Chapter 4
Supporting Students' Productive Collaboration and Mathematics Learning in Online Environments

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Abstract Digital technologies provide a wide range of tools and functions that can support students' learning of mathematics as well as the development of their mathematical and collaborative practices. Bringing such technologies to mathematics classrooms often do not parallel students' previous classroom experiences, especially when collaborative practices are emphasized. When facilitating mathematics learning, discrepancies between students' previous classroom experiences and their expected engagement with new collaborative technologies result in challenges to which teachers need to attend. In this chapter, we describe how a high school mathematics teacher engaged his students in an online collaborative environment, Virtual Math Team with GeoGebra (VMTwG), and how he addressed students' technological and collaborative challenges to support growth in their geometrical understanding. From a cultural historical perspective, we present a model of how teachers can support students' instrumentation of collaborative environments and mathematical understanding. In our model, during a mathematical activity, teachers progressively decentralize their role and, simultaneously, support students' development and performance of collaborative practices. This model informs the theory of instrumental orchestration (Trouche L, Interact Comput 15(6):783–800, 2003; Trouche L, Int J Comput Math Learn 9(3):281–307, 2004; Trouche L, Instrumental genesis, individual and social aspects. The didactical challenge of symbolic calculators. Springer,

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New York, pp 197–230, 2005) by providing a pedagogical intervention trajectory that supports students' instrumental genesis (Rabardel P, Beguin P, Theor Issues in Ergon Sci 6(5): 29–461, 2005) of collaborative mathematical environments and shifts students' focus from their teacher to their peer collaborators.

4.1 Introduction

The functionalities and tools of Web 2.0 applications offer potential support for mathematics learning by providing virtual spaces for individuals to perform collaborative and mathematical practices. Mathematical practices (Common Core State Standards Initiative, 2010) can be performed and made visible with dynamic mathematics software such as dynamic geometry environments (DGEs). These environments afford learners' abilities to construct, visualize, and manipulate geometric objects and relations and dependencies. These affordances support empirical explorations and theoretical justifications or proofs (Christou, Mousoulides, & Pittalis, 2004). In DEGs, empirical explorations are experienced immediately, while the need to formulate proofs is latent and to be realized requires either learners' disposition toward justification or pedagogical intervention. Pedagogically motivated transitions from empirical explorations to theoretical justifications depend on carefully designed tasks, teacher guidance, and classroom climates that support conjecturing and deductive justifications (Öner, 2008).

Conjecturing and deductive reasoning or formal proofs have been regarded as the pinnacle of geometry education (Wu, 1996). Students taking a formal geometry course at the high school level are expected to construct (in Euclidean sense) geometric objects and use the relations among objects (or parts of objects) to prove why certain properties or relations are true (Common Core State Standards Initiative, 2010). In contrast, at the middle school level, students are primarily expected to solve basic geometric problems (numerical and algebraic) using given formulas, and at best, they may be expected to describe or verify properties through experiments (Common Core State Standards Initiative, 2010). Noticeably, students are not expected to provide arguments for the properties and relations; however, at the high school level, this expectation changes dramatically. Without prior experiences justifying mathematical statements, this dramatic change causes difficulties for students to understand basic tenets of mathematical proofs (Miyazaki, Fujita, & Jones, 2016). One objective of STEM education concerns helping students develop meaningful use of tools to investigate phenomena and construct viable arguments. To address this objective in mathematics education, teachers need to support students' explorations and thinking about mathematical objects and relations among them. DGEs, uniquely designed to promote explorations, can be used to transition middle school students from the current geometric-properties focused learning to relational reasoning focused learning (Jones, 2000). This relational understanding is what enables students to move away from empirical explanations toward deductive arguments. In addition to geometric constructions, DGEs provide seamless access to both graphical and algebraic representations as well as present immediate, visual feedback. Appropriate and strategic use of DGEs as vehicles for representing mathematical situations supports STEM education in mathematics classrooms.

