

$$N'^i_{\ jk} = 2\sum_{h=1}^{8}\left(J'^h_{\ j}\frac{\partial J'^i_{\ k}}{\partial x^h} - J'^h_{\ k}\frac{\partial J'^i_{\ j}}{\partial x^h} - J'^i_{\ h}\frac{\partial J'^h_{\ k}}{\partial x^j} + J'^i_{\ h}\frac{\partial J'^h_{\ j}}{\partial x^k}\right) \tag{23}$$

all vanish (cf. [10, p. 124]), with $J'^j_{\ i}$ as in (15). Since $N'^i_{\ jk} = -N'^i_{\ kj}$, we need to consider $N'^i_{\ jk}$ $(j < k)$. By explicit calculation, the nonzero components of the Nijenhuis tensor are as follows:

$$\begin{aligned}
&N'^1_{\ 15} = -N'^1_{\ 37} = -N'^3_{\ 17} = -N'^3_{\ 35} = p_1 - r_1, \quad N'^3_{\ 57} = -\frac{p-r}{2}(p_1 - r_1),\\
&N'^1_{\ 25} = N'^1_{\ 78} = -N'^3_{\ 27} = N'^3_{\ 58} = p_2 - r_2,\\
&N'^1_{\ 17} = N'^1_{\ 35} = N'^3_{\ 15} = -N'^3_{\ 37} = p_3 - r_3, \quad N'^1_{\ 57} = \frac{p-r}{2}(p_3 - r_3),\\
&N'^1_{\ 45} = N'^1_{\ 67} = -N'^3_{\ 47} = -N'^3_{\ 56} = p_4 - r_4,\\
&N'^1_{\ 47} = N'^1_{\ 56} = N'^3_{\ 45} = N'^3_{\ 67} = -p_6 + r_6,\\
&N'^1_{\ 27} = -N'^1_{\ 58} = N'^3_{\ 25} = N'^3_{\ 78} = p_8 - r_8.
\end{aligned} \tag{24}$$

Then we have the following

Proposition 2 *The Nijenhuis tensor $N'^i_{\ jk}$ vanishes, and therefore J' is integrable if and only if the following PDEs hold.*

$$p_1 - r_1 = p_2 - r_2 = p_3 - r_3 = p_4 - r_4 = p_6 - r_6 = p_8 - r_8 = 0. \tag{25}$$

Corollary 2 *J' is integrable, i.e., (g, J') is opposite Hermitian if and only if*

$$p - r = p(x^1, \ldots, x^8) - r(x^1, \ldots, x^8) = -\eta(x^5, x^7). \tag{26}$$

where η is an arbitrary function of x^5 and x^7 only.

Remark Comparing Corollaries 1 and 2, there is a certain reciprocity between the symplectic condition of Ω' and the integrability condition of J' in opposite orientation, similarly to the case of normal orientation [18].

8 Opposite Kähler Structures

The opposite almost-Kähler structure (g, J', Ω') is **opposite Kähler** if J' is integrable.

Theorem 1 *The opposite almost-Hermitian Walker 8-manifold* (M, g, J')*, with* g *in (4) and* J' *in (14), is opposite Kähler if and only if* p *and* r *are both arbitrary functions of* (x^5, x^7) *only, or explicitly*

$$p = p(x^5, x^7), \qquad r = r(x^5, x^7). \tag{27}$$

Proof p and r must satisfy (22) and (26). Therefore we have that $p = \frac{1}{2}\{\xi(x^5, x^7) - \eta(x^5, x^7)\}$, and $r = \frac{1}{2}\{\xi(x^5, x^7) + \eta(x^5, x^7)\}$, both of which are functions of x^5 and x^7 only. □

9 Strict Opposite Almost-Kähler Structures

We say that (g, J', Ω') is strictly opposite almost-Kähler if it is opposite almost-Kähler, but not opposite Kähler. Since our purpose is to find counterexamples to the Goldberg conjecture with opposite orientation, we must find functions p and r such that they satisfy the opposite almost-Kähler condition (22) but not the integrability condition (26) for J'.

On the basis of the above results, we can derive conditions for the triple (g, J', Ω') to be an opposite almost Kähler structure, which is not opposite Kähler. For p and r restricted by (22) only, they have the following forms

$$\begin{aligned} p &= p(x^1, \dots, x^8) = f(x^1, \dots, x^8) + g(x^5, x^7) \\ r &= r(x^1, \dots, x^8) = -f(x^1, \dots, x^8) + h(x^5, x^7), \\ &g(x^5, x^7) + h(x^5, x^7) = \xi(x^5, x^7), \end{aligned} \tag{28}$$

where $f = f(x^1, \dots, x^8)$ $(= (p - r)/2)$ is an arbitrary function of all variables x^i, and $g(x^5, x^7)$, $h(x^5, x^7)$ are both arbitrary functions of two variables x^5, x^7, with their sum $\xi(x^5, x^7)$. However, the case of of the form $f = f(x^5, x^7)$ must be excluded to avoid satisfying (26).

Proposition 3 *The triple* (g, J', Ω') *is strict opposite almost-Kähler, if at least one of the derivatives* f_1*,* f_2*,* f_3*,* f_4*,* f_6 *and* f_8 *does not vanish.*

Proof Since the PDEs (25) can be rewritten as $f_1 = f_2 = f_3 = f_4 = f_6 = f_8 = 0$, the assertion is clear. □

In what follows, we assume that $g(x^5, x^7) = h(x^5, x^7) = \frac{1}{2}\xi(x^5, x^7)$ for simplicity.

10 Einstein Condition

Let S be the scalar curvature of the metric (4). Then, we have $S = p_{11} + r_{33}$, which is an important observation for considering the Einstein condition. For the problem of the Goldberg conjecture, g must be an Einstein metric, i.e., g must be a solution to the Einstein equation $G_{ij} = R_{ij} - (S/8)g_{ij} = 0$, where R_{ij} is the Ricci curvature, and S is the scalar curvature of g. For the block matrix Q of in (4), with p, r in (28):

$$Q = \begin{bmatrix} f(x^1,\ldots,x^8) + \frac{1}{2}\xi(x^5,x^7) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -f(x^1,\ldots,x^8) + \frac{1}{2}\xi(x^5,x^7) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{29}$$

the nonzero components of G_{ij} are given as follows:

$$\begin{aligned} &G_{25} = \frac{1}{2} f_{12}, \quad G_{17} = -G_{35} = -\frac{1}{2} f_{13}, \quad G_{45} = \frac{1}{2} f_{14}, \quad G_{56} = \frac{1}{2} f_{16}, \\ &G_{58} = \frac{1}{2} f_{18}, \quad G_{27} = -\frac{1}{2} f_{23}, \quad G_{47} = -\frac{1}{2} f_{34}, \quad G_{67} = -\frac{1}{2} f_{36}, \\ &G_{78} = -\frac{1}{2} f_{38}, \; G_{15} = \frac{1}{8}(3 f_{11} + f_{33}), \; G_{26} = G_{48} = -\frac{1}{8}(f_{11} - f_{33}), \\ &G_{37} = -\frac{1}{8}(f_{11} + 3 f_{33}), \quad G_{57} = \frac{1}{2}(f_{17} + f_1 f_3 - f_{35}), \\ &G_{55} = -f_{26} - f_{37} - f_{48} + \frac{3}{8} f (f_{11} - f_{33}) + \frac{1}{8}\xi(3 f_{11} + 5 f_{33}) - \frac{1}{2} {f_3}^2, \\ &G_{77} = f_{15} + f_{26} + f_{48} - \frac{3}{8} f (f_{11} - f_{33}) - \frac{1}{8}\xi(5 f_{11} + 3 f_{33}) - \frac{1}{2} f_1^2. \end{aligned} \tag{30}$$

We note that the Einstein condition is common to the issue for the Goldberg conjecture with the normal orientation in Walker 8-manifold, since g is common, as treated in [18]. It may be very hard to solve the Einstein equation above. Since our purpose is to find a counterexample, we assume that f is independent of x^1 and x^3. Such an assumption implies that $S = 0$.

Then, the Einstein equation (30) becomes drastically simplified form as follows:

$$G_{55} = -G_{77} = -f_{26} - f_{48} = 0. \tag{31}$$

Then, we have

Proposition 4 *Let the block matrix Q be of the form (29). If the function $f = f(x^1,\ldots,x^8)$ is independent of x^1 and x^3, then the metric g is scalar flat, i.e., $S =$*

0. *Moreover, if* f *is a sum of four functions* F^1*,* F^2*,* F^3 *and* F^4*, each a function of four arguments as follows:*

$$\begin{aligned} f &= f(x^2, x^4, x^5, x^6, x^7, x^8) \\ &= F^1(x^2, x^4, x^5, x^7) + F^2(x^2, x^5, x^7, x^8) + F^3(x^4, x^5, x^6, x^7) + F^4(x^5, x^6, x^7, x^8), \end{aligned} \tag{32}$$

then the metric g *is Einstein, in fact, Ricci flat.*

Proof It is easy to see that if f is independent of x^1 and x^3, then f of the form in (32) is a solution to (31). □

Remark We need not to obtain general solutions to the Einstein equation, but certain class of Einstein metrics, which may give rise to candidates of counterexamples to the Goldberg conjecture. In fact, there are some specific solutions to (31), e.g., $f(x^2, x^4, x^6, x^8) = x^2x^6 - x^4x^8$, which is not of the type (32). However, the class of solutions (32) suffice for our present purpose.

11 Strict Opposite Almost-Kähler Einstein Structure

At this stage, we show explicitly a metric and an opposite almost complex structure, which give rise to counterexamples to the Goldberg conjecture constructed on an 8-dimensional Walker Einstein manifold.

Let f be a function of variables x^2, x^4, x^5, x^6, x^7, and x^8, characterized in Proposition 4, and at least one of the derivatives f_2, f_3, f_4, f_6 and f_8 does not vanish. With such a function f, let g be the Walker metric of the form

$$g = \left[\, g_{ij} \,\right] = \begin{bmatrix} 0 & I_4 \\ I_4 & Q \end{bmatrix}, \quad \text{with } Q = \begin{bmatrix} f + \frac{1}{2}\xi(x^5, x^7) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -f + \frac{1}{2}\xi(x^5, x^7) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

and an opposite almost complex structure J' be as follows:

$$J' = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

Then, it is easy to see that the opposite Kähler form, given by

$$\begin{aligned}\Omega' &= \sum_{i<j} \Omega'(\partial_i, \partial_j)\, dx^i \wedge dx^j \\ &= dx^1 \wedge dx^7 + dx^2 \wedge dx^4 - dx^3 \wedge dx^5 + dx^6 \wedge dx^8 + \frac{1}{2}\,\xi(x^5, x^7)\, dx^5 \wedge dx^7,\end{aligned} \tag{35}$$

is a closed form, and therefore a triple (g, J', Ω') is a strictly almost-Kähler structure.

Theorem 2 *Let g, J' and Ω' be as characterized by (33)–(35), respectively. Then, for an 8-dimensional Walker Einstein manifold $M = (M, g, J', \Omega')$, the triple (g, J', Ω') is an opposite almost-Kähler Einstein structure, with J' not integrable. In fact, the triple (g, J', Ω') is a strict opposite almost-Kähler Einstein structure.*

Remark If f is a function of two variables x^5 and x^7 only, then J' is integrable.

12 Counterexamples to the Goldberg Conjecture of Indefinite and Opposite Version

Finally, we must consider if the 8-dimensional Walker Einstein metric g constructed above can descend to a metric on some compact 8-manifold. We see easily that all coordinates $x^1, \ldots, x^8$ can be chosen as cyclic coordinates, by means of identification: a point $(x^1, \ldots, x^8)$ with a point $(x^1 + 2\pi, \ldots, x^8 + 2\pi)$ on $\mathbb{R}^8$, which gives an 8-torus T^8.

We now state our main

Theorem 3 *Let $M = (M, g, J', \Omega')$ be an 8-dimensional Walker Einstein manifold $M = (M, g, J', \Omega')$ as in Theorem 2. Then, with identification of all coordinates as $x^i + 2\pi \equiv x^i$ $(i = 1, \ldots, 8)$, the triple (g, J', Ω') is a strict opposite almost-Kähler Einstein structure on an 8-torus T^8, and hence this is a counterexample to the Goldberg conjecture of indefinite and opposite version.*

We end this paper with a simple nontrivial counterexample to the Goldberg conjecture of indefinite and opposite version on an 8-torus T^8, characterized by the triple (g, J', Ω') as follows:

1. an 8-dimensional Walker Einstein metric g:

$$g = [\, g_{ij} \,] = \begin{bmatrix} 0 & I_4 \\ I_4 & Q \end{bmatrix}, \quad \text{with } Q = \begin{bmatrix} \sin x^7 + \sin x^8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sin x^7 - \sin x^8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{36}$$

2. a nonintegrable opposite almost complex structure J':

$$J' = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & \sin x^8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & \sin x^8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{37}$$

with the nonzero components of its Nijenhuis tensor ${N'}^i_{jk}$ in (24) as follows:

$${N'}^1_{27} = -{N'}^1_{58} = {N'}^3_{25} = {N'}^3_{78} = \cos x^8, \tag{38}$$

3. the opposite Kähler form (symplectic form):

$$\begin{gathered} \Omega' = dx^1 \wedge dx^7 + dx^2 \wedge dx^4 - dx^3 \wedge dx^5 + dx^6 \wedge dx^8 + \tfrac{1}{2} \sin x^7 \, dx^5 \wedge dx^7, \\ d\Omega' = 0, \end{gathered} \tag{39}$$

4. the nonzero components of the curvature tensor $R^i{}_{jkl}$:

$$\begin{gathered} R^1{}_{757} = -R^3{}_{557} = -\tfrac{1}{2} \sin x^7, \\ R^1{}_{858} = -R^3{}_{878} = -R^4{}_{558} = R^4{}_{778} = \tfrac{1}{2} \sin x^8, \end{gathered} \tag{40}$$

5. the Ricci curvature R_{ij}, the scalar curvature S, and the Einstein tensor $G_{ij} = R_{ij} - \dfrac{S}{8} g_{ij}$ all vanish.

Acknowledgements We would like to express our cordial thanks to Kouei Sekigawa for helpful suggestions and encouragement.

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A Construction of Weakly Reflective Submanifolds in Compact Symmetric Spaces

Shinji Ohno

Abstract In this paper, we give sufficient conditions for orbits of Hermann actions to be weakly reflective in terms of symmetric triads, that is a generalization of irreducible root systems. Using these sufficient conditions, we obtain new examples of weakly reflective submanifolds in compact symmetric spaces.

1 Introduction

Ikawa, Sakai, and Tasaki [5] proposed the notion of weakly reflective submanifold as a generalization of the notion of reflective submanifold [7]. In [5], they detected a certain global symmetry of several austere submanifolds in a hypersphere, and classified austere orbits and weakly reflective orbits of the linear isotropy representation of irreducible symmetric spaces. They gave a necessary and sufficient condition for orbits of the linear isotropy representations of irreducible symmetric spaces to be austere submanifolds (further, weakly reflective submanifolds) in the hypersphere in terms of root systems. We would like to generalize this fact to compact Riemannian symmetric spaces. However, it is known that austere orbits of the isotropy action of compact symmetric spaces are reflective submanifolds. Therefore, we consider Hermann actions which are a generalization of isotropy actions of compact symmetric spaces. Ikawa [3] introduced the notion of symmetric triad as a generalization of the notion of irreducible root system to study orbits of Hermann actions. Ikawa expressed orbit spaces of Hermann actions by using symmetric triads, and gave a characterization of the minimal, austere and totally geodesic orbits of Hermann actions in terms of symmetric triads. However, weakly reflective orbits have not been classified yet. In this paper, we give sufficient conditions for orbits of Hermann actions to be weakly reflective in terms of symmetric triads.

S. Ohno (✉)
Osaka City University Advanced Mathematical Institute (OCAMI), 3-3-138 Sugimoto, Sumiyoshi-ku Osaka 558-8585, Japan
e-mail: kudamono.shinji@gmail.com

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*, Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_10

Let G be a compact, connected, semisimple Lie group, and K_1, K_2 be symmetric subgroups of G. We consider the following three Lie group actions:

1. $(K_2 \times K_1) \curvearrowright G : (k_2, k_1)g = k_2 g k_1^{-1}((k_2, k_1) \in K_2 \times K_1)$,
2. $K_2 \curvearrowright G/K_1 : k_2\pi_1(g) = \pi_1(k_2 g)(k_2 \in K_2)$,
3. $K_1 \curvearrowright K_2\backslash G : k_1\pi_2(g) = \pi_2(g k_1^{-1})(k_1 \in K_1)$.

The K_2-action and the K_1-action are called Hermann actions. Orbits of the $(K_2 \times K_1)$-action have properties which are similar to orbits of Hermann actions. In particular, by using Ikawa's method, we can characterize a minimal orbit and an austere orbit of the $(K_2 \times K_1)$-action in terms of the symmetric triad determined by (G, K_1, K_2). Since totally geodesic orbits of Hermann actions are reflective submanifolds, we only consider austere orbits which are not totally geodesic.

2 Preliminaries

2.1 Weakly Reflective Submanifolds

We recall the definitions of reflective submanifold and weakly reflective submanifold. Let $(\tilde{M}, \langle , \rangle)$ be a complete Riemannian manifold.

Definition 1 Let M be a submanifold of $\tilde{M}$. Then M is a *reflective submanifold* of $\tilde{M}$ if there exists an involutive isometry σ_M of $\tilde{M}$ such that M is a connected component of the fixed point set of σ_M. Then, we call σ_M the reflection of M.

Definition 2 Let M be a submanifold of $\tilde{M}$. For each normal vector $\xi \in T_x^{\perp}M$ at each point $x \in M$, if there exists an isometry σ_ξ on $\tilde{M}$ which satisfies $\sigma_\xi(x) = x$, $\sigma_\xi(M) = M$ and $(d\sigma_\xi)_x(\xi) = -\xi$, then we call M a *weakly reflective submanifold* and σ_ξ a reflection of M with respect to ξ.

If M is a reflective submanifold of $\tilde{M}$, then σ_M is a reflection of M with respect to each normal vector $\xi \in T_x^{\perp}M$ at each point $x \in M$. Thus, a reflective submanifold of $\tilde{M}$ is a weakly reflective submanifold of $\tilde{M}$. Notice that a reflective submanifold is totally geodesic, but a weakly reflective submanifold is not necessarily totally geodesic.

Definition 3 ([2]) Let M be a submanifold of $\tilde{M}$. We denote the shape operator of M by A. M is called an *austere submanifold* if for each normal vector $\xi \in T_x^{\perp}M$, the set of eigenvalues with their multiplicities of A^ξ is invariant under the multiplication by -1.

It is clear that an austere submanifold is a minimal submanifold. Ikawa, Sakai and Tasaki proved that a weakly reflective submanifold is an austere submanifold.

Lemma 1 ([5]) *Let G be a Lie group acting isometrically on a Riemannian manifold $\tilde{M}$. For $x \in \tilde{M}$, we consider the orbit Gx. If for each $\xi \in T_x^\perp Gx$, there exists a reflection of Gx at x with respect to ξ, then Gx is a weakly reflective submanifold of $\tilde{M}$.*

Proposition 1 ([5]) *Any singular orbit of a cohomogeneity one action on a Riemannian manifold is a weakly reflective submanifold.*

2.2 The Actions and Geometric Properties of Orbits

In this section, we consider Hermann actions and associated actions on Lie groups which are hyperpolar actions on compact symmetric spaces. An isometric action of a compact Lie group on a Riemannian manifold M is called hyperpolar if there exists a closed, connected and flat submanifold S of M that meets all orbits orthogonally. Then, the submanifold S is called a section. A. Kollross [6] classified the hyperpolar actions on compact irreducible symmetric spaces. By the classification, we can see that a hyperpolar action on a compact symmetric space whose cohomogeneity is two or greater is orbit-equivalent to some Hermann action.

Let G be a compact, connected, semisimple Lie group, and K_1, K_2 be closed subgroups of G. For each $i = 1, 2$, assume that there exists an involutive automorphism θ_i of G which satisfies $(G_{\theta_i})_0 \subset K_i \subset G_{\theta_i}$, where G_{θ_i} is the set of fixed points of θ_i and $(G_i)_0$ is the identity component of G_{θ_i}. Then the triple (G, K_1, K_2) is called a compact symmetric triad. The pair (G, K_i) is a compact symmetric pair for $i = 1, 2$. We denote the Lie algebras of G, K_1 and K_2 by $\mathfrak{g}$, $\mathfrak{k}_1$ and $\mathfrak{k}_2$, respectively. The involutive automorphism of $\mathfrak{g}$ induced from θ_i will be also denoted by θ_i. Take an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Then the inner product $\langle\cdot,\cdot\rangle$ induces a bi-invariant Riemannian metric on G and G-invariant Riemannian metrics on the coset manifolds $M_1 := G/K_1$ and $M_2 := K_2\backslash G$. We denote these Riemannian metrics on G, M_1 and M_2 by the same symbol $\langle\cdot,\cdot\rangle$. These Riemannian manifolds G, M_1 and M_2 are Riemannian symmetric spaces with respect to $\langle\cdot,\cdot\rangle$. We denote by π_i the natural projection from G to M_i $(i = 1, 2)$, and consider the following three Lie group actions:

- $(K_2 \times K_1) \curvearrowright G \ : (k_2, k_1)g = k_2 g k_1^{-1} \ ((k_2, k_1) \in K_2 \times K_1)$,
- $K_2 \curvearrowright M_1 \ : k_2\pi_1(g) = \pi_1(k_2 g) \ (k_2 \in K_2)$,
- $K_1 \curvearrowright M_2 \ : k_1\pi_2(g) = \pi_2(g k_1^{-1}) \ (k_1 \in K_1)$,

for $g \in G$. The three actions have the same orbit space $K_2\backslash G/K_1$. Ikawa computed the second fundamental form of orbits of Hermann actions in the case $\theta_1\theta_2 = \theta_2\theta_1$. We can apply Ikawa's method to the geometry of orbits of the $(K_2 \times K_1)$-action. For $g \in G$, we denote the left (resp. right) transformation of G by L_g (resp. R_g). The isometry on M_1 (resp. M_2) induced by L_g (resp. R_g) will be also denoted by the same symbol L_g (resp. R_g).

For $i = 1, 2$, we set

$$\mathfrak{m}_i = \{X \in \mathfrak{g} \mid \theta_i(X) = -X\}.$$

Then we have an orthogonal direct sum decomposition of $\mathfrak{g}$ that is the canonical decomposition:

$$\mathfrak{g} = \mathfrak{k}_i \oplus \mathfrak{m}_i.$$

Let e denotes the identity element of G. The tangent space $T_{\pi_i(e)}M_i$ of M_i at the origin $\pi_i(e)$ is identified with $\mathfrak{m}_i$ in a natural way. We define a closed subgroup G_{12} of G by

$$G_{12} = \{g \in G \mid \theta_1(g) = \theta_2(g)\}.$$

Hence $((G_{12})_0, K_{12})$ is a compact symmetric pair, where K_{12} is a closed subgroup of $(G_{12})_0$ defined by

$$K_{12} = \{k \in (G_{12})_0 \mid \theta_1(k) = k\}.$$

The canonical decomposition of $((G_{12})_0, K_{12})$ is given by

$$\mathfrak{g}_{12} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2).$$

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}_1 \cap \mathfrak{m}_2$. Then $\exp(\mathfrak{a})$ is a toral subgroup in $(G_{12})_0$. Then $\exp(\mathfrak{a})$, $\pi_1(\exp(\mathfrak{a}))$ and $\pi_2(\exp(\mathfrak{a}))$ are sections of the $(K_2 \times K_1)$-action, the K_2-action and the K_1-action, respectively. To investigate the orbit spaces of the three actions, we consider a equivalent relation $\sim$ on $\mathfrak{a}$ defined as follows: For $H_1, H_2 \in \mathfrak{a}$, $H_1 \sim H_2$ if $K_2 \exp(H_1)K_1 = K_2 \exp(H_2)K_1$. Clearly, we have $H_1 \sim H_2$ if and only if $K_2\pi_1(\exp(H_1)) = K_2\pi_1(\exp(H_2))$, and similarly, $H_1 \sim H_2$ if and only if $K_1\pi_2(\exp(H_1)) = K_1\pi_2(\exp(H_2))$. Then we have $\mathfrak{a}/\!\sim\, = K_2\backslash G/K_1$. For each subgroup L of G, we define

$$N_L(\mathfrak{a}) = \{k \in L \mid \mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}\},$$
$$Z_L(\mathfrak{a}) = \{k \in L \mid \mathrm{Ad}(k)H = H \ (H \in \mathfrak{a})\}.$$

Then $Z_L(\mathfrak{a})$ is a normal subgroup of $N_L(\mathfrak{a})$. We define a group $\tilde{J}$ by

$$\tilde{J} = \{([s], Y) \in N_{K_2}(\mathfrak{a})/Z_{K_1\cap K_2}(\mathfrak{a}) \ltimes \mathfrak{a} \mid \exp(-Y)s \in K_1\}.$$

The group $\tilde{J}$ naturally acts on $\mathfrak{a}$ by the following:

$$([s], Y)H = \mathrm{Ad}(s)H + Y \quad (([s], Y) \in \tilde{J},\ H \in \mathfrak{a}).$$

Matsuki [8] proved that

$$K_2\backslash G/K_1 \cong \mathfrak{a}/\tilde{J}.$$

Hereafter, we suppose $\theta_1\theta_2 = \theta_2\theta_1$. Then we have an orthogonal direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

We define subspaces of $\mathfrak{g}$ as follows:

$$\begin{aligned} \mathfrak{k}_0 &= \{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = \{0\}\}, \\ V(\mathfrak{k}_1 \cap \mathfrak{m}_2) &= \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid [\mathfrak{a}, X] = \{0\}\}, \\ V(\mathfrak{m}_1 \cap \mathfrak{k}_2) &= \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = \{0\}\}. \end{aligned}$$

For $\lambda \in \mathfrak{a}$,

$$\begin{aligned} \mathfrak{k}_\lambda &= \{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ (H \in \mathfrak{a})\}, \\ \mathfrak{m}_\lambda &= \{X \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ (H \in \mathfrak{a})\}, \\ V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) &= \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ (H \in \mathfrak{a})\}, \\ V_\lambda^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) &= \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ (H \in \mathfrak{a})\}. \end{aligned}$$

We set

$$\begin{aligned} \Sigma &= \{\lambda \in \mathfrak{a} \setminus \{0\} \mid \mathfrak{k}_\lambda \neq \{0\}\}, \\ W &= \{\alpha \in \mathfrak{a} \setminus \{0\} \mid V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) \neq \{0\}\}, \\ \tilde{\Sigma} &= \Sigma \cup W. \end{aligned}$$

It is known that $\dim \mathfrak{k}_\lambda = \dim \mathfrak{m}_\lambda$ and $\dim V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) = \dim V_\lambda^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ for each $\lambda \in \tilde{\Sigma}$. Thus we set $m(\lambda) := \dim \mathfrak{k}_\lambda,\ n(\lambda) := \dim V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$.

We assume that (G, K_1, K_2) satisfies one of the following conditions (A), (B) or (C).

(A) G is simple and θ_1 and θ_2 can not transform each other by an inner automorphism of $\mathfrak{g}$.

(B) There exist a compact connected simple Lie group U and a symmetric subgroup $\overline{K}$ of U such that

$$G = U \times U, \quad K_1 = \Delta G = \{(u, u) \mid u \in U\}, \quad K_2 = \overline{K} \times \overline{K}.$$

(C) There exist a compact connected simple Lie group U and an involutive outer automorphism σ such that

$$\begin{aligned} G = U \times U, \quad K_1 &= \Delta G = \{(u, u) \mid u \in U\}, \\ K_2 &= \{(u_1, u_2) \mid (\sigma(u_2),\ \sigma(u_1)) = (u_1, u_2)\}. \end{aligned}$$

Then Ikawa proved the following theorem.

Theorem 1 ([4]) *Let (G, K_1, K_2) be a compact symmetric triad which satisfies one of the conditions (A), (B) or (C). Then the triple $(\tilde{\Sigma}, \Sigma, W)$ defined as above is a* *symmetric triad with multiplicities. Conversely every symmetric triad is obtained in this way.*

Notice that Σ is the root system of the pair $((G_{12})_0, K_{12})$, and $\tilde{\Sigma}$ is a root system of $\mathfrak{a}$ (see [3]). We take a basis of $\mathfrak{a}$ and the lexicographic ordering $>$ on $\mathfrak{a}$ with respect to the basis. We set

$$\tilde{\Sigma}^+ = \{\lambda \in \tilde{\Sigma} \mid \lambda > 0\},\ \Sigma^+ = \Sigma \cap \tilde{\Sigma}^+,\ W^+ = W \cap \tilde{\Sigma}^+.$$

Then we have an orthogonal direct sum decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k}_0 \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda \oplus \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda \oplus V(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus \sum_{\alpha \in W^+} V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\alpha \in W^+} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

For $H \in \mathfrak{a}$, we set

$$\Sigma_H = \{\lambda \in \Sigma \mid \langle \lambda, H\rangle \in \pi\mathbb{Z}\},\ W_H = \{\alpha \in W \mid \langle \alpha, H\rangle \in (\pi/2) + \pi\mathbb{Z}\},$$
$$\tilde{\Sigma}_H = \Sigma_H \cup W_H,\ \Sigma_H^+ = \Sigma^+ \cap \Sigma_H,\ W_H^+ = W^+ \cap W_H,\ \tilde{\Sigma}_H^+ = \Sigma_H^+ \cup W_H^+.$$

Using the symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ with multiplicities (m, n), we can describe geometric properties of orbits of these Lie group actions.

Theorem 2 ([3] Corollaries 4.23, 4.29, 4.24, and [1] Theorem 5.3) *Let $g = \exp(H)$ $(H \in \mathfrak{a})$. Denote the mean curvature vector of $K_2\pi_1(g) \subset M_1$ at $\pi_1(g)$ by m_H^1. Then we have:*

(1)

$$dL_g^{-1} m_H^1 = -\sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H\rangle \notin \pi\mathbb{Z}}} m(\lambda)\cot\langle\lambda, H\rangle\lambda + \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H\rangle \notin (\pi/2)+\pi\mathbb{Z}}} n(\alpha)\tan\langle\alpha, H\rangle\alpha.$$

(2) The orbit $K_2\pi_1(g) \subset M_1$ is austere if and only if the finite subset of $\mathfrak{a}$ defined by

$$\{-\lambda\cot\langle\lambda, H\rangle\ (\textit{multiplicity} = m(\lambda)) \mid \lambda \in \Sigma^+, \langle\lambda, H\rangle \notin \pi\mathbb{Z}\}$$
$$\cup\{\alpha\tan\langle\alpha, H\rangle\ (\textit{multiplicity} = n(\alpha)) \mid \alpha \in W^+, \langle\alpha, H\rangle \notin (\pi/2) + \pi\mathbb{Z}\}$$

is invariant under the multiplication by -1 with multiplicities.

(3) The orbit $K_2\pi_1(g) \subset M_1$ is totally geodesic if and only if $\langle\lambda, H\rangle \in (\pi/2)\mathbb{Z}$ for each $\lambda \in \tilde{\Sigma}^+$.

We can apply Theorem 2 for orbits $K_1\pi_2(g) \subset M_2$. Thus, we have the following corollary.

Corollary 1 (Corollary 4.30 in [3], Corollaries 2 and 4 in [9]) *(i) The orbit $K_2\pi_1(g)$ is minimal (resp. austere, totally geodesic) if and only if $K_1\pi_2(g)$ is minimal (resp. austere, totally geodesic).*

(ii) The orbit $(K_2 \times K_1)g$ is minimal (resp. austere) if and only if $K_1\pi_2(g)$ is minimal (resp. austere).

Remark 1 There is no correspondence in totally geodesic orbits. For example, when θ_1 and θ_2 cannot be transformed each other by an inner automorphism of $\mathfrak{g}$, $K_2eK_1 \subset G$ is not totally geodesic, but $K_2\pi_1(e) \subset M_1$ is totally geodesic.

3 Main Theorem

In the previous section, we saw a correspondence of austereness of orbits of the $(K_2 \times K_1)$-action and the K_2-action. In this section, we consider weakly reflective orbits of the $(K_2 \times K_1)$-action, the K_2-action and the K_1-action, and give two sufficient conditions for an orbit to be weakly reflective. The first sufficient condition is the following:

Theorem 3 *Assume K_1 and K_2 are connected. Let $g = \exp(H)$ $(H \in \mathfrak{a})$. If $\langle \lambda, H \rangle \in (\pi/2)\mathbb{Z}$ for any $\lambda \in \tilde{\Sigma}$, that is, $H \in \Gamma$, then the orbit $K_2gK_1 \subset G$ is weakly reflective.*

Proof We set $\sigma = L_g\theta_1L_g^{-1}$. Then σ satisfies the following conditions:

1. $\sigma(g) = g$,
2. $\sigma(K_2gK_1) = K_2gK_1$,
3. $d\sigma(\xi) = -\xi$ $(\xi \in T_g^{\perp}(K_2gK_1))$.

Clearly, $\sigma(g) = g$ holds. By Lemmas 4.10 and 4.16 in [3], we have $\mathrm{Ad}(g^2)\mathfrak{k}_2 = \mathfrak{k}_2$. Since K_2 is connected, we have $g^2K_2g^{-2} = K_2$. In addition, since $\theta_1\theta_2 = \theta_2\theta_1$, we have $\theta_1\mathfrak{k}_2 = \mathfrak{k}_2$. Thus, we also have $\theta_1(K_2) = K_2$. Therefore, for $(k_2, k_1) \in K_2 \times K_1$,

$$\sigma(k_2gk_1^{-1}) = (g^2\theta_1(k_2)g^{-2})gk_1^{-1} \in K_2gK_1.$$

Hence, $\sigma(K_2gK_1) = K_2gK_1$. Since $T_g^{\perp}(K_2gK_1) = dL_g(\mathrm{Ad}(g)^{-1}(\mathfrak{m}_2) \cap \mathfrak{m}_1)$, we have

$$d\sigma(\xi) = dL_g\theta_1(dL_g^{-1}(\xi)) = -dL_gdL_g^{-1}(\xi) = -\xi$$

Therefore, σ is a reflection of K_2gK_1 at g with respect to each normal vector $dL_g\xi \in T_g^{\perp}(K_2gK_1)$.

Corollary 2 *The orbit $K_2eK_1 \subset G$ is weakly reflective.*

Remark 2 Under the same condition as Theorem 3, we can prove that $K_2\pi_1(g) \subset M_1$ and $K_1\pi_2(g) \subset M_2$ are weakly reflective. However, Ikawa proved $K_2\pi_1(g) \subset M_1$

and $K_1\pi_2(g) \subset M_2$ are reflective. Hence $K_2\pi_1(g) \subset M_1$ and $K_1\pi_2(g) \subset M_2$ are totally geodesic, but $K_2 g K_1$ is not necessarily totally geodesic. In fact, when θ_1 and θ_2 cannot be transformed each other by inner automorphism of $\mathfrak{g}$, then there is no totally geodesic orbit of the $(K_2 \times K_1)$-action on G.

Let $\tilde{W}(\tilde{\Sigma}, \Sigma, W)$ be a subgroup of the affine group $\mathrm{O}(\mathfrak{a}) \ltimes \mathfrak{a}$ which is generated by

$$\left\{ \left(s_\lambda, \frac{2n\pi}{\langle \lambda, \lambda \rangle}\lambda \right) \;\middle|\; \lambda \in \Sigma,\ n \in \mathbb{Z} \right\} \cup \left\{ \left(s_\alpha, \frac{(2n+1)\pi}{\langle \alpha, \alpha \rangle}\alpha \right) \;\middle|\; \alpha \in W,\ n \in \mathbb{Z} \right\}.$$

Then, we have the following lemma.

Lemma 2 ([3] Lemmas 4.4 and 4.21)

$$\tilde{W}(\tilde{\Sigma}, \Sigma, W) \subset \tilde{J}$$

Using the above lemma, we have the following lemma.

Lemma 3 ([9]) *Let* $g = \exp(H)$ $(H \in \mathfrak{a})$. *Then, for each* $\lambda \in \tilde{\Sigma}_H$, *there exists* $k_\lambda \in N_{K_2}(\mathfrak{a})$, *such that*

1.
$$\left(k_\lambda,\ \exp\left(-\frac{2\langle \lambda, H \rangle}{\langle \lambda, \lambda \rangle}\lambda \right) k_\lambda \right) \in (K_2 \times K_1)_g,$$

2.
$$d\left(k_\lambda,\ \exp\left(-\frac{2\langle \lambda, H \rangle}{\langle \lambda, \lambda \rangle}\lambda \right) k_\lambda \right)_g (dL_g \xi) = dL_g(s_\lambda \xi) \quad (\xi \in \mathfrak{a}).$$

Proposition 2 *For any* $H \in \mathfrak{a}$, *if* $\tilde{\Sigma}_H$ *is nonempty, then* $\tilde{\Sigma}_H$ *is a root system of* $\mathrm{Span}(\tilde{\Sigma}_H)$.

Proof We set $g = \exp(H)$. We consider the orthogonal symmetric Lie algebra

$$((\mathrm{Ad}(g)^{-1}\mathfrak{k}_2) \cap \mathfrak{k}_1) \oplus ((\mathrm{Ad}(g)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1).$$

Then, we can decompose the Lie algebra as the following:

$$\left(\mathfrak{k}_0 \oplus \sum_{\lambda \in \Sigma_H^+} \mathfrak{k}_\lambda \oplus \sum_{\alpha \in W_H^+} V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) \right) \oplus \left(\mathfrak{a} \oplus \sum_{\lambda \in \Sigma_H^+} \mathfrak{m}_\lambda \oplus \sum_{\alpha \in W_H^+} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) \right).$$

It is the root space decomposition of the orthogonal symmetric Lie algebra with respect to $\mathfrak{a}$.

Remark 3 By Proposition 2 and Theorem 1, for any symmetric triad of $\mathfrak{a}$ and $H \in \mathfrak{a}$, if $\tilde{\Sigma}_H$ is nonempty, then $\tilde{\Sigma}_H$ is a root system of $\mathrm{Span}(\tilde{\Sigma}_H)$.

For each $H \in \mathfrak{a}$, denote by $W(\tilde{\Sigma}_H)$ the Weyl group of $\tilde{\Sigma}_H$. The second sufficient condition is the following:

Theorem 4 *Let* $g = \exp(H)$ $(H \in \mathfrak{a})$. *If* $\mathrm{Span}(\tilde{\Sigma}_H) = \mathfrak{a}$ *and* $-\mathrm{id}_{\mathfrak{a}} \in W(\tilde{\Sigma}_H)$, *then* $K_2gK_1 \subset G$, $K_2\pi_1(g) \subset M_1$ *and* $K_1\pi_2(g) \subset M_2$ *are weakly reflective.*

Proof Since the slice representations of these orbits are equivalent to the linear isotropy representation of the compact symmetric pair which corresponds the orthogonal symmetric Lie algebra $((\mathrm{Ad}(g)^{-1}\mathfrak{k}_2) \cap \mathfrak{k}_1) \oplus ((\mathrm{Ad}(g)^{-1}\mathfrak{m}_2) \cap \mathfrak{m}_1)$, it is sufficient to prove the existence of a reflection with respect to $dL_g\xi$ for each $\xi \in \mathfrak{a}$. Since $-\mathrm{id}_{\mathfrak{a}} \in W(\tilde{\Sigma}_H)$, there exist $\mu_1, \ldots, \mu_l \in \tilde{\Sigma}_H$ such that $s_{\mu_1} \cdots s_{\mu_l} = -\mathrm{id}_{\mathfrak{a}}$. By Lemma 3, there exists $k_{\mu_i} \in N_{K_2}(\mathfrak{a})$ for each μ_i $(1 \leq i \leq l)$. We set

$$k'_{\mu_i} = \exp\left(-2\frac{\langle \mu_i, H\rangle}{\langle \mu_i, \mu_i\rangle}\mu_i\right)k_{\mu_i} \in K_1,$$

and

$$\sigma = (k_{\mu_1},\ k'_{\mu_1})\cdots(k_{\mu_l},\ k'_{\mu_l}) \in (K_2 \times K_1)_g.$$

Then, σ is a reflection of K_2gK_1 with respect to $dL_g\xi$ for each $\xi \in \mathfrak{a}$. Indeed,

$$\sigma(g) = g, \quad \sigma(K_2gK_1) = K_2gK_1, \quad d\sigma(dL_g(\xi)) = dL_g s_{\mu_1} \cdots s_{\mu_l}(\xi) = -dL_g\xi$$

hold. Similarly, $\sigma_1 = k_{\mu_1} \cdots k_{\mu_l}$ is a reflection of $K_2\pi_1(g)$ at $\pi_1(g)$ with respect to $dL_g\xi$. The isometry $\sigma_2 = k'_{\mu_1} \cdots k'_{\mu_l}$ is a reflection of $K_1\pi_2(g)$ at $\pi_2(g)$ with respect to $dR_g\xi$.

In [5], they mainly studied weakly reflective submanifolds in S^n and $\mathbb{C}P^n$. The cohomogeneity of Hermann actions on rank one symmetric spaces must be one. Therefore, by Proposition 1, singular orbits of Hermann actions on rank one symmetric spaces are weakly reflective. However, when the cohomogeneity of Hermann action is two or greater, applying Theorems 3 and 4, we have new examples of weakly reflective submanifolds in compact symmetric spaces.

For each symmetric triad of $\mathfrak{a}$, austere points are classified in [3]. Using the classification, we investigate $\tilde{\Sigma}_{H_i}$ $(1 \leq i \leq r)$ for each type of symmetric triads.

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Dual Orlicz Mixed Quermassintegral

Jia He, Denghui Wu and Jiazu Zhou

Abstract We study the dual Orlicz mixed Quermassintegral. For arbitrary monotone continuous function ϕ, the dual Orlicz radial sum and dual Orlicz mixed Quermassintegral are introduced. Then the dual Orlicz–Minkowski inequality and dual Orlicz–Brunn–Minkowski inequality for dual Orlicz mixed Quermassintegral are obtained. These inequalities are just the special cases of their L_p analogues (including cases $-\infty < p < 0$, $p = 0$, $0 < p < 1$, $p = 1$, and $1 < p < +\infty$). These inequalities for $\phi = \log t$ are related to open problems including log-Minkowski problem and log-Brunn-Minkowski problem. Moreover, the equivalence of the dual Orlicz–Minkowski inequality for dual Orlicz mixed Quermassintegral and dual Orlicz–Brunn–Minkowski inequality for dual Orlicz mixed Quermassintegral is shown.

Keywords Star body · Orlicz radial sum · Dual Orlicz mixed Quermassintegral · Dual Orlicz–Minkowski inequality · Dual Orlicz–Brunn–Minkowski inequality

2010 Mathematics Subject Classification Primary: 52A30 · 52A40

1 Introduction

The classical Brunn–Minkowski theory for convex bodies (compact convex sets with nonempty interior) is known as consequences of the combination of Minkowski addition and volume, which constitutes the core of convex geometry. Significant results in this theory, for instance the Minkowski's first inequality and the

J. He · D. Wu · J. Zhou (✉)
School of Mathematics and Statistics, Southwest University, Chongqing 400715, China
e-mail: zhoujz@swu.edu.cn

J. He
e-mail: 13608343695@163.com

D. Wu
e-mail: wudenghui66@163.com

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_11

Brunn–Minkowski inequality, have important applications in analysis, geometry, random matrices, and many other fields (see [28]).

In 1960s, Firey extended Minkowski addition to L_p addition in [2]. Since then, the Brunn–Minkowski theory has gained amazing developments. This extended theory is called L_p Brunn–Minkowski theory, which connects volumes with L_p addition (see e.g. [7, 9, 19–23, 31]). As a development of L_p Brunn–Minkowski theory, Orlicz–Brunn–Minkowski theory is a new blossom in recent years, which is motivated by [8, 15, 16, 24, 25]. For more references, see [3, 11, 14, 34, 35, 38]. Specifically, Xiong and Zou studied Orlicz mixed Quermassintegral in [35].

In [17, 18], Lutwak introduced duality of the Brunn–Minkowski theory, in which the research object substitutes star bodies for convex bodies, obtained dual counterparts of the several wonderful results in the Brunn–Minkowski theory. Intersection body is a useful geometrical object in dual Brunn–Minkowski theory, introduced by Lutwak in [18]. The class of intersection bodies and mixed intersection bodies are valuable in geometry, especially in answering the known Busemann–Petty problem (see [12]). We refer the reader to [5, 6, 13, 26, 32, 33] for the extended intersection bodies and their applications.

In [37], a dual Orlicz–Brunn–Minkowski theory was presented and the dual Orlicz–Brunn–Minkowski inequality for volume was established. An Orlicz radial sum and dual Orlicz mixed volumes were introduced. The dual Orlicz–Minkowski inequality and the dual Orlicz–Brunn–Minkowski inequality were established. The variational formula for the volume with respect to the Orlicz radial sum was proved. The equivalence between the dual Orlicz–Minkowski inequality and the dual Orlicz–Brunn–Minkowski inequality was demonstrated. Orlicz intersection bodies were introduced and the Orlicz–Busemann–Petty problem was posed. It should noted that analog theory was also discussed in [4]. Following ideas of [4, 37], the dual Orlicz–Brunn–Minkowski inequality for dual mixed Quermassintegral was also discussed in [35].

Motivated by works of [4, 37], we consider the dual Orlicz–Brunn–Minkowski inequality for dual mixed Quermassintegral in the n-dimensional Euclidean space $\mathbb{R}^n$. We denote by $\mathscr{C}^{in}$ the set of all increasing continuous functions $\phi: (0, \infty) \to (-\infty, \infty)$ and by $\mathscr{C}^{de}$ the set of all decreasing continuous functions $\phi: (0, \infty) \to (-\infty, \infty)$. Let $\mathscr{C}$ denote the union of $\mathscr{C}^{in}$ and $\mathscr{C}^{de}$. The n dimensional unit ball and the unit sphere are denoted by B and S^{n-1} respectively.

A set K in $\mathbb{R}^n$ is star-shaped set with respect to $z \in K$ if the intersection of every line through z with K is a line segment. The radial function, $\rho_K: S^{n-1} \to [0, \infty)$, of a compact star-shaped set (about the origin) is defined by

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \qquad u \in S^{n-1}. \tag{1}$$

If $\rho(K, \cdot)$ is positive and continuous, K is called a star body. Let $\mathscr{S}^n$ and $\mathscr{S}_0^n$ denote the set of start bodies and the set of start bodies about the origin in $\mathbb{R}^n$, respectively.

Definition 1 Let $K, L \in \mathscr{S}_0^n, a, b > 0$.
If $\phi \in \mathscr{C}^{in}$, then Orlicz radial sum $a \cdot K \tilde{+}_\phi b \cdot L$ is defined by

$$\rho_{a\cdot K \tilde{+}_\phi b\cdot L}(u) = \inf\left\{t > 0 : a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) \le \phi(1)\right\}, \forall u \in S^{n-1}. \tag{2}$$

If $\phi \in \mathscr{C}^{de}$, then Orlicz radial sum $a \cdot K \tilde{+}_\phi b \cdot L$ is defined by

$$\rho_{a\cdot K \tilde{+}_\phi b\cdot L}(u) = \sup\left\{t > 0 : a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) \le \phi(1)\right\}, \forall u \in S^{n-1}. \tag{3}$$

The dual mixed Quermassintegral $\tilde{W}_i(K, L)$, defined in [17], is

$$\tilde{W}_i(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n-i-1}\rho_L(u)dS(u). \tag{4}$$

Motivated by this, we define the following dual Orlicz mixed Quermassintegral.

Definition 2 Let $K, L \in \mathscr{S}_0^n$, $i \in \mathbb{R}$, $\phi \in \mathscr{C}$. The dual Orlicz mixed Quermassintegral $\tilde{W}_{\phi,i}(K, L)$ is defined by

$$\tilde{W}_{\phi,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)dS(u). \tag{5}$$

When $\phi(t) = t^p$, with $p \neq 0$, the dual Orlicz mixed volume reduces to L_p dual mixed Quermassintegral (see [20] for the case $p \ge 1$ and $i = 0$)

$$\tilde{W}_{p,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-i-p}(u)\rho_L^p(u)dS(u).$$

When $\phi(t) = \log t$, one has

$$\tilde{W}_{\phi,i}(K, L) = \frac{1}{n}\int_{S^{n-1}} \log\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)dS(u).$$

In Sect. 2, we introduce some basic concepts. In Sect. 3, the Orlicz radial sum and some related properties are discussed. Some important properties of dual Orlicz mixed Quermassintegral are investigated in Sect. 4.

In Sect. 5, dual Orlicz–Minkowski inequality and dual Orlicz–Brunn–Minkowski inequality are established for dual Orlicz mixed Quermassintegral. As special cases, these inequalities are just the L_p counterparts, including the cases $-\infty < p < 0$, $p = 0$, $0 < p < 1$, $p = 1$ and $1 < p < +\infty$. These inequalities for $\phi = \log t$ are related to open problems, such as, the log-Brunn–Minkowski problem and the log-Minkowski problem. Moreover, we prove the equivalence of dual Orlicz–Minkowski inequality and dual Orlicz–Brunn–Minkowski inequality.

2 Preliminaries

Let $K, L \in \mathscr{S}_0^n$. By (1), one has

$$K \subset L \quad \text{if and only if} \quad \rho_K(u) \leq \rho_L(u). \tag{6}$$

Two star bodies K and L are dilates (of each other) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. If $t > 0$, we have

$$\rho(tK, u) = t\rho(K, u), \qquad \text{for all } u \in S^{n-1}.$$

We write A^{-1} for the inverse matrix of A where $A \in GL(n)$. So associated with the definition of the radial function, for $A \in GL(n)$, the radial function of the image $AK = \{Ay : y \in K\}$ of K is shown by

$$\rho(AK, u) = \rho(K, A^{-1}u), \qquad \text{for all } u \in S^{n-1}. \tag{7}$$

The radial Hausdorff metric between the star bodies K and L is

$$\tilde{\delta}(K, L) = \max_{u \in S^{n-1}} \mid \rho_K(u) - \rho_L(u) \mid .$$

A sequence $\{K_i\}$ of star bodies is said to be convergent to K if

$$\tilde{\delta}(K_i, K) \to 0, \qquad \text{as } i \to \infty.$$

Therefore, a sequence of star bodies K_i converges to K if and only if the sequence of the radial function $\rho(K_i, \cdot)$ converges uniformly to $\rho(K, \cdot)$ [27].

Let $K, L \in \mathscr{S}^n$. We have

$$K \tilde{+}_\phi \varepsilon L \to K$$

in the radial Hausdorff metric as $\varepsilon \to 0^+$ [36].

The radial Minkowski linear combination of sets $K_1, \ldots, K_r$ in $\mathbb{R}^n$ is defined by

$$\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r\}, \qquad \text{for all } \lambda_i \in \mathbb{R}, \ \ i = 1, \ldots, r.$$

If $K, L \in \mathscr{S}_0^n$ and $a, b > 0$, $aK \tilde{+} bL$ can be defined as a star body with satisfying that

$$\rho_{aK\tilde{+}bL}(u) = a\rho_K(u) + b\rho_L(u), \qquad \text{for all } u \in S^{n-1}. \tag{8}$$

Write $V(K)$ for the volume of the compact set K in $\mathbb{R}^n$. In fact, the volume of the radial Minkowski linear combination $\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r$ is a homogeneous n-th polynomial in λ_i (see [17, 18]).

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum_{r \leq n} \tilde{V}(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficient $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ is called the dual mixed volume of $K_{i_1}, \ldots, K_{i_n}$, it is nonnegative and only depends on the sets $K_{i_1}, \ldots, K_{i_n}$. Or write $\tilde{V}_i(K, L) = \tilde{V}(\underbrace{K, \ldots, K}_{n-i}, \underbrace{L, \ldots, L}_{i})$. If $L = B$, the dual mixed volume $\tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$ which is called the dual Quermassintegral of K.

If $K_1, \ldots, K_n \in S_0^n$, the dual mixed volume $\tilde{V}(K_1, \ldots, K_n)$ is defined [17]

$$\tilde{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}(u) \cdots \rho_{K_n}(u) dS(u),$$

where S is the Lebesgue measure on S^{n-1} (i.e., the (n-1)-dimensional Hausdorff measure). Let $K \in \mathscr{S}_0^n$ and $i \in \mathbb{R}$. A slight extension (see [29]) of the notation $\tilde{W}_i(K)$ is

$$\tilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-i} dS(u). \tag{9}$$

In (4), let $i = 0$, we immediately get the following integral representation for the first dual mixed volume proved by Lutwak in [17]: if $K, L \in \mathscr{S}_0^n$, then

$$\tilde{V}_1(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-1} \rho_L(u) dS(u).$$

The integral representation (4), together with the Hölder inequality and (9), immediately lead to the following dual Minkowski inequality about the dual mixed Quermassintegral $\tilde{W}_i(K, L)$.

Lemma 1 *If $K, L \in \mathscr{S}_0^n$ and $i < n-1$, then*

$$\tilde{W}_i(K, L)^{n-i} \leq \tilde{W}_i(K)^{n-i-1} \tilde{W}_i(L), \tag{10}$$

with equality if and only if K and L are dilates of each other.

If $i > n-1$ and $i \neq n$, (10) is reversed, with equality if and only if K and L are dilates.

We shall obtain the dual Brunn–Minkowski inequality for the dual Quermassintegral $\tilde{W}_i(aK \tilde{+} bL)$.

Lemma 2 *If $K, L \in \mathscr{S}_0^n$, $i < n-1$ and $a, b > 0$, then*

$$\tilde{W}_i(aK \tilde{+} bL)^{\frac{1}{n-i}} \leq a \tilde{W}_i(K)^{\frac{1}{n-i}} + b \tilde{W}_i(L)^{\frac{1}{n-i}}, \tag{11}$$

with equality if and only if K and L are dilates of each other.

If $i > n-1$ and $i \neq n$, (11) is reversed, with equality if and only if K and L are dilates.

Upon the definition of the function ϕ, suppose that μ is a probability measure on a space X and $g : X \to I \subset \mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality (see [10]) shows that if $\phi : I \to \mathbb{R}$ is a concave function, then

$$\int_X \phi(g(x))d\mu(x) \leq \phi\bigg(\int_X g(x)d\mu(x)\bigg). \tag{12}$$

If $\phi \in \mathscr{C}_2$, the inequality is reverse, that is

$$\int_X \phi(g(x))d\mu(x) \geq \phi\bigg(\int_X g(x)d\mu(x)\bigg). \tag{13}$$

If ϕ is strictly concave or convex, each equality in (12) and (13) holds if and only if $g(x)$ is constant for μ-almost all $x \in X$.

3 Orlicz Radial Sum

From (7) and the definition of the Orlicz radial sum, we have

Proposition 1 *Let $K, L \in \mathscr{S}_0^n$, and $a, b > 0$. If $\phi \in \mathscr{C}$, then for $A \in GL(n)$,*

$$A(a \cdot K \tilde{+}_\phi\, b \cdot L) = a \cdot AK \tilde{+}_\phi\, b \cdot AL. \tag{14}$$

Proof For $\phi \in \mathscr{C}^{in}$, $u \in S^{n-1}$, by (7)

$$\begin{aligned}
\rho_{a \cdot AK \tilde{+}_\phi\, b \cdot AL}(u) &= \inf\left\{t > 0 : a\phi\left(\frac{\rho_{AK}(u)}{t}\right) + b\phi\left(\frac{\rho_{AL}(u)}{t}\right) \leq \phi(1)\right\} \\
&= \inf\left\{t > 0 : a\phi\left(\frac{\rho_K(A^{-1}u)}{t}\right) + b\phi\left(\frac{\rho_L(A^{-1}u)}{t}\right) \leq \phi(1)\right\} \\
&= \rho_{a \cdot K \tilde{+}_\phi\, b \cdot L}(A^{-1}u) \\
&= \rho_{A(a \cdot K \tilde{+}_\phi\, b \cdot L)}(u).
\end{aligned}$$

If $\phi \in \mathscr{C}^{de}$, in the same way, we also have (14).

Since $K, L \in \mathscr{S}_0^n$ and $u \in S^{n-1}$, $0 < \rho_K(u) < \infty$ and $0 < \rho_L(u) < \infty$, hence $\frac{\rho_K(u)}{t} \to 0$ and $\frac{\rho_L(u)}{t} \to 0$ as $t \to \infty$. By the assumption that ϕ is monotone increasing (or decreasing) in $(0, \infty)$, so the function

$$t \mapsto a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right),$$

is monotone decreasing (or increasing) in $(0, \infty)$. Since it is also continuous, we have

Lemma 3 *Let $K, L \in \mathscr{S}_0^n$, $a, b > 0$, and $u \in S^{n-1}$. If $\phi \in \mathscr{C}$, then*

$$a\phi\left(\frac{\rho_K(u)}{t}\right) + b\phi\left(\frac{\rho_L(u)}{t}\right) = \phi(1),$$

if and only if

$$\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u) = t.$$

Remark 1 We shall provide several special examples of the Orlicz radial sum. Let $K, L \in \mathscr{S}_0^n$, $a, b > 0$.

(1) When $\phi(t) = t^p$, with $p \neq 0$, it is easy to show that the Orlicz radial sum reduces to an analogue form of Lutwak's L_p radial combination ($p \geq 1$, see [20])

$$\rho(a \cdot K\tilde{+}_\phi b \cdot L, u)^p = a\rho(K, u)^p + b\rho(L, u)^p.$$

(2) When $\phi(t) = \log t$, we obtain

$$(a + b)\log \rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u) = a \log \rho_K(u) + b \log \rho_L(u).$$

This sum is dual of the logarithm sum which is an important notion (see [1, 30]).

(3) When $\phi(t) = \log(t + 1)$, we have

$$\left(\frac{\rho_K(u)}{\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)} + 1\right)^a \left(\frac{\rho_L(u)}{\rho_{a\cdot K\tilde{+}_\phi b\cdot L}(u)} + 1\right)^b = 2,$$

and $\phi(0) = 0$.

Lemma 4 *Let $K, L \in \mathscr{S}_0^n$, for $0 < \lambda < 1$,*

(1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, then

$$(1 - \lambda) \cdot K\tilde{+}_\phi\lambda \cdot L \subseteq (1 - \lambda)K\tilde{+}\lambda L. \tag{15}$$

When ϕ is strictly concave or convex, the equality holds if and only if $K = L$.

(2) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$ or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, then

$$(1 - \lambda) \cdot K\tilde{+}_\phi\lambda \cdot L \supseteq (1 - \lambda)K\tilde{+}\lambda L. \tag{16}$$

When ϕ is strictly concave or convex, the equality holds if and only if $K = L$.

Proof Let $K_\lambda = (1 - \lambda) \cdot K\tilde{+}_\phi\lambda \cdot L$.

(1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$, by Lemma 3 and concavity of ϕ, we have

$$\begin{aligned}\phi(1) &= (1-\lambda)\phi\left(\frac{\rho_K(u)}{\rho_{K_\lambda}(u)}\right) + \lambda\phi\left(\frac{\rho_L(u)}{\rho_{K_\lambda}(u)}\right) \\ &\le \phi\left(\frac{(1-\lambda)\rho_K(u) + \lambda\rho_L(u)}{\rho_{K_\lambda}(u)}\right).\end{aligned}$$

Since ϕ is monotone increasing on $(0, \infty)$, hence we have

$$(1-\lambda)\rho_K(u) + \lambda\rho_L(u) \ge \rho_{K_\lambda}(u),$$

that is,

$$\rho_{(1-\lambda)K\tilde{+}\lambda L}(u) \ge \rho_{K_\lambda}(u). \tag{17}$$

If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, by Lemma 3 convexity of ϕ and ϕ is monotone decreasing on $(0, \infty)$, in the same way, we can obtain (17). Then by (6), (17) deduces the helpful conclusion (15).

(2) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, by Lemma 3 and convexity of ϕ, we have

$$\begin{aligned}\phi(1) &= (1-\lambda)\phi\left(\frac{\rho_K(u)}{\rho_{K_\lambda}(u)}\right) + \lambda\phi\left(\frac{\rho_L(u)}{\rho_{K_\lambda}(u)}\right) \\ &\ge \phi\left(\frac{(1-\lambda)\rho_K(u) + \lambda\rho_L(u)}{\rho_{K_\lambda}(u)}\right).\end{aligned}$$

Since ϕ is monotone increasing on $(0, \infty)$, hence we also have

$$(1-\lambda)\rho_K(u) + \lambda\rho_L(u) \le \rho_{K_\lambda}(u),$$

that is,

$$\rho_{(1-\lambda)K\tilde{+}\lambda L}(u) \le \rho_{K_\lambda}(u). \tag{18}$$

If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, by Lemma 3, concavity of ϕ and ϕ is monotone decreasing on $(0, \infty)$, in the same way, we can obtain (18). Then by (6), (18) deduce the helpful conclusion (16).

From the equality condition in the concavity (or convexity) of ϕ, then each equality in (15) and (16) holds if and only if $K = L$.

Corollary 1 *Let $K, L \in \mathscr{S}_0^n$, $0 < \lambda < 1$ and $\tilde{W}_i(K) = \tilde{W}_i(L)$.*

(1) If $i < n-1$, $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, then

$$\tilde{W}_i((1-\lambda)\cdot K \tilde{+}_\phi \lambda \cdot L) \le \tilde{W}_i(K), \tag{19}$$

with equality if and only if $K = L$.

(2) If $i > n-1$ and $i \ne n$, $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$ or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, then

$$\tilde{W}_i((1-\lambda)\cdot K \tilde{+}_\phi \lambda \cdot L) \ge \tilde{W}_i(K), \tag{20}$$

with equality if and only if $K = L$.

Proof (1) By Lemmas 4 and 2, we have

$$\begin{aligned}\tilde{W}_i((1-\lambda)\cdot K\tilde{+}_\phi\lambda\cdot L)^{\frac{1}{n-i}} &\le \tilde{W}_i((1-\lambda)\cdot K\tilde{+}\lambda\cdot L)^{\frac{1}{n-i}}\\ &\le (1-\lambda)\tilde{W}_i(K)^{\frac{1}{n-i}}+\lambda\tilde{W}_i(L)^{\frac{1}{n-i}}\\ &= \tilde{W}_i(K)^{\frac{1}{n-i}}.\end{aligned}$$

The equality condition in (19) can be obtained from the equality condition of (11).
(2) Similarly, from Lemmas 3.4 and 2.2, we can obtain (20).

4 Dual Orlicz Mixed Quermassintegral

We denote the right derivative of a real-valued function f by f_r'. In the following Lemma 5 the function ϕ is different from ϕ in Lemma 4.1 of [37]. However, we can use the similar argument to prove Lemma 5, so we omit the details.

Lemma 5 *Let* $\phi \in \mathscr{C}$ *and* $K, L \in \mathscr{S}_0^n$. *Then*

$$\lim_{\varepsilon\to 0^+}\frac{\rho_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho_K(u)}{\varepsilon}=\frac{\rho_K(u)}{\phi_r'(1)}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),$$

uniformly for all $u \in S^{n-1}$.

Theorem 1 *Let* $\phi \in \mathscr{C}$, $K, L \in \mathscr{S}_0^n$ *and* $i \neq n$. *Then*

$$\frac{n}{n-i}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}_\phi\varepsilon\cdot L)-\tilde{W}_i(K)}{\varepsilon}=\frac{1}{\phi_r'(1)}\int_{S^{n-1}}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho_K^{n-i}(u)dS(u).$$

Proof Let $\varepsilon > 0$, $K, L \in \mathscr{S}_0^n$, $i \neq n$ and $u \in S^{n-1}$. By Lemma 5, it follows that

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{\rho^{n-i}_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho_K^{n-i}(u)}{\varepsilon} &= (n-i)\rho_K^{n-i-1}(u)\lim_{\varepsilon\to 0^+}\frac{\rho_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho_K(u)}{\varepsilon}\\ &= \frac{(n-i)\rho_K^{n-i}(u)}{\phi_r'(1)}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right),\end{aligned}$$

uniformly on S^{n-1}. Hence

$$\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}_\phi\varepsilon\cdot L)-\tilde{W}_i(K)}{\varepsilon}=\lim_{\varepsilon\to 0^+}\left(\frac{1}{n}\int_{S^{n-1}}\frac{\rho^{n-i}_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho^{n-i}_K(u)}{\varepsilon}dS(u)\right)$$
$$=\frac{1}{n}\int_{S^{n-1}}\lim_{\varepsilon\to 0^+}\frac{\rho^{n-i}_{K\tilde{+}_\phi\varepsilon\cdot L}(u)-\rho^{n-i}_K(u)}{\varepsilon}dS(u)$$
$$=\frac{n-i}{n\phi'_r(1)}\int_{S^{n-1}}\phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right)\rho^{n-i}_K(u)dS(u),$$

we complete the proof of Theorem 1.

From Definition 2 and Theorem 1, we have

$$\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}_\phi\varepsilon\cdot L)-\tilde{W}_i(K)}{\varepsilon}=\frac{n-i}{\phi'_r(1)}\tilde{W}_{\phi,i}(K,L). \tag{21}$$

An immediate consequence of Proposition 1 and (21) is contained in:

Proposition 2 *If $\phi\in\mathscr{C}$, $K,L\in\mathscr{S}^n_0$ and $i\neq n$, then for $A\in SL(n)$,*

$$\tilde{W}_{\phi,i}(AK,AL)=\tilde{W}_{\phi,i}(K,L).$$

Proof From Proposition 1 and (21), for $A\in SL(n)$, we have

$$\tilde{W}_{\phi,i}(AK,AL)=\frac{\phi'_r(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(AK\tilde{+}_\phi\varepsilon\cdot AL)-\tilde{W}_i(AK)}{\varepsilon}$$
$$=\frac{\phi'_r(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(A(K\tilde{+}_\phi\varepsilon\cdot L))-\tilde{W}_i(K)}{\varepsilon}$$
$$=\frac{\phi'_r(1)}{n-i}\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K\tilde{+}_\phi\varepsilon\cdot L)-\tilde{W}_i(K)}{\varepsilon}$$
$$=\tilde{W}_{\phi,i}(K,L).$$

5 Geometric Inequalities

For $K\in\mathscr{S}^n_0$ and $i\in\mathbb{R}$, it will be rather good to use the volume-normalized dual conical measure $\tilde{W}^*_i(K)$ defined by

$$d\tilde{W}^*_i(K)=\frac{1}{n\tilde{W}_i(K)}\rho^{n-i}_K dS, \tag{22}$$

where S is the Lebesgue measure on S^{n-1} and $\tilde{W}^*_i(K)$ is a probability measure on S^{n-1}. When i=0, this is same as the definition in [4].

We now set up the dual Orlicz–Minkowski inequality for the dual Quermassintegral as follows:

Theorem 2 *Suppose* $K, L \in \mathscr{S}_0^n$.

(1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ *and* $i < n-1$, *then*

$$\tilde{W}_{\phi,i}(K, L) \le \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right). \tag{23}$$

(2) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$ *and* $i < n-1$, *then*

$$\tilde{W}_{\phi,i}(K, L) \ge \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right). \tag{24}$$

(3) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, $i > n-1$ *and* $i \neq n$, *then*

$$\tilde{W}_{\phi,i}(K, L) \ge \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right). \tag{25}$$

(4) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, $i > n-1$ *and* $i \neq n$, *then*

$$\tilde{W}_{\phi,i}(K, L) \le \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right). \tag{26}$$

Each equality in (23)–(26) holds if and only if K *and* L *are dilates of each other.*

Proof (1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$, then by dual Orlicz mixed Quermassintegral (5), and $\tilde{W}_i^*(K)$ defined by (22) is a probability measure on S^{n-1}, Jensen's inequality (12), the integral formulas of dual mixed Quermassintegral (4), dual Minkowski inequality (10), and the fact that ϕ is increasing on $(0, \infty)$, we have

$$\begin{aligned}
\frac{\tilde{W}_{\phi,i}(K, L)}{\tilde{W}_i(K)} &= \frac{1}{n\tilde{W}_i(K)} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K^{n-i}(u) dS(u) \\
&\le \phi\left(\frac{1}{n\tilde{W}_i(K)} \int_{S^{n-1}} \frac{\rho_L(u)}{\rho_K(u)} \rho_K^{n-i}(u) dS(u)\right) \\
&= \phi\left(\frac{\tilde{W}_i(K, L)}{\tilde{W}_i(K)}\right) \\
&\le \phi\left(\frac{\tilde{W}_i(K)^{\frac{n-i-1}{n-i}} \tilde{W}_i(L)^{\frac{1}{n-i}}}{\tilde{W}_i(K)}\right)
\end{aligned}$$

$$= \phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),$$

(2) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, from (5), (22), Jensen's inequality (13), (4), (10), and ϕ is decreasing on $(0, \infty)$, we have

$$\begin{aligned}
\frac{\tilde{W}_{\phi,i}(K, L)}{\tilde{W}_i(K)} &= \frac{1}{n\tilde{W}_i(K)} \int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_K(u)}\right) \rho_K^{n-i}(u) dS(u) \\
&\geq \phi\left(\frac{1}{n\tilde{W}_i(K)} \int_{S^{n-1}} \frac{\rho_L(u)}{\rho_K(u)} \rho_K^{n-i}(u) dS(u)\right) \\
&= \phi\left(\frac{\tilde{W}_i(K, L)}{\tilde{W}_i(K)}\right) \\
&\geq \phi\left(\frac{\tilde{W}_i(K)^{\frac{n-i-1}{n-i}} \tilde{W}_i(L)^{\frac{1}{n-i}}}{\tilde{W}_i(K)}\right) \\
&= \phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),
\end{aligned}$$

(3) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, $i > n-1$ and $i \neq n$, proof as similar above, we can immediately obtain (25) which have the same form with (24).

(4) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, $i > n-1$ and $i \neq n$, similarly, we can immediately obtain (26) which have the same form with (23).

Each equality in (23)–(26) holds if and only if K and L are dilates of each other. Thus we get the significant dual Orlicz–Minkowski inequality.

Remark 2 It immediately follows a few cases for all $K, L \in \mathscr{S}_0^n$.

(1) Let $\phi(t) = t^p$ with $p < 0$. Equation (24) is just a similar result of Lutwak's L_p dual Minkowski inequality for the L_p dual mixed volume (see [20]): for $i < n-1$,

$$\tilde{W}_{p,i}(K, L)^{n-i} \geq \tilde{W}_i(K)^{n-i-p} \tilde{W}_i(L)^p.$$

(2) Let $\phi(t) = \log t$, we have

$$\tilde{W}_{p,i}(K, L) \leq \frac{\tilde{W}_i(K)}{n-i} \log \frac{\tilde{W}_i(L)}{\tilde{W}_i(K)},$$

it is a very meaningful result, see [1, 30].

(3) Let $\phi(t) = t^p$ with $0 < p < 1$. For $i < n-1$, (23) is just

$$\tilde{W}_{p,i}(K, L)^{n-i} \leq \tilde{W}_i(K)^{n-i-p} \tilde{W}_i(L)^p.$$

(4) Let $\phi(t) = t$. From (23) and (25), we have for $i < n-1$,

$$\tilde{W}_i(K, L)^{n-i} \leq \tilde{W}_i(K)^{n-i-1}\tilde{W}_i(L),$$

and for $i > n-1$, $i \neq n$, the above inequality is reversed.

(5) Let $\phi(t) = t^p$ with $p \geq 1$. It follows from (25) that for $i > n-1$, $i \neq n$,

$$\tilde{W}_i(K, L)^{n-i} \geq \tilde{W}_i(K)^{n-i-p}\tilde{W}_i(L)^p.$$

Corollary 2 *Let $K, L \in \mathscr{S}_0^n$, $i < n-1$, $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ (or $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$). If*

$$\tilde{W}_{\phi,i}(M, K) = \tilde{W}_{\phi,i}(M, L), \quad \textit{for all } M \in S_0^n, \tag{27}$$

or

$$\frac{\tilde{W}_{\phi,i}(K, M)}{\tilde{W}_i(K)} = \frac{\tilde{W}_{\phi,i}(L, M)}{\tilde{W}_i(L)}, \quad \textit{for all } M \in S_0^n, \tag{28}$$

then $K = L$.

Proof Whatever $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$, *or* $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$, the process of proof is almost identical, so we next just prove the situation that $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$.

Suppose (27) holds, if we take K for M, then from Definition 2 and (9), we have

$$\tilde{W}_{\phi,i}(K, L) = \tilde{W}_{\phi,i}(K, K) = \phi(1)\tilde{W}_i(K).$$

However, from (23), we have

$$\tilde{W}_{\phi,i}(K, K) = \tilde{W}_{\phi,i}(K, L) \leq \tilde{W}_i(K)\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),$$

then

$$\phi(1) \leq \phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K)}\right)^{\frac{1}{n-i}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is monotone increasing on $(0, \infty)$, we get

$$\tilde{W}_i(L) \geq \tilde{W}_i(K),$$

with equality if and only if K and L are dilates of each other. If we take L for M, similarly we get $\tilde{W}_i(K) \geq \tilde{W}_i(L)$ which shows there is in fact equality in both inequalities and that $\tilde{W}_i(K) = \tilde{W}_i(L)$, hence the equality implies that $K = L$.

Next, assume (28) holds, if we take K for M, then from Definition 2 and (9), we have

$$\frac{\tilde{W}_{\phi,i}(K,K)}{\tilde{W}_i(K)} = \phi(1) = \frac{\tilde{W}_{\phi,i}(L,K)}{\tilde{W}_i(L)}.$$

But from (23), we have

$$\frac{\tilde{W}_{\phi,i}(L,K)}{\tilde{W}_i(L)} \leq \frac{\tilde{W}_i(L)\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(L)}\right)^{\frac{1}{n-i}}\right)}{\tilde{W}_i(L)},$$

then

$$\phi(1) \leq \phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(L)}\right)^{\frac{1}{n-i}}\right),$$

with equality if and only if K and L are dilates of each other. Since ϕ is strictly increasing on $(0, \infty)$, we have

$$\tilde{W}_i(K) \geq \tilde{W}_i(L),$$

with equality if and only if K and L are dilates of each other.

On the other hand, taking L for M, similarly we have $\tilde{W}_i(L) \geq \tilde{W}_i(K)$, which shows that in fact equality holds in both inequalities and $\tilde{W}_i(K) = \tilde{W}_i(L)$. Hence the equality implies $K = L$.

We now establish the following dual Orlicz–Brunn–Minkowski inequality for dual Quermassintegral:

Theorem 3 *Let $K, L \in \mathscr{S}_0^n$ and $a, b > 0$.*

(1) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ and $i < n-1$ then

$$\phi(1) \leq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right). \tag{29}$$

(2) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$ and $i < n-1$, then

$$\phi(1) \geq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right). \tag{30}$$

(3) If $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, $i > n-1$ and $i \neq n$, then

$$\phi(1) \geq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a \cdot K \tilde{+}_\phi b \cdot L)}\right)^{\frac{1}{n-i}}\right). \tag{31}$$

(4) If $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, $i > n-1$ *and* $i \neq n$, *then*

$$\phi(1) \leq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)}\right)^{\frac{1}{n-i}}\right). \tag{32}$$

Each equality in (29)–(32) holds if and only if K and L are dilates of each other.

Proof Note $K_\phi = a\cdot K\tilde{+}_\phi b\cdot L$.

(1) When $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_1$ and $i < n-1$, by (9), Lemma 3, Definition 2 and (23), then

$$\begin{aligned}
\phi(1) &= \frac{1}{n\tilde{W}_i(K_\phi)}\int_{S^{n-1}} \phi(1)\rho_{K_\phi}^{n-i}(u)dS(u)\\
&= \frac{1}{n\tilde{W}_i(K_\phi)}\int_{S^{n-1}} \left[a\phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right) + b\phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right)\right]\rho_{K_\phi}^{n-i}(u)dS(u)\\
&= \frac{a}{n\tilde{W}_i(K_\phi)}\int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_{K_\phi}(u)}\right)\rho_{K_\phi}^{n-i}(u)dS(u)\\
&\quad + \frac{b}{n\tilde{W}_i(K_\phi)}\int_{S^{n-1}} \phi\left(\frac{\rho_L(u)}{\rho_{K_\phi}(u)}\right)\rho_{K_\phi}^{n-i}(u)dS(u)\\
&= \frac{a}{\tilde{W}_i(K_\phi)}\tilde{W}_{\phi,i}(K_\phi, K) + \frac{b}{\tilde{W}_i(K_\phi)}\tilde{W}_{\phi,i}(K_\phi, L)\\
&\leq a\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\phi)}\right)^{\frac{1}{n-i}}\right) + b\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\phi)}\right)^{\frac{1}{n-i}}\right).
\end{aligned}$$

(2) When $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_2$ and $i < n-1$, by (9), Lemma 3, Definition 2 and (24), we obtain (30).

(3) When $\phi \in \mathscr{C}^{in} \bigcap \mathscr{C}_2$, $i > n-1$ and $i \neq n$, by (9), Lemma 3, Definition 2 and (25), we obtain (31).

(4) When $\phi \in \mathscr{C}^{de} \bigcap \mathscr{C}_1$, $i > n-1$ and $i \neq n$, by (9), Lemma 3, Definition 2 and (26), we obtain (32).

Each equality in (29)–(32) holds as an equality if and only if K and L are dilates of each other. We obtain the desired dual Orlicz–Brunn–Minkowski inequality (29)–(32).

Remark 3 For $K, L \in \mathscr{S}_0^n$, $a, b > 0$, some particular cases are as follows: each equality holds if and only if K and L are dilates of each other.

(1) Let $\phi(t) = t^p$ with $p < 0$. From (30) we can deduces to the analogous form of Lutwak's L_p dual Brunn–Minkowski inequality (see [20]): for $i < n-1$,

$$\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)^{\frac{p}{n-i}} \geq a\tilde{W}_i(K)^{\frac{p}{n-i}} + b\tilde{W}_i(L)^{\frac{p}{n-i}}.$$

(2) Let $\phi(t) = \log t$, from (29), we obtain

$$\frac{a}{n-i}\log\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)}\right)+\frac{b}{n-i}\log\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)}\right)\geq 0.$$

(3) Let $\phi(t) = t^p$ with $0 < p < 1$. For $i < n-1$, (29) is just

$$\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)^{\frac{p}{n-i}} \leq a\tilde{W}_i(K)^{\frac{p}{n-i}} + b\tilde{W}_i(L)^{\frac{p}{n-i}}.$$

(4) Let $\phi(t) = t$. From (29) and (31), we have for $i < n-1$,

$$\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)^{\frac{1}{n-i}} \leq a\tilde{W}_i(K)^{\frac{1}{n-i}} + b\tilde{W}_i(L)^{\frac{1}{n-i}},$$

and for $i > n-1$, $i \neq n$, the above inequality reversed.

(5) Let $\phi(t) = t^p$ with $p > 1$. From (31), it follows that for $i > n-1$, $i \neq n$,

$$\tilde{W}_i(a\cdot K\tilde{+}_\phi b\cdot L)^{\frac{p}{n-i}} \geq a\tilde{W}_i(K)^{\frac{p}{n-i}} + b\tilde{W}_i(L)^{\frac{p}{n-i}}.$$

We derive the equivalence between the dual Orlicz–Minkowski inequalities (23)–(26) and the dual Orlicz–Brunn–Minkowski inequalities (29)–(32), respectively. Since we proved that (23)–(26) implies (29)–(32), respectively, so now we just need to prove that (29)–(32) can deduce (23)–(26), respectively. Since all the process are similar, so we just prove (23) by (29).

Proof of the implication (29) *to* (23). For $\varepsilon \geq 0$, let $K_\varepsilon = K\tilde{+}_\phi \varepsilon\cdot L$. By (29), the following function

$$G(\varepsilon) = \phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) + \varepsilon\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) - \phi(1),$$

is non-negative and it easily get $G(0) = 0$. Then,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon} &= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) + \varepsilon\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) - \phi(1)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) - \phi(1)}{\varepsilon} + \phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)\\ &= \lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right) - \phi(1)}{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}} - 1}\cdot\lim_{\varepsilon\to 0^+}\frac{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}} - 1}{\varepsilon}\end{aligned}$$

$$+\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right). \tag{33}$$

Let $t=\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}$ and note that $t\to 1^+$ as $\varepsilon\to 0^+$, consequently,

$$\lim_{\varepsilon\to 0^+}\frac{\phi\left(\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)-\phi(1)}{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}-1}=\lim_{t\to 1^+}\frac{\phi(t)-\phi(1)}{t-1}=\phi_r'(1). \tag{34}$$

By (21), we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{\left(\frac{\tilde{W}_i(K)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}-1}{\varepsilon}&=-\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K_\varepsilon)^{\frac{1}{n-i}}-\tilde{W}_i(K)^{\frac{1}{n-i}}}{\varepsilon}\cdot\lim_{\varepsilon\to 0^+}\tilde{W}_i(K_\varepsilon)^{-\frac{1}{n-i}}\\&=-\frac{1}{n-i}\tilde{W}_i(K)^{\frac{1-n+i}{n-i}}\cdot\lim_{\varepsilon\to 0^+}\frac{\tilde{W}_i(K_\varepsilon)-\tilde{W}_i(K)}{\varepsilon}\cdot\tilde{W}_i(K)^{-\frac{1}{n-i}}\\&=-\frac{\tilde{W}_{\phi,i}(K,L)}{\phi_r'(1)\tilde{W}_i(K)}.\end{aligned} \tag{35}$$

From (33), (34), (35) and since $G(\varepsilon)$ is non-negative, thus

$$\lim_{\varepsilon\to 0^+}\frac{G(\varepsilon)-G(0)}{\varepsilon}=-\frac{\tilde{W}_{\phi,i}(K,L)}{\tilde{W}_i(K)}+\phi\left(\left(\frac{\tilde{W}_i(L)}{\tilde{W}_i(K_\varepsilon)}\right)^{\frac{1}{n-i}}\right)\geq 0. \tag{36}$$

Therefore, we have the formula (23). The equality holds as an equality in (36) if and only if $G(\varepsilon)=G(0)=0$, and this means that the equality case in (23) can be obtained from the equality condition of (29). □

Acknowledgements This research is supported in part by National Natural Science Foundation of China (Grant No. 11271302) and Fundamental Research Funds for the Central Universities (No. XDJK2016D026). We like to thank referees for helpful suggestions.

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Characterizations of a Clifford Hypersurface in a Unit Sphere

Keomkyo Seo

Abstract The Clifford hypersurface is one of the simplest compact hypersurfaces in a unit sphere. We give two different characterizations of Clifford hypersurfaces among constant m-th order mean curvature hypersurfaces with two distinct principal curvatures. One is obtained by assuming embeddedness and by comparing two distinct principal curvatures. The proof uses the maximum principle to the two-point function, which was used in the proof of Lawson conjecture by Brendle (Acta Math. 211(2):177–190, 2013, [6]). The other is given by obtaining a sharp curvature integral inequality for hypersurfaces in a unit sphere with constant m-th order mean curvature and with two distinct principal curvatures, which generalizes Simons integral inequality (Simons, Ann. Math. (2) 88:62–105, 1968, [30]). This article is based on joint works (Min and Seo, Math. Res. Lett. 24(2):503–534, 2017, [18], Min and Seo, Monatsh. Math. 181(2):437–450, 2016, [19]) with Sung-Hong Min.

1 Introduction and Results

Recently minimal surface theory in a 3-dimensional unit sphere $\mathbb{S}^3$ has been extensively studied by many geometers. Among compact minimal surfaces in $\mathbb{S}^3$, the simplest one is the equator, which is totally geodesic. In 1966, Almgren [2] obtained the uniqueness theorem, which states that any immersed 2-sphere in $\mathbb{S}^3$ is totally geodesic. Thereafter Lawson [16] constructed compact embedded minimal surfaces in $\mathbb{S}^3$ with any genus. Moreover he conjectured that the only compact embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus. Brendle [6] proved ingeniously this famous conjecture by using the maximum principle for the two-point function.

Theorem 1 ([6]) *The only embedded minimal torus in $\mathbb{S}^3$ is the Clifford torus.*

K. Seo (✉)
Department of Mathematics, Sookmyung Women's University, Cheongpa-ro 47-gil 100, Yongsan-ku, Seoul 04310, South Korea
e-mail: kseo@sookmyung.ac.kr
URL: http://sites.google.com/site/keomkyo/

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_12

In 1989, Pinkall and Sterling [29] proposed the conjecture that any embedded constant mean curvature(CMC) torus is rotationally symmetric, which is a CMC-version of Lawson conjecture. Applying Brendle's argument in [6], Andrews and Li [3] gave an affirmative answer to Pinkall–Sterling's conjecture.

Theorem 2 ([3]) *Every embedded CMC torus in $\mathbb{S}^3$ is rotationally symmetric.*

It would be interesting to obtain an analogue in higher-dimensional cases. However, the situation is more complicated in higher-dimensional cases. In the following we give brief historical review in this direction.

Let M be a compact minimal hypersurface in $\mathbb{S}^{n+1}$. Simons [30] obtained the following identity:

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + |A|^2(n - |A|^2),$$

where Δ, ∇, and A denote the Laplacian, the Levi-Civita connection, and the second fundamental form on M, respectively. Integrating this identity over M, Simons was able to prove the following integral inequality:

$$\int_M |A|^2 \left(|A|^2 - n\right) \geq 0. \tag{1}$$

It follows from the above integral inequality that there are three possibilities: Such M is either totally geodesic, or $|A|^2 \equiv n$, or $|A|^2(x) > n$ at some point $x \in M$. Regarding the second case, Chern, do Carmo and Kobayashi [10] in 1968 and Lawson [15] in 1969 independently proved

Theorem 3 ([10, 15]) *For $n \geq 3$, if $|A|^2 \equiv n$ on M, then M is isometric to a Clifford minimal hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-1}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{1}{n}}\right)$.*

For higher-dimensional cases, Otsuki deeply investigated minimal hypersurfaces with two distinct principal curvatures as follows:

Theorem 4 ([23–25]) *Let M be a minimal hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ.*

- *The distribution of the space of principal vectors corresponding to each principal curvature is completely integrable.*
- *If the multiplicity of a principal curvature is greater than* 1*, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*
- *If one of λ and μ is simple, then there are infinitely many immersed minimal hypersurfaces other than Clifford minimal hypersurfaces.*
- *If M is embedded, then M is locally congruent to a Clifford minimal hypersurface.*

Hence one sees that the only compact embedded minimal hypersurfaces with two distinct principal curvatures in $\mathbb{S}^{n+1}$ is a Clifford minimal hypersurface. However this uniqueness result does not hold in general. For example, Hsiang [14] constructed infinitely many mutually noncongruent embedded minimal hypersurfaces in $\mathbb{S}^{n+1}$ which are homeomorphic to the Clifford hypersurface using equivariant differential geometry. Furthermore it is well-known that a lot of isoparametric hypersurfaces exist in $\mathbb{S}^{n+1}$, which are all embedded. See [1, 7–9, 12, 13, 21, 22, 26] for examples and [20] for more references.

Otsuki's result was extended to higher-order mean curvature cases for hypersurfaces with two distinct principal curvatures. Wu [34] proved that if the multiplicities of two distinct principal curvatures are at least 2, then a compact hypersurface with constant m-th order mean curvature is congruent to a Clifford hypersurface. Thus we shall consider hypersurfaces with constant m-th order mean curvature satisfying that one of the two distinct principal curvatures is simple. We remark that if M is a hypersurface in a space form with two distinct principal curvatures such that one of two distinct principal curvatures is simple, then M is a part of rotationally symmetric hypersurface, which was proved by do Carmo and Dajzer [11]. Recall that the m-th order mean curvature H_m of an n-dimensional hypersurface $M \subset \mathbb{S}^{n+1}$ is defined by the elementary symmetric polynomial of degree m in the principal curvatures $\lambda_1, \lambda_2, \cdots, \lambda_n$ on M as follows:

$$\binom{n}{m} H_m = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \ldots \lambda_{i_m}.$$

We also recall that if an n-dimensional Clifford hypersurface in $\mathbb{S}^{n+1}$ has two distinct principal curvatures λ and μ of multiplicities $n-k$ and k respectively, then it is given by

$$\mathbb{S}^{n-k}\left(\frac{1}{\sqrt{1+\lambda^2}}\right) \times \mathbb{S}^{k}\left(\frac{1}{\sqrt{1+\mu^2}}\right)$$

with $\lambda\mu + 1 = 0$.

Assume that M is a compact hypersurface in a unit sphere with constant m-th order mean curvature H_m and with two distinct principal curvatures with multiplicities $n-1, 1$. Without loss of generality, we may assume that $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$. Choose the orthonormal frame tangent to M such that $h_{ij} = \lambda_i \delta_{ij}$, that is,

$$\begin{aligned} Ae_i &= \lambda e_i \quad \text{for } i = 1, \ldots, n-1, \\ Ae_n &= \mu e_n. \end{aligned}$$

Then

$$\binom{n}{m} H_m = \binom{n-1}{m} \lambda^m + \binom{n-1}{m-1} \lambda^{m-1} \mu,$$

which gives

$$H_m = \frac{m}{n}\lambda^{m-1}\left(\frac{n-m}{m}\lambda + \mu\right). \tag{2}$$

For some Weingarten hypersurfaces with two distinct principal curvatures, Andrews, Huang, and Li obtained the following:

Theorem 5 ([4]) *Let Σ be a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures λ and μ, whose multiplicities are m and $n-m$ respectively for $1 \le m \le n-1$. If $\lambda + \alpha\mu = 0$ for some positive constant α, Σ is congruent to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{1}{\alpha+1}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{\alpha}{1+\alpha}}\right)$.*

Using the identity (2) and Theorem 5, we see that any compact embedded hypersurfaces with vanishing m-th order mean curvature and with two distinct principal curvatures is congruent to a Clifford hypersurface. On the other hand, Perdomo [28] constructed compact embedded CMC hypersurfaces in $\mathbb{S}^{n+1}$, which have two distinct principal curvatures, one of them being simple.

Theorem 6 ([28]) *For any integer $m \ge 2$ and H between $\cot\frac{\pi}{m}$ and $\frac{(m^2-2)\sqrt{n-1}}{n\sqrt{m^2-1}}$, there exists a compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature H other than the totally geodesic n-spheres and Clifford hypersurfaces.*

We note that the two distinct principal curvatures λ and μ satisfy $\lambda > \mu$ in Theorem 6, where μ is simple. In case where $\lambda < \mu$, it is natural to ask whether one can obtain the uniqueness of Clifford hypersurface or not. In [18], Sung-Hong Min and the author gave the affirmative answer to this question as follows:

Theorem 7 ([18]) *Let Σ be an $n(\ge 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature $H \ge 0$ and with two distinct principal curvatures λ and μ, μ being simple. If $\mu > \lambda$, then Σ is congruent to a Clifford hypersurface.*

In Sect. 3, we give another characterization of Clifford hypersurfaces using Simons-type integral inequality for a compact hypersurface in a unit sphere with constant higher-order mean curvature and with two distinct principal curvatures.

2 Proof of Theorem 7

Here we give the brief sketch of the proof of Theorem 7. If $H = 0$, then it is already known that Σ is congruent to a Clifford minimal hypersurfaces by the work due to Otsuki. We now assume that $H > 0$. Since Σ is a compact embedded hypersurface, Σ divides $\mathbb{S}^{n+1}$ into two connected components. We may assume that $H > 0$ by the suitable choice of the orientation of Σ. Let R be the region satisfying that ν points out of R. The *mean curvature vector* $\mathbf{H}$ satisfies that $\mathbf{H} = -nH\nu(x)$. For a positive function Ψ on Σ, we denote by $B_T(x, \frac{1}{\Psi(x)})$ a geodesic ball with radius $\frac{1}{\Psi(x)}$ which touches Σ at $F(x)$ inside the region R in $\mathbb{S}^{n+1}$. Then $B_T(x, \frac{1}{\Psi(x)})$ is given by the intersection of $\mathbb{S}^{n+1}$ and a ball of radius $\frac{1}{\Psi(x)}$ centered at $p(x) = F(x) - \frac{1}{\Psi(x)}\nu(x)$ in $\mathbb{R}^{n+2}$. Define the two-point function $Z : \Sigma \times \Sigma \to \mathbb{R}$ by

$$Z(x, y) := \Psi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle. \tag{3}$$

We introduce the notion of the interior ball curvature at $x \in \Sigma$, which was originally given by Andrews-Langford-McCoy [5] (see also [3]).

Definition 1 The *interior ball curvature* k is a positive function on Σ defined by

$$k(x) := \inf\left\{\frac{1}{r} : B_T(x, r) \cap \Sigma = \{x\},\ r > 0\right\}.$$

Since Σ is compact and embedded in $\mathbb{S}^{n+1}$, the function k is a well-defined positive function on Σ. From the definition of $k(x)$ for every point $x \in \Sigma$, we have

$$k(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geq 0$$

for all $y \in \Sigma$.

Let $\Phi(x) := \max\{\lambda(x), \mu(x)\}$ be the maximum value of the principal curvatures of Σ in $\mathbb{S}^{n+1}$ at $F(x)$. It is easy to see that $\Phi(x) - H > 0$. Motivated by the works of Brendle [6] and Andrews-Li [3], we introduce the constant κ as follows:

$$\kappa := \sup_{x\in\Sigma} \frac{k(x) - H}{\Phi(x) - H}.$$

For convenience, we will write $\varphi(x) := \Phi(x) - H$. It follows that there exists a constant $K > 0$ satisfying

$$1 \leq \kappa < K.$$

By definition, we see that $\Phi(x) \leq k(x)$ for every $x \in \Sigma$ in general. Indeed, we have the equality case under our setting.

Proposition 1 *Let Σ be an $n(\geq 3)$-dimensional compact embedded hypersurface in $\mathbb{S}^{n+1}$ with constant mean curvature H with two distinct principal curvatures, one of them being simple. If $H > 0$. Then*

$$k(x) = \Phi(x)$$

for all $x \in \Sigma$.

Proof See [18] for the proof. □

From this observation, it follows that $k(x) = \Phi(x)$ and hence

$$\Phi(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle \geq 0,$$

for all $x, y \in \Sigma$. Fix $x \in \Sigma$ and choose an orthonormal frame $\{e_1, \ldots, e_n\}$ in a neighborhood of x such that $h(e_n, e_n) = \Phi$. Let $\gamma(t)$ be a geodesic on Σ such that $\gamma(0) = F(x)$ and $\gamma'(0) = e_n$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) := Z(F(x), \gamma(t)) = \Phi(x)(1 - \langle F(x), \gamma(t)\rangle) + \langle \nu(x), \gamma(t)\rangle.$$

Then one sees that $f(t) \geq 0$ and $f(0) = 0$. Moreover

$$f'(t) = -\langle \Phi(x)F(x) - \nu(x), \gamma'(t)\rangle,$$

$$f''(t) = \langle \Phi(x)F(x) - \nu(x), \gamma(t) + h(\gamma'(t), \gamma'(t))\nu(\gamma(t))\rangle,$$

$$\begin{aligned} f'''(t) = \langle \Phi(x)F(x) - \nu(x), \gamma'(t) + (\nabla^{\Sigma}_{\gamma'(t)} h)(\gamma'(t), \gamma'(t))\nu(\gamma(t))\rangle \\ + \langle \Phi(x)F(x) - \nu(x), h(\gamma'(t), \gamma'(t))\nabla_{\gamma'(t)}\nu(\gamma(t))\rangle, \end{aligned}$$

where ∇ is the covariant derivative of $\mathbb{R}^{n+2}$. In particular, at $t = 0$,

$$\begin{aligned} f(0) &= f'(0) = 0, \\ f''(0) &= \langle \Phi(x)F(x) - \nu(x), F(x) + \Phi(x)\nu(x)\rangle = 0. \end{aligned}$$

Because f is nonnegative, we get $f'''(0) = 0$. Hence

$$0 = f'''(0) = \langle \Phi(x)F(x) - \nu(x), e_n + h_{nnn}(x)\nu(x)\rangle = -h_{nnn}(x).$$

Therefore we see that $e_n\lambda = h_{11n} = -\frac{1}{n-1}h_{nnn} = 0$, which implies that λ and μ are constant on Σ by Ostuki. It follows that Σ is an isoparametric hypersurface in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. From the classification of isoparametric hypersurfaces with two principal curvatures due to Cartan [7], Σ is congruent to the Clifford hypersurface.

3 Sharp Curvature Integral Inequality

In this section, we give another uniqueness result of Clifford hypersurfaces in terms of curvature integral inequality. Perdomo [27] and Wang [31] independently obtained a curvature integral inequality for minimal hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures, which characterizes a Clifford minimal hypersurface. Later, Wei [32] showed that the similar curvature integral inequality holds for hypersurfaces with the vanishing m-th order mean curvature (i.e., $H_m \equiv 0$).

Theorem 8 ([27, 31, 32]) *Let M be an $n(\geq 3)$-dimensional compact hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple. Then*

$$\int_M |A|^2 \leq \frac{n(m^2-2m+n)}{m(n-m)}\mathrm{Vol}(M),$$

where equality holds if and only if M is isometric to a Clifford hypersurface $\mathbb{S}^{n-1}\left(\sqrt{\frac{n-m}{n}}\right) \times \mathbb{S}^1\left(\sqrt{\frac{m}{n}}\right)$.

In [19], Sung-Hong Min and the author obtained a sharp curvature integral inequality for compact hypersurfaces in $\mathbb{S}^{n+1}$ with $H_m \equiv constant$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple.

Theorem 9 ([19]) *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with constant m-th order mean curvature H_m and with two distinct principal curvatures λ and μ, μ being simple (i.e., multiplicity 1). For the unit principal direction vector e_n corresponding to μ, we have*

$$\int_M \mathrm{Ric}(e_n, e_n) \geq 0,$$

where Ric *denotes the Ricci curvature. Moreover, equality holds if and only if M is isometric to a Clifford hypersurface.*

We remark that if $H_m \equiv 0$ for $1 \leq m < n$, then

$$\mathrm{Ric}(e_n, e_n) = (n-1)\left(1 - \frac{m(n-m)}{n(m^2-2m+n)}|A|^2\right).$$

Theorem 9 can be regarded as an extension of [27, 31, 32]. From this theorem, one sees that if $\mathrm{Ric}(e_n, e_n) \leq 0$ on such a hypersurface M, then M is congruent to a Clifford hypersurface.

Proof of Theorem 9 Here we give a brief idea of the proof of Theorem 9 (See [19] for more details). Note that for $\lambda = \lambda_1 = \cdots = \lambda_{n-1}$ and $\mu = \lambda_n$, the function $w = |\lambda^m - H_m|^{-\frac{1}{n}}$ is well-defined. From this notion, the Laplacian of $f = f(w)$ on M is given by

$$\Delta f = -\frac{1}{n-1} f'(w) w \,\mathrm{Ric}(e_n, e_n) + \left[f''(w) + (n-1)\frac{f'(w)}{w}\right](e_n w)^2. \tag{4}$$

If we let a function $f(w) = \log w$ in (4), then

$$\Delta f = -\frac{\mathrm{Ric}(e_n, e_n)}{n-1} + \frac{n-2}{w^2}(e_n w)^2.$$

Integrating Δf over M gives

$$\int_M \mathrm{Ric}(e_n, e_n) = (n-1)(n-2)\int_M \frac{(e_n w)^2}{w^2} \geq 0.$$

Equality holds if and only if $e_n\lambda \equiv 0$. Thus both λ and μ are constant, which shows that M is congruent to a Clifford hypersurface by Cartan [7]. □

In the following, we generalize Simons' integral inequality into closed hypersurfaces with two distinct principal curvatures.

Theorem 10 ([19]) *Let M be an $n(\geq 3)$-dimensional closed hypersurface in $\mathbb{S}^{n+1}$ with $H_m \equiv 0$ $(1 \leq m < n)$ and with two distinct principal curvatures, one of them being simple. Then we have*

$$\begin{cases} \int_M |A|^p \left(|A|^2 - \dfrac{n(m^2-2m+n)}{m(n-m)}\right) \leq 0 \;\; if \;\; p < \frac{n-2}{n}m, \\ \int_M |A|^p \left(|A|^2 - \dfrac{n(m^2-2m+n)}{m(n-m)}\right) \geq 0 \;\; if \;\; p > \frac{n-2}{n}m. \end{cases}$$

Equalities in the above hold if and only if M is congruent to a Clifford hypersurface.

Proof See [19] for the proof. □

We remark that if $m = 1$ and $p = 2$, then Theorem 10 is exactly the same as Simons' integral inequality (1), which was mentioned in the introduction. We also remark that when $m = 2$ and $p = 2$, Li [17] obtained some pointwise estimates on $|A|^2$, which gives the above theorem. For $p = 2$ and $3 < m < n$, Wei [33] proved the above theorem for compact and rotational hypersurfaces in a unit sphere with $H_m \equiv 0$.

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3-Dimensional Real Hypersurfaces with η-Harmonic Curvature

Mayuko Kon

Abstract We classify real hypersurfaces with η-harmonic curvature of a non-flat complex space form of complex dimension 2 under the condition that the Ricci tensor S satisfies $S\xi = \beta\xi$ where β is a function and ξ is the structure vector field.

1 Introduction

For any Riemannian manifold, the divergence δR of the curvature tensor R and the Ricci tensor S satisfy $\delta R = dS$. If $\delta R = 0$, that is,

$$(\nabla_X S)Y - (\nabla_Y S)X = 0$$

for any vector fields X and Y then the manifold is said to have a harmonic curvature. If the Ricci tensor S is parallel, then the manifold has harmonic curvature. In 1980, Derdziński [1] constructed examples of Riemannian manifolds with harmonic curvature and nonparallel Ricci tensor.

In [2], Ki proved that there are no real hypersurfaces with parallel Ricci tensor of a complex space form $M^n(c)$, $c \neq 0$. Moreover, there are no real hypersurfaces with harmonic curvature in a complex space form $M^n(c)$, $c \neq 0$, $n \geq 3$ (see Kim [5], Kwon–Nakagawa [6]). So, Ki, Kim and Nakagawa [3] defined the notion of η-harmonic curvature for real hypersurfaces. If the Ricci tensor satisfies

$$g((\nabla_X S)Y - (\nabla_Y S)X, Z) = 0$$

for any X, Y and Z orthogonal to ξ, then M is said to have η-harmonic curvature. They classified Hopf hypersurfaces of a non-flat complex space form $M^n(c)$, $n \geq 3$, with η-harmonic curvature under the additional condition on the shape operator.

M. Kon (✉)
Faculty of Education, Shinshu University, 6-Ro, Nishinagano,
Nagano City 380-8544, Japan
e-mail: mayuko_k@shinshu-u.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_13

The purpose of this paper is to study real hypersurfaces of $M^2(c)$, $c \neq 0$, with η-harmonic curvature and $S\xi = \beta\xi$, where β is a function. As an application, we prove the non-existence of real hypersurface with harmonic curvature of $M^2(c)$ under the condition that $S\xi = \beta\xi$.

2 Preliminaries

Let $M^n(c)$ denote the complex space form of complex dimension n (real dimension $2n$) with constant holomorphic sectional curvature $4c$. We denote by J the almost complex structure of $M^n(c)$. The Hermitian metric of $M^n(c)$ will be denoted by G.

Let M be a real $(2n-1)$-dimensional hypersurface immersed in $M^n(c)$. We denote by g the Riemannian metric induced on M from G. We take the unit normal vector field N of M in $M^n(c)$. For any vector field X tangent to M, we define ϕ, η and ξ by

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where ϕX is the tangential part of JX, ϕ is a tensor field of type (1,1), η is a 1-form, and ξ is the unit vector field on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0$$

for any vector field X tangent to M. Moreover, we have

$$g(\phi X, Y) + g(X, \phi Y) = 0, \quad \eta(X) = g(X, \xi),$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus (ϕ, ξ, η, g) defines an almost contact metric structure on M.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by ∇ the one in M determined by the induced metric. Then the *Gauss and Weingarten formulas* are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M. We call A the *shape operator* of M. If the shape operator A of M satisfies $A\xi = \alpha\xi$ for some functions α, then M is said to be *Hopf*. We use the following (cf. [10]).

Lemma 1 *Let M be a Hopf hypersurface of $M^n(c)$, $n \geq 2$, $c \neq 0$. If a vector field X is orthogonal to ξ and $AX = \lambda X$, then*

$$(2\lambda - \alpha)A\phi X = (\lambda\alpha + 2c)\phi X,$$

where $\alpha = g(A\xi, \xi)$, and α is constant.

For the almost contact metric structure on M, we have

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$

We denote by R the Riemannian curvature tensor field of M. Then the *equation of Gauss* is given by

$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X \\ &- g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY,Z)AX - g(AX,Z)AY, \end{aligned}$$

and the *equation of Codazzi* by

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$

From the equation of Gauss, the Ricci tensor S of M satisfies

$$\begin{aligned} g(SX,Y) = (2n+1)cg(X,Y) - 3c\eta(X)\eta(Y) \\ + \mathrm{Tr}Ag(AX,Y) - g(AX,AY), \end{aligned} \tag{1}$$

where $\mathrm{Tr}A$ is the trace of A. By (2.1), we have

$$\begin{aligned} (\nabla_X S)Y = &-3cg(\phi AX, Y)\xi - 3c\eta(Y)\phi AX \\ &+ (X\,\mathrm{Tr}\,A)AY + \mathrm{Tr}A(\nabla_X A)Y - A(\nabla_X A)Y \\ &- (\nabla_X A)AY, \end{aligned} \tag{2}$$

from this equation and the equation of Codazzi, we obtain

$$\begin{aligned} &(\nabla_X S)Y - (\nabla_Y S)X \\ &= -3cg(\phi AX + A\phi X, Y)\xi - 3c\eta(Y)\phi AX + 3c\eta(X)\phi AY \\ &\quad + X(\mathrm{Tr}\,A)AY - Y(\mathrm{Tr}\,A)AX \\ &\quad + c\,\mathrm{Tr}\,A(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi) \\ &\quad - c(\eta(X)A\phi Y - \eta(Y)A\phi X - 2g(\phi X, Y)A\xi) \\ &\quad - (\nabla_X A)AY + (\nabla_Y A)AX. \end{aligned} \tag{3}$$

If the Ricci tensor S satisfies

$$g((\nabla_X S)Y, Z) = 0$$

for any vector fields X, Y and Z orthogonal to ξ, then S is said to be η-*parallel* (Suh [11]). When M is Hopf hypersurface in $M_n(c)$, $c \neq 0$, $n \geq 2$, the classification theorem is given by Suh [11] and Maeda [9]).

Theorem A ([9, 11]) *Let M be a connected Hopf hypersurface in $CP^n(c)$, $n \geq 2$. Suppose that M has η-parallel Ricci tensor. Then M is either locally congruent to one of homogeneous real hypersurfaces of type (A_1), (A_2) and (B) in $CP^n(c)$, $n \geq 2$, or a non-homogeneous real hypersurface with $A\xi = 0$ in $CP^2(c)$. This non-homogeneous real hypersurface M is locally congruent a tube of radius $\pi/(4\sqrt{c})$ over a non-totally geodesic complex curve which does not have a principal curvature $\pm\sqrt{c}$ in $CP^2(c)$.*

Theorem B ([9, 11]) *Let M be a connected Hopf hypersurface in $CH^n(c)$, $n \geq 2$. Suppose that M has η-parallel Ricci tensor. Then M is either locally congruent to one of homogeneous real hypersurfaces of types (A_0), $(A_{1,0})$, $(A_{1,1})$, (A_2) and (B) in $CH^n(c)$, $n \geq 2$, or a non-homogeneous real hypersurface with $A\xi = 0$ in $CH^2(c)$.*

3 η-Harmonic Curvature

In this section, we prove the following theorem.

Theorem 1 *Let M be a real hypersurface of $M^2(c)$, $c \neq 0$, with η-harmonic curvature. If the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β, then M is a Hopf hypersurface with η-parallel Ricci tensor.*

Let M be a real hypersurface of a complex 2-dimensional space form $M^2(c)$. If M is not Hopf, then there is a point x, and hence an open neighborhood U of x, where $A\xi \neq \alpha\xi, \alpha = g(A\xi, \xi)$. So there is a function h which is non-zero on U and a unit vector field e_1 orthogonal to ξ such that

$$A\xi = \alpha\xi + he_1.$$

We take an orthonormal frame $\{\xi, e_1, e_2\}$, where we have put $e_2 = \phi e_1$. Then we have

$$g(Ae_2, \xi) = g(A\xi, e_2) = g(\alpha\xi + he_1, \phi e_1) = 0.$$

Thus there are smooth functions a_1, a_2 and k defined near x such that A is represented by a matrix

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & a_1 & k \\ 0 & k & a_2 \end{pmatrix}$$

with respect to $\{\xi, e_1, e_2\}$. We use the following lemmas (see [7]).

Lemma 2 *Let M be a real hypersurface of $M^2(c)$, $c \neq 0$. Suppose that the Ricci tensor S of M satisfies $S\xi = \beta\xi$ for some function β, then M is a Hopf hypersurface or the shape operator A is represented by a matrix*

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & a_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to some orthonormal basis $\{\xi, e_1, e_2\}$*, locally.*

Lemma 3 *Let* M *be a real hypersurface of* $M^2(c)$*,* $c \neq 0$*. If there exists an orthonormal frame* $\{\xi, e_1, e_2\}$ *on a sufficiently small neighborhood* U *of* $x \in M$ *such that the shape operator* A *is represented as*

$$A = \begin{pmatrix} \alpha & h & 0 \\ h & a_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then we have

$$a_1 g(\nabla_{e_2} e_1, e_2) = 0, \tag{4}$$
$$-2c - \alpha a_1 + h g(\nabla_{e_1} e_2, e_1) + (e_2 h) = 0, \tag{5}$$
$$-h a_1 + a_1 g(\nabla_{e_1} e_2, e_1) + (e_2 a_1) = 0, \tag{6}$$
$$h g(\nabla_{e_2} e_1, e_2) = 0, \tag{7}$$
$$-c - h^2 + a_1 g(\nabla_\xi e_2, e_1) + (e_2 h) = 0, \tag{8}$$
$$-h\alpha + h g(\nabla_\xi e_2, e_1) + (e_2 \alpha) = 0. \tag{9}$$

To prove Theorem 1, first we show the following lemma.

Lemma 4 *Let* M *be a real hypersurface of* $M^2(c)$*,* $c \neq 0$*, with* η*-harmonic curvature. If the Ricci tensor* S *of* M *satisfies* $S\xi = \beta\xi$ *for some function* β*, then* M *is a Hopf hypersurface.*

Proof We suppose that M is not a Hopf. Putting $X = e_1$ and $Y = e_2$ into (3), by Lemma 2, we obtain

$$\begin{aligned} &(\nabla_{e_1} S)e_2 - (\nabla_{e_2} S)e_1 \\ &= -3ca_1\xi - e_2(a_1 + \alpha)(a_1 e_1 + h\xi) - 2(a_1 + \alpha)c\xi \\ &\quad + 2c(he_1 + \alpha\xi) + (\nabla_{e_2} A)(a_1 e_1 + h\xi) \\ &= (2ch - (e_2\alpha)a_1 + h(e_2 h))e_1 \\ &\quad + (-5a_1 c - (e_2 a_1)h + a_1(e_2 h))\xi \\ &\quad + a_1^2 \nabla_2 e_1 - a_1 A\nabla_{e_2} e_1 + h^2 \nabla_{e_2} e_1. \end{aligned}$$

Taking an inner product with e_1,

$$2ch - a_1(e_2\alpha) + h(e_2 h) = 0. \tag{10}$$

By (8) and (9), we have

$$-hc - h^3 + ha_1\alpha + h(e_2h) - a_1(e_2\alpha) = 0.$$

Using (10), we obtain

$$h^2 - a_1\alpha = -3c. \tag{11}$$

From this equation, we have

$$2h(e_2h) - (e_2a_1)\alpha - a_1(e_2\alpha) = 0.$$

Using (6), (8), (9) and (11), we have

$$\begin{aligned} 0 &= 2ch + 2h^3 - 2ha_1\alpha - ha_1g(\nabla_\xi e_2, e_1) + a_1\alpha g(\nabla_1 e_2, e_1) \\ &= -4hc - ha_1g(\nabla_\xi e_1, e_1) + a_1\alpha g(\nabla_{e_1} e_1, e_2). \end{aligned} \tag{12}$$

On the other hand, (5), (8) and (11) imply

$$4hc + a_1hg(\nabla_\xi e_2, e_1) - h^2 g(\nabla_{e_1} e_2, e_1) = 0.$$

From these equations, we have

$$g(\nabla_{e_1} e_2, e_1) = 0. \tag{13}$$

Substituting this into (12), we obtain

$$a_1 g(\nabla_\xi e_1, e_2) = 4c.$$

Since $c \neq 0$, we see that $a_1 \neq 0$. By (13) and the following

$$g(\nabla_{e_1} e_1, e_1) = 0, \quad g(\nabla_{e_1} e_1, \xi) = -g(e_1, \phi A e_1) = 0,$$

we have $\nabla_{e_1} e_1 = 0$. By the similar computation, we have

$$\begin{aligned} &\nabla_{e_2} e_2 = 0,\ \nabla_{e_2} e_1 = 0,\ \nabla_{e_1} e_2 = -a_1\xi, \\ &\nabla_\xi e_1 = \frac{4c}{a_1} e_2,\ \nabla_\xi e_2 = -\frac{4c}{a_1} e_1 - h\xi. \end{aligned}$$

Using these equations, we compute the sectional curvature for the plane spanned by e_1 and e_2.

$$\begin{aligned}
&g(R(e_1, e_2)e_2, e_1)\\
&= g(\nabla_{e_1}\nabla_{e_2}e_2 - \nabla_{e_2}\nabla_{e_1}e_2 - \nabla_{[e_1,e_2]}e_2, e_1)\\
&= g(\nabla_{e_2}(a_1\xi) + \nabla_{a_1\xi}e_2, e_1)\\
&= a_1 g(\nabla_\xi e_2, e_1) = -4c.
\end{aligned}$$

On the other hand, by the equation of Gauss, we obtain $g(R(e_1, e_2)e_2, e_1) = 4c$. This contradicts to the assumption that $c \neq 0$. So we have our lemma.

When M is a Hopf hypersurface, the shape operator A is represented as

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$$

for an suitable orthonormal frame $\{\xi, e_1, e_2\}$ on a sufficiently small neighborhood. We use the following lemma ([8]).

Lemma 5 *Let M be a Hopf hypersurface of $M^2(c)$, $c \neq 0$. Then we have*

$$(a_1 - a_2)g(\nabla_{e_2}e_1, e_2) - (e_1a_2) = 0, \tag{14}$$

$$2c - 2a_1a_2 + \alpha(a_1 + a_2) = 0, \tag{15}$$

$$(a_1 - a_2)g(\nabla_{e_1}e_2, e_1) + (e_2a_1) = 0, \tag{16}$$

$$-(c + a_2\alpha - a_1a_2) + (a_1 - a_2)g(\nabla_\xi e_1, e_2) = 0, \tag{17}$$

$$\xi a_1 = \xi a_2 = 0. \tag{18}$$

We remark that if M is Hopf, then the Ricci tensor S satisfies $S\xi = \beta\xi$, $\beta = 2c + \alpha(a_1 + a_2)$.

Lemma 6 *If a Hopf hypersurface M of $M^2(c)$ has η-harmonic curvature, then the Ricci tensor S is η-parallel.*

Proof By (3), we have

$$\begin{aligned}
&(\nabla_{e_1}S)e_2 - (\nabla_{e_2}S)e_1\\
&= -3cg(\phi Ae_1 + A\phi e_1, e_2)\xi + e_1(a_1 + a_2 + \alpha)a_2e_2\\
&\quad - e_2(a_1 + a_2 + \alpha)a_1e_1 - 2c(a_1 + a_2 + \alpha)g(\phi e_1, e_2)\xi\\
&\quad + 2cg(\phi e_1, e_2)\alpha\xi + (\nabla_{e_1}A)a_2e_2 - (\nabla_{e_2}A)a_1e_1\\
&= -5c(a_1 + a_2)\xi + (e_1a_1)a_2e_2 - (e_2a_2)a_1e_1\\
&\quad - a_2^2\nabla_{e_1}e_2 + a_2A\nabla_{e_1}e_2 + a_1^2\nabla_{e_2}e_1 - a_1A\nabla_{e_2}e_2.
\end{aligned}$$

Taking an inner product with e_1, by (16),

$$
\begin{aligned}
0 &= g((\nabla_{e_1} S)e_2 - (\nabla_{e_2} S)e_1, e_1) \\
&= -(e_2 a_2)a_1 + a_2(a_1 - a_2)g(\nabla_{e_1} e_2, e_1) \\
&= -e_2(a_1 a_2).
\end{aligned}
$$

Similarly, taking an inner product with e_2, we obtain $e_1(a_1 a_2) = 0$. Moreover, from (18), we see that $a_1 a_2$ is constant. Since M is Hopf hypersurface, from Lemma 1, we obtain

$$
2a_1 a_2 - (a_1 + a_2)\alpha - 2c = 0.
$$

If $\alpha \neq 0$, then $a_1 + a_2$ is constant, so a_1 and a_2 are also constant. On the other hand, if $\alpha = 0$, then $a_1 a_2 = c$. From these, together with (2), we obtain

$$
\begin{aligned}
& g((\nabla_{e_1} S)e_1, e_1) \\
&= (e_1 \operatorname{Tr} A)a_1 + (\operatorname{Tr} A - 2a_1)g((\nabla_{e_1} A)e_1, e_1) \\
&= (e_1 \alpha)a_1 + (e_1 a_2)a_1 + \alpha(e_1 a_1) + a_2(e_1 a_1) \\
&= e_1(a_1 \alpha) + e_1(a_1 a_2) \\
&= 0,
\end{aligned}
$$

$$
g((\nabla_{e_1} S)e_1, e_2) = \alpha(e_2 a_1) = 0.
$$

Similarly, the straightforward computation shows that $g((\nabla_X S)Y, Z) = 0$ for $X,\ Y,\ Z \in \{e_1, e_2\}$. Thus the Ricci tensor S is η-parallel.

4 Harmonic Curvature

In this section, we study real hypersurfaces with harmonic curvature. Then they have η-harmonic curvature. Therefore, by Lemma 3, a real hypersurface with harmonic curvature of $M^2(c)$, $c \neq 0$, is a Hopf hypersurface.

Theorem 2 *There are no real hypersurface with harmonic curvature of $M^2(c)$, $c \neq 0$, on which the Ricci tensor S satisfies $S\xi = \beta\xi$, where β is a function.*

Proof From the proof of Lemma 5, if $\alpha \neq 0$, then a_1 and a_2 are constant. By (3), we have

$$
\begin{aligned}
& (\nabla_{e_1} S)\xi - (\nabla_\xi S)e_1 \\
&= (-4ca_1 - \alpha c - a_1\alpha^2 + a_1 a_2 \alpha)e_1 \\
&\quad + a_1^2 \nabla_\xi e_1 - a_1 A \nabla_\xi e_1 \qquad (19) \\
&= 0,
\end{aligned}
$$

$$\begin{aligned}&(\nabla_{e_2}S)\xi-(\nabla_\xi S)e_2\\&=(4ca_2+\alpha c+a_2\alpha^2-a_1a_2\alpha)e_1\\&\quad+a_2^2\nabla_\xi e_2-a_2A\nabla_\xi e_2\\&=0.\end{aligned}\tag{20}$$

Thus we have

$$\begin{aligned}&g((\nabla_{e_1}S)e_2-(\nabla_{e_2}S)e_1,\xi)\\&=-5c(a_1+a_2)+a_1a_2^2+a_1^2a_2-2a_1a_2\alpha\\&=0.\end{aligned}$$

Next, taking an inner product of (19) with e_2, we obtain

$$-4ca_1-\alpha c-a_1\alpha^2+a_1a_2\alpha+a_1(a_1-a_2)g(\nabla_\xi e_1,e_2)=0.$$

Combining this equation with (17), we have

$$-5ca_1-\alpha c-a_1\alpha^2+a_1^2a_2=0.\tag{21}$$

On the other hand, by (20), we have

$$\begin{aligned}&g((\nabla_{e_2}S)\xi-(\nabla_\xi S)e_2,e_1)\\&=4ca_2+\alpha c+a_2\alpha^2-a_1a_2\alpha\\&\quad-a_2(a_1-a_2)g(\nabla_\xi e_2,e_1).\end{aligned}$$

Using (17), we have

$$5ca_2+\alpha c+a_2\alpha^2-a_1a_2^2=0.\tag{22}$$

From (21) and (22), we obtain

$$(a_1-a_2)(-5c-\alpha^2+a_1a_2)=0.$$

First we consider the case that $-5c-\alpha^2+a_1a_2=0$. By (21), we have $\alpha c=0$, and hence $\alpha=0$. Then we have $-5c+a_1a_2=0$. On the other hand, by Lemma 1, we have

$$(2a_1-\alpha)A\phi e_1=(a_1\alpha+2c)\phi e_1.$$

So we obtain $a_1a_2-c=0$. This contradicts to the assumption that $c\neq 0$.

Next, we consider the case that $a_1=a_2$. We put $a_1=a_2=a$. Since M is Hopf, again using Lemma 1,

$$a^2-a\alpha-c=0.\tag{23}$$

From this equation, we see that a is non-zero constant. On the other hand, by (3), we have

$$
\begin{aligned}
&(\nabla_{e_1} S)e_2 - (\nabla_{e_2} S)e_1 \\
&= -10ca\xi - a^2\nabla_{e_1}e_2 + aA\nabla_{e_1}e_2 + a^2\nabla_{e_2}e_1 - aA\nabla_{e_2}e_1.
\end{aligned}
$$

So we have

$$
g((\nabla_{e_1} S)e_2 - (\nabla_{e_2} S)e_1, \xi) = -10ca + 2a^3 - 2a^2\alpha.
$$

By (23), we see that $g((\nabla_{e_1} S)e_2 - (\nabla_{e_2} S)e_1, \xi) = -8ca \neq 0$. Hence we have our result.

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Gromov–Witten Invariants on the Products of Almost Contact Metric Manifolds

Yong Seung Cho

Abstract We investigate Gromov–Witten invariants and quantum cohomologies on the products of almost contact metric manifolds. The product of two cosymplectic manifolds has a Kähler structure. We compute some cohomology classes of compact cosymplectic manifolds and show that any compact simply connected Kähler manifold cannot be a product of two cosymplectic manifolds. On the products we get some geometric properties, Gromov–Witten invariants and quantum cohomologies. We have some relations between Gromov–Witten invariants of the products and the ones of two cosymplectic manifolds.

1 Introduction

Let M be a real $(2n+1)$-dimensional smooth manifold and $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M. An almost co-complex structure on M is defined by a smooth $(1, 1)$-tensor field φ, a smooth vector field ξ, and a smooth 1-form η on M such that $\varphi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$, where I denotes the identity transformation of the tangent bundle TM. Manifolds with an almost co-complex structure (φ, ξ, η) are called almost contact manifolds. An almost contact manifold (M, φ, ξ, η) with a Riemannian metric tensor g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(M)$ is called an almost contact metric manifold, and denote it by $(M, g, \varphi, \xi, \eta)$. An almost contact metric manifold has its structure group of the form $U(n) \oplus (1)$, and the fundamental 2-form ϕ defined by

$$\phi(X, Y) = g(X, \varphi Y)$$

Y.S. Cho (✉)
Department of Mathematics, Ewha Womans University, 52, Ewhayeodae-gil,
Seodaemun-gu, Seoul 03760, Republic of Korea
e-mail: yescho@ewha.ac.kr

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_14

for all $X, Y \in \mathfrak{X}(M)$. An almost cosymplectic structure (η, ϕ) on M is called cosymplectic if $d\eta = 0 = d\phi$, in this case M is called an almost co-Kähler manifold. When $\phi = d\eta$ the associated almost cosymplectic structure is called a contact structure on M and M an almost Sasakian manifold. The Nijenhuis tensor N_φ of φ is the $(1, 2)$-tensor field defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - [X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y]$$

for all $X, Y \in \mathfrak{X}(M)$, where $[X, Y]$ is the Lie bracket of X and Y. An almost cocomplex structure (φ, ξ, η) is called integrable if $N_\varphi = 0$, and normal if $N_\varphi + 2d\eta \otimes \xi = 0$. An integrable almost cocomplex structure is called a cocomplex structure. An integrable almost co-Kähler manifold is called a co-Kähler manifold, while a Sasakian manifold is a normal almost Sasakian manifold. In this paper we follow definitions and notations on almost contact metric manifolds in the references [1–4].

Let (M, g, ω, J) be a symplectic manifold with an almost complex structure J which is compatible with a symplectic structure ω. To study symplectic manifolds many geometers [5–7] have studied the theory of pseudo-holomorphic maps from a Riemann surface to M. Let $A \in H_2(M; \mathbb{Z})$ be an integral homology class, and $\mathfrak{M}_{g,k}(M, A, J)$ be the moduli space of stable J-holomorphic maps which represent A from a Riemann surface with genus g and k marked points to M. The moduli spaces define the Gromov–Witten invariants via evaluation maps. Using the Gromov–Witten invariants we can define quantum product on cohomologies and have the quantum cohomology ring $QH^*(M; \Lambda)$ [6, 7] with coefficients in some Novikov ring Λ. In [8, 9] we have studied Gromov–Witten invariants and quantum cohomologies on symplectic manifolds, in [10] geodesic surface congruences. We have extended the notion of pseudo-holomorphic map in symplectic manifolds to the one of pseudo-co-holomorphic map in almost contact metric manifolds. We have had some results on Gromov–Witten type invariants and quantum type cohomologies on contact manifolds [2], and on the products of cosymplectic manifolds and circle [11].

In this paper we consider the products of almost contact metric manifolds. On the products we investigate some geometric structures, Gromov–Witten invariants, and quantum cohomologies. In Sect. 2, we induce the fundamental 2-form and almost complex structure on the product of two almost contact metric manifolds. In particular, the product of two cosymplectic manifolds is Kähler. In Sect. 3, we have some topological properties of the product of two cosymplectic manifolds. We show that the cosymplectic structure (η, ϕ) of a compact cosymplectic manifold contributes to each Betti numbers. As a consequence we have that any compact simply connected Kähler manifold can not be a product of two cosymplectic manifolds. In Sect. 4, we study Gromow-Witten invariants on the product of two cosymplectic manifolds. We show that the Gromov–Witten invariant of the product is equal to the product of Gromov–Witten type invariants of two cosymplectic manifolds.

2 The Product of Two Almost Contact Metric Manifolds

Let $(M_i^{2n_i+1}, g_i, \varphi_i, \xi_i, \eta_i, \phi_i)$, $i = 1, 2$, be almost contact metric manifolds. Then the product $M := M_1 \times M_2$ is a smooth manifold of dimension $2n$, where $n = n_1 + n_2 + 1$. Let g be a Riemannian metric on M defined by

$$g((X_1, X_2), (Y_1, Y_2)) = g_1(X_1, Y_1) + g_2(X_2, Y_2)$$

for every $(X_1, X_2), (Y_1, Y_2) \in \mathfrak{X}(M)$, and J a $(1, 1)$-tensor field on M defined by

$$J(X_1, X_2) = (\varphi_1 X_1 - \eta_2(X_2)\xi_1, \varphi_2 X_2 + \eta_1(X_1)\xi_2)$$

for every $(X_1, X_2) \in \mathfrak{X}(M)$.

The tangent bundles are decomposed as

$$TM_1 = \mathscr{D}_1 \oplus \langle \xi_1 \rangle, \quad TM_2 = \mathscr{D}_2 \oplus \langle \xi_2 \rangle,$$

where $\mathscr{D}_1 = \{X \in TM_1 \mid \eta_1(X) = 0\}$, $\mathscr{D}_2 = \{X \in TM_2 \mid \eta_2(X) = 0\}$, and $\langle \xi_i \rangle$, $i = 1, 2$ are trivial real line bundles on M_i.

Lemma 1 *Let M be the product of almost contact metric manifolds M_1 and M_2. Then we have*

(1) $TM \simeq \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle$ *is isomorphic to a sum of complex vector bundles.*
(2) $J^2 = -I$.
(3) $J = \varphi_1$ *on* $\mathscr{D}_1$, $J = \varphi_2$ *on* $\mathscr{D}_2$, *and* $J := \varphi_3$ *on* $\langle \xi_1, \xi_2 \rangle$, *where* $\varphi_3(\xi_1) = \xi_2$ *and* $\varphi_3(\xi_2) = -\xi_1$.
(4) $g = J^* g$.

Proof By the definitions of the almost contact metric manifold, the metric g, and the $(1, 1)$-tensor J, we can easily prove Lemma 1. □

By Lemma 1 the product of two almost contact metric manifolds is an almost complex manifold. The fundamental 2-form on the product M is given by

$$\phi((X_1, X_2), (Y_1, Y_2)) = g((X_1, X_2), J(Y_1, Y_2))$$

for every $(X_1, X_2), (Y_1, Y_2) \in \mathfrak{X}(M)$.

Lemma 2 *The fundamental 2-form ϕ on the product M is*

$$\phi = \phi_1 + \phi_2 - \eta_1 \wedge \eta_2.$$

Proof For every $(X_1, X_2), (Y_1, Y_2) \in \mathfrak{X}(M)$,

$$\phi((X_1, X_2), (Y_1, Y_2)) = g((X_1, X_2), J(Y_1, Y_2))$$

$$
\begin{aligned}
&= g((X_1, X_2), (\varphi_1 Y_1 - \eta_2(Y_2)\xi_1, \varphi_2 Y_2 + \eta_1(Y_1)\xi_2)) \\
&= g_1(X_1, \varphi_1 Y_1 - \eta_2(X_2)\xi_1) + g_2(X_2, \varphi_2 Y_2 + \eta_1(Y_1)\xi_2) \\
&= g_1(X_1, \varphi_1 Y_1) - \eta_2(X_2) g_1(X_1, \xi_1) + g_2(X_2, \varphi_2 Y_2) + \eta_1(Y_1) g_2(X_2, \xi_2) \\
&= \phi_1(X_1, Y_1) + \phi_2(X_2, Y_2) - \eta_2(X_2)\eta_1(X_1) + \eta_1(Y_1)\eta_2(X_2) \\
&= (\phi_1 + \phi_2 - \eta_1 \wedge \eta_2)((X_1, X_2), (Y_1, Y_2)).
\end{aligned}
$$

Thus we have $\phi = \phi_1 + \phi_2 - \eta_1 \wedge \eta_2$. □

Recall that an almost contact metric manifold $(M, g, \varphi, \xi, \eta, \phi)$ is almost cosymplectic or almost co-Kähler (cosymplectic or co-Kähler) if and only if $d\eta = 0 = d\phi$ $(d\eta = 0 = d\phi = N_\varphi)$, respectively.

Theorem 1 *Let* $(M_i^{2n_i+1}, g_i, \varphi_i, \xi_i, \eta_i, \phi_i)$, $i = 1, 2$, *be almost contact metric manifolds and* (M^{2n}, g, J, ϕ) *their product, where* $n = n_1 + n_2 + 1$. *Then we have*

(1) if M_i, $i = 1, 2$, *are almost cosymplectic, then* M *is symplectic.*
(2) if M_i, $i = 1, 2$, *are cosymplectic, then* M *is Kähler.*

Proof By Lemma 1, the product (M, g, J, ϕ) is an almost complex manifold. By Lemma 2 the fundamental 2-form on the product is $\phi = \phi_1 + \phi_2 + \eta_2 \wedge \eta_1$.

The exterior derivative of ϕ is

$$d\phi = d\phi_1 + d\phi_2 + d\eta_2 \wedge \eta_1 - \eta_2 \wedge d\eta_1.$$

(1) Let $M_i, i = 1, 2$, be almost cosymplectic. Then $d\phi_i = 0 = d\eta_i, i = 1, 2$. and so $d\phi = 0$. Thus ϕ is closed. The n times exterior product of ϕ is

$$\phi^n = (\phi_1 + \phi_2 + \eta_2 \wedge \eta_1)^n = \phi_1^{n_1} \wedge \phi_2^{n_2} \wedge \eta_2 \wedge \eta_1$$

which does not vanish on M.
Thus the fundamental 2-form ϕ is a nondegenerate closed 2-form on M.

(2) Let $M_i, i = 1, 2$, be cosymplectic. Then by (1) M is symplectic and J is almost complex structure J is compatible with ϕ.
Since the Nigenhuis tensor on M_i is $N_{\varphi_i} = 0, i = 1, 2$, the Nijenhuis tensor N_J on M is zero. Thus (M, g, J, ϕ) is Kähler.

□

Remark 1 The odd dimensional spheres S^{2n_1+1} and S^{2n_2+1}, $n_i > 0$, are contact. The product $S^{2n_1+1} \times S^{2n_2+1}$ is a complex manifold but not symplectic [12].

3 The Product of Two Cosymplectic Manifolds

Let $(M^{2n+1}, g, \varphi, \xi, \eta, \phi)$ be a cosymplectic manifold, and $\bigtriangledown$ the Levi-Civita connection which is compatible with the metric g. Define Two operators L and $\wedge$ on M by the exterior product $L = \varepsilon(\phi)$ and the interior product $\wedge = \iota(\phi)$.

Recall the cohomologies of cosymplectic manifolds.

Lemma 3 ([1]) *For a cosymplectic manifold $(M, g, \varphi, \xi, \eta, \phi)$*

(1) $\bigtriangledown_X \phi = 0$ for every $X \in \mathfrak{X}(M)$.
(2) L commutes with the Laplace-Beltrami operator $\triangle$.
(3) L maps the space of harmonic p-forms into the space of harmonic $(p+2)$-forms.

Theorem 2 ([1]) *Let $(M^{2n+1}, g, \varphi, \xi, \eta, \phi)$ be a compact cosymplectic manifold. Then the each Betti number $B_i(M)$ of M is nonzero, i.e.,*

$$B_i(M) \neq 0 \quad i = 0, 1, \ldots, 2n+1.$$

Recall the topology of compact cosymplectic manifolds. Since the fundamental 2-form ϕ satisfies $\bigtriangledown_X \phi = 0$ for every $X \in \mathfrak{X}(M)$ we have $d\phi = 0$ and $d^*\phi = 0$. Thus $\triangle\phi = (d^*d + dd^*)\phi = 0$, and ϕ is harmonic.

Suppose ϕ^p is harmonic, then we have

$$\triangle(\phi^{p+1}) = \triangle(L\phi^p) = L(\triangle\phi^p) = L(0) = 0.$$

Thus ϕ^{p+1} is harmonic for every p.

Since $\phi^n \neq 0$ and $\phi^p \neq 0$ for every $1 \leq p \leq n$, the Betti numbers $B_{2p}(M) \neq 0$, $0 \leq p \leq n$. By Poincaré duality the odd dimensional Betti numbers

$$B_{2p+1}(M) \neq 0, \quad 0 \leq p \leq n.$$

Let $\{\xi, e_i, \varphi e_i \mid i = 1, \ldots, n\}$ be a local φ-basis and $\{\eta, \omega_i, \omega_i^* \mid i = 1, \ldots, n\}$ its dual basis in M. Then we have

$$\begin{aligned}
\phi &= \sum_{i=1}^{n} \omega_i \wedge \omega_i^*, \\
\phi^n &= n!\omega_1 \wedge \omega_1^* \wedge \cdots \wedge \omega_n \wedge \omega_n^*, \\
* \phi^n &= n! * (\omega_1 \wedge \omega_1^* \wedge \cdots \wedge \omega_n \wedge \omega_n^*) = n!\eta,
\end{aligned}$$

and $\phi^n \wedge \eta$ is a nowhere vanishing $(2n+1)$-form.

Since the Hodge star $*$ operator commutes to $\triangle$, i.e., $*\triangle = \triangle*$,

$$n!\triangle\eta = \triangle n!\eta = \triangle * \phi^n = *\triangle\phi^n = *0 = 0.$$

Thus the η is a nonzero harmonic 1-form in M.

For every $1 \leq p \leq n$, since $\triangle(\phi^p \wedge \eta) = (\triangle\phi^p) \wedge \eta + \phi^p \wedge (\triangle\eta) = 0$, the $\phi^p \wedge \eta$ are nonzero harmonic $(2p+1)$-forms.

Theorem 3 *Let $(M^{2n+1}, g, \varphi, \xi, \eta, \phi)$ be a compact cosymplectic manifold. Then we have*

(1) the cohomology classes, $1, \eta, \phi, \phi \wedge \eta, \phi^2, \ldots, \phi^n, \phi^n \wedge \eta$ contribute the Betti numbers $B_i(M)$, $i = 0, \ldots, 2n+1$, respectively.
(2) every Morse function $f : M \to \mathbb{R}$ has critical points more than $n+2$ points such that there are critical points $x_k \in M$ of f satisfying $ind_f(x_k) = k$ for $k = 0, 1, \ldots, 2n+1$.

Let $(M^{2n_i}, g_i, \varphi_i, \xi_i, \eta_i, \phi_i)$ be compact cosymplectic manifolds, $i = 1, 2$ and $(M = M_1 \times M_2, g, J, \phi)$ the product of M_1 and M_2. By Theorem 1 M is a Kähler manifold. By the Künneth Theorem the cohomology of M is

$$H^*(M, \mathbb{Q}) = H^*(M_1, \mathbb{Q}) \otimes H^*(M_2, \mathbb{Q}).$$

The k-dimensional cohomology of M is

$$H^k(M, \mathbb{Q}) = \sum_{k_1+k_2=k} H^{k_1}(M_1, \mathbb{Q}) \otimes H^{k_2}(M_2, \mathbb{Q}).$$

and the kth Betti number of M,

$$B_k(M) = \sum_{k_1+k_2=k} B_{k_1}(M_1) B_{k_2}(M_2).$$

By Theorem 3 the first Betti number of M is $B_1(M) = B_1(M_1) + B_1(M_2) \geq 2$.

Theorem 4 *Let M be a product of two compact cosymplectic manifolds. Then the $B_1(M)$ is even and greater than or equal to 2.*

Theorem 5 *A compact simply connected Kähler manifold cannot be the product of two cosymplectic manifolds.*

4 Gromov–Witten Invariants on the Products

Let $(M^{2n_i+1}, g_i, \varphi_i, \xi_i, \eta_i, \phi_i)$, $i = 1, 2$, compact cosymplectic manifolds and $\mathscr{D}_i = \{X \in TM_i \mid \eta_i(X) = 0\}$, $i = 1, 2$, the distribution bundles associated with η_i on M_i, respectively. As in Sect. 2 we denote (M, g, J, ϕ) the product of M_1 and M_2. We decompose the tangent bundle TM into three complex subbundles as follows:

$$\begin{array}{ccc} TM = \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle & \xrightarrow{\quad J \quad} & TM = \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle \\ & \searrow \quad \swarrow & \\ & M, & \end{array}$$

for every $(X_1, X_2, r_1\xi_1, r_2\xi_2) \in \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle$.

In the decomposition $(\mathscr{D}_1, \varphi_1)$, $(\mathscr{D}_2, \varphi_2)$, $(\langle \xi_1, \xi_2 \rangle, \varphi_3)$ are Hermitian vector bundles of rank n_1, n_2, and 1, respectively. By the Künneth formula the 2-dimensional homology of M is

$$H_2(M) = H_2(M_1) \oplus H_2(M_2) \oplus (H_1(M_1) \otimes H_1(M_2)).$$

The first Chern class of M is

$$\begin{aligned} c_1(TM) &= c_1(\mathscr{D}_1) + c_1(\mathscr{D}_2) + c_1(\langle \xi_1, \xi_2 \rangle) \\ &= c_1(\mathscr{D}_1) + c_1(\mathscr{D}_2), \end{aligned}$$

where $\langle \xi_1, \xi_2 \rangle$ is a trivial complex line bundle.

Assume that an integral curve of ξ_i in M_i is a circle S^1_i, $i = 1, 2$. Then the torus $T := S^1_1 \times S^1_2 \subset M_1 \times M_2$ is an integral surface of $\{\xi_1, \xi_2\}$ whose tangent bundle is $T = \langle \xi_1, \xi_2 \rangle$. For example, $M_i = N_i \times S^1_i$ are the products of Kähler manifolds N_i and circles S^1_i, $i = 1, 2$ [11].

Let $A \in H_2(M)$ be decomposed as $A = A_1 + A_2 + A_3$, where $A_1 \in H_2(M_1)$, $A_2 \in H_2(M_2)$, $A_3 \in H_1(M_1) \otimes H_1(M_2)$ and let $\pi_i : M \to M_1, M_2, T$, $i = 1, 2, 3$ be the projections, respectively. Recall that a smooth map $u : (\Sigma, j) \to (M, J)$ from a Riemann surface (Σ, j) to (M, J) is J-holomorphic if $du \circ j = J \circ du$. For each $i = 1, 2, 3$ the map $u_i := \pi_i \circ u$ is φ_i-holomorphic if $du_i \circ j = \varphi_i \circ du_i$.

Lemma 4 *A smooth map $u : (\Sigma, j) \to (M, J)$ is J-holomorphic if and only if $u_i : (\Sigma, j) \to (M_i, J_i)$ is φ_i-holomorphic $i = 1, 2, 3$, where $(M_3, J_3) = (T, \varphi_3)$ and $J = \varphi_1 \oplus \varphi_2 \oplus \varphi_3$ on $TM = \mathscr{D}_1 \oplus \mathscr{D}_2 \oplus \langle \xi_1, \xi_2 \rangle$.*

Let $\mathfrak{M}_{0,3}(M; A, J) := \{u : (\Sigma, j) \to (M, J) \mid u \text{ is } J\text{-holomorphic}, u_*([\Sigma]) = A\}$ be the moduli space of stable J-holomorphic maps from a sphere with 3 marked points to M which represent the 2-dimensional homology class A.

Note that there is no nontrivial rational map to a torus [5, 9].

Lemma 5 *The moduli space of T is*

$$\mathfrak{M}_{0,3}(T; A, \varphi_3) = \begin{cases} T & \text{if } A = 0 \\ \phi & \text{otherwise.} \end{cases}$$

Theorem 6 *(1) The moduli space $\mathfrak{M}_{0,3}(M; A, J)$ is a compact stratified space of dimension $2[c_1(\mathscr{D}_1)A_1 + c_1(\mathscr{D}_2)A_2 + n]$.*

(2) *If* $A_3 = 0$*, then there is a canonical homeomorphism*

$$\mathfrak{M}_{0,3}(M; A, J) \to \mathfrak{M}_{0,3}(M; A_1, \varphi_1) \times \mathfrak{M}_{0,3}(M_2; A_2, \varphi_2) \times T.$$

Proof (1) By the stability of J-holomorphic maps the moduli space $\mathfrak{M}_{0,3}(M; A, J)$ is compact. The dimension of $\mathfrak{M}_{0,3}(M; A, J)$ is

$$\begin{aligned}\dim \mathfrak{M}_{0,3}(M; A, J) &= 2c_1(TM)A + 2n \\ &= 2(c_1(\mathfrak{D}_1) + c_1(\mathfrak{D}_2) + c_1(\langle \xi_1, \xi_2 \rangle))(A_1 + A_2 + A_3) + 2(n_1 + n_2 + 1) \\ &= (2c_1(\mathfrak{D}_1)A_1 + 2n_1) + (2c_1(\mathfrak{D}_2)A_2 + 2n_2) + 2 \\ &= \dim \mathfrak{M}_{0,3}(M_2; A_2, \varphi_2) + \dim \mathfrak{M}_{0,3}(M_1; A_1, \varphi_1) + \dim T.\end{aligned}$$

(2) By Lemmas 4 and 5, (2) is clear. □

There are the canonical evaluation maps given by as follows:

$$\begin{aligned} &ev : \mathfrak{M}_{0,3}(M; A, J) \to M^3, && ev([u; z_1, z_2, z_3]) = (u(z_1), u(z_2), u(z_3)), \\ &ev_1 : \mathfrak{M}_{0,3}(M_1; A_1, \varphi_1) \to M_1^3, && ev([u_1; z_1, z_2, z_3]) = (u_1(z_1), u_1(z_2), u_1(z_3)), \\ &ev_2 : \mathfrak{M}_{0,3}(M_2; A_2, \varphi_2) \to M_2^3, && ev([u_2; z_1, z_2, z_3]) = (u_2(z_1), u_2(z_2), u_2(z_3)), \\ &ev_3 : \mathfrak{M}_{0,3}(T; A_3, \varphi_3) \to T^3, && ev_3([u_3; z_1, z_2, z_3]) = (u_3(z_1), u_3(z_2), u_3(z_3)). \end{aligned}$$

The Gromov–Witten invariants are defined by

$$\begin{aligned} &\Phi_{0,3}^{M,A,J} : H^*(M^3) \to \mathbb{Q}, && \Phi_{0,3}^{M,A,J}(\alpha) = \int_{\mathfrak{M}_{0,3}(M;A,J)} ev^*(\alpha), \\ &\Phi_{0,3}^{M_1,A_1,\varphi_1} : H^*(M_1^3) \to \mathbb{Q}, && \Phi_{0,3}^{M_1,A_1,\varphi_1}(\alpha_1) = \int_{\mathfrak{M}_{0,3}(M_1;A_1,\varphi_1)} ev_1^*(\alpha_1), \\ &\Phi_{0,3}^{M_2,A_2,\varphi_2} : H^*(M_2^3) \to \mathbb{Q}, && \Phi_{0,3}^{M_2,A_2,\varphi_2}(\alpha_2) = \int_{\mathfrak{M}_{0,3}(M_2;A_2,\varphi_2)} ev_2^*(\alpha_2), \\ &\Phi_{0,3}^{T,A_3,\varphi_3} : H^*(T^3) \to \mathbb{Q}, && \Phi_{0,3}^{T,A_3,\varphi_3}(\alpha_3) = \int_T ev_3^*(\alpha_3). \end{aligned}$$

By Lemma 5 we have

Lemma 6 *If* $A_3 = 0$*, then the Gromov–Witten invariants of* T *are*

$$\Phi_{0,3}^{T,A_3,\varphi_3} : H^*(T^3) \to \mathbb{Q}, \quad \Phi_{0,3}^{T,A_3,\varphi_3}(\alpha_{31} \otimes \alpha_{32} \otimes \alpha_{33}) = \int_T (\alpha_{31} \cup \alpha_{32} \cup \alpha_{33}),$$

where $\alpha_{3i} \in H^*(T)$*,* $i = 1, 2, 3$.

Theorem 7 *Under the above assumptions we have*

$$\Phi_{0,3}^{M,A,J}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = \Phi_{0,3}^{M_1,A_1,\varphi_1}(\alpha_1) \cdot \Phi_{0,3}^{M_2,A_2,\varphi_2}(\alpha_2) \cdot \int_T ev_3^*(\alpha_3),$$

where $\alpha_1 \in H^*(M_1^3)$, $\alpha_2 \in H^*(M_2^3)$, $\alpha_3 \in H^*(T^3)$, *and* $A_3 = 0$.

Proof Let $\alpha_1 \in H^{d_1}(M_1^3)$, $\alpha_2 \in H^{d_2}(M_2^3)$, and $\alpha_3 \in H^2(T^3)$, where $d_i = \dim \mathfrak{M}_{0,3}(M_i; A_i, \varphi_i) = 2c_i(\mathscr{D}_i) + 2n_i$, $i = 1, 2$. Then we have

$$\begin{aligned}
&\Phi_{0,3}^{M;A,J}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = \int_{\mathfrak{M}_{0,3}(M,A,J)} ev^*(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \\
&= \int_{\mathfrak{M}_{0,3}(M_1,A_1,\varphi_1)} ev_1^*(\alpha_1) \cdot \int_{\mathfrak{M}_{0,3}(M_2,A_2,\varphi_2)} ev_2^*(\alpha_2) \cdot \int_{\mathfrak{M}_{0,3}(T,0,\varphi_3)} ev_3^*(\alpha_3) \\
&= \Phi_{0,3}^{M_1,A_1,\varphi_1}(\alpha_1) \cdot \Phi_{0,3}^{M_2,A_2,\varphi_2}(\alpha_2) \cdot \int_T ev_3^*(\alpha_3).
\end{aligned}$$

□

Acknowledgements I would like to express my sincere application to professor T. Sakai for his hospitalities during staying at Tokyo Metropolitan University on April 2016.

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On LVMB, but Not LVM, Manifolds

Jin Hong Kim

Abstract The aim of this paper is to survey the constructions of the so-called LVM or LVMB manifolds after López de Medrano, Verjovsky, Meersseman, and Bosio, and to discuss some recent results as well as interesting related open questions.

1 Introduction

One of the most well-known examples of a compact, complex, non-Kählerian manifold is Hopf manifold, diffeomorphic to the product $S^{2n-1} \times S^1$ of spheres, which can be obtained by taking the quotient of $\mathbb{C}^n \backslash \{0\}$ by a holomorphic totally discontinuous action of $\mathbb{Z}$ ([8]). Another example is the Calabi-Eckmann manifold which is given by the existence of complex structures on $S^{2k-1} \times S^{2l-1}$ [4]. To achieve it, Calabi and Eckmann consider the smooth fibration

$$S^{2k-1} \times S^{2l-1} \to \mathbb{CP}^{k-1} \times \mathbb{CP}^{l-1},$$

equipped with the torus fiber of the bundle with a structure of an elliptic curve. In the paper [11], López de Medrano and Verjovsky constructed a family of compact, complex, non-symplectic manifolds which can be obtained by taking the quotient of a open dense subset U of $\mathbb{CP}^n$ by the holomorphic action of $\mathbb{C}$. This construction was extended to the case of a holomorphic action of $\mathbb{C}^m$ by Meersseman in [12]. These non-Kählerian manifolds are called *LVM manifolds*. Meersseman also constructed a holomorphic foliation $\mathscr{F}$ on each LVM manifold, and showed that $\mathscr{F}$ is transverse Kähler with respect to the Euler class of a certain S^1-bundle (refer to [12, Theorem 7]).

Finally, in his paper [3] Bosio showed that Meersseman's construction can be generalized to more general holomorphic actions of $\mathbb{C}^m$, so that he obtained the

J.H. Kim (✉)
Department of Mathematics Education, Chosun University, Gwangju 61452, Republic of Korea
e-mail: jinkim@chosun.ac.kr

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*, Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_15

so-called *LVMB manifolds* $N = N(\mathscr{L}, \mathscr{E}, m, n)$ (see Sect. 2 for a precise definition). The class of these manifolds properly includes the family of LVM manifolds. So there exists an LVMB manifold which is not biholomorphic to any LVM manifold (see, e.g., [5, Example 1.2]). It turns out that many interesting properties of LVM manifolds continue to hold for LVMB manifolds. In addition, as in the case of LVM manifolds there exists a holomorphic foliation $\mathscr{F}$ on each LVMB manifold.

We say that an LVMB manifold $N(\mathscr{L}, \mathscr{E}, m, n)$ satisfies *condition (K)* if there exists a real affine automorphism of the dual space $(\mathbb{C}^m)^*$ of $\mathbb{C}^m$ as a real vector space $\mathbb{R}^{2m}$ sending each component of an admissible configuration $\mathscr{L}$ to a vector with integer coefficients. In the paper [5], Cupit-Foutou and Zaffran showed that if the holomorphic foliation $\mathscr{F}$ on an LVMB manifold $N = N(\mathscr{L}, \mathscr{E}, m, n)$ is transverse Kähler and N satisfies the condition (K), then N is actually an LVM manifold.

As mentioned above, the main aim of this survey paper is to explain the constructions of the so-called LVM or LVMB manifolds after López de Medrano, Verjovsky, Meersseman, and Bosio, as well as some recent results and interesting related open questions.

We organize this paper, as follows. In Sect. 2, we briefly recollect the constrictions of the LVM and LVMB manifolds, and give some interesting examples. Section 3 is devoted to giving two open problems related to LVMB manifolds which are not LVM. In the same section, we also present some recent attempt to prove one of the open problems given in Sect. 3.

2 LVM and LVMB Manifolds

In this section, we briefly review the constructions of LVM and LVMB manifolds given in [3, 12] and collect some basic facts necessary for explaining the open problems given in Sect. 4.

2.1 LVM Manifolds

Let m and n be two positive integers such that $n > 2m$. Let $\mathscr{L} = (l_1, l_2, \dots, l_n)$ be an ordered n-tuple of linear forms (or vectors) on $\mathbb{C}^m$, and let

$$l_i = (l_i^1, l_i^2, \dots, l_i^m), \quad 1 \le i \le n.$$

Each l_i can be also thought of a vector in $\mathbb{R}^{2m}$ by using the identification of l_i with

$$(\mathrm{Re}\, l_i, \mathrm{Im}\, l_i) \in \mathbb{R}^m \times \mathbb{R}^m \cong \mathbb{R}^{2m}.$$

Let us denote by $\mathscr{H}(l_1, \ldots, l_n)$ the convex hull generated by $l_1, \ldots, l_n$ in $\mathbb{C}^m$. We say that an ordered n-tuple $(l_1, l_2, \ldots, l_n)$ is *admissible* if the following two conditions hold:

(1) (Siegel condition) $0 \in \mathscr{H}(l_1, \ldots, l_n)$.
(2) (Weak hyperbolicity condition) For every $2m$-tuple $(i_1, \ldots, i_{2m})$ of integers such that $1 \leq i_1 < \cdots < i_{2m} \leq n$, we have $0 \notin \mathscr{H}(l_{i_1}, \ldots, l_{i_{2m}})$.

Let $l'_i = (l_i, 1)$ be a vector in $\mathbb{C}^{m+1}$ whose last coordinate is 1. An admissible configuration then implies that for every set J of integers between 1 and n such that $0 \in \mathscr{H}((l_j)_{j \in J})$, the complex rank of the matrix whose columns are vectors $(l'_j)_{j \in J}$ is equal to $m + 1$.

We say that two admissible configurations $(l_1, l_2, \ldots, l_n)$ and $(l'_1, l'_2, \ldots, l'_n)$ are *equivalent* if there is a continuous map $H : [0, 1] \to (\mathbb{C}^m)^n$ such that

(1) $H(0) = (l_1, l_2, \ldots, l_n)$,
(2) $H(1) = (l''_1, l''_2, \ldots, l''_n)$, where $(l''_1, l''_2, \ldots, l''_n)$ is an arbitrary permutation of $(l'_1, l'_2, \ldots, l'_n)$,
(3) for all $t \in [0, 1]$, the set $H(t)$ is an admissible configuration.

To each admissible configuration $(l_1, \ldots, l_n)$, one can associate the linear foliation $\mathscr{F}$ of $\mathbb{C}^m$ generated by m holomorphic vector fields ξ_j $(1 \leq j \leq m)$ such that

$$\xi_j(z_1, \ldots, z_n) = \sum_{i=1}^{n} l_i^j z_i \frac{\partial}{\partial z_i}$$

which can be obtained from the following holomorphic action

$$\mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^n, \quad (T, (z_1, \ldots, z_n)) \mapsto (z_1 e^{\langle l_1, T \rangle}, \ldots, z_n e^{\langle l_n, T \rangle}).$$

Here $\langle l_i, T \rangle$ denotes the standard scalar product. Note that by their constructions m holomorphic vector fields ξ_j $(1 \leq j \leq m)$ are commuting to each other and that he foliation $\mathscr{F}$ is degenerate, since 0 is a singular point. Recall that a leaf L of $\mathscr{F}$ is called a *Poincaré leaf* if 0 belongs to L, while L is called a *Siegel leaf*, otherwise. The union S of the Siegel leaves is given by

$$S = \{z \in \mathbb{C}^n \backslash \{0\} \mid 0 \in \mathscr{H}(l_j)_{j \in I_z}\},$$

where $j \in I_z$ if and only if $z_j \neq 0$. Note that by the Siegel condition S contains $(\mathbb{C}^*)^n$ as a dense subset of $\mathbb{C}^n$. In fact, S can be written as

$$S = \mathbb{C}^n \backslash E,$$

where E denotes an analytic set whose different components correspond to subspaces of $\mathbb{C}^n$ whose some coordinates are zero. From now on, let us denote by d the minimal complex codimension of E.