Learning environments that support conjecturing and deductive reasoning can be virtual as well as presential, focused on the individual or collaborative groups. Support for social conjecturing and justification can occur in computer-supported collaborative learning (CSCL) environments (Öner, 2008; Silverman, 2011). Longitudinal investigations suggest that learners' dispositions toward conjecturing and deductive reasoning can emerge from collaborative interactions among learners in online environments (Alqahtani, 2016; Alqahtani & Powell, 2016, 2017; Stahl, 2015). However, in such CSCL settings, mathematics education researchers and mathematics teachers remain unsure of how to orchestrate students' instrumentation of collaborative environments so as to support students' mathematical practices and movement between exploration and deductive justifications. Knowing how to orchestrate and promote this movement will enable mathematics education researchers and mathematics teachers to realize the potential of DGEs to improve geometry learning and of CSCL environments to engage learners in developing mathematical ideas through online collaboration that parallel the real-world online, collaborative work of mathematicians, including Fields Medal recipients (Alagic & Alagic, 2013).

In this chapter, reporting from a larger iterative project,[1] informed by design-based research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), we address practical and theoretical challenges concerning the orchestration of students' collaborative mathematical interactions in an online environment. After positioning our work in the literature and presenting our conceptual framework, we describe an online environment for collaborative learning, called Virtual Math Team with GeoGebra (VMTwG). Following these, we illustrate the case of a teacher, working with early high school students (15-year-olds), whose pedagogical orchestrations shape students' movement between exploration and deductive justification by focusing on students' collaborative practices. We understand pedagogical orchestrations to be instructional actions initiated by teachers that precede, invite, sustain, monitor, or reflect on students' activity. By movement between exploration and deductive justifications, we mean discursive, recursive trajectories in which students are motivated by mathematical relations that they notice while manipulating mathematical objects to develop and communicate convincing arguments about the relations that satisfy their peers. Finally, we propose a model of how teachers can support students' instrumentation of collaborative environments and mathematical understanding. In our model, during a mathematical activity, teachers progressively decentralize their role and, simultaneously, support students' development and performance of collaborative practices. This model informs the theory of instrumental

[1] The project—Computer-Supported Math Discourse among Teachers and Students—is an NSF-funded collaboration among researchers affiliated with The Math Forum at the National Council of Teachers of Mathematics (NCTM) and Rutgers University-Newark.

orchestration (Trouche, 2003, 2004, 2005) by providing a pedagogical intervention trajectory that supports students' instrumental genesis (Lonchamp, 2012; Rabardel & Beguin, 2005) of collaborative mathematical environments and shifts students' focus from their teacher to their peer collaborators.

4.2 Positioning Within the Literature

Our online environment, VMTwG,[2] is an interactional, synchronous space, containing support for chat rooms with collaborative tools for mathematical explorations, including a multiuser, dynamic version of GeoGebra. This dynamic geometry environment within VMTwG provides affordances typical and beyond most DGEs. From different perspectives and foci, how DGEs influence learning has been the object of research. Some studies focused on affordances of DGEs and how learners use them, while others discussed how DGEs mediate mathematical activity and shape mathematical understanding. Early research noticed differences between pencil-and-paper geometric constructions and dynamic geometry constructions. Laborde (1993) distinguished between *drawing* and *figure* in DGEs to emphasize these differences. A drawing refers to the perceptual image as drawn on paper, while a figure is the theoretical object, constructed in DGE, and whose defining properties remains invariant under the drag test. Focusing on the dragging affordance of DGEs, researchers investigated how learners understand and use dragging and identified different dragging modalities that shape learners' interactions with the environment and their mathematical understanding (Alqahtani & Powell, 2016, 2017; Arzarello, Bairral, & Danè, 2014; Arzarello, Olivero, Paola, & Robutti, 2002; Baccaglini-Frank & Mariotti, 2010; Hollebrands, 2007; Hölzl, 1996; Lopez-Real & Leung, 2006). Measurement affordance of DGEs was also investigated to understand how it influences learners' mathematical understanding (González & Herbst, 2009; Hollebrands, 2007; Olivero & Robutti, 2007; Sinclair, 2004). In DGEs that provide multiple representations of objects such as GeoGebra, Alqahtani and Powell (2017) found that the analytical information offered in Algebra view provided additional support for learners' discussion of properties and relations of geometric figures. These studies of affordances of DGEs show that learners' cognitive processes relate to how learners use these affordances.