It can be shown that the leaf space of the foliation $\mathscr{F}$ restricted to S is given by

$$\mathscr{T} = \{z \in \mathbb{C}^n \backslash \{0\} \mid \sum_{i=1}^{n} l_i |z_i|^2 = 0\}.$$

But then the weak hyperbolicity condition implies that $\mathscr{T}$ is a smooth manifold of complex dimension $n - m$. It is also possible to think of m holomorphic vector fields ξ_j $(1 \leq j \leq m)$ as those defined over the complex projective space $\mathbb{CP}^{n-1}$ by projectivization by $\mathbb{C}^*$ resulted from the action induced by the vector field

$$R(z) = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}, \quad z \in \mathbb{C}^n \backslash \{0\}$$

which is commuting with the vector fields ξ_j $(1 \leq j \leq m)$ above. The leaf space N of the projectivized foliation on $\mathbb{CP}^{n-1}$ given by

$$N = \mathscr{T}/\mathbb{C}^* \subset \mathbb{CP}^{n-1}$$

is called an *LVM manifold* of complex dimension $n - m - 1$.

Now, let $M_1 = \mathscr{T} \cap S^{2n-1}$. Then M_1 is a compact smooth manifold of real dimension $2n - 2m - 1$. It can be shown that two equivalent admissible configurations give rise to diffeomorphic manifolds M_1 and N, but the converse is not true, in general (see [12, p. 102]). Note also that there is a natural action of real torus $(S^1)^n$ on M_1 given by

$$(S^1)^n \times M_1 \to M_1 \quad (e^{i\theta}, z) \mapsto (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n),$$

where θ denotes $(\theta_1, \ldots, \theta_n)$. The quotient of M_1 by the action of $(S^1)^n$ can be written as

$$P = \{r = (r_1, \ldots, r_n) \in (\mathbb{R}^+)^n \mid \sum_{i=1}^{n} r_i l_i = 0, \ \sum_{i=1}^{n} r_i = 1\},$$

which is a simple convex polytope of dimension $n - 2m - 1$. This polytope will be called the *associated polytope* of M_1. There is a natural map from the set of admissible configurations to the set of simple convex polytopes obtained by using the associated polytopes. As mentioned above, two equivalent admissible configurations bijectively give rise to two diffeomorphic manifolds, two diffeomorphic open sets S, and so two combinatorially equivalent associated polytopes (refer to [12, Theorem 13]).

Example 1 Let $m = 0$. Then clearly we have the following

$$S = \mathbb{C}^n \backslash \{0\} = \mathscr{T} \text{ and } M_1 = S^{2n-1}.$$

Thus $N = \mathscr{T}/\mathbb{C}^* = \mathbb{CP}^{n-1}$.

Next, let $m = 1, n = 4$, and $l_1 = l_2 = 1$, $l_3 = -3 + i$, $l_4 = -i$. Then a simple computation shows that

$$M_1 = \{z \in \mathbb{C}^4 \backslash \{0\} \mid \sum_{i=1}^{2} |z_i|^2 = \frac{3}{5}, |z_4|^2 = |z_5|^2 = \frac{1}{5}\} \cong S^3 \times S^1 \times S^1.$$

Thus, N is diffeomorphic to $S^3 \times S^1$ (Hopf manifold).

Finally, let $m = 1, n = 5$, and $l_1 = l_2 = l_3 = 1, l_4 = -3 + i, l_5 = -i$. Similarly, it is easy to obtain

$$M_1 = \{z \in \mathbb{C}^5 \backslash \{0\} \mid \sum_{i=1}^{3} |z_i|^2 = \frac{3}{5}, |z_4|^2 = |z_5|^2 = \frac{1}{5}\} \cong S^5 \times S^1 \times S^1.$$

Thus, N is diffeomorphic to $S^5 \times S^1$ (Hopf or Calabi-Eckmann manifold).

Finally, we close this subsection with an important fact regarding the convex polytope $\mathscr{H}(l_1, \ldots, l_n)$.

Lemma 1 ([12, Lemma VII.2]) *Let $\mathscr{L} = (l_1, \ldots, l_n)$ be an admissible configuration. Then the convex polytope $\mathscr{H}(l_1, \ldots, l_n)$ is a Gale diagram of the dual polytope P^* of the associate polytope P.*

2.2 LVMB Manifolds

The aim of this subsection is to give a definition of LVMB manifolds introduced by Bosio after the constructions of LV by López de Madrano and Verjovsky and later LVM manifolds by Meersseman. The material of this subsection is largely taken from [5, Sect. 1].

As before, let m and n be positive integers with $n > 2m$, and let $\mathscr{L} = (l_1, l_2, \ldots, l_n)$ be an ordered n-tuple of linear forms (or vectors) on $\mathbb{C}^m$. Let $\mathscr{E} = \{\mathscr{E}_\alpha\}_{\alpha \in \Gamma}$ be a family of subsets of $[n] := \{1, 2, \ldots, n\}$ whose cardinality is equal to $2m + 1$. Given a 4-tuple $(\mathscr{L}, \mathscr{E}, m, n)$, we then define $U = U(\mathscr{E}) \subset \mathbb{CP}^{n-1}$ by the following condition:

$$[x_1, \ldots, x_n] \in U(\mathscr{E}) \text{ if and only if } x_i \neq 0 \text{ for some } \mathscr{E}_\alpha \in \mathscr{E} \text{ and all } i \in \mathscr{E}_\alpha.$$

Notice that there is a natural action of $\mathbb{C}^m$ on U given by

$$\mathbb{C}^m \times U \mapsto U, \quad (z, [x_1, \ldots, x_n]) \mapsto [e^{l_1(z)} x_1, \ldots, e^{l_n(z)} x_n],$$

where $l_i(z)$ means the standard scalar product $\langle l_i, z \rangle$. For each $\alpha \in \Gamma$, we also denote by $\mathscr{H}_\alpha$ the convex hull generated by $l_i \in \mathscr{E}_\alpha$.

For the rest of the paper, we shall assume that the following two conditions hold:

(1) For any two distinct α and β in Γ, $\mathscr{H}_\alpha^\circ \cap \mathscr{H}_\beta^\circ$ is non-empty.
(2) For every $\mathscr{E}_\alpha \in \mathscr{E}$ and every $i \in [n]$, there exists $j \in \mathscr{E}_\alpha$ such that

$$(\mathscr{E}_\alpha \backslash \{j\}) \cup \{i\} \in \mathscr{E}.$$

When a 4-tuple $(\mathscr{L}, \mathscr{E}, m, n)$ satisfies the above two conditions, we say that $(\mathscr{L}, \mathscr{E}, m, n)$ is an *LVMB data*, and we will write $(\mathscr{L}, \mathscr{E}, m, n) \in LVMB$ and denote by $N(\mathscr{L}, \mathscr{E}, m, n)$ the quotient $U/\mathbb{C}^m$. It has been shown in [3, p. 1261] that the quotient $N(\mathscr{L}, \mathscr{E}, m, n)$ is a compact complex manifold of complex dimension $n - m - 1$, thanks to the above two conditions. In this case, $N(\mathscr{L}, \mathscr{E}, m, n)$ will be called an *LVMB manifold*.

Let $\mathscr{H}_\alpha^\circ$ denote the relative interior of the convex hull $\mathscr{H}_\alpha$. If all the intersections $\cap_{\alpha \in \Gamma} \mathscr{H}_\alpha^\circ$ is not empty, then the action of $\mathbb{C}^m$ on U is called an *LVM action*. Moreover, in this case we will write $(\mathscr{L}, \mathscr{E}, m, n) \in LVM$ and call $N(\mathscr{L}, \mathscr{E}, m, n)$ an *LVM manifold*. It can be shown as in [3, Proposition 1.3] that every LVM manifold can be obtained in this way from an LVMB data. We remark that there is an LVMB manifold which is not an LVM manifold (see, e.g., [5, Example 1.2]).

Example 2 For the sake of simplicity, let us write abc for $\{a, b, c\}$. Then let

$$\mathscr{E} := \{125, 145, 235, 345\} \subset 2^{[5]}.$$

Then clearly $\mathscr{E}_{1,5}$ satisfies the imbrication and SEP conditions. Note that the open set $\mathscr{S}$ is given by

$$\begin{aligned}\mathscr{S} &= \{(z_1, \dots, z_5) \in \mathbb{C}^5 \mid (z_1, z_3) \neq 0, (z_2, z_4) \neq 0, z_5 \neq 0\} \\ &\cong (\mathbb{C}^2 \backslash \{0\})^2 \times \mathbb{C}^*.\end{aligned}$$

Now, set $l_1 = l_3 = 1$, $l_2 = l_4 = i$, $l_5 = 0$. Then $\mathscr{L}$ satisfies the Imbrication condition and an LVM datum (see Fig. 1). It is easy to see that we have $N(\mathscr{L}, \mathscr{E}, 1, 5) = S^3 \times S^3$ whose the associated complex $\mathscr{P}$ is given by

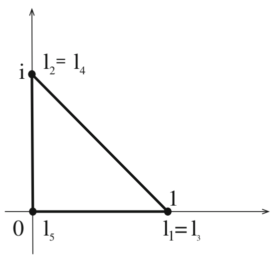

Fig. 1 $(\mathscr{L}, \mathscr{E}, 1, 5)$

$$\mathscr{P} = \{\emptyset, 1, 2, 3, 4, 12, 23, 34, 14\},$$

which is the boundary of the square.

Finally, we close this subsection with one more definition. Indeed, we say that $(\mathscr{L}, \mathscr{E}, m, n)$ satisfies the *condition (K)*, denoted by $(\mathscr{L}, \mathscr{E}, m, n) \in$ (K), if there exists a real affine automorphism of the dual space $(\mathbb{C}^m)^*$ of $\mathbb{C}^m$ as a real vector space $\mathbb{R}^{2m}$ which maps each l_i to a vector with integer coefficients.

3 Foliations and Leaf Spaces

The aim of this section is to quickly review the foliation $\mathscr{F}$ defined on an LVMB manifold as in an LVM manifold, and collect several important properties about its leaf space.

To do so, for $1 \leq i \leq n$ let $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^{2m}) \in \mathbb{Z}^{2m}$, and consider an algebraic and effective action of $(\mathbb{C}^*)^{2m}$ on $\mathbb{CP}^{n-1}$ or the restriction to U given by

$$(\mathbb{C}^*)^{2m} \times \mathbb{CP}^{n-1} \to \mathbb{CP}^{n-1}, \quad (t, [x_1, \ldots, x_n]) \mapsto [t^{\lambda_1} x_0, \ldots, t^{\lambda_n} x_n],$$

where t^{λ_i} means $t_1^{\lambda_i^1} \cdots t_{2m}^{\lambda_i^{2m}}$. Let G be a closed co-compact complex Lie subgroup of $(\mathbb{C}^*)^{2m}$ isomorphic to $\mathbb{C}^m$. Then the restricted action of G of $(\mathbb{C}^*)^{2m}$ to U is free, and U/G is a compact complex manifold of dimension $n - m - 1$. Moreover, it has been proved in [5, Theorem 2.1] that if a 4-tuple $(\mathscr{L}, \mathscr{E}, m, n) \in$ (K), then the LVM manifold $N(\mathscr{L}, \mathscr{E}, m, n)$ can be obtained as the quotient U/G for some choice of a $(\mathbb{C}^*)^{2m}$-action on $\mathbb{CP}^{n-1}$ and a subgroup G of $(\mathbb{C}^*)^{2m}$. As a consequence, we have the following commutative diagram which will play an important role in the proof of our main Theorem 1:

$$\begin{array}{ccc} U & = & U \\ \downarrow & & \downarrow \\ U/(\mathbb{C}^*)^{2m} =: X(\mathscr{L}, \mathscr{E}, m, n) & \xleftarrow{\pi} & N(\mathscr{L}, \mathscr{E}, m, n) := U/G. \end{array} \qquad (1)$$

Notice that if $(\mathscr{L}, \mathscr{E}, m, n) \in LVMB \cap (K)$, then the map $\pi : N \to X$ gives a Seifert principal fibration whose fibers are $(\mathbb{C}^*)^{2m}/G$ isomorphic to the compact real torus $\mathbb{T}^{2m} = S^1 \times \cdots \times S^1$ ($2m$ times), and the fibers of π defines a foliation $\mathscr{F}$ on $N(\mathscr{L}, \mathscr{E}, m, n)$.

On the other hand, it is worth mentioning that even if $(\mathscr{L}, \mathscr{E}, m, n) \in LVMB$ does not satisfies the condition (K), there is still a foliation $\mathscr{F}$ whose leaves are generated by the m holomorphic vector fields ξ_i $(1 \leq i \leq m)$ defined as in Sect. 2.1.

We believe that the following proposition ([5, Proposition 3.2]) will play a certain role in resolving the open problems given in Sect. 4.

Proposition 1 *Let $(\mathscr{L}, \mathscr{E}, m, n) \in LVMB \cap (K)$, and let N and X be the same as in (1). Then X is projective if and only if $(\mathscr{L}, \mathscr{E}, m, n) \in LVM$.*

4 Open Problems and Related Results

The main aim of this section is to give some interesting open problems related to LVMB manifolds which are not LVM, and to discuss some recent related result.

One of the most interesting problems posed in [5, Sect. 1] is

Question 1 Let $(\mathscr{L}, \mathscr{E}, m, n) \in LVMB$, but not in LVM. Is it possible to find $(\mathscr{L}', \mathscr{E}', m', n') \in LVM$ such that $N(\mathscr{L}, \mathscr{E}, m, n)$ is isomorphic to $N(\mathscr{L}', \mathscr{E}', m', n')$?

In fact, a proof of Question 1 has already been provided by Battisti in [2]. However, it would be much nicer to give another proof that is easier to follow (e.g., Lemmas 2.5 and 4.1 in [2] seem to contain some erroneous claims and arguments, respectively).

As mentioned in Sect. 1, Cupit-Foutou and Zaffran showed in [5] that if the holomorphic foliation $\mathscr{F}$ on an LVMB manifold $N = N(\mathscr{L}, \mathscr{E}, m, n)$ is transverse Kähler and N satisfies the condition (K), then N is actually an LVM manifold. So it is natural to ask if, when the holomorphic foliation $\mathscr{F}$ on an LVMB manifold N is simply transverse Kähler, N is actually an LVM manifold.

In view of this result, another interesting question posed in the papers [1, 5] (see also [9, 10, 13]) is

Question 2 let $N(\mathscr{L}, \mathscr{E}, m, n)$ be an LVMB manifold, and let $\mathscr{F}$ be the holomorphic foliation on N. If $\mathscr{F}$ is transverse Kähler, is $N(\mathscr{L}, \mathscr{E}, m, n)$ actually an LVM manifold?

We think that Question 2 is true. Indeed, we currently have the following claim (a work in progress):

Theorem 1 *Let $N := N(\mathscr{L}, \mathscr{E}, m, n)$ be an LVMB manifold, and let $\mathscr{F}$ be the holomorphic foliation on N. If $\mathscr{F}$ is transverse Kähler with respect to a basic and closed real 2-form, then N is actually an LVM manifold.*

One key ingredient to prove Theorem 1 is

Proposition 2 *Let M be a complex k-dimensional manifold equipped with a transverse Kähler foliation $\mathscr{F}$ generated by l-dimensional leaves with respect to a basic and closed real 2-form ω. Then the leaf space X admits the structure of a Kähler orbifold of complex dimension $k - l$ in the natural way.*

With Proposition 2 in place, we can prove Theorem 1 by contradiction. So suppose first that N admits the transverse Kaehler foliation $\mathscr{F}$ with respect to the basic and closed 2-form induced from the standard Kähler form on $\mathbb{CP}^{n-1}$, and let X denote the leaf space of the foliation $\mathscr{F}$ on N. Then we can show that

$$H^2_B(\mathscr{S}, \mathscr{O}_{\mathscr{S}}) \cong H^2_{\bar{\partial}}(X, \mathscr{O}_X),$$

where $\mathscr{O}_{\mathscr{S}}$ (resp. $\mathscr{O}_X$) denotes the sheaf of germs of holomorphic functions on $\mathscr{S}$ (resp. X) (refer to [6]). But it turns out that

$$H^2_{\bar{\partial}}(X, \mathscr{O}_X) = 0.$$

Some computations show that we have

$$H^2(X, \mathbb{R}) \cong H^{1,1}_{\bar{\partial}}(X) \cong \mathbb{R} \text{ or } \mathbb{R} \oplus \mathbb{R}.$$

Since $H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R}) \cong H^{1,1}_{\bar{\partial}}(X)$, we can conclude that there should be an integral Kähler form on X whose lift to N is a basic and transverse Kähler form. Our orbit space X is, in fact, a Kähler orbifold with an integral Kähler form, so X is projective by the Kodaira embedding theorem [7]. Recall now that the associated polytope of a projective toric manifold is a Delzant (or moment) polytope which is, in particular, polytopal. Clearly this fact applies to our toric projective manifold (or orbifold) X, and so the associated polytope of X is polytopal. Therefore, N would be actually an LVM manifold.

Acknowledgements This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1B03930639).

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Inequalities for Algebraic Casorati Curvatures and Their Applications II

Young Jin Suh and Mukut Mani Tripathi

Abstract Different kind of algebraic Casorati curvatures are introduced. A result expressing basic Casorati inequalities for algebraic Casorati curvatures is presented and equality cases are discussed. As their applications, basic Casorati inequalities for different δ-Casorati curvatures for different kind of submanifolds of quaternionic space forms are presented.

1 Introduction

In 1889, Felice Casorati defined a curvature, well known as the Casorati curvature, for a regular surface in Euclidean 3-space which turns out to be the normalized sum of the squared principal curvatures (cf. [6–8]). Casorati preferred this curvature over the traditional Gaussian curvature because the Casorati curvature vanishes for a surface in Euclidean 3-space if and only if both Euler normal curvatures (or principal curvatures) of the surface vanish simultaneously and thus corresponds better with the common intuition of curvature. For a hypersurface of a Riemannian manifold the Casorati curvature is defined to be the normalized sum of the squared principal normal curvatures of the hypersurface, and in general, the Casorati curvature of a submanifold of a Riemannian manifold is defined to be the normalized squared length of the second fundamental form [14]. Geometrical meaning and the importance of the Casorati curvature, discussed by several geometers, can be visualized in several research/survey papers including [11, 15, 16, 18–20, 22, 23, 30, 36, 37].

Y.J. Suh
Department of Mathematics and RIRCM,
Kyungpook National University, Daegu 41566, Republic of Korea
e-mail: yjsuh@knu.ac.kr

M.M. Tripathi (✉)
Department of Mathematics, Institute of Science, Banaras Hindu University,
Varanasi 221005, India
e-mail: mmtripathi66@yahoo.com

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_16

In a previous paper [35], the second author introduced the notion of different kind of algebraic Casorati curvatures, obtained some basic Casorati inequalities for algebraic Casorati curvatures, applied those results to obtain basic Casorati inequalities for different δ-Casorati curvatures for Riemannian submanifolds of a Riemannian manifold and in particular of a real space form, and presented some problems for further studies.

The present paper is organized as follows. In Sect. 2, we recall curvature like tensors and algebraic Casorati curvatures $\widehat{\delta}_{\mathscr{C}^{T,\zeta}}(n-1)$, $\delta_{\mathscr{C}^{T,\zeta}}(n-1)$, $\delta_{\mathscr{C}^{T,\zeta}}(r;n-1)$, $\widehat{\delta}_{\mathscr{C}^{T,\zeta}}(r;n-1)$, which in special cases of Riemannian submanifolds reduce to already known δ-Casorati curvatures. In Sect. 3, we recall some basic preliminaries about quaternionic space forms and its submanifolds. In the last Sect. 4, we obtain basic Casorati inequalities for Casorati curvatures $\delta(r;n-1)$, $\widehat{\delta}(r;n-1)$, $\delta(n-1)$, $\widehat{\delta}(n-1)$ for Riemannian submanifolds, in particular for slant and totally real submanifolds of a quaternionic space form with very short proofs. Some problems are also presented for further studies.

2 Algebraic Casorati Curvatures

Let (M,g) be an n-dimensional Riemannian submanifolds of an m-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$. The equation of Gauss is given by

$$\begin{aligned} R(X,Y,Z,W) = {} & \widetilde{R}(X,Y,Z,W) + \widetilde{g}(\sigma(Y,Z),\sigma(X,W)) \\ & - \widetilde{g}(\sigma(X,Z),\sigma(Y,W)) \end{aligned} \tag{1}$$

for all $X,Y,Z,W \in TM$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M, respectively and σ is the second fundamental form of the immersion of M in $\widetilde{M}$.

Let M be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian manifold $\widetilde{M}$. A point $p \in M$ is said to be an *invariantly quasi-umbilical point* if there exist $m-n$ mutually orthogonal unit normal vectors $N_{n+1},\dots,N_m$ such that the shape operators with respect to all directions N_α have an eigenvalue of multiplicity $n-1$ and that for each N_α the distinguished eigendirection is the same. The submanifold is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point. For details, we refer to [4].

Let (M,g) be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of the tangent space T_pM and e_α belongs to an orthonormal basis $\{e_{n+1},\dots,e_m\}$ of the normal space $T_p^{\perp}M$. We let

$$\sigma_{ij}^{\alpha} = \widetilde{g}\left(\sigma\left(e_i,e_j\right),e_\alpha\right), \quad i,j \in \{1,\dots,n\}, \quad \alpha \in \{n+1,\dots,m\}.$$

Then, the squared mean curvature and the squared norm of second fundamental form σ of the submanifold M in $\widetilde{M}$ are defined by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} \sigma_{ii}^{\alpha} \right)^2, \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \widetilde{g}\left(\sigma\left(e_i, e_j\right), \sigma\left(e_i, e_j\right)\right),$$

respectively. Let K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$, respectively. In view of (1), we have

$$\tau_{\mathrm{Nor}}(p) = \widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right) + \frac{n}{n-1}\|H\|^2 - \frac{1}{n(n-1)}\|\sigma\|^2, \tag{2}$$

where

$$\tau_{\mathrm{Nor}}(p) = \frac{2\tau(p)}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} K_{ij},$$

$$\widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right) = \frac{2\widetilde{\tau}\left(T_pM\right)}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \widetilde{K}_{ij}$$

are the normalized scalar curvature of M at p and the normalized scalar curvature of the n-plane section T_pM in the ambient manifold $\widetilde{M}$, respectively.

The *Casorati curvature* $\mathscr{C}$ [14] of the Riemannian submanifold M is defined to be the normalized squared length of the second fundamental form σ, that is,

$$\mathscr{C} = \frac{1}{n}\|\sigma\|^2 = \frac{1}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} \left(\sigma_{ij}^{\alpha}\right)^2. \tag{3}$$

For a k-dimensional subspace Π_k of T_pM, $k \geq 2$ spanned by $\{e_1, \ldots, e_k\}$, the Casorati curvature $\mathscr{C}(\Pi_k)$ of the subspace Π_k is defined to be [13]

$$\mathscr{C}(\Pi_k) = \frac{1}{k} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{k} \left(\sigma_{ij}^{\alpha}\right)^2.$$

The (modified) *normalized* δ*-Casorati curvatures* $\delta_{\mathscr{C}}(n-1)$ (cf. [26, 38]) and the *normalized* δ*-Casorati curvatures* $\widehat{\delta}_{\mathscr{C}}(n-1)$ [13] of the Riemannian submanifold M are given by

$$[\delta_{\mathscr{C}}(n-1)]_p = \frac{1}{2}\mathscr{C}_p + \frac{n+1}{2n} \inf\left\{\mathscr{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}, \tag{4}$$

$$[\widehat{\delta}_{\mathscr{C}}(n-1)]_p = 2\mathscr{C}_p - \frac{2n-1}{2n} \sup\left\{\mathscr{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}, \tag{5}$$

respectively. For a positive real number $r \neq n(n-1)$, letting

$$\mathrm{a}(r) = \frac{1}{nr}(n-1)\,(n+r)\left(n^2 - n - r\right),$$

the *normalized δ-Casorati curvatures* $\delta_{\mathscr{C}}(r; n-1)$ and $\widehat{\delta}_{\mathscr{C}}(r; n-1)$ of a Riemannian submanifold M are given by [14]

$$[\delta_{\mathscr{C}}(r; n-1)]_p = r\,\mathscr{C}_p + \mathrm{a}(r) \inf \left\{\mathscr{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}, \tag{6}$$

if $0 < r < n(n-1)$, and

$$[\widehat{\delta}_{\mathscr{C}}(r; n-1)]_p = r\,\mathscr{C}_p + \mathrm{a}(r) \sup \left\{\mathscr{C}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}, \tag{7}$$

if $n(n-1) < r$, respectively. In [26] the normalized δ-Casorati curvatures $\widehat{\delta}_{\mathscr{C}}(r; n-1)$ and $\delta_{\mathscr{C}}(r; n-1)$ are called as the generalized normalized δ-Casorati curvatures $\widehat{\delta}_{\mathscr{C}}(r; n-1)$ and $\delta_{\mathscr{C}}(r; n-1)$, respectively. We see that [28]

$$[\delta_{\mathscr{C}}(n-1)]_p = \frac{1}{n(n-1)}\left[\delta_{\mathscr{C}}\left(\frac{n(n-1)}{2}; n-1\right)\right]_p, \tag{8}$$

$$\left[\widehat{\delta}_{\mathscr{C}}(n-1)\right]_p = \frac{1}{n(n-1)}\left[\widehat{\delta}_{\mathscr{C}}\left(2n(n-1); n-1\right)\right]_p \tag{9}$$

for all $p \in M$.

Now, let (M, g) be an n-dimensional Riemannian manifold and T a curvature-like tensor (cf. [24, Sect. 8 of Chap. 1], [25]) so that it satisfies

$$T(X, Y, Z, W) = -\,T(Y, X, Z, W), \tag{10}$$

$$T(X, Y, Z, W) = T(Z, W, X, Y), \tag{11}$$

$$T(X, Y, Z, W) + T(Y, Z, X, W) + T(Z, X, Y, W) = 0 \tag{12}$$

for all vector fields X, Y, Z and W on M. Let $\{e_1, e_2, \ldots, e_n\}$ be any orthonormal basis of T_pM. We denote

$$(K_T)_{ij} = T\left(e_i, e_j, e_j, e_i\right).$$

If $i \neq j$, then $(K_T)_{ij} = K_T(e_i \wedge e_j)$ is the *T-sectional curvature* of the 2-plane section Π_2 spanned by e_i and e_j at $p \in M$ [5]. The *T-Ricci curvature* $\mathrm{Ric}_T(e_i)$ is given by

$$\mathrm{Ric}_T(e_i) = \sum_{j=1,\, j\neq i}^{k} K_T(e_i \wedge e_j).$$

The T*-scalar curvature* is given by [5]

$$\tau_T(p) = \sum_{1 \le i < j \le n} T\left(e_i, e_j, e_j, e_i\right) = \frac{1}{2} \sum_{i=1}^{n} \mathrm{Ric}_T(e_i). \tag{13}$$

Now, let Π_k be a k-plane section of T_pM. If $k = n$ then $\Pi_n = T_pM$; and if $k = 2$ then Π_2 is a plane section of T_pM. We choose an orthonormal basis $\{e_1, \dots, e_k\}$ of Π_k. Then we define the T*-*k*-Ricci curvature* of Π_k at e_i, $i \in \{1, \dots, k\}$, denoted $(\mathrm{Ric}_T)_{\Pi_k}(e_i)$, by

$$(\mathrm{Ric}_T)_{\Pi_k}(e_i) = \sum_{j=1, j \ne i}^{k} K_T(e_i \wedge e_j). \tag{14}$$

We note that the T*-*n*-Ricci curvature* $(\mathrm{Ric}_T)_{T_pM}(e_i)$ is the usual T*-Ricci curvature* of e_i, denoted $\mathrm{Ric}_T(e_i)$. The T*-*k*-scalar curvature* $\tau_T(\Pi_k)$ of the k-plane section Π_k is given by

$$\tau_T(\Pi_k) = \sum_{1 \le i < j \le k} K_T(e_i \wedge e_j). \tag{15}$$

The T-scalar curvature of M at p is identical with the T-n-scalar curvature of the tangent space T_pM of M at p, that is, $\tau_T(p) = \tau_T(T_pM)$. If Π_2 is a 2-plane section, $\tau_T(\Pi_2)$ is nothing but the T-sectional curvature $K_T(\Pi_2)$ of Π_2. The T*-*k*-normalized scalar curvature* of a k-plane section Π_k at p is defined as

$$(\tau_T)_{\mathrm{Nor}}(\Pi_k) = \frac{2}{k(k-1)}\, \tau_T(\Pi_k).$$

The T*-normalized scalar curvature* at p is defined as

$$(\tau_T)_{\mathrm{Nor}}(p) = (\tau_T)_{\mathrm{Nor}}(T_pM) = \frac{2}{n(n-1)}\, \tau_T(p).$$

If T is replaced by the Riemann curvature tensor R, then T-sectional curvature K_T, T-Ricci tensor S_T, T-Ricci curvature Ric_T, T-scalar curvature τ_T, T-normalized scalar curvature $(\tau_T)_{\mathrm{Nor}}$, T-k-Ricci curvature $(\mathrm{Ric}_T)_{\Pi_k}$, T-k-scalar curvature $\tau_T(\Pi_k)$, T-k-normalized scalar curvature $(\tau_T)_{\mathrm{Nor}}(\Pi_k)$ and T-normalized scalar curvature $(\tau_T)_{\mathrm{Nor}}$ become the sectional curvature K, the Ricci tensor S, the Ricci curvature Ric, the scalar curvature τ, the normalized scalar curvature τ_{Nor}, k-Ricci curvature Ric_{Π_k}, k-scalar curvature $\tau(\Pi_k)$, k-normalized scalar curvature $\tau_{\mathrm{Nor}}(\Pi_k)$ and normalized scalar curvature τ_{Nor}, respectively.

Let (M, g) be an n-dimensional Riemannian manifold and (B, g_B) a Riemannian vector bundle over M. If ζ is a B-valued symmetric $(1, 2)$-tensor field and T a $(0, 4)$-tensor field on M such that

$$T(X, Y, Z, W) = g_B(\zeta(X, W), \zeta(Y, Z)) - g_B(\zeta(X, Z), \zeta(Y, W)) \tag{16}$$

for all vector fields X, Y, Z, W on M, then the Eq. (16) is said to be an *algebraic Gauss equation* [10]. Every $(0, 4)$-tensor field T on M, which satisfies (16), becomes a curvature-like tensor.

Now, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_α belong to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the Riemannian vector bundle (B, g_B) over M at p. We put

$$\zeta_{ij}^{\alpha} = g_B\left(\zeta\left(e_i, e_j\right), e_\alpha\right), \quad \|\zeta\|^2 = \sum_{i,j=1}^{n} g_B\left(\zeta\left(e_i, e_j\right), \zeta\left(e_i, e_j\right)\right),$$

$$\operatorname{trace}\zeta = \sum_{i=1}^{n} \zeta\left(e_i, e_i\right), \quad \|\operatorname{trace}\zeta\|^2 = g_B(\operatorname{trace}\zeta, \operatorname{trace}\zeta).$$

The *algebraic Casorati curvature* $\mathscr{C}^{T,\zeta}$ with respect to T and ζ is defined by [35]

$$\mathscr{C}^{T,\zeta} = \frac{1}{n}\|\zeta\|^2 = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(\zeta_{ij}^{\alpha}\right)^2. \tag{17}$$

For a k-dimensional subspace Π_k of T_pM, $k \geq 2$, spanned by $\{e_1, \ldots, e_k\}$, the *algebraic Casorati curvature* $\mathscr{C}^{T,\zeta}(\Pi_k)$ of the subspace Π_k is defined to be

$$\mathscr{C}^{T,\zeta}(\Pi_k) = \frac{1}{k}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{k}\left(\zeta_{ij}^{\alpha}\right)^2. \tag{18}$$

We note that

$$\mathscr{C}_p^{T,\zeta} = \mathscr{C}^{T,\zeta}(T_pM), \qquad p \in M.$$

The *algebraic Casorati curvatures* $\delta_{\mathscr{C}^{T,\zeta}}(r; n-1)$ and $\widehat{\delta}_{\mathscr{C}^{T,\zeta}}(r; n-1)$ are defined by [35]

$$\begin{aligned}[\delta_{\mathscr{C}^{T,\zeta}}(r; n-1)]_p &= r\,\mathscr{C}_p^{T,\zeta} \\ &\quad + \mathrm{a}(r)\inf\left\{\mathscr{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\},\end{aligned} \tag{19}$$

if $0 < r < n(n-1)$,

$$\begin{aligned}[\widehat{\delta}_{\mathscr{C}^{T,\zeta}}(r; n-1)]_p &= r\,\mathscr{C}_p^{T,\zeta} \\ &\quad + \mathrm{a}(r)\sup\left\{\mathscr{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\},\end{aligned} \tag{20}$$

if $n(n-1) < r$, where

$$\mathrm{a}(r) = \frac{1}{nr}(n-1)(n+r)\left(n^2 - n - r\right)$$

for any positive real number $r \neq n(n-1)$. Also we have [35]

$$\begin{aligned}\left[\delta_{\mathscr{C}^{T,\zeta}}(n-1)\right]_p &= \frac{1}{n(n-1)}\left[\delta_{\mathscr{C}^{T,\zeta}}\left(\frac{n(n-1)}{2}; n-1\right)\right]_p \\ &= \frac{1}{2}\mathscr{C}_p^{T,\zeta} + \frac{n+1}{2n}\inf\left\{\mathscr{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}, \quad (21)\end{aligned}$$

$$\begin{aligned}\left[\widehat{\delta}_{\mathscr{C}^{T,\zeta}}(n-1)\right]_p &= \frac{1}{n(n-1)}\left[\widehat{\delta}_{\mathscr{C}^{T,\zeta}}(2n(n-1); n-1)\right]_p \\ &= 2\mathscr{C}_p^{T,\zeta} - \frac{2n-1}{2n}\sup\{\mathscr{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}. \quad (22)\end{aligned}$$

Let (M, g) be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Let the Riemannian vector bundle (B, g_B) over M be replaced by the normal bundle $T^{\perp}M$, and the B-valued symmetric $(1, 2)$-tensor field ζ be replaced by the second fundamental form of immersion σ. Then the algebraic Casorati curvature $\mathscr{C}^{T,\zeta}$ becomes the *Casorati curvature* $\mathscr{C}$ of the Riemannian submanifold M given by (3). The algebraic Casorati curvatures$\delta_{\mathscr{C}^{T,\zeta}}(n-1)$ and $\widehat{\delta}_{\mathscr{C}^{T,\zeta}}(n-1)$ become *normalized δ-Casorati curvatures* $\delta_{\mathscr{C}}(n-1)$ and $\widehat{\delta}_{\mathscr{C}}(n-1)$ of the Riemannian submanifold M given by (4) and (5), respectively. Finally, algebraic Casorati curvatures $\delta_{\mathscr{C}^{T,\zeta}}(r; n-1)$ and $\widehat{\delta}_{\mathscr{C}^{T,\zeta}}(r; n-1)$ become normalized δ-Casorati curvatures $\delta_{\mathscr{C}}(r; n-1)$ and $\widehat{\delta}_{\mathscr{C}}(r; n-1)$ of the Riemannian submanifold M given by (6) and (7), respectively.

3 Quaternionic Space Forms

Let $(\widetilde{M}, \widetilde{g})$ be a $4m$-dimensional Riemannian manifold equipped with a 3-dimensional vector bundle $\mathscr{V}$ of tensors of type $(1, 1)$ with a local basis formed by Hermitian structures $\{J_1, J_2, J_3\}$ such that

$$J_1 \circ J_2 = -J_2 \circ J_1 = J_3 \quad \text{and} \quad \widetilde{\nabla}_X J_a = \sum_{b=1}^{3} Q_{ab} J_b, \qquad a \in \{1, 2, 3\}$$

for any vector field X, where $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{g}$ and Q_{ab} are certain local 1-forms on $\widetilde{M}$ such that $Q_{ab} + Q_{ba} = 0$. Then $(\widetilde{g}, \mathscr{V})$ is said to be a *quaternionic Kaehler structure* on $\widetilde{M}$ and $(\widetilde{M}, \widetilde{g}, \mathscr{V})$ is said to be a *quaternionic Kaehler manifold*.

Let $(\widetilde{M}, \widetilde{g}, \mathscr{V})$ be a quaternionic Kaehler manifold and let X be a non-null vector field on $\widetilde{M}$. Then the 4-dimensional plane $Q(X)$, spanned by $\{X, J_1X, J_2X, J_3X\}$, is called a *quaternionic 4-plane*. Any 2-plane in $Q(X)$ is called a *quaternionic plane*. The sectional curvature of a quaternionic plane is called a *quaternionic sectional* *curvature*. If the quaternionic sectional curvatures of a quaternionic Kaehler manifold $(\widetilde{M}, \widetilde{g}, \mathscr{V})$ are equal to a real constant c, then it is said to be a *quaternionic space form*, and is denoted by $\widetilde{M}(c)$. It is well known that a quaternionic Kaehler manifold is a quaternionic space form $\widetilde{M}(c)$ if and only if its Riemann curvature tensor is given by

$$\begin{aligned}\widetilde{R}(X, Y, Z, W) = \frac{c}{4}\{&\widetilde{g}(Y, Z)\,\widetilde{g}(X, W) - \widetilde{g}(X, Z)\,\widetilde{g}(Y, W) \\ &+ \sum_{a=1}^{3}\{\widetilde{g}(J_a Y, Z)\,\widetilde{g}(J_a X, W) - \widetilde{g}(J_a X, Z)\,\widetilde{g}(J_a Y, W) \\ &- 2\widetilde{g}(J_a X, Y)\,\widetilde{g}(J_a Z, W)\}\end{aligned} \tag{23}$$

for all vector fields X, Y, Z, W on $\widetilde{M}$ and any local basis $\{J_1, J_2, J_3\}$. For details we refer to [21].

Let M be an n-dimensional submanifold of a quaternionic Kaehler manifold $(\widetilde{M}, \widetilde{g}, \mathscr{V})$. For any $X \in T_pM$ we decompose J_aX, $a \in \{1, 2, 3\}$, into tangential and normal parts given by

$$J_a X = P_a X + F_a X, \qquad P_a X \in T_p M, \quad F_a X \in T_p^{\perp} M. \tag{24}$$

The squared norm of P_a at $p \in M$ is

$$\|P_a\|^2 = \sum_{i,j=1}^{n} g\left(P_a e_i, e_j\right)^2,$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of the tangent space T_pM.

An n-dimensional submanifold M of a $4m$-dimensional quaternionic Kaehler manifold is called a *quaternionic submanifold* (or *an invariant submanifold*) if $J_a\left(T_pM\right) = T_pM,\ p \in M$, and it is called a *totally real submanifold* (or *an anti-invariant submanifold*) if $J_a\left(T_pM\right) \subseteq T_p^{\perp}M,\ p \in M$. Thus, M is a quaternionic submanifold if each $F_a = 0$, and it is totally real if each $P_a = 0$. In more general, a submanifold M of a quaternionic Kaehler manifold is called a *quaternionic CR-submanifold* if there exist two orthogonal complementary distributions D and $D^{\perp}$ such that $J_a(D) = D$, and $J_a\left(D^{\perp}\right) \subseteq T^{\perp}M$. Thus a quaternionic CR-submanifold is a quaternionic submanifold (resp. totally real submanifold) if $D^{\perp} = \{0\}$ (resp. $D = \{0\}$). Moreover, a totally real submanifold is known as a *Lagrangian submanifold* if $n = m$. For more details we refer to [1]. Analogous to the θ-slant submanifolds [9] of an almost Hermitian manifold, there is the concept of θ-slant submanifolds [31] of a quaternionic Kaehler manifold, which is another generalization of quaternionic and totally real submanifolds. A *θ-slant submanifold* of a quaternionic Kaehler manifold is a submanifold M such that the angle between J_aX and T_pM, $a \in \{1, 2, 3\}$ is the same for all $p \in M$ and for all $X \in T_pM$. Thus a θ-slant submanifold is quaternionic or totally real according as $\theta = 0$ or $\theta = \pi/2$. A θ-slant submanifold is said to be a proper θ-slant submanifold if it neither quaternionic nor totally real.

4 Casorati Inequalities

First, we recall the following:

Proposition 1 ([35]) *Let (M, g) be an n-dimensional Riemannian manifold, (B, g_B) a Riemannian vector bundle over M and ζ a B-valued symmetric $(1, 2)$-tensor field. Let T be a curvature-like tensor field satisfying the algebraic Gauss equation (16). Then*

$$(\tau_T)_{\mathrm{Nor}}(p) \leq \frac{1}{n(n-1)} \left[\delta_{\mathscr{C}^{T,\zeta}}(r; n-1)\right]_p, \qquad 0 < r < n(n-1), \tag{25}$$

$$(\tau_T)_{\mathrm{Nor}}(p) \leq \frac{1}{n(n-1)} [\widehat{\delta}_{\mathscr{C}^{T,\zeta}}(r; n-1)]_p, \qquad n(n-1) < r. \tag{26}$$

If

$$\inf\{\mathscr{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}$$
$$\big(\text{resp. } \sup\{\mathscr{C}^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}\big)$$

is attained by a hyperplane Π_{n-1} of T_pM, $p \in M$, then the equality sign holds in (25) (resp. (26)) if and only if with respect to a suitable orthonormal tangent frame $\{e_1, \ldots, e_n\}$ and a suitable orthonormal frame $\{e_{n+1}, \ldots, e_m\}$ of the Riemann vector bundle (B, g_B), the components of ζ satisfy

$$\zeta^\alpha_{ij} = 0 \qquad i, j \in \{1, \ldots, n\}, \quad i \neq j \quad \alpha \in \{n+1, \ldots, m\}, \tag{27}$$

$$\zeta^\alpha_{11} = \zeta^\alpha_{22} = \cdots = \zeta^\alpha_{n-1\,n-1} = \frac{r}{n(n-1)}\,\zeta^\alpha_{nn} \qquad \alpha \in \{n+1, \ldots, m\}. \tag{28}$$

We shall need the following Lemma.

Lemma 1 *Let M be an n-dimensional submanifold of a $4m$-dimensional quaternionic space form $\widetilde{M}(4c)$. Then*

$$\widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right) = \frac{c}{4} + \frac{3c}{4n(n-1)} \sum_{a=1}^{3} \|P_a\|^2. \tag{29}$$

Proof Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_{4m}\}$ of the normal space $T_p^\perp M$. Then, from (23), we get

$$\widetilde{K}_{ij} = \frac{c}{4} + \frac{3c}{4} \sum_{a=1}^{3} g(P_a e_i, e_j)^2. \tag{30}$$

In view of $\widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) = \sum_{j=1, j\neq i}^{n} \widetilde{K}_{ij}$, from (30) we get

$$\widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) = (n-1)\frac{c}{4} + \frac{3c}{4}\sum_{a=1}^{3} \| P_a e_i \|^2 \tag{31}$$

Next, in view of $2\widetilde{\tau}\left(T_pM\right) = \sum_{i=1}^{n} \widetilde{\mathrm{Ric}}_{(T_pM)}(e_i)$, from (31) we get

$$2\widetilde{\tau}\left(T_pM\right) = n(n-1)\frac{c}{4} + \frac{3c}{4}\sum_{a=1}^{3} \| P_a \|^2, \tag{32}$$

and consequently, we have (29).

Now, we present the following Theorem and Corollaries, which include Casorati inequalities for submanifolds of quaternionic space forms.

Theorem 1 *Let M be an n-dimensional submanifold of a $4m$-dimensional quaternionic space form $\widetilde{M}(c)$. Then the generalized normalized δ-Casorati curvatures $\delta_{\mathscr{C}}(r; n-1)$ and $\widehat{\delta}_{\mathscr{C}}(r; n-1)$ satisfy*

$$\tau_{\mathrm{Nor}}(p) \leq \frac{[\delta_{\mathscr{C}}(r; n-1)]_p}{n(n-1)} + \frac{c}{4}\left\{1 + \frac{3}{n(n-1)}\sum_{a=1}^{3} \| P_a \|^2\right\}, \quad 0 < r < n(n-1), \tag{33}$$

and

$$\tau_{\mathrm{Nor}}(p) \leq \frac{[\widehat{\delta}_{\mathscr{C}}(r; n-1)]_p}{n(n-1)} + \frac{c}{4}\left\{1 + \frac{3}{n(n-1)}\sum_{a=1}^{3} \| P_a \|^2\right\}, \quad n(n-1) < r, \tag{34}$$

respectively. The equality sign holds in (33) (*resp.* (34)) *if and only if (M, g) is an invariantly quasi-umbilical submanifold, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \ldots, e_{4m}\}$, the shape operators $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, \ldots, 4m\}$, take the following forms:*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \dfrac{n(n-1)}{r}a \end{pmatrix}, \quad A_{n+2} = \cdots = A_{4m} = 0. \tag{35}$$

Proof Let (M, g) be an n-dimensional Riemannian submanifold of a $4m$-dimensional quaternionic space form $\widetilde{M}(c)$. Let the Riemannian vector bundle (B, g_B) over M

be replaced by the normal bundle $T^{\perp}M$, and the B-valued symmetric $(1,2)$-tensor field ζ be replaced by the second fundamental form of immersion σ. In (16), we set

$$T(X,Y,Z,W) = R(X,Y,Z,W) - \widetilde{R}(X,Y,Z,W)$$

with R the Riemann curvature tensor on M and $\zeta = \sigma$. Then we see that

$$(\tau_T)_{\mathrm{Nor}}(p) = \tau_{\mathrm{Nor}}(p) - \widetilde{\tau}_{\mathrm{Nor}}\left(T_pM\right),$$

$$\delta_{\mathscr{C}^{T,\zeta}}(r;n-1) = \delta_{\mathscr{C}}(r;n-1), \quad \widehat{\delta}_{\mathscr{C}^{T,\zeta}}(r;n-1) = \widehat{\delta}_{\mathscr{C}}(r;n-1).$$

Using these facts along with (29) in (25) and (26), we get (33) and (34), respectively.

The conditions of equality cases (27) and (28) become

$$\sigma_{ij}^{\alpha} = 0 \qquad i,j \in \{1,\ldots,n\}, \quad i \neq j \quad \alpha \in \{n+1,\ldots,m\} \tag{36}$$

and

$$\sigma_{11}^{\alpha} = \sigma_{22}^{\alpha} = \cdots = \sigma_{n-1\,n-1}^{\alpha} = \frac{r}{n(n-1)}\sigma_{nn}^{\alpha}, \qquad \alpha \in \{n+1,\ldots,m\}, \tag{37}$$

respectively. Thus the equality sign holds in both the inequalities (33) and (34) if and only if (36) and (37) are true.

The interpretation of the relations (36) is that the shape operators with respect to all normal directions e_α commute, or equivalently, that the normal connection $\nabla^{\perp}$ is flat, or still, that the *normal curvature tensor* $R^{\perp}$, that is, the curvature tensor of the normal connection, is trivial. Furthermore, the interpretation of the relations (37) is that there exist $m-n$ mutually orthogonal unit normal vectors $\{e_{n+1},\ldots,e_m\}$ such that the shape operators with respect to all directions e_α ($\alpha \in \{e_{n+1},\ldots,e_m\}$) have an eigenvalue of multiplicity $n-1$ and that for each e_α the distinguished eigendirection is the same (namely e_n), that is, the submanifold is *invariantly quasi-umbilical* [4].

Thus from the relations (36) and (37), we conclude that the equality holds in (33) and/or (34) for all $p \in M$ if and only if the Riemannian submanifold M is invariantly quasi-umbilical, such that with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operators take the form given by (35).

Corollary 1 *Let M be an n-dimensional submanifold of a $4m$-dimensional quaternionic space form $\widetilde{M}(c)$. Then the normalized δ-Casorati curvature $\delta_{\mathscr{C}}(n-1)$ satisfies*

$$\tau_{\mathrm{Nor}}(p) \leq [\delta_{\mathscr{C}}(n-1)]_p + \frac{c}{4}\left\{1 + \frac{3}{n(n-1)}\sum_{a=1}^{3}\|P_a\|^2\right\}. \tag{38}$$

The equality sign holds in (38) for all $p \in M$ if and only if (M,g) is an invariantly quasi-umbilical submanifold, such that with respect to suitable tangent orthonor-

mal frame $\{e_1, \ldots, e_n\}$ *and normal orthonormal frame* $\{e_{n+1}, \ldots, e_{4m}\}$*, the shape operators* $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, \ldots, 4m\}$*, take the following forms:*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & 2a \end{pmatrix}, \quad A_{n+2} = \cdots = A_{4m} = 0. \tag{39}$$

Proof Using (8) in (33), we get (38). Putting $2r = n(n-1)$ in (35) we get (39).

Corollary 2 *Let* M *be an* n*-dimensional submanifold of a* $4m$*-dimensional quaternionic space form* $\widetilde{M}(c)$*. Then the normalized* δ*-Casorati curvature* $\widehat{\delta}_{\mathscr{C}}(n-1)$ *satisfies*

$$\tau_{\text{Nor}}(p) \leq [\widehat{\delta}_{\mathscr{C}}\,(n-1)]_p + \frac{c}{4}\left\{1 + \frac{3}{n(n-1)}\sum_{a=1}^{3}\|P_a\|^2\right\}. \tag{40}$$

The equality sign holds in (40) *if and only if* (M, g) *is an invariantly quasi-umbilical submanifold, such that with respect to suitable tangent orthonormal frame* $\{e_1, \ldots, e_n\}$ *and normal orthonormal frame* $\{e_{n+1}, \ldots, e_{4m}\}$*, the shape operators* $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, \ldots, 4m\}$*, take the following forms:*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{a}{2} \end{pmatrix}, \quad A_{n+2} = \cdots = A_{4m} = 0. \tag{41}$$

Proof Using (9) in (34), we get (40). Putting $r = 2n(n-1)$ in (35) we get (41).

Corollary 3 *Let* M *be an* n*-dimensional* θ*-slant submanifold of a* $4m$*-dimensional quaternionic space form* $\widetilde{M}(c)$*. Then the generalized normalized* δ*-Casorati curvatures* $\delta_{\mathscr{C}}(r; n-1)$ *and* $\widehat{\delta}_{\mathscr{C}}(r; n-1)$ *satisfy*

$$\tau_{\text{Nor}}(p) \leq \frac{[\delta_{\mathscr{C}}(r; n-1)]_p}{n(n-1)} + \frac{c}{4}\left\{1 + \frac{9}{n-1}\cos^2\theta\right\}, \quad 0 < r < n(n-1), \tag{42}$$

and

$$\tau_{\text{Nor}}(p) \leq \frac{[\widehat{\delta}_{\mathscr{C}}\,(r; n-1)]_p}{n(n-1)} + \frac{c}{4}\left\{1 + \frac{9}{n-1}\cos^2\theta\right\}, \quad n(n-1) < r, \tag{43}$$

respectively. The equality sign holds in (42) (*resp.* (43)) *if and only if* (M, g) *is an invariantly quasi-umbilical submanifold, such that with respect to suitable tangent orthonormal frame* $\{e_1, \ldots, e_n\}$ *and normal orthonormal frame* $\{e_{n+1}, \ldots, e_{4m}\}$*, the shape operators* $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, \ldots, 4m\}$*, take the forms given by* (35).

Proof Using

$$\|P_a\|^2 = n\cos^2\theta, \qquad a \in \{1, 2, 3\}, \tag{44}$$

in (33) and (34) we get (42) and (43), respectively.

Remark 1 The inequality (42) is the inequality (3) of Theorem 3.1 of [28], the inequality (1) of Theorem 2.1 of [27] and the inequality (2) of Theorem 4.1 of [12]. The inequality (43) is the inequality (4) of Theorem 3.1 of [28], the inequality (2) of Theorem 2.1 of [27] and the inequality (3) of Theorem 4.1 of [12]. The Eq. (35) is the Eq. (5) of Theorem 3.1 of [28], the Eq. (3) of Theorem 2.1 of [27] and the inequality (4) of Theorem 4.1 of [12].

Corollary 4 *Let* M *be an* n*-dimensional* θ*-slant submanifold of a* $4m$*-dimensional quaternionic space form* $\widetilde{M}(c)$*. Then the normalized* δ*-Casorati curvature* $\delta_{\mathscr{C}}(n-1)$ *satisfies*

$$\tau_{\mathrm{Nor}}(p) \le [\delta_{\mathscr{C}}(n-1)]_p + \frac{c}{4}\left\{1 + \frac{9}{n-1}\cos^2\theta\right\}. \tag{45}$$

The equality sign holds in (45) *for all* $p \in M$ *if and only if* (M, g) *is an invariantly quasi-umbilical submanifold, such that with respect to suitable tangent orthonormal frame* $\{e_1, \ldots, e_n\}$ *and normal orthonormal frame* $\{e_{n+1}, \ldots, e_{4m}\}$*, the shape operators* $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, \ldots, 4m\}$*, take the forms given by* (39).

Proof Using (8) in (42), we get (45).

Remark 2 The inequality (45) is the inequality (1) of Theorem 1.1 of [32] (or the inequality (18) of Corollary 3.2 of [28]). The Eq. (39) is the Eq. (2) of Theorem 1.1 of [32] (or the Eq. (19) of Corollary 3.2 of [28]).

Corollary 5 *Let* M *be an* n*-dimensional* θ*-slant submanifold of a* $4m$*-dimensional quaternionic space form* $\widetilde{M}(c)$*. Then the normalized* δ*-Casorati curvature* $\widehat{\delta}_{\mathscr{C}}(n-1)$ *satisfies*

$$\tau_{\mathrm{Nor}}(p) \le [\widehat{\delta}_{\mathscr{C}}(n-1)]_p + \frac{c}{4}\left\{1 + \frac{9}{n-1}\cos^2\theta\right\}. \tag{46}$$

The equality sign holds in (46) *if and only if* (M, g) *is an invariantly quasi-umbilical submanifold, such that with respect to suitable tangent orthonormal frame* $\{e_1, \ldots, e_n\}$ *and normal orthonormal frame* $\{e_{n+1}, \ldots, e_{4m}\}$*, the shape operators* $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, \ldots, 4m\}$*, take the forms given by* (41).

Proof Using (9) in (43), we get (46).

Remark 3 The inequality (46) is the inequality (3) of Theorem 1.1 of [32] (or the inequality (20) of Corollary 3.2 of [28]). The Eq. (41) is the Eq. (4) of Theorem 1.1 of [32] (or the Eq. (21) of Corollary 3.2 of [28]).

Remark 4 Using $\theta = \pi/2$ in (42), (43), (45) and (46) (or $P_a = 0$ in (33), (34), (38) and (40)) we get corresponding results for an n-dimensional totally real submanifold of a $4m$-dimensional quaternionic space form $\widetilde{M}(c)$.

Remark 5 Using $\theta = 0$ in (42), (43), (45) and (46) we get corresponding results for an n-dimensional quaternionic submanifold of a $4m$-dimensional quaternionic space form $\widetilde{M}(c)$.

Finally, we list some problems for further studies.

Problem 1 An n-dimensional totally real submanifold M of a $4n$-dimensional quaternionic space form $\widetilde{M}(c)$ is called a Lagrangian submanifold of $\widetilde{M}(c)$. Like the improved Chen–Ricci inequalities [34], to improve Casorati inequalities for Lagrangian submanifolds of a quaternionic space form, if possible.

Problem 2 Like a Kaehler manifold of quasi constant holomorphic sectional curvatures (cf. [2, 17]), to define and study a quaternionic Kaehler manifold of quasi constant quaternionic sectional curvatures and its submanifolds, if possible.

Problem 3 Like a generalized complex space form (cf. [29, 33]), to define and study a generalized quaternionic Kaehler space form and its submanifolds, if possible.

Problem 4 To obtain Casorati inequalities for different kind of submanifolds of complex two plane Grassmannians [3].

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Volume-Preserving Mean Curvature Flow for Tubes in Rank One Symmetric Spaces of Non-compact Type

Naoyuki Koike

Abstract First we investigate the evolutions of the radius function and its gradient along the volume-preserving mean curvature flow starting from a tube (of nonconstant radius) over a closed geodesic ball in an invariant submanifold in a rank one symmetric space of non-compact type, where we impose some boundary condition to the flow and the invariancy of the submanifold means the total geodesicness in the case where the ambient symmetric space is a (real) hyperbolic space. Next, we prove that the tubeness is preserved along the flow in the case where the radius function of the initial tube is radial with respect to the center of the closed geodesic ball. Furthermore, in this case, we prove that the flow reaches to the invariant submanifold or it exists in infinite time and converges to a tube of constant mean curvature over the closed geodesic ball in the C^∞-topology in infinite time.

1 Introduction

Let f_t's ($t \in [0, T)$) be a one-parameter C^∞-family of immersions of an n-dimensional compact manifold M into an $(n+1)$-dimensional Riemannian manifold $\overline{M}$, where T is a positive constant or $T = \infty$. Define a map $\widetilde{f} : M \times [0, T) \to \overline{M}$ by $\widetilde{f}(x, t) = f_t(x)$ $((x, t) \in M \times [0, T))$. Denote by π_M the natural projection of $M \times [0, T)$ onto M. For a vector bundle E over M, denote by $\pi_M^* E$ the induced bundle of E by π_M. Also, denote by H_t, g_t and N_t the mean curvature, the induced metric and the outward unit normal vector of f_t, respectively. Define the function H over $M \times [0, T)$ by $H_{(x,t)} := (H_t)_x$ $((x, t) \in M \times [0, T))$, the section g of $\pi_M^*(T^{(0,2)}M)$ by $g_{(x,t)} := (g_t)_x$ $((x, t) \in M \times [0, T))$ and the section N of $\widetilde{f}^*(T\overline{M})$ by $N_{(x,t)} := (N_t)_x$ $((x, t) \in M \times [0, T))$, where $T^{(0,2)}M$ is the tensor bundle of degree $(0, 2)$ of M and $T\overline{M}$ is the tangent bundle of $\overline{M}$. The average mean curvature $\overline{H}(: [0, T) \to \mathbb{R})$ is defined by

N. Koike (✉)
Faculty of Science, Department of Mathematics, Tokyo University of Science,
1-3 Kagurazaka Shinjuku-ku, Tokyo 162-8601, Japan
e-mail: koike@rs.kagu.tus.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_17

$$\overline{H}_t := \frac{\int_M H_t dv_{g_t}}{\int_M dv_{g_t}}, \tag{1.1}$$

where dv_{g_t} is the volume element of g_t. The flow f_t's ($t \in [0, T)$) is called a *volume-preserving mean curvature flow* if it satisfies

$$\widetilde{f}_* \left(\frac{\partial}{\partial t} \right) = (\overline{H} - H)N. \tag{1.2}$$

In particular, if f_t's are embeddings, then we call $M_t := f_t(M)$'s ($0 \in [0, T)$) rather than f_t's ($0 \in [0, T)$) a volume-preserving mean curvature flow. Note that, if M has no boundary and if f is an embedding, then, along this flow, the volume of (M, g_t) decreases but the volume of the domain D_t surrounded by $f_t(M)$ is preserved invariantly.

First we shall recall the result by M. Athanassenas [1, 2]. Let P_i ($i = 1, 2$) be affine hyperplanes in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ meeting an affine line l orthogonally and E a closed domain of $\mathbb{R}^{n+1}$ with $\partial E = P_1 \cup P_2$. Also, let M be a hypersurface of revolution in $\mathbb{R}^{n+1}$ such that $M \subset E$, $\partial M \subset P_1 \cup P_2$ and that M meets P_1 and P_2 orthogonally. Let D be the closed domain surrounded by P_1, P_2 and M, and d the distance between P_1 and P_2. She [1, 2] proved the following fact.

Known fact. Let M_t ($t \in [0, T)$) be the volume-preserving mean curvature flow starting from M such that M_t meets P_1 and P_2 orthogonally for all $t \in [0, T)$. Then the following statements (i) and (ii) hold:

(i) M_t ($t \in [0, T)$) remain to be hypersurfaces of revolution.

(ii) If $\mathrm{Vol}(M) \leq \frac{\mathrm{Vol}(D)}{d}$ holds, then $T = \infty$ and as $t \to \infty$, the flow M_t converges to the cylinder C such that the volume of the closed domain surrounded by P_1, P_2 and C is equal to $\mathrm{Vol}(D)$.

E. Cabezas–Rivas and V. Miquel [3–5] proved the similar result in certain kinds of rotationally symmetric spaces. Let $\overline{M}$ be an $(n+1)$-dimensional rotationally symmetric space (i.e., $SO(n)$ acts on $\overline{M}$ isometrically and its fixed point set is a one-dimensional submanifold). Note that real space forms are rotationally symmetric spaces.

A *symmetric space of compact type* (resp. *non-compact type*) is a naturally reductive Riemannian homogeneous space $\overline{M}$ such that, for each point p of $\overline{M}$, there exists an isometry of $\overline{M}$ having p as an isolated fixed point and that the isometry group of $\overline{M}$ is a semi-simple Lie group each of whose irreducible factors is compact (resp. not compact) (see [6]). Note that symmetric spaces of compact type other than a sphere and symmetric spaces of non-compact type other than a (real) hyperbolic space are not rotationally symmetric.

In this paper, we shall derive results similar to those of M. Athanassenas [1, 2] and E. Cabezas–Rivas and V. Miquel [3, 4] in rank one symmetric spaces of non-compact type. The setting in this paper is as follows.

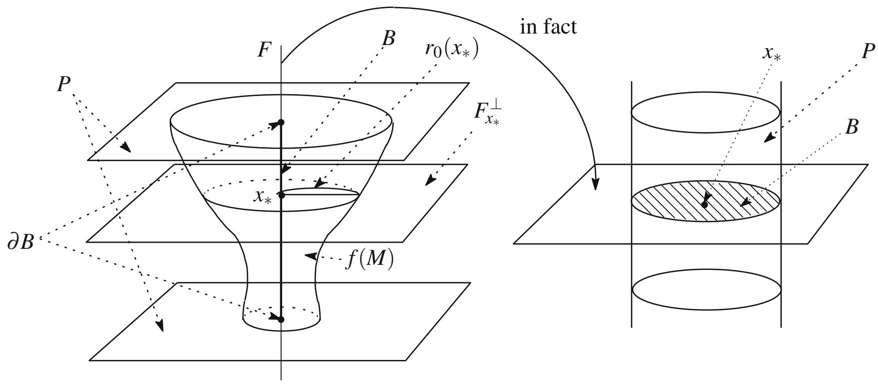

Fig. 1 Setting (S)

Setting (S). Let F be an invariant submanifold in an $(n+1)$-dimensional rank one symmetric space $\overline{M}$ of non-compact type (i.e., $\overline{M} = \mathbb{R}H^{n+1}$, $\mathbb{C}H^{\frac{n+1}{2}}$, $\mathbb{Q}H^{\frac{n+1}{4}}$ or, $\mathbb{O}H^2$ $(n=7)$) and B be the closed geodesic ball of radius r_B centered at some point $x_*(\in F)$ in F, where the invariancy of F means the total geodesicness in the case where $\overline{M} = \mathbb{R}H^{n+1}$. Set $P := \underset{x\in\partial B}{\cup} F^\perp_x$ and denote by E the closed domain in $\overline{M}$ surrounded by P. Let $M := t_{r_0}(B)$ and $f := \exp^\perp|_{t_{r_0}(B)}$, where r_0 is a non-constant positive C^∞-function over B such that $\mathrm{grad}\, r_0 = 0$ holds along ∂B. Denote by D the closed domain surrounded by P and $f(M)$. See Fig. 1 about this setting.

The above setting (S) includes the setting in [3, 4]. Under the above setting (S), we consider the volume-preserving mean curvature flow f_t $(t \in [0, T))$ starting from f and satisfying the following boundary condition:

(B) $\mathrm{grad}\, r_t = 0$ holds along ∂B for all $t \in [0, T)$, where r_t is the radius function of $M_t := f_t(M)$ (i.e., $M_t = \exp^\perp(t_{r_t}(B))$) ($r_t$ is possible to be multi-valued),

It is shown that there uniquely exists the volume-preserving mean curvature flow $f_t : M \hookrightarrow \overline{M}$ starting from f as in the above setting (S) and satisfying the boundary condition (B) in short time (see Proposition 2.2). Under these assumptions, we can derive the evolution equations for the radius functions of the flow and some quantities related to the gradients of the functions (see Sects. 2 and 3). We obtain the following preservability theorem for the tubeness along the flow by using the evolution equations.

Theorem A ([7]) *Let f be as in the above setting (S) and f_t $(t \in [0, T))$ the volume-preserving mean curvature flow starting from f and satisfying the boundary condition (B). If r_0 is radial with respect to x_* (i.e., r_0 is constant along each geodesic sphere centered at x_* in F), then M_t $(t \in [0, T))$ remain to be tubes over B such that the volume of the closed domain surrounded by M_t and P is equal to* $\mathrm{Vol}(D)$.

Furthermore, we obtain the following results.

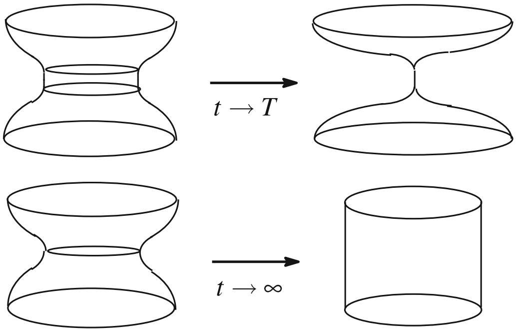

Fig. 2 Blowing up and convergence of the volume-preserving mean curvature flow

Theorem B ([7]) *Under the hypothesis of Theorem A, one of the following statements* (a) *and* (b) *holds:*

(a) $M_t := f_t(M)$ *reaches* B *as* $t \to T$*,*

(b) $T = \infty$ *and* M_t *converges to a tube of constant mean curvature over* B *(in* C^∞*-topology) as* $t \to \infty$*.*

Theorem C ([7]) *Under the hypothesis of Theorem A, assume that*

$$\mathrm{Vol}(M_0) \leq v_{m_F-1} v_{m^V} (\delta_2 \circ \delta_1^{-1}) \left(\frac{\mathrm{Vol}(D)}{v_{m^V} \mathrm{Vol}(B)} \right),$$

where $m_F := \dim F$*,* $m^V := \operatorname{codim} F - 1$*,* v_{m_F-1} *(resp.* v_{m^V}*) is the volume of the* $m_F - 1$ *sl (resp.* m^V*)-dimensional Euclidean unit sphere and* δ_i $(i = 1, 2)$ *are increasing functions over* $\mathbb{R}$ *explicitly described (see Sect. 4). Then* $T = \infty$ *and* M_t *converges to a tube of constant mean curvature over* B *(in* C^∞*-topology) as* $t \to \infty$*.*

Remark 1.1 Let $\overline{g}$ be the metric of $\overline{M}$ and c a positive constant. As $c \to \infty$, $c\overline{g}$ approaches to a flat metric and δ_i $(i = 1, 2)$ approaches to the identity transformation of $[0, \infty)$ and hence $v_{m^V} (\delta_2 \circ \delta_1^{-1}) \left(\frac{\mathrm{Vol}(D)}{v_{m^V} \mathrm{Vol}(B)} \right)$ approaches to $\frac{\mathrm{Vol}(D)}{\mathrm{Vol}(B)}$. Thus, as $c \to \infty$, the condition $\mathrm{Vol}(M_0) \leq v_{m_F-1} v_{m^V} (\delta_2 \circ \delta_1^{-1}) \left(\frac{\mathrm{Vol}(D)}{v_{m^V} \mathrm{Vol}(B)} \right)$ approaches to the condition $\mathrm{Vol}(M) \leq \frac{\mathrm{Vol}(D)}{d}$ in the statement (ii) of Known Fact in the case of $\dim F = 1$ (Fig. 2).

2 The Evolution of the Radius Function

We use the notations in Introduction. Denote by κ the maximum sectional curvature of $\overline{M}$. Then we have $\mathrm{Sec}(\mathrm{Gr}_2(T\overline{M})) = [4\kappa, \kappa]$ in the case where $\overline{M}$ is other than a (real) hyperbolic space, where $\mathrm{Gr}_2(T\overline{M})$ is the Grassmann bundle of 2-planes of $\overline{M}$ and Sec is the sectional curvature function of $\overline{M}$. Let F, B, $M = t_{r_0}(B)$ and f be as in Setting (S) of Introduction. Set $m_F := \dim F$, $m_1^V := n - m_F - m_2^V$ and $m_2^V := 0$ (when $\overline{M} = \mathbb{R}H^{n+1}$), 1 (when $\overline{M} = \mathbb{C}H^{\frac{n+1}{2}}$), 3 (when $\overline{M} = \mathbb{Q}H^{\frac{n+1}{4}}$ or 7 (when $\overline{M} = \mathbb{O}H^2$ $(n = 7)$). Assume that the volume-preserving mean curvature flow f_t $(t \in [0, T))$ starting from f and satisfying the boundary condition (B). Denote by $S^\perp B$ the unit normal bundle of of B and $S_x^\perp B$ the fibre of this bundle over $x \in B$. Define a positive-valued function $\widehat{r_t} : M \to \mathbb{R}$ $(t \in [0, T))$ and a map $w_t^1 : M \to S^\perp B$ $(t \in [0, T))$ by $f_t(\xi) = \exp^\perp(\widehat{r_t}(\xi) w_t^1(\xi))$ $(\xi \in M)$. Also, define a map $c_t : M \to B$ by $c_t(\xi) := \pi(w_t^1(\xi))$ $(\xi \in M)$ and a map $w_t : M \to T^\perp B$ $(t \in [0, T))$ by $w_t(\xi) := \widehat{r_t}(\xi) w_t^1(\xi)$ $(\xi \in M)$. Here we note that c_t is surjective by the boundary condition in Theorem A, $\widehat{r_0}(\xi) = r_0(\pi(\xi))$ and that $c_0(\xi) = \pi(\xi)$ $(\xi \in M)$. Define a function $\bar{r}_t$ over B by $\bar{r}_t(x) := \widehat{r_t}(\xi)$ $(x \in B)$ and a map $\bar{c}_t : B \to B$ by $\bar{c}_t(x) := c_t(\xi)$ $(x \in B)$, where ξ is an arbitrary element of $M \cap S_x^\perp B$. It is clear that they are well-defined. This map $\bar{c}_t$ is not necessarily a diffeomorphism. In particular, if $\bar{c}_t$ is a diffeomorphism, then $M_t := f_t(M)$ is equal to the tube $\exp^\perp(t_{r_t}(B))$, where $r_t := \bar{r}_t \circ \bar{c}_t^{-1}$. It is easy to show that, if $c_t(\xi_1) = c_t(\xi_2)$, then $\widehat{r_t}(\xi_1) = \widehat{r_t}(\xi_2)$ and $\pi(\xi_1) = \pi(\xi_2)$ hold. In this section, we shall calculate the evolution equations for the functions r_t and $\widehat{r_t}$. Define $r : B \times [0, T) \to \mathbb{R}$, $w^1 : M \times [0, T) \to M$ and $c : M \times [0, T) \to B$ by $r(x, t) := r_t(x)$, $w^1(\xi, t) := w_t^1(\xi)$, $w(\xi, t) := w_t(\xi)$ and $c(\xi, t) := c_t(\xi)$, respectively, where $\xi \in M$, $x \in B$ and $t \in [0, T)$ (see Fig. 3).

Set $T_1 := \sup\{t' \in [0, T) \mid M_t := f_t(M)\ (0 \le t \le t') : \text{tubes over } B\}$. Note that $\bar{c}_t$ $(0 \le t < T_1)$ are diffeomorphisms. Then we can derive the following evolution equation for r_t.

Lemma 2.1 ([7]) *The radius functions r_t's satisfies the following evolution equation:*

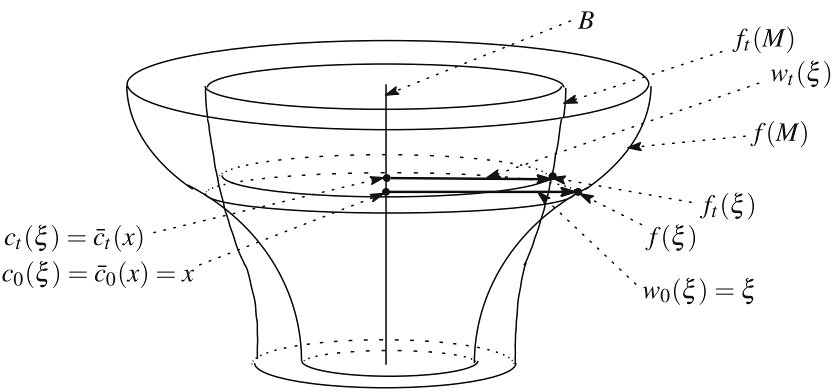

Fig. 3 The definitions of w_t and c_t

$$\begin{aligned}
&\frac{\partial r}{\partial t}(x,t) - \frac{(\triangle_F r_t)(x)}{\cosh^2(\sqrt{-\kappa} r_t(x))} \\
&= \frac{\sqrt{\cosh^2(\sqrt{-\kappa} r_t(x)) + ||(\mathrm{grad}\, r_t)_x||^2}}{\cosh(\sqrt{-\kappa} r_t(x))} \cdot (\overline{H}_t - \rho_{r_t}(x)) \\
&- \frac{(\nabla^F dr_t)_x((\mathrm{grad}\, r_t)_x, (\mathrm{grad}\, r_t)_x)}{\cosh^2(\sqrt{-\kappa} r_t(x))(\cosh^2(\sqrt{-\kappa} r_t(x)) + ||(\mathrm{grad}\, r_t)_x||^2)}
\end{aligned} \tag{2.2}$$

$((x,t) \in M \times [0, T_1))$, *where* ρ_{r_t} *is some function defined in terms of* $\sqrt{-\kappa}$ *and* r_t.

Replacing $\overline{H}$ in (2.2) to any $C^{1,\alpha/2}$ real-valued function ϕ such that $\phi(0) = \overline{H}(0)$, we obtain a parabolic equation, which has a unique solution r_t such that $\mathrm{grad}\, r_t = 0$ along ∂B in short time for any initial data r_0 such that $\mathrm{grad}\, r_0 = 0$ holds along ∂B. By using a routine fixed point argument (see [8]), we can establish the short time existence and uniqueness also for (2.2) with the same boundary condition. From this fact, we can derive the following statement.

Proposition 2.2 ([7]) *Under Setting (S), there uniquely exists the volume-preserving mean curvature flow* $f_t : M \hookrightarrow \overline{M}$ *starting from* f *and satisfying the boundary condition (B) in short time.*

3 The Evolution of the Gradient of the Radius Function

We use the notations in Introduction and Sects. 1 and 2. Let T_1 be as in Sect. 2. Define a function $\widehat{u}_t : M \to \mathbb{R}$ $(t \in [0, T_1))$ by

$$\widehat{u}_t(\xi) := \bar{g}(N_{(\xi,t)}, \tau_{\gamma_{w(\xi,t)}|_{[0,1]}}(w^1(\xi,t))) \quad (\xi \in M)$$

and a map $\widehat{v}_t : M \to \mathbb{R}$ by $\widehat{v}_t := \frac{1}{\widehat{u}_t}$ $(0 \le t < T_1)$. Define a map $\widehat{u} : M \times [0, T_1) \to \mathbb{R}$ by $\widehat{u}(\xi, t) := \widehat{u}_t(\xi)$ $((\xi,t) \in M \times [0, T_1))$ and a map $\widehat{v} : M \times [0, T_1) \to \mathbb{R}$ by $\widehat{v}(\xi, t) := \widehat{v}_t(\xi)$ $((\xi,t) \in M \times [0, T_1))$. Define a function $\bar{u}_t$ (resp. $\bar{v}_t$) over B by $\bar{u}_t(x) := \widehat{u}_t(\xi)$ $(x \in B)$ (resp. $\bar{v}_t(x) := \widehat{v}_t(\xi)$ $(x \in B)$, where ξ is an arbitrary element of $M \cap S^\perp_x B$. It is clear that these functions are well-defined. Set $u_t := \bar{u}_t \circ \bar{c}_t^{-1}$ and $v_t := \bar{v}_t \circ \bar{c}_t^{-1}$. We have only to show $\inf_{(x,t)\in B\times[0,T_1)} u(x,t) > 0$, that is, $\sup_{(x,t)\in B\times[0,T_1)} v(x,t) < \infty$. In the sequel, assume that $t < T_1$. Then we can show

$$\widehat{v}_t(\xi) = \frac{1}{\cosh(\sqrt{-\kappa}\widehat{r}(\xi,t))} \cdot \sqrt{\cosh^2(\sqrt{-\kappa}\widehat{r}(\xi,t)) + ||(\mathrm{grad}\, r_t)_{c(\xi,t)}||^2}. \tag{3.1}$$

In order to investigate te evolution of the gradient $\mathrm{grad}\, r_t$ of the radius function r_t, we suffice to investigate the evolution of $\widehat{v}_t$.

We can derive the following evolution equation for $\widehat{v}_t$.

Lemma 3.1 ([7]) *The functions $\widehat{v}_t$'s ($t \in [0,T)$) satisfy the following evolution equation:*

$$\begin{aligned}
&\frac{\partial \widehat{v}}{\partial t}(\xi,t) - (\triangle_t \widehat{v}_t)(\xi) \\
= & -\overline{H}_t \sqrt{-\kappa}\tanh(\sqrt{-\kappa}\widehat{r}(\xi,t))(\widehat{v}(\xi,t)^2 - 1) \\
& + \widehat{v}(\xi,t)\left(1 - \frac{1}{\widehat{v}(\xi,t)^2}\right)\sum_{k=1}^{2}\frac{m_k^V (k\sqrt{-\kappa})^2}{\sinh^2(k\sqrt{-\kappa}\widehat{r}(\xi,t))} \\
& + \widehat{v}(\xi,t)\left(1 - \frac{1}{\widehat{v}(\xi,t)^2}\right)\sqrt{-\kappa}\tanh(\sqrt{-\kappa}\widehat{r}(\xi,t)) \\
& \quad \times \sum_{k=1}^{2}\frac{m_k^V k\sqrt{-\kappa}}{\tanh(k\sqrt{-\kappa}\widehat{r}(\xi,t))} \\
& - m_F \widehat{v}(\xi,t)\left(1 - \frac{1}{\widehat{v}(\xi,t)^2}\right)(\sqrt{-\kappa})^2\tanh^2(\sqrt{-\kappa}\widehat{r}(\xi,t)) \\
& - \widehat{v}(\xi,t)\left(\lambda_t(\xi) + \frac{m_F}{\widehat{v}(\xi,t)}\sqrt{-\kappa}\tanh(\sqrt{-\kappa}\widehat{r}(\xi,t))\right)^2 \\
& - m_F \widehat{v}(\xi,t)\left(1 - \frac{1}{\widehat{v}(\xi,t)^2}\right)\frac{(\sqrt{-\kappa})^2 ||(\mathrm{grad}\, r_t)_{c(\xi,t)}||}{\cosh^2(\sqrt{-\kappa}\widehat{r}(\xi,t))} \\
& - \frac{2}{\widehat{v}(\xi,t)}||(\mathrm{grad}_t \widehat{v}_t)_\xi||^2 \qquad ((\xi,t) \in M \times [0,T_1)).
\end{aligned} \tag{3.2}$$

4 Estimate of the Volume

We use the notations in Introduction and Sects. 1, 2 and 3. Assume that r_0 is radial with respect to x_*. Then it is easy to show that so are also r_t. For $X \in \widetilde{S}'(x_*, 1)$, denote by γ_X the geodesic in F having X as the initial velocity vector (i.e., $\gamma_X(z) = \exp^F_{x_*}(zX)$). Then, since r_t is radial, it is described as $r_t(\gamma_X(z)) = r_t^\circ(z)$ ($X \in \widetilde{S}'(x_*, 1)$, $z \in [0, r_B)$) for some function r_t° over $[0, r_B)$, where $\widetilde{S}'(x_*, 1)$ denotes the unit sphere in $T_{x^*}F$ centered $\mathbf{0}$. Denote by ∇^t the Levi-Civita connection of g_t. In the sequel, assume that $t < T_1$. Define a function $\overline{\psi}$ over $[0, \infty)$ by

$$\overline{\psi}(s) := \left(\mathop{\Pi}_{k=1}^{2}\left(\frac{\sinh(k\sqrt{-\kappa}s)}{k\sqrt{-\kappa}}\right)^{m_k^V}\right)\cosh^{m_F}(\sqrt{-\kappa}s).$$

Note that $\overline{\psi}$ is positive. Since r_t is described as above by the radiality of r_t, the volume $\mathrm{Vol}(D_t)$ is described as

$$\mathrm{Vol}(D_t) = v_{m^V}\int_{x\in B}\left(\int_0^{r_t(x)} s^{m^V}\overline{\psi}(s)ds\right)dv_F. \tag{4.1}$$

Define a function δ_1 and δ_2 over $[0, \infty)$ by

$$\delta_1(s) := \int_0^s s^{m^V}\overline{\psi}(s)ds \quad \text{and} \quad \delta_2(s) := \int_0^s \frac{s^{m^V}\overline{\psi}(s)}{\cosh(\sqrt{-\kappa}s)}ds,$$

respectively. According to (4.1), we have $\frac{\mathrm{Vol}(D_t)}{v_{m^V}\mathrm{Vol}(B)} \in \delta_1([0, \infty))$. Note that δ_i $(i = 1, 2)$ are increasing. Set $\hat{r}_1 := \delta_1^{-1}\left(\frac{\mathrm{Vol}(D)}{v_{m^V}\mathrm{Vol}(B)}\right)$. Denote by $(r_t)_{\max}$ (resp. $(r_t)_{\min}$) the maximum (resp. the minimum) of r_t.

We can estimate the volume of M_t from below as follows:

$$\begin{aligned}\mathrm{Vol}(M_t) &\geq v_{m^V} v_{m_F-1}\int_{(r_t)_{\min}}^{(r_t)_{\max}} \frac{s^{m^V}\overline{\psi}(s)}{\cosh(\sqrt{-\kappa}s)}ds \\ &= v_{m^V} v_{m_F-1}(\delta_2((r_t)_{\max}) - \delta_2((r_t)_{\min})).\end{aligned} \tag{4.2}$$

For uniform boundedness of the average mean curvatures $|\overline{H}|$, we can derive the following result.

Lemma 4.1 *If* $0 < a_1 \leq r_t(x) \leq a_2 < r_F$ *holds for all* $(x, t) \in M \times [0, T_0]$ *(*$T_0 < T_1$*), then* $\max_{t\in[0,T_0]} \overline{H}_t \leq C(a_1, a_2)$ *holds for some constant* $C(a_1, a_2)$ *depending only on* a_1 *and* a_2.

For uniform positivity of the average mean curvatures $|\overline{H}|$*, we can derive the following result.*

Lemma 4.2 *Assume that* $\overline{M}$ *is of non-compact type. If* $r_t(x) \geq a > 0$ *holds for all* $(x, t) \in M \times [0, T_0]\,(T_0 < T_1)$*, then* $\min_{t\in[0,T_0]} \overline{H}_t \geq \widehat{C}(a)$ *holds for some constant* $\widehat{C}(a)$ *depending only on* a.

5 Proof of Theorems A, B and C

In this section, we shall state the outline of the proofs of Theorems A, B and C. We use the notations in Introduction and Sects. 1, 2, 3 and 4.

Proof of Theorem A. Suppose that $T_1 < T$. Take any $t_0 \in (T_1, T)$. Set

$$\beta_1(t_0) := \min_{(x,t)\in B\times[0,t_0]} r_t(x) \ (> 0) \quad \text{and} \quad \beta_2(t_0) := \max_{(x,t)\in B\times[0,t_0]} r_t(x) \ (< \infty).$$

According to Lemmas 4.1 and 4.2, we have

$$0 < \widehat{C}(\beta_1(t_0)) < \overline{H}_t < C(\beta_1(t_0), \beta_2(t_0)) \qquad (t \in [0, T_1)).$$

By using Lemma 3.1, we can derive

$$\frac{\partial \widehat{v}}{\partial t}(\xi,t)-(\triangle_t\widehat{v}_t)(\xi)\leq \widehat{v}(\xi,t)K_1(\beta_1(t_0),\beta_2(t_0))-\widehat{v}(\xi,t)^2K_2(\beta_1(t_0)) \tag{5.1}$$

($t\in[0,T_1)$), where $K_1(\beta_1(t_0),\beta_2(t_0))$ and $K_2(\beta_1(t_0))$ are defined by

$$K_1(\beta_1(t_0),\beta_2(t_0)):=\left(C(\beta_1(t_0),\beta_2(t_0))+m_F b\tanh(b\widehat{r}(\xi,t))+\sum_{k=1}^{2}\frac{m_k^V kb}{\tanh(kb\widehat{r}(\xi,t))}\right)\times b\tanh(b\widehat{r}(\xi,t))$$

and

$$K_2(\beta_1(t_0)):=\widehat{C}(\beta_1(t_0))b\tanh(b\widehat{r}(\xi,t)).$$

Take any $t_1\in[0,T_1)$. Let $(\xi_2,t_2)\in M\times[0,t_1]$ be a point attaining the maximum of $\widehat{v}$ over $M\times[0,t_1]$. Since $\widehat{v}_{t_2}=1$ along ∂M, (ξ_2,t_2) belongs to the interior of $M\times[0,t_1]$. Then we have $\frac{\partial\widehat{v}}{\partial t}(\xi_2,t_2)=0$ and $(\triangle_{t_2}\widehat{v}_{t_2})(\xi_2)\leq 0$, that is, $\frac{\partial\widehat{v}}{\partial t}(\xi_2,t_2)-(\triangle_{t_2}\widehat{v}_{t_2})(\xi_2)\geq 0$. Hence, from (5.1), we can derive

$$\max_{(\xi,t)\in M\times[0,t_1]}\widehat{v}(\xi,t)=\widehat{v}(\xi_2,t_2)\leq\frac{K_1(\beta_1(t_0),\beta_2(t_0))}{K_2(\beta_1(t_0))}.$$

From the arbitrariness of t_1, we obtain

$$\sup_{(\xi,t)\in M\times[0,T_1)}\widehat{v}(\xi,t)\leq\frac{K_1(\beta_1(t_0),\beta_2(t_0))}{K_2(\beta_1(t_0))},$$

which implies that M_t's ($t\in[T_1,T_1+\varepsilon)$) are tubes over B for a sufficiently small positive number ε. This contradicts the definition of T_1. Therefore we obtain $T_1=T$. That is, M_t ($t\in[0,T)$) remain to be tubes over B. q.e.d.

Outline of proof of Theorem B. Define a function φ over $[0,1/\sqrt{\kappa})$ by $\varphi(s):=\frac{s^2}{1-\kappa s^2}$ and a function Φ_t over M by $\Phi_t(\xi):=(\varphi\circ\widehat{v}_t)(\xi)||(A_t)_\xi||_t^2$ $(\xi\in M)$, where $\kappa:=\frac{1}{2\sup_{(\xi,t)\in M\times[0,T)}\widehat{v}(\xi,t)^2}$. Also, define a function Φ over $M\times[0,T)$ by $\Phi(\xi,t):=\Phi_t(\xi)$ ($(\xi,t)\in M\times[0,T)$). By using Lemmas 3.1 and 4.1, we can derive $\sup\limits_{t\in[0,T)}\max\limits_{M}\Phi_t<\infty$. By using this uniform boundedness, we can derive $\sup\limits_{t\in[0,T)}\max\limits_{M}||A_t||^2<\infty$. Furthermore, by using this uniform boundedness, we can derive $\sup_{t\in[0,T)}\max_M||(\nabla^t)^kA_t||^2<\infty$. By using this fact, we can derive that $T=\infty$ and M_t converges to a tube of constant mean curvature over B (in C^∞-topology) as $t\to\infty$. q.e.d.

Proof of Theorem C Suppose that M_t reaches to B. Then, by using (4.2), we can derive

$$\mathrm{Vol}(M_0) \geq v_{m^V} v_{m_F - 1}(\delta_2 \circ \delta_1^{-1}) \left(\frac{\mathrm{Vol}(D)}{v_{m^V} \mathrm{Vol}(B)} \right).$$

Thus the statement of Theorem C is derived. q.e.d.

Acknowledgements The author is grateful to JSPS for support (Grant-in-Aid for Science Research (C), no.25400076).

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A Duality Between Compact Symmetric Triads and Semisimple Pseudo-Riemannian Symmetric Pairs with Applications to Geometry of Hermann Type Actions

Kurando Baba, Osamu Ikawa and Atsumu Sasaki

Abstract This is a survey paper of not-yet-published papers listed in the reference as [1–3]. We introduce the notion of a duality between commutative compact symmetric triads and semisimple pseudo-Riemannian symmetric pairs, which is a generalization of the duality between compact/noncompact Riemannian symmetric pairs. As its application, we give an alternative proof for Berger's classification of semisimple pseudo-Riemannian symmetric pairs from the viewpoint of compact symmetric triads. More precisely, we give an explicit description of a one-to-one correspondence between commutative compact symmetric triads and semisimple pseudo-Riemannian symmetric pairs by using the theory of symmetric triads introduced by the second author. We also study the action of a symmetric subgroup of G on a pseudo-Riemannian symmetric space G/H, which is called a Hermann type action. For more details, see [1–3].

K. Baba (✉)
Faculty of Science and Technology, Tokyo University of Science,
Noda, Chiba 278-8510, Japan
e-mail: baba_kurando@ma.noda.tus.ac.jp

O. Ikawa
Faculty of Arts and Sciences, Kyoto Institute of Technology, Matsugasaki, Sakyoku, Kyoto 606-8585, Japan
e-mail: ikawa@kit.ac.jp

A. Sasaki
Faculty of Science, Department of Mathematics, Tokai University, 4-1-1, Kitakaname, Hiratsuka-shi, Kanagawa 259-1292, Japan
e-mail: atsumu@tokai-u.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_18

1 A Generalized Duality

1.1 Basic Notion

Let $\mathfrak{g}_u$ be a semisimple compact Lie algebra, and θ_1, θ_2 be involutions on $\mathfrak{g}_u$. The triplet $(\mathfrak{g}_u, \theta_1, \theta_2)$ is called a *semisimple compact symmetric triad*. We say that $(\mathfrak{g}_u, \theta_1, \theta_2)$ is *commutative* if $\theta_1\theta_2 = \theta_2\theta_1$ holds. Denote by $\mathscr{A}$ the set of all commutative semisimple compact symmetric triads. We define an equivalence relation $\equiv$ on $\mathscr{A}$ as follows: For two semisimple compact symmetric triads $(\mathfrak{g}_u, \theta_1, \theta_2)$ and $(\mathfrak{g}'_u, \theta'_1, \theta'_2)$, the relation $(\mathfrak{g}_u, \theta_1, \theta_2) \equiv (\mathfrak{g}'_u, \theta'_1, \theta'_2)$ holds, if there exists a Lie algebra isomorphism $\varphi : \mathfrak{g}_u \to \mathfrak{g}'_u$ satisfying $\theta'_i = \varphi\theta_i\varphi^{-1}$ for $i = 1, 2$. We regard a Riemannian symmetric pair $(\mathfrak{g}_u, \theta)$ of compact type as a commutative semisimple compact symmetric triad $(\mathfrak{g}_u, \theta, \theta)$. Let $\mathfrak{g}$ be a semisimple Lie algebra, and σ be an involution on $\mathfrak{g}$. The pair $(\mathfrak{g}, \sigma)$ is called a *semisimple pseudo-Riemannian symmetric pair*. Denote by $\mathscr{B}$ the set of all semisimple pseudo-Riemannian symmetric pairs. We define an equivalent relation $\equiv$ on $\mathscr{B}$ as follows: For two semisimple pseudo-Riemannian symmetric pairs $(\mathfrak{g}, \sigma)$ and $(\mathfrak{g}', \sigma')$, the relation $(\mathfrak{g}, \sigma) \equiv (\mathfrak{g}', \sigma')$ holds, if there exists a Lie algebra isomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ satisfying $\sigma' = \varphi\sigma\varphi^{-1}$. We note that a pseudo-Riemannian symmetric pair $(\mathfrak{g}, \sigma)$ is a Riemannian symmetric pair of noncompact type, if σ is a non-trivial Cartan involution of $\mathfrak{g}$.

Notation 1. Throughout this paper, we denote by $\mathfrak{l}^f$ the fixed point subset of a set $\mathfrak{l}$ for a map $f : \mathfrak{l} \to \mathfrak{l}$.

1.2 A Generalized Duality

1.2.1 Construction of a Map $\Phi : \mathscr{A} \to \mathscr{B}$

In this subsection, we give a map $\Phi : \mathscr{A} \to \mathscr{B}$. Let $(\mathfrak{g}_u, \theta_1, \theta_2)$ be a commutative semisimple compact symmetric triad. Denote by $\mathfrak{g}_u^{\mathbb{C}}$ the complexification of $\mathfrak{g}_u$. We set

$$\mathfrak{g} = \mathfrak{g}_u^{\theta_1} \oplus \sqrt{-1}\mathfrak{g}_u^{-\theta_1} (\subset \mathfrak{g}_u^{\mathbb{C}}).$$

Then $\mathfrak{g}$ is a noncompact real form of $\mathfrak{g}_u^{\mathbb{C}}$. We extend θ_1, θ_2 to $\mathbb{C}$-linear involutions on $\mathfrak{g}_u^{\mathbb{C}}$, denoted by the same symbols θ_1 and θ_2, respectively. Since θ_2 commutes with θ_1, we have $\theta_2(\mathfrak{g}) = \mathfrak{g}$. Therefore $(\mathfrak{g}, \sigma := \theta_2)$ is a semisimple pseudo-Riemannian symmetric pair. From above argument the following map is defined:

$$\Phi : \mathscr{A} \to \mathscr{B}; (\mathfrak{g}_u, \theta_1, \theta_2) \mapsto (\mathfrak{g}, \sigma). \tag{1}$$

Here, we note that θ_1 gives a Cartan involution of $\mathfrak{g}$ commuting with $\sigma (= \theta_2)$.

1.2.2 Construction of a Map $\Psi : \mathscr{B} \to \mathscr{A}$

In this subsection, we give a map $\Psi : \mathscr{B} \to \mathscr{A}$. Let $(\mathfrak{g}, \sigma)$ be a semisimple pseudo-Riemannian symmetric pair, and θ be a Cartan involution of $\mathfrak{g}$ commuting with σ (see [4] for the existence of such a Cartan involution). Then a semisimple compact real form $\mathfrak{g}_u$ of $\mathfrak{g}^{\mathbb{C}}$ is given by

$$\mathfrak{g}_u = \mathfrak{g}^{\theta} \oplus \sqrt{-1}\mathfrak{g}^{-\theta} (\subset \mathfrak{g}^{\mathbb{C}}).$$

We extend θ, σ to $\mathbb{C}$-linear involutions on $\mathfrak{g}^{\mathbb{C}}$, denoted by the same symbols θ and σ, respectively. Therefore $(\mathfrak{g}_u, \theta_1 := \theta, \theta_2 := \sigma)$ is a commutative semisimple compact symmetric triad. From above argument the following map is defined:

$$\Psi = \Psi_\theta : \mathscr{B} \to \mathscr{A}; (\mathfrak{g}, \sigma) \mapsto (\mathfrak{g}_u, \theta_1, \theta_2). \tag{2}$$

1.2.3 Induced Maps $\widetilde{\Phi}$ and $\widetilde{\Psi}$

In this subsection, we give a one-to-one correspondence between $\mathscr{A}/_\equiv$ and $\mathscr{B}/_\equiv$. First, the map Φ defined by (1) induces the map $\widetilde{\Phi}$ from $\mathscr{A}/_\equiv$ to $\mathscr{B}/_\equiv$. Indeed, we can prove that $\widetilde{\Phi}$ is well-defined as follows: Suppose that $(\mathfrak{g}_u, \theta_1, \theta_2) \equiv (\mathfrak{g}'_u, \theta'_1, \theta'_2)$. Then there exists a Lie algebra isomorphism $\varphi : \mathfrak{g}_u \to \mathfrak{g}'_u$ satisfying $\theta'_i = \varphi\theta_i\varphi^{-1}$ for $i = 1, 2$. This implies that $(\mathfrak{g}'_u)^{\pm\theta'_i} = \varphi(\mathfrak{g}_u^{\pm\theta_i})$ holds for each $i = 1, 2$. Therefore we have $\Phi(\mathfrak{g}'_u, \theta'_1, \theta'_2) \equiv \Phi(\mathfrak{g}_u, \theta_1, \theta_2)$.

Next, the map $\Psi = \Psi_\theta$ defined by (2) induces the map $\widetilde{\Psi}$ from $\mathscr{B}/_\equiv$ to $\mathscr{A}/_\equiv$. By a similar argument for the definition of $\widetilde{\Phi}$ we can prove that $\widetilde{\Psi}$ is well-defined. Moreover, we obtain that $\widetilde{\Psi}$ coincides with the inverse of $\widetilde{\Phi}$. Here, we remark that $\widetilde{\Psi}$ is independent of choosing θ, which is proved as follows: Let $(\mathfrak{g}, \sigma)$ be a semisimple pseudo-Riemannian symmetric pair, and θ, θ' be Cartan involutions of $\mathfrak{g}$ commuting with σ. It follows from [15] that there exists an $X \in \mathfrak{g}^{\sigma}$ satisfying $\theta' = \exp(\mathrm{ad}\, X)\theta \exp(-\mathrm{ad}\, X)$. Therefore we have $\Psi_{\theta'}(\mathfrak{g}, \sigma) \equiv \Psi_\theta(\mathfrak{g}, \sigma)$. Hence we have the following result.

Theorem 1 (A generalized duality,[2]) *The induced maps $\widetilde{\Phi}$ and $\widetilde{\Psi}$ give a (natural) one-to-one correspondence between $\mathscr{A}/_\equiv$ and $\mathscr{B}/_\equiv$. In particular, $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are generalizations of the duality between Riemannian symmetric pairs of compact type and Riemannian symmetric pairs of noncompact type.*

Remark 1 We remark that one can find the generalized duality in the literature (for example, Helminck [11], etc.). We believe that the generalized duality in Theorem 1 is more useful for the study of Hermann (type) actions than Helminck's one (see Sect. 2.2 for more details).

Notation 2. In Sect. 2, we use the notation as follows: For any $(\mathfrak{g}_u, \theta_1, \theta_2) \in \mathscr{A}/_\equiv$, $(\mathfrak{g}_u, \theta_1, \theta_2)^* = \widetilde{\Phi}(\mathfrak{g}_u, \theta_1, \theta_2)$.

2 Applications

Retain the notation as in Sect. 1. In this section, we focus our attention to the case where $\mathfrak{g}$ is simple.

2.1 An Alternative Proof for Berger's Classification

The classification of local isomorphism classes of semisimple pseudo-Riemannian symmetric spaces was given by Berger [4], which is called Berger's classification. This classification corresponds uniquely to that of equivalence classes of semisimple pseudo-Riemannian symmetric pairs. In this subsection, we give an alternative proof for Berger's classification from the viewpoint of the duality in Theorem 1 between $\mathscr{A}/_{\equiv}$ and $\mathscr{B}/_{\equiv}$.

2.1.1 The Classification of $\mathscr{A}/_{\equiv}$

First, we review Matsuki's classification of simple compact symmetric triads [16]. He defined an equivalence relation $\sim$ on the set $\mathscr{C}$ of all (not necessary commutative) semisimple compact symmetric triads as follows: For two semisimple compact symmetric triads $(\mathfrak{g}_u, \theta_1, \theta_2)$ and $(\mathfrak{g}'_u, \theta'_1, \theta'_2) \in \mathscr{C}$, the relation $(\mathfrak{g}_u, \theta_1, \theta_2) \sim (\mathfrak{g}'_u, \theta'_1, \theta'_2)$ holds, if there exists a Lie algebra isomorphism $\varphi : \mathfrak{g}_u \to \mathfrak{g}'_u$ and $\tau \in \mathrm{Int}(\mathfrak{g}'_u)$ satisfying $\theta'_1 = \varphi\theta_1\varphi^{-1}$ and $\theta'_2 = \tau(\varphi\theta_2\varphi^{-1})\tau^{-1}$. He also gave the classification of $\mathscr{C}/_{\sim}$ (see [16]).

Remark 2 The study of the classification of compact symmetric triads was initiated by Conlon [6, 8]. However, we couldn't obtain the paper [6], which contains his proof. Therefore, we cite Matsuki's classification for the description of $\mathscr{C}/_{\sim}$.

On the other hand, the second author [12] introduced the notion of (abstract) symmetric triads with multiplicities as a generalization of root systems and restricted root systems with multiplicities. In [12], he also gave an equivalence relation on the set of all symmetric triads (Definition 2.6 in [12]), and classified symmetric triads (Theorem 2.19 in [12]). By using the notion of (abstract) symmetric triads with multiplicities we have the following result (See [3], for the proof).

Theorem 2 *Let $(\mathfrak{g}_u, \theta_1, \theta_2)$, $(\mathfrak{g}'_u, \theta'_1, \theta'_2)$ be commutative simple compact symmetric triads. Denote by $(\tilde{\Sigma}, \Sigma, W; m, n)$ (resp. $(\tilde{\Sigma}', \Sigma', W'; m'n')$) the symmetric triad with multiplicities corresponding to $(\mathfrak{g}_u, \theta_1, \theta_2)$ (resp. $(\mathfrak{g}'_u, \theta'_1, \theta'_2)$). Then $(\mathfrak{g}_u, \theta_1, \theta_2) \sim (\mathfrak{g}'_u, \theta'_1, \theta'_2)$ or $(\mathfrak{g}_u, \theta_2, \theta_1) \sim (\mathfrak{g}'_u, \theta'_1, \theta'_2)$ if and only if $(\tilde{\Sigma}, \Sigma, W; m, n) \sim (\tilde{\Sigma}', \Sigma', W'; m'n')$.*

Remark 3 See [3] for the definition of the equivalence relation $\sim$ on the set of all symmetric triads with multiplicities.

We note that, for each equivalence class of a commutative compact symmetric triads in the sense of Matsuki, the equivalence class of the corresponding symmetric triad with multiplicities was completely determined by the first author and the second author [1]. Therefore, it follows from Theorem 2 that $\mathscr{A}/_{\equiv}$ can be determined by using the classification of abstract symmetric triads with multiplicities.

2.1.2 An Explicit Description of the Generalized Duality in Theorem 1

The following is a recipe to determine the duality in Theorem 1 explicitly.

(Step 1) Iterate the following for all commutative simple compact symmetric triads $(\mathfrak{g}_u, \theta_1, \theta_2)$ according to the classification due to Matsuki [16].

(Step 2) By using the results in [1, 3] we give the equivalence class $[(\tilde{\Sigma}, \Sigma, W; m, n)]$ of the symmetric triad with multiplicities corresponding to $(\mathfrak{g}_u, \theta_1, \theta_2)$.

(Step 3) Iterate the following steps (Step 4)–(Step 6) for each $(\tilde{\Sigma}', \Sigma', W'; m', n') \in [(\tilde{\Sigma}, \Sigma, W; m, n)]$.

(Step 4) Let $(\mathfrak{g}'_u, \theta'_1, \theta'_2)$ be a commutative simple compact symmetric triad corresponding to $(\tilde{\Sigma}', \Sigma', W'; m', n')$. We determine $\mathfrak{g}'$, $(\mathfrak{g}')^d$ by calculating the duals $(\mathfrak{g}', \theta')$, $((\mathfrak{g}')^d, (\theta')^d)$ of $(\mathfrak{g}'_u, \theta'_1)$, $(\mathfrak{g}'_u, \theta'_2)$, respectively.

(Step 5) We clarify $(\mathfrak{g}'_u)^{\theta'_1\theta'_2}$ by calculating $((\mathfrak{g}'_u)^{\theta'_1\theta'_2}, (\mathfrak{g}'_u)^{\theta'_1} \cap (\mathfrak{g}'_u)^{\theta'_2})$. Indeed, we can determine $((\mathfrak{g}'_u)^{\theta'_1\theta'_2}, (\mathfrak{g}'_u)^{\theta'_1} \cap (\mathfrak{g}'_u)^{\theta'_2})$ by using the data of the restricted root system $(\Sigma'; m')$ with multiplicity and the classification of Riemannian symmetric pairs of compact type.

(Step 6) We determine σ', $(\sigma')^d$ by calculating the duals $((\mathfrak{g}')^{\sigma'}, \theta')$, $(((\mathfrak{g}')^d)^{(\sigma')^d}, (\theta')^d)$ of $((\mathfrak{g}'_u)^{\theta'_2}, (\mathfrak{g}'_u)^{\theta'_1} \cap (\mathfrak{g}'_u)^{\theta'_2})$ $((\mathfrak{g}'_u)^{\theta'_1}, (\mathfrak{g}'_u)^{\theta'_1} \cap (\mathfrak{g}'_u)^{\theta'_2})$, respectively. Therefore, we have $(\mathfrak{g}'_u, \theta'_1, \theta'_2)^* = (\mathfrak{g}', \sigma')$ and $(\mathfrak{g}'_u, \theta'_2, \theta'_1)^* = ((\mathfrak{g}')^d, (\sigma')^d)$.

Here, we note that $((\mathfrak{g}')^d, (\sigma')^d)$ is the dual of $(\mathfrak{g}', \sigma')$ in the sense of [17]. Therefore, by using above recipe we have the following result.

Theorem 3 *All simple pseudo-Riemannian symmetric pairs are explicitly determined as the generalized duality in Theorem 1.*

In order to prove Theorem 3 we apply Theorem 1 in the case when $\mathfrak{g}_u$ and $\mathfrak{g}$ are simple.

Example 1 We will give examples of the generalized duality of commutative compact symmetric triads $(\mathfrak{g}_u, \theta_1, \theta_2)$ at the end of this paper. In Tables 1 and 2, we determine the generalized duality of $(\mathfrak{g}_u, \theta_1, \theta_2)$ in the case when $\mathfrak{g}_u$ is simple and exceptional (see [3], for other cases).

Remark 4 It is known that other alternative proofs for Berger's classification were investigated by [7, 9, 11]. Conlon's proof [7] is also based on the classification of commutative compact symmetric triads. However, his correspondence between commutative compact symmetric triads and semisimple pseudo-Riemannian symmetric pairs is implicit.

Table 1 The dual of $(\mathfrak{g}_u, \theta_1, \theta_2)$ ($\mathfrak{g}_u$: simple, exceptional, $\theta_1 \not\sim \theta_2$)

$(\mathfrak{g}_u, \theta_1, \theta_2) = (\mathfrak{g}_u, \mathfrak{g}_u^{\theta_1}, \mathfrak{g}_u^{\theta_2})$	$(\tilde{\Sigma}, \Sigma, W; m, n)$	$(\mathfrak{g}_u, \theta_1, \theta_2)^*$ / $(\mathfrak{g}_u, \theta_2, \theta_1)^*$
$(\mathfrak{e}_6, \mathfrak{su}(6) \oplus \mathfrak{su}(2), \mathfrak{sp}(4))$	(I-F_4)	$(\mathfrak{e}_{6(2)}, \mathfrak{sp}(3, 1))$
		$(\mathfrak{e}_{6(6)}, \mathfrak{su}^*(6) \oplus \mathfrak{su}(2))$
	(I'-F_4)	$(\mathfrak{e}_{6(2)}, \mathfrak{sp}(4, \mathbb{R}))$
		$(\mathfrak{e}_{6(6)}, \mathfrak{sl}(6, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}))$
$(\mathfrak{e}_6, \mathfrak{so}(10) \oplus \mathfrak{u}(1), \mathfrak{sp}(4))$	(II-BC_2)	$(\mathfrak{e}_{6(-14)}, \mathfrak{sp}(2, 2))$
		$(\mathfrak{e}_{6(6)}, \mathfrak{so}(5, 5) \oplus \mathbb{R})$
$(\mathfrak{e}_6, \mathfrak{f}_4, \mathfrak{sp}(4))$	(III-A_2)	$(\mathfrak{e}_{6(-26)}, \mathfrak{sp}(1, 3))$
		$(\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$
$(\mathfrak{e}_6, \mathfrak{so}(10) \oplus \mathfrak{u}(1), \mathfrak{su}(6) \oplus \mathfrak{su}(2))$	(I-BC_2-B_2; basic)	$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(1, 5) \oplus \mathfrak{sl}(2, \mathbb{R}))$
		$(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}(2))$
	(I-BC_2-B_2; non-basic)	$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(2, 4) \oplus \mathfrak{su}(2))$
		$(\mathfrak{e}_{6(2)}, \mathfrak{so}(6, 4) \oplus \mathfrak{so}(2))$
$(\mathfrak{e}_6, \mathfrak{su}(6) \oplus \mathfrak{su}(2), \mathfrak{f}_4)$	(III-BC_1)	$(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$
		$(\mathfrak{e}_{6(-26)}, \mathfrak{su}^*(6) \oplus \mathfrak{su}(2))$
$(\mathfrak{e}_6, \mathfrak{so}(10) \oplus \mathfrak{u}(1), \mathfrak{f}_4)$	(III-BC_1)	$(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$
		$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(1, 9) \oplus \mathbb{R})$
$(\mathfrak{e}_7, \mathfrak{so}(12) \oplus \mathfrak{su}(2), \mathfrak{su}(8))$	(I-F_4)	$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(6, 2))$
		$(\mathfrak{e}_{7(7)}, \mathfrak{so}^*(12) \oplus \mathfrak{su}(2))$
	(I'-F_4)	$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4, 4))$
		$(\mathfrak{e}_{7(7)}, \mathfrak{so}(6, 6) \oplus \mathfrak{sl}(2, \mathbb{R}))$
$(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathfrak{u}(1), \mathfrak{su}(8))$	(I-C_3)	$(\mathfrak{e}_{7(-25)}, \mathfrak{su}(6, 2))$
		$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2))$
	(I'-C_3)	$(\mathfrak{e}_{7(-25)}, \mathfrak{su}^*(8))$
		$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R})$
$(\mathfrak{e}_7, \mathfrak{so}(12) \oplus \mathfrak{su}(2), \mathfrak{e}_6 \oplus \mathfrak{u}(1))$	(I-BC_2-B_2; basic)	$(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2))$
		$(\mathfrak{e}_{7(-25)}, \mathfrak{so}(10, 2) \oplus \mathfrak{sl}(2, \mathbb{R}))$
	(I-BC_2-B_2; non-basic)	$(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2))$
		$(\mathfrak{e}_{7(-25)}, \mathfrak{so}^*(12) \oplus \mathfrak{su}(2))$
$(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}(2), \mathfrak{so}(16))$	(I-F_4)	$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(12, 4))$
		$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2))$
	(I'-F_4)	$(\mathfrak{e}_{8(-24)}, \mathfrak{so}^*(16))$
		$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(7)} \oplus \mathfrak{sl}(2, \mathbb{R}))$
$(\mathfrak{f}_4, \mathfrak{so}(9), \mathfrak{su}(2) \oplus \mathfrak{sp}(3))$	(III-BC_1)	$(\mathfrak{f}_{4(-20)}, \mathfrak{sp}(1, 2) \oplus \mathfrak{su}(2))$
		$(\mathfrak{f}_{4(4)}, \mathfrak{so}(4, 5))$

See [12] for the notation (I-F_4), etc. of symmetric triads
See [1] for the definition of “basic” and “non-basic”

Table 2 (ii) The dual of $(\mathfrak{g}_u, \theta_1, \theta_2)$ ($\mathfrak{g}_u$: simple, exceptional, $\theta_1 \sim \theta_2$ (i.e., $(\mathfrak{g}_u, \theta_1, \theta_2) \sim (\mathfrak{g}_u, \theta, \theta)$ for some involution θ))

$(\mathfrak{g}_u, \theta) = (\mathfrak{g}_u, \mathfrak{g}_u^\theta)$	$(\tilde{\Sigma}, \Sigma, W; m, n)$	$(\mathfrak{g}_u, \theta_1, \theta_2)^*$ $(\mathfrak{g}_u, \theta_1, \theta_1\theta_2)^*$ $(\mathfrak{g}_u, \theta_1\theta_2, \theta_2)^*$
$(\mathfrak{e}_6, \mathfrak{sp}(4))$	$(E_6, E_6, \emptyset)$	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))$
		$(\mathfrak{e}_{6(6)}, \mathfrak{e}_{6(6)})$
		$(\mathfrak{e}_6, \mathfrak{sp}(4))$
	$(E_6, D_5, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(2, 2))$
		$(\mathfrak{e}_{6(6)}, \mathfrak{so}(5, 5) \oplus \mathbb{R})$
		$(\mathfrak{e}_{6(-14)}, \mathfrak{sp}(2, 2))$
	$(E_6, A_1 \oplus A_5, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4, \mathbb{R}))$
		$(\mathfrak{e}_{6(6)}, \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(6, \mathbb{R}))$
		$(\mathfrak{e}_{6(2)}, \mathfrak{sp}(4, \mathbb{R}))$
$(\mathfrak{e}_6, \mathfrak{su}(6) \oplus \mathfrak{su}(2))$	$(F_4, F_4, \emptyset)$	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(6) \oplus \mathfrak{su}(2))$
		$(\mathfrak{e}_{6(2)}, \mathfrak{e}_{6(2)})$
		$(\mathfrak{e}_6, \mathfrak{su}(6) \oplus \mathfrak{su}(2))$
	$(F_4, A_1 \oplus C_3, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_6, \mathfrak{su}(3, 3) \oplus \mathfrak{sl}(2, \mathbb{R}))$
	$(F_4, B_4, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(4, 2) \oplus \mathfrak{su}(2))$
		$(\mathfrak{e}_{6(2)}, \mathfrak{so}(6, 4) \oplus \mathfrak{so}(2))$
		$(\mathfrak{e}_{6(-14)}, \mathfrak{su}(4, 2) \oplus \mathfrak{su}(2))$
$(\mathfrak{e}_6, \mathfrak{so}(10) \oplus \mathfrak{u}(1))$	$(BC_2, BC_2, \emptyset)$	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \oplus \mathfrak{u}(1))$
		$(\mathfrak{e}_{6(-14)}, \mathfrak{e}_{6(-14)})$
		$(\mathfrak{e}_6, \mathfrak{so}(10) \oplus \mathfrak{u}(1))$
	$(BC_2, A_1 \oplus BC_1, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}(2))$
		$(\mathfrak{e}_{6(-14)}, \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(5, 1))$
		$(\mathfrak{e}_{6(2)}, \mathfrak{so}^*(10) \oplus \mathfrak{so}(2))$
	$(BC_2, B_2, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(8, 2) \oplus \mathfrak{so}(2))$
$(\mathfrak{e}_6, \mathfrak{f}_4)$	$(A_2, A_2, \emptyset)$	$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_4)$
		$(\mathfrak{e}_{6(-26)}, \mathfrak{e}_{6(-26)})$
		$(\mathfrak{e}_6, \mathfrak{f}_4)$
	$(A_2, A_1, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$
		$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9, 1) \oplus \mathbb{R})$
		$(\mathfrak{e}_{6(-14)}, \mathfrak{f}_{4(-20)})$
$(\mathfrak{e}_7, \mathfrak{su}(8))$	$(E_7, E_7, \emptyset)$	$(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$
		$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{7(7)})$
		$(\mathfrak{e}_7, \mathfrak{su}(8))$
	$(E_7, A_1 \oplus D_6, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{7(7)}, \mathfrak{su}(4, 4))$
		$(\mathfrak{e}_{7(7)}, \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(6, 6))$
		$(\mathfrak{e}_{7(-5)}, \mathfrak{su}(4, 4))$

(continued)

Table 2 (continued)

$(\mathfrak{g}_u, \theta) = (\mathfrak{g}_u, \mathfrak{g}_u^\theta)$	$(\tilde{\Sigma}, \Sigma, W; m, n)$	$(\mathfrak{g}_u, \theta_1, \theta_2)^*$ $(\mathfrak{g}_u, \theta_1, \theta_1\theta_2)^*$ $(\mathfrak{g}_u, \theta_1\theta_2, \theta_2)^*$
	$(E_7, A_7, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{7(7)}, \mathfrak{sl}(8, \mathbb{R}))$
	$(E_7, E_6, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{7(7)}, \mathfrak{su}^*(8))$
		$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \mathbb{R})$
		$(\mathfrak{e}_{7(-25)}, \mathfrak{su}^*(8))$
$(\mathfrak{e}_7, \mathfrak{so}(12) \oplus \mathfrak{su}(2))$	$(F_4, F_4, \emptyset)$	$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(12) \oplus \mathfrak{su}(2))$
		$(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{7(-5)})$
		$(\mathfrak{e}_7, \mathfrak{so}(12) \oplus \mathfrak{su}(2))$
	$(F_4, A_1 \oplus C_3, \emptyset)$	$(\mathfrak{e}_{7(-5)}, \mathfrak{so}^*(12) \oplus \mathfrak{sl}(2, \mathbb{R}))$
	$(F_4, B_4, \emptyset)$	$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(8, 4) \oplus \mathfrak{su}(2))$
$(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathfrak{u}(1))$	$(C_3, C_3, \emptyset)$	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 \oplus \mathfrak{so}(2))$
		$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{7(-25)})$
		$(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathfrak{so}(2))$
	$(C_3, C_1 \oplus C_2, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2))$
	$(C_3, A_2, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \mathbb{R})$
$(\mathfrak{e}_8, \mathfrak{so}(16))$	$(E_8, E_8, \emptyset)$	$(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$
		$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{8(8)})$
		$(\mathfrak{e}_8, \mathfrak{so}(16))$
	$(E_8, D_8, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{8(8)}, \mathfrak{so}(8, 8))$
	$(E_8, A_1 \oplus A_7, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{8(8)}, \mathfrak{so}^*(16))$
$(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}(2))$	$(F_4, F_4, \emptyset)$	$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7 \oplus \mathfrak{su}(2))$
		$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{8(-24)})$
		$(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}(2))$
	$(F_4, A_1 \oplus C_3, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{7(-25)} \oplus \mathfrak{sl}(2, \mathbb{R}))$
	$(F_4, B_4, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2))$
		$(\mathfrak{e}_{8(-24)}, \mathfrak{so}(12, 4))$
		$(\mathfrak{e}_{8(8)}, \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2))$
$(\mathfrak{f}_4, \mathfrak{su}(2) \oplus \mathfrak{sp}(3))$	$(F_4, F_4, \emptyset)$	$(\mathfrak{f}_{4(4)}, \mathfrak{su}(2) \oplus \mathfrak{sp}(3))$
		$(\mathfrak{f}_{4(4)}, \mathfrak{f}_{4(4)})$
		$(\mathfrak{f}_4, \mathfrak{su}(2) \oplus \mathfrak{sp}(3))$
	$(F_4, A_1 \oplus C_3, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3, \mathbb{R}))$
	$(F_4, B_4, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2))$
		$(\mathfrak{f}_{4(4)}, \mathfrak{so}(5, 4))$
		$(\mathfrak{f}_{4(-20)}, \mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2))$

(continued)

Table 2 (continued)

$(\mathfrak{g}_u, \theta) = (\mathfrak{g}_u, \mathfrak{g}_u^{\theta})$	$(\tilde{\Sigma}, \Sigma, W; m, n)$	$(\mathfrak{g}_u, \theta_1, \theta_2)^*$
		$(\mathfrak{g}_u, \theta_1, \theta_1\theta_2)^*$
		$(\mathfrak{g}_u, \theta_1\theta_2, \theta_2)^*$
$(\mathfrak{f}_4, \mathfrak{so}(9))$	$(BC_1, BC_1, \emptyset)$	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$
		$(\mathfrak{f}_{4(-20)}, \mathfrak{f}_{4(-20)})$
		$(\mathfrak{f}_4, \mathfrak{so}(9))$
	$(BC_1, B_1, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(8, 1))$
$(\mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathfrak{su}(2))$	$(G_2, G_2, \emptyset)$	$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2) \oplus \mathfrak{su}(2))$
		$(\mathfrak{g}_{2(2)}, \mathfrak{g}_{2(2)})$
		$(\mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathfrak{su}(2))$
	$(G_2, A_1 \oplus A_1, \tilde{\Sigma} - \Sigma)$	$(\mathfrak{g}_{2(2)}, \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}))$

2.2 The Geometry of Hermann Type Actions

The notion of Hermann type actions was given by Koike [13, 14], which gives examples of isometric group actions on pseudo-Riemannian symmetric spaces. In this subsection, we investigate the geometric structures of orbits for Hermann type actions. Let G be a connected semisimple noncompact Lie group, and H be a closed subgroup of G satisfying $(G^\sigma)_0 \subset H \subset G^\sigma$, where σ is an involution of G which is not Cartan one, and $(G^\sigma)_0$ denotes the identity component of G^σ. Let θ be a Cartan involution of G commuting with σ, and denote by $K = G^\theta$. We note that the Killing form of $\mathfrak{g} := \mathrm{Lie}(G)$ induces the structure of a pseudo-Riemannian symmetric space on G, G/K and G/H, respectively. In particular, G/K is a Riemannian symmetric space of noncompact type. The following natural actions are called Hermann type actions: (i) the H-action on G/K; (ii) the K-action on G/H; (iii) the $(H \times K)$-action on G.

In this paper, we focus on the study of orbits for the K-action on the pseudo-Riemannian symmetric space G/H, which is an example of a compact group action on a pseudo-Riemannian manifold. Denote by $(G_u, \theta_1, \theta_2)$ the generalized duality of (G, H) as Lie group level, and by $(\tilde{\Sigma}, \Sigma, W; m, n)$ the symmetric triad with multiplicities corresponding to $(G_u, \theta_1, \theta_2)$. Then $(\Sigma; m)$ gives the restricted root system with multiplicity of the (reductive) Riemannian symmetric pair $(\mathfrak{g}_u^{\theta_1\theta_2}, \mathfrak{g}_u^{\theta_1} \cap \mathfrak{g}_u^{\theta_2})$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{g}_u^{-\theta_1} \cap \mathfrak{g}_u^{-\theta_2}$. It is clear that $\sqrt{-1}\mathfrak{a}$ is contained in $\mathfrak{g}$. Set, for any $\lambda \in \Sigma (\subset \mathfrak{a})$,

$$\mathfrak{k}_\lambda = \{X \in \mathfrak{g}_u^{\theta_1} \cap \mathfrak{g}_u^{\theta_2} \mid \mathrm{ad}(A)^2(X) = -\langle \lambda, A \rangle^2 X, \forall A \in \mathfrak{a}\},$$
$$\mathfrak{m}_\lambda = \{X \in \mathfrak{g}_u^{-\theta_1} \cap \mathfrak{g}_u^{-\theta_2} \mid \mathrm{ad}(A)^2(X) = -\langle \lambda, A \rangle^2 X, \forall A \in \mathfrak{a}\},$$

where $\langle , \rangle$ denotes the invariant inner product on $\mathfrak{a}$. Then we have the following decompositions of $\mathfrak{g}^{\theta_1} \cap \mathfrak{g}_u^{\theta_2}$ and $\mathfrak{g}_u^{-\theta_1} \cap \mathfrak{g}_u^{-\theta_2}$, respectively:

$$\mathfrak{g}_u^{\theta_1} \cap \mathfrak{g}_u^{\theta_2} = \mathfrak{k}_0 \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda, \quad \mathfrak{g}_u^{-\theta_1} \cap \mathfrak{g}_u^{-\theta_2} = \mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+} \mathfrak{m}_\lambda,$$

where $\mathfrak{k}_0$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{g}_u^{\theta_1} \cap \mathfrak{g}_u^{\theta_2}$ and Σ^+ is the set of positive roots of Σ. The following fact was proved by Flensted–Jensen and Rossmann, independently.

Proposition 1 [10, 18] $G = K(\exp(\sqrt{-1}\mathfrak{a}))H = H(\exp(\sqrt{-1}\mathfrak{a}))K$.

It follows from Proposition 1 that all K-orbits meet $A = \pi_H(\exp(\sqrt{-1}\mathfrak{a}))$, where $\pi_H : G \to G/H$ is the natural projection. Denote by $K(gH)$ the K-orbit through $gH \in G/H$. Without loss of generality, we assume that $g = \exp(\sqrt{-1}Z) \in \exp(\sqrt{-1}\mathfrak{a})$ (because of Proposition 1). Then the tangent space and the normal space of $K(gH)$ at gH are expressed as follows:

$$g_*^{-1} T_{gH}(K(gH)) = \sqrt{-1}\left(\sum_{\lambda \in \Sigma^+; \langle \lambda, Z \rangle \neq 0} \mathfrak{m}_\lambda\right) \oplus (\mathfrak{g}_u^{\theta_1} \oplus \mathfrak{g}_u^{-\theta_2}),$$

$$g_*^{-1} T_{gH}^{\perp}(K(gH)) = \sqrt{-1}\left(\mathfrak{a} \oplus \sum_{\lambda \in \Sigma^+; \langle \lambda, Z \rangle = 0} \mathfrak{m}_\lambda\right).$$

From above argument we have the following result.

Proposition 2 *For the K-action on the pseudo-Riemannian symmetric space G/H, the following statements hold.*

(1) Any K-orbit is a pseudo-Riemannian submanifold in G/H.
(2) The induced symmetric bilinear form on $T_{gH}^{\perp} K(gH)$ is positive definite for all $g \in G$.
(3) $A = \pi_H(\exp(\sqrt{-1}\mathfrak{a}))$ is a totally geodesic submanifold in G/H.
(4) A meets every K-orbit orthogonally with respect to the pseudo-Riemannian metric on G/H.

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Transversally Complex Submanifolds of a Quaternion Projective Space

Kazumi Tsukada

Abstract We study a kind of complex submanifolds in a quaternion projective space $\mathbb{H}P^n$, which we call transversally complex submanifolds, from the viewpoint of quaternionic differential geometry. There are several examples of transversally complex immersions of Hermitian symmetric spaces. For a transversally complex immersion $f : M \to \mathbb{H}P^n$, a key notion is a Gauss map associated with f, which is a map $S : M \to \mathrm{End}(\mathbb{H}^{n+1})$ with $S^2 = -\mathrm{id}$. Our theory is an attempt of a generalization of the theory "Conformal geometry of surfaces in S^4 and quaternions" by Burstall, Ferus, Leschke, Pedit, and Pinkall [4].

1 Introduction

We study complex submanifolds of a quaternion projective space. It is an attempt of a generalization of a theory by Burstall, Ferus, Leschke, Pedit, and Pinkall [4]. They studied conformal geometry of Riemann surfaces in S^4 using a "quaternionic valued function theory", whose meromorphic functions are conformal maps into $\mathbb{H}$. Our proposal of a generalization is explained as in the following diagram:

the theory of [4]		Its generalization
$GL(2, \mathbb{H})$-geometry of $\mathbb{H}P^1$ $=$ Conformal geometry of S^4	$\Rightarrow$	$GL(n+1, \mathbb{H})$-geom. of $\mathbb{H}P^n$ $=$ Quaternionic diff. geom. of $\mathbb{H}P^n$
Riemann surfaces of S^4	$\Rightarrow$	Half-dim. complex submanifolds of $\mathbb{H}P^n$

K. Tsukada (✉)
Department of Mathematics, Ochanomizu University, 2-1-1 Otsuka Bunkyo-ku, Tokyo 112-8610, Japan
e-mail: tsukada.kazumi@ocha.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*, Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_19

We expect that our proposal of a generalization is an interesting subject in the field of quaternionic holomorphic differential geometry where the quaternionic differential geometry and the holomorphic differential geometry interact.

This is a survey article which explains our recent results of transversally complex submanifolds of $\mathbb{H}P^n$. The details including proofs will be discussed in the forthcoming paper.

2 Quaternionic Manifolds

In this section, we review basic definitions of quaternionic manifolds.

Definition 1 (*cf.* [1]) Let M be a $4n(n \geq 2)$-dimensional manifold and Q be a rank 3 subbundle of End TM which satisfies the following conditions:

(a) For an arbitrary point $p \in M$, there is a neighborhood U of p over which there exists a local frame field $\{I, J, K\}$ of Q satisfying

$$I^2 = J^2 = K^2 = -\mathrm{id}, \quad IJ = -JI = K,$$
$$JK = -KJ = I, \quad KI = -IK = J.$$

(b) There exists a torsionfree affine connection which preserves Q.

Then Q is called *a quaternionic structure* on M and the manifold (M, Q) *a quaternionic manifold*. We say that such torsionfree affine connection which preserves Q is a Q-connection and that $\{I, J, K\}$ is *a local admissible frame field*.

We remark that for a quaternionic structure Q, a Q-connection is not unique. More precisely we have the following.

Proposition 1 (cf. [1] Proposition 5.1) *Let ∇ be a Q-connection. Then for any other Q-connection ∇', there exists a 1-form θ such that ∇' is written as*

$$\nabla'_X Y = \nabla_X Y + \theta(X)Y + \theta(Y)X - \theta(IX)IY - \theta(IY)IX$$
$$-\theta(JX)JY - \theta(JY)JX - \theta(KX)KY - \theta(KY)KX. \qquad (1)$$

Conversely for a 1-form θ, the connection ∇' defined by (1) is a Q-connection.

We study properties and quantities which depend on a quaternionic structure Q and do not depend on a choice of a Q-connection.

From the view point of Riemannian geometry, we introduce the notion of a quaternionic (pseudo-) Kähler manifold (cf. [7]). Let g be a (pseudo-) Riemannian metric on a quaternionic manifold (M, Q). If any element of Q_p is skew-symmetric with respect to g_p at any point $p \in M$ and the Riemannian connection associated with g preserves Q, then (Q, g) is called *a quaternionic (pseudo-) Kähler structure* and (M, Q, g) *a quaternionic (pseudo-) Kähler manifold*.

3 Transversally Complex Submanifolds of a Quaternionic Manifold

In this section, we study fundamental properties of transversally complex submanifolds of a quaternionic manifold.

Let $(\tilde{M}^{4n}, \tilde{Q})$ be a $4n(n \geq 2)$-dimensional quaternionic manifold with a quaternionic structure $\tilde{Q}$ and M^{2m} be an immersed submanifold in $\tilde{M}$. Then M is said to be *an almost complex submanifold* if there exists a section $\tilde{I}$ of the bundle $\tilde{Q}|_M$ which satisfies $\tilde{I}^2 = -\mathrm{id}$ and $\tilde{I}TM = TM$ (cf. Alekseevsky and Marchiafava [2]). We denote by I the almost complex structure on M induced from the section $\tilde{I}$. We have the decomposition:

$$\tilde{Q}|_M = \mathbb{R}\tilde{I} + Q', \tag{2}$$

where Q' is defined by $[\tilde{I}, \tilde{Q}|_M]$. We take a local frame field $\{\tilde{J}, \tilde{K}\}$ of Q' such that $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ is a local admissible frame field of $\tilde{Q}|_M$. For each point $p \in M$, we define a subspace $\bar{T}_pM$ of T_pM by $\bar{T}_pM = T_pM \cap \tilde{J}(T_pM)$. Then the subspace $\bar{T}_pM$ is $\tilde{Q}$-invariant and I-invariant. Then we have the following:

Proposition 2 (cf. [2] Theorem 1.1) *Let M^{2m} be a $2m(m \geq 2)$-dimensional almost complex submanifold of a quaternionic manifold $(\tilde{M}^{4n}, \tilde{Q})$. If for each point $p \in M$, $\dim T_pM/\bar{T}_pM > 2$, then I is integrable.*

Now we define a transversally complex submanifold. Let M^{2m} be an almost complex submanifold of a quaternionic manifold $(\tilde{M}^{4n}, \tilde{Q})$ together with the section $\tilde{I}$ of $\tilde{Q}|_M$. Then M is said to be *a transversally complex submanifold* if at each point $p \in M$, $LT_pM \cap T_pM = \{0\}$ for any $L \in Q'_p$ or equivalently $\bar{T}_pM = \{0\}$. Under this assumption, we see that the induced complex structure I is integrable by Proposition 2. We can define the special class of transversally complex submanifolds in the Riemannian setting. Let $(\tilde{M}, \tilde{Q}, \tilde{g})$ be a quaternionic pseudo-Kähler manifold and M^{2m} be an almost complex submanifold whose tangent spaces are nondegenerate with respect to the pseudo-Kähler metric $\tilde{g}$. Then M is said to be *a totally complex submanifold* if at each point $p \in M$, LT_pM is orthogonal to the tangent space T_pM for any $L \in Q'_p$ (cf. Funabashi [5]). Several authors have studied totally complex submanifolds of quaternionic Kähler manifolds [2, 3, 9] and interesting examples are known.

Example 1 It is well-known that an n-dimensional quaternion projective space $\mathbb{H}P^n$ is a quaternionic Kähler manifold which is a symmetric space [10]. The author [9] constructed and classified half-dimensional totally complex submanifolds of $\mathbb{H}P^n$ whose second fundamental forms are parallel. They are locally congruent to one of the following:

(1) $\mathbb{C}P^n \hookrightarrow \mathbb{H}P^n$ (totally geodesic),
(2) $Sp(3)/U(3) \hookrightarrow \mathbb{H}P^6$,
(3) $SU(6)/S(U(3) \times U(3)) \hookrightarrow \mathbb{H}P^9$,
(4) $SO(12)/U(6) \hookrightarrow \mathbb{H}P^{15}$,
(5) $E_7/E_6 \cdot T^1 \hookrightarrow \mathbb{H}P^{27}$,
(6) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}/2) \hookrightarrow \mathbb{H}P^2$,
(7) $\mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \times \mathbb{C}P^1(\tilde{c}) \hookrightarrow \mathbb{H}P^3$, or
(8) $\mathbb{C}P^1(\tilde{c}) \times SO(n+1)/SO(2) \cdot SO(n-1) \hookrightarrow \mathbb{H}P^n$ $(n \geq 4)$,

where $\mathbb{H}P^n$ has the scalar curvature $4n(n+2)\tilde{c}$ and $\mathbb{C}P^1(\tilde{c})$ is of constant curvature $\tilde{c}$.

From now on we assume that M is a transversally complex submanifold of a quaternionic manifold $\tilde{M}$ with $\dim_{\mathbb{R}} M = \frac{1}{2}\dim_{\mathbb{R}} \tilde{M} \geq 4$. We denote by $\tilde{I}$ the corresponding section of $\tilde{Q}|_M$ and by $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ a local admissible frame field as usual. We put $T^{\perp}M = \tilde{J}TM$. It is an $\tilde{I}$-invariant subbundle of $T\tilde{M}|_M$. We have the direct sum decomposition:

$$T\tilde{M}|_M = TM + T^{\perp}M. \tag{3}$$

We take some Q-connection $\tilde{\nabla}$ of $\tilde{M}$. According to the decomposition (3), we define the induced connection ∇ on M and the second fundamental form σ following the usual submanifold theory in the affine differential geometry. Then ∇ is torsionfree and σ is a $T^{\perp}M$-valued symmetric tensor field. Moreover we have $\nabla I = 0$. We denote by σ_+ and σ_-, the $(2,0)+(0,2)$-part and the $(1,1)$-part of σ with respect to the complex structure I, respectively. That is, they satisfy

$$\sigma_+(IX, IY) = -\sigma_+(X, Y), \sigma_-(IX, IY) = \sigma_-(X, Y) \quad \text{for } X, Y \in TM.$$

Then the following holds:

Proposition 3 *Let $(\tilde{M}^{4n}, \tilde{Q}, \tilde{g})$ be a quaternionic Kähler manifold whose scalar curvature does not vanish and M^{2n} $(n \geq 2)$ be a half-dimensional transversally complex submanifold of $\tilde{M}$. We denote by σ_- the* $(1,1)$*-part of the second fundamental form with respect to the Riemannian connection $\tilde{\nabla}$. Then M is totally complex if and only if σ_- vanishes.*

We take another Q-connection $\tilde{\nabla}'$ of $\tilde{M}$. By Proposition 1, there exists a 1-form θ which satisfies (1). We denote by ∇' the induced connection on M and by σ', σ'_+, σ'_-, the second fundamental form, its $(2,0)+(0,2)$-part, its $(1,1)$-part, respectively. The following is easily seen:

Proposition 4 (1) $\nabla'_X Y = \nabla_X Y + \theta(X)Y + \theta(Y)X - \{\theta(IX)IY + \theta(IY)IX\}$.
(2) $\sigma'_+(X, Y) = \sigma_+(X, Y)$.

(3) $\sigma'_-(X, Y) = \sigma_-(X, Y) - \{\theta(\tilde{J}X)\tilde{J}Y + \theta(\tilde{J}Y)\tilde{J}X + \theta(\tilde{K}X)\tilde{K}Y + \theta(\tilde{K}Y)\tilde{K}X\}$.

Remark 1 (1) The relation between the two induced connections ∇ and ∇' in Proposition 4 (1) is known as the holomorphically projective change (cf. [6]).

(2) By Proposition 4 (2), the $(2, 0) + (0, 2)$-part σ_+ does not depend on the choice of Q-connections and hence it is an invariant in the quaternionic differential geometry.

(3) Let $M_1, M_2 \subset \tilde{M}$ be transversally complex submanifolds of $\tilde{M}$ which have a common point p. We assume that $T_pM_1 = T_pM_2$ at this point p. Then Proposition 4 (3) means that whether or not σ_- of M_1, M_2 at this point coincide does not depend on the choice of Q-connections.

4 A Quaternionic Structure and Q-Connections on $\mathbb{H}P^n$

In this section, we give a description of a quaternionic structure and Q-connections on a quaternion projective space.

Let $\mathbb{H}^{n+1}$ be the space of column $n+1$-tuples with entries in $\mathbb{H}$. The space $\mathbb{H}^{n+1}$ is considered as a right quaternionic vector space. The quaternion projective space $\mathbb{H}P^n$ is defined as the set of quaternionic lines in $\mathbb{H}^{n+1}$. We study the quaternionic differential geometry of $\mathbb{H}P^n$ by the theory of the quaternionic vector bundles following [4]. We denote by $\underline{\mathbb{H}^{n+1}} = \mathbb{H}P^n \times \mathbb{H}^{n+1}$ the product bundle. The tautological line bundle L is defined as follows:

$$L = \{(l, v) \in \mathbb{H}P^n \times \mathbb{H}^{n+1} \mid v \in l\}.$$

This line subbundle induces a quotient bundle $\underline{\mathbb{H}^{n+1}}/L$. We denote by $\pi_L : \underline{\mathbb{H}^{n+1}} \to \underline{\mathbb{H}^{n+1}}/L$ the projection and by $\mathrm{Hom}(L, \underline{\mathbb{H}^{n+1}}/L)$ the real vector bundle whose fibres are the spaces of $\mathbb{H}$-linear homomorphisms of l into $\mathbb{H}^{n+1}/l$ at each line $l \in \mathbb{H}P^n$. Let d be the trivial connection of the trivial bundle $\underline{\mathbb{H}^{n+1}} = \mathbb{H}P^n \times \mathbb{H}^{n+1}$. For $l \in \mathbb{H}P^n$ and $v \in l$, we take a local section s of L such that $s(l) = v$. For $X \in T_l\mathbb{H}P^n$, we define $\alpha(X) : l \to \mathbb{H}^{n+1}/l$ as follows:

$$\alpha(X)v = \pi_L(d_X s) \tag{4}$$

It is defined independently of the choice of local sections and $\alpha(X)$ is a $\mathbb{H}$-linear homomorphism. The following is well-known (cf. Sects. 3, 4 in [4])

Proposition 5 *The map* $\alpha : T\mathbb{H}P^n \to \mathrm{Hom}(L, \underline{\mathbb{H}^{n+1}}/L)$ *is a bundle isomorphism as real vector bundles.*

Using Proposition 5, we introduce a quaternionic structure Q on $\mathbb{H}P^n$. At each $l \in \mathbb{H}P^n$, the tangent space $T_l\mathbb{H}P^n$ is real linear isomorphic to the space $\mathrm{Hom}(l, \mathbb{H}^{n+1}/l)$. Let U be an open subset of $\mathbb{H}P^n$ and we take a local section $s_0 \in \Gamma(L)$ defined on U without the zero points. Then s_0 induces a bundle isomorphism of $T\mathbb{H}P^n$ onto

$\underline{\mathbb{H}^{n+1}}/L$ on U, i.e., $T\mathbb{H}P^n \ni X \mapsto \alpha(X)s_0 \in \underline{\mathbb{H}^{n+1}}/L$. For $X \in \Gamma(T\mathbb{H}P^n|_U)$, we define

$$\alpha(\tilde{I}X)(s_0) = (\alpha(X)(s_0))i, \ \ \alpha(\tilde{J}X)(s_0) = (\alpha(X)(s_0))j$$

and put $\tilde{K} = \tilde{I}\tilde{J}$. We denote by Q the rank 3-subbundle of End $T\mathbb{H}P^n$ spanned by $\tilde{I}, \tilde{J}, \tilde{K}$. The subbundle Q is defined independently of the choice of local sections s_0. The group $GL(n+1, \mathbb{H})$ acts on $\mathbb{H}P^n$ transitively. The quaternionic structure Q is invariant by the action of $GL(n+1, \mathbb{H})$.

Next we describe Q-connections on $\mathbb{H}P^n$ in terms of quaternionic vector bundles.

Theorem 1 *There exists the natural one to one correspondence of Q-connections on $\mathbb{H}P^n$ with the direct sum decompositions $\underline{\mathbb{H}^{n+1}} = L + L^c$ as quaternionic vector bundles, where L^c denotes a complementary bundle of L in $\underline{\mathbb{H}^{n+1}}$.*

Proof We explain how to construct the Q-connection from a direct sum decomposition:

$$\underline{\mathbb{H}^{n+1}} = L + L^c. \tag{5}$$

We consider the decomposition of the trivial connection d according to (5). For $X \in \Gamma(T\mathbb{H}P^n)$, $s \in \Gamma(L)$, and $\xi \in \Gamma(L^c)$, we have

$$\begin{aligned} d_X s &= D_X s + \alpha(X)s \\ d_X \xi &= \tau(X)\xi + D^c_X \xi \quad . \end{aligned}$$

Here D and D^c are connections of the quaternionic vector bundles L and L^c, respectively and $\tau : T\mathbb{H}P^n \to \mathrm{Hom}(L^c, L)$ is a bundle homomorphism as real vector bundles. Using the isomorphism $\alpha : T\mathbb{H}P^n \cong \mathrm{Hom}(L, \underline{\mathbb{H}^{n+1}}/L) \cong \mathrm{Hom}(L, L^c)$, we induce the affine connection ∇ from D and D^c. That is, for $X, Y \in \Gamma(T\mathbb{H}P^n)$, $s \in \Gamma(L)$

$$\alpha(\nabla_X Y)(s) = D^c_X(\alpha(Y)(s)) - \alpha(Y)(D_X s).$$

Then we see that the induced connection ∇ is a Q-connection. □

Example 2 (*Affine coordinates*) Let $\{e_1, \ldots, e_{n+1}\}$ be the standard basis of $\mathbb{H}^{n+1}$ and $\{\theta^1, \ldots, \theta^{n+1}\}$ be its dual basis. We put $M' = \{[v] \in \mathbb{H}P^n \mid \theta^1(v) \neq 0\}$ and define the complementary bundle L^c of L by the quaternionic vector subbundle of $\underline{\mathbb{H}^{n+1}}$ spanned by $e_2, \ldots, e_{n+1}$ on M'. Then the Q-connection which corresponds to the decomposition is a canonical connection of $\mathbb{H}^n \cong \mathbb{R}^{4n}$.

Example 3 (*Quaternionic pseudo-Kähler metrics*) Let $\langle\ ,\ \rangle$ be a nondegenerate quaternionic Hermitian inner product on $\mathbb{H}^{n+1}$. We put $M' = \{[v] \in \mathbb{H}P^n \mid \langle v, v\rangle \neq 0\}$. Then we have the orthogonal decomposition on M':

$$\underline{\mathbb{H}^{n+1}} = L + L^{\perp}, \tag{6}$$

where $L^\perp$ denotes the orthogonal complement of L with respect to $\langle\ ,\ \rangle$. From $\langle\ ,\ \rangle$, we define a pseudo-Riemannian metric g on M' (cf. [4] Sect. 3.2). Then we see that the Q-connection defined by the decomposition (6) is a pseudo-Riemannian connection with respect to g. Hence (M', Q, g) is a quaternionic pseudo-Kähler manifold.

5 The Gauss Maps of Transversally Complex Submanifolds of $\mathbb{H}P^n$

In this section, we generalize the theory of the mean curvature sphere (or conformal Gauss map) introduced by [4] to transversally complex submanifolds of $\mathbb{H}P^n$.

First we prepare the linear algebra of quaternionic vector spaces. Let V and W be right quaternionic vector spaces with $\dim_{\mathbb{H}} V = 1$ and $\dim_{\mathbb{H}} W = n$, respectively. We define a quaternionic structure $Q \subset \mathrm{End}(\mathrm{Hom}_{\mathbb{H}}(V, W))$ on the space $\mathrm{Hom}_{\mathbb{H}}(V, W)$ as follows: We take a non-zero vector $v \in V$ and define the map φ_v by $\varphi_v(F) = F(v)$ for $F \in \mathrm{Hom}_{\mathbb{H}}(V, W)$. Then φ_v is a real linear isomorphism of $\mathrm{Hom}_{\mathbb{H}}(V, W)$ onto W. We define I,J, and K in $\mathrm{End}(\mathrm{Hom}_{\mathbb{H}}(V, W))$ by $\varphi_v(IF) = \varphi_v(F)i$, $\varphi_v(JF) = \varphi_v(F)j$, $K = IJ$. We denote by Q the subspace of $\mathrm{End}(\mathrm{Hom}_{\mathbb{H}}(V, W))$ which is real linearly spanned by I,J, and K. Then Q is a quaternionic structure. Clearly Q is independent on the choice of $v \in V$.

We shall show a characterization of half-dimensional transversally complex subspaces of $\mathrm{Hom}_{\mathbb{H}}(V, W)$. It is a higher-dimensional analogue to Lemma 3 in [4].

Lemma 1 (1) *Let U be a transversally complex subspace of* $\mathrm{Hom}_{\mathbb{H}}(V, W)$ *with* $\dim_{\mathbb{R}} U = \frac{1}{2} \dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{H}}(V, W)$. *Then there exist complex structures* $J_1 \in \mathrm{End}_{\mathbb{H}}(V)$ *and* $J_2 \in \mathrm{End}_{\mathbb{H}}(W)$ *such that*

$$J_2 U = U, \quad U J_1 = U$$

$$U = \{F \in \mathrm{Hom}_{\mathbb{H}}(V, W) | J_2 F = F J_1\}.$$

Moreover the pair (J_1, J_2) *of complex structures is unique up to its sign.*

(2) *Conversely given complex structures* $J_1 \in \mathrm{End}_{\mathbb{H}}(V)$ *and* $J_2 \in \mathrm{End}_{\mathbb{H}}(W)$, *we put*

$$U = \{F \in \mathrm{Hom}_{\mathbb{H}}(V, W) | J_2 F = F J_1\}.$$

Then U is a transversally complex subspace of $\mathrm{Hom}_{\mathbb{H}}(V, W)$ *with* $\dim_{\mathbb{R}} U = \frac{1}{2} \dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{H}}(V, W)$. *Moreover we have*

$$U^\perp = \{F \in \mathrm{Hom}_{\mathbb{H}}(V, W) | J_2 F = -F J_1\}.$$

We describe maps $f : M \to \mathbb{H}P^n$ of a manifold M into $\mathbb{H}P^n$ and their differentials $df : TM \to T\mathbb{H}P^n$ in terms of quaternionic vector bundles. For a map f, we consider the pull-back bundle f^*L of the tautological bundle L over $\mathbb{H}P^n$. For simplicity, we use the same notation L instead of f^*L. Then we obtain the $\mathbb{H}$-line subbundle L of the product bundle $\mathscr{H} = M \times \mathbb{H}^{n+1}$. Conversely a $\mathbb{H}$-line subbundle $L \subset \mathscr{H} = M \times \mathbb{H}^{n+1}$ defines a map $f : M \to \mathbb{H}P^n$. Therefore we obtain an identification of maps $f : M \to \mathbb{H}P^n$ with $\mathbb{H}$-line subbundles $L \subset \mathscr{H} = M \times \mathbb{H}^{n+1}$. By Proposition 5, the differential df of f is viewed as a real linear homomorphism $df : TM \to \mathrm{Hom}(L, \mathscr{H}/L)$.

From now on, we study the following setting: Let M be a complex n-dimensional complex manifold and $f : M \to \mathbb{H}P^n$ a transversally complex immersion. This means that for the complex structure I of M and the section $\tilde{I} \in \Gamma(Q|_M)$ we have $df(IX) = \tilde{I} df(X)$ $X \in TM$. By the definition, the differential $df_p : T_pM \to \mathrm{Hom}(L_p, (\mathscr{H}/L)_p)$ is injective and the image $df_p(T_pM)$ is a transversally complex subspace of $\mathrm{Hom}(L_p, (\mathscr{H}/L)_p)$ with $\dim_{\mathbb{R}} df_p(T_pM) = \frac{1}{2} \dim_{\mathbb{R}} \mathrm{Hom}(L_p, (\mathscr{H}/L)_p)$. Therefore we can apply Lemma 1. Given a (vector-valued) 1-form ω on M, we define $*\omega$ by $*\omega(X) = \omega(IX)$. We consider higher dimensional analogues of two spheres S^2 in $\mathbb{H}P^1$ defined by [4] Sects. 3.4 and 4.2 Example 12.

Example 4 (*Complex projective spaces*) $\mathbb{C}P^n$. For $S \in \mathrm{End}(\mathbb{H}^{n+1})$ with $S^2 = -\mathrm{id}$, we define

$$S' = \{l \in \mathbb{H}P^n \mid Sl = l\} \subset \mathbb{H}P^n.$$

Then S' is a complex n-dimensional transversally complex submanifold and it is holomorphically diffeomorphic to a complex projective space $\mathbb{C}P^n$.

We attempt a generalization of the theory in [4] Sect. 5. Let $S : M \to \mathrm{End}(\mathbb{H}^{n+1})$ (or $S \in \Gamma(\mathrm{End}\,\mathscr{H})$) with $S^2 = -\mathrm{id}$ be a complex structure of $\mathscr{H} = M \times \mathbb{H}^{n+1}$. For this S, we define $\mathrm{End}(\mathbb{H}^{n+1})$ (or $\mathrm{End}\,\mathscr{H}$)-valued 1-forms A^+, A^- on M as follows:

$$A^+ = \frac{1}{4}(SdS + *dS), \quad A^- = \frac{1}{4}(SdS - *dS). \tag{7}$$

Since $f : M \to \mathbb{H}P^n$ is a transversally complex immersion, by Lemma 1 there exists a pair of complex structures $J_1 \in \Gamma(\mathrm{End}\,L)$, $J_2 \in \Gamma(\mathrm{End}\,\mathscr{H}/L)$ such that

$$df(IX) = df(X)J_1 = J_2 df(X) \qquad \text{for } X \in TM \quad .$$

We extend the pair (J_1, J_2) to a complex structure S of $\mathscr{H} = M \times \mathbb{H}^{n+1}$ i.e., find an $S \in \Gamma(\mathrm{End}\,\mathscr{H})$ such that $S^2 = -\mathrm{id}$ and

$$SL = L, \ \ S|_L = J_1, \ \ \pi_L S = J_2 \pi_L. \tag{8}$$

This implies $dS(L) \subset L$. In fact, for $\psi \in \Gamma(L)$, we have

$$\pi_L((d_X S)(\psi)) = \pi_L(d_X(S\psi) - S(d_X\psi)) = df(X)J_1\psi - J_2 df(X)\psi = 0 \ .$$

We show a theorem which is a higher dimensional version of Theorem 2 in [4].

Theorem 2 *Let $f : M \to \mathbb{H}P^n$ be a transversally complex immersion of a complex n-dimensional complex manifold M and $L \subset \mathscr{H} = M \times \mathbb{H}^{n+1}$ be its corresponding $\mathbb{H}$-line bundle. Then there exists a unique complex structure $S \in \Gamma(\mathrm{End}\ \mathscr{H})$ such that*
(1) $SL = L,\ dS(L) \subset L$,
(2) $*df = df S|_L = \pi_L S df$,
(3) $A^-|_L = 0$.

The complex structure $S \in \Gamma(\mathrm{End}\ \mathscr{H})$ or $S : M \to \mathrm{End}(\mathbb{H}^{n+1})$ defined by Theorem 2 is called a *Gauss map* of a transversally complex immersion $f : M \to \mathbb{H}P^n$. The Gauss map depends on the quaternionic structure of $\mathbb{H}P^n$ and does not depend on $Q-$connections. Moreover it is invariant by the action of $GL(n+1, \mathbb{H})$. We expect that Gauss maps shall be useful for studying transversally complex immersions.

We denote by $\mathscr{S}$ the set of all complex structures of $\mathbb{H}^{n+1}$, i.e.,

$$\mathscr{S} = \{\ S \in \mathrm{End}(\mathbb{H}^{n+1}) \mid S^2 = -\mathrm{id}\ \}.$$

Then $\mathscr{S}$ is a closed submanifold of $\mathrm{End}(\mathbb{H}^{n+1})$ with real dimension $2(n+1)^2$. Clearly $\mathscr{S}$ is invariant by the action of $GL(n+1, \mathbb{H})$ and the group $GL(n+1, \mathbb{H})$ acts on $\mathscr{S}$ transitively. The structure of a pseudo-Hermitian symmetric space is defined on $\mathscr{S}$ naturally. It has the signature $((n+1)(n+2), n(n+1))$. We view the Gauss map S of a transversally complex immersion $f : M \to \mathbb{H}P^n$ as a map of M to $\mathscr{S}$. Then S is holomorphic (resp. anti-holomorphic) if and only if $A^- = 0$ (resp. $A^+ = 0$).

Given a Q-connection $\tilde{\nabla}$ on $\mathbb{H}P^n$, we study the Gauss map of a transversally complex immersion. Let $\underline{\mathbb{H}^{n+1}} = L + L^c$ be the corresponding decomposition to the Q-connection $\tilde{\nabla}$.

Proposition 6 *Let M be a complex n-dimensional complex manifold and $f : M \to \mathbb{H}P^n$ a transversally complex immersion with the Gauss map $S : M \to \mathscr{S} \subset \mathrm{End}(\mathbb{H}^{n+1})$. When we are given a Q-connection $\tilde{\nabla}$ on $\mathbb{H}P^n$, the following three conditions are mutually equivalent:*
(1) The complex structure $\tilde{I}$ is parallel with respect to the induced connection $\tilde{\nabla}$ on $Q|_M$.
(2) The $(1,1)$-part σ_- of the second fundamental form vanishes.
(3) The Gauss map S preserves the complementary bundle L^c, i.e., $S(L^c) = L^c$.

Proposition 7 *We equip $\mathbb{H}P^n$ with the quaternionic Kähler structure $(Q, \tilde{g})$. Let $f : M \to \mathbb{H}P^n$ be a totally complex immersion of a complex n-dimensional complex manifold M with the Gauss map $S : M \to \mathscr{S}$.Then we have $A^+ = 0$ and hence the Gauss map $S : M \to \mathscr{S}$ is anti-holomorphic.*

We shall show a characterization of complex projective spaces $\mathbb{C}P^n$ which are transversally complex submanifolds of $\mathbb{H}P^n$.

Theorem 3 *Let M be a complex n-dimensional transversally complex submanifold of $\mathbb{H}P^n$. If the $(2,0)+(0,2)$-part σ_+ of the second fundamental form vanishes, then the Gauss map $S : M \to \mathscr{S}$ is constant. In particular, M is an open submanifold of the complex projective space $\mathbb{C}P^n$ defined by S as in Example 4.*

Remark 2 (*A geometric meaning of a Gauss map*) Let M be a complex n-dimensional transversally complex submanifold of $\mathbb{H}P^n$ and $S : M \to \mathscr{S}$ be its Gauss map defined by Theorem 2. We consider the complex structure S_p at each point $p \in M$. We denote by S'_p the complex projective space defined by S_p as in Example 4. Then S'_p contains p. The tangent spaces of S'_p and M at p coincide and σ_- of S'_p and M at p coincide. In [4] for a surface M of $S^4 = \mathbb{H}P^1$, the complex structure $S \in \Gamma(\mathrm{End}\,\mathscr{H})$ is called *the mean curvature sphere* of M.

6 Problems

We propose some problems related with the geometry of transversally complex submanifolds of $\mathbb{H}P^n$.

1. Construct good examples of transversally complex submanifolds of $\mathbb{H}P^n$.

1–1. Homogeneous ones, that is, the orbits by the action of closed subgroups of $GL(n+1,\mathbb{H})$. In [3], they proved that maximal totally complex submanifolds of $\mathbb{H}P^n$ which are the orbits of compact Lie groups of isometries are exhausted by the ones in Example 1. Find homogeneous transversally complex submanifolds which are not totally complex.

1–2. Compact non-homogeneous ones. It is not easy to construct examples of compact totally complex submanifolds of $\mathbb{H}P^n$ (see [8]).

2. Characterize totally complex submanifolds which are given in Example 1 by the $(2,0)+(0,2)$-part σ_+ or their Gauss maps.

3. Define a functional for compact transversally complex submanifolds of $\mathbb{H}P^n$ by their Gauss maps and study their variational problems. In [4], they discuss Willmore functional in terms of the mean curvature spheres. Attempt to generalize their results.

Acknowledgements The author is supported by JSPS KAKENHI Grant Number 25400065.

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On Floer Homology of the Gauss Images of Isoparametric Hypersurfaces

Yoshihiro Ohnita

Abstract The Gauss images of isoparametric hypersurfaces in the unit standard sphere provide compact minimal (thus monotone) Lagrangian submanifolds embedded in complex hyperquadrics. Recently we used the Floer homology and the lifted Floer homology for monotone Lagrangian submanifolds in order to study their Hamiltonian non-displaceability in our recent joint paper with Hiroshi Iriyeh, Hui Ma and Reiko Miyaoka. In this note we will explain the spectral sequences for the Floer homology and the lifted Floer homology of monotone Lagrangian submanifolds and their applications to the Gauss images of isoparametric hypersurfaces. They are the main technical part in our joint work. Moreover we will suggest some related open problems for further research.

1 Introduction

Let N^n be an isoparametric hypersurface in the unit standard sphere $S^{n+1} \subset \mathbb{R}^{n+2}$. By the structure theory of isoparametric hypersurfaces (see [16]), N^n is nothing but a hypersurface of constant principal curvatures and if we denote by g the distinct number of principal curvatures of N^n and by $m_1, m_2, \ldots, m_g$ their multiplicities, then we know $m_i = m_{i+2}$ $(i \mod g)$. Moreover, it is known that N^n can be extended to a compact oriented hypersurface embedded in S^{n+1}.

We know that isoparametric hypersurfaces in S^{n+1} provide a nice class of Lagrangian submanifolds in the complex hyperquadric $Q_n(\mathbb{C})$. Note that a complex hyperquadric $Q_n(\mathbb{C})$ can be identified with a real Grassmann manifold $\widetilde{Gr}_2(\mathbb{R}^{n+2})$ of oriented 2-dimensional vector subspaces of $\mathbb{R}^{n+2}$ in the standard way:

Y. Ohnita (✉)
Department of Mathematics and OCAMI, Osaka City University,
Osaka 558-8585, Japan
e-mail: ohnita@sci.osaka-cu.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_20

$$\begin{aligned} Q_n(\mathbb{C}) &\cong \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong SO(n+2)/(SO(2)\times SO(n)) \\ [\mathbf{a}+\sqrt{-1}\mathbf{b}] \leftrightarrow &\quad \mathbf{a}\wedge\mathbf{b} \quad \leftrightarrow \left[\mathbf{a},\mathbf{b},\ldots\right](SO(2)\times SO(n)) \end{aligned}$$

It is an irreducible compact Hermitian symmetric space of rank 2 for $n \geq 3$. If $n = 2$, then $Q_2(\mathbb{C}) \cong S^2 \times S^2$, and if $n = 1$, then $Q_1(\mathbb{C}) \cong \mathbb{C}P^1$. We denote by ω_{std} the standard Kähler form of $Q_n(\mathbb{C})$.

In general the Gauss map $\mathcal{G}$ of an oriented hypersurface N^n immersed in the unit standard sphere S^{n+1} is defined by

$$\mathcal{G} : N^n \ni p \longmapsto [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] \in Q_n(\mathbb{C}).$$

Then we know that $\mathcal{G} : N^n \to (Q_n(\mathbb{C}), \omega_{\mathrm{std}})$ is always a Lagrangian immersion. Palmer [23] gave a formula expressing the mean curvature form of $\mathcal{G}$ in terms of principal curvatures of N^n in S^{n+1} and from this formula he observed that if N^n has constant principal curvatures, then the Gauss map $\mathcal{G} : N^n \to (Q_n(\mathbb{C}), \omega_{\mathrm{std}})$ is a minimal Lagrangian immersion. Note that the Gauss map is not necessary an embedding into $Q_n(\mathbb{C})$.

For each isoparametric hypersurface N^n in S^{n+1} the image of the Gauss map, which is called the *Gauss image* $\mathcal{G}(N^n)$, has the following nice properties.

Theorem 1 ([12, 14, 21]) *Suppose that N^n is a compact oriented isoparametric hypersurface embedded in S^{n+1} with g distinct principal curvatures and multiplicities (m_1, m_2). Then the Gauss image $\mathcal{G}(N^n)$ has the following properties:*

(1) *The Gauss image $L^n = \mathcal{G}(N^n)$ is a compact smooth minimal Lagrangian submanifold embedded in $Q_n(\mathbb{C})$.*
(2) *The Gauss map $\mathcal{G}$ into the Gauss image $L^n = \mathcal{G}(N^n)$ gives a covering map $\mathcal{G} : N^n \to L^n = \mathcal{G}(N^n) \cong N^n/\mathbb{Z}_g$ with the covering transformation group $\mathbb{Z}_g$.*
(3) *The Gauss image $L^n = \mathcal{G}(N^n)$ is monotone in $Q_n(\mathbb{C})$ and its minimal Maslov number Σ_L is given by*

$$\Sigma_L = \frac{2n}{g} = \begin{cases} m_1 + m_2, & \text{if } g \text{ is even}, \\ 2m_1, & \text{if } g \text{ is odd}. \end{cases} \tag{1}$$

The Gauss image $\mathcal{G}(N^n)$ is orientable (resp. non-orientable) if and only if $2n/g$ is even (resp. odd).

It is a natural and interesting problem to study the properties of the Guass image of isoparametric hypersurfaces in S^{n+1} as Lagrangian submanifolds embedded in $Q_n(\mathbb{C})$ [12–15, 23].

Recently in our recent joint paper with Hiroshi Iriyeh, Hui Ma and Reiko Miyaoka [11] in order to study their Hamiltonian non-displaceability we used the Floer homology and the lifted Floer homology for monotone Lagrangian submanifolds. In this note we will explain the construction of the Floer homology and the spectral sequences for monotone Lagrangian submanifolds, and also their lifted Floer homology (Floer, Y.-G. Oh, Biran, Damian), and their applications to the Gauss images of isoparametric hypersurfaces. They are the main technical part in our joint work. The ideas to use the spectral sequence and the lifted Floer homology in this case are due to H. Iriyeh [9]. Moreover we will suggest some related open problems for further research.

Throughout this article any manifold is smooth and connected.

2 Floer Homology of Monotone Lagrangian Submanifolds

Let (M, ω) be a symplectic manifold. If a diffeomorphism ϕ of M is given by $\phi = \phi_1$ for some time-dependent Hamiltonian H_t ($t \in [0, 1]$) and an isotopy $\phi_t : M \to M$ ($t \in [0, 1]$) of M satisfying the Hamiltonian equation

$$\frac{d\phi_t(x)}{dt} = (X_{H_t})_{\phi_t(x)} \quad \text{and} \quad \phi_0(x) = x \qquad (x \in M),$$

then ϕ is called a *Hamiltonian diffeomorphism* of (M, ω) and $\{\phi_t\}_{t\in[0,1]}$ is called a *Hamiltonian isotopy* of (M, ω). Here X_{H_t} denotes a Hamiltonian vector field corresponding to a Hamiltonian H_t with respect to ω. Let $\mathrm{Haml}(M, \omega)$ denote a set of all Hamiltonian diffeomorphisms of (M, ω). Then $\mathrm{Haml}(M, \omega)$ is a subgroup of the identity component $\mathrm{Symp}^0(M, \omega)$ of the symplectic diffeomorphism group of (M, ω). A Lagrangian submanifold L in M is called *Hamiltonian non-displaceable* if $L \cap \phi(L) \neq \emptyset$ for any $\phi \in \mathrm{Haml}(M, \omega)$, and it is called *Hamiltonian displaceable* otherwise. It is well-known that in the 2-dimensional standard sphere a small circle is Hamiltonian displaceable but a great circle is Hamiltonian non-displaceable. Then the following is one of elementary questions in symplectic geometry:

Question. What Lagrangian submanifolds are Hamiltonian non-displaceable?

Let (M, ω) be a closed symplectic manifold and L be a closed (i.e. compact without boundary) Lagrangian submanifold embedded in M. Let $\phi \in \mathrm{Haml}(M, \omega)$ such that L and $\phi(L)$ intersects transversally (denoted by the symbol $L \pitchfork \phi(L)$). Let $(\phi_t)_{t\in[0,1]}$ be a Hamiltonian isotopy of (M, ω) with $\phi_1 = \phi$ and set $L_t := \phi_t(L)$ ($t \in [0, 1]$).

Choose an almost complex structure J on M compatible with ω (i.e. $\omega(\,\cdot\,, J\,\cdot\,)$ is a Riemannian metric on M). Define

$$\begin{aligned} \mathscr{M}(L_0, L_1) := \Big\{ u \in C^\infty(\mathbb{R} \times [0, 1]) \mid \frac{\partial u}{\partial \tau} + J_t \frac{\partial u}{\partial t} &= 0, \\ u(\tau, 0) \subset L,\ u(\tau, 1) &\subset \phi_1(L), \\ E(u) := \frac{1}{2} \int_{\mathbb{R}\times[0,1]} \|du\|^2 d\tau dt &< \infty \Big\} \end{aligned}$$

For $x, y \in L_0 \cap L_1$, set

$$\mathscr{M}(x, y) := \{ u \in \mathscr{M}(L_0, L_1) \mid \lim_{\tau \to -\infty} u(\tau, \cdot) = x, \ \lim_{\tau \to \infty} u(\tau, \cdot) = y \}.$$

Floer [6] showed

$$\mathscr{M}(L_0, L_1) = \bigcup_{x,y \in L_0 \cap L_1} \mathscr{M}(x, y).$$

For a generic choice J, a neighborhood of each $u \in \mathscr{M}(x, y)$ is a smooth manifold of finite dimension equal to $\mu_u(x, y) =$ the Maslov–Viterbo index of u [5, 24].

Let $\widehat{\mathscr{M}}(x, y) := \mathscr{M}(x, y)/\mathbb{R}$ be the moduli space of holomorphic strips joining from x to y modulo translations with respect to τ. Then note that for such a choice J, a neighborhood of each $[u] \in \widehat{\mathscr{M}}(x, y)$ is a smooth manifold of finite dimension equal to $\mu_u(x, y) - 1$. Denote by $\widehat{\mathscr{M}}^0(x, y)$ (resp. $\widehat{\mathscr{M}}^1(x, y)$) the 0-dimensional (resp. 1-dimensional) component of $\widehat{\mathscr{M}}(x, y)$.

For a given Lagrangian submanifold L of a symplectic manifold (M, ω), two kinds of group homomorphisms $I_{\mu,L} : \pi_2(M, L) \to \mathbb{Z}$ and $I_{\omega,L} : \pi_2(M, L) \to \mathbb{R}$ are defined as follows: For a smooth map $u : (D^2, \partial D^2) \to (M, L)$ belonging to $A \in \pi_2(M, L)$, choose a trivialization of the pull-back bundle $u^{-1}TM \cong D^2 \times \mathbb{C}^n$ as a symplectic vector bundle, which is unique up to the homotopy. This gives a smooth map $\tilde{u}$ from $S^1 = \partial D^2$ to $\Lambda(\mathbb{C}^n)$. Here $\Lambda(\mathbb{C}^n)$ denotes a Grassmann manifold of Lagrangian vector subspaces of $\mathbb{C}^n$. Using the Moslov class $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z})$, we can define $I_{\mu,L}(A) := \mu(\tilde{u})$. Next $I_{\omega,L}$ is defined by $I_{\omega,L}(A) := \int_{D^2} u^*\omega$. $I_{\mu,L}$ is invariant under symplectic diffeomorphisms and $I_{\omega,L}$ is invariant under Hamiltonian diffeomorphisms but not under symplectic diffeomorphisms.

A Lagrangian submanifold L is called *monotone* if $I_{\mu,L} = \lambda I_{\omega,L}$ for some constant $\lambda > 0$. We denote by $\Sigma_L \in \mathbb{Z}_+$ a positive generator of an additive subgroup $\mathrm{Im}(I_{\mu,L}) \subset \mathbb{Z}$ and Σ_L is called the *minimal Maslov number* of L. It is known that any compact minimal Lagrangian submanifold in an Einstein–Kähler manifold of positive Einstein constant is monotone [22].

The compactness and compactification of 0-dimensional and 1-dimensional moduli spaces of holomorphic strips are due to Gromov [8], Floer [4], Y.-G. Oh [17].

Theorem 2 (Compactness) *Suppose that L is compact and monotone with $\Sigma_L \geq 2$. Let $x, y \in L_0 \cap L_1$ and $A > 0$. Let $\{u_\alpha\} \subset \mathscr{M}(x, y)$ be a sequence with constant index $\mu(u_\alpha) = \mu_0 \leq 2$ and with $E(u_\alpha) < A$. Then there exists a finite subset $\{z_0, \ldots, z_k\} \subset L_0 \cap L_1$ with $z_0 = x$ and $z_k = y$, some $u^i \in \mathscr{M}(z_{i-1}, z_i)$ for $i = 1, \ldots, k$, and a sequence of real numbers $\{\sigma_\alpha^i\}_\alpha$ for $i = 1, \ldots, k$, such that for each $i = 1, \ldots, k$ the sequence $\{u_\alpha(\tau + \sigma_\alpha^i, t)\}_\alpha$ converges to $u^i(\tau, t)$ in C^∞_{loc}, and moreover $\sum_{i=1}^k \mu(u^k) = \mu_0$.*

If $\mu_0 = 1$, then we have $k = 1$ and $\{u_\alpha(\tau + \sigma_\alpha, t)\}_\alpha$ converges to $u^1(\tau, t)$ in C^∞_{loc}, because of $\mu(u^k) \geq 1$. It implies that the 0-dimensional component of the moduli space $\widehat{\mathscr{M}}^0(x, y)$ is compact and thus a finite set.

Theorem 3 (Compactification) *Suppose that L is compact and monotone with* $\Sigma_L \geq 3$. *Then*

$$\overline{\widehat{\mathscr{M}}^1(x,y)} := \widehat{\mathscr{M}}^1(x,y) \cup \bigcup_{z \in L_0 \cap L_1} (\widehat{\mathscr{M}}^0(x,z) \times \widehat{\mathscr{M}}^0(z,y))$$

is a compact 1*-dimensional smooth manifold whose boundary is*

$$\bigcup_{z \in L_0 \cap L_1} (\widehat{\mathscr{M}}^0(x,z) \times \widehat{\mathscr{M}}^0(z,y)).$$

Let

$$CF(L,\phi) := \bigoplus_{x \in L \cap \phi(L)} \mathbb{Z}_2\, x$$

denote a free $\mathbb{Z}_2$-module over all intersection points of L and $\phi(L)$ where $\phi \in \mathrm{Haml}(M,\omega)$. By Theorem 2, since $\widehat{\mathscr{M}}^0(x,y)$ is finite, we can define $n(x,y) := \sharp \widehat{\mathscr{M}}^0(x,y) \mod 2$. Then the Floer boundary operator ∂_J is defined by

$$\partial_J(x) := \sum_{y \in L \cap \phi(L)} n(x,y)\, y \qquad (y \in L \cap \phi(L)).$$

Assume that L is monotone with minimal Maslov number $\Sigma_L \geq 2$. Then we know that $\partial_J \circ \partial_J = 0$, by Theorem 3 if $\Sigma_L \geq 3$, or by [18] if $\Sigma_L \geq 2$. The homology $H_*(CF(L,\phi), \partial_J)$ for the chain complex $(CF(L,\phi), \partial_J)$ does not depend on the choice of $J \in \mathscr{J}_{reg}(M,\omega)$ and $\phi \in \mathrm{Haml}(M,\omega)$ (Floer [7], Y.-G. Oh [17]). The *Floer homology* of L is defined by

$$HF(L) := H_*(CF(L,\phi), \partial_J).$$

Now fix an element $x_0 \in L \cap \phi(L)$. We define a grading of $x \in L \cap \phi(L)$ by $\mu_u(x,x_0) \mod \Sigma_L$. Here we use a fact that $\mu_u(x,x_0) - \mu_v(x,x_0)$ is a multiple of Σ_L for arbitrary smooth maps $u, v : [0,1] \times [0,1] \to M$ with $u(\tau,0), v(\tau,0) \in L$, $u(\tau,1), v(\tau,1) \in \phi(L)$ and $u(\tau,0), v(\tau,0) \in L$, $u(\tau,1), v(\tau,1) \in \phi(L)$ ([17], p. 973, Lemma 4.7). Thus the Floer complex $CF(L,\phi)$ has a $\mathbb{Z}/\Sigma_L$-grading, which depends on a choice of a base intersection point $x_0 \in L \cap \phi(L)$. Denote this grading by

$$CF(L,\phi) = \bigoplus_{i=0}^{\Sigma_L - 1} CF_{i \bmod \Sigma_L}(L,\phi,x_0),$$

where

$$CF_{i \bmod \Sigma_L}(L,\phi,x_0) := \bigoplus_{x \in L \cap \phi(L),\, \mu_u(x,x_0) \equiv i \bmod \Sigma_L} \mathbb{Z}_2\, x.$$

Let $x, y \in L \cap \phi(L)$. For $u \in \widehat{\mathscr{M}^0}(x, y)$, since by a composition formula ([5], p. 406)

$$\mu_{uv}(x, x_0) = \mu_u(x, y) + \mu_v(y, x_0)$$

where uv is a composition of u and v, we have

$$\mu_v(y, x_0) = \mu_{uv}(x, x_0) - \mu_u(x, y) = \mu_{uv}(x, x_0) - 1.$$

Thus the Floer boudary operator decreases the grading by 1:

$$\partial_J : CF_{i \bmod \Sigma_L}(L, \phi, x_0) \longrightarrow CF_{i-1 \bmod \Sigma_L}(L, \phi, x_0).$$

Hence it induces a $\mathbb{Z}/\Sigma_L$-grading of the Floer homology as

$$HF(L, \phi) = \bigoplus_{i=0}^{\Sigma_L - 1} HF_{i \bmod \Sigma_L}(L, \phi.x_0).$$

The grading of the Floer homology of L is also preserved under any Hamiltonian diffeomorphism of (M, ω) (Floer, Oh [17]).

3 Spectral Sequence for Floer Homology and Lifted Floer Homology

Let (M^{2n}, ω) be a compact symplectic manifold of dimension $2n$. Let L be a compact Lagrangian submanifold embedded in M. For a Hamiltonian isotopy $(\phi_t)_{t \in [0,1]}$ of M with $L \pitchfork \phi(L)$, set $\phi = \phi_1 \in \mathrm{Haml}(M, \omega)$ and $L' = \phi(L) = \phi_1(L)$.

Consider a Morse–Smale function f on L and a particular Hamiltonian isotopy $(\phi_t)_{t \in [0,1]}$ which maps L to $\phi_t(L) = d(tf)(L) \subset \mathscr{W} \subset M$, as in [7]. Here $\mathscr{W}$ is a Weinstein neighborhood of L in M which is symplectically diffeomorphic to a tubular neighborhood of the zero section of T^*L. In this situation note that $L \cap \phi_1(L)$ coincide with the critical point set $\mathrm{Crit}(f)$ of f on L. We may assume that f has exactly one relative minimum point x_0 on L. We choose x_0 as a base intersection point of $L \cap \phi(L)$. Let $\mathrm{Crit}(f)$ denote the set of all critical points of f and $\mathrm{Crit}_k(f)$ the subset of all critical points of f with index $\mathrm{ind}(f)_x = k$. Denote by (C^f_*, ∂^f) the Morse complex for f, where

$$C^f_* = \bigoplus_{k=0}^{n} C^f_k, \quad \text{where} \quad C^f_k = \bigoplus_{x \in \mathrm{Crit}_k(f)} \mathbb{Z}_2\, x$$

and $\partial^f : C^f_k \to C^f_{k-1}$ is the Morse boudary operator of f. In this situation note that the Maslov–Viterbo index of u coincides with the Morse index of f: For each $x \in L \cap$

$\phi(L) = \mathrm{Crit}(f)$ and a smooth map $v : [0,1] \times [0,1] \to \mathscr{W} \subset M$ with $v(\tau, 0) \in L$, $v(\tau, 1) \in \phi(L)$, $v(0, t) = x$ and $v(1, t) = y$,

$$\mu_v(x, x_0) = \mathrm{ind}(f)_x - \mathrm{ind}(f)_{x_0} = \mathrm{ind}(f)_x$$

([24, p. 370, Proposition 5], [20, p. 318, Lemma 4]). Thus by the definition of a grading of $CF(L, \phi)$ we have

$$CF_{i \mod \Sigma_L}(L, \phi, x_0) = \bigoplus_{k \in \mathbb{Z},\, k \equiv i \mod \Sigma_L} C_k^f = \bigoplus_{\ell \in \mathbb{Z}} C^f_{i + \ell \Sigma_L}.$$

We follow the argument of [20]. For a sufficiently small t we can take a disk $w : (D^2, \partial D^2) \to (M, L)$ with $I_{\omega, L}([w]) > 0$ by gluing $u \in \widehat{\mathscr{M}^0}(x, y)$ with $\mathrm{Im}\, u \not\subset \mathscr{W}$ to a thin strip v between $\phi_t(L)$ and L connecting y and x. By the monotonicity of L note that $I_{\mu, L}([w]) = \lambda\, I_{\omega, L}([w]) > 0$, and thus $I_{\mu, L}([w]) = \ell \Sigma_L$ for some $\ell \in \mathbb{N}$. Thus we have

$$\begin{aligned} 0 < I_{\mu, L}([w]) &= \mu_u(x, y) - \mu_{u'}(x, y) \\ &= 1 - (\mathrm{ind}(f)_x - \mathrm{ind}(f)_y) \\ &= 1 - \mathrm{ind}(f)_x + \mathrm{ind}(f)_y \leq n + 1. \end{aligned}$$

Hence we obtain $1 \leq \ell \leq \frac{n+1}{\Sigma_L}$ and $\mathrm{ind}(f)_y = \mathrm{ind}(f)_x - 1 + \ell \Sigma_L$. Set $\nu := \left[\frac{n+1}{\Sigma_L}\right]$. Since we see that $\partial_J(C_k^f) \subset \bigoplus_{\ell=0}^{\nu} C^f_{k-1+\ell\Sigma_L}$, the Floer boudary operator ∂_J can be decomposed as

$$\partial_J = \partial_0 + \partial_1 + \cdots + \partial_\nu,$$

where $\partial_\ell : C_*^f \to C^f_{*-1+\ell\Sigma_L}$ ($\ell = 1, \ldots, \nu$). Here note that ∂_0 counts *small* isolated Floer trajectries (J-holomorphic strips) contained in a Weinstein neighborhood of L and it coincides with the Morse boudary operator ∂^f of f (local Floer homology [20]). The operator $\partial_1 + \cdots + \partial_\nu$ expresses a contribution of *large* isolated Floer trajectries.

Y.-G. Oh [20] and Biran [1] showed the existence of a spectral sequence $\{E_r^{p,q}, d_r\}$ converging towards the Floer homology. Such a spectral sequence was constructed by Biran in the following way [1].

Let $A := \mathbb{Z}_2[T, T^{-1}] = \bigoplus_{k \in \mathbb{Z}} A^{k\Sigma_L}$ be the algebra of Laurent polynomials over $\mathbb{Z}_2$ with the variable T. Here define $\deg(T) = \Sigma_L$ and $A^{k\Sigma_L} := \mathbb{Z}_2 T^k$. Now let

$$\widetilde{C} := C^f \otimes A = \bigoplus_{k \in \mathbb{Z}} C^f \otimes A^{k\Sigma_L} = \bigoplus_{i \in \mathbb{Z}} \widetilde{C}_i,$$

where

$$\widetilde{C}_i := \bigoplus_{k \in \mathbb{Z}} C^f_{i - k\Sigma_L} \otimes A^{k\Sigma_L} = \bigoplus_{k \in \mathbb{Z},\, \frac{i-n}{\Sigma_L} \leq k \leq \frac{i}{\Sigma_L}} C^f_{i-k\Sigma_L} \otimes A^{k\Sigma_L}.$$

Define $\widetilde{\partial_J} : \widetilde{C}_* \to \widetilde{C}_{*-1}$ by

$$\widetilde{\partial_J} := \partial_0 \otimes 1 + \partial_1 \otimes \tau + \cdots + \partial_\nu \otimes \tau^\nu,$$

where each $\tau^\ell : A^* \to A^{*-\ell\Sigma_L}$ is the multiplication by $T^{-\ell}$. Then we know that $\widetilde{\partial_J} : \widetilde{C} \to \widetilde{C}$ satisfies $\widetilde{\partial_J} \circ \widetilde{\partial_J} = 0$, that is, $(\widetilde{C}, \widetilde{\partial})$ is a chain complex. Moreover, as vector spaces over $\mathbb{Z}_2$, we obtain

$$HF_{i \bmod \Sigma_L}(L) \cong H_i(\widetilde{C}, \widetilde{\partial_J}) = \frac{\mathrm{Ker}(\widetilde{\partial_J} : \widetilde{C}_i \to \widetilde{C}_{i-1})}{\mathrm{Im}(\widetilde{\partial_J} : \widetilde{C}_{i+1} \to \widetilde{C}_i)} \quad (\forall i \in \mathbb{Z})$$

and

$$H(\widetilde{C}, \widetilde{\partial_J}) = \frac{\mathrm{Ker}(\widetilde{\partial_J} : \widetilde{C} \to \widetilde{C})}{\mathrm{Im}(\widetilde{\partial_J} : \widetilde{C} \to \widetilde{C})} = \bigoplus_{i \in \mathbb{Z}} H_i(\widetilde{C}, \widetilde{\partial_J}) \cong HF(L) \otimes_{\mathbb{Z}_2} A.$$

For each $p \in \mathbb{Z}$, set

$$A_p := \bigoplus_{k \le p} A^{k\Sigma_L} \subset A$$

and define

$$F^p\widetilde{C} := C^f \otimes A_p.$$

Then we have an increasing filtration on $\widetilde{C} = \bigcup_{p \in \mathbb{Z}} F^p\widetilde{C}$:

$$\cdots \subset F^{-1}\widetilde{C} \subset F^0\widetilde{C} \subset \cdots \subset F^p\widetilde{C} \subset F^{p+1}\widetilde{C} \subset \cdots$$

For each $p, l \in \mathbb{Z}$, define

$$F^p\widetilde{C}_l := F^p\widetilde{C} \cap \widetilde{C}_l = \bigoplus_{k \le p} C^f_{l-k\Sigma_L} \otimes A^{k\Sigma_L}.$$

Then for each $l \in \mathbb{Z}$ we have the increasing filtration on $\widetilde{C}_l = \bigcup_{p \in \mathbb{Z}} F^p\widetilde{C}_l$

$$\cdots \subset F^{-1}\widetilde{C}_l \subset F^0\widetilde{C}_l \subset \cdots \subset F^p\widetilde{C}_l \subset F^{p+1}\widetilde{C}_l \subset \cdots$$

which satisfies $F^p\widetilde{C}_l = \{0\}$ for any $p < \dfrac{l-n}{\Sigma_L}$ and $F^p\widetilde{C}_l = \widetilde{C}_l$ for any $p \ge \dfrac{l}{\Sigma_L}$. And for each $p \in \mathbb{Z}$, $F^p\widetilde{C}$ has a grading

$$F^p\widetilde{C} = \bigoplus_{l \in \mathbb{Z}} F^p\widetilde{C}_l.$$

For each $p \in \mathbb{Z}$ with $p > \dfrac{i}{\Sigma_L}$,

$$H_i(F^p\widetilde{C}, \widetilde{\partial_J}) = \frac{\mathrm{Ker}(\widetilde{\partial_J} : F^p\widetilde{C}_i \to F^p\widetilde{C}_{i-1})}{\mathrm{Im}(\widetilde{\partial_J} : F^p\widetilde{C}_{i+1} \to F^p\widetilde{C}_i)}$$
$$= \frac{\mathrm{Ker}(\widetilde{\partial_J} : F\widetilde{C}_i \to F\widetilde{C}_{i-1})}{\mathrm{Im}(\widetilde{\partial_J} : F\widetilde{C}_{i+1} \to F\widetilde{C}_i)} = H_i(\widetilde{C}, \widetilde{\partial_J}) \cong HF_{i \bmod \Sigma_L}(L)$$

and

$$\begin{aligned} H(F^p\widetilde{C}, \widetilde{\partial_J}) &= \frac{\mathrm{Ker}(\widetilde{\partial_J} : F^p\widetilde{C} \to F^p\widetilde{C})}{\mathrm{Im}(\widetilde{\partial_J} : F^p\widetilde{C} \to F^p\widetilde{C})} \\ &= \bigoplus_{i\in\mathbb{Z}} H_i(F^p\widetilde{C}, \widetilde{\partial_J}) \\ &= \bigoplus_{i<p\Sigma_L} H_i(F^p\widetilde{C}, \widetilde{\partial_J}) \oplus \bigoplus_{p\Sigma_L \le i} H_i(F^p\widetilde{C}, \widetilde{\partial_J}) \\ &= \bigoplus_{i<p\Sigma_L} H_i(\widetilde{C}, \widetilde{\partial_J}) \oplus \bigoplus_{p\Sigma_L \le i \le p\Sigma_L+n} H_i(F^p\widetilde{C}, \widetilde{\partial_J}) \\ &\cong \bigoplus_{i<p\Sigma_L} HF_{i \bmod \Sigma_L}(L) \oplus \bigoplus_{p\Sigma_L \le i \le p\Sigma_L+n} H_i(F^p\widetilde{C}, \widetilde{\partial_J}). \end{aligned}$$

For each $p \in \mathbb{Z}$, since $(F^p\tilde{C} = \bigoplus_{l\in\mathbb{Z}} F^p\widetilde{C}_l, \widetilde{\partial_J})$ is a graded filtered complex with filtration $\{\cdots \subset F^{p-2}\widetilde{C} \subset F^{p-1}\widetilde{C} \subset F^p\widetilde{C}\}$ and for each $l \in \mathbb{Z}$ the filtration $\{\cdots \subset F^{p-2}\widetilde{C}_l \subset F^{p-1}\widetilde{C}_l \subset F^p\widetilde{C}_l\}$ has finite length, there exists a spectral sequence which converges to $H_*(F^p\tilde{C}, \widetilde{\partial_J})$ (cf. Bott-Tu [2, p. 160, Theorem 14.6]). The following spectral sequence $\{E_r^{p,q}, d_r\}$ converging to $HF(L)$ was constructed by Biran [1]:

(1)
$$E_0^{p,q} = C^f_{p+q-p\Sigma_L} \otimes A^{p\Sigma_L}, \quad d_0 = [\partial_0] \otimes 1,$$

(2)
$$E_1^{p,q} = H_{p+q-p\Sigma_L}(L, \mathbb{Z}_2) \otimes A^{p\Sigma_L}, \quad d_1 = [\partial_1] \otimes T^{-\Sigma_L},$$

where
$$[\partial_1] : H_{p+q-p\Sigma_L}(L; \mathbb{Z}_2) \to H_{p+q-1-(p-1)\Sigma_L}(L; \mathbb{Z}_2)$$

is induced by the operator ∂_1.

(3) For each $r \ge 1$, $E_r^{p,q}$ has the form $E_r^{p,q} = V_r^{p,q} \otimes A^{p\Sigma_L}$ and
$$d_r = \delta_r \otimes T^{-r\Sigma_L} : E_r^{p,q} \to E_r^{p-r,q+r-1},$$

where each $V_r^{p,q}$ is a vector space over $\mathbb{Z}_2$ and $\delta_r : V_r^{p,q} \to V_r^{p-r,q+r-1}$ are homomorphisms defined for all p, q satisfying $\delta_r \circ \delta_r = 0$. Moreover, it holds

$$V_{r+1}^{p,q} = \frac{\mathrm{Ker}(\delta_r : V_r^{p,q} \to V_r^{p-r,q+r-1})}{\mathrm{Im}(\delta_r : V_r^{p+r,q-r+1} \to V_r^{p,q})}.$$

In particular, we have $V_0^{p,q} = C^f_{p+q-p\Sigma_L}$, $V_1^{p,q} = H_{p+q-p\Sigma_L}(L;\mathbb{Z}_2)$, $\delta_1 = [\partial_1]$.

(4) $E_r^{p,q}$ collapses to $E_{\nu+1}^{p,q} = E_{\nu+2}^{p,q} = \cdots = E_\infty^{p,q}$ at $(\nu+1)$-step and for each $p \in \mathbb{Z}$ it holds

$$\bigoplus_{q\in\mathbb{Z}} E_\infty^{p,q} \cong HF(L) \otimes_{\mathbb{Z}_2} A,$$

where we know that $\nu = \left[\frac{\dim L + 1}{\Sigma_L}\right]$.

Damian [3] provided the theory of the *lifted Floer homology* $HF^{\bar{L}}(L)$ for an arbitrary covering $\bar{L} \to L$. Let $p : \bar{L} \to L$ be a covering map of a compact Lagrangian submanifold L embedded in M. We need to assume that L is monotone with $N_L \geq 3$. By lifting to the covering space $\bar{L}$ all data on L necessary to define the Floer homology $HF(L)$, Damian defined the *lifted Floer complex* $CF^{\bar{L}}$ and the *lifted Floer homology* $HF^{\bar{L}}(L)$ of L. The Hamiltonian invariance of the lifted Floer homology also holds. Moreover he constructed the spectral sequence converging to $HF^{\bar{L}}(L)$ with the Morse homology of $\bar{L}$ as the first step. Note that the lifted Floer homology $HF^{\bar{L}}(L)$ is not well-defined in the case of $\Sigma_L = 2$. See [3] for the details. By definition the non-vanishing of the lifted Floer homology $HF^{\bar{L}}(L)$ also implies the Hamiltonian non-displaceability of L. However there seems to be no direct relationship between the original Floer homology and the lifted Floer homology.

4 Floer Homology and Lifted Floer Homology of Gauss Images of Isoparametric Hypersurfaces

Suppose that $L^n = \mathcal{G}(N^n) \subset Q_n(\mathbb{C})$ is the Guass image of an isoparametric hypersurface N^n in S^{n+1} with g distinct principal curvatures and multiplicities (m_1, m_2). Since it follows from Theorem 1 (3) that $\nu = \left[\frac{\dim L + 1}{\Sigma_L}\right] = \left[\frac{(n+1)g}{2n}\right]$, we get

Lemma 1 ([11]) *For each $p, q \in \mathbb{Z}$ it holds*

(0) $E_1^{p,q} = E_\infty^{p,q}$ $(\nu = 0)$ *if and only if* $g = 1$ *and* $n \geq 2$.
(1) $E_2^{p,q} = E_\infty^{p,q}$ $(\nu = 1)$ *if and only if* $(g, n) = (1, 1)$*,* $g = 2$ *or* $(g, m_1, m_2) = (3, 2, 2), (3, 4, 4), (3, 8, 8)$.
(2) $E_3^{p,q} = E_\infty^{p,q}$ $(\nu = 2)$ *if and only if* $(g, m_1, m_2) = (3, 1, 1)$ *or* $g = 4$.
(3) $E_4^{p,q} = E_\infty^{p,q}$ $(\nu = 3)$ *if and only if* $(g, m_1, m_2) = (6, 1, 1)$ *or* $(6, 2, 2)$.

In the case when $g = 1$ or $g = 2$, since the Gauss image of isoparametric hypersurfaces are nothing but real forms of complex hyperquadrics, it is well-known that $HF(L) \cong H_*(L;\mathbb{Z}_2)$ [10, 19].

In the case when $g = 3$, that is, N^n is a Cartan hypersurface, we proved

Lemma 2 ([11]) *The Gauss image of $L^n = \mathscr{G}(N^n)$ of each isoparametric hypersurface with $g = 3$ is a $\mathbb{Z}_2$-homology sphere (i.e. $H_k(L^n; \mathbb{Z}_2) = 0$ for each $0 < k < n$) satisfying $H_1(L^n; \mathbb{Z}) \cong \mathbb{Z}_3$.*

The Gauss images of Cartan hypersurfaces provide new examples of Lagrangian $\mathbb{Z}_2$-homology spheres embedded in compact Hermitian symmetric spaces.

This result is quite essential for the proof of main theorem [11] in the case when $g = 3$. When $g = 3$ and $m = m_1 = m_2 = 2, 4$ or 8, by Lemma 1 we have $\nu = 1$. By Lemma 2 the spectral sequence

$$[\partial_1] : H_{p+q-2mp}(L; \mathbb{Z}_2) \to H_{p+q-2mp+2m-1}(L; \mathbb{Z}_2),$$

namely

$$[\partial_1] : H_k(L; \mathbb{Z}_2) \to H_{k+2m-1}(L; \mathbb{Z}_2) \quad (k = 0, 1, \ldots, n)$$

implies $[\partial_1] = 0$, because we see that, $H_k(L; \mathbb{Z}_2) = \{0\}$ or $H_{k+2m-1}(L; \mathbb{Z}_2) = \{0\}$ since L is a $\mathbb{Z}_2$-homology sphere. Thus $d_1 = 0$. The spectral sequence becomes

$$V_2^{p,q} = \frac{\mathrm{Ker}([\partial_1] : V_1^{p,q} \to V_1^{p-1,q})}{\mathrm{Im}([\partial_1] : V_1^{p+1,q} \to V_1^{p,q})} = V_1^{p,q} = H_{p+q-p\Sigma_L}(L; \mathbb{Z}_2).$$

and $E_\infty^{p,q} = E_2^{p,q} = V_1^{p,q} \otimes A^{p\Sigma_L} \cong H_{p+q-p\Sigma_L}(L; \mathbb{Z}_2) \otimes A^{p\Sigma_L}$. Hence we obtain

$$HF(L) \cong \bigoplus_{q \in \mathbb{Z}} E_\infty^{p,q} = \bigoplus_{q \in \mathbb{Z}} E_2^{p,q} \cong \bigoplus_{q \in \mathbb{Z}} H_{p+q-p\Sigma_L}(L; \mathbb{Z}_2) \otimes A^{p\Sigma_L} \cong H_*(L; \mathbb{Z}_2).$$

Concerned with the lifted Floer homology to $\mathscr{G} : N \to L = \mathscr{G}(N)$, similarly using Damian's spectral sequence and the homological data of isoparametric hypersurfaces N [16] we obtain $HF^N(L) \cong H_*(N; \mathbb{Z}_2)$.

Theorem 4 (IMMO [11]) *In the case when $g = 3$ and $m = m_1 = m_2 = 2, 4$ or 8, the Floer homology $HF(L)$ (resp. the lifted Floer homology $HF^N(L)$) is isomorphic to $H_*(L; \mathbb{Z}_2)$ (resp. $H_*(N; \mathbb{Z}_2)$).*

In particular, $HF(L) \neq \{0\}$ and thus we see that for any $\phi \in \mathrm{Haml}(Q_n(\mathbb{C}), \omega_{\mathrm{std}})$ with $L \pitchfork \phi(L)$, it holds $\sharp(L \cap \phi(L)) \geq \mathrm{rank}\, H_*(L^n; \mathbb{Z}_2) = 2$.

In the case when $g = 4$ or 6, we use homological data on isoparametric hypersurfaces N^n [16] and the spectral sequence for the lifted Floer homology $HF^{\bar{L}}(L)$ applied to the covering map $\mathscr{G} : \bar{L} = N \to L = \mathscr{G}(N)$ (Damian [3]) in order to discuss the non-vanishing of the lifted Floer homology.

Theorem 5 (IMMO [11]) *In the case when $g = 4$ or $g = 6$ except for the remaining three cases as below, the lifted Floer homology $HF^N(L)$ is non-zero:*

$$(g, n, m_1, m_2) = (3, 3, 1, 1), \qquad N = \frac{SO(3)}{\mathbb{Z}_2 + \mathbb{Z}_2},$$
$$(g, n, m_1, m_2) = (4, 2k+2, 1, k), \; N = \frac{SO(2) \times SO(k+2)}{\mathbb{Z}_2 \times SO(k)} \; (k \geq 1),$$
$$(g, n, m_1, m_2) = (6, 6, 1, 1), \qquad N = \frac{SO(4)}{\mathbb{Z}_2 + \mathbb{Z}_2},$$

Notice that $(g, n, m_1, m_2) = (1, 1, 1, -), (2, 2, 1, 1), (3, 3, 1, 1), (4, 4, 1, 1)$ or $(6, 6, 1, 1)$ if and only if the minimal Maslov number of the Gauss image L of isoparametric hypersurface has $\Sigma_L = 2$, then any lifted Floer homology $HF^{\bar{L}}(L)$ is not well-defined.

Problem 1 Determine whether the lifted Floer homology $HF^{\bar{L}}(L)$ is nonzero or not in the case when $(g, n, m_1, m_2) = (4, 2k+2, 1, k)$ $(k \geq 2)$ (then $\Sigma_L = k + 1 \geq 3$).

Problem 2 Determine whether the Floer homology $HF(L)$ is nonzero or not in the case when $(g, n, m_1, m_2) = (3, 3, 1, 1), (4, 4, 1, 1)$ or $(6, 6, 1, 1)$ (then $\Sigma_L = 2$). When is the Floer homology $HF(L)$ isomorphic to $H_*(L; \mathbb{Z}_2)$?

More generally we should pose the following problem as our goal:

Problem 3 Determine explicitly the Floer homology $HF(L)$ of the Gauss images of isoparametric hypersurfaces in the case when $(g, m) = (3, 1)$, $g = 4$ or $g = 6$.

Since the Floer homology is based on the Mores homology, it is quite natural to study the following problems:

Problem 4 Determine explicitly the homology $HF_*(L; \mathbb{Z}_2)$ of the Gauss images of isoparametric hypersurfaces in the case when $g = 4$ or $g = 6$.

Problem 5 Construct concretely the Morse homology of the Gauss images of isoparametric hypersurfaces in the case when $g = 3, 4$ or $g = 6$.

Acknowledgements The author sincerely would like to thank Professor Young Jin Suh (Director of RIRCM) for his leading the international joint project of submanifold theory in differential geometry and his organization of twenty times successful workshops. The author also would like to thank Dr. Hyunjin Lee for her excellent supports to those workshops. The author was partly supported by JSPS Grant-in-Aid for Scientific Research (C) No. 15K04851 and JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers "Mathematical Science of Symmetry, Topology and Moduli, Evolution of International Research Network based on OCAMI".

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On the Pointwise Slant Submanifolds

Kwang-Soon Park

Abstract In this survey paper, we consider several kinds of submanifolds in Riemannian manifolds, which are obtained by many authors. (i.e., slant submanifolds, pointwise slant submanifolds, semi-slant submanifolds, pointwise semi-slant submanifolds, pointwise almost h-slant submanifolds, pointwise almost h-semi-slant submanifolds, etc.) And we deal with some results, which are obtained by many authors at this area. Finally, we give some open problems at this area.

1 Introduction

Given a Riemannian manifold $(\overline{M}, g)$ with some additional structures, there are several kinds of submanifolds:

(Almost) complex submanifolds, totally real submanifolds, slant submanifolds, pointwise slant submanifolds, semi-slant submanifolds, pointwise semi-slant submanifolds, etc.

In 1990, Chen [3] defined the notion of slant submanifolds of an almost Hermitian manifold as a generalization of almost complex submanifolds and totally real submanifolds.

In 1994, Papaghiuc [7] introduced a semi-slant submanifold of an almost Hermitian manifold as a generalization of CR-submanifolds and slant submanifolds.

In 1996, Lotta [6] introduced a slant submanifold of an almost contact metric manifold.

In 1998, Etayo [5] defined the notion of pointwise slant submanifolds of an almost Hermitian manifold under the name of quasi-slant submanifolds as a generalization of slant submanifolds.

In 1999, Cabrerizo, Carriazo, Fernandez, Fernandez [2] defined the notion of semi-slant submanifolds of an almost contact metric manifold.

K.-S. Park (✉)
Division of General Mathematics, University of Seoul, 163 Seoulsiripdaero,
Dongdaemun-gu, Seoul 02504, Korea
e-mail: parkksn@gmail.com

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_21

In 2012, Chen and Garay [4] studied deeply pointwise slant submanifolds of an almost Hermitian manifold.

In 2013, Sahin [10] introduced pointwise semi-slant submanifolds of an almost Hermitian manifold.

In 2014, Park [8] defined the notion of pointwise almost h-slant submanifolds and pointwise almost h-semi-slant submanifolds of an almost quaternionic Hermitian manifold.

In 2015, Park [9] introduced pointwise slant and pointwise semi-slant submanifolds of an almost contact metric manifold.

In this paper, we consider some results, which are obtained by many authors at this area. And we give some open problems at this area.

2 Preliminaries

Let $(\overline{M}, g, J)$ be an *almost Hermitian manifold*, where $\overline{M}$ is a C^∞-manifold, g is a Riemannian metric on $\overline{M}$, and J is an almost complex structure on $\overline{M}$ which is compatible with g.

I.e., $J \in \mathrm{End}(T\overline{M})$, $J^2 = -id$, $g(JX, JY) = g(X, Y)$ for $X, Y \in \Gamma(T\overline{M})$.

Let M be a submanifold of $\overline{M} = (\overline{M}, g, J)$. We have the following notions.

We call M an *almost complex submanifold* of $\overline{M}$ if $J(T_xM) \subset T_xM$ for $x \in M$.

The submanifold M is said to be a *totally real submanifold* if $J(T_xM) \subset T_xM^\perp$ for $x \in M$.

The submanifold M is called a *CR-submanifold* if there exists a distribution $\mathscr{D} \subset TM$ on M such that $J(\mathscr{D}_x) = \mathscr{D}_x$ and $J(\mathscr{D}_x^\perp) \subset T_xM^\perp$ for $x \in M$, where $\mathscr{D}^\perp$ is the orthogonal complement of $\mathscr{D}$ in TM.

The almost Hermitian manifold $\overline{M} = (\overline{M}, g, J)$ is said to be *Kähler* if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

Now we recall other notions. Let N be a $(2n + 1)$-dimensional C^∞-manifold with a tensor field ϕ of type $(1, 1)$, a vector field ξ, and a 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where I denotes the identity endomorphism of TN.

Then we have [1]

$$\phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2}$$

And we call (ϕ, ξ, η) an *almost contact structure* and (N, ϕ, ξ, η) an *almost contact manifold*.

If there is a Riemannian metric g on N such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{3}$$

for $X, Y \in \Gamma(TN)$, then we call (ϕ, ξ, η, g) an *almost contact metric structure* and (N, ϕ, ξ, η, g) an *almost contact metric manifold*.

The metric g is called a *compatible metric*.

Then we obtain

$$\eta(X) = g(X, \xi). \tag{4}$$

Define $\Phi(X, Y) := g(X, \phi Y)$ for $X, Y \in \Gamma(TN)$.

Since ϕ is anti-symmetric with respect to g, the tensor Φ is a 2-form on N and is called the *fundamental 2-form* of the almost contact metric structure (ϕ, ξ, η, g).

An almost contact metric manifold (N, ϕ, ξ, η, g) is said to be a *contact metric manifold* (or *almost Sasakian manifold*) if it satisfies

$$\Phi = d\eta. \tag{5}$$

It is easy to check that given a contact metric manifold (N, ϕ, ξ, η, g), we get

$$(d\eta)^n \wedge \eta \neq 0. \tag{6}$$

The *Nijenhuis tensor* of a tensor field ϕ is defined by

$$N(X, Y) := \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \tag{7}$$

for $X, Y \in \Gamma(TN)$.

We call the almost contact metric structure (ϕ, ξ, η, g) *normal* if

$$N(X, Y) + 2d\eta(X, Y)\xi = 0 \tag{8}$$

for $X, Y \in \Gamma(TN)$.

A contact metric manifold (N, ϕ, ξ, η, g) is said to be a *K-contact manifold* if the characteristic vector field ξ is Killing.

It is well-known that for a contact metric manifold (N, ϕ, ξ, η, g), ξ is Killing if and only if the tensor $\bar{h} := \frac{1}{2}L_\xi \phi$ vanishes, where L denotes the Lie derivative [1].

An almost contact metric manifold (N, ϕ, ξ, η, g) is called a *Sasakian manifold* if it is contact and normal.

Given an almost contact metric manifold (N, ϕ, ξ, η, g), we know that it is Sasakian if and only if

$$(\overline{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{9}$$

for $X, Y \in \Gamma(TN)$ [1].

If an almost contact metric manifold (N, ϕ, ξ, η, g) is Sasakian, then we have

$$\overline{\nabla}_X \xi = -\phi X \tag{10}$$

for $X \in \Gamma(TN)$ [1].

Moreover, a Sasakian manifold is a K-contact manifold [1].

An almost contact metric manifold (N, ϕ, ξ, η, g) is said to be a *Kenmotsu manifold* if it satisfies

$$(\overline{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{11}$$

for $X, Y \in \Gamma(TN)$ [1].

Then we easily obtain

$$\overline{\nabla}_X \xi = X - \eta(X)\xi \tag{12}$$

for $X \in \Gamma(TN)$ [1].

An almost contact metric manifold (N, ϕ, ξ, η, g) is called an *almost cosymplectic manifold* if η and Φ are closed.

An almost cosymplectic manifold (N, ϕ, ξ, η, g) is said to be a *cosymplectic manifold* if it is normal.

Given an almost contact metric manifold (N, ϕ, ξ, η, g), we also know that it is cosymplectic if and only if ϕ is parallel (i.e., $\overline{\nabla}\phi = 0$) [1].

Given a cosymplectic manifold (N, ϕ, ξ, η, g), we easily get

$$\overline{\nabla}\phi = 0, \ \overline{\nabla}\eta = 0, \ \text{and } \overline{\nabla}\xi = 0. \tag{13}$$

Let $\overline{M}$ be a $4m$-dimensional C^∞-manifold and let E be a rank 3 subbundle of $\text{End}(T\overline{M})$ such that for any point $p \in \overline{M}$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call E an *almost quaternionic structure* on $\overline{M}$ and $(\overline{M}, E)$ an *almost quaternionic manifold*.

Moreover, let g be a Riemannian metric on $\overline{M}$ such that for any point $p \in \overline{M}$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{14}$$

$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \tag{15}$$

for $X, Y \in \Gamma(T\overline{M})$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call $(\overline{M}, E, g)$ an *almost quaternionic Hermitian manifold*.

Conveniently, the above basis $\{J_1, J_2, J_3\}$ satisfying (14) and (15) is said to be a *quaternionic Hermitian basis*.

Let $(\overline{M}, E, g)$ be an almost quaternionic Hermitian manifold.

We call $(\overline{M}, E, g)$ a *quaternionic Kähler manifold* if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$$

for $X \in \Gamma(T\overline{M})$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on $\overline{M}$ (i.e., $\nabla J_\alpha = 0$ for $\alpha \in \{1, 2, 3\}$, where ∇ is the Levi-Civita connection of the metric g), then $(\overline{M}, E, g)$ is said to be a *hyperkähler manifold*.

Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on $\overline{M}$ and g a *hyperkähler metric*.

Let $\overline{M} = (\overline{M}, E, g)$ be an almost quaternionic Hermitian manifold and M a submanifold of $\overline{M}$.

We call M a *QR-submanifold* (quaternionic-real submanifold) of $\overline{M}$ if there exists a vector subbundle $\mathscr{D}$ of $TM^\perp$ on M such that given a local quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of E, we have $J_\alpha \mathscr{D} = \mathscr{D}$ and $J_\alpha(\mathscr{D}^\perp) \subset TM$ for $\alpha \in \{1, 2, 3\}$, where $\mathscr{D}^\perp$ is the orthogonal complement of $\mathscr{D}$ in $TM^\perp$.

The submanifold M is said to be a *quaternion CR-submanifold* if there exists a distribution $\mathscr{D} \subset TM$ on M such that given a local quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of E, we get $J_\alpha \mathscr{D} = \mathscr{D}$ and $J_\alpha(\mathscr{D}^\perp) \subset TM^\perp$ for $\alpha \in \{1, 2, 3\}$, where $\mathscr{D}^\perp$ is the orthogonal complement of $\mathscr{D}$ in TM.

Throughout this paper, we will use the above notations.

3 Some Results

In this section, we consider some results at this area.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and M a submanifold of $\overline{M}$.

We call M a *slant submanifold* [3] of $\overline{M}$ if the angle $\theta = \theta(X)$ between JX and the tangent space $T_x M$ is constant for nonzero $X \in T_x M$ and any $x \in M$.

Given $X \in \Gamma(TM)$, we have

$$JX = PX + FX, \tag{16}$$

where $PX \in \Gamma(TM)$ and $FX \in \Gamma(TM^\perp)$.

Lemma 1 ([3]) *Let M be a submanifold of an almost Hermitian manifold $\overline{M}$.*

Then $\nabla P = 0$ if and only if M is locally the Riemannian product $M_1 \times \cdots \times M_k$, where each M_i is either a Kähler submanifold, a totally real submanifold, or a Kählerian slant submanifold.

Theorem 1 ([3]) *Let M be a surface in C^2 which is neither holomorphic nor totally real.*

Then M is a minimal slant surface if and only if $\nabla F = 0$.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and M a submanifold of $\overline{M}$.

The submanifold M is said to be a *semi-slant submanifold* [7] if there is a distribution $\mathscr{D} \subset TM$ on M such that $J(\mathscr{D}_x) = \mathscr{D}_x$ for $x \in M$ and the angle $\theta = \theta(X)$ between JX and the space $\mathscr{D}_x^{\perp}$ is constant for nonzero $X \in \mathscr{D}_x^{\perp}$ and any $x \in M$, where $\mathscr{D}^{\perp}$ is the orthogonal complement of $\mathscr{D}$ in TM.

Proposition 1 ([7]) *Let M be a semi-slant submanifold of a Kähler manifold $(\overline{M}, g, J)$.*

Then the complex distribution $\mathscr{D}$ is integrable if and only if we have $h(X, JY) = h(JX, Y)$ for $X, Y \in \Gamma(\mathscr{D})$.

Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and M a submanifold of N.

We call M a *slant submanifold* [6] of N if the angle $\theta = \theta(X)$ between ϕX and the tangent space $T_x M$ is constant for nonzero $X \in T_x M$ with X, ξ being linearly independent and any $x \in M$.

Given $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX, \tag{17}$$

where $PX \in \Gamma(TM)$ and $FX \in \Gamma(TM^{\perp})$.

Theorem 2 ([6]) *Let M be a m-dimensional slant submanifold of an almost contact metric manifold N and suppose $\theta \neq \frac{\pi}{2}$.*

Then we have

$$m \text{ is even } \Leftrightarrow \xi \text{ is orthogonal to } N$$

$$m \text{ is odd } \Leftrightarrow \xi \text{ is tangent to } N.$$

Theorem 3 ([6]) *Let M be an immersed submanifold of a K-contact manifold N such that the characteristic vector field ξ is tangent to M. Let $\theta \in [0, \frac{\pi}{2}]$. The following statements are equivalent:*

(a) M is slant in N with the slant angle θ.

(b) For any $x \in M$ the sectional curvature of any 2-plane of $T_x M$ containing ξ_x equals $\cos^2 \theta$.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and M a submanifold of $\overline{M}$.

The submanifold M is called a *pointwise slant submanifold* [4, 5] of $\overline{M}$ if at each given point $x \in M$, the angle $\theta = \theta(X)$ between JX and the tangent space $T_x M$ is constant for nonzero $X \in T_x M$.

Proposition 2 ([5]) *Let M be a pointwise slant submanifold of an almost Hermitian manifold $(\overline{M}, g, J)$.*

If M has odd dimension, then M is a totally real submanifold.

Theorem 4 ([5]) *Let M be a submanifold of an almost Hermitian manifold $(\overline{M}, g, J)$.*

Then M is a pointwise slant submanifold if and only if P_x is a homothety for $x \in M$.

Theorem 5 ([5]) *Let M be a pointwise slant complete totally geodesic submanifold of a Kähler manifold $(\overline{M}, g, J)$.*

Then M is a slant submanifold.

Define $\Omega(X, Y) := g(X, PY)$ for $X, Y \in \Gamma(TM)$.

Theorem 6 ([4]) *Let M be a proper pointwise slant submanifold of a Kähler manifold $(\overline{M}, g, J)$.*

Then Ω is a non-degenerate closed 2-form on M.

Consequently, Ω defines a canonical cohomology class of Ω:

$$[\Omega] \in H^2(M; R).$$

Theorem 7 ([4]) *Let M be a compact $2n$-dimensional differentiable manifold with $H^{2i}(M; R) = 0$ for some $i \in \{1, \ldots, n\}$.*

Then M cannot be immersed in any Kähler manifold as a pointwise proper slant submanifold.

Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and M a submanifold of N.

We call M a *semi-slant submanifold* [2] of N if there is a distribution $\mathscr{D} \subset TM$ on M such that $\phi(\mathscr{D}_x) = \mathscr{D}_x$ for $x \in M$ and the angle $\theta = \theta(X)$ between ϕX and the space $\mathscr{D}_x^{\perp}$ is constant for nonzero $X \in \mathscr{D}_x^{\perp}$ with X, ξ being linearly independent and any $x \in M$, where $\mathscr{D}^{\perp}$ is the orthogonal complement of $\mathscr{D}$ in TM.

Theorem 8 ([2]) *Let M be a submanifold of an almost contact metric manifold $N = (N, \phi, \xi, \eta, g)$ such that $\xi \in \Gamma(TM)$.*

Then M is semi-slant if and only if there exists a constant $\lambda \in [0, 1)$ such that (i) $D = \{X \in TM | P^2 X = -\lambda X\}$ is a distribution. (ii) For any $X \in TM$, orthogonal to D, $FX = 0$.

Furthermore, in this case, $\lambda = \cos^2\theta$, where θ denotes the slant angle of M.

Let $(\overline{M}, g, J)$ be an almost Hermitian manifold and M a submanifold of $\overline{M}$.

We call M a *pointwise semi-slant submanifold* [10] of $\overline{M}$ if there is a distribution $\mathscr{D} \subset TM$ on M such that $J(\mathscr{D}_x) = \mathscr{D}_x$ for $x \in M$ and at each given point $x \in M$, the angle $\theta = \theta(X)$ between JX and the space $\mathscr{D}_x^{\perp}$ is constant for nonzero $X \in \mathscr{D}_x^{\perp}$, where $\mathscr{D}^{\perp}$ is the orthogonal complement of $\mathscr{D}$ in TM.

Theorem 9 ([10]) *Let $\overline{M}$ be a Kähler manifold.*

Then there exist no non-trivial warped product submanifolds $M = M_\theta \times_f M_T$ of a Kähler manifold $\overline{M}$ such that M_T is a holomorphic submanifold and M_θ is a proper pointwise slant submanifold of $\overline{M}$.

Theorem 10 ([10]) *Let M be an $m + n$-dimensional non-trivial warped product pointwise semi-slant submanifold of the form $M_T \times_f M_\theta$ in a Kähler manifold $\overline{M}^{m+2n}$, where M_T is a holomorphic submanifold and M_θ is a proper pointwise slant submanifold of $\overline{M}^{m+2n}$.*

Then we have

(i) The squared norm of the second fundamental form of M satisfies

$$||h||^2 \geq 2n(\csc^2\theta + \cot^2\theta)||\nabla(\ln f)||^2, \quad \dim(M_\theta) = n. \tag{18}$$

(ii) If the equality of (18) holds identically, then M_T is a totally geodesic submanifold and M_θ is a totally umbilical submanifold of $\overline{M}^{m+2n}$.

Moreover, M is a minimal submanifold of $\overline{M}^{m+2n}$.

Let $(\overline{M}, E, g)$ be an almost quaternionic Hermitian manifold and M a submanifold of $(\overline{M}, g)$.

The submanifold M is called a *pointwise almost h-slant submanifold* [8] if given a point $p \in M$ with a neighborhood V, there exist an open set $U \subset \overline{M}$ with $U \cap M = V$ and a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, at each given point $q \in V$ the angle $\theta_R = \theta_R(X)$ between RX and the tangent space T_qM is constant for nonzero $X \in T_qM$.

We call such a basis $\{I, J, K\}$ a *pointwise almost h-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-slant functions* as functions on V.

The submanifold M is called a *pointwise almost h-semi-slant submanifold* [8] if given a point $p \in M$ with a neighborhood V, there exist an open set $U \subset \overline{M}$ with $U \cap M = V$ and a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$, there is a distribution $\mathscr{D}_1^R \subset TM$ on V such that

$$TM = \mathscr{D}_1^R \oplus \mathscr{D}_2^R, \; R(\mathscr{D}_1^R) = \mathscr{D}_1^R,$$

and at each given point $q \in V$ the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathscr{D}_2^R)_q$ is constant for nonzero $X \in (\mathscr{D}_2^R)_q$, where $\mathscr{D}_2^R$ is the orthogonal complement of $\mathscr{D}_1^R$ in TM.

We call such a basis $\{I, J, K\}$ a *pointwise almost h-semi-slant basis* and the angles $\{\theta_I, \theta_J, \theta_K\}$ *almost h-semi-slant functions* as functions on V.

Let M be a pointwise almost h-semi-slant submanifold of a hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $\{I, J, K\}$ is a pointwise almost h-semi-slant basis. We call M *proper* if $\theta_R(p) \in [0, \frac{\pi}{2})$ for $p \in M$ and $R \in \{I, J, K\}$.

Let M be a proper pointwise almost h-slant submanifold of a hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $\{I, J, K\}$ is a pointwise almost h-slant basis.

Define

$$\Omega_R(X, Y) := g(\phi_R X, Y) \tag{19}$$

for $X, Y \in \Gamma(TM)$ and $R \in \{I, J, K\}$.

Theorem 11 ([8]) *Let M be a proper pointwise almost h-slant submanifold of a hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $\{I, J, K\}$ is a pointwise almost h-slant basis. Then the 2-form Ω_R is closed for each $R \in \{I, J, K\}$.*

Theorem 12 ([8]) *Let M be a $2n$-dimensional compact proper pointwise almost h-slant submanifold of a $4m$-dimensional hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $\{I, J, K\}$ is a pointwise almost h-slant basis.*

Then

$$H^*(M, R) \supseteq \widetilde{H}, \tag{20}$$

where $\widetilde{H}$ is the algebra spanned by $\{[\Omega_I], [\Omega_J], [\Omega_K]\}$.

Theorem 13 ([8]) *Let $(\overline{M}, I, J, K, g)$ be a hyperkähler manifold. Then given $R \in \{I, J, K\}$, there do not exist any non-trivial warped product submanifolds $M = B \times_f F$ of a Kähler manifold $(\overline{M}, R, g)$ such that B is a proper pointwise slant submanifold of $(\overline{M}, R, g)$ and F is a holomorphic submanifold of $(\overline{M}, R, g)$.*

Theorem 14 ([8]) *Let $M = B \times_f F$ be a non-trivial warped product proper pointwise h-semi-slant submanifold of a hyperkähler manifold $(\overline{M}, I, J, K, g)$ such that $TB = \mathscr{D}_1$, $TF = \mathscr{D}_2$, $\dim B = 4n_1$, $\dim F = 2n_2$, $\dim \overline{M} = 4m$, $\theta_I(p)\theta_J(p)\theta_K(p) \neq 0$ for any $p \in M$, and $\{I, J, K\}$ is a pointwise h-semi-slant basis.*

Assume that $m = n_1 + n_2$.

Then given $R \in \{I, J, K\}$, we get

$$||h||^2 \geq 4n_2(\csc^2 \theta_R + \cot^2 \theta_R)||\nabla(\ln f)||^2 \tag{21}$$

with equality holding if and only if $g(h(V, W), Z) = 0$ for $V, W \in \Gamma(TF)$ and $Z \in \Gamma(TM^{\perp})$.

Let $N = (N, \phi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional almost contact metric manifold and M a submanifold of N.

The submanifold M is called a *pointwise slant submanifold* [9] if at each given point $p \in M$ the angle $\theta = \theta(X)$ between ϕX and the space M_p is constant for nonzero $X \in M_p$, where $M_p := \{X \in T_pM \mid g(X, \xi(p)) = 0\}$.

The submanifold M is called a *pointwise semi-slant submanifold* [9] if there is a distribution $\mathscr{D}_1 \subset TM$ on M such that

$$TM = \mathscr{D}_1 \oplus \mathscr{D}_2, \quad \phi(\mathscr{D}_1) \subset \mathscr{D}_1,$$

and at each given point $p \in M$ the angle $\theta = \theta(X)$ between ϕX and the space $(\mathscr{D}_2)_p$ is constant for nonzero $X \in (\mathscr{D}_2)_p$, where $\mathscr{D}_2$ is the orthogonal complement of $\mathscr{D}_1$ in TM.

Theorem 15 ([9]) *Let M be a pointwise slant connected totally geodesic submanifold of a cosymplectic manifold (N, ϕ, ξ, η, g).*

Then M is a slant submanifold of N.

Theorem 16 ([9]) *Let M be a $2m$-dimensional compact proper pointwise slant submanifold of a $(2n + 1)$-dimensional cosymplectic manifold (N, ϕ, ξ, η, g) such that ξ is normal to M.*

Then $[\Omega] \in H^2(M, R)$ is non-vanishing.

Theorem 17 ([9]) *Let M be a $(2m+1)$-dimensional compact proper pointwise slant submanifold of a $(2n+1)$-dimensional cosymplectic manifold (N, ϕ, ξ, η, g) such that ξ is tangent to M.*

Then both $[\eta] \in H^1(M, R)$ and $[\Omega] \in H^2(M, R)$ are non-vanishing.

Let M be a submanifold of a Riemannian manifold (N, g). We call M a *totally umbilic submanifold* of (N, g) if

$$h(X, Y) = g(X, Y)H \quad \text{for } X, Y \in \Gamma(TM), \tag{22}$$

where H is the mean curvature vector field of M in N.

Lemma 2 ([9]) *Let M be a pointwise semi-slant totally umbilic submanifold of an almost contact metric manifold (N, ϕ, ξ, η, g).*

Assume that ξ is tangent to M and N is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.

Then

$$H \in \Gamma(F\mathscr{D}_2). \tag{23}$$

Theorem 18 ([9]) *Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $M = B \times_f \overline{F}$ a nontrivial warped product submanifold of N. Assume that ξ is normal to M and N is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

Then there does not exist a proper pointwise semi-slant submanifold M of N such that $\mathscr{D}_1 = T\overline{F}$ and $\mathscr{D}_2 = TB$.

Theorem 19 ([9]) *Let $N = (N, \phi, \xi, \eta, g)$ be an almost contact metric manifold and $M = B \times_f \overline{F}$ a nontrivial warped product submanifold of N. Assume that ξ is tangent to M and N is one of the following three manifolds: cosymplectic, Sasakian, Kenmotsu.*

Then there does not exist a proper pointwise semi-slant submanifold M of N such that $\mathscr{D}_1 = T\overline{F}$ and $\mathscr{D}_2 = TB$.

Theorem 20 ([9]) *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional Sasakian manifold (N, ϕ, ξ, η, g) with the semi-slant function θ such that $\mathscr{D}_1 = TB$, $\mathscr{D}_2 = T\overline{F}$, and ξ is tangent to M.*

Assume that $n = m_1 + 2m_2$.

Then we have

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\phi\nabla(\ln f)||^2 + 4m_2\sin^2\theta \tag{24}$$

with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.

Theorem 21 ([9]) *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional cosymplectic manifold (N, ϕ, ξ, η, g) with the semi-slant function θ such that $\mathscr{D}_1 = TB$, $\mathscr{D}_2 = T\overline{F}$, and ξ is tangent to M.*

Assume that $n = m_1 + 2m_2$.

Then we have

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\phi\nabla(\ln f)||^2 \tag{25}$$

with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.

Theorem 22 ([9]) *Let $M = B \times_f \overline{F}$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold (N, ϕ, ξ, η, g) with the semi-slant function θ such that $\mathscr{D}_1 = TB$, $\mathscr{D}_2 = T\overline{F}$, and ξ is normal to M with $\xi \in \Gamma(\mu)$.*

Assume that $n = m_1 + 2m_2$.

Then we have

$$||h||^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)||\nabla(\ln f)||^2 + 2m_1 \tag{26}$$

with equality holding if and only if $g(h(Z, W), V) = 0$ for $Z, W \in \Gamma(T\overline{F})$ and $V \in \Gamma(TM^{\perp})$.

4 Open Questions

Question 1. Let M be a (pointwise) slant (or (pointwise) semi-slant) submanifold of a Riemannian manifold $(\overline{M}, g)$ with some geometric structures.

Then

1. Give some examples of the manifold M when $\dim M \geq 3$.
2. What kind of rigidity problems can we do on $M \subset \overline{M}$?

Question 2. Let M be a pointwise almost h-semi-slant submanifold of an almost quaternionic Hermitian manifold $(\overline{M}, E, g)$ with the almost h-semi-slant functions $\{\theta_I, \theta_J, \theta_K\}$.

Then

1. Can we give a characterization of the almost h-semi-slant functions $\{\theta_I, \theta_J, \theta_K\}$?
2. What kind of rigidity problems can we do on $M \subset \overline{M}$?
3. Since the quaternionic Kähler manifolds have applications in physics, what is the relation between this notion and physics?
4. Using this notion, what are the advantages for studying quaternionic geometry?

Question 3. Let M be a pointwise slant (or pointwise semi-slant) submanifold of an almost contact metric manifold (N, ϕ, ξ, η, g) with the slant (or semi-slant) function θ.

Then

1. Can we give a characterization of the slant (or semi-slant) function θ?
2. What kind of rigidity problems can we do on $M \subset N$?
3. Using these notions, what are the advantages for studying contact geometry?

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Riemannian Hilbert Manifolds

Leonardo Biliotti and Francesco Mercuri

Abstract In this article we collect results obtained by the authors jointly with other authors and we discuss old and new ideas. In particular we discuss singularities of the exponential map, completeness and homogeneity for Riemannian Hilbert quotient manifolds. We also extend a Theorem due to Nomizu and Ozeki to infinite dimensional Riemannian Hilbert manifolds.

1 Introduction

Let $\mathbb{H}$ be a Hilbert space. A Riemannian Hilbert manifold $(M, \langle\cdot,\cdot\rangle)$, RH manifold for short, is a smooth manifold modeled on the Hilbert space $\mathbb{H}$, equipped with an inner product $\langle\cdot,\cdot\rangle_p$ on any tangent space T_pM depending smoothly on p and defining on $T_pM \cong \mathbb{H}$ a norm equivalent to the one of $\mathbb{H}$.

The local Riemannian geometry of RH manifolds goes in the same way as in the finite dimensional case. We can prove, just like in the finite dimensional case, the existence and uniqueness of a symmetric connection, compatible with the Riemannian metric, the Levi-Civita connection, characterized by the Koszul formula

$$\begin{aligned}2\langle\nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &\quad + \langle[X, Y], Z\rangle + \langle[Z, X], Y\rangle - \langle[Y, Z], X\rangle.\end{aligned}$$

Hence we can define covariant differentiation of a vector field along a smooth curve, parallel translation, geodesics, exponential map, the curvature tensor R, its sectional curvature K etc., just like in the finite dimensional case (see [10, 18, 22] for details).

L. Biliotti (✉)
Dipartimento di Matematica e Informatica, Parco Area Delle Scienze 53/A, 43124 Parma, Italy
e-mail: leonardo.biliotti@unipr.it

F. Mercuri (✉)
Departamento de Matemática, IMECC, UNICAMP, C.P. 6065, Campinas, SP 13081-970, Brazil
e-mail: mercuri@ime.unicamp.br

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_22

The investigation of global properties in infinite dimensional geometry is harder than in the finite dimensional case essentially because of the lack of local compactness. For example, there exist complete RH manifolds with points that cannot be connected by minimal geodesics, complete connected RH manifolds for which the exponential map is not surjective etc. (see Sect. 3). Moreover, on some RH manifolds one can construct finite geodesic segments containing infinitely many conjugate points [13]. A complete description of conjugate points along finite geodesic segment is given in [7] and similar questions have been studied in [3, 16, 17, 25–30].

The aim of this survey is to describe results obtained by the authors jointly with D. Tausk, R. Exel and P. Piccione and others authors [1–7, 11, 13, 24]. We have tried to avoid technical results in order to make the paper more readable also by non experts in this field. The interested reader will find details and further results in papers and books quoted in the bibliography.

This paper is organized as follows. In Sect. 2, we investigate complete Riemannian Hilbert manifolds. We extend a Theorem due to Nomizu and Ozeki [31] to Riemannian Hilbert manifolds. We also investigate Hopf–Rinow manifolds, i.e., Riemannian Hilbert manifolds such that there exists minimal geodesic between any two points of M, properly discontinuous actions on Riemannian Hilbert manifolds and homogeneity for Riemannian Hilbert quotient manifolds. We also point out that if M has constant sectional curvature then completeness is equivalent to geodesically completeness and there are not non trivial Clifford translations on a Hadamard manifold. In Sect. 3, following the point of view used by Karcher [15], we introduce the Jacobi flow, we discuss singularities of the exponential map and the main result proved in [7].

2 Complete Riemannian Hilbert Manifolds

Let M be a RH manifold. If $\gamma : [a, b] \subseteq \mathbb{R} \longrightarrow M$ is a piecewise smooth curve, the length of γ is defined, as in the finite dimensional case, $L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt$. Then, if M is connected, we can define a distance function

$$\mathrm{d}(p, q) = inf\{L(\gamma) : \gamma \text{ is a piecewise smooth curve joining } p \text{ and } q\}.$$

The function d is, in fact, a distance and induces the original topology of M [22, 33].

Definition 2.1 We will say that a RH manifold M is *complete* if it is complete as a metric space.

Let M be a Hilbert manifold. A natural question is if there exists a Riemannian metric $\langle \cdot, \cdot \rangle$ such that $(M, \langle \cdot, \cdot \rangle)$ is a complete RH manifold. McAlpin [24] proved that any separable Hilbert manifold modeled on a separable Hilbert space can be embedded as a closed submanifold of a separable Hilbert space. Hence, if $f : M \longrightarrow \mathbb{H}'$ is such an embedding, M, with the induced metric, is a complete RH manifold. The following result is an extension to the infinite dimensional case of a Theorem due to Nomizu and Ozeki [31].

Theorem 2.1 *Let $(M, \langle\cdot,\cdot\rangle)$ be a separable RH manifold modeled on a separable Hilbert space. Then there exists a positive smooth function $f : M \longrightarrow \mathbb{R}$ such that $(M, f\langle\cdot,\cdot\rangle)$ is a complete RH manifold.*

Proof Consider the geodesic ball $B(p,\varepsilon) = \{q \in M : \mathrm{d}(p,q) < \varepsilon\}$. By a result of Ekeland [11] there exists $\varepsilon > 0$ such that $\overline{B(p,\varepsilon)}$ is a complete metric space. We define

$$r : M \longrightarrow \mathbb{R}, \qquad r(p) = \sup\{r > 0 : \ \overline{B(p,\varepsilon)} \text{ is a complete metric space}\}.$$

If $r(p) = \infty$ for some $p \in M$ then $(M, \langle\cdot,\cdot\rangle)$ is complete. Hence we may assume $r(p) < +\infty$ for every $p \in M$. We claim $|r(p) - r(q)| \leq \mathrm{d}(p,q)$ and so it is a continuous function. Indeed, if $\mathrm{d}(p,q) \geq \max(r(p), r(q))$ then the above inequality holds true. Hence, we may assume without lost of generality that $\mathrm{d}(p,q) < r(p)$. Pick $0 < \varepsilon < \frac{r(p)-d(p,q)}{2}$. The triangle inequality implies $B(q, r(p) - \mathrm{d}(p,q) - \varepsilon) \subset B(p, r(p) - \varepsilon)$ and so $\overline{B(q, d(p,q) - r(p) - \varepsilon)} \subset \overline{B(p, r(p) - \varepsilon)}$. Hence $r(q) \geq r(p) - d(p,q)$ and so $|r(p) - r(q)| \leq d(p,q)$. Applying a result of [23], see also [39], there exists a smooth function $f : M \longrightarrow \mathbb{R}$ such that $f(x) > \frac{1}{r(x)}$ for any $x \in M$. Pick $\langle\cdot,\cdot\rangle'(x) = f^2(x)\langle\cdot,\cdot\rangle(x)$. Then $(M, \langle\cdot,\cdot\rangle')$ is a RH manifold. We denote by d' the distance defined by $\langle\cdot,\cdot\rangle'$.

Let $x, y \in M$ and let $\gamma : [0,1] \longrightarrow M$ be a piecewise smooth curve joining x and y. We denote by L, respectively L', be the length of γ with respect to $\langle\cdot,\cdot\rangle$, respectively the length of γ with respect to $\langle\cdot,\cdot\rangle'$. Then

$$L' = \int_0^1 f(\gamma(t))\langle\dot\gamma(t), \dot\gamma(t)\rangle^{1/2}\mathrm{d}t \geq f(\gamma(c))L > \frac{1}{r(\gamma(c))}L$$

where $0 \leq c \leq 1$. Since $r(\gamma(c)) \leq r(x) + d(x, \gamma(c)) \leq r(x) + L$, it follows $L' > \frac{L}{r(x)+L}$. Therefore, as in [31], for any $0 < \varepsilon < 1$ and for any $x \in M$, we get $B^{\langle\cdot,\cdot\rangle'}(x, \frac{1}{3-\varepsilon})$ is contained in $B(x, \frac{r(x)}{2-\varepsilon})$. Hence $\overline{B^{\langle\cdot,\cdot\rangle'}(x, \frac{1}{3-\varepsilon})}$ is a complete metric space, with respect to d. We claim that $\overline{B^{\langle\cdot,\cdot\rangle'}(x, \frac{1}{3})}$ is a complete metric space with respect to d' as well.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of $\overline{B^{\langle\cdot,\cdot\rangle'}(x, \frac{1}{3})}$ with respect to d'. Let $0 < \varepsilon < \frac{2}{21}$. Then there exists n_o such that for any $n, m \geq n_o$ we get $\mathrm{d}'(x_n, x_m) \leq \frac{\varepsilon}{4}$. We claim that if $\gamma : [0,1] \longrightarrow M$ is a curve joining x_n and x_m, for any $n, m \geq n_o$, such that $L'(\gamma) < \frac{\varepsilon}{2}$, then $\gamma([0,1]) \subset B(x, \frac{3r(x)}{4})$. Indeed, let $t \in [0,1]$. Then

$$\mathrm{d}'(\gamma(t), x) \leq \mathrm{d}'(\gamma(t), x_n) + \mathrm{d}'(x_n, x_m) + \mathrm{d}'(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{1}{3} < \frac{1}{3} + \varepsilon = \frac{1}{3-\varepsilon'},$$

where $\varepsilon' = \frac{9\varepsilon}{1+3\varepsilon}$. Hence $\mathrm{d}'(\gamma(t), x) < \frac{1}{3-\varepsilon'}$ and so $\mathrm{d}(\gamma(t), x) < \frac{r(x)}{2-\varepsilon'} < \frac{3r(x)}{4}$.

Now, $L' \geq \frac{1}{r(\gamma(c))} L$, for some $0 \leq c \leq 1$. Since $d(\gamma(c), x) < \frac{3r(x)}{4}$, it follows $r(\gamma(c)) \leq r(x) + d(x, \gamma(c)) \leq r(x) + \frac{3r(x)}{4} = K_o$ and so $L' \geq \frac{1}{K_o} L \geq \frac{1}{K_o} \mathrm{d}(x_n, x_m)$. Hence $\mathrm{d}'(x_n, x_m) \geq \frac{1}{K_o} \mathrm{d}(x_n, x_m)$ and so $\{x_n\}_{n \geq n_o}$ is a Cauchy sequence of $\overline{B(x, \frac{3r(x)}{4})}$ with respect to d. Therefore it converges proving $\overline{B^{\langle \cdot, \cdot \rangle'}(x, \frac{1}{3})}$ is complete with respect to d', for any $x \in M$.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence with respect to the distance d'. Then there exists n_o such that $x_n \in \overline{B^{\langle \cdot, \cdot \rangle'}(x_{n_o}, \frac{1}{3})}$ for $n \geq n_o$. Hence $\{x_n\}_{n \geq n_o}$ is a Cauchy sequence of $\overline{B^{\langle \cdot, \cdot \rangle'}(x_{n_o}, \frac{1}{3})}$ and so it converges.

Remark 1 In [31] the authors consider the function $r(x)$ to be the supremum of positive numbers r such that the neighborhood $B(x, r)$ is relative compact. This function does not work if M has infinite dimension due of the lack of the local compactness. Moreover, in the finite dimensional case $\overline{B^{\langle \cdot, \cdot \rangle'}(x, \frac{1}{3})}$ is a complete metric space with respect to d' since it is contained in $\overline{B(x, \frac{r(x)}{2})}$ and so it is compact. In our case, we have to check directly that $\overline{B^{\langle \cdot, \cdot \rangle'}(x, \frac{1}{3})}$ is complete.

If M is a connected finite dimensional RH manifold, then M is complete if and only if it is *geodesically complete* at some point $p \in M$, i.e., there exists $p \in M$ such that the maximal interval of definition of any geodesics starting at p is all of $\mathbb{R}$ and so the exponential map $\exp_p$ is defined on all of T_pM. This also implies that the exponential map $\exp_q$ is defined in all of T_qM for any $q \in M$ and any two points can be joined by a minimal geodesic. These facts are not true, in general, for infinite dimensional RH manifolds. The following example is due to Grossman [13].

Example 2.1 Let $\mathbb{H}$ be a separable Hilbert space with an orthonormal basis $\{e_i, i \in \mathbb{N}\}$. consider

$$M = \left\{ \sum_{i=1}^{\infty} x_i e_i \in \mathbb{H} : \; x_1^2 + \sum_{i=2}^{\infty} \left(1 - \frac{1}{i}\right)^2 x_i^2 = 1 \right\}.$$

Then M is a complete RH manifold such that e_1 and $-e_1 \in M$ can be connected by infinitely many geodesics but there are not a minimal geodesics between the two points.

Remark 2 Atkin [1] modified the above example to construct a complete RH manifold such that the exponential map at some point fails to be surjective.

On the other hand the following result holds.

Theorem 2.2 *Let M be a complete RH manifold and $p \in M$. Then the exponential map $\exp_p$ is defined on all of T_pM. Moreover, the set $\mathscr{M}_p = \{q \in M$: there exists a unique minimal geodesic joining p and q $\}$ is dense in M*

The first part of the Theorem can be proved as in the finite dimensional case. The second part is a result due to Ekeland [11]. He proved $\mathscr{M}_p$ contained a countable intersection of open and dense subsets of M. By the Baire's Theorem it follows $\mathscr{M}_p$ is dense.

The next result proves that a RH manifold of constant sectional curvature which is geodesically complete it is also complete.

Proposition 2.1 *Let M be a RH manifold of constant sectional curvature K_o. Then M is a complete RH manifold if and only there exists $p \in M$ such that* $\exp_p$ *is defined in all of T_pM.*

Proof By Theorem 2.2, completeness implies geodesically completeness. Vice-versa, if the sectional curvature is non positive then geodesic completeness is equivalent to completeness. This is a consequence of a version of the Cartan–Hadamard Theorem due to McAlpin [24] and Grossman [13, 22]. Hence we may assume $K_o > 0$. Let $p \in M$ and let $S_{\sqrt{K_o}}(T_pM \times \mathbb{R})$ the sphere of $T_pM \times \mathbb{R}$ of radius $\frac{1}{\sqrt{K_o}}$. Let $N = (0, \frac{1}{\sqrt{K_o}}) \in S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R})$ and let $T : T_N S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \longrightarrow T_pM$ be an isometry. By Proposition 3.1 and a Theorem of Cartan [18, Theorem 1.12.8], the map

$$F = \exp_p \circ T \circ \exp_N^{-1} : S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \setminus \{-N\} \longrightarrow M$$

is a local isometry. Let $v \in T_N S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R})$ be a unit vector. Then $\gamma^v(t) = F(\exp_N(tv))$ is a geodesic in M. Let $q(v) = \gamma^v(\pi)$. It is easy to see that $q(v) = q(w)$ for any unit vector $w \in T S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R})$. Hence we may extend $F : S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R}) \longrightarrow M$ and it is easy to check that it is still an isometry. Since $S_{\sqrt{K_o}}(\mathbb{H} \times \mathbb{R})$ is complete, by [22, Theorem 6.9 p. 228] we get F is a Riemannian covering map, and so F is surjective, and M is complete.

Definition 2.2 A Hopf–Rinow manifold is a complete RH manifold such that any two points $x, y \in M$ can be joined by a minimal geodesic.

The unit sphere $S(\mathbb{H})$ is Hopf–Rinow. The Stiefel manifolds of orthonormal p frames in a Hilbert space $\mathbb{H}$ and the Grassmann manifolds of p subspaces of $\mathbb{H}$ are Hopf–Rinow manifolds [5, 14]. These manifolds are homogeneous, i.e., the isometry group acts transitively on M. It is easy to see that homogeneity implies completeness [5] but it does not imply the existence of path of minimal length between two points. We also point out that the isometry group of a complete RH manifold can be turned in a Banach Lie group and its Lie algebra is given by the complete Killing vector fields, i.e., vector fields X such that $L_X\langle \cdot, \cdot \rangle = 0$. Moreover the natural action of the isometry group on M is smooth (see [20]).

In [2, 5] properly isometric discontinuous actions on the unit sphere of a Hilbert space $\mathbb{H}$ and on the Stiefel and Grassmannian manifolds are studied. We recall that a group Γ of isometries acts properly discontinuously on M if for any $f \in \Gamma$, the condition $f(x) = x$ for some $x \in M$ implies $f = \mathrm{e}$ and the orbit throughout any element $x \in M$ is closed and discrete [21]. We completely classify properly discontinuous actions of a finitely generated abelian group on the unit sphere of a separable Hilbert space and we give new examples of complete RH manifolds, respectively Kähler RH manifolds, with non negative and non positive sectional curvature with infinite fundamental group, respectively with non negative holomorphic sectional curvature with infinite fundamental group ([2, 5]). These new examples of RH manifolds are Hopf–Rinow manifolds due the following simple fact.

Proposition 2.2 *Let M be a Hopf–Rinow manifold. Let Γ be a group acting isometrically and properly discontinuously on M. Then M/Γ is also Hopf–Rinow.*

Proof Since Γ acts isometrically and properly discontinuously on M, it follows that M/Γ admits a Riemannian metric such that M/Γ is complete and $\pi : M \longrightarrow M/\Gamma$ is a Riemannian covering map [2, 22]. Let $p, q \in M/\Gamma$. Since Γ acts properly discontinuously on M, then both $\pi^{-1}(p)$ and $\pi^{-1}(q)$ are Γ orbits, and also closed and discrete subsets of M [21]. Hence given $z \in \pi^{-1}(p)$, there exists a unique $w \in \pi^{-1}(q)$ such that $\mathrm{d}(z, w) \leq \mathrm{d}(r, s)$ for every $r \in \pi^{-1}(p)$ and $s \in \pi^{-1}(q)$, i.e., $\mathrm{d}(z, w) = \mathrm{d}(\pi^{-1}(p), \pi^{-1}(q))$. Let γ be a minimal geodesic joining z and w. We claim that $\pi \circ \gamma$ is a minimal geodesic. Since π is a Riemannian covering map, then $\mathrm{d}(p, q) \leq L(\pi \circ \gamma) = L(\gamma) = \mathrm{d}(z, w)$. On the other hand pick a sequence $\gamma_n : [0, 1] \longrightarrow M/\Gamma$ joining p and q such that $\lim_{n \mapsto +\infty} L(\gamma_n) = \mathrm{d}(p, q)$. Since π is a Riemannian covering map there exists a lift $\tilde{\gamma}_n$ starting at z satisfying $L(\gamma_n) = L(\tilde{\gamma}_n)$. Therefore

$$L(\gamma) = \mathrm{d}(z, w) \leq L(\gamma_n) \mapsto \mathrm{d}(p, q),$$

and so $L(\pi \circ \gamma) = \mathrm{d}(p, q)$.

In [5] we prove a homogeneity result for Riemannian Hilbert manifolds of constant sectional curvature. In finite dimension this result was proved by Wolf [35, 38].

An isometry $f : M \longrightarrow M$ is called a *Clifford translation* if $\delta_f(x) = \mathrm{d}(x, f(x))$ is a constant function. As in the finite dimensional case, if M is a homogeneous Riemannian manifold and Γ a group acting on M isometrically and properly discontinuously on M, then M/Γ is homogeneous if and only if the centralizer of Γ, that we denote by $\mathrm{Z}(\Gamma)$, acts transitively on M [5, 38]. In particular if M/Γ is homogeneous then any element $g \in \Gamma$ is a Clifford translation. Indeed,

$$\mathrm{d}(x, g(x)) = \mathrm{d}(h(x), hg(x)) = \mathrm{d}(h(x), g(h(x))),$$

for any $h \in \mathrm{Z}(\Gamma)$. Hence if $\mathrm{Z}(\Gamma)$ acts transitively on M we get f is a Clifford translation.

In the finite dimensional case, the homogeneity conjecture says that if M is a homogeneous simply connected Riemannian manifold then M/Γ is homogeneous if and only if all the elements of Γ are Clifford translations. We point out that the conjecture is true for locally homogeneous symmetric spaces [36] and also for locally homogeneous Finsler symmetric spaces [8]. In [5] we proved the homogeneity conjecture for complete RH manifolds of constant sectional curvature. We leave the investigation of locally homogeneous symmetric space of infinite dimension for future investigation (see [9, 19, 22] for basic references of symmetric space in infinite dimension.) The following result proves there are not non trivial Clifford translations on a Hadamard manifold, i.e., a simply connected Riemannian Hilbert manifold with negative sectional curvature.

Proposition 2.3 *Let M be a simply connected RH manifold of negative sectional curvature. If $f : M \longrightarrow M$ is a Clifford translation then $f = \mathrm{Id}$.*

Proof Assume $f(p) \neq p$ for some $p \in M$, hence for every $p \in M$. By Cartan–Hadamard Theorem M is a Hopf–Rinow manifold and so by Lemma 5.2 p. 448 in [5], see also [32], f preserves the minimal geodesic, that we denote by γ_p, joining p and $f(p)$. Let $p \in M$ and let θ be a geodesic different from γ_p. As in the Proof of Theorem 1 p. 16 in [37], one can prove that the union $\gamma_{\theta(t)}$ is a flat totally geodesic surface which is a contradiction.

3 Jacobi Fields and Conjugate Points

Let M be a RH manifold and let $\gamma : [0, b) \longrightarrow M$ be a geodesic with $\gamma(0) = p$. Without lost of generality we assume that $\gamma(t) = \exp_p(tv)$, with $\|v\| = 1$. A Jacobi field along γ is a smooth vector field J along γ satisfying

$$\nabla_{\dot{\gamma}(t)} \nabla_{\dot{\gamma}(t)} J(t) + R(\dot{\gamma}(t), J(t))J(t) = 0.$$

In the sequel we will denote by $J'(t)$ the covariant derivative $\nabla_{\dot{\gamma}(t)} J(t)$. If J_1 and J_2 are Jacobi fields along γ, then

$$\langle J_1'(t), J_2(t)\rangle - \langle J_1(t), J_2'(t)\rangle = \text{Constant}. \tag{1}$$

This formula is due to Ambrose (see [22]). The Jacobi field along γ satisfying $J(0) = 0$ and $J'(0) = \nabla_{\dot{\gamma}(t)} J(0) = w$ is given by $J(t) = (\mathrm{d}\exp_p)_{tv}(tw)$. Hence $(\mathrm{d}\exp_p)_v(w) = 0$ if and only if there exists a Jacobi field J along $\gamma(t)$ such that $J(0) = 0$ and $J(1) = 0$.

In infinite dimension there exist two types of singularities of the exponential map.

Definition 3.1 We will say that $q = \gamma(t_o)$, $t_o \in (0, b)$, is

- *monoconjugate* to p along γ if $(\mathrm{d}\exp_p)_{t_o v}$ is not injective,
- *epiconjugate*, to p along γ if $(\mathrm{d}\exp_p)_{t_o v}$ is not surjective.

We also say $q = \gamma(t_o)$ is conjugate of p along γ if $(\mathrm{d}\exp_p)_{t_o v}$ is not an isomorphism and $t_o \in (0, b)$ is a conjugate, monoconjugate, respectively epiconjugate instant if $\gamma(t_o)$ is conjugate, monoconjugate, respectively epiconjugate of p along γ.

Let $\tau_t^s : T_{\gamma(t)}M \longrightarrow T_{\gamma(s)}M$ be the isometry between the tangent spaces given by the parallel transport along the geodesic γ. The following result is easy to check.

Lemma 1 *If* $V : [0, b) \longrightarrow T_pM$*, then* $\nabla_{\dot\gamma(t)} \tau_0^t(V(t)) = \tau_0^t(\dot V(t))$.

By the above Lemma, a Jacobi field along γ such that $J(0) = 0$ is given by $J(t) = \tau_0^t(T(t)(V))$, where $V \in T_pM$, and $T(t)$ is a family of endomorphism of T_pM satisfying

$$\begin{cases} T''(t) + R_t(T(t)) = 0; \\ T(0) = 0, \ T'(0) = \mathrm{Id}, \end{cases}$$

where $R_t : T_pM \longrightarrow T_pM$ is a one parameter family od endomorphism of T_pM defined by $R_t(X) = \tau_t^0(R(\tau_0^t(X), \dot\gamma(t))\dot\gamma(t))$. We call the above differential equation the *Jacobi flow* of γ.

Example 3.1 Assume that M is a RH manifold with constant sectional curvature K_o. Then

$$T(t)(w) = \begin{cases} \frac{\sinh(t\sqrt{-K_o})}{\sqrt{-K_o}} w & K_o < 0 \\ tw & K_o = 0 \\ \frac{\sin(t\sqrt{K_o})}{\sqrt{K_o}} & K_o > 0 \end{cases}$$

Karcher used the Jacobi flow to get Jacobi fields estimates [15]. By standard properties of the curvature, it follows R_t is a symmetric endomorphism of T_pM. Since $\tau_0^t \circ T(t) = t(\mathrm{d}\exp_p)_{tv}$ we may thus equivalently state the definitions of monoconjugate, epiconjugate in terms of injectivity, respectively surjectivity of $T(t)$. Moreover, conjugate instants are also discussed in terms of Lagrangian curves [7]. Indeed, the Hilbert space $T_pM \times T_pM$ has a natural symplectic structure given by $\omega((X, Y), (Z, W)) = \langle X, W\rangle - \langle Y, Z\rangle$. It is easy to check that $\Psi(t) : T_pM \times T_pM \longrightarrow T_pM \times T_pM$ defined by $\Psi(t)(X, Y) = (\tau_t^0(J(t)), \tau_t^0(J'(t)))$, where $J(t)$ is the Jacobi field along γ such that $J(0) = X$ and $J'(0) = Y$, is a symplectomorphism of $(T_pM \times T_pM, \omega)$. Then $E_t = \Phi_t(\{0\} \times T_pM)$ is a curve of Lagrangian subspaces of $T_pM \times T_pM$. Moreover $t_o \in (0, b)$ is a monoconjugate instant, respectively a epiconjugate instant, if and only if $E_t \cap (\{0\} \times T_pM) \neq \{0\}$, respectively if and only if $E_t + (\{0\} \times T_pM) \neq T_pM \times T_pM$.

Let $t_o \in (0, b)$. We compute the transpose of $T(t_o)$. Let $J_1(t) = \tau_0^t(T(t)(v))$ and let $u \in T_pM$. Let J_2 be the Jacobi field along the geodesic γ such that $J_2(t_o) = 0$, $\nabla_{\dot\gamma(t_o)} J_2(t_o) = \tau_0^{t_o}(u)$. By (1), we have $\langle J_1(t_o), J_2'(t_o)\rangle = \langle J_1'(0), J_2(0)\rangle$ and so $\langle T(t_o)(v), u\rangle = \langle v, \tau_{t_o}^0(J_2(t_0))\rangle$. Let $\overline{\gamma}(t) = \gamma(t_o - t)$ and let

$$\begin{cases} \tilde T''(t) + R_t(\tilde T(t)) = 0; \\ \tilde T(0) = 0, \ \tilde T'(0) = id, \end{cases}$$

be the Jacobi flow along $\overline{\gamma}$. Summing up we have proved that $T^*(t_o) = \tau_{t_o}^0 \circ \tilde T(t_o) \circ \circ \tau_0^{t_o}$. As a corollary, keeping in mind Example 3.1, we get the following result.

Proposition 3.1 *The kernel of* $T(t_o)$ *and the kernel of* $T^*(t_o)$ *are isomorphic. Hence a monoconjugate point is also epiconjugate. Moreover, if* M *has constant sectional* *curvature K_o, then $T(t)$ is an isomorphism for any $t > 0$ whether $K_o \leq 0$, and $T(t)$ is an isomorphism for $0 < t < \frac{\pi}{\sqrt{K_o}}$ whether $K_o > 0$.*

The above result was proven by McAlpin [24] and Grossmann in [13]. Since both Rauch and Berger Comparison Theorems work for RH manifolds [4, 22], they also work for a weak Riemannian Hilbert manifold [3], the second part of Proposition 3.1 holds for any RH manifold with negative sectional curvature and for any RH manifold with sectional curvature bounded above for a constant $K_o > 0$.

Proposition 3.1 implies that if $\operatorname{Im} T(t_o)$ is closed then monoconjugate implies epiconjugate and vice-versa. This holds, for example, if $\exp_p$ is Fredholm. We recall that a smooth map between Hilbert manifolds $f : M \longrightarrow N$ is called Fredholm if for each $p \in M$ the derivative $(\mathrm{d}f)_p : T_p M \longrightarrow T_{f(p)} N$ is a Fredholm operator. If M is connected then the ind $(df)_p$ is independent of p, and one defines the index of f by setting $\operatorname{ind}(f) = \operatorname{ind}(df)_p$ (see [12, 34]). Misiolek proved that the exponential map of a free loop space with its natural Riemannian metric is Fredholm [27]. Misiolek also pointed out that if the curvature is a compact operator, i.e., for any $X \in T_p M$, the map $Z \mapsto R(Z, X)X$ is a compact operator, then $T(t)$ is Fredholm of index zero and so the exponential map is Fredholm as well [28]. Indeed,

$$T(t) = tId - \int_0^t \left(\int_0^h R_s(T(s)) ds \right) dh$$

and so $T(t) = tId + K(t)$ where $K(t)$ is a compact operator. Hence $T(t)$ is Fredholm [34] and so $\exp_p$ is Fredholm.

It is convenient to introduce the notion of *strictly epiconjugate* instant, to denote an instant $t \in]0, b[$ for which the range of $T(t)$ fails to be closed. Unlike finite-dimensional Riemannian geometry, conjugate instants can accumulate. The classical example of this phenomenon is given by an infinite dimensional ellipsoid in ℓ^2 whose axes form a non discrete subset of the real line given by Grossman ([13]).

Let $M = \{x \in \ell^2 : \, x_1^2 + x_2^2 + \sum_{i=3}^{\infty} (1 - \frac{1}{i})^4 x_i^2 = 1\}$. M is a closed submanifold of ℓ^2 and the curve $\gamma(t) = \cos te_1 + \sin te_2$ is a geodesic of M since it is the set of fixed points of the isometry

$$F\left(\sum_{i=1}^{\infty} x_i e_i \right) = x_1 e_1 + x_2 e_2 + \sum_{i=3}^{\infty} (-x_i) e_i .$$

For any $k \geq 3$, $E_k := \{x_1^2 + x_2^2 + (1 - \frac{1}{k})^4 x_k^2 = 1 \} \hookrightarrow M$ is a totally geodesic submanifold of M since it is the fixed points set of the isometry $F(\sum_{i=1}^{\infty} x_i e_i) = x_1 e_1 - x_2 e_2 + x_k e_k + \sum_{i=3, i \neq k}^{\infty} (-x_i) e_i$. Hence $K(\dot{\gamma}(s), e_k) = (1 - \frac{1}{k})^2$, $J_k(t) = \sin(t(1 - \frac{1}{k})) e_k$ is the Jacobi field along γ satisfying $J(0) = 0$ and $J'(0) = e_k$. Consider $q_k = \frac{k\pi}{k-1}$. Then q_k is a sequence of monoconjugate instant such that $\lim_{k \mapsto \infty} q_k = \pi$. We claim that $-e_1 = \gamma(\pi)$ is a strictly epiconjugate point. Indeed,

$$T(\pi)\left(e_2+\sum_{k=3}^{\infty}b_ke_k\right)=e_2+\sum_{k=3}^{\infty}b_k\sin\left(\left(\frac{k-1}{k}\right)\pi\right)e_k$$

which implies $T(\pi)$ is injective. On the other hand $\sum_{i=3}^{\infty}\frac{1}{k}e_k$ does not lie in Im $T(\pi)$ and so $\gamma(\pi)$ is strictly epiconjugate. Indeed if $\sum_{i=3}^{\infty}\frac{1}{k}e_k\in$ Im $T(\pi)$ then $\sum_{k=3}^{\infty}\frac{1}{k}e_k=\sum_{k=3}^{\infty}b_k\sin((1-\frac{1}{k})\pi)e_k$ and so $-\sin(\pi\frac{1}{k})b_k=\frac{1}{k}$. Hence

$$\lim_{k\mapsto+\infty}b_k=-\lim_{k\mapsto+\infty}k\sin\left(\pi\frac{1}{k}\right)=-\pi$$

which is a contradiction. Hence $\gamma(\pi)$ is a strictly epiconjugate point along γ and it is an accumulation point of sequence of monoconjugate points.

In [7] the authors give a complete characterization of the conjugate instants along a geodesic. In particular the set of conjugate instants is closed and the set of strictly epiconjugate points are limit of conjugate points as before. Hence if there is no strictly epiconjugate instant along γ then the set of conjugate instants along any compact segment of γ is finite. Under these circumstances a Morse Index Theorem for geodesics in RH manifolds holds true.

Acknowledgements The authors thank the referee for useful suggestions and remarks. The authors were partially supported by FIRB 2012 "Geometria differenziale e teoria geometrica delle funzioni" grant RBFR12W1AQ003. The first author was partially supported by GNSAGA of INdAM and PRIN Varietà reali e complesse: geometria, topologia e analisi armonica. The second author was partially supported by Fapesp and CNPq (Brazil).

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Real Hypersurfaces in Hermitian Symmetric Space of Rank Two with Killing Shape Operator

Ji-Eun Jang, Young Jin Suh and Changhwa Woo

Abstract We have considered a new notion of the shape operator A satisfies Killing tensor type for real hypersurfaces M in complex Grassmannians of rank two. With this notion we prove the non-existence of real hypersurfaces M in complex Grassmannians of rank two.

1 Introduction

A typical example of Hermitian symmetry spaces of rank two is the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Another one is complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$.

Characterizing model spaces of real hypersurfaces under certain geometric conditions is one of our main interests in the classification theory in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ or complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$. In this paper, we use the same geometric condition on real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ as used in $G_2(\mathbb{C}^{m+2})$ to compare the results.

$G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2{\cdot}U_m)$ has a compact transitive group SU_{2+m}, however $SU_{2,m}/S(U_2{\cdot}U_m)$ has a noncompact indefinite transitive group $SU_{2,m}$. This

J.-E. Jang
Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea
e-mail: ji-eun82@daum.net

Y.J. Suh
Department of Mathematics and RIRCM,
Kyungpook National University, Daegu 41566, Republic of Korea
e-mail: yjsuh@knu.ac.kr

C. Woo (✉)
Department of Mathematics Education, Woosuk University, Wanju, Jeonbuk 565-701, Republic of Korea
e-mail: legalgwch@woosuk.ac.kr

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_23

distinction gives various remarkable results. Riemannian symmetric space $SU_{2,m}/S(U_2 \cdot U_m)$ has a remarkable geometrical structure. It is the unique noncompact, Kähler, irreducible, quaternionic Kähler manifold with negative curvature.

Suppose that M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2 \cdot U_m)$). Let N be a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2 \cdot U_m)$). Since $G_2(\mathbb{C}^{m+2})$(or $SU_{2,m}/S(U_2 \cdot U_m)$) has the Kähler structure J, we may define the *Reeb vector field* $\xi = -JN$ and a one dimensional distribution $[\xi] = \mathscr{C}^{\perp}$ where $\mathscr{C}$ denotes the orthogonal complement in T_xM, $x \in M$, of the Reeb vector field ξ. The Reeb vector field ξ is said to be *Hopf* if $\mathscr{C}$ (or $\mathscr{C}^{\perp}$) is invariant under the shape operator A of M. The one dimensional foliation of M defined by the integral curves of ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* if and only if the Hopf foliation of M is totally geodesic. By the formulas in [5, Sect. 2], it can be checked that ξ is Hopf vector field if and only if M is Hopf hypersurface.

From the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2 \cdot U_m)$), there naturally exist *almost contact 3-structure* vector fields $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Put $\mathscr{Q}^{\perp} = \text{Span}\{\xi_1, \xi_2, \xi_3\}$. It is a 3-dimensional distribution in the tangent bundle TM of M. In addition, we denoted by $\mathscr{Q}$ the orthogonal complement of $\mathscr{Q}^{\perp}$ in TM. It is the quaternionic maximal subbundle of TM. Thus the tangent bundle of M is expressed as a direct sum of $\mathscr{Q}$ and $\mathscr{Q}^{\perp}$.

For any geodesic γ in M, a (1,1) type tensor field T is said to be Killing if $T\dot{\gamma}$ is parallel displacement along γ, which gives $0 = \nabla_{\dot{\gamma}}(T\dot{\gamma}) = (\nabla_{\dot{\gamma}}T)\dot{\gamma} + T(\nabla_{\dot{\gamma}}\dot{\gamma}) = (\nabla_{\dot{\gamma}}T)\dot{\gamma}$. That is, $(\nabla_X T)X = 0$ for any tangent vector field X on M (see [2]).

$$\begin{aligned} 0 &= (\nabla_{X+Y}T)(X+Y) \\ &= (\nabla_X T)X + (\nabla_X T)Y + (\nabla_Y T)X + (\nabla_Y T)Y \\ &= (\nabla_X T)Y + (\nabla_Y T)X \end{aligned}$$

for any vector fields X and Y on M.

Thus the Killing tensor field T is equivalent to $(\nabla_X T)Y + (\nabla_Y T)X = 0$.

From this notion, in this paper we consider a new condition related to the shape operator A of M defined in such a way that

$$(\nabla_X A)Y + (\nabla_Y A)X = 0 \tag{C-1}$$

for any vector fields X on M.

In this paper, we give a complete classification for real hypersurfaces in $\bar{M}$ ($G_2(\mathbb{C}^{m+2})$ or $SU_{2,m}/S(U_2 \cdot U_m)$) with Killing shape operator. In order to do it, we consider a problem related to the following:

Theorem 1 *There does not exist any real hypersurface in $\bar{M}$ complex Grassmannians of rank two, $m \geq 3$, with Killing shape operator.*

Since the notion of Killing tensor field is weaker than the notion of parallel tensor field, by Theorem 1, we naturally have the following:

quotation There does not exist any real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel shape operator (see [11]).

On the other hand, by virtue of Theorem 2 we can assert the following:

Corollary 1 *There does not exist any hypersurface in* $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$ *with parallel shape operator.*

2 Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$ and $SU_{2,m}/S(U_2{\cdot}U_m)$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [5, 6, 11, 12]. The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B, this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way, $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight.

When $m = 2$, we note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ of K has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\Big\} \\ & + \sum_{\nu=1}^{3}\Big\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\Big\},\end{aligned} \tag{2.1}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

Now we summarize basic material about complex hyperbolic two-plane Grassmann manifolds $SU_{2,m}/S(U_2{\cdot}U_m)$, for details we refer to [14, 16].

The Riemannian symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$, which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, is a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank 2. Let $G = SU_{2,m}$ and $K = S(U_2{\cdot}U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K, respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{A}ut(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m}AI_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$$

I_2 (resp., I_m) denotes the identity 2×2-matrix (resp., $m \times m$-matrix). Then $< X, Y >= -B(X, \theta Y)$ becomes a positive definite $\mathrm{A}d(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2{\cdot}U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2{\cdot}U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2{\cdot}U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2}I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2}I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2{\cdot}U_m)$. By construction, J commutes with each almost Hermitian structure J_ν in $\mathfrak{J}$ for $\nu = 1, 2, 3$. Recall that a canonical local basis $\{J_1, J_2, J_3\}$ of a quaternionic Kähler structure $\mathfrak{J}$ consists of three almost

Hermitian structures J_1, J_2, J_3 in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is to be taken modulo 3. The tensor field JJ_ν, which is locally defined on $SU_{2,m}/S(U_2{\cdot}U_m)$, is self-adjoint and satisfies $(JJ_\nu)^2 = I$ and $tr(JJ_\nu) = 0$, where I is the identity transformation. For a nonzero tangent vector X we define $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

We identify the tangent space $T_oSU_{2,m}/S(U_2{\cdot}U_m)$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ at o with $\mathfrak{p}$ in the usual way. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since $SU_{2,m}/S(U_2{\cdot}U_m)$ has rank 2, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_oSU_{2,m}/S(U_2{\cdot}U_m) \cong \mathfrak{p}$ is contained in some maximal abelian subspace of $\mathfrak{p}$. Generically this subspace is uniquely determined by X, in which case X is called regular. If there exists more than one maximal abelian subspaces of $\mathfrak{p}$ containing X, then X is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Up to scaling there exists a unique $SU_{2,m}$-invariant Riemannian metric g on complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$. Equipped with this metric $SU_{2,m}/S(U_2{\cdot}U_m)$ is a Riemannian symmetric space of rank 2 which is both Kähler and quaternionic Kähler. For computational reasons we normalize g such that the minimal sectional curvature of $(SU_{2,m}/S(U_2{\cdot}U_m), g)$ is -4. The sectional curvature K of the noncompact symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$ equipped with the Killing metric g is bounded by $-4 \leq K \leq 0$. The sectional curvature -4 is obtained for all 2-planes $\mathbb{C}X$ when X is a non-zero vector with $JX \in \mathfrak{J}X$.

When $m = 1$, $G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1{\cdot}U_2)$ is isometric to the two-dimensional complex hyperbolic space $\mathbb{C}H^2$ with constant holomorphic sectional curvature -4.

When $m = 2$, we note that the isomorphism $SO(4, 2) \simeq SU_{2,2}$ yields an isometry between $G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2{\cdot}U_2)$ and the indefinite real Grassmann manifold $G_2^*(\mathbb{R}_2^6)$ of oriented two-dimensional linear subspaces of an indefinite Euclidean space $\mathbb{R}_2^6$. For this reason we assume $m \geq 3$ from now on, although many of the subsequent results also hold for $m = 1, 2$.

Hereafter X, Y and Z always stand for any tangent vector fields on M.

The Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ is locally given by

$$\begin{aligned}\bar{R}(X, Y)Z = -\frac{1}{2}\Big[& g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ & - g(JX, Z)JY - 2g(JX, Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y \\ & - 2g(J_\nu X, Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY, Z)J_\nu JX - g(J_\nu JX, Z)J_\nu JY\}\Big],\end{aligned} \tag{2.2}$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathfrak{J}$.

3 Basic Formulas

In this section we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$) (see [3, 5, 7, 10–12, 18]).

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$). The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.1}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$) there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{3.2}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$), together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in Sect. 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0,\\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\
&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned} \tag{3.3}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$ in Sect. 2 and (3.1), the relation between these two almost contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\begin{aligned}
&\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu,\\
&\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi.
\end{aligned} \tag{3.4}$$

On the other hand, from the parallelism of Kähler structure J, that is, $\bar\nabla J = 0$ and the quaternionic Kähler structure $\mathfrak{J}$, together with Gauss and Weingarten formulas it follows that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{3.5}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{3.6}$$

$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \tag{3.7}$$

Combining these formulas, we find the following:

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu)\\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu)\\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX\\ &\quad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{3.8}$$

Using the above expression (2.1) for the curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$), the equations of Codazzi is given by

$$\begin{aligned}k\Big\{(\nabla_X A)Y - (\nabla_Y A)X\Big\} &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\Big\}\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\Big\}\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\Big\}\xi_\nu,\end{aligned} \tag{3.9}$$

where in the case of $G_2(\mathbb{C}^{m+2})$ (resp., $SU_{2,m}/S(U_2{\cdot}U_m)$), the constant $k = 1$ and $SU_{2,m}/S(U_2{\cdot}U_m)$ (resp., $k = -2$).

4 Proof of Theorems

In this section, we classify real hypersurfaces in $\bar{M}$ ($G_2(\mathbb{C}^{m+2})$ or $SU_{2,m}/S(U_2{\cdot}U_m)$) whose shape operator has Killing tensor field.

From (C-1) and the Codazzi equation (3.9), we have

$$\begin{aligned}-2k(\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\Big\}\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\Big\}\\ &\quad + \sum_{\nu=1}^{3}\Big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\Big\}\xi_\nu\end{aligned} \tag{4.1}$$

Putting $Y = \xi$ into (4.1),

$$-2k(\nabla_\xi A)X = -\phi X + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 3\eta_\nu(\phi X)\xi_\nu\big\}. \tag{4.2}$$

Lemma 1 *Let M be a real hypersurface in complex Grassmannians of rank two $\bar{M}$, $m \geq 3$ with Killing shape operator. Then the Reeb vector field ξ belongs to either the distribution $\mathscr{Q}$ or the distribution $\mathscr{Q}^\perp$.*

Proof Without loss of generality, ξ is written as

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1, \tag{$**$}$$

where X_0 (resp., ξ_1) is a unit vector in $\mathscr{Q}$ (resp., $\mathscr{Q}^\perp$).

Taking the inner product of (4.2) with ξ, we have

$$-2kg\big((\nabla_\xi A)X, \xi\big) = -4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi X). \tag{4.3}$$

Since $(\nabla_\xi A)$ is self-adjoint, it follows from (C-1) that $-4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi X) = 0$. By putting $X = \phi X_0$ and using (**), we have $-4\eta_1^2(\xi)\eta(X_0) = 0$.

Thus we have only two cases: $\xi \in \mathscr{Q}^\perp$ or $\xi \in \mathscr{Q}$.

- **Case 1.** $\xi \in \mathscr{Q}^\perp$.

Without loss of generality, we may put $\xi = \xi_1 \in \mathscr{Q}^\perp$. Then (4.2) is reduced into

$$-2k(\nabla_\xi A)X = -\phi X - \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3. \tag{4.4}$$

The symmetric endomorphism of (4.4) with respect to the metric g, we have

$$-2k(\nabla_\xi A)X = \phi X + \phi_1 X - 2\eta_3(X)\xi_2 + 2\eta_2(X)\xi_3. \tag{4.5}$$

Combining (4.4) with (4.5), we have $\phi X + \phi_1 X - 2\eta_3(X)\xi_2 + 2\eta_2(X)\xi_3 = 0$. By putting $X = \xi_3$ into the equation above, we have $2\xi_3 = 0$. This is a contradiction.

Thus, there does not exist any hypersurface in $\bar{M}$, $m \geq 3$, with Killing shape operator and $\xi \in \mathscr{Q}^\perp$ everywhere.

- **Case 2.** $\xi \in \mathscr{Q}$.

Equation (4.2) becomes

$$-2k(\nabla_\xi A)X = -\phi X + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu\xi - 3\eta_\nu(\phi X)\xi_\nu\big\}. \tag{4.6}$$

The symmetric endomorphism of (4.6) with respect to the metric g, we have

$$-2k(\nabla_\xi A)X = \phi X + \sum_{\nu=1}^{3}\big\{-\eta_\nu(\phi X)\xi_\nu + 3\eta_\nu(X)\phi\xi_\nu\big\}. \tag{4.7}$$

Combining (4.6) with (4.7), we have $2\phi X + 2\sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi\xi_\nu + \eta_\nu(\phi X)\xi_\nu\big\} = 0$. By putting $X = \xi_1$ into above equation, we have $4\phi\xi_1 = 0$. This is a contradiction, too. Thus, there does not exist any hypersurface in $\bar{M}$, $m \geq 3$, with Killing shape operator and $\xi \in \mathscr{Q}$ everywhere.

Accordingly, we complete the proof of Theorem 1 in the introduction.

Usually, the notion of parallel is stronger than the notion of Killing, we also have a non-existence of parallel hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. Then Corollary 1 in the introduction is naturally proved.

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The Chern-Moser-Tanaka Invariant on Pseudo-Hermitian Almost CR Manifolds

Jong Taek Cho

Abstract We study on the Chern-Moser-Tanaka invariant (Chern, Acta Math 133:219–271, 1974, [5], Tanaka, Japan J Math 12:131–190, 1976, [14]) of pseudo-conformal transformations on pseudo-Hermitian almost CR manifolds.

1 Introduction

A contact manifold (M, η) admits the fundamental structures which enrich the geometry. One is a Riemannian metric g compatible to the contact form η and we obtain a *contact Riemannian manifold* $(M; \eta, g)$. The other is a *pseudo-Hermitian and strictly pseudo-convex structure* (η, L) (or (η, J)), where L is the *Levi form* associated with an endomorphism J on D (= kernel of η) such that $J^2 = -I$. $(M; \eta, J)$ is called a *strictly pseudo-convex, pseudo-Hermitian manifold (or almost CR manifold)*. Then we have a one-to-one correspondence between the two associated structures by the relation $g = L + \eta \otimes \eta$, where we denote by the same letter L the natural extension ($i_\xi L = 0$) of the Levi form to a (0,2)-tensor field on M. So, we treat contact Riemannian structures together with strictly pseudo-convex almost CR structures. In earlier works [6–8, 10], the present author started the intriguing study of the interactions between them. For complex analytical considerations, it is desirable to have integrability of the almost complex structure J (on D). If this is the case, we speak of an *(integrable) CR structure* and of a *CR manifold*. Indeed, S. Webster [21, 22] introduced the term *pseudo-Hermitian structure* for a CR manifold with a non-degenerate Levi-form. In the present paper, we treat the pseudo-Hermitian structure as an extension to the case of non-integrable $\mathscr{H}$.

There is a canonical affine connection in a non-degenerate CR manifold, the so-called pseudo-Hermitian connection (or the Tanaka-Webster connection). S. Tanno [16] extends the Tanaka-Webster connection for strictly pseudo-convex almost CR manifolds (in which $\mathscr{H}$ is in general non-integrable). We call it the *generalized Tanaka-Webster connection*. Using this we have the *pseudo-Hermitian Ricci*

J.T. Cho (✉)
Department of Mathematics, Chonnam National University, Gwangju 61186, Korea
e-mail: jtcho@chonnam.ac.kr

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_24

curvature tensor. If the pseudo-Hermitian Ricci curvature tensor is a scalar (field) multiple of the Levi form in a strictly pseudo-convex almost CR manifold, then it is said to have the *pseudo-Einstein structure*. A *pseudo-Hermitian CR space form* is a strictly pseudo-convex CR manifold of constant holomorphic sectional curvature (for Tanaka-Webster connection). Then we have that a pseudo-Hermitian CR space form is pseudo-Einstein. In Sect. 4, we study the generalized Chern-Moser-Tanaka curvature tensor C as a pseudo-conformal invariant in a strictly pseudo-convex almost CR manifold. Then we first prove that the Chern-Moser-Tanaka curvature tensor vanishes for a pseudo-Hermitian CR space form. Moreover, we prove that for a strictly pseudo-convex almost CR manifold M^{2n+1} ($n > 1$) with vanishing C, M is pseudo-Einstein if and only if M is of pointwise constant holomorphic sectional curvature.

2 Preliminaries

We start by collecting some fundamental materials about contact Riemannian geometry and strictly pseudo-convex pseudo-Hermitian geometry. All manifolds in the present paper are assumed to be connected, oriented and of class C^∞.

2.1 Contact Riemannian Structures

A *contact manifold* (M, η) is a smooth manifold M^{2n+1} equipped with a global one-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. For a contact form η, there exists a unique vector field ξ, called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there also exist a Riemannian metric g and a $(1, 1)$-tensor field φ such that

$$\eta(X) = g(X, \xi),\ d\eta(X, Y) = g(X, \varphi Y),\ \varphi^2 X = -X + \eta(X)\xi, \tag{1}$$

where X and Y are vector fields on M. From (1), it follows that

$$\varphi\xi = 0,\ \eta \circ \varphi = 0,\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (1) is said to be a *contact Riemannian manifold* or *contact metric manifold* and it is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M, we define a $(1, 1)$-tensor field h by $h = \frac{1}{2}\pounds_\xi\varphi$, where $\pounds_\xi$ denotes Lie differentiation for the characteristic direction ξ. Then we may observe that h is self-adjoint and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h, \tag{3}$$

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{4}$$

where ∇ is Levi-Civita connection. From (3) and (4) we see that ξ generates a geodesic flow. Furthermore, we know that $\nabla_\xi \varphi = 0$ in general (cf. p. 67 in [1]). From the second equation of (3) it follows also that

$$(\nabla_\xi h)\varphi = -\varphi(\nabla_\xi h). \tag{5}$$

A contact Riemannian manifold for which ξ is Killing is called a *K-contact manifold.* It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. For further details on contact Riemannian geometry, we refer to [1].

2.2 Pseudo-Hermitian Almost CR Structures

For a contact manifold M, the tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_pM | \eta(v) = 0\}$. Then the $2n$-dimensional distribution (or subbundle) $D : p \to D_p$ is called the *contact distribution (or contact subbundle)*. Its associated almost CR structure is given by the holomorphic subbundle

$$\mathscr{H} = \{X - iJX : X \in \Gamma(D)\}$$

of the complexification $\mathbb{C}TM$ of the tangent bundle TM, where $J = \varphi|D$, the restriction of φ to D. Then we see that each fiber $\mathscr{H}_p$ $(p \in M)$ is of complex dimension n and $\mathscr{H} \cap \bar{\mathscr{H}} = \{0\}$. Furthermore, we have $\mathbb{C}D = \mathscr{H} \oplus \bar{\mathscr{H}}$. For the real representation $\{D, J\}$ of $\mathscr{H}$ we define the Levi form by

$$L : \Gamma(D) \times \Gamma(D) \to \mathscr{F}(M), \quad L(X, Y) = -d\eta(X, JY)$$

where $\mathscr{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is Hermitian and positive definite. We call the pair (η, L) (or (η, J)) a *strictly pseudo-convex, pseudo-Hermitian structure* on M. We say that *the almost CR structure is integrable* if $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H}$. Since $d\eta(JX, JY) = d\eta(X, Y)$, we see that $[JX, JY] - [X, Y] \in \Gamma(D)$ and $[JX, Y] + [X, JY] \in \Gamma(D)$ for $X, Y \in \Gamma(D)$, further if M satisfies the condition $[J, J](X, Y) = 0$ for $X, Y \in \Gamma(D)$, then the pair (η, J) is called a *strictly pseudo-convex (integrable) CR structure* and $(M; \eta, J)$ is called a *strictly pseudo-convex CR manifold* or a *strictly pseudo-convex integrable pseudo-Hermitian manifold*. A *pseudo-Hermitian torsion* is defined by $\tau = \varphi h$ (cf. [2]).

For a given strictly pseudo-convex pseudo-Hermitian manifold M, the almost CR structure is integrable if and only if M satisfies the integrability condition $\Omega = 0$, where Ω is a (1,2)-tensor field on M defined by

$$\Omega(X,Y) = (\nabla_X\varphi)Y - g(X+hX,Y)\xi + \eta(Y)(X+hX) \tag{6}$$

for all vector fields X, Y on M (see [16], Proposition 2.1]). It is well known that for 3-dimensional contact Riemannian manifolds their associated CR structures are always integrable (cf. [16]).

A *Sasakian manifold* is a strictly pseudo-convex CR manifold whose characteristic flow is isometric (or equivalently, vanishing the pseudo-Hermitian torsion). From (6) it follows at once that a Sasakian manifold is also determined by the condition

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X \tag{7}$$

for all vector fields X and Y on the manifold.

Now, we review the *generalized Tanaka-Webster connection* [16] on a strictly pseudo-convex almost CR manifold $M = (M;\eta,J)$. The generalized Tanaka-Webster connection $\hat{\nabla}$ is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X\eta)(Y)\xi - \eta(Y)\nabla_X\xi$$

for all vector fields X, Y on M. Together with (4), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + B(X,Y), \tag{8}$$

where we have put

$$B(X,Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi hX) - g(\varphi X + \varphi hX, Y)\xi. \tag{9}$$

Then, we see that the generalized Tanaka-Webster connection $\hat{\nabla}$ has the torsion $\hat{T}(X,Y) = 2g(X,\varphi Y)\xi + \eta(Y)\varphi hX - \eta(X)\varphi hY$. In particular, for a K-contact manifold we get

$$B(X,Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi. \tag{10}$$

Furthermore, it was proved that

Proposition 1 ([16]) *The generalized Tanaka-Webster connection $\hat{\nabla}$ on a strictly pseudo-convex almost CR manifold $M = (M;\eta,J)$ is the unique linear connection satisfying the following conditions:*

(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$, *where g is the associated Riemannian metric;*
$(iii-1)$ $\hat{T}(X,Y) = 2L(X,JY)\xi$, $X,\ Y \in \Gamma(D)$;
$(iii-2)$ $\hat{T}(\xi,\varphi Y) = -\varphi\hat{T}(\xi,Y)$, $Y \in \Gamma(D)$;
(iv) $(\hat{\nabla}_X\varphi)Y = \Omega(X,Y)$, $X,\ Y \in \Gamma(TM)$.

The pseudo-Hermitian connection (or The Tanaka-Webster connection) [14, 22] on a non-degenerate (integrable) CR manifold is defined as the unique linear connection satisfying (i), (ii), (iii-1), (iii-2) and $\Omega = 0$. We refer to [2] for more details about pseudo-Hermitian geometry in strictly pseudo-convex almost CR manifolds.

2.3 Pseudo-homothetic Transformations

In this subsection, we first review

Definition 1 Let $(M; \eta, \xi.\varphi, g)$ be a contact Riemannian manifold. Then a diffeomorphism f on M is said to be a *pseudo-homothetic transformation* if there exists a positive constant a such that

$$f^*\eta = a\eta,\ \ f_*\xi = \xi/a,\ \ \varphi \circ f_* = f_* \circ \varphi,\ \ f^*g = ag + a(a-1)\eta \otimes \eta.$$

Due to S. Tanno [15], we have

Theorem 1 *If a diffeomorphism f on a contact Riemannian manifold M is φ-holomorphic, i.e.,*

$$\varphi \circ f_* = f_* \circ \varphi,$$

then f is a pseudo-homothetic transformation.

Here, the new contact Riemannian manifold $(M; \bar\eta, \bar\xi.\bar\varphi, \bar g)$ defined by

$$\bar\eta = a\eta,\ \ \bar\xi = \xi/a,\ \ \bar\varphi = \varphi,\ \ \bar g = ag + a(a-1)\eta \otimes \eta, \tag{11}$$

is called a *pseudo-homothetic deformation* of $(M, \eta, \xi.\varphi, g)$. Then we have

$$\bar\nabla_X Y = \nabla_X Y + A(X, Y), \tag{12}$$

where A is the $(1, 2)$-type tensor defined by

$$A(X, Y) = -(a-1)[\eta(Y)\varphi X + \eta(X)\varphi Y] - \frac{a-1}{a} g(\varphi h X, Y)\xi.$$

Then we have

Proposition 2 ([9]) *The generalized Tanaka-Webster connection is pseudo-homothetically invariant.*

The so-called (k, μ)-spaces are defined by the condition

$$R(X, Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y)$$

for $(k, \mu) \in \mathbb{R}^2$, where I denotes the identity transformation. This class involves the Sasakian case for $k = 1$ $(h = 0)$. For a non-Sasakian contact Riemannian manifold,

h has the only two eigenvalues $\sqrt{1-k}$ and $-\sqrt{1-k}$ on D with their multiplicities n respectively. The (k, μ)-spaces have integrable CR structures and further, this class of spaces is invariant under pseudo-homothetic transformations. Indeed, a pseudo-homothetic transformation with constant $a(>0)$ transforms a (k, μ)-space into a $(\bar{k}, \bar{\mu})$-space where $\bar{k} = \frac{k+a^2-1}{a^2}$ and $\bar{\mu} = \frac{\mu+2a-2}{a}$ (cf. [1] or [3]). In particular, we find that $k = 1$ and $\mu = 2$ are the only two invariants under pseudo-homothetic transformations for all $a \neq 1$.

3 Pseudo-Einstein Structures

We define the pseudo-Hermitian curvature tensor (or the generalized Tanaka-Webster curvature tensor) on a strictly pseudo-convex almost CR manifold $\hat{R}$ of $\hat{\nabla}$ by

$$\hat{R}(X, Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X,Y]}Z$$

for all vector fields X, Y, Z in M. We remark that the generalized Tanaka-Webster connection is not torsion-free, and then the Jacobi- or Bianchi-type identities do not hold, in general. From the definition of $\hat{R}$, we have

$$\hat{R}(X, Y)Z = R(X, Y)Z + H(X, Y)Z, \tag{13}$$

and

$$\begin{aligned} H(X, Y)Z = {} & \eta(Y)\big((\nabla_X\varphi)Z - g(X + hX, Z)\xi\big) - \eta(X)\big((\nabla_Y\varphi)Z - g(Y + hY, Z)\xi\big) \\ & + \eta(Z)\big((\nabla_X\varphi)Y - (\nabla_Y\varphi)X + (\nabla_X\varphi h)Y - (\nabla_Y\varphi h)X \\ & + \eta(Y)(X + hX) - \eta(X)(Y + hY)\big) - 2g(\varphi X, Y)\varphi Z \\ & - g(\varphi X + \varphi hX, Z)(\varphi Y + \varphi hY) + g(\varphi Y + \varphi hY, Z)(\varphi X + \varphi hX) \\ & - g((\nabla_X\varphi)Y - (\nabla_Y\varphi)X + (\nabla_X\varphi h)Y - (\nabla_Y\varphi h)X, Z)\xi \end{aligned} \tag{14}$$

for all vector fields X, Y, Z in M.

Now, we introduce the pseudo-Hermitian Ricci (curvature) tensor:

$$\hat{\rho}(X, Y) = \frac{1}{2}\text{trace of } \{V \mapsto J\hat{R}(X, JY)V\},$$

where X, Y are vector fields orthogonal to ξ. This definition was referred as a 2nd kind in the author's earlier work [9]. Indeed, the pseudo-Hermitian Ricci (curvature) tensor of the 1st kind $\hat{\rho}_1$ is defined by

$$\hat{\rho}_1(X, Y) = \text{trace of } \{V \mapsto \hat{R}(V, X)Y\},$$

where V is any vector field on M and X, Y are vector fields orthogonal to ξ. Then we can find the following useful relation between the two notions in general:

$$\begin{aligned}\hat{\rho}(X, Y) =& \hat{\rho}_1(X, Y) - 2(n-1)g(hX, Y) \\ &+ \sum_{i=1}^{2n} \Big(g((\hat{\nabla}_{e_i}\Omega)(X, Y), \varphi e_i) - g((\hat{\nabla}_X \Omega)(e_i, Y), \varphi e_i) \Big)\end{aligned} \tag{15}$$

for $X, Y \in \Gamma(D)$ (cf. [17]). We define the corresponding pseudo-Hermitian Ricci operator $\hat{Q}$ is defined by $L(\hat{Q}X, Y) = \hat{\rho}(X, Y)$. The Tanaka-Webster (or the pseudo-Hermitian) scalar curvature $\hat{r}$ is given by

$$\hat{r} = \text{trace of } \{V \mapsto \hat{Q}V\}.$$

Then, from Proposition 2, we get

Corollary 1 *The pseudo-Hermitian curvature tensor* (*or The generalized Tanaka-Webster curvature tensor*) $\hat{R}$ *and the pseudo-Hermitian Ricci tensor* $\hat{Q}$ *are pseudo-homothetic invariants.*

Definition 2 Let $(M; \eta, J)$ be a strictly pseudo-convex almost CR manifold. Then the pseudo-Hermitian structure (η, J) is said to be pseudo-Einstein if the pseudo-Hermitian Ricci tensor is proportional to the Levi form, namely,

$$\hat{\rho}(X, Y) = \lambda L(X, Y),$$

where $X, Y \in \Gamma(D)$, where $\lambda = \hat{r}/2n$.

Remark 1 N. Tanaka [13] and J.M. Lee [11] defined the pseudo-Hermitian Ricci tensor on a non-degenerate CR manifold in a complex fashion. Further, J.M. Lee defined and intensively studied the pseudo-Einstein structure. Then every 3-dimensional strictly pseudo-convex CR manifold is pseudo-Einstein.

Remark 2 From (15), we at once see that for the Sasakian case or the 3-dimensional case $\hat{\rho} = \hat{\rho}_1$.

Moreover, we have

Proposition 3 ([9]) *A non-Sasakian contact* (k, μ)*-space* $(k < 1)$ *is pseudo-Einstein with constant pseudo-Hermitian scalar curvature* $\hat{r} = 2n^2(2 - \mu)$.

In [3] they proved that unit tangent sphere bundles with standard contact metric structures are (k, μ)-spaces if and only if the base manifold is of constant curvature b with $k = b(2 - b)$ and $\mu = -2b$. Thus, we have

Corollary 2 *The standard contact metric structure of* $T_1M(b)$ *of a space of constant curvature b is pseudo-Einstein. Its pseudo-Hermitian scalar curvature* $\hat{r} = 4n^2(1 + b)$.

The class of contact (k, μ)-spaces, whose associated CR structures are integrable as stated at the end of Sect. 2, contains non-unimodular Lie groups with left-invariant contact metric structure other than unit tangent bundles of a space of constant curvature (see [4]).

4 Pseudo-Hermitian CR Space Forms

In this section, we give

Definition 3 ([7]) Let $(M; \eta, J)$ be a strictly pseudo-convex almost CR manifold. Then M is said to be of constant holomorphic sectional curvature c (with respect to the generalized Tanaka-Webster connection) if M satisfies

$$L(\hat{R}(X, \varphi X)\varphi X, X) = c$$

for any unit vector field X orthogonal to ξ. In particular, for the CR integrable case we call M a pseudo-Hermitian (strictly pseudo-convex) CR space form.

Then for a strictly pseudo-convex almost CR manifold M, from (13) and (14) we get

$$g(\hat{R}(X, \varphi X)\varphi X, X) = g(R(X, \varphi X)\varphi X, X) + 3g(X, X)^2 - g(hX, X)^2 - g(\varphi hX, X)^2 \tag{16}$$

for any X orthogonal to ξ. From this, we easily see that s Sasakian space form $M^{2n+1}(c_0)$ of constant φ-holomorphic sectional curvature c_0 (with respect to the Levi-Civita connection) is a strictly pseudo-convex CR space form of constant holomorphic sectional curvature (with respect to the Tanaka-Webster connection) $c = c_0 + 3$. Simply connected and complete Sasakian space forms are the unit sphere S^{2n+1} with the natural Sasakian structure with $c_0 = 1$ $(c = 4)$, the Heisenberg group H^{2n+1} with Sasakian φ-holomorphic sectional curvature $c_0 = -3$ $(c = 0)$, or $B^n \times R$ with Sasakian φ-holomorphic sectional curvature $c_0 = -7$ $(c = -4)$, where B^n is a simply connected bounded domain in C^n with constant holomorphic sectional curvature -4.

For a class of the contact (k, μ)-spaces, we proved the following results.

Theorem 2 ([7]) *Let M be a contact (k, μ)-space. Then M is of constant holomorphic sectional curvature c for Tanaka-Webster connection if and only if* (1) *M is Sasakian space of constant φ-holomorphic sectional curvature $c_0 = c - 3$,* (2) *$\mu = 2$ and $c = 0$, or* (3) *dim M=3 and $\mu = 2 - c$.*

Corollary 3 ([7]) *The standard strictly pseudo-convex CR structure on a unit tangent sphere bundle $T_1M(b)$ of $(n + 1)$-dimensional space of constant curvature b has constant holomorphic sectional curvature c if and only if $b = -1$ and $c = 0$, or $n = 1$ and $b = (c - 2)/2$.*

Remark 3 (1) The standard contact metric structure of the unit tangent sphere bundle $T_1\mathbb{S}^{n+1}(1)$ is Sasakian [20], but it has not constant holomorphic sectional curvature for both Levi-Civita and Tanaka-Webster connection.

(2) The unit tangent sphere bundle $T_1\mathbb{H}^{n+1}(-1)$ of a hyperbolic space $\mathbb{H}^{n+1}(-1)$ is a non-Sasakian example of constant holomorphic sectional curvature for Tanaka-Webster connection but not for Levi-Civita connection.

In [7] we determined the Riemannian curvature tensor explicitly for a strictly pseudo-convex CR space of constant holomorphic sectional curvature c. Then we have

$$g(\hat{R}(X,Y)Z,W) = g(H(X,Y)Z,W) + \frac{c}{4}\Big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W)\Big\} \tag{17}$$

for all vector fields $X, Y, Z, W \perp \xi$, where

$$\begin{aligned} g(H(X,Y)Z,W) = {} & g(Y,Z)g(hX,W) - g(X,Z)g(hY,W) \\ & - g(Y,W)g(hX,Z) + g(X,W)g(hY,Z) \\ & + g(\varphi Y,Z)g(\varphi hX,W) - g(\varphi X,Z)g(\varphi hY,W) \\ & - g(\varphi Y,W)g(\varphi hX,Z) + g(\varphi X,W)g(\varphi hY,Z). \end{aligned} \tag{18}$$

Then from (17) we get

$$\hat{\rho}(X,Y) = c(n+1)/2\; g(X,Y). \tag{19}$$

Proposition 4 ([9]) *A strictly pseudo-convex CR space form of constant holomorphic sectional curvature c is pseudo-Einstein with constant pseudo-Hermitian scalar curvature* $\hat{r} = n(n+1)c$.

5 The Chern-Moser-Tanaka Invariant

Now, we review the pseudo-conformal transformations of a strictly pseudo-convex almost CR structure. Given a contact form η, we consider a 1-form $\bar{\eta} = \sigma\eta$ for a positive smooth function σ. By assuming $\bar{\phi}|D = \phi|D$ ($\bar{J} = J$), the associated Riemannian structure $\bar{g}$ of $\bar{\eta}$ is determined in a natural way. Namely, we have

$$\bar{\xi} = (1/\sigma)(\xi + \zeta),\ \zeta = (1/2\sigma)\phi(\operatorname{grad}\sigma),\ \bar{\phi} = \phi + (1/2\sigma)\eta \otimes (\operatorname{grad}\sigma - \xi\sigma \cdot \xi),$$
$$\bar{g} = \sigma g - \sigma(\eta \otimes \nu + \nu \otimes \eta) + \sigma(\sigma - 1 + \|\zeta\|^2)\eta \otimes \eta,$$

where ν is dual to ζ with respect to g. We call the transformation $(\eta, J) \to (\bar{\eta}, \bar{J})$ a *pseudo-conformal transformation (or gauge transformation)* of the strictly pseudo-convex almost CR structure. We remark in particular that when σ is a constant, then a gauge transformation reduces to a pseudo-homothetic transformation.

Let ω be a nowhere vanishing $(2n+1)$-form on M and fix it. Let $dM(g) = ((-1)^n/2^n n!)\eta \wedge (d\eta)^n$ denote the volume element of (M, η, g). We define β by $dM(g) = \pm e^{\beta}\omega$ and $\theta \in \Gamma(D^*)$ by $\theta(X) = X\beta$ for $X \in \Gamma(D)$. For a strictly pseudo-convex almost CR manifold, the generalized Chern-Moser-Tanaka curvature tensor $C \in \Gamma(D \otimes D^{*3})$ is defined by S. Tanno in [18] (see also, [8]).

$$\begin{aligned}
&(2n+4)g(C(X,Y)Z,W)\\
&\quad = (2n+4)g(\hat{R}(X,Y)Z,W)\\
&\qquad - \hat{\rho}(Y,Z)g(X,W) + \hat{\rho}(X,Z)g(Y,W) - g(Y,Z)\hat{\rho}(X,W) + g(X,Z)\hat{\rho}(Y,W)\\
&\qquad + \hat{\rho}(Y,\varphi Z)g(\varphi X,W) - \hat{\rho}(X,\varphi Z)g(\varphi Y,W) - [\hat{\rho}(X,\varphi Y) - \hat{\rho}(\varphi X,Y)]g(\varphi Z,W)\\
&\qquad + \hat{\rho}(X,\varphi W)g(\varphi Y,Z) - \hat{\rho}(Y,\varphi W)g(\varphi X,Z) - [\hat{\rho}(Z,\varphi W) - \hat{\rho}(\varphi Z,W)]g(\varphi X,Y)\\
&\qquad + [\hat{r}/(2n+2)][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\\
&\qquad + g(\varphi Y,Z)g(\varphi X,W) - g(\varphi X,Z)g(\varphi Y,W) - 2g(\varphi X,Y)g(\varphi Z,W)]\\
&\qquad - (2n+4)[g(hY,Z)g(X,W) - g(hX,Z)g(Y,W) + g(Y,Z)g(hX,W)\\
&\qquad - g(X,Z)g(hY,W) + g(\varphi hY,Z)g(\varphi X,W) - g(\varphi hX,Z)g(\varphi Y,W)\\
&\qquad + g(\varphi hX,W)g(\varphi Y,Z) - g(\varphi hY,W)g(\varphi X,Z)]\\
&\qquad - (n+2)/(n+1)g(U(X,Y,Z;\theta),W)).
\end{aligned} \tag{20}$$

Here $U \in \Gamma(D^2 \otimes D^{*3})$ and $U(X,Y,Z;\theta) = (\theta_j U^{ji}_{lhk} X^h Y^k Z^l)$ in terms of an adapted frame $\{e_\alpha\} = \{e_j,\ e_0 = \xi;\ 1 \le j \le 2n\}$. For a full understanding, we may describe it by using the components of U in terms of $\{e_j, e_0\}$ (cf. [18]). That is,

$$\begin{aligned}
U^{ji}_{lhk} = 2\Big[&1/(n+2)\{-\delta^i_h(\Omega^j_{km} + \Omega^j_{mk})\phi^m_l - \phi^i_h(\Omega^j_{lk} + \Omega^j_{kl}) + g_{hl}(\Omega^j_{km} + \Omega^j_{mk})\phi^{mi}\\
&- \phi_{hl}(\Omega^j_{km} + \Omega^j_{mk})g^{mi}\} + \Omega^j_{lk}\phi^i_h + \phi_{hl}\Omega^j_{mk}g^{im} + \Omega^j_{hk}\phi^i_l\\
&+ (1/2)(\Omega^j_{ml} - \Omega^j_{lm})g^{mi}\phi_{hk} + \phi^j_l\Omega^i_{hk} + \phi^j_h\Omega^i_{lk} - (1/2)\phi^{ij}\Omega^m_{kl}g_{hm}\Big]_{hk},
\end{aligned}$$

where $[\cdots]_{hk}$ denotes the skew-symmetric part of $[\cdots]$ with respect to h, k.

Remark 4 (1) If $n = 1$ (dim M=3), then we always have $C = 0$ (see Remark in [18]).

(2) When $(M; \eta, g)$ is Sasakian, then ($h = 0$ and) C reduces to the C-Bochner curvature tensor, which is the corresponding (through the Boothby-Wang fibration) to the Bochner curvature tensor in a Kähler manifold [12].

Using (17) and (19), from the Eq. (9) we find

Proposition 5 *On a pseudo-Hermitian CR space form, the Chern-Moser-Tanaka invariant C vanishes.*

Moreover we have

Theorem 3 *Let $(M^{2n+1}; \eta, J)$ $(n > 1)$ be a strictly pseudo-convex almost CR manifold with vanishing C. Then M is pseudo-Einstein if and only if M is of pointwise constant holomorphic sectional curvature for the Tanaka-Webster connection.*

The argument and computation of present paper gives a simpler proof of [9, Theorem 22].

Remark 5 The unit tangent sphere bundle $T_1\mathbb{H}^{n+1}(-1)$ of a hyperbolic space $\mathbb{H}^{n+1}(-1)$ is a non-Sasakian example which supports Theorem 3 well. It was proved that the Chern-Moser-Tanaka curvature tensor C on $T_1\mathbb{H}^{n+1}(-1)$ vanishes [19] and within the class of (k, μ)-spaces, it is the only such an example [8].

Acknowledgements This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2016R1D1A1B03930756).

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Bott Periodicity, Submanifolds, and Vector Bundles

Jost Eschenburg and Bernhard Hanke

Abstract We sketch a geometric proof of the classical theorem of Atiyah, Bott, and Shapiro [3] which relates Clifford modules to vector bundles over spheres. Every module of the Clifford algebra Cl_k defines a particular vector bundle over $\mathbb{S}^{k+1}$, a generalized Hopf bundle, and the theorem asserts that this correspondence between Cl_k-modules and stable vector bundles over $\mathbb{S}^{k+1}$ is an isomorphism modulo Cl_{k+1}-modules. We prove this theorem directly, based on explicit deformations as in Milnor's book on Morse theory [8], and without referring to the Bott periodicity theorem as in [3].

Introduction

Topology and Geometry are related in various ways. Often topological properties of a specific space are obtained by assembling its local curvature invariants, like in the Gauss-Bonnet theorem. Bott's periodicity theorem is different: A detailed investigation of certain totally geodesic submanifolds in specific symmetric spaces leads to fundamental insight not just for these spaces but for whole areas of mathematics. This geometric approach was used originally by Bott [4, 5] and Milnor in his book on Morse theory [8] where the stable homotopy of the classical groups was computed. Later Bott's periodicity theorem was re-interpreted as a theorem on K-theory [1–3], but the proofs were different and less geometric. However we feel that the original approach of Bott and Milnor can prove also the K-theoretic versions of the periodicity theorem. As an example we discuss Theorem 11.5 from the fundamental paper [3] by Atiyah, Bott and Shapiro, which relates Clifford modules to vector bundles over spheres. The argument in [3] uses explicit computations of the right and left hand sides of the stated isomorphism, and depends on the Bott periodicity theorem for the orthogonal groups. Instead we prove bijectivity of the relevant comparison

J. Eschenburg · B. Hanke (✉)
Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany
e-mail: hanke@math.uni-augsburg.de

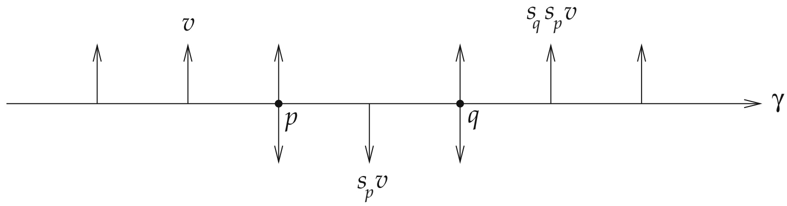
J. Eschenburg
e-mail: eschenburg@math.uni-augsburg.de

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_25

map directly. In consequence the Bott periodicity theorem for the orthogonal groups is now implied by its algebraic counterpart in the representation theory of Clifford algebras [3]. This gives a positive response to the remark in [3, p. 4]: "It is to be hoped that Theorem (11.5) can be give a more natural and less computational proof", cf. also [7, p. 69]. We will concentrate on the real case which is more interesting and less well known than the complex theory. Much of the necessary geometry was explained to us by Peter Quast [12].

1 Poles and Centrioles

We start with the geometry. A *symmetric space* is a Riemannian manifold P with an isometric point reflection s_p (called *symmetry*) at any point $p \in P$, that is $s_p \in \hat{G}$ = isometry group of P with $s_p(\exp_p(v)) = \exp_p(-v)$ for all $v \in T_pP$. The map $s : p \mapsto s_p : P \to \hat{G}$ is called *Cartan map*; it is a covering onto its image $s(P) \subset \hat{G}$ which is also symmetric.[1] The composition of any two symmetries, $\tau = s_q s_p$ is called a *transvection*. It translates the geodesic γ connecting $p = \gamma(0)$ to $q = \gamma(r)$ by $2r$ and acts by parallel translation along γ, see next figure. The subgroup of $\hat{G}$ generated by all transvections (acting transitively on P) will be called G.



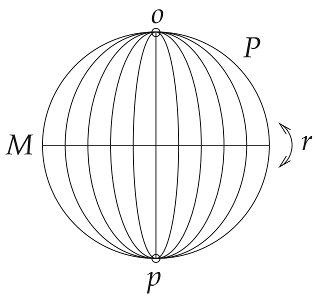
Two points $o, p \in P$ will be called *poles* if $s_p = s_o$. The notion was coined for the north and south pole of a round sphere, but there are many other spaces with poles; e.g. $P = SO_{2n}$ with $o = I$ and $p = -I$, or the Grassmannian $P = \mathbb{G}_n(\mathbb{R}^{2n})$ with $o = \mathbb{R}^n$ and $p = (\mathbb{R}^n)^\perp$. A geodesic γ connecting $o = \gamma(0)$ to $p = \gamma(1)$ is reflected into itself at o and p and hence it is closed with period 2.

Now we consider the *midpoint set* M between poles o and p,

$$M = \{m = \gamma\left(\tfrac{1}{2}\right) : \gamma \text{ geodesic in } P \text{ with } \gamma(0) = o,\ \gamma(1) = p\}.$$

For the sphere $P = \mathbb{S}^n$ with north pole o, this set would be the equator, see figure below.

[1] $s(P) \subset \hat{G}$ is a connected component of the set $\{g \in \hat{G} : g^{-1} = g\}$. When we choose a symmetric metric on $\hat{G}$ such that $g \mapsto g^{-1}$ is an isometry, $s(P)$ is a reflective submanifold and hence totally geodesic, thus symmetric.

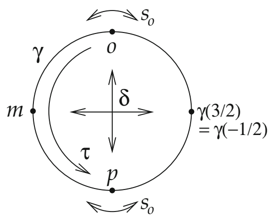


Theorem 1 ([11]) *M is the fixed set of an isometric involution r on P.*

Proof In the example of the sphere $P = \mathbb{S}^n$, the equator M is the fixed set of $-s_o = -I \circ s_o$. Here, $-I$ is the deck transformation[2] of the covering $\mathbb{S}^n \to \mathbb{RP}^n = \mathbb{S}^n/\{\pm I\}$. In the general case we consider the covering $P \to s(P)$. Since $s(P)$ is again symmetric, we have $s(P) = P/\Delta$ for some discrete freely acting group $\Delta \subset \hat{G}$ normalized by all symmetries and centralized by all transvections.[3] Since $s_o = s_p$, the points o and p are identified in $s(P)$. Thus there is a unique $\delta \in \Delta$ with $\delta(o) = p$. This will be the analogue of $-I$ in the case $P = \mathbb{S}^n$. We will show that δ has order 2 and preserves any geodesic γ with $\gamma(0) = o$ and $\gamma(1) = p$. In fact, let τ be the transvection along γ from o to p. Then $\tau^2(o) = o$ and therefore

$$\delta(p) = \delta(\tau(o)) = \tau(\delta(o)) = \tau(p) = o.$$

Thus δ^2 fixes o which shows $\delta^2 = \text{id}$ since Δ acts freely. Hence $\{I, \delta\} \subset \Delta$ is a subgroup and $\bar{P} = P/\{\text{id}, \delta\}$ a symmetric space. Under the projection $\pi : P \to \bar{P}$, the geodesic γ is mapped onto a closed geodesic doubly covered by γ, thus δ preserves γ and shifts its parameter by 1, and γ has period 2.



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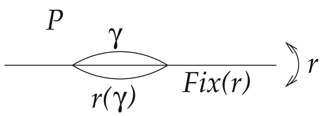
[2]A *deck transformation* of $\pi : P \to \bar{P}$ is an isometry δ of P with $\pi \circ \delta = \pi$.

[3]Consider a symmetric space P and a covering $\pi : P \to P/\Delta$ for some discrete freely acting group Δ of isometries on P. Then P/Δ is again symmetric if and only if each symmetry s_p of P maps Δ-orbits onto Δ-orbits. Thus for each $\delta \in \Delta$ we have $s_p(\delta x) = \tilde{\delta} s_p(x)$ for all $x \in P$, and $\tilde{\delta} \in \Delta$ is independent of x, by discreteness. Thus $s_p \delta = \tilde{\delta} s_p$, in particular $s_p \delta s_p = \tilde{\delta} \in \Delta$. For any other symmetry s_q we have the same equation $s_q \delta = \tilde{\delta} s_q$ with the same $\tilde{\delta} \in \Delta$, again by discreteness. Thus $\delta^{-1} s_p s_q \delta = s_p \tilde{\delta}^{-1} \tilde{\delta} s_q = s_p s_q$, and δ commutes with the transvection $s_p s_q$ (see also [14, Theorem 8.3.11]).

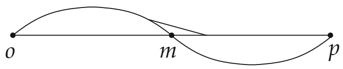
We put $r = s_o\delta$. This is an involution since s_o and δ commute: $\delta' = s_o\delta s_o \in \Delta$ sends o to p like δ, thus $\delta' = \delta$. Then r fixes the midpoint $m = \gamma(\frac{1}{2})$ of any geodesic γ from o to p since $s_o(\delta(\gamma(\frac{1}{2})) = s_o(\gamma(\frac{3}{2})) = s_o(\gamma(-\frac{1}{2})) = \gamma(\frac{1}{2})$. Thus $M \subset \mathrm{Fix}(r)$.

Vice versa, assume that $m \in P$ is a fixed point of r. Thus $s_o m = \delta m$. Join o to m by a geodesic γ with $\gamma(0) = o$ and $\gamma(\frac{1}{2}) = m$. Then $\gamma(-\frac{1}{2}) = s_o(m) = \delta(m) = \delta(\gamma(\frac{1}{2}))$, and the projection $\pi : P \to \bar{P} = P/\{\mathrm{id}, \delta\}$ maps $\gamma : [-\frac{1}{2}, \frac{1}{2}] \to P$ onto a geodesic loop $\bar{\gamma} = \pi \circ \gamma$, that is a closed geodesic of period 1 (since $\bar{P}$ is symmetric). Thus γ extends to a closed geodesic of period 2 doubly covering $\bar{\gamma}$, and δ shifts the parameter of γ by 1. Therefore $\gamma(1) = \delta(o) = p$. Hence m is the midpoint of $\gamma|_{[0,1]}$ from o to p. Thus $M \supset \mathrm{Fix}(r)$.

Connected components of the midpoint set M are called *centrioles* [6]. Connected components of the fixed set of an isometry are totally geodesic (otherwise shortest geodesic segments in the ambient space with end points in the fixed set were not unique, see figure below); if the isometry is an involution, its fixed components are called *reflective*.



Most interesting are connected components containing midpoints of geodesics with *minimal* length between o and p ("*minimal centrioles*"). Each such midpoint $m = \gamma(\frac{1}{2})$ determines its geodesic γ uniquely: if there were two geodesics of equal length from o to p through m, they could be made shorter by cutting the corner.



There exist chains of minimal centrioles (centrioles in centrioles),

$$P \supset P_1 \supset P_2 \supset \dots \tag{1}$$

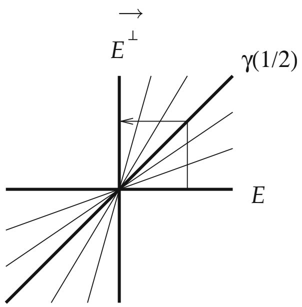
Peter Quast [12, 13] classified all such chains with at least 3 steps starting with a compact simple Lie group $P = G$. Up to group coverings, the result is as follows. The chains 1, 2, 3 occur in Milnor [8].

No.	G	P_1	P_2	P_3	P_4	restr.
1	$(S)O_{4n}$	SO_{4n}/U_{4n}	U_{2n}/Sp_n	$\mathbb{G}_p(\mathbb{H}^n)$	Sp_p	$p = \frac{n}{2}$
2	$(S)U_{2n}$	$\mathbb{G}_n(\mathbb{C}^{2n})$	U_n	$\mathbb{G}_p(\mathbb{C}^n)$	U_p	$p = \frac{n}{2}$
3	Sp_n	Sp_n/U_n	U_n/SO_n	$\mathbb{G}_p(\mathbb{R}^n)$	SO_p	$p = \frac{n}{2}$
4	$Spin_n$	Q_n	$(\mathbb{S}^1 \times \mathbb{S}^{n-3})/\pm$	$\mathbb{S}^{n-4}$	$\mathbb{S}^{n-5}$	$n \geq 5$
5	E_7	$E_7/(\mathbb{S}^1 E_6)$	$\mathbb{S}^1 E_6/F_4$	$\mathbb{OP}^2$	$-$	

(2)

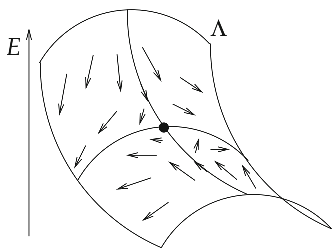
By $\mathbb{G}_p(\mathbb{K}^n)$ we denote the Grassmannian of p-dimensional subspaces in $\mathbb{K}^n$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Further, Q_n denotes the complex quadric in $\mathbb{CP}^{n+1}$ which is isomorphic to the real Grassmannian $\mathbb{G}_2^+(\mathbb{R}^{n+2})$ of oriented 2-planes, and $\mathbb{OP}^2$ is the octonionic projective plane $F_4/Spin_9$.

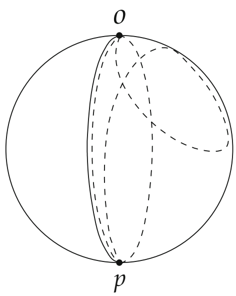
A chain is extendible beyond P_k if and only if P_k contains poles again. E.g. among the Grassmannians $P_3 = \mathbb{G}_p(\mathbb{K}^n)$ only those of half dimensional subspaces ($p = \frac{n}{2}$) enjoy this property: Then $(E, E^\perp)$ is a pair of poles for any $E \in \mathbb{G}_{n/2}(\mathbb{K}^n)$, and the corresponding midpoint set is the group $O_{n/2}$, $U_{n/2}$, $Sp_{n/2}$ since its elements are the graphs of orthogonal $\mathbb{K}$-linear maps $E \to E^\perp$, see figure below.



2 Centrioles with Topological Meaning

Points in minimal centrioles are in 1:1 correspondence to minimal geodesics between the corresponding poles o and p. Thus minimal centrioles sometimes can be viewed as low-dimensional approximations of the full *path space* Λ, the space of all H^1-curves[4] $\lambda : [0, 1] \to P$ with $\lambda(0) = o$ and $\lambda(1) = p$. This is due to the Morse theory for the energy function E on Λ where $E(\lambda) = \int_0^1 |\lambda'(t)|^2 dt$. We may decrease the energy of any path λ by applying the gradient flow of $-E$ (left figure).

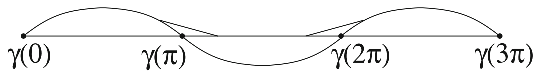




[4] H^1 means that λ has a derivative almost everywhere which is square integrable. Replacing any path λ by a geodesic polygon with N vertices, we may replace Λ by a finite dimensional manifold, cf. [8].

Most elements of Λ will be flowed to the minima of E which are the shortest geodesics between o and p. The only exceptions are the domains of attraction ("unstable manifolds") for the other critical points, the non-minimal geodesics between o and p. The codimension of the unstable manifold is the *index* of the critical point, the maximal dimension of any subspace where the second derivative of E (taken at the critical point) is negative. If β denotes the smallest index of all non-minimal critical points, any continuous map $f : X \to \Lambda$ from a connected cell complex X of dimension $<\beta$ can be moved away from these unstable manifolds and flowed into a connected component of the minimum set, that is into some centriole P_1. Thus f is homotopic to a map $\tilde{f} : X \to P_1$.

But this works only if all non-minimal geodesics from o to p have high index ($\geq\beta$). Which symmetric spaces P have this property? An easy example is the sphere, $P = \mathbb{S}^n$. A nonminimal geodesic γ between poles o and p covers a great circle at least one and a half times and can be shortened within any 2-sphere in which it lies (right figure above). There are $n-1$ such 2-spheres perpendicular to each other since the tangent vector $\gamma'(0) = e_1$ is contained in $n-1$ perpendicular planes Span (e_1, e_i) with $i \geq 2$ in the tangent space. Thus the index is $\geq n-1$, in fact $\geq 2(n-1)$ since any such geodesic contains at least 2 conjugate points where it can be shortened by cutting the corner, see figure.



For the classical groups we can argue similarly. E.g. in SO_{2n}, a shortest geodesic from I to $-I$ is a product of n half turns, planar rotations by the angle π in n perpendicular 2-planes in $\mathbb{R}^{2n}$. A non-minimal geodesic must make an additional full turn and thus a 3π-rotation in at least one of these planes, say in the x_1x_2-plane. This rotation belongs to the rotation group $SO_3 \subset SO_{2n}$ in the $x_1x_2x_k$-space for any $k \in \{3, \dots, 2n\}$. Using $SO_3 = \mathbb{S}^3/\pm$, we lift the 3π-rotation to $\mathbb{S}^3$ and obtain a 3/4 great circle which can be shortened. There are $2n-2$ coordinates x_k and therefore $2n-2$ independent contracting directions, hence the index of a nonminimal geodesic in SO_{2n} is $\geq 2n-2$ (compare [8, Lemma 24.2]). The index of the spaces P_k can be bounded from below in a similar way, see next section for the chain of SO_n. This implies the homotopy version of the periodicity theorem:

Theorem 2 *When n is even and sufficiently large, we have for* $G = SO_{4n}, SU_{2n}, Sp_n$ *(notations of table 2):*

$$\pi_k(G) = \pi_{k-1}(P_1) = \pi_{k-2}(P_2) = \pi_{k-3}(P_3) = \pi_{k-4}(P_4).$$

Together with table 2 this implies the following periodicities:

$$\pi_{k+2}(SU_n) = \pi_k(SU_{n/2}),$$
$$\pi_{k+4}(SO_n) = \pi_k(Sp_{n/8}),$$
$$\pi_{k+4}(Sp_n) = \pi_k(SO_{n/2}).$$

3 Clifford Modules

For compact matrix groups G containing $-I$, there is a linear algebra interpretation for the iterated midpoint sets M_j and their components P_j. A geodesic γ in G with $\gamma(0) = I$ is a one-parameter subgroup, and when $\gamma(1) = -I$, then $\gamma(\frac{1}{2}) = J$ is a *complex structure*, $J^2 = -I$. Thus the midpoint set M_1 is the set of complex structures in G. When the connected component P_1 of M_1 contains antipodal points J_1 and $-J_1$, there is a next midpoint set $M_2 \subset P_1$. It consists of points $J_1\gamma(\frac{1}{2})$ where γ is a one-parameter subgroup in G with $\gamma(1) = -I$ such that $J_1\gamma(t)$ is a complex structure for all t,

$$J_1\gamma J_1\gamma = -I\,. \tag{$*$}$$

In particular the midpoint $J = \gamma(\frac{1}{2})$ anticommutes with J_1 (since $J_1JJ_1J = -I \iff J_1J = -JJ_1$), and when γ is minimal, this condition is sufficient for ($*$): then both $J_1\gamma J_1$ and $-\gamma^{-1}$ are shortest geodesics from $-I$ to I with midpoint J, so they must agree. By induction hypothesis, we have anticommuting complex structures $J_u \in G$ with $J_i \in P_i$ for $i < k$, and P_k is a connected component of the set

$$M_k = \{J \in G : J^2 = -I,\ JJ_i = -J_iJ \text{ for } i < k\} \tag{3}$$

of complex structures $J \in G$ which anticommute with $J_1, \dots, J_{k-1}$. To finish the induction step we choose some $J_k \in P_k$.

Recall that the *real Clifford algebra* Cl_k is the associative real algebra with 1 which is generated by $\mathbb{R}^k$ with the relations $vw + wv = -2\langle v, w\rangle$. Equivalently, an orthonormal basis $e_1, \dots, e_k$ of $\mathbb{R}^k \subset Cl_k$ satisfies

$$e_ie_j + e_je_i = -2\delta_{ij}\,.$$

A *representation* of Cl_k is an algebra homomorphism from Cl_k into some matrix algebra $\mathbb{K}^{n\times n}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$; the space $\mathbb{K}^n$ on which the matrices operate is called *Clifford module* S. A representation maps the vectors e_i onto matrices J_i with the same relations $J_i^2 = -I$ and $J_iJ_j = -J_jJ_i$ for $i \neq j$. Thus a Cl_k module is nothing but a *Clifford system*, a family of k are anticommuting complex structures, and the midpoint set $M_{k+1} \subset P_k$ between J_k and $-J_k$ can be viewed as the set of extensions of a given Cl_k-module (defined by $J_1, \dots, J_k$) to a Cl_{k+1}-module.

The algebraic theory of the Clifford representations is rather easy (cf. [7]). They are direct sums of irreducible representations, and in the real case there is just one irreducible Cl_k-module S_k (up to isomorphisms) when $k \not\equiv 3 \mod 4$, while there are two with equal dimensions when $k \equiv 3 \mod 4$. For $k = 0, \ldots, 8$ we have

Theorem 3

$$\begin{array}{cccccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ S_k & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{H} & \mathbb{H}^2 & \mathbb{C}^4 & \mathbb{O} & \mathbb{O} & \mathbb{O}^2 \end{array} \tag{4}$$

and further we have the periodicity theorem for Clifford modules,

$$S_{k+8} = S_k \otimes S_8. \tag{5}$$

For $k = 3$ and $k = 7$, the two different module structures are given by left and right multiplications of $\mathbb{R}^k = \mathbb{K}' := \mathbb{K} \ominus \mathbb{R} \cdot 1$ on $S_k = \mathbb{K}$ for $\mathbb{K} = \mathbb{H}, \mathbb{O}$.

4 Index of Nonminimal Geodesics

From (3) we have gained a uniform description for all iterated centrioles P_k of G in terms of Clifford systems. This can be used for a calculation of the lower bound β for the index of nonminimal geodesics in all P_k, cf. [8].[5]

Theorem 4 *Let $SO_n = G \supset P_1 \supset P_2 \supset \cdots \supset P_k \supset \ldots$ be the chain (1) of iterated centrioles where n is divisible by a high power of 2. Then for each k there is some lower bound β depending on n such that the index of nonminimal geodesics from J_k to $-J_k$ is $\geq \beta$, and $\beta \to \infty$ as $n \to \infty$.*

Proof Let $\tilde{\gamma} = J_k \gamma : [0, 1] \to P_k$ be a non-minimal geodesic from J_k to $-J_k$. Then $\gamma(t) = e^{\pi t A}$ for some $A \in \mathfrak{so}_n$. Since $\tilde{\gamma}(t)$ anticommutes with J_i for all $i < k$, it follows that $\gamma(t)$ and A commute with J_i. Further, from $\tilde{\gamma}(t)^2 = -I$ we obtain $J_k e^{\pi t A} J_k^{-1} = e^{-\pi t A}$ and therefore A anticommutes with J_k. Thus we have computed the tangent space of P_k at J_k:

$$T_{J_k} P_k = \{J_k A : A \in \mathfrak{so}_n,\ A J_k = -J_k A,\ A J_i = J_i A \text{ for } i < k\}. \tag{6}$$

Since $\gamma(1) = -I$, the (complex) eigenvalues of A have the form $a\mathrm{i}$ with $\mathrm{i} = \sqrt{-1}$ and a an odd integer.

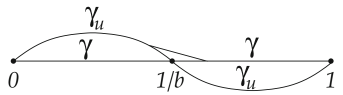
To relate these eigenvalues to the index we argue similar as in [8, pp. 144–147]. We split $\mathbb{R}^n$ into a sum of subspaces V_j being invariant under the linear maps $A, J_1, \ldots, J_k$ and being minimal with respect to this property. All J_i, $i < k$, preserve the (complex) eigenspaces E_a of A, corresponding to the nonzero eigenvalue

[5] A different argument using root systems was given by Bott (6.7) [4] and in more detail by Mitchell [9, 10].

ai, while J_k interchanges E_a and E_{-a}. Thus by minimality of V_j, there is just one pair $\pm a$ such that V_j is the real part of $E_a + E_{-a}$. Therefore $J' := A/a$ is an additional complex structure on V_j commuting with J_i $(i < k)$ and anticommuting with J_k, and $J_{k+1} := J_k J'$ is a complex structure which anticommutes with *all* $J_1, \dots, J_k$. Hence V_j is an irreducible Cl_k-module. Moreover, $A = a_j J'$ on V_j for some nonzero integer a_j while $A = 0$ on V_0. By choice of the sign of $J'|V_j$ we may assume that all $a_j > 0$. hence $a_j \in \{1, 3, 5, \dots\}$.

Choose two of these irreducible modules, say V_j and V_h. By (4), there is a module isomorphism $V_j \to V_h$ as Cl_{k+1}-modules when $k+1 \not\equiv 3 \mod 4$ (Case 1) and as Cl_k-modules when $k+1 \equiv 3 \mod 4$ (Case 2). This remains true when we alter the Cl_{k+1}-module structure of V_h in Case 1 by changing the sign of J_{k+1} (and thus that of J') on V_h. With this identification we have $V_j + V_h = V_j \otimes \mathbb{R}^2$ and $B = I \otimes \begin{pmatrix} & -I \\ I & \end{pmatrix}$ (with $B = 0$ on the other submodules) commutes with all J_j, $j \le k$, and the same is true for e^{uB}. Putting $A_u = e^{uB} A e^{-uB}$, we have $J_k A_u \in T_{J_k} P_k$ by (6).

Case 1: $\underline{k+1 \not\equiv 3 \mod 4}$: We have modified our identification of V_j and V_h by changing the sign of J_{k+1} on V_h. Thus on $V_j + V_h = V_j \otimes \mathbb{R}^2$ we have $A = J' \otimes D$ where $D = \operatorname{diag}(a_j, -a_h) = cI + D'$ with $D' = \operatorname{diag}(b, -b)$ for $b = \frac{1}{2}(a_j + a_h)$ and $c = \frac{1}{2}(a_j - a_h)$. Let us consider the family of geodesics $J_k \gamma_u$ from J_k to $-J_k$ in P_k with $\gamma_u(t) = e^{t\pi A_u} = e^{uB}\gamma(t)e^{-uB}$. The point $\gamma(t) = e^{\pi t c} e^{\pi t D'}$ is fixed under conjugation with the rotation matrix $e^{uB} = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}$ precisely when $e^{\pi t D'} = \operatorname{diag}(e^{\pi t b}, e^{-\pi t b})$ is a multiple of the identity matrix which happens for $t = 1/b$. If one of the eigenvalues a of A is > 1, say $a_h \ge 3$, then $b = \frac{1}{2}(a_j + a_h) \ge 2$ and $1/b \in (0, 1)$. All γ_u are geodesics connecting I to $-I$ on $[0, 1]$. By "cutting the corner" it follows that γ can no longer be locally shortest beyond $t = 1/b$, see figure. If there is at least one eigenvalue $a_h > 1$, we have $r - 1$ index pairs (j, h), hence the index of non-minimal geodesics is at least $r - 1$.



Case 2: $\underline{k+1 \equiv 3 \mod 4}$: In this case, the product $J_o := J_1 J_2 \dots J_{k-1}$ is a complex structure[6] which commutes with A and anticommutes with J_k (since $k-1$ is odd). Thus A can be viewed as a complex matrix, using J_o as the multiplication by i.

[6]Putting $S_n = (J_1 \dots J_n)^2$ we have

$$S_n = J_1 \dots J_n J_1 \dots J_n = (-1)^{n-1} S_{n-1} J_n^2 = (-1)^n S_{n-1},$$

thus $S_n = (-1)^s I$ with $s = n + (n-1) + \dots + 1 = \frac{1}{2}n(n+1)$. When $n = k-1 \equiv 1 \mod 4$, then s is odd, hence $S_n = -I$.

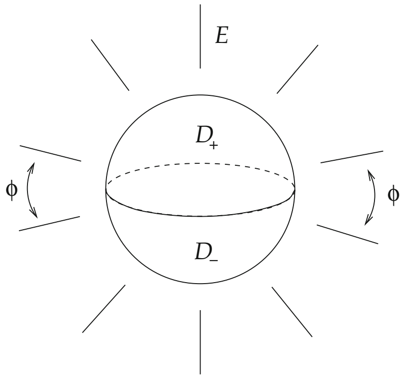
Let $E_a \subset V_j$ be the eigenspace of A corresponding to the eigenvalue $\mathrm{i}a$ where a is any odd integer. Then E_a is invariant under the J_i, $i < k$, which commute with A, but is it also invariant under J_k which anticommutes with A and with $\mathrm{i} = J_o$ (since $k-1$ is odd). By minimality we have $V_j = E_a$, hence $A = aJ_o$. As before, we consider the linear map $B = \left(\begin{smallmatrix} & -I \\ I & \end{smallmatrix}\right)$ on $V_j + V_h = V_j \otimes \mathbb{R}^2$ and the family of geodesics $\gamma_u(t) = e^{t\pi A_u} = e^{uB}\gamma(t)e^{-uB}$. This time, $A = J' \otimes D$ where $D = \mathrm{diag}(a_j, a_h) = cI + D'$ with $c = \frac{1}{2}(a_j + a_h)$ and $D' = \mathrm{diag}(b, -b)$ with $b = \frac{1}{2}(a_j - a_h)$. Thus the element $\gamma(t) = e^{\pi tc}e^{\pi tD'}$ is fixed under conjugation with the rotation matrix $e^{uB} = \left(\begin{smallmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{smallmatrix}\right)$ precisely when $e^{\pi tD'} = \mathrm{diag}(e^{\pi tb}, e^{-\pi tb})$ is a multiple of the identity matrix which happens for $t = 1/b$. If $b > 1$, we obtain an energy-decreasing deformation by cutting the corner, see figure above. We need to show that there are enough pairs (j, h) with $b > 1$ when γ is non-minimal.

Any $J \in P_k$ defines a $\mathbb{C}$-linear map J_kJ since J_kJ commutes with J_i and hence with J_o. Thus a path $\lambda : I \to P_k$ from J_k to $-J_k$ defines a family of $\mathbb{C}$-linear maps, and its complex determinant $\det(J_k\lambda)$ is a path in $\mathbb{S}^1$ starting and ending at $\det(\pm I) = 1$ (recall that the dimension n is even). This loop in $\mathbb{S}^1$ has a mapping degree which is apparently invariant under homotopy; it decomposes the path space ΛP_k into infinitely many connected components. If λ is a geodesic, $\lambda(t) = J_k e^{\pi tA}$, then $\det J_k^{-1}\lambda(t) = e^{\pi t \,\mathrm{trace}\, A}$, hence its mapping degree is $\frac{1}{2}\,\mathrm{trace}\, A/\mathrm{i}$. Since $\mathrm{trace}\, A/\mathrm{i} = m\sum_j a_j$, we may fix $c := \sum_j a_j$ (which means fixing the connected component of ΛP_k) and we may assume that $|c|$ is much smaller than r (the number of submodules V_j). Let p denote the sum of the positive a_j and $-q$ the sum of the negative a_j. Then $p + q \ge r$ since all $|a_j| \ge 0$, and $p - q = c$ which means roughly $p \approx q \approx r/2$. Assume for the moment $c = 0$. If there is some eigenvalue a_h with $|a_h| > 1$, say $a_h = -3$, there are many positive a_j with $a_j - a_h \ge 4$, more precisely $\sum_{a_j>0}(a_j - a_h) \ge 4 \cdot r/2 = 2r$, and this is a lower bound for the index. In the general case this result has to be corrected by the comparably small number c. In contrast, if all $a_j = \pm 1$, the geodesic γ consists of simultaneous half turns in $n/2$ perpendicular planes; these are shortest geodesics from I to $-I$ in SO_n.[7]

5 Vector Bundles over Spheres

Clifford representations have a direct connection to vector bundles over spheres and hence to K-theory. Every vector bundle $E \to \mathbb{S}^{k+1}$ is trivial over each of the two closed hemispheres $D_+, D_- \subset \mathbb{S}^{k+1}$, but along the equator $\mathbb{S}^k = D_+ \cap D_-$ the fibres over ∂D_+ and ∂D_- are identified by some map $\phi : \mathbb{S}^k \to O_n$ called *clutching map*.

[7] Any one-parameter subgroup γ in SO_n is a family of planar rotations in perpendicular planes. When $\gamma(1) = -I$, all rotation angles are odd multiples of π. The squared length of γ is the sum of the squared rotation angles. Thus the length is minimal if all rotation angles are just $\pm\pi$.



Homotopic clutching maps define equivalent vector bundles. Thus vector bundles over $\mathbb{S}^{k+1}$ are classified by the homotopy group $\pi_k(O_n)$. When we allow adding of trivial bundles (stabilization), n may be arbitrarily high. Let $\mathscr{V}_k$ be the set of vector bundles over $\mathbb{S}^{k+1}$ up to equivalence and adding of trivial bundles ("stable vector bundles"). Then

$$\mathscr{V}_k = \lim_{n\to\infty} \pi_k(O_n). \tag{7}$$

Hence we could apply Theorem 2 in order to classify stable vector bundles over spheres. However, a separate argument based on the same ideas but also using Clifford modules will give more information.

A Cl_k module $S = \mathbb{R}^n$ or the corresponding Clifford system $J_1, \dots, J_k \in O_n$ defines a peculiar map $\phi = \phi_S : \mathbb{S}^k \to O_n$ which is *linear*, that is a restriction of a linear map $\phi : \mathbb{R}^{k+1} \to \mathbb{R}^{n\times n}$, where we put

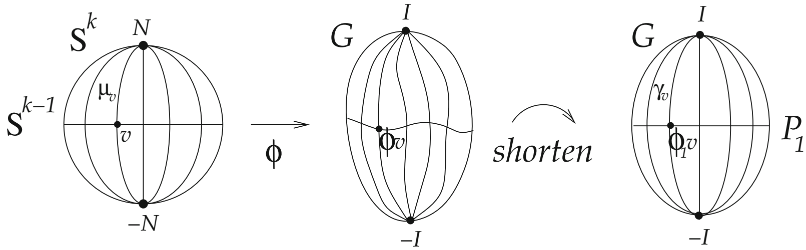
$$\phi_S(e_{k+1}) = I, \quad \phi_S(e_i) = J_i \text{ for } i \le k. \tag{8}$$

The bundles defined by such clutching maps ϕ_S are called *generalized Hopf bundles*. In the cases $k = 1, 3, 7$, these are the classical complex, quaternionic, and octonionic Hopf bundles over $\mathbb{S}^{k+1}$.

In fact, Cl_k-modules are in 1:1 correspondence to linear maps $\phi : \mathbb{S}^k \to O_n$ with the identity matrix in the image. To see this, let ϕ be such map and $W = \phi(\mathbb{R}^{k+1})$ its image. Then $\mathbb{S}_W := \phi(S^k) \subset O_n$. Thus ϕ is an isometry for the inner product $\langle A, B\rangle = \frac{1}{n}\,\mathrm{trace}\,(A^T B)$ on $\mathbb{R}^{n\times n}$ since $\phi(\mathbb{S}^k) \subset O_n$ and O_n lies in the unit sphere of $\mathbb{R}^{n\times n}$. For all $A, B \in \mathbb{S}_W$ we have $(A + B) \in \mathbb{R}\cdot O_n$. On the other hand, $(A + B)^T(A + B) = 2I + A^T B + B^T A$, thus $A^T B + B^T A = tI$ for some $t \in \mathbb{R}$. From the inner product with I we obtain $t = 2\langle A, B\rangle$. Inserting $A = I$ and $B \perp I$ yields $B + B^T = 0$, and for any $A, B \perp I$ we obtain $AB + BA = -2\langle A, B\rangle I$. Thus $\phi|\mathbb{R}^k$ defines a Cl_k-representation on $\mathbb{R}^n$.

Atiyah, Bott and Shapiro [3] reduced the theory of vector bundles over spheres to the simple algebraic structure of Clifford modules by showing that all vector bundles over spheres are generalized Hopf bundles plus trivial bundles, see Theorem 5 below. We sketch a different proof of this theorem using the original ideas of Bott and Milnor. We will homotopically deform the clutching map $\phi : \mathbb{S}^k \to G = SO_n$ of the given bundle $E \to \mathbb{S}^{k+1}$ step by step into a linear map. Since adding of trivial bundles is allowed, we may assume that the rank n of E is divisible by a high power of 2.

We declare $N = e_{k+1}$ to be the "north pole" of $\mathbb{S}^k$. First we deform ϕ such that $\phi(N) = I$ and $\phi(-N) = -I$. Thus ϕ maps each meridian from N to $-N$ in $\mathbb{S}^k$ onto some path from I to $-I$ in G, an element of ΛG. The meridians μ_v are parametrized by $v \in \mathbb{S}^{k-1}$ where $\mathbb{S}^{k-1}$ is the equator of $\mathbb{S}^k$. Therefore ϕ can be considered as a map $\phi : \mathbb{S}^{k-1} \to \Lambda G$. Using the negative gradient flow for the energy function E on the path space ΛG as in Sect. 2 we may shorten all $\phi(\mu_v)$ simultaneously to minimal geodesics from I to $-I$ and obtain a map $\tilde{\phi} : \mathbb{S}^{k-1} \to \Lambda_o G$ where $\Lambda_o G$ is the set of shortest geodesics from I to $-I$, the minimum set of E on ΛG. Let $m(\gamma) = \gamma(\frac{1}{2})$ be the midpoint of any geodesic $\gamma : [0, 1] \to G$. Thus we obtain a map $\phi_1 = m \circ \tilde{\phi} : \mathbb{S}^{k-1} \to P_1$, and we may replace ϕ by the geodesic suspension over ϕ_1 from I and $-I$.

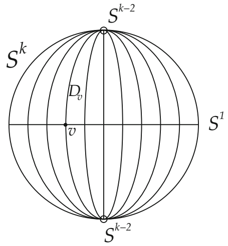


We repeat this step replacing G by P_1 and ϕ by ϕ_1. Again we choose a "north pole" $N_1 = e_k \in \mathbb{S}^{k-1}$ and deform ϕ_1 such that $\phi_1(\pm N_1) = \pm J_1$ for some $J_1 \in P_1$. Now we deform the curves $\phi_1(\mu_1)$ for all meridians $\mu_1 \subset \mathbb{S}^{k-1}$ to shortest geodesics, whose midpoints define a map $\phi_2 : \mathbb{S}^{k-2} \to P_1$, and then we replace ϕ_1 by a geodesic suspension from $\pm J_1$ over ϕ_2. This step is repeated $(k-1)$-times until we reach a map $\phi_{k-1} : \mathbb{S}^1 \to P_{k-1}$. This loop can be shortened to a geodesic loop $\tilde{\gamma} = J_{k-1}\gamma : [0, 1] \to P_{k-1}$ (which is a closed geodesic since P_{k-1} is symmetric) starting and ending at J_{k-1}, such that $\tilde{\gamma}$ and γ are shortest in their homotopy class.

We have $\gamma(t) = e^{2\pi t A}$ for some $A \in T_{J_{k-1}} P_{k-1}$. Since γ is closed, the (complex) eigenvalues of A have the form $a\mathrm{i}$ with $a \in \mathbb{Z}$ and $\mathrm{i} = \sqrt{-1}$. To compute these eigenvalues we argue as in Sect. 4. We split $\mathbb{R}^n$ into $V_0 = \ker A$ and a sum of subspaces V_j which are invariant under the linear maps $A, J_1, \dots, J_{k-1}$ and minimal with respect to this property. As before, $A = aJ'$ for some nonnegative integer a, and $J_k = J_{k-1}J'$ is a complex structure anticommuting with $J_1, \dots, J_{k-1}$. Hence V_j is an irreducible Cl_k-module with dimension m_k, see (4) and (5).

Since the clutching map of the given vector bundle $E \to \mathbb{S}^{k+1}$ (after the deformation) is determined by $\gamma, J_1, \dots, J_{k-1}$ which leave all V_j, $j \geq 0$, invariant, the vector bundle splits accordingly as $E = E_0 \oplus \sum_{j>0} E_j$ where E_0 is trivial.[8]

We claim that the minimality of γ implies $a_j = 1$ for all j and hence $A = J_k$. In fact, the geodesic variation γ_u of Sect. 4 shows that $|a_j - a_h| < 2$ for all j, h, otherwise we could shorten γ by cutting the corner.



Now suppose that, say, $a_1 \geq 2$. We may assume that $V_0 = \ker A$ contains another copy $\tilde{V}_1$ of V_1 as a Cl_{k-1}-module: if not, we extend E_0 by the trivial bundle $\mathbb{S}^{k+1} \times \tilde{V}_1$. Thus we have eigenvalues 0 and a_1 on $\tilde{V}_1 \oplus V_1$ with difference ≥ 2, in contradiction to the minimality of the geodesic.

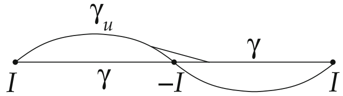
We have shown $E = E_0 \oplus E_1$ where E_0 is trivial and E_1 is a generalized Hopf bundle for the Clifford system $J_1, \dots, J_k$ on $\sum_{j>0} V_j$.

Let $\mathscr{M}_k$ the set of equivalence classes of Cl_k-modules, modulo trivial Cl_k-representations. We have studied the map

$$\hat{\alpha} : \mathscr{M}_k \to \mathscr{V}_k$$

which assigns to each $S \in \mathscr{M}_k$ the corresponding generalized Hopf bundle over $\mathbb{S}^{k+1}$. It is additive with respect to direct sums. We have just proved that $\hat{\alpha}$ is onto. But it is not 1:1. In fact, every Cl_{k+1}-module is also a Cl_k module since $Cl_k \subset Cl_{k+1}$. This defines a restriction map $\rho : \mathscr{M}_{k+1} \to \mathscr{M}_k$. Any Cl_k-module S which is really a Cl_{k+1}-module gives rise to a contractible clutching map $\phi_S : \mathbb{S}^k \to SO_n$ and hence to a trivial vector bundle since ϕ_S can be extended to $\mathbb{S}^{k+1}$ and thus contracted over one of the half spheres $D_+, D_- \subset \mathbb{S}^{k+1}$. Thus $\hat{\alpha}$ sends $\rho(\mathscr{M}_{k+1})$ into trivial bundles

[8] The clutching map $\phi : \mathbb{S}^k \to SO_n$ splits into components $\phi_j : \mathbb{S}^k \to SO(V_j)$. The domain $\mathbb{S}^k$ is the union of totally geodesic spherical $(k-1)$-discs D_v, $v \in \mathbb{S}^1$, centered at v and perpendicular to $\mathbb{S}^1$. All D_v have a common boundary $\mathbb{S}^{k-2}$. Since $\phi_0|_{D_v}$ is constant in v, it is contractable along D_v to a constant map.

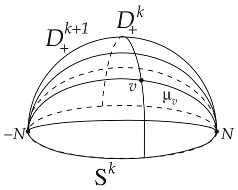


and hence it descends to an additive map[9]

$$\alpha : \mathscr{A}_k := \mathscr{M}_k/\rho(\mathscr{M}_{k+1}) \to \mathscr{V}_k.$$

We claim that this map is injective: if a stable bundle $\hat{\alpha}(S)$ is trivial for some Cl_k-module S, then S is (the restriction of) a Cl_{k+1}-module.

Proof of the claim. Let S be a Cl_k-module and $\phi = \phi_S : \mathbb{S}^k \to G$ the corresponding clutching map (that is $\phi(e_{k+1}) = I$, $\phi(e_i) = J_i$). We assume that ϕ is contractible, that is it extends to $\hat{\phi} : D^{k+1} \to G$, possibly after adding to S an element in $\rho(\mathscr{M}_{k+1})$. The closed disk D^{k+1} will be considered as the northern hemisphere $D_+^{k+1} \subset \mathbb{S}^{k+1}$. Repeating the argument above for the surjectivity, we consider the meridians μ_v between $N = e_{k+1} \in \mathbb{S}^k$ and $-N$, but this time there are much more such meridians, not only those in $\mathbb{S}^k$ but also those through the hemisphere D_+^{k+1}. They are labeled by $v \in D_+^k := D_+^{k+1} \cap N^\perp$.



Applying the negative energy gradient flow we deform the curves $\phi(\mu_v)$ to minimal geodesics without changing those in $\phi(\mathbb{S}^k)$ which are already minimal. Then we obtain the midpoint map $\hat{\phi}_1 : D_+^k \to P_1$ with $\phi_1(v) = m(\hat{\phi}(\mu_v))$ which extends the given midpoint map ϕ_1 of ϕ. This step is repeated k times until we reach $\hat{\phi}_k : D_+^1 \to P_k$ which is a path from J_k to $-J_k$ in P_k. This path can be shortened to a minimal geodesic in P_k whose midpoint is a complex structure J_{k+1} anticommuting with $J_1, \dots, J_k$. Thus we have shown that our Cl_k-module S is extendible to a Cl_{k+1}-module, that is $S \in \rho(\mathscr{M}_{k+1})$. This finishes the proof of the injectivity.

Theorem 5 ([3]) *Every vector bundle over $\mathbb{S}^k$ splits stably into a trivial bundle and a generalized Hopf bundle. More precisely, the map $\alpha : \mathscr{A}_k = \mathscr{M}_k/\rho(\mathscr{M}_{k+1}) \to \mathscr{V}_k$ sending the equivalence class of a Cl_k-module S onto its generalized Hopf bundle is an isomorphism.*

From (4) one easily obtains the groups $\mathscr{A}_k$ since the modules S_k in (4) are the (one or two) generators of $\mathscr{M}_k$. If $S_k = \rho(S_{k+1})$, then $\mathscr{A}_k = 0$. This happens for $k = 2, 4, 5, 6$. For $k = 0, 1$ we have

$$\rho(S_{k+1}) = S_k \oplus S_k = 2S_k,$$

[9]In fact, both $\mathscr{V}_k$ and $\mathscr{A}_k$ are abelian groups with respect to direct sums, not just semigroups, and α is a group homomorphism. Using the tensor product, $\mathscr{V} = \sum_k \mathscr{V}_k$ and $\mathscr{A} = \sum_k \mathscr{A}_k$ become rings and α a ring homomorphism, see [3].

hence $\mathscr{A}_0 = \mathscr{A}_1 = \mathbb{Z}_2$. For $k = 3, 7$ there are two generators for $\mathscr{M}_k$, say S_k and S'_k, and $\rho(S_{k+1}) = S_k \oplus S'_k$, thus $\mathscr{A}_3 = \mathscr{A}_7 = \mathbb{Z}$. Hence

$$\begin{array}{c|cccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathscr{A}_k & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} \end{array} \tag{9}$$

and because of the periodicity (5) we have $\mathscr{A}_{k+8} = \mathscr{A}_k$.

Consequently, the list (9) for $\mathscr{A}_k$ is the same as that for $\mathscr{V}_k$ and for $\pi_k(O_n)$, n large (see (7)). Thus we have also computed the stable homotopy of O_n.

We have seen that the following objects are closely related and obey the same periodicity theorem:

- Iterated centrioles of O_n,
- stable homotopy groups of O_n,
- Clifford modules,
- stable vector bundles over spheres.

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The Solvable Models of Noncompact Real Two-Plane Grassmannians and Some Applications

Jong Taek Cho, Takahiro Hashinaga, Akira Kubo, Yuichiro Taketomi and Hiroshi Tamaru

Abstract Every Riemannian symmetric space of noncompact type is isometric to some solvable Lie group equipped with a left-invariant Riemannian metric. The corresponding metric solvable Lie algebra is called the solvable model of the symmetric space. In this paper, we give explicit descriptions of the solvable models of noncompact real two-plane Grassmannians, and mention some applications to submanifold geometry, contact geometry, and geometry of left-invariant metrics.

1 Introduction

In the studies on Riemannian symmetric spaces of noncompact type, the solvable models have played important roles. Let $M = G/K$ be a Riemannian symmetric space of noncompact type, where G is the identity component of the isometry group $\mathrm{Isom}(M)$. Let $G = KAN$ be an Iwasawa decomposition, where K is maximal compact, A is abelian, and N is nilpotent. Then M is isometric to the solvable Lie group $S := AN$, by being equipped with a suitable left-invariant metric $\langle , \rangle$. The

J.T. Cho
Department of Mathematics, Chonnam National University, Gwangju 61186, Korea
e-mail: jtcho@chonnam.ac.kr

T. Hashinaga
National Institute of Technology, Kitakyushu College, Kitakyushu 802-0985, Japan
e-mail: hashinaga@kct.ac.jp

A. Kubo
Faculty of Economic Sciences, Hiroshima Shudo University, Hiroshima 731-3195, Japan
e-mail: akubo@shudo-u.ac.jp

Y. Taketomi · H. Tamaru (✉)
Department of Mathematics, Hiroshima University, Higashi-Hiroshima 739-8526, Japan
e-mail: tamaru@math.sci.hiroshima-u.ac.jp

Y. Taketomi
e-mail: y-taketomi@hiroshima-u.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_26

solvmanifold $(S, \langle , \rangle)$, or the corresponding metric solvable Lie algebra $(\mathfrak{s}, \langle , \rangle)$, is called the *solvable model* of the symmetric space $M = G/K$.

For a real hyperbolic space $\mathbb{R}\mathrm{H}^n$, its solvable model is the so-called Lie algebra of $\mathbb{R}\mathrm{H}^n$, which is of a quite simple form and has several interesting properties (see [13, 16, 18]). For other (complex, quaternion, and octonion) hyperbolic spaces, which are of rank one, their solvable models are given by Damek–Ricci spaces [1, 5]. Particularly in the case of a complex hyperbolic space $\mathbb{C}\mathrm{H}^n$, the solvable model provides a lot of interesting examples of isometric actions and homogeneous submanifolds. We refer to a survey paper [11] and references therein. These studies have still continued, for examples, the third author [14] studied the geometry of polar foliations on $\mathbb{C}\mathrm{H}^n$, and the second author and Kajigaya [10] studied homogeneous Lagrangian submanifolds in $\mathbb{C}\mathrm{H}^n$.

For higher rank cases, the solvable models are theoretically known, and can be described in terms of the root systems. They have played fundamental roles in the studies on symmetric spaces of noncompact type. Among others, successive examples would be the studies on homogeneous codimension one foliations [3] and hyperpolar foliations [4]. However, we sometimes need more explicit descriptions of the solvable models, in order to study more detailed properties, as in the case of complex hyperbolic spaces $\mathbb{C}\mathrm{H}^n$.

In this paper, we concentrate on a noncompact real two-plane Grassmannian $G_2^*(\mathbb{R}^{n+2})$, and explicitly describe its solvable model according to [8]. It is not difficult to determine the structure of the solvable model, but as far as we know, it is hard to find it in the literature. We also give several applications of the solvable model of $G_2^*(\mathbb{R}^{n+2})$. The topics contain cohomogeneity one actions (homogeneous codimension one foliations), geometry of Lie hypersurfaces, particular contact metric manifolds, and left-invariant Einstein and Ricci soliton metrics on Lie groups. We believe that our solvable model would play a fundamental role in further studies on geometry of $G_2^*(\mathbb{R}^{n+2})$.

2 The Solvable Model

In this section, we recall a description of the solvable models of noncompact real two-plane Grassmannians $G_2^*(\mathbb{R}^{n+2}) = \mathrm{SO}^0(2, n)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(n))$, according to the description given in [8].

2.1 A Description of the Solvable Model

In this subsection we give a definition of the solvable model of $G_2^*(\mathbb{R}^{n+2})$. We usually assume $n \geq 3$, since, in the case of $n = 2$, the symmetric space $G_2^*(\mathbb{R}^4)$ is not irreducible and has different features.

Definition 1 Let $c > 0$ and $n \geq 3$. We call $(\mathfrak{s}(c), \langle , \rangle, J)$ the *solvable model* of $G_2^*(\mathbb{R}^{n+2})$ if

(1) $\mathfrak{s}(c) := \mathrm{span}\{A_1, A_2, X_0, Y_1, \ldots, Y_{n-2}, Z_1, \ldots, Z_{n-2}, W_0\}$ is a $2n$-dimensional Lie algebra whose nonzero bracket relations are defined by

- $[A_1, X_0] = cX_0, [A_1, Y_i] = -(c/2)Y_i, [A_1, Z_i] = (c/2)Z_i, [A_1, W_0] = 0,$
- $[A_2, X_0] = 0, [A_2, Y_i] = (c/2)Y_i, [A_2, Z_i] = (c/2)Z_i, [A_2, W_0] = cW_0,$
- $[X_0, Y_i] = cZ_i, [Y_i, Z_i] = cW_0.$

(2) $\langle , \rangle$ is an inner product on $\mathfrak{s}(c)$ so that the above basis is orthonormal,
(3) J is a complex structure on $\mathfrak{s}(c)$ given by

$$J(A_1) = -X_0, \quad J(A_2) = W_0, \quad J(Y_i) = Z_i.$$

Let $S(c)$ denote the connected and simply-connected Lie group with Lie algebra $\mathfrak{s}(c)$, equipped with the induced left-invariant metric $\langle , \rangle$ and the induced complex structure J. The triplet $(S(c), \langle , \rangle, J)$ is also called the solvable model.

Theorem 2 ([8]) *The solvable model* $(S(c), \langle , \rangle, J)$ *is isomorphic to* $G_2^*(\mathbb{R}^{n+2})$ *with minimal sectional curvature* $-c^2$.

The proof is given by describing the Iwasawa decomposition of $\mathfrak{so}(2, n)$ explicitly, in terms of matrices. This is long but a straightforward calculation.

We here see the structure of the Lie algebra $\mathfrak{s}(c)$. One can directly see that

$$\mathfrak{n} := [\mathfrak{s}(c), \mathfrak{s}(c)] = \mathrm{span}\{X_0, Y_1, \ldots, Y_{n-2}, Z_1, \ldots, Z_{n-2}, W_0\}.$$

Furthermore, by the given bracket relations, we have

$$[\mathfrak{n}, \mathfrak{n}] = \mathrm{span}\{Z_1, \ldots, Z_{n-2}, W_0\}, \quad [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \mathrm{span}\{W_0\}, \quad [\mathfrak{n}, [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]]] = 0.$$

Therefore, $\mathfrak{s}(c)$ is solvable, whose derived subalgebra is three-step nilpotent. This is compatible with the root space decomposition, mentioned in the next subsection.

2.2 A Description in Terms of Root Spaces

In this subsection, we describe the root space decomposition of the solvable model $(\mathfrak{s}(c), \langle , \rangle, J)$. We need such description in order to translate some general results stated in terms of the root spaces.

Let us put $\mathfrak{a} := \mathrm{span}\{A_1, A_2\} \subset \mathfrak{s}(c)$, which is an abelian subalgebra. Then, for each $\alpha \in \mathfrak{a}^*$, the root space $\mathfrak{s}_\alpha$ of $\mathfrak{s}(c)$ with respect to $\mathfrak{a}$ is defined by

$$\mathfrak{s}_\alpha := \{X \in \mathfrak{s}(c) \mid [H, X] = \alpha(H)X \quad (\forall H \in \mathfrak{a})\}.$$

Proposition 3 *Let us define $\varepsilon_i \in \mathfrak{a}^*$ by*

$$\varepsilon_1(A_1) := c/2, \quad \varepsilon_2(A_1) := -c/2, \quad \varepsilon_1(A_2) := c/2, \quad \varepsilon_2(A_2) := c/2.$$

Then the nontrivial root spaces can be described as follows:

$$\mathfrak{s}_{\varepsilon_1-\varepsilon_2} = \mathrm{span}\{X_0\}, \qquad \mathfrak{s}_{\varepsilon_2} = \mathrm{span}\{Y_1, \dots, Y_{n-2}\},$$
$$\mathfrak{s}_{\varepsilon_1} = \mathrm{span}\{Z_1, \dots, Z_{n-2}\}, \qquad \mathfrak{s}_{\varepsilon_1+\varepsilon_2} = \mathrm{span}\{W_0\}.$$

Proof It follows directly from the bracket relations of the solvable model. □

As usual, we put $\alpha_1 := \varepsilon_1 - \varepsilon_2$ and $\alpha_2 := \varepsilon_2$. We then have the root space decomposition of $\mathfrak{s}(c)$ with respect to $\mathfrak{a}$,

$$\mathfrak{s}(c) = \mathfrak{a} \oplus \mathfrak{s}_{\alpha_1} \oplus \mathfrak{s}_{\alpha_2} \oplus \mathfrak{s}_{\alpha_1+\alpha_2} \oplus \mathfrak{s}_{\alpha_1+2\alpha_2}.$$

Therefore the set of roots is of type B_2, and $\{\alpha_1, \alpha_2\}$ is the set of simple roots. This agrees with the root system of $G_2^*(\mathbb{R}^{n+2})$.

3 Applications

In this section, we mention several applications of the solvable models $(S(c), \langle, \rangle, J)$ of noncompact real two-plane Grassmannians $G_2^*(\mathbb{R}^{n+2})$.

3.1 Cohomogeneity One Actions

In this subsection, we study cohomogeneity one actions on $G_2^*(\mathbb{R}^{n+2})$ in terms of the solvable model.

Definition 4 For an isometric action on a Riemannian manifold, maximal dimensional orbits are said to be *regular*, and other orbits *singular*. The codimension of a regular orbit is called the *cohomogeneity* of the action.

Therefore, a cohomogeneity one action is an isometric action whose regular orbits are of codimension one. For irreducible symmetric spaces of noncompact type, cohomogeneity one actions without singular orbit (equivalently, homogeneous codimension one foliations) have been classified in [3]. The classification result is described in terms of the root systems, but one can translate it into the solvable models as follows.

Theorem 5 ([3]) *An isometric action of a connected group on $G_2^*(\mathbb{R}^{n+2})$ is a cohomogeneity one action without singular orbit if and only if it is orbit equivalent to one of the actions given by*

(N) $\mathfrak{h} = \mathrm{span}\{a_1 A_1 + a_2 A_2\} \oplus \mathfrak{n}$ *with* $a_1^2 + a_2^2 = 1$,
(A_1) $\mathfrak{h} = \mathfrak{s}(c) \ominus \mathrm{span}\{X_0\}$,
(A_2*)* $\mathfrak{h} = \mathfrak{s}(c) \ominus \mathrm{span}\{Y_1\}$.

We refer these actions as the actions of type (N), (A_1), and (A_2), respectively. Note that there exist continuously many actions of type (N). The orbits of these actions play leading roles throughout this section.

Remark 6 Let H be a Lie subgroup of the solvable model $S(c)$. We identify $G_2^*(\mathbb{R}^{n+2}) \cong S(c)$, and hence H acts on $S(c)$ by the multiplication from the left. In this paper we consider H is acting on $G_2^*(\mathbb{R}^{n+2})$ in this way. On the other hand, one has $G_2^*(\mathbb{R}^{n+2}) = \mathrm{SO}^0(2, n)/\mathrm{S}(\mathrm{O}(2) \times \mathrm{O}(n))$, and H acts on this homogeneous space since $H \subset S(c) \subset \mathrm{SO}^0(2, n)$. We note that these two actions are equivariant, by the identification $F : S(c) \to G_2^*(\mathbb{R}^{n+2}) : g \mapsto g.o$, where o denotes the origin.

3.2 Lie Hypersurfaces

In this subsection, we study extrinsic geometry of orbits of cohomogeneity one actions on $G_2^*(\mathbb{R}^{n+2})$ without singular orbits. These orbits are sometimes called *Lie hypersurfaces*.

Proposition 7 ([3, 15]) *For the cohomogeneity one actions on* $G_2^*(\mathbb{R}^{n+2})$ *described in Theorem 5, we have the following:*

(1) For each action of type (N)*, all orbits are isometrically congruent to each other.*
(2) For each of the actions of type (A_1) *and* (A_2)*, there exists the unique minimal orbit.*

It depends on the choice of $a_1 A_1 + a_2 A_2$ whether a cohomogeneity one action of type (N) has minimal orbits or not. In order to study it, we have only to study the minimality of the orbit $H.e$ through the identity $e \in S(c)$. This is equivalent to the minimality of the Lie subgroup $H \subset S(c)$.

We here recall some general facts on the minimality of Lie subgroups. Let $(G, \langle , \rangle)$ be a Lie group with a left-invariant Riemannian metric, which we identify with the corresponding metric Lie algebra $(\mathfrak{g}, \langle , \rangle)$. First of all, we define the symmetric bilinear form $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$2\langle U(X, Y), Z\rangle = \langle [Z, X], Y\rangle + \langle X, [Z, Y]\rangle \quad (\forall X, Y, Z \in \mathfrak{g}).$$

Then, the Koszul formula yields that the Levi-Civita connection $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of $(\mathfrak{g}, \langle , \rangle)$ can be written as

$$\nabla_X Y = (1/2)[X, Y] + U(X, Y).$$

Let H be a Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then the second fundamental form $h : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g} \ominus \mathfrak{h}$ of the submanifold $H \subset G$ is defined by

$$h(X, Y) := (\nabla_X Y)^{\perp} := U(X, Y)_{\mathfrak{g}\ominus\mathfrak{h}},$$

which means the $(\mathfrak{g} \ominus \mathfrak{h})$-component of $U(X, Y)$ (and $\ominus$ denotes the orthogonal complement). The trace of h is called the *mean curvature vector* of the submanifold H in G, and H is said to be *minimal* if the mean curvature vector vanishes. In order to study the minimality of some Lie subgroups, the following notion is convenient.

Definition 8 A vector $H_0 \in \mathfrak{g}$ is called the *mean curvature vector* of $(\mathfrak{g}, \langle , \rangle)$ if it satisfies

$$\langle H_0, X \rangle = \mathrm{tr}(\mathrm{ad}_X) \quad (\forall X \in \mathfrak{g}).$$

Note that one has to distinguish the mean curvature vector of $(\mathfrak{g}, \langle , \rangle)$ and the mean curvature vector of a submanifold H in G. These two mean curvature vectors are related in the following particular cases.

Proposition 9 *Let H_0 be the mean curvature vector of $(\mathfrak{g}, \langle , \rangle)$, and H be a Lie subgroup of G whose Lie algebra $\mathfrak{h}$ contains $[\mathfrak{g}, \mathfrak{g}]$. Then the mean curvature vector of the submanifold H in G coincides with $(H_0)_{\mathfrak{g}\ominus\mathfrak{h}}$.*

Proof Since $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{h}$, one has a decomposition $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}] \oplus (\mathfrak{h} \ominus [\mathfrak{g}, \mathfrak{g}])$. Let $\{e_i\}$ and $\{e'_j\}$ be orthonormal bases of $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{h} \ominus [\mathfrak{g}, \mathfrak{g}]$, respectively. Then, the mean curvature vector H'_0 of the submanifold H in G is given by

$$H'_0 = \sum h(e_i, e_i) + \sum h(e'_j, e'_j) = \sum U(e_i, e_i)_{\mathfrak{g}\ominus\mathfrak{h}} + \sum U(e'_j, e'_j)_{\mathfrak{g}\ominus\mathfrak{h}}.$$

Here, since $e'_j \perp [\mathfrak{g}, \mathfrak{g}]$, one has $U(e'_j, e'_j) = 0$. We thus have

$$H'_0 = \sum U(e_i, e_i)_{\mathfrak{g}\ominus\mathfrak{h}}.$$

Our claim is $H'_0 = (H_0)_{\mathfrak{g}\ominus\mathfrak{h}}$. Take any $X \in \mathfrak{g} \ominus \mathfrak{h}$. Then we have

$$\langle H'_0, X \rangle = \langle \sum U(e_i, e_i), X \rangle = \sum \langle [X, e_i], e_i \rangle = \mathrm{tr}(\mathrm{ad}_X|_{[\mathfrak{g},\mathfrak{g}]}).$$

On the other hand, by the definition of H_0, one knows

$$\langle H_0, X \rangle = \mathrm{tr}(\mathrm{ad}_X) = \mathrm{tr}(\mathrm{ad}_X|_{[\mathfrak{g},\mathfrak{g}]}),$$

where the last equality follows from $\mathrm{ad}_X(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$. This completes the proof. □

Remark 10 In general, the mean curvature vector H_0 of $(\mathfrak{g}, \langle , \rangle)$ satisfies

$$\langle H_0, [\mathfrak{g}, \mathfrak{g}] \rangle = 0,$$

since $\langle H_0, [X, Y]\rangle = \mathrm{tr}(\mathrm{ad}_{[X,Y]}) = \mathrm{tr}([\mathrm{ad}_X, \mathrm{ad}_Y]) = 0$. Therefore, if we consider the particular case $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$, the mean curvature vector of the submanifold $H = [G, G]$ coincides with H_0. This is a reason why H_0 is called the mean curvature vector.

We apply this general theory to the actions of type (N) on $G_2^*(\mathbb{R}^{n+2}) \cong S(c)$. By the given bracket relations of $\mathfrak{s}(c)$, one can directly calculate H_0.

Proposition 11 *$H_0 := cA_1 + c(n-1)A_2$ is the mean curvature vector of the solvable model $(\mathfrak{s}(c), \langle , \rangle)$.*

For an action of type (N), there exist no minimal orbits in a generic case. However, if $a_1A_1 + a_2A_2$ is a particular one, then the action has minimal orbit (and hence all orbits are minimal). Such phenomenon has been known in [3, Corollary 3.2], but we here point out which action has a minimal orbit.

Proposition 12 *A cohomogeneity one action of type (N) on $G_2^*(\mathbb{R}^{n+2})$ has a minimal orbit (and hence all orbits are minimal) if and only if it is given by*

$$\mathfrak{h} := \mathrm{span}\{A_1 + (n-1)A_2\} \oplus \mathfrak{n}.$$

Proof Let $\mathfrak{h} := \mathrm{span}\{a_1A_1 + a_2A_2\} \oplus \mathfrak{n}$, and H be the connected Lie subgroup of $S(c)$ with Lie algebra $\mathfrak{h}$. We study the condition for the submanifold H in $S(c)$ to be minimal. Note that $[\mathfrak{s}(c), \mathfrak{s}(c)] = \mathfrak{n} \subset \mathfrak{h}$ holds. Therefore, by Proposition 9, H is minimal in $S(c)$ if and only if $(H_0)_{\mathfrak{s}(c)\ominus\mathfrak{h}} = 0$. This is equivalent to $\mathfrak{h} = \mathrm{span}\{H_0\} \oplus \mathfrak{n}$. We thus complete the proof by Proposition 11. □

3.3 Einstein Solvmanifolds

In this subsection, we study intrinsic geometry of orbits of cohomogeneity one actions on $G_2^*(\mathbb{R}^{n+2})$ without singular orbits. In particular, they provide examples of Einstein solvmanifolds. First of all we recall the following notation.

Definition 13 A metric solvable Lie algebra $(\mathfrak{s}, \langle , \rangle)$ is said to be of *Iwasawa type* if

(i) $\mathfrak{a} := \mathfrak{s} \ominus [\mathfrak{s}, \mathfrak{s}]$ is abelian,
(ii) for every $A \in \mathfrak{a}$, ad_A is symmetric with respect to $\langle , \rangle$, and $\mathrm{ad}_A \neq 0$ if $A \neq 0$,
(iii) there exists $A_0 \in \mathfrak{a}$ such that $\mathrm{ad}_{A_0}|_{[\mathfrak{s},\mathfrak{s}]}$ is positive definite.

One can easily see that the solvable model $(\mathfrak{s}(c), \langle , \rangle)$ of $G_2^*(\mathbb{R}^{n+2})$ is of Iwasawa type. More generally, the solvable parts of Iwasawa decompositions of semisimple Lie algebras are of Iwasawa type.

Proposition 14 ([12], Theorem 4.18) *Let $(\mathfrak{s}, \langle , \rangle)$ be an Einstein solvable Lie algebra of Iwasawa type, and H_0 be the mean curvature vector of $(\mathfrak{s}, \langle , \rangle)$. We put $\mathfrak{n} := [\mathfrak{s}, \mathfrak{s}]$, $\mathfrak{a} := \mathfrak{s} \ominus \mathfrak{n}$, and take a nonzero subspace $\mathfrak{a}' \subset \mathfrak{a}$. Then $(\mathfrak{s}' := \mathfrak{a}' \oplus \mathfrak{n}, \langle , \rangle|_{\mathfrak{s}'\times\mathfrak{s}'})$ is Einstein if and only if $H_0 \in \mathfrak{a}'$.*

The above procedure is called the rank reduction of an Einstein solvable Lie algebra $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle , \rangle)$. Note that our solvable model is Einstein, since it is isometric to an irreducible symmetric space. Hence, by applying the above procedure, we immediately have the following.

Proposition 15 *Let* $(\mathfrak{s}(c), \langle , \rangle, J)$ *be the solvable model of* $G_2^*(\mathbb{R}^{n+2})$*, and put* $\mathfrak{h} := \mathrm{span}\{A_1 + (n-1)A_2\} \oplus \mathfrak{n}$*. Then, for the corresponding cohomogeneity one action of type* (N)*, all orbits are Einstein hypersurfaces with respect to the induced metrics.*

In particular, $G_2^*(\mathbb{R}^{n+2})$ admits (homogeneous) real hypersurfaces which are Einstein. This is an easy observation, but would be interesting from the viewpoint of submanifold geometry. In fact, this is in contrast to the case of $\mathbb{C}\mathrm{H}^n$, namely, $\mathbb{C}\mathrm{H}^n$ do not admit any Einstein real hypersurfaces (see [19]).

3.4 Contact Metric Manifolds

In this subsection, we apply the solvable model $(\mathfrak{s}(c), \langle , \rangle, J)$ of $G_2^*(\mathbb{R}^{n+2})$ to study contact metric manifolds. Let M be a smooth manifold and $\mathfrak{X}(M)$ denote the set of all smooth vector field. A contact metric structure is denoted by (η, ξ, φ, g). The following notion has been introduced in [6].

Definition 16 Let $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold $(M, \eta, \xi, \varphi, g)$ is called a (κ, μ)*-space* if the Riemannian curvature tensor R satisfies

$$R(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y) \quad (\forall X, Y \in \mathfrak{X}(M)),$$

where I denotes the identity transformation and $h := (1/2)\mathscr{L}_\xi \varphi$ is the Lie derivative of φ along ξ.

It has been known that $\kappa \leq 1$ always holds. Furthermore, a contact metric manifold is Sasakian if and only if it is a $(1, \mu)$-space [6]. Therefore, the class of (κ, μ)-spaces is a kind of generalization of Sasakian manifolds. Typical examples of non-Sasakian (κ, μ)-spaces are the unit tangent sphere bundles $T_1(M(c))$ over Riemannian manifolds $M(c)$ of constant curvature $c \neq 1$. Non-Sasakian (κ, μ)-spaces have been studied deeply by Boeckx [7], but a geometric understanding seems to be not enough. The following gives a realization of $(0, 4)$-spaces.

Theorem 17 ([8]) *Let* $(\mathfrak{s}(2\sqrt{2}), \langle , \rangle, J)$ *be the solvable model of* $G_2^*(\mathbb{R}^{n+2})$ *with normalization* $c = 2\sqrt{2}$*, where* $n \geq 3$*. Then,* $\mathfrak{h} := \mathfrak{s}(2\sqrt{2}) \ominus \mathrm{span}\{A_1 + A_2\}$ *is a subalgebra, and the corresponding Lie group* H *equipped with the standard almost contact metric structure is a* $(0, 4)$*-space of dimension* $2n - 1$*.*

Recall that every real hypersurface in a Kähler manifold admits an almost contact metric structure. Note that $G_2^*(\mathbb{R}^{n+2})$ is a Hermitian symmetric space, which is Kähler, of dimension $2n$. Therefore, the above Lie subgroup H is equipped with an almost contact metric structure, and of dimension $2n - 1$. The proof is given by showing that $\mathfrak{h}$ is isomorphic to the example constructed by Boeckx [7].

We also note that this result is relevant to the study by Berndt and Suh [2], who classified contact real hypersurfaces in $G_2^*(\mathbb{R}^{n+2})$ with constant principal curvatures. The above $(0, 4)$-space is an example of such hypersurfaces, and hence is contained in their classification list (which is called a horosphere).

3.5 Ricci Soliton Solvmanifolds

In this subsection, we see that the orbits of cohomogeneity one actions of type (N) provide examples of Ricci soliton solvmanifolds. Recall that a Riemannian manifold (M, g) is called a *Ricci soliton* if there exist $c \in \mathbb{R}$ and $X \in \mathfrak{X}(M)$ such that the Ricci tensor Ric_g satisfies

$$\mathrm{Ric}_g = cg + \mathscr{L}_X g,$$

where $\mathscr{L}_X g$ denotes the Lie derivative of g along X.

Definition 18 A metric Lie algebra $(\mathfrak{g}, \langle , \rangle)$ is called an *algebraic Ricci soliton with constant* $c \in \mathbb{R}$ *if there exists* a derivation $D \in \mathrm{Der}(\mathfrak{g})$ such that

$$\mathrm{Ric} = c \cdot \mathrm{id} + D.$$

An algebraic Ricci soliton is called a *solvsoliton* if $\mathfrak{g}$ is solvable, and a *nilsoliton* if $\mathfrak{g}$ is nilpotent. Note that any algebraic Ricci soliton gives rise to a Ricci soliton metric on the corresponding simply-connected Lie group (see [17]).

Proposition 19 ([8, 17]) *Let* $(\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}, \langle , \rangle)$ *be a solvsoliton with constant* $c < 0$. *Take any subspace* $\mathfrak{a}'$ *of* $\mathfrak{a}$, *and put* $\mathfrak{s}' := \mathfrak{a}' \oplus \mathfrak{n}$. *Then,* $\mathfrak{s}'$ *is a subalgebra, and* $(\mathfrak{s}', \langle , \rangle|_{\mathfrak{s}' \times \mathfrak{s}'})$ *is also a solvsoliton with constant* c.

Recall that our solvable model $\mathfrak{s}(c)$ is Einstein with negative scalar curvature and solvable, which is a special case of solvsolitons with constant $c < 0$. Therefore, the above proposition yields the following.

Proposition 20 *All orbits of cohomogeneity one actions of type* (N) *on* $G_2^*(\mathbb{R}^{n+2})$ *are Ricci soliton solvmanifolds.*

Recall that a particular choice of $\mathfrak{a}'$, that is $\mathfrak{h} := \mathrm{span}\{A_1 + (n-1)A_2\} \oplus \mathfrak{n}$, gives rise to an Einstein solvmanifold (see Proposition 15). Other choices of $\mathfrak{a}'$ provide nontrivial (not Einstein) Ricci soliton solvmanifolds.

Corollary 21 *The connected, simply-connected and complete* $(0, 4)$*-space with dimension* ≥ 5 *is a nontrivial Ricci soliton.*

Proof It has been known in [7] that non-Sasakian (κ, μ)-spaces are locally determined by its dimension and the values $(\kappa, \mu) \in \mathbb{R}^2$. Therefore, a connected, simply-connected and complete $(0, 4)$-space is isometric to the one given in Theorem 17 by

$$\mathfrak{h} = \mathfrak{s}(2\sqrt{2}) \ominus \mathrm{span}\{A_1 + A_2\} = \mathrm{span}\{A_1 - A_2\} \oplus \mathfrak{n}.$$

In particular, it is an orbit of a cohomogeneity one action of type (N). By Proposition 20, it must be Ricci soliton. Furthermore, it is not Einstein, since $A_1 - A_2$ is not proportional to H_0. □

Note that Ghosh–Sharma [9] have studied non-Sasakian (κ, μ)-spaces which are Ricci soliton. In fact, they have proved the following classification result.

Theorem 22 ([9]) *Let M be a non-Sasakian (κ, μ)-space whose metric is a Ricci soliton. Then M is locally isometric to either* $(0, 0)$*-space or* $(0, 4)$*-space as a contact metric manifold.*

For $(0, 4)$-spaces with dimension ≥ 5, the converse statement would not be explicitly examined (they have used the software MATLAB). Our argument above complements the theorem of Ghosh–Sharma, by giving a Lie-theoretic proof of the converse direction, which can be checked by hand.

Acknowledgements The first author was supported in part by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2014R1A1A2053665). The second author was supported in part by JSPS KAKENHI (16K17603). The fifth author was supported in part by JSPS KAKENHI (26287012, 16K13757).

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Biharmonic Homogeneous Submanifolds in Compact Symmetric Spaces

Shinji Ohno, Takashi Sakai and Hajime Urakawa

Abstract This paper is a survey of our recent works on biharmonic homogeneous submanifolds in compact symmetric spaces (Biharmonic homogeneous submanifolds in compact symmetric spaces and compact Lie groups (in preparation), Biharmonic homogeneous hypersurfaces in compact symmetric spaces. Differ Geom Appl 43, 155–179 (2015)) [12, 13]. We give a necessary and sufficient condition for an isometric immersion whose tension field is parallel to be biharmonic. By this criterion, we study biharmonic orbits of commutative Hermann actions in compact symmetric spaces, and give some classifications.

1 Introduction

A harmonic map is a smooth map between Riemannian manifolds which is a critical point of the energy functional, hence it is a natural generalization of geodesics and minimal immersions. The Euler-Lagrange equation of the energy functional is the vanishing of the tension field, that is a second order elliptic PDE. The theory of harmonic maps relates to various subject in mathematics and plays an important role in differential geometry. As a generalization of harmonic maps, J. Eells and L. Lemaire [4] introduced the notion of biharmonic map between Riemannian manifolds, which is defined as a critical point of the bienergy functional. G.Y. Jiang [10] studied the first and second variation formulas of the bienergy functional and

S. Ohno
Advanced Mathematical Institute (OCAMI), Osaka City University, 3-3-138 Sugimoto Sumiyoshi-ku, Osaka 558-8585, Japan
e-mail: kudamono.shinji@gmail.com

T. Sakai (✉)
Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Minami-Osawa 1-1, Hachioji, Tokyo 192-0397, Japan
e-mail: sakai-t@tmu.ac.jp

H. Urakawa
Global Learning Center, Tohoku University, Kawauchi 41, Sendai 980-8576, Japan
e-mail: urakawa@math.is.tohoku.ac.jp

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_27

obtained the Euler-Lagrange equation, which is a fourth order PDE. By definition a harmonic map is always biharmonic. One of the most important problems is to ask whether the converse is true. B.Y. Chen [3] raised the following conjecture:

Every biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ must be harmonic (minimal).

Although many results supporting B.Y. Chen's conjecture have been obtained [1], it is still open. Furthermore, R. Caddeo, S. Montaldo, P. Piu and C. Oniciuc [2] raised the generalized B.Y. Chen's conjecture:

Every biharmonic submanifold of a Riemannian manifold of non-positive curvature must be harmonic (minimal).

This conjecture is false in general, in fact Ou and Tang [15] gave a counter example in a Riemannian manifold of negative curvature. However, under some additional conditions, the generalized B.Y. Chen's conjecture can be true (see Corollary 1).

On the contrary, in the case where the target space (N, h) has non-negative sectional curvature, the theory of biharmonic maps is quite different. There exist examples of proper biharmonic maps into Riemannian manifold of non-negative curvature. Here, proper biharmonic means biharmonic, but not harmonic.

In this paper, we study biharmonic submanifolds in compact symmetric spaces. First we give a necessary and sufficient condition for a submanifold whose tension field is parallel to be biharmonic. For orbits of commutative Hermann actions in compact symmetric spaces, this condition can be described in terms of symmetric triads, which is introduced by Ikawa [7]. By using this criterion, we determine all proper biharmonic hypersurfaces in irreducible symmetric spaces of compact type which are regular orbits of commutative Hermann actions of cohomogeneity one. Moreover, we construct higher codimensional proper biharmonic submanifolds in compact symmetric spaces and show a classification result. Also, we will give some concrete examples of proper biharmonic homogeneous submanifolds in Grassmannian manifolds.

2 Biharmonic Isometric Immersions

We first recall the definition and fundamentals of harmonic maps and biharmonic maps. Let $\varphi : (M, g) \to (N, h)$ be a smooth map from an m-dimensional compact Riemannian manifold (M, g) into an n-dimensional Riemannian manifold (N, h). Then φ is said to be harmonic if it is a critical point of the energy functional defined by

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v_g.$$

By the first variation formula for E, the Euler-Lagrange equation is given as the vanishing of the tension field $\tau(\varphi) := \operatorname{trace} B_\varphi \in \Gamma(\varphi^{-1}TN)$, where B_φ is the second fundamental form of φ defined by

$$B_\varphi(X, Y) = (\widetilde{\nabla} d\varphi)(X, Y) = \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y),$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, ∇ is the Levi-Civita connection on TM and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$ and $T^*M \otimes \varphi^{-1}TN$, respectively.

Eells and Lemaire [4] proposed the notion of biharmonic maps. We define the bienergy functional by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g.$$

A smooth map φ is said to be biharmonic if it is a critical point of E_2. Jiang [10] studied the first and second variation formulas of biharmonic maps, and showed that φ is biharmonic if and only if $\tau_2(\varphi) = 0$. Here $\tau_2(\varphi)$ is called the bitension field of φ defined by

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathscr{R}(\tau(\varphi)).$$

Here J is the Jacobi operator acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta}V - \mathscr{R}(V),$$

where $\overline{\Delta}V = \overline{\nabla}^*\overline{\nabla}V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i}V - \overline{\nabla}_{\nabla_{e_i}e_i}V\}$ is the rough Laplacian and $\mathscr{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathscr{R}(V) = \sum_{i=1}^m R^h(V, d\varphi(e_i))d\varphi(e_i)$, where R^h is the curvature tensor of (N, h) and $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g).

By definition, every harmonic map is biharmonic. We say that a smooth map $\varphi : (M, g) \to (N, h)$ is proper biharmonic if it is biharmonic but not harmonic.

Example 1 (*Oniciuc*) A small sphere $S^{n-1}(1/\sqrt{2})$ is a proper biharmonic hypersurface in the unit sphere $S^n(1)$, that is, its inclusion map is proper biharmonic.

Example 2 ([10]) A Clifford hypersurface $S^p(1/\sqrt{2}) \times S^q(1/\sqrt{2})$ $(p + q = n - 1,\ p \neq q)$ is a proper biharmonic hypersurface in $S^n(1)$.

Inoguchi and Urakawa [5, 6] showed that the above two examples are the only proper biharmonic isoparametric hypersurfaces in $S^n(1)$. Furthermore, they gave a classification of all proper biharmonic homogeneous hypersurfaces in S^n, $\mathbb{C}P^n$ and $\mathbb{H}P^n$.

Now we shall give a characterization theorem for an isometric immersion φ of a Riemannian manifold (M, g) into another Riemannian manifold (N, h) whose tension field $\tau(\varphi)$ satisfies $\overline{\nabla}^{\perp}_X\tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$ to be biharmonic, where $\overline{\nabla}^{\perp}$ is the normal connection on the normal bundle $T^{\perp}M$. From Jiang's theorem [10], we obtain the following theorem.

Theorem 1 ([12, 13]) *Let $\varphi : (M, g) \to (N, h)$ be an isometric immersion. Assume that $\overline{\nabla}^{\perp}_X\tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$. Then, φ is biharmonic if and only if*

$$\sum_{i=1}^m R^h\big(\tau(\varphi), d\varphi(e_i)\big)d\varphi(e_i) = \sum_{i=1}^m B_\varphi\big(A_{\tau(\varphi)}e_i, e_i\big)$$

holds, where A_ξ *denotes the shape operator of* φ *with respect to a normal vector* $\xi \in T^\perp M$.

Form Theorem 1, we obtain:

Corollary 1 ([13]) *Assume that the sectional curvature of the target space* (N, h) *is non-positive. Let* $\varphi : (M, g) \to (N, h)$ *be an isometric immersion whose tension field satisfies* $\overline{\nabla}^\perp_X \tau(\varphi) = 0$ *for all* $X \in \mathfrak{X}(M)$*. Then, if* φ *is biharmonic, then it is harmonic.*

3 Commutative Hermann Actions and Symmetric Triads

We will review some basics of Hermann actions on compact symmetric spaces and symmetric triad due to Ikawa [7].

Let (G, K_1) and (G, K_2) be compact symmetric pairs with respect to involutive automorphisms θ_1 and θ_2 of a compact connected Lie group G, respectively. Here we assume that K_1 and K_2 are connected, i.e. K_1 (resp. K_2) is the identity component of the fixed point set of θ_1 (resp. θ_2) in G. Then the triple (G, K_1, K_2) is called a compact symmetric triad. We denote the Lie algebras of G, K_1 and K_2 by $\mathfrak{g}$, $\mathfrak{k}_1$ and $\mathfrak{k}_2$, respectively. The involutive automorphism of $\mathfrak{g}$ induced from θ_i will be also denoted by θ_i. Take an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. Then the inner product $\langle \cdot, \cdot \rangle$ induces a bi-invariant Riemannian metric on G and G-invariant Riemannian metrics on the coset manifolds G/K_1 and $K_2\backslash G$. We denote these Riemannian metrics on G, G/K_1 and $K_2\backslash G$ by the same symbol $\langle \cdot, \cdot \rangle$. Then G, G/K_1 and $K_2\backslash G$ are compact Riemannian symmetric spaces. We denote by π_1 (resp. π_2) the natural projections from G onto G/K_1 (resp. $K_2\backslash G$). The isometric action of K_2 on G/K_1 and the isometric action of K_1 on $K_2\backslash G$ defined by

- $K_2 \curvearrowright G/K_1; \quad k_2\pi_1(x) = \pi_1(k_2 x) \quad (k_2 \in K_2, x \in G)$
- $K_2\backslash G \curvearrowleft K_1; \quad \pi_2(x)k_1 = \pi_2(x k_1) \quad (k_1 \in K_1, x \in G)$

are called Hermann actions. Under this setting, we can also consider the isometric action of $K_2 \times K_1$ on G defined by

- $K_2 \times K_1 \curvearrowright G; \quad (k_2, k_1) \cdot x = k_2 x k_1^{-1} \quad (k_2 \in K_2, k_1 \in K_1, x \in G)$

These three Lie group actions have the same orbit space, in fact the following diagram is commutative:

$$\begin{array}{ccc} & K_2 \times K_1 \curvearrowright G & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ K_2 \curvearrowright G/K_1 & & K_2\backslash G \curvearrowleft K_1 \\ \searrow & & \swarrow \\ & K_2\backslash G/K_1 \cong \overline{P} & \end{array}$$

Now we have two canonical decompositions of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{m}_1 = \mathfrak{k}_2 \oplus \mathfrak{m}_2,$$

where $\mathfrak{m}_i = \{X \in \mathfrak{g} \mid \theta_i(X) = -X\}$ $(i = 1, 2)$. Then $T_{\pi_1(e)}(G/K_1)$ (resp. $T_{\pi_2(e)}(K_2\backslash G)$) is identified with $\mathfrak{m}_1$ (resp. $\mathfrak{m}_2$) in a natural way. Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{m}_1 \cap \mathfrak{m}_2$. Then $\exp \mathfrak{a}$ is a torus subgroup in G.

A Hermann action of K_2 on G/K_1 is hyperpolar, in fact the totally geodesic flat torus $\pi_1(\exp \mathfrak{a})$ is a section, i.e. all orbits of K_2-action on G/K_1 meet $\pi_1(\exp \mathfrak{a})$ perpendicularly. Similarly K_1-action on $K_2\backslash G$ is also hyperpolar, since $\pi_2(\exp \mathfrak{a})$ is a flat section. We note that the cohomogeneity of K_2-action on G/K_1 and that of K_1-action on $K_2\backslash G$ are equal to $\dim \mathfrak{a}$.

Henceforth we assume that G is semisimple and two involutions θ_1 and θ_2 on G commute with each other, i.e. $\theta_1\theta_2 = \theta_2\theta_1$. Then (G, K_1, K_2) is called a commutative compact symmetric triad, and K_2-action on G/K_1 and K_1-action on $K_2\backslash G$ are called commutative Hermann actions. Then we have a direct sum decomposition

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{m}_2) \oplus (\mathfrak{m}_1 \cap \mathfrak{k}_2).$$

We define subspaces in $\mathfrak{g}$ as follows:

$$\begin{aligned}
\mathfrak{k}_0 &= \{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = \{0\}\},\\
V(\mathfrak{k}_1 \cap \mathfrak{m}_2) &= \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid [\mathfrak{a}, X] = \{0\}\},\\
V(\mathfrak{m}_1 \cap \mathfrak{k}_2) &= \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [\mathfrak{a}, X] = \{0\}\},
\end{aligned}$$

and for $\lambda \in \mathfrak{a}$

$$\begin{aligned}
\mathfrak{k}_\lambda &= \{X \in \mathfrak{k}_1 \cap \mathfrak{k}_2 \mid [H, [H, X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\},\\
\mathfrak{m}_\lambda &= \{X \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \mid [H, [H, X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\},\\
V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) &= \{X \in \mathfrak{k}_1 \cap \mathfrak{m}_2 \mid [H, [H, X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\},\\
V_\lambda^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2) &= \{X \in \mathfrak{m}_1 \cap \mathfrak{k}_2 \mid [H, [H, X]] = -\langle \lambda, H\rangle^2 X \ (H \in \mathfrak{a})\}.
\end{aligned}$$

We set

$$\begin{aligned}
\Sigma &= \{\lambda \in \mathfrak{a} \setminus \{0\} \mid \mathfrak{k}_\lambda \neq \{0\}\},\\
W &= \{\alpha \in \mathfrak{a} \setminus \{0\} \mid V_\alpha^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) \neq \{0\}\},\\
\tilde{\Sigma} &= \Sigma \cup W.
\end{aligned}$$

It is known that $\dim \mathfrak{k}_\lambda = \dim \mathfrak{m}_\lambda$ and $\dim V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2) = \dim V_\lambda^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2)$ for each $\lambda \in \tilde{\Sigma}$. Thus we define $m(\lambda) := \dim \mathfrak{k}_\lambda$ and $n(\lambda) := \dim V_\lambda^\perp(\mathfrak{k}_1 \cap \mathfrak{m}_2)$.

Ikawa [7] introduced the notion of symmetric triad with multiplicities as a generalization of an irreducible root system, and obtained the following proposition.

Proposition 1 ([7] Lemma 4.12, Theorem 4.33) *Let (G, K_1, K_2) be a commutative compact symmetric triad where G is semisimple. Then $\tilde{\Sigma}$ is a root system of $\mathfrak{a}$. In addition, if G is simple and $\theta_1 \not\sim \theta_2$, then $(\tilde{\Sigma}, \Sigma, W)$ is a symmetric triad of $\mathfrak{a}$, moreover $m(\lambda)$ and $n(\alpha)$ are multiplicities of $\lambda \in \Sigma$ and $\alpha \in W$. Here $\theta_1 \not\sim \theta_2$ means that θ_1 and θ_2 cannot be transformed each other by an inner automorphism of $\mathfrak{g}$.*

Now we consider an orbit $K_2\pi_1(x)$ of the action of K_2 on G/K_1 for $x \in G$. Without loss of generalities we can assume that $x = \exp H$ where $H \in \mathfrak{a}$, since $\pi_1(\exp \mathfrak{a})$ is a section of the action. We define an open subset $\mathfrak{a}_{\mathrm{reg}}$ of $\mathfrak{a}$ by

$$\mathfrak{a}_{\mathrm{reg}} = \bigcap_{\lambda \in \Sigma, \alpha \in W} \left\{ H \in \mathfrak{a} \;\middle|\; \langle \lambda, H \rangle \notin \pi\mathbb{Z},\ \langle \alpha, H \rangle \notin \frac{\pi}{2} + \pi\mathbb{Z} \right\}.$$

Then, for $x = \exp H$ $(H \in \mathfrak{a})$, $K_2\pi_1(x)$ is a regular orbit if and only if $H \in \mathfrak{a}_{\mathrm{reg}}$. Here we call an orbit of the maximal dimension a regular orbit. We take a connected component P, which is called a cell, of $\mathfrak{a}_{\mathrm{reg}}$. Then the closure $\overline{P}$ of P can be identified with the orbit space $K_2\backslash G/K_1$. More precisely, for each orbit $K_2\pi_1(x)$, there exists $H \in \overline{P}$ uniquely so that $x = \exp H$. An interior point H in $\overline{P}$ corresponds to a regular orbit, and a point H in the boundary of $\overline{P}$ corresponds to a singular orbit. Indeed, P is a simplex in $\mathfrak{a}$, and the cell decomposition of $\overline{P}$ gives a stratification of orbit types of the action.

We identify the tangent space $T_{\pi_1(e)}(G/K_1)$ with $\mathfrak{m}_1$ via $(d\pi_1)_e$. For $x = \exp H$ $(H \in \mathfrak{a})$, the tangent space and the normal space of $K_2\pi_1(x)$ at $\pi_1(x)$ are given as

$$\begin{aligned} dL_x^{-1}(T_{\pi_1(x)}(K_2\pi_1(x))) &\cong (\mathrm{Ad}(x^{-1})\mathfrak{k}_2)_{\mathfrak{m}_1} \\ &= \sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H \rangle \notin \pi\mathbb{Z}}} \mathfrak{m}_\lambda \oplus V(\mathfrak{m}_1 \cap \mathfrak{k}_2) \oplus \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H \rangle \notin (\pi/2) + \pi\mathbb{Z}}} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2), \\ dL_x^{-1}(T_{\pi_1(x)}^\perp(K_2\pi_1(x))) &\cong (\mathrm{Ad}(x^{-1})\mathfrak{m}_2) \cap \mathfrak{m}_1 \\ &= \mathfrak{a} \oplus \sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H \rangle \in \pi\mathbb{Z}}} \mathfrak{m}_\lambda \oplus \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H \rangle \in (\pi/2) + \pi\mathbb{Z}}} V_\alpha^\perp(\mathfrak{m}_1 \cap \mathfrak{k}_2), \end{aligned}$$

where $X_{\mathfrak{m}_1}$ denotes $\mathfrak{m}_1$-component of $X \in \mathfrak{g}$ with respect to the canonical decomposition $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{m}_1$, and L_k denote the isometry on G/K_1 by the action of $k \in G$. Using the above decompositions of the tangent space and the normal space of the orbit $K_2\pi_1(x)$, one can express the second fundamental form B_H and the tension field (mean curvature vector field) τ_H in terms of the symmetric triad $(\tilde{\Sigma}, \Sigma, W)$.

$$dL_x^{-1}(\tau_H) = -\sum_{\substack{\lambda \in \Sigma^+ \\ \langle \lambda, H \rangle \notin \pi\mathbb{Z}}} m(\lambda) \cot\langle \lambda, H \rangle \lambda + \sum_{\substack{\alpha \in W^+ \\ \langle \alpha, H \rangle \notin (\pi/2) + \pi\mathbb{Z}}} n(\alpha) \tan\langle \alpha, H \rangle \alpha.$$

From this expression of τ_H, Ikawa [7] obtained the following.

Theorem 2 ([7])

1. *In each strata of the orbit space $K_2\backslash G/K_1 \cong \overline{P}$, there exists a unique minimal orbit $K_2\pi_1(x)$ in G/K_1.*
2. *An orbit $K_2\pi_1(x)$ is minimal in G/K_1 if and only if $\pi_2(x)K_1$ is minimal in $K_2\backslash G$.*

4 Biharmonic Orbits of Commutative Hermann Actions

Every orbits of Hermann actions satisfy $\overline{\nabla}^{\perp}_X \tau(\varphi) = 0$ for all $X \in \mathfrak{X}(M)$ [8]. Therefore we can apply Theorem 1 to study the biharmonicity of orbits of Hermann actions. Moreover, as in the previous section, the second fundamental form and the shape operator of an orbit of a commutative Hermann action can be described in terms of the symmetric triad. Consequently, we obtain the following characterization theorem.

Theorem 3 ([12]) *Let (G, K_1, K_2) be a commutative compact symmetric triad. For $x = \exp H$ $(H \in \mathfrak{a})$, the orbit $K_2\pi_1(x)$ is biharmonic in G/K_1 if and only if*

$$\sum_{\substack{\lambda\in\Sigma^+ \\ \langle\lambda,H\rangle\notin\pi\mathbb{Z}}} m(\lambda)\langle dL_x^{-1}(\tau_H), \lambda\rangle\big(1 - (\cot\langle\lambda, H\rangle)^2\big)\lambda$$
$$+ \sum_{\substack{\alpha\in W^+ \\ \langle\alpha,H\rangle\notin(\pi/2)+\pi\mathbb{Z}}} n(\alpha)\langle dL_x^{-1}(\tau_H), \alpha\rangle\big(1 - (\tan\langle\alpha, H\rangle)^2\big)\alpha = 0.$$

Since a symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ is determined by the pair of compact symmetric pairs (G, K_1) and (G, K_2), from the above theorem, we obtain the following immediately.

Corollary 2 *An orbit $K_2\pi_1(x)$ is biharmonic in G/K_1 if and only if $\pi_2(x)K_1$ is biharmonic in $K_2\backslash G$.*

Let us consider the case of $\dim \mathfrak{a} = 1$. Then the Hermann action of K_2 on G/K_1 is of cohomogeneity one, hence also K_1-action on $K_2\backslash G$ is. In this case, the orbit space $K_2\backslash G/K_1$ can be identified with a closed interval $[0, 1]$. Two end points of $K_2\backslash G/K_1$ correspond to two singular orbits, and the interior points of $K_2\backslash G/K_1$ correspond to regular orbits, which are homogeneous hypersurfaces in G/K_1. According to the classification of commutative compact symmetric triads with $\dim \mathfrak{a} = 1$, we obtain the following result.

Theorem 4 ([13]) *Let (G, K_1, K_2) be a commutative compact symmetric triad where G is simple, and suppose that K_2-action on G/K_1 is cohomogeneity one. Then all the proper biharmonic hypersurfaces which are regular orbits of K_2-action (resp. K_1-action) in the compact symmetric space G/K_1 (resp. $K_2\backslash G$) are classified into the following lists:*

(1) *When* (G, K_1, K_2) *is one of the following cases, there exists a unique proper biharmonic hypersurface which is a regular orbit of* K_2*-action on* G/K_1 *(resp.* K_1*-action on* $K_2\backslash G$*).*

(1-1) $(\mathrm{SO}(1+b+c),\ \mathrm{SO}(1+b)\times\mathrm{SO}(c),\ \mathrm{SO}(b+c))\quad (b>0,\ c>1,\ c-1\neq b)$
(1-2) $(\mathrm{SU}(4),\ \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2)),\ \mathrm{Sp}(2))$
(1-3) $(\mathrm{Sp}(2),\ \mathrm{U}(2),\ \mathrm{Sp}(1)\times\mathrm{Sp}(1))$

(2) *When* (G, K_1, K_2) *is one of the following cases, there exist exactly two distinct proper biharmonic hypersurfaces which are regular orbits of of* K_2*-action on* G/K_1 *(resp.* K_1*-action on* $K_2\backslash G$*).*

(2-1) $(\mathrm{SO}(2+2q),\ \mathrm{SO}(2)\times\mathrm{SO}(2q),\ \mathrm{U}(1+q))\quad (q>1)$
(2-2) $(\mathrm{SU}(1+b+c),\quad \mathrm{S}(\mathrm{U}(1+b)\times\mathrm{U}(c)),\quad \mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(b+c))(b\geq 0,\ c>1)$
(2-3) $(\mathrm{Sp}(1+b+c),\quad \mathrm{Sp}(1+b)\times\mathrm{Sp}(c),\quad \mathrm{Sp}(1)\times\mathrm{Sp}(b+c))\quad (b\geq 0,\ c>1)$
(2-4) $(\mathrm{SO}(8),\ \mathrm{U}(4),\ \mathrm{U}(4)')$
(2-5) $(\mathrm{E}_6,\ \mathrm{SO}(10)\cdot\mathrm{U}(1),\ \mathrm{F}_4)$
(2-6) $(\mathrm{SO}(1+q),\ \mathrm{SO}(q),\ \mathrm{SO}(q))\quad (q>1)$
(2-7) $(\mathrm{F}_4,\ \mathrm{Spin}(9),\ \mathrm{Spin}(9))$

(3) *When* (G, K_1, K_2) *is one of the following cases, any biharmonic regular orbit of* K_2*-action on* G/K_1 *(resp.* K_1*-action on* $K_2\backslash G$*) is harmonic.*

(3-1) $(\mathrm{SO}(2c),\ \mathrm{SO}(c)\times\mathrm{SO}(c),\ \mathrm{SO}(2c-1))\quad (c>1)$
(3-2) $(\mathrm{SU}(4),\ \mathrm{Sp}(2),\ \mathrm{SO}(4))$
(3-3) $(\mathrm{SO}(6),\ \mathrm{U}(3),\ \mathrm{SO}(3)\times\mathrm{SO}(3))$
(3-4) $(\mathrm{SU}(1+q),\ \mathrm{SO}(1+q),\ \mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(q)))\quad (q>1)$
(3-5) $(\mathrm{SU}(2+2q),\ \mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(2q)),\ \mathrm{Sp}(1+q))\quad (q>1)$
(3-6) $(\mathrm{Sp}(1+q),\ \mathrm{U}(1+q),\ \mathrm{Sp}(1)\times\mathrm{Sp}(q))\quad (q>1)$
(3-7) $(\mathrm{E}_6,\ \mathrm{SU}(6)\cdot\mathrm{SU}(2),\ \mathrm{F}_4)$
(3-8) $(\mathrm{F}_4,\ \mathrm{Sp}(3)\cdot\mathrm{Sp}(1),\ \mathrm{Spin}(9))$

Next we shall consider the case of $\dim\mathfrak{a}=2$. In this case, we can determine all proper biharmonic singular orbits of K_2-action on G/K_1. In [12], we will give a list of proper biharmonic singular orbits. From the list, we obtain the following.

Theorem 5 ([12]) *Assume that* G *is simple and* $\dim\mathfrak{a}=2$*. In each singular orbit type, the existences of proper biharmonic orbits* $K_2\pi_1(x)\subset G/K_1$ *is classified into the following three classes:*

(1) *There exists a unique proper biharmonic orbit.*
(2) *There exist exactly two proper biharmonic orbits.*
(3) *All the biharmonic orbits must be harmonic.*

Example 3 $(G, K_1, K_2) = \big(\mathrm{SO}(n+1),\ \mathrm{SO}(n),\ \mathrm{SO}(p+1) \times \mathrm{SO}(q+1)\big)$ $(p, q \geq 1,\ p + q = n - 1)$, that is the case (1.1) in Theorem 4.

In this case, $G/K_1 = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is homothetic to the unit sphere $S^n(1)$, and a principal orbit of $K_2 = \mathrm{SO}(p+1) \times \mathrm{SO}(q+1)$-action on $G/K_1 \cong S^n(1)$ is a Clifford hypersurfaces $K_2\pi_1(x) \cong S^p(r_1) \times S^q(r_2)$ $(r_1^2 + r_2^2 = 1)$, which is an isoparametric hypersurface with 2 distinct principal curvatures. It is well-known that $S^p(r_1) \times S^q(r_2) \subset S^n(1)$ is minimal if and only if $r_1 = \sqrt{\frac{p}{n-1}},\ r_2 = \sqrt{\frac{q}{n-1}}$. Moreover, by Theorem 3, $S^p(r_1) \times S^q(r_2) \subset S^n(1)$ is biharmonic if and only if $r_1 = r_2 = \frac{1}{\sqrt{2}}$, hence it is proper biharmonic when $p \neq q$. This was given in Example 2.

On the other hand, $K_2\backslash G = (\mathrm{SO}(p+1) \times \mathrm{SO}(q+1))\backslash \mathrm{SO}(n+1)$ is the Grassmannian manifold $\widetilde{G_{p+1}}(\mathbb{R}^{n+1})$ of oriented $(p+1)$-planes in $\mathbb{R}^{n+1}$, and a principal orbit of $K_1 = \mathrm{SO}(n)$-action on $K_2\backslash G \cong \widetilde{G_{p+1}}(\mathbb{R}^{n+1})$ is diffeomorphic to $\mathrm{SO}(n)/(\mathrm{SO}(p) \times \mathrm{SO}(q))$, i.e. the universal covering of a real flag manifold, embedded in $\widetilde{G_{p+1}}(\mathbb{R}^{n+1})$ as the tube over the totally geodesic sub-Grassmannian $\widetilde{G_p}(\mathbb{R}^n)$. From Corollary 2, there exists a unique proper biharmonic orbit at the midpoint of the orbit space, when $p \neq q$.

Example 4 $(G, K_1, K_2) = \big(\mathrm{SO}(2+n),\ \mathrm{SO}(2) \times \mathrm{SO}(n),\ \mathrm{SO}(2) \times \mathrm{SO}(n)\big)$ $(n \geq 3)$

In this case, both G/K_1 and $K_2\backslash G$ are isometric to the Grassmannian manifold $\widetilde{G_2}(\mathbb{R}^{n+2})$ of oriented 2-planes in $\mathbb{R}^{n+2}$, and it is isomorphic to the complex quadric $Q_n(\mathbb{C})$, which is a compact Hermitian symmetric space of rank two. Then K_2-action on G/K_1 (and so K_1-action on $K_2\backslash G$) is the isotropy action of $\widetilde{G_2}(\mathbb{R}^{n+2}) \cong Q_n(\mathbb{C})$, therefore the symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ reduces to the restricted root system of type B_2, more precisely, $\tilde{\Sigma} = \Sigma$ are the root system of type B_2 and $W = \emptyset$. The orbit space $K_2\backslash G/K_1 \cong \overline{P}$ can be expressed as the triangular region in Fig. 1. Each vertex of the triangle corresponds to an isolated orbit, which is minimal. Edges of the triangle correspond to singular orbit types. By Theorem 2, there exists a unique minimal orbit in each edge. Corresponding to the midpoint H_1 of the hypotenuse, the orbit $K_2\pi_1(x)$ $(x = \exp H_1)$ is a real form of $Q_n(\mathbb{C})$, which is a totally geodesic Lagrangian submanifold and diffeomorphic to $(S^1 \times S^{n-1})/\mathbb{Z}_2$ and obtained as the image of the Gauss map of the Clifford hypersurface $S^1 \times S^{n-1}$ in S^{n+1}. This orbit type is the case of (2) in Theorem 5, thus there exist two proper biharmonic Lagrangian orbits corresponding to two $\circ$ points on the hypotenuse. In equilateral edges, H_2 and H_3

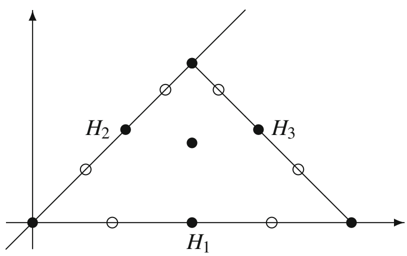

Fig. 1 Minimal orbits and biharmonic orbits in $K_2\backslash G/K_1 \cong \overline{P}$

correspond to minimal orbits of codimension 3, which are diffeomorphic to a Stiefel manifold $SO(n)/SO(n-2)$. These orbit types are the cases of (2) in Theorem 5, thus there exist two proper biharmonic orbits corresponding to two ∘ points on each edges.

Concluding Remarks and Further Problems

In the present paper, we studied biharmonic submanifolds in compact symmetric spaces, especially orbits of commutative Hermann actions. In our arguments, the condition $\theta_1\theta_2 = \theta_2\theta_1$ is crucial to define symmetric triads. As a further problem, we should study orbits of Hermann actions in the case of $\theta_1\theta_2 \neq \theta_2\theta_1$.

Homogeneous hypersurfaces in irreducible compact symmetric spaces were classified by Kollross [11]. There exist exceptional cohomogeneity one actions, that is, non-Hermann type. To obtain the complete classification of all biharmonic homogeneous hypersurfaces in irreducible compact symmetric spaces, we may have to study orbits of each exceptional cohomogeneity one action individually. Recently, Inoguchi and Sasahara [9] also obtained some results on biharmonic homogeneous hypersurfaces in compact symmetric spaces.

Finally, we mention that our method can be also applied to study orbits of $K_2 \times K_1$-action on G associated to Hermann actions. In the next paper [12], we will give some results on biharmonic homogeneous submanifolds in compact Lie groups.

Acknowledgements The second author was supported by Grant-in-Aid for Scientific Research (C) No. 26400073, Japan Society for the Promotion of Science. The third author was supported by Grant-in-Aid for Scientific Research (C) No. 25400154, Japan Society for the Promotion of Science.

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Recent Results on Real Hypersurfaces in Complex Quadrics

Young Jin Suh

Abstract In this survey article, first we introduce the classification of homogeneous hypersurfaces in some Hermitian symmetric spaces of rank 2. Second, by using the isometric Reeb flow, we give a complete classification for hypersurfaces M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2U_m)$, complex hyperbolic two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$, complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ and its dual $Q^{m*} = SO^o_{m,2}/SO_mSO_2$. As a third, we introduce the classifications of contact hypersurfaces with constant mean curvature in the complex quadric Q^m and its noncompact dual Q^{m*} for $m \geq 3$. Finally we want to mention some classifications of real hypersurfaces in the complex quadrics Q^m with Ricci parallel, harmonic curvature, parallel normal Jacobi, pseudo-Einstein, pseudo-anti commuting Ricci tensor and Ricci soliton etc.

1 Introduction

Let us denote by $(\bar{M}, g)$ a Riemannian manifold and $I(\bar{M}, g)$ the set of all isometries defined on $\bar{M}$. Here, a homogeneous submanifold of $(\bar{M}, g)$ is a connected submanifold M of $\bar{M}$ which is an orbit of some closed subgroup G of $I(\bar{M}, g)$. If the codimension of M is one, then M is called a *homogeneous hypersurface*. When M becomes a homogeneous hypersurface of $\bar{M}$, there exists some closed subgroup G of $I(\bar{M}, g)$ having M as an orbit. Since the codimension of M is one, the regular orbits of the action of G on $\bar{M}$ have codimension one, that is, the action of G on $\bar{M}$ is of cohomogeneity one. This means that the classification of homogeneous hypersurfaces is equivalent to the classification of cohomogeneity one actions up to orbit equivalence.

The orbit space $\bar{M}/G$ with quotient topology for a closed subgroup G of $I(\bar{M}, g)$ with cohomogeneity one becomes a one dimensional Hausdorff space homeomorphic

Y.J. Suh (✉)
Department of Mathematics and RIRCM, Kyungpook National University,
Daegu 41566, Republic of Korea
e-mail: yjsuh@knu.ac.kr

Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0_28

to the real line $\mathbb{R}$, the circle S^1, the half-open interval $[0, \infty)$, or the closed interval $[0, 1]$. This was proved by Mostert [28] for the case G is compact and in general by Bérard-Bergery.

When $\bar{M}$ is simply connected and compact, the quotient space $\bar{M}/G$ must be homeomorphic to $[0, 1]$ and each singular orbit must have codimension greater than one. This means that each regular orbit is *a tube around any of the two singular orbits*, and each singular orbit is a focal set of any regular orbit. This fact will be applied in Sects. 2, 4 and 6 for complex projective space $\mathbb{C}P^m$, complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and complex quadric Q^m which are Hermitian symmetric spaces of compact type with rank 1 and rank 2 respectively.

When $\bar{M}$ is simply connected and non-compact, the quotient space $\bar{M}/G$ must be homeomorphic to $\mathbb{R}$ or $[0, \infty)$. In the latter case the singular orbit must have codimension greater than one, and each regular orbit is *a tube around the singular one*. This fact will be applied and discussed in detail in Sects. 2 and 4 for dual complex two-plane Grassmannians $G_2^*(\mathbb{C}^{m+2})$ and dual complex quadric Q^{*m} which are Hermitian symmetric spaces of non compact type with rank 2.

Hereafter let us note that HSSP denotes a Hermitian Symmetric Space. For HSSP with rank one we have complex projective spaces $\mathbb{C}P^m$, complex hyperbolic spaces $\mathbb{C}H^m$. For HSSP of compact type with rank 2 we have $SU_{2+m}/S(U_2U_m)$, SO_8/U_4, $G_2(\mathbb{R}^{2+m})$, Sp_2/U_2 and $E_6/Spin_{10}U_1$, and for HSSP of non-compact type with rank 2 we can give $SU_{2,m}/S(U_2U_m)$, SO_8^*/U_4, $G_2^*(\mathbb{R}^{2+m})$, $Sp(2, \mathbb{R})/U_2$ and $E_6^{-14}/Spin_{10}U_1$ (See Helgason [12, 13]).

One of the motivations of this article is to suggest the problem of classifying all orientable real hypersurfaces M in almost Hermitian manifold $\bar{M}$ for which the Reeb flow is isometric. The almost Hermitian structure on almost Hermitian manifold $\bar{M}$ induces an almost contact metric structure on M. The corresponding unit tangent vector field on M is the *Reeb* vector field, and its flow is said to be the *Reeb flow* on M.

The classification of all real hypersurfaces in complex projective space $\mathbb{C}P^m$ with isometric *Reeb flow* has been obtained by Okumura [30]. The corresponding classification in complex hyperbolic space $\mathbb{C}H^m$ is due to Montiel and Romero [26] and in quaternionic projective space $\mathbb{H}P^m$ due to Martinez and Pérez [24] respectively.

In complex hyperbolic space $\mathbb{C}H^m$ we consider the anti-de Sitter sphere H_1^{2m-1} in $\mathbb{C}^m$, where the orbits of the *Reeb flow* induce the Hopf foliation on H_1^{2m-1} with principal S^1-bundle of time-like totally geodesic fibres. It is well known that H_1^{2m-1} is a principal S^1-bundle over a complex hyperbolic space $\mathbb{C}H^m$ with projection $\pi : H_1^{2m+1} \to \mathbb{C}H^m$. Moreover, in a paper due to Montiel and Romero [26] it was proved that the second fundamental tensor A' of a Lorentzian hypersurface in H_1^{2m-1} is parallel if and only if the corresponding hypersurface in $\mathbb{C}H^m$ has isometric *Reeb flow*, that is, $\phi A = A\phi$, where $\pi^* A = A'$, $\pi^* A$ is called a *pullback* of the shape operator A for a hypersurface in $\mathbb{C}H^m$ by the projection π and ϕ denotes the structure tensor induced from the Kähler structure J of $\mathbb{C}H^m$.

2 Compact Hermitian Symmetric Space with Rank 2

The study of real hypersurfaces in non-flat complex space forms or quaternionic space forms which belong to HSSP with rank 1 of compact type is a classical topic in differential geometry. For instance, there have been many investigations for homogeneous hypersurfaces of type A_1, A_2, B, C, D and E in complex projective space $\mathbb{C}P^m$. They are completely classified by Cecil and Ryan [10], Kimura [18] and Takagi [58]. Here, explicitly, we mention that A_1: Geodesic hyperspheres, A_2: tubes around a totally geodesic complex projective space $\mathbb{C}P^k$, B: tubes around a complex quadric Q^{m-1} and can be viewed as a tube around a real projective space $\mathbb{R}P^m$, C: tubes around the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^k$ into $\mathbb{C}P^{2k+1}$ for some $k \geq 2$, D: tubes around the Plücker embedding into $\mathbb{C}P^9$ of the complex Grassmannian manifold $G_2(\mathbb{C}^5)$ of complex 2-planes in $\mathbb{C}^5$ and E: tubes around the half spin embedding into $\mathbb{C}P^{15}$ of the Hermitian symmetric space SO_{10}/U_5.

Now let us study hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ which is a kind of HSSP with rank two of compact type. The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J.

On the other hand, Cecil and Ryan [10] proved that any tube M around a complex submanifold in complex projective space $\mathbb{C}P^m$ is characterized by the invariance of $A\xi = \alpha\xi$, where the Reeb vector ξ is defined by $\xi = -JN$ for a Kähler structure J and a unit normal N to a hypersurface M in $\mathbb{C}P^m$. Moreover, the corresponding geometrical feature for hypersurfaces in $\mathbb{H}P^m$ is the invariance of the distribution $\mathfrak{D}^\perp = \text{Span}\,\{\xi_1, \xi_2, \xi_3\}$ by the shape operator, where $\xi_i = -J_i N$, $J_i \in \mathfrak{J}$. In fact every tube around a quaternionic submanifold $\mathbb{H}P^m$ satisfies such kind of geometrical feature (See [24, 32, 34]).

From such a view point, we consider two natural geometric conditions for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, that the maximal complex subbundle $\mathscr{C}$ and the maximal quaternionic subbundle $\mathscr{Q}$ of TM are both invariant under the shape operator of M, where the maximal complex subbundle $\mathscr{C}$ of the tangent bundle TM of M is defined by $\mathscr{C} = \{X \in TM | JX \in TM\}$, and the maximal quaternionic subbundle $\mathscr{Q}$ of TM is defined by $\mathscr{Q} = \{X \in TM | \mathfrak{J}X \in TM\}$ respectively. By using such conditions and the result in Alekseevskii [1], Berndt and Suh [2] proved the following:

Theorem A *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the maximal complex subbundle $\mathscr{C}$ and the maximal quaternionic subbundle $\mathscr{Q}$ of TM are both invariant under the shape operator of M if and only if*

(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or

(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

When the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*, we have that the Reeb vector field ξ on M is Killing. Moreover, the Reeb vector field ξ is said to be *Hopf* if it is invariant by the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the *Hopf foliation* of M is totally geodesic.

By using Theorem A, in a paper due to Berndt and Suh [3] we have given a complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with isometric Reeb flow as follows:

Theorem 1 *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

3 Complex Hyperbolic Two-Plane Grassmannian $SU_{2,m}/S(U_2U_m)$

Now let us consider the case that the Riemannian manifold $\bar{M}$ becomes a Riemannian symmetric space of non compact type with rank 1 or rank 2. As some examples of non compact type with rank 1 we have a real hyperbolic space $\mathbb{R}H^m = SO^0_{1,m}/SO_m$, a complex hyperbolic space $\mathbb{C}H^m = SU_{1,m}/S(U_1U_m)$, a quaternionic hyperbolic space $\mathbb{H}H^m = Sp_{1,m}/Sp_1Sp_m$, and a Caley projective plane $\mathbb{O}P^2 = F_4/Spin_9$. The study of homogeneous hypersurfaces in such a symmetric spaces of noncompact type with rank 1 was investigated in Berndt [4], Berndt and Tamaru [8].

In this section we consider a hypersurface in HSSP of noncompact type with rank 2. Among some examples of noncompact type with rank 2 given in Sect. 2 we focus on a dual complex two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$. The Riemannian symmetric space $SU_{2,m}/S(U_2U_m)$ is a connected, simply connected, irreducible Riemannian symmetric space of noncompact type with rank 2.

Let $G = SU_{2,m}$ and $K = S(U_2U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{A}ut(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m}AI_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$$

I_2 and I_m denote the identity (2×2)-matrix and $(m \times m)$-matrix respectively. Then $< X, Y >= -B(X, \theta Y)$ becomes a positive definite $\mathrm{A}d(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2U_m)$. By construction, J commutes with each almost Hermitian structure J_ν in $\mathfrak{J}$ for $\nu = 1, 2, 3$. Recall that a canonical local basis J_1, J_2, J_3 of a quaternionic Kähler structure $\mathfrak{J}$ consists of three almost Hermitian structures J_1, J_2, J_3 in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is to be taken modulo 3. The tensor field JJ_ν, which is locally defined on $SU_{2,m}/S(U_2U_m)$, is selfadjoint and satisfies $(JJ_\nu)^2 = I$ and $tr(JJ_\nu) = 0$, where I is the identity transformation. For a nonzero tangent vector X we define $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

Then by the argument asserted in Sect. 2, we note that any homogeneous hypersurface in $SU_{2,m}/S(U_2U_m)$ becomes a tube around one singular orbit. By virtue of this fact and using geometric tools given in Helgason [12, 13], Eberlein [11], Berndt and Suh [4] proved a characterization of homogeneous hypersurfaces in $SU_{2,m}/S(U_2U_m)$ as follows:

Theorem 2 *Let M be a connected real hypersurface in the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. Then the maximal complex subbundle $\mathscr{C}$ and the maximal quaternionic subbundle $\mathscr{Q}$ of TM are both invariant under the shape operator of M if and only if M is congruent to an open part of one of the following hypersurfaces:*

(A) a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$,
(B) a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}H^n$ in $SU_{2,2}/S(U_2U_m)$, $m = 2n$,
(C) a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular.

The horosphere mentioned in Theorem 2(C) can be described as in see Fig. 1.

In this section we give a classification of all real hypersurfaces with isometric *Reeb flow* in complex hyperbolic two-plane Grassmann manifold $SU_{2,m}/S(U_2U_m)$ as follows (see Suh [47]).

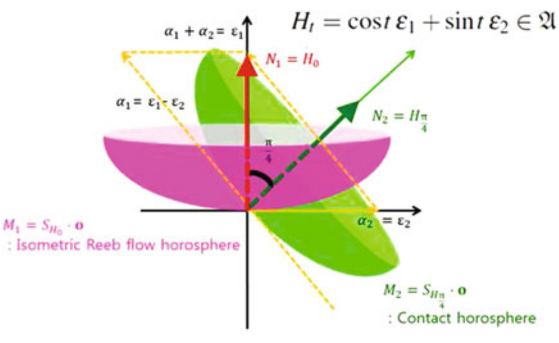


Fig. 1 Horospheres in $SU_{2,m}/S(U_2U_m)$

Theorem 3 *Let M be a connected orientable real hypersurface in the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ or a horosphere whose center at infinity is singular.*

A tube around $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ is a principal orbit of the isometric action of the maximal compact subgroup $SU_{1,m+1}$ of SU_{m+2}, and the orbits of the *Reeb flow* corresponding to the orbits of the action of U_1. The action of $SU_{1,m+1}$ has two kinds of singular orbits. One is a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ and the other is a totally geodesic $\mathbb{C}H^m$ in $SU_{2,m}/S(U_2U_m)$.

A remarkable consequence of Theorem 3 is that a connected complete real hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$ with isometric *Reeb flow* is homogeneous. This was also true in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, which could be identified with symmetric space of compact type $SU_{m+2}/S(U_2 \cdot U_m)$, as follows from the classification. It would be interesting to understand the actual reason for it (see [2, 3, 35, 43]).

4 Isometric Reeb Flow in Complex Quadric $\mathbb{Q}^m$

The homogeneous quadratic equation $z_1^2 + \cdots + z_{m+2}^2 = 0$ on $\mathbb{C}^{m+2}$ defines a complex hypersurface Q^m in the $(m+1)$-dimensional complex projective space $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. The hypersurface Q^m is known as the m-dimensional complex quadric. The complex structure J on $\mathbb{C}P^{m+1}$ naturally induces a complex structure on Q^m which we will denote by J as well. We equip Q^m with the Riemannian metric g which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The 1-dimensional quadric Q^1 is isometric to the round 2-sphere S^2. For $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of rank two and its maximal sectional curvature is equal to 4. The 2-dimensional quadric Q^2 is isometric to the Riemannian product $S^2 \times S^2$.

For a nonzero vector $z \in \mathbb{C}^{m+1}$ we denote by $[z]$ the complex span of z, that is, $[z] = \{\lambda z \mid \lambda \in \mathbb{C}\}$. Note that by definition $[z]$ is a point in $\mathbb{C}P^{m+1}$. As usual, for each $[z] \in \mathbb{C}P^{m+1}$ we identify $T_{[z]}\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus [z]$ of $[z]$ in $\mathbb{C}^{m+2}$. For $[z] \in Q^m$ the tangent space $T_{[z]}Q^m$ can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus ([z] \oplus [\bar{z}])$ of $[z] \oplus [\bar{z}]$ in $\mathbb{C}^{m+2}$. Note that $\bar{z} \in \nu_{[z]}Q^m$ is a unit normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$.

We denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to $\bar{z}$. Then we have $A_{\bar{z}}w = \overline{w}$ for all $w \in T_{[z]}Q^m$, that is, $A_{\bar{z}}$ is just complex conjugation restricted to $T_{[z]}Q^m$. The shape operator $A_{\bar{z}}$ is an antilinear involution on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (+1)-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (−1)-eigenspace of $A_{\bar{z}}$. Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$. Recall that a real structure on a complex vector space V is by definition an antilinear involution $A : V \to V$. Since the normal space $\nu_{[z]}Q^m$ of Q^m in $\mathbb{C}P^{m+1}$ at $[z]$ is a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ of complex dimension one, every normal vector in $\nu_{[z]}Q^m$ can be written as $\lambda\bar{z}$ with some $\lambda \in \mathbb{C}$. The shape operators $A_{\lambda\bar{z}}$ of Q^m define a rank two vector subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$. Since the second fundamental form of the embedding $Q^m \subset \mathbb{C}P^{m+1}$ is parallel (see e.g. [51]), $\mathfrak{A}$ is a parallel subbundle of $\mathrm{End}(TQ^m)$. For $\lambda \in S^1 \subset \mathbb{C}$ we again get a real structure $A_{\lambda\bar{z}}$ on $T_{[z]}Q^m$ and also becomes an antilinear involution as follows:

It satisfies the following for any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A^2_{\lambda\bar{z}}w &= A_{\lambda\bar{z}}A_{\lambda\bar{z}}w = A_{\lambda\bar{z}}\lambda\bar{w} \\ &= \lambda A_{\bar{z}}\lambda\bar{w} = \lambda\bar{\nabla}_{\lambda\bar{w}}\bar{z} = \lambda\bar{\lambda}\bar{\bar{w}} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. So we thus have an S^1-subbundle of $\mathfrak{A}$ consisting of real structures on the tangent spaces of Q^m.

The Gauss equation for the complex hypersurface $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor R of Q^m can be expressed in terms of the Riemannian metric g, the complex structure J and a generic real structure A in $\mathfrak{A}$:

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y \\ &+ g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &+ g(AY, Z)AX - g(AX, Z)AY \\ &+ g(JAY, Z)JAX - g(JAX, Z)JAY. \end{aligned}$$

Note that the complex structure J anti-commutes with each endomorphism $A \in \mathfrak{A}$, that is, $AJ = -JA$.

A nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a real structure $A \in \mathfrak{A}_{[z]}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a real structure $A \in \mathfrak{A}_{[z]}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

Basic complex linear algebra shows that for every unit tangent vector $W \in T_{[z]}Q^m$ there exist a real structure $A \in \mathfrak{A}_{[z]}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$.

Let M be a real hypersurface in a Kähler manifold $\bar{M}$. The complex structure J on $\bar{M}$ induces locally an almost contact metric structure (ϕ, ξ, η, g) on M. In the context of contact geometry, the unit vector field ξ is often referred to as the Reeb vector field on M and its flow is known as the Reeb flow. The Reeb flow has been of significant interest in recent years, for example in relation to the Weinstein Conjecture. We are interested in the Reeb flow in the context of Riemannian geometry, namely in the classification of real hypersurfaces with isometric Reeb flow in homogeneous Kähler manifolds.

For the complex projective space $\mathbb{C}P^m$ a full classification was obtained by Okumura in [30]. He proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_m U_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_m U_2)$ the classification was obtained by Berndt and the author in [3]. We have proved that the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. Finally, related to the isometric Reeb flow, we give a mention for our recent work due to Berndt and Suh [5]. In this lecture we want to investigate this problem for the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$. In view of the previous two results a natural expectation could involve at least the totally geodesic $Q^{m-1} \subset Q^m$. But for real hypersurfaces in Q^m with isometric *Reeb flow* the situations are quite different from the above. Now we state the following.

Theorem 4 (see [5]) *Let M be a real hypersurface in the complex quadric Q^m, $m \geq 3$. Then the Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

Every tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$ is a homogeneous hypersurface. In fact, the closed subgroup U_{k+1} of SO_{2k+2} acts on Q^{2k} with cohomogeneity one. The two singular orbits are totally geodesic $\mathbb{C}P^k \subset Q^{2k}$ and the principal orbits are the tubes around any of these two singular orbits. So as a corollary we get:

Corollary 1 *Let M be a connected complete real hypersurface in the complex quadric Q^{2k}, $k \geq 2$. If the Reeb flow on M is isometric, then M is a homogeneous hypersurface of Q^{2k}.*

It is remarkable that in this situation the existence of a particular one-parameter group of isometries implies transitivity of the isometry group. As another interesting consequence we get:

Corollary 2 *There are no real hypersurfaces with isometric Reeb flow in the odd-dimensional complex quadric Q^{2k+1}, $k \geq 1$.*

To our knowledge the odd-dimensional complex quadrics are the first examples of homogeneous Kähler manifolds which do not admit a real hypersurface with isometric Reeb flow.

5 Contact Hypersurfaces in Complex Quadric $\mathbb{Q}^m$ and Non-compact Dual $\mathbb{Q}^{m*}$

This section is a recent work due to Berndt and the author [6]. A contact manifold is a smooth $(2m-1)$-dimensional manifold M together with a one-form η satisfying $\eta \wedge (d\eta)^{m-1} \neq 0$, $m \geq 2$. The one-form η on a contact manifold is called a contact form. The kernel of η defines the so-called contact distribution $\mathscr{C}$ in the tangent bundle TM of M. Note that if η is a contact form on a smooth manifold M, then $\rho\eta$ is also a contact form on M for each smooth function ρ on M which is nonzero everywhere. The origin of contact geometry can be traced back to Hamiltonian mechanics and geometric optics. The standard example of a contact manifold is $\mathbb{R}^3$ together with the contact form $\eta = dz - y\,dx$.

Another standard example is a round sphere in an even-dimensional Euclidean space. Consider the sphere $S^{2m-1}(r)$ with radius $r \in \mathbb{R}_+$ in $\mathbb{C}^m$ and denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{C}^m$ given by $\langle z, w \rangle = \mathrm{Re} \sum_{\nu=1}^{n} z_\nu \bar{w}_\nu$. By defining $\xi_z = -\frac{1}{r} iz$ for $z \in S^{2m-1}(r)$ we obtain a unit tangent vector field ξ on $S^{2m-1}(r)$. We denote by η the dual one-form given by $\eta(X) = \langle X, \xi \rangle$ and by ω the Kähler form on $\mathbb{C}^m$ given by $\omega(X, Y) = \langle iX, Y \rangle$. A straightforward calculation shows that $d\eta(X, Y) = -\frac{2}{r}\omega(X, Y)$. Since the Kähler form ω has rank $2(m-1)$ on the kernel of η it follows that $\eta \wedge (d\eta)^{m-1} \neq 0$. Thus $S^{2m-1}(r)$ is a contact manifold with contact form η. This argument for the sphere motivates a natural generalization to Kähler manifolds.

Let $(\bar{M}, J, g)$ be a Kähler manifold of complex dimension n and let M be a connected oriented real hypersurface of $\bar{M}$. The Kähler structure on $\bar{M}$ induces an almost contact metric structure (ϕ, ξ, η, g) on M. The Riemannian metric on M is the one induced from the Riemannian metric on $\bar{M}$, both denoted by g. The orientation on M determines a unit normal vector field N of M. The so-called Reeb vector field ξ on M is defined by $\xi = -JN$ and η is the dual one form on M, that is, $\eta(X) = g(X, \xi)$. The tensor field ϕ on M is defined by $\phi X = JX - g(JX, N)N = JX - \eta(X)N$, so that ϕX is just the tangential component of JX. The tensor field ϕ determines the fundamental 2-form ω on M by $\omega(X, Y) = g(\phi X, Y)$. M is said to be a contact hypersurface if there exists an everywhere nonzero smooth function ρ on M such that $d\eta = 2\rho\omega$. It is clear that if $d\eta = 2\rho\omega$ holds then $\eta \wedge (d\eta)^{m-1} \neq 0$, that is, every contact hypersurface in a Kähler manifold is a contact manifold.

Contact hypersurfaces in complex space forms of complex dimension $m \geq 3$ have been investigated and classified by Okumura [30] (for the complex Euclidean space $\mathbb{C}^m$ and the complex projective space $\mathbb{C}P^m$) and Vernon [59] (for the complex hyperbolic space $\mathbb{C}H^m$). In this paper we carry out a systematic study of contact hypersurfaces in Kähler manifolds. We will then apply our results to the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ and its noncompact dual space $Q^{m*} = SO^o_{m,2}/SO_m SO_2$ to prove the following two classifications:

Theorem 5 (see [6]) *Let M be a connected orientable real hypersurface with constant mean curvature in the complex quadric $Q^m = SO^o_{m+2}/SO_m SO_2$ and $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of the tube of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around a real form S^m of Q^m.*

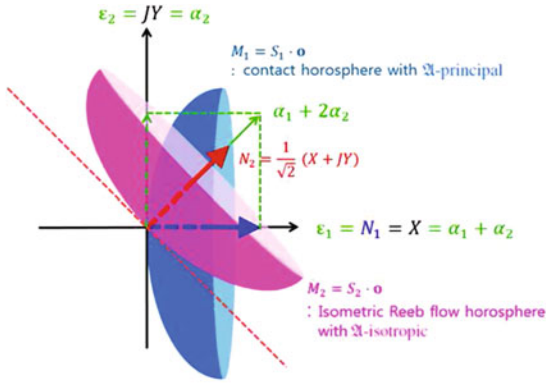


Fig. 2 Horospheres in Q^{m*}

When we consider a real hypersurface in complex hyperbolic quadric Q^{m*}, naturally we have one focal (singular) submanifold in Q^{m*}, which is different from the situation of Theorem 5. In this case we give a complete classification of contact real hypersurfaces in Q^{m*} as follows:

Theorem 6 (see [6]) *Let M be a connected orientable real hypersurface with constant mean curvature in the noncompact dual $Q^{m*} = SO^o_{m,2}/SO_m SO_2$ of the complex quadric and $m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of one of the following contact hypersurfaces in Q^{n*}:*

(i) the tube of radius $r \in \mathbb{R}_+$ around the totally geodesic $Q^{(m-1)*}$ in Q^{m*};
(ii) a horosphere in Q^{m*} whose center at infinity is determined by an $\mathfrak{A}$-principal geodesic in Q^{m*};
(iii) the tube of radius $r \in \mathbb{R}_+$ around a real form $\mathbb{R}H^m$ in Q^{m*}.

In this complex hyperbolic quadric Q^{m*} we have two kinds of horospheres. One is a horosphere in Q^{m*} with $\mathfrak{A}$-isotropic geodesic in Q^{m*} and the other is mentioned in Theorem 6(ii). Now let us explain these two kinds of horospheres in Q^{m*} in see Fig. 2.

6 Real Hypersurfaces in Complex Quadric $\mathbb{Q}^m$ with Commuting and Parallel Ricci Tensor

When we consider some Hermitian symmetric spaces of rank 2, usually we can give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [2, 4, 35, 36, 45]). Those are said to be Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$. The rank of $SU_{2,m}/S(U_2U_m)$ is 2 and there are exactly two types of singular tangent vectors X

of $SU_{2,m}/S(U_2U_m)$ which are characterized by the geometric properties $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$ respectively.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we have the example of complex quadric $Q^m = SO_{m+2}/SO_2SO_m$, which is a complex hypersurface in complex projective space $\mathbb{C}P^m$ (see Berndt and Suh [3], and Smyth [40]). The complex quadric also can be regarded as a kind of real Grassmann manifold of compact type with rank 2 (see Kobayashi and Nomizu [23]). Accordingly, the complex quadric admits two important geometric structures as a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [16] and Reckziegel [37]).

Apart from the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m. Here the notion of parallel vector bundle $\mathfrak{A}$ means that $(\bar{\nabla}_X A)Y = q(X)JAY$ for any vector fields X and Y on Q^m, where $\bar{\nabla}$ and q denote a connection and a certain 1-form defined on T_zQ^m, $z \in Q^m$ respectively.

For the complex projective space $\mathbb{C}P^m$ and the quaternionic projective space $\mathbb{H}P^m$ some characterizations was obtained by Okumura [30], and Pérez and Suh [34] respectively. In particular Okumura [29] proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. Here the isometric Reeb flow means that $\mathcal{L}_\xi g = 0$ for the Reeb vector field $\xi = -JN$, where N denotes a unit normal vector field of M in $\mathbb{C}P^m$. Moreover, in [47] we have asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$.

By the Kähler structure J of the complex quadric Q^m, we can transfer any tangent vector fields X on M in Q^m as follows:

$$JX = \phi X + \eta(X)N,$$

where $\phi X = (JX)^T$ denotes the tangential component of JX and N a unit normal vector field on M in Q^m.

When the Ricci tensor Ric of M in Q^m commutes with the structure tensor ϕ, that is, $Ric\cdot\phi = \phi\cdot Ric$, M is said to be *Ricci commuting*. When the Ricci tensor Ric of M in Q^m is parallel, that is, $\nabla Ric = 0$, let us say M has a *parallel Ricci tensor*. Then first with the notion of *commuting Ricci tensor* for a hypersurface M in the complex quadric Q^m, we can prove the following

Theorem 7 (see [53]) *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$, with commuting Ricci tensor. Then the unit normal vector field N of M is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

In the first class where M has an $\mathfrak{A}$-isotropic unit normal N, we have asserted in Berndt and the author [5] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k} if the shape operator commutes with the structure tensor, that is $S{\cdot}\phi = \phi{\cdot}S$. In the second class for N $\mathfrak{A}$-principal we have proved that M is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m if M is a contact hypersurface, that is, $S\phi + \phi S = k\phi$, $k{\neq}0$ constant (see [6]).

We now assume that M is a Hopf hypersurface. Then the shape operator S of M in Q^m satisfies

$$S\xi = \alpha\xi$$

for the Reeb vector field ξ and the Reeb function $\alpha = g(S\xi, \xi)$ on M in Q^m. Then in this section we give a complete classification for real hypersurfaces in the complex quadric Q^m with commuting and parallel Ricci tensor as follows:

Theorem 8 (see [53]) *There do not exist any Hopf real hypersurfaces in the complex quadric Q^m, $m{\geq}4$, with commuting and parallel Ricci tensor.*

Now let us consider an Einstein hypersurface in complex quadric Q^m. Then the Ricci tensor of type (1, 1) on M becomes $Ric = \lambda I$, where λ is constant on M and I denotes the identity tensor on M. Accordingly, the Ricci tensor is parallel and commuting, that is $\mathrm{Ric}{\cdot}\phi = \phi{\cdot}\mathrm{Ric}$. Moreover, M has an $\mathfrak{A}$-isotropic unit normal vector field N in Q^m. So we assert a corollary as follows:

Corollary 3 *There do not exist any Hopf Einstein real hypersurfaces in the complex quadric Q^m, $m \geq 4$.*

7 Real Hypersurfaces in Complex Quadric $\mathbb{Q}^m$ with Parallel Ricci Tensor

For the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{Q}P^{m+1}$ some classifications related to parallel Ricci tensor were investigated in Kimura [19], and Pérez [33], respectively. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_m U_2)$ a new classification was obtained by Berndt and Suh [2]. By using this classification Pérez and Suh [35] proved a non-existence property for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. Suh [45] strengthened this result to hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor. Moreover, Suh and Woo [57] studied another non-existence property for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 U_m)$ with parallel Ricci tensor.

When we consider a hypersurface M in the complex quadric Q^m, the unit normal vector field N of M in Q^m can be divided into two classes if either N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [3, 4, 49]). In the first case where N is $\mathfrak{A}$-isotropic, we have shown in [3] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [4]). Now we consider the notion of *Ricci parallelism* for hypersurfaces in Q^m, that is, $\nabla Ric = 0$. Then motivated by the result obtained when N is $\mathfrak{A}$-principal for contact hypersurfaces in Q^m, we assert the following:

Theorem 9 (see [51]) *There does not exist any Hopf hypersurfaces in the complex quadric Q^m with parallel Ricci tensor and $\mathfrak{A}$-principal normal vector field.*

For a case where the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^{\perp} = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^{\perp} = \text{Span}\{A\xi, AN\}$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then motivated by Theorem 9, in this section we give another theorem for real hypersurfaces in the complex quadric Q^m with parallel Ricci tensor and $\mathfrak{A}$-isotropic unit normal as follows:

Theorem 10 (see [51]) *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with parallel Ricci tensor and $\mathfrak{A}$-isotropic unit normal N. If the shape operator commutes with the structure tensor on the distribution $\mathcal{Q}^{\perp}$, then M has 3 distinct constant principal curvatures given by*

$$\alpha = \sqrt{\frac{2m-1}{2}},\ \gamma = 0,\ \lambda(=\alpha) = \sqrt{\frac{2m-1}{2}},\ \lambda = 0 \text{ and } \mu = -\frac{2\sqrt{2}}{\sqrt{2m-1}}$$

with corresponding principal curvature spaces

$$T_\alpha = [\xi],\ T_\gamma = [A\xi, AN],\ \phi(T_\lambda) = T_\mu,\ \dim T_\lambda = \dim T_\mu = m-2.$$

8 Real Hypersurfaces in Complex Quadric $\mathbb{Q}^m$ with Harmonic Curvature

Usually, for a Riemannian manifold (N, g) the Ricci tensor Ric can be regarded as a 1-form with values in the cotangent bundle T^*N. Then a Riemannian manifold N is said to have *harmonic curvature* or *harmonic Weyl tensor*, if Ric_N or $Ric_N - r_N g_N/2(n-1)$ for the scalar curvature r is a Codazzi tensor, that is, it satisfies

$$dRic = 0 \quad \text{or} \quad d\{Ric - rg/2(n-1)\} = 0,$$

where d denotes the exterior differential. For the harmonic Weyl tensor, it is seen that in the case of $n \geq 4$ the Weyl curvature tensor W which is regarded as a 2-form with values in the bundle $\Lambda^2 T^* N$ is closed and coclosed, namely it is harmonic. In the case of $n = 3$ the Riemannian manifold N is confomally flat (See Besse [9]).

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms it can be easily checked that there does not exist any real hypersurface with parallel shape operator A by virtue of the equation of Codazzi.

From this point of view many differential geometers have considered a notion weaker than the parallel Ricci tensor, that is, $\nabla Ric = 0$. In particular, Kwon and Nakagawa [22] have proved that there are no Hopf real hypersurfaces M in a complex projective space $\mathbb{C}P^m$ with *harmonic curvature*, that is, $(\nabla_X Ric)Y = (\nabla_Y Ric)X$ for any X, Y in M. Moreover, Ki, Nakagawa and Suh [17] have also proved that there are no real hypersurface with *harmonic Weyl tensor* in non-flat complex space forms $M_n(c)$, $c \neq 0$, $n \geq 3$.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the above situation is not so simple if we consider a real hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. Suh [45] has shown that there does not exist any hypersurface in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor, that is, $\nabla Ric = 0$, and have investigated the problem related to the Reeb parallel Ricci tensor Ric for real hypersurfaces M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, that is, $\nabla_\xi Ric = 0$ for the Reeb vector field ξ tangent to M (See [46]).

In the proof of Theorem A we proved that the 1-dimensional distribution $[\xi]$ is contained in either the 3-dimensional distribution $\mathfrak{D}^\perp$ or in the orthogonal complement $\mathfrak{D}$ such that $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$. The case (A) in Theorem A is just the case that the 1- dimensional distribution $[\xi]$ is contained in the distribution $\mathfrak{D}^\perp$. Of course, it is not difficult to check that the Ricci tensor of any real hypersurface mentioned in Theorem A is not parallel. Then it is a natural question to ask whether real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with conditions weaker than parallel Ricci tensor can exist or not.

From such a view point, Besse [9] has introduced the notion of harmonic curvature which is given by $\triangle Ric = (d\delta + \delta d)Ric = 0$ for the Ricci tensor Ric. Then the notion of harmonic curvature is equivalent to $\delta Ric = 0$, because $dRic = 0$ always holds from the contraction of the 2nd Bianchi identity.

Then a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with harmonic curvature satisfies

$$(\nabla_X Ric)Y = (\nabla_Y Ric)X$$

for any tangent vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$.

But considering real hypersurfaces of *harmonic curvature* in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, the situation is quite different from the complex projective space $\mathbb{C}P^m$. Instead of the non-existence results in $\mathbb{C}P^m$, we [48] gave a classification of all Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with harmonic or Weyl harmonic tensor. First for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with harmonic curvature tensor, we asserted the following:

Theorem B *Let M be a Hopf real hypersurface of harmonic curvature in $G_2(\mathbb{C}^{m+2})$ with constant scalar and mean curvatures. If the shape operator commutes with the structure tensor on the distribution $\mathfrak{D}^\perp$, then M is locally congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius r,* $\cot^2\sqrt{2}r = \frac{4}{3}(m-1)$.

On the other hand, a $(4m-1)$-dimensional real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to have *harmonic Weyl tensor* if $\Delta W = 0$ for Weyl curvature tensor W defined by $W = Ric - rg/4(2m-1)$, where Ric and r denotes respectively the Ricci tensor and the scalar curvature of M in $G_2(\mathbb{C}^{m+2})$. Then from the 2nd Bianchi identity this is equivalent to $\delta W = 0$, that is, $(\nabla_X W)Y = (\nabla_Y W)X$. Naturally it means that

$$(\nabla_X Ric)Y - (\nabla_Y Ric)X = \{dr(X)Y - dr(Y)X\}/4(2m-1).$$

Now we consider the notion of *harmonic curvature* for hypersurfaces in Q^m, that is, $(\nabla_X Ric)Y = (\nabla_Y Ric)X$ for any vector fields X and Y on M in Q^m. Then motivated by the result in the case of $\mathfrak{A}$-principal normal for contact hypersurfaces in Q^m, we assert the following

Theorem 11 (see [52]) *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with harmonic curvature. If the unit normal N is $\mathfrak{A}$-principal, then M has at most 5 distinct constant principal curvatures, five of which are given by*

$$\alpha, \quad \lambda_1, \quad \mu_1, \quad \lambda_2, \quad \textit{and} \quad \mu_2$$

with corresponding principal curvature spaces

$$T_\alpha = [\xi], \quad \phi T_{\lambda_1} = T_{\mu_1}, \quad \phi T_{\lambda_2} = T_{\mu_2},$$

$$\dim T_{\lambda_1} + \dim T_{\lambda_2} = m-1, \quad \dim T_{\mu_1} + \dim T_{\mu_2} = m-1.$$

Here four roots λ_i and μ_i, $i = 1, 2$ satisfy the quadratic equation

$$2x^2 - 2\beta x + 2 + \alpha\beta = 0,$$

where the function β is denoted by $\beta = \frac{\alpha^2+1\pm\sqrt{(\alpha^2+1)^2+4\alpha h}}{\alpha}$ and the function h denotes the mean curvature of M in Q^m.

Now at each point $z \in M$ let us consider a maximal $\mathfrak{A}$-invariant subspace $\mathscr{Q}_z$ of T_zM, $z \in M$, defined by

$$\mathscr{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}$$

of T_zM, $z \in M$. Thus if the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathscr{Q}_z^\perp = \mathscr{C}_z \ominus \mathscr{Q}_z$, $z \in M$, of the distribution $\mathscr{Q}$ in the complex subbundle $\mathscr{C}$, becomes $\mathscr{Q}_z^\perp = \text{Span}\,[A\xi, AN]$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then motivated by the above result, we have another theorem for real hypersurfaces in the complex quadric Q^m with harmonic curvature and $\mathfrak{A}$-isotropic unit normal vector field as follows:

Theorem 12 (see [52]) *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with harmonic curvature and $\mathfrak{A}$-isotropic unit normal N. If the shape operator commutes with the structure tensor on the distribution $\mathcal{Q}^{\perp}$, then M is locally congruent to an open part of a tube around k-dimensional complex projective space $\mathbb{C}P^k$ in Q^m, $m = 2k$, or M has at most 6 distinct constant principal curvatures given by*

$$\alpha, \quad \gamma = 0(\alpha), \quad \lambda_1, \quad \mu_1, \quad \lambda_2 \quad \textit{and} \quad \mu_2$$

with corresponding principal curvature spaces

$$T_\alpha = [\xi], \quad T_\gamma = [A\xi,\ AN], \quad \phi(T_{\lambda_1}) = T_{\mu_1}, \quad \phi T_{\lambda_2} = T_{\mu_2}.$$

$$\dim T_{\lambda_1} + \dim T_{\lambda_2} = m - 2, \quad \dim T_{\mu_1} + \dim T_{\mu_2} = m - 2.$$

Here four roots λ_i and μ_i, $i = 1, 2$ satisfy the equation

$$2x^2 - 2\beta x + 2 + \alpha\beta = 0,$$

where the function β denotes $\beta = \frac{\alpha^2+2\pm\sqrt{(\alpha^2+2)^2+4\alpha h}}{\alpha}$ and the function h denotes the mean curvature of M in Q^m. In particular, $\alpha = \sqrt{\frac{2m-1}{2}}$, $\gamma(=\alpha) = \sqrt{\frac{2m-1}{2}}$, $\lambda = 0$, $\mu = -\frac{2\sqrt{2}}{\sqrt{2m-1}}$, with multiplicities 1, 2, $m-2$ and $m-2$ respectively.

The particular case mentioned in Theorem 12 can occur for real hypersurfaces in Q^m with parallel Ricci tensor, that is, $\nabla Ric = 0$. Naturally harmonic curvature $\delta Ric = 0$ includes the notion of Ricci parallelism.

9 Real Hypersurfaces in Complex Quadric $\mathbb{Q}^m$ with Commuting Ricci Tensor

In the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{H}P^{m+1}$ some classifications related to commuting Ricci tensor or commuting structure Jacobi operator were investigated by Kimura [18, 19], Pérez [32] and Pérez and Suh [34, 35] respectively. Under the invariance of the shape operator along some distributions a new classification in the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ was investigated. By using this classification Pérez and Suh [35] proved a non-existence property for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. Recently, Hwang, Lee and Woo [14] considered the notion of semi-parallelism with respect to some symmetric operators, that is, shape operator and structure (or normal) Jacobi operator, and obtained a complete classifications for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with such operators. Moreover, Suh [43] strengthened this result to hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor and gave a characterization of real hypersurfaces in $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_m U_2)$ as follows:

Theorem C *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Moreover, Suh [50] studied another classification for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 U_m)$ with commuting Ricci tensor as follows:

Theorem D *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 U_m)$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2 U_{m-1})$ in $SU_{2,m}/S(U_2 U_m)$ or a horosphere whose center at infinity with $JX \in \mathfrak{J}X$ is singular.*

It is known that the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. Corresponding to this result, in [47] we asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2 U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2 U_{m-1}) \subset SU_{2,m}/S(U_2 U_m)$. Here, the Reeb flow on real hypersurfaces in $SU_{m+2}/S(U_m U_2)$ or $SU_{2,m}/S(U_2 U_m)$ is said to be *isometric* if the shape operator commutes with the structure tensor. In papers due to Berndt and Suh [5] and Suh [51], we have introduced this problem for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ and obtained Theorem 4 in Sect. 4.

From the assumption of harmonic curvature, it was impossible to derive the fact that either the unit normal N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So in [52] we gave a complete classification with the further assumption of $\mathfrak{A}$-isotropic as in Theorem D. For the case where the unit normal vector field N is $\mathfrak{A}$-principal we have proved that real hypersurfaces in Q^m with harmonic curvature can not exist.

But fortunately when we consider Ricci commuting, that is, $\mathrm{Ric}\cdot\phi = \phi\cdot\mathrm{Ric}$ for hypersurfaces M in Q^m, we can assert that the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. Then motivated by such a result and using Theorem C, we have a complete classification for real hypersurfaces in the complex quadric Q^m with commuting Ricci tensor, that is, $\mathrm{Ric}\cdot\phi = \phi\cdot\mathrm{Ric}$ as follows:

Theorem 13 (see [55]) *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with commuting Ricci tensor. If the shape operator commutes with the structure tensor on the distribution $\mathcal{Q}^{\perp}$, then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$ or M has 3 distinct constant principal curvatures given by*

$$\alpha = \sqrt{2(m-3)},\ \gamma = 0,\ \lambda = 0,\ and\ \mu = -\frac{2}{\sqrt{2(m-3)}}\ or$$

$$\alpha = \sqrt{\frac{2}{3}(m-3)},\ \gamma = 0,\ \lambda = 0,\ and\ \mu = -\frac{\sqrt{6}}{\sqrt{m-3}}$$

with corresponding principal curvature spaces respectively

$$T_\alpha = [\xi],\ T_\gamma = [A\xi,\ AN],\ \phi(T_\lambda) = T_\mu,\ and\ \dim T_\lambda = \dim T_\mu = m-2.$$

Remark 1 In Theorem 13 the second and the third ones can be explained geometrically as follows: the real hypersurface M is locally congruent to $M_1 \times \mathbb{C}$, where M_1 is a tube of radius $r = \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{m-3}$ or respectively, of radius $r = \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{\frac{m-3}{3}}$, around $(m-1)$-dimensional sphere S^{m-1} in Q^{m-1}. That is, M_1 is a contact hypersurface defined by $S\phi + \phi S = k\phi$, $k = -\frac{2}{\sqrt{2(m-3)}}$, and $k = -\frac{\sqrt{6}}{\sqrt{m-3}}$ respectively (see Suh [55]). By the Segre embedding, the embedding $M_1 \times \mathbb{C} \subset Q^{m-1} \times \mathbb{C} \subset Q^m$ is defined by $(z_0, z_1, \ldots, z_m, w) \to (z_0 w, z_1 w, \ldots, z_m w, 0)$. Here $(z_0 w)^2 + (z_1 w)^2 + \cdots + (z_m w)^2 = (z_0^2 + \cdots + z_m^2) w^2 = 0$, where $\{z_0, \ldots, z_m\}$ denotes a coordinate system in Q^{m-1} satisfying $z_0^2 + \cdots + z_m^2 = 0$.

10 Pseudo-Einstein Real Hypersurfaces in Complex Quadric $\mathbb{Q}^m$

In complex space forms or in quaternionic space forms many differential geometers have discussed real Einstein hypersurfaces, complex Einstein hypersuraces or more generally real hypersurfaces with parallel Ricci tensor, that is $\nabla Ric = 0$, where ∇ denotes the Riemannian connection of M (see Cecil-Ryan [10], Kimura [18, 19], Romero [38, 39] and Martinez and Pérez [24]).

From such a view point Kon [20] has considered the notion of pseudo-Einstein real hypersurfaces M in complex projective space $\mathbb{C}P^m$ with Kähler structure J, which are defined in such a way that

$$Ric(X) = aX + b\eta(X)\xi,$$

where a, b are constants, $\eta(X) = g(\xi, X)$ and $\xi = -JN$ for any tangent vector field X and a unit normal vector field N defined on M. In [20] Kon has also given a complete classification of pseudo-Einstein real hypersurfaces in $\mathbb{C}P^m$ by using the work of Takagi [58] and proved that there do not exist Einstein real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$. Moreover, Kon [21] has considered a new notion of the Ricci tensor $\hat{Ric}$ in the generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$.

The notion of pseudo-Einstein was generalized by Cecil-Ryan [10] to any smooth functions a and b defined on M. By using the theory of tubes, Cecil-Ryan [10] have given a complete classification of such pseudo-Einstein real hypersurfaces and proved that there do not exist Einstein real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$.

On the other hand, Montiel [25] considered pseudo-Einstein real hypersurfaces in complex hyperbolic space $\mathbb{C}H^m$ and gave a complete classification of such hypersurfaces and also proved that there do not exist Einstein real hypersurfaces in $\mathbb{C}H^m$, $m \geq 3$.

For real hypersurfaces in quaternionic projective space $\mathbb{H}P^m$ the notion of pseudo Einstein was considered by Martinez and Pérez [24]. But in [32] Pérez proved that the unique Einstein real hypersurfaces in $\mathbb{H}P^m$ are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$ and $\cot^2 r = \frac{1}{2m}$.

The situation mentioned above is not so simple if we consider a real hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *pseudo-Einstein* if the Ricci tensor Ric of M satisfies

$$Ric(X) = aX + b\eta(X)\xi + c\sum_{i=1}^{3}\eta_i(X)\xi_i$$

for any constants a, b and c on M. In a paper due to Pérez, Suh and Watanabe [36] we have defined the notion of pseudo-Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with the assumption that b and c are *non-vanishing constants*. In this case the meaning of pseudo-Einstein is *proper pseudo-Einstein*. So in [36] we have given a complete classification of *proper Hopf* pseudo-Einstein as follows.

Theorem E *Let M be a pseudo-Einstein Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then M is congruent to*

(a) a tube of radius r, $\cot^2\sqrt{2}r = \frac{m-1}{2}$, over $G_2(\mathbb{C}^{m+1})$, where $a = 4m+8$, $b+c = -2(m+1)$, provided that $c \neq -4$.
(b) a tube of radius r, $\cot r = \frac{1+\sqrt{4m-3}}{2(m-1)}$, over $\mathbb{H}P^m$, $m = 2n$, where $a = 8n+6$, $b = -16n+10$, $c = -2$.

For the real hypersurfaces of type (a) or of type (b) in Theorem E the constants b and c of pseudo-Einstein real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ never vanish at the same time on M, that is, at least one of them is non-vanishing at any point of M. As a direct consequence of Theorem E, we have also asserted that there are no Einstein Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Now let us consider the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ which is a Kähler manifold and a kind of Hermitian symmetric space of rank 2. For real hypersurfaces M in the complex quadric Q^m we have classified the isometric Reeb flow which is defined by $\mathscr{L}_\xi g = 0$, where $\mathscr{L}_\xi$ denotes a Lie derivative along the Reeb direction ξ. The Lie invariant $\mathscr{L}_\xi g = 0$ along the direction ξ is equivalent to the commuting shape operator S of M in Q^m, that is, $S\phi = \phi S$. In order to give a complete classification of pseudo-Einstein hypersurfaces in the complex quadric Q^m we need the classification of isometric Reeb flow in a theorem due to Berndt and Suh [5].

Motivated by above two Theorems G and Theorem 5 in Sect. 5, let us consider a new notion of pseudo-Einstein for real hypersurfaces in the complex quadric Q^m. When the Ricci tensor Ric of a real hypersurface M in Q^m satisfies

$$Ric(X) = aX + b\eta(X)\xi,$$

for constants $a, b \in \mathbb{R}$ and the Reeb vector field $\xi = -JN$, then M is said to be *pseudo-Einstein*.

First, we obtained that any pseudo-Einstein real hypersurfaces in the complex quadric Q^m satisfies the following property

Theorem 14 *Let M be a pseudo-Einstein real hypersurface in the complex quadric Q^m, $m \geq 3$. Then the unit normal vector field N of M is singular, that is, N is either $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic.*

Theorem 15 (see [41]) *Let M be a pseudo-Einstein Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then M is locally congruent to one of the following:*

(i) M is an open part of a tube of radius r around a totally real and totally geodesic m-dimensional unit sphere S^m in Q^m, with $a = 2m$, and $b = -2m$.

(ii) $m = 2k$, M is an open part of a tube of radius r, $r = \cot^{-1}\sqrt{\frac{k}{k-1}}$ around a totally geodesic k-dimensional complex projective space $\mathbb{C}P^k$ in Q^{2k} with $a = 4k$ and $b = -4 + \frac{2}{k}$.

Now let us consider an Einstein hypersurface in the complex quadric Q^m. Then the Ricci tensor of M becomes $Ric = \lambda g$. In case (i) in above Theorem 15, there do not exist any Einstein hypersurfaces in Q^m, because $b = -2m$ is non-vanishing. In this case, the unit normal N on M is $\mathfrak{A}$-principal.

Moreover, in (ii), if M is assumed to be Einstein, then the constant should be $b = 0$. This gives $4 = \frac{2}{k}$, which implies a contradiction. In this case M has an $\mathfrak{A}$-isotropic unit normal vector field N in Q^m. So we conclude a corollary as follows:

Corollary 4 (see [41]) *There do not exist any Einstein Hopf real hypersurfaces in the complex quadric Q^m, $m \geq 3$.*

11 Pseudo-anti Commuting Ricci Tensor and Ricci Soliton in Complex Quadric $\mathbb{Q}^m$

When the Ricci tensor S commutes or anti-commutes with the structure tensor ϕ such as $S\phi = \phi S$ or $S\phi = -\phi S$, the Ricci tensor is said to be commuting or anti-commuting respectively. Motivated by such notion of commuting and anti-commuting Ricci tensor, we consider a new notion of *pseudo-anti commuting Ricci tensor* which was well introduced in a paper due to Jeong and Suh [15]. It is defined by

$$Ric{\cdot}\phi + \phi{\cdot}Ric = \kappa\phi, \quad \kappa \neq 0 : \text{constant},$$

where the structure tensor ϕ is induced from the Kähler structure J of Hermitian symmetric space.

It is known that Einstein, pseudo-Einstein real hypersurfaces in the sense of Besse [9], Kon [22], and Cecil and Ryan [10], satisfy the condition of pseudo-anti commuting. Real hypersurfaces of type (B) in $\mathbb{C}P^m$, which are characterized by $S\phi + \phi S = k\phi$, $k \neq 0$ and tubes over a totally real totally geodesic real projective space $\mathbb{R}P^n$, $m = 2n$, satisfy the formula of pseudo-anti commuting (see Yano and Kon [60]). Moreover, it can be easily checked that Einstein hyperspaces and some special kind of pseudo Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$, and hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$, which are tubes over a totally real totally geodesic quaternionic projective space $\mathbb{H}P^n$, $m = 2n$, satisfy this formula (see Pérez, Suh and Watanabe [36], Suh [42] and [45]).

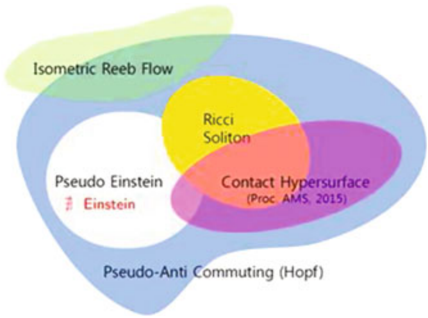
Recently, we have known that a solution of the Ricci flow equation $\frac{\partial}{\partial t}g(t) = -2Ric(g(t))$ is given by

$$\frac{1}{2}(\mathfrak{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) = \rho g(X, Y),$$

where ρ is a constant and $\mathfrak{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Morgan and Tian [27]). Then the solution is said to be a *Ricci soliton* with potential vector field V and Ricci soliton constant ρ, and surprisingly, it satisfies the pseudo-anti commuting condition $S\phi + \phi S = \kappa\phi$, where $\kappa = 2\rho$ is non-zero constant (Fig. 3).

In the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, Jeong and Suh [15] gave a classification of Ricci solitons for real hypersurfaces. From such a view point, we want to give a complete classification of pseudo-anti commuting Hopf real hypersurfaces in the complex quadric Q^m. In order to do this we want to introduce some backgrounds for the study of real hypersurfaces in Hermitian symmetric spaces including complex projective space $\mathbb{C}P^m = SU_{m+1}/S(U_1U_m)$, complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$, complex hyperbolic two-plane Grassmannian $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2U_m)$ and complex quadric Q^m.

Fig. 3 Pseudo-anti commuting



Theorem 16 (see [54]) *Let M be a pseudo-anti commuting Hopf real hypersurfaces in the complex quadric Q^m, $m \geq 3$. Then M is locally congruent to one of the following:*

(i) M is an open part of a tube of radius r, $0 < r < \frac{\pi}{2\sqrt{2}}$, around a totally real and totally geodesic m-dimensional unit sphere S^m in Q^m, with $\mathfrak{A}$-principal unit normal vector field.

(ii) M is an open part of a tube of radius r, $0 < r < \frac{\pi}{2}$, $r \neq \frac{\pi}{4}$, around a totally geodesic k-dimensional complex projective space $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$. Here the unit normal N is $\mathfrak{A}$-isotropic.

Here we note that the unit normal N is said to be $\mathfrak{A}$-Principal if N is invariant under the complex conjugation A, that is, $AN = N$. When we consider the Ricci soliton (M, g, ξ, ρ) on a real hypersurface in the complex quadric Q^m, it can be easily checked that the Ricci soliton (M, g, ξ, ρ) satisfies the condition of pseudo-anti commuting, that is, $Ric \cdot \phi + \phi \cdot Ric = \kappa\phi, \kappa = 2\rho \neq 0$ constant. So, naturally the classification result in Theorem 16 can be used to study Ricci solitons (M, g, ξ, ρ). Then by virtue of Theorems 4, 5 and 16 we can assert another theorem on Ricci solitons as follows:

Theorem 17 (see [54]) *Let (M, g, ξ, ρ) be a Ricci soliton on a Hopf real hypersurface in the complex quadric Q^m, $m \geq 3$. Then M is locally congruent to one of the following:*

(i) M is an open part of a tube of radius r around a totally real and totally geodesic m-dimensional unit sphere S^m in Q^m, with radii $r = \frac{1}{\sqrt{2}} \cot^{-1}\left(\frac{1}{2\sqrt{2}(m-1)}\right)$ and $r = \frac{1}{\sqrt{2}} \cot^{-1}\left(\frac{1}{2\sqrt{2}m}\right)$. Here the unit normal N is $\mathfrak{A}$-principal.

(ii) M is an open part of a tube of radius $r = \tan^{-1}\sqrt{\frac{k}{k-1}}$ around a totally geodesic k-dimensional complex projective space $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$. Here the unit normal N is $\mathfrak{A}$-isotropic.

Acknowledgements The present author would like to express his sincere gratitude to the referee for his careful reading of our manuscript and giving some comments. This work is supported by NRF-2015-R1A2A1A-01002459 from National Research Foundations of KOREA.

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Y.J. Suh et al. (eds.), *Hermitian–Grassmannian Submanifolds*,
Springer Proceedings in Mathematics & Statistics 203,
DOI 10.1007/978-981-10-5556-0