Other researchers studied how DGEs mediate learners' activities to justify and prove mathematical propositions. With DGEs, learners justify and prove relations

[2] The environment, Virtual Math Teams (VMT), has been the focus of years of development by a team led by Gerry Stahl, Drexel University, and Stephen Weimar, The Math Forum at the National Council of Teachers of Mathematics (NCTM) (formerly, The Math Forum @ Drexel University), and the target of considerable research (see, e.g., Powell & Lai, 2009; Stahl, 2008; Stahl, 2009b). This chapter is part of a recent body of investigations centered on an updated VMT with a multiuser version of GeoGebra (see, for instance, Alqahtani & Powell, 2016, 2017; Grisi-Dicker, Powell, Silverman, & Fetter, 2012; Powell, 2014; Powell, Grisi-Dicker, & Alqahtani, 2013; Stahl, 2013, 2015).

using empirical and deductive reasoning (Hadas, Hershkowitz, & Schwarz, 2000; Jones, 2000; Lachmy & Koichu, 2014; Leung & Lopez-Real, 2002; Mariotti, 2000, 2006, 2012; Marrades & Gutiérrez, 2000; Powell & Pazuch, 2016). DGEs allow learners to identify properties of mathematical objects, notice relations and dependencies among them, make and test conjectures, and develop proofs. In addition, DGEs provide systems of tools, such as dragging and trace, and signs associated with these tools that learners can internalize and use to build mathematical meaning (Falcade, Laborde, & Mariotti, 2007; Mariotti, 2000). This internalization influences teachers' and students' mathematical discourse and activity (Alqahtani & Powell, 2015a; Powell & Alqahtani, 2015; Sinclair & Yurita, 2008; Stahl, 2015).

Some studies that investigated how DGEs support learning of mathematics used collaborative settings in which learners share and discuss their ideas as they manipulate and construct objects (Alqahtani & Powell, 2016, 2017; Arzarello et al., 2014; Baccaglini-Frank & Mariotti, 2010; Jones, 2000; Lachmy & Koichu, 2014; Leung & Lopez-Real, 2002; Mariotti, 2000, 2012; Marrades & Gutiérrez, 2000; Stahl, 2015). However, few studies attended to collaborative practices that learners develop while working in small groups with DGEs (Alqahtani & Powell, 2016, 2017; Stahl, 2015). Affordances of DGEs enrich learners' mathematical discourse when learners are working collaboratively (Oner, 2008, 2013; Wei & Ismail, 2010).

In the literature, some studies attend to how teachers organize instruction to support students' learning with digital technologies. Using technology in their classroom, teachers often have to manage new pedagogical situations and develop "a new repertory of appropriate teaching practices for these technology-rich settings" (Drijvers et al., 2014, p. 190). To understand these situations and practices for a given instructional setting, Trouche (2004) introduced the construct of instrumental orchestration that explains how teachers organize available artifacts and engage students with them. Later, Drijvers, Doorman, Boon, Reed, and Gravemeijer (2010) and Drijvers (2012) highlighted the complexity of teaching processes and further developed Trouche's construct. Adding to his two components of instrumental orchestration, didactic configuration and exploitation mode, they distinguish a third component, didactical performance. The didactical configuration concerns how teachers arrange learning environment such as tools, materials, and seating. In the exploitation mode, they plan how to engage students with tasks and tools and in discussions. The third component of instrumental orchestration captures how teachers make instructional decisions in real time under changing circumstances. Together, these three components focus on the design, the didactical context, and the use of the technological tools in a classroom. Drijvers et al. (2010) identified various instrumental orchestrations for whole-class teaching and orchestrations for settings in which students work individually or in pairs with technology, distinguishing between teacher-centered and student-centered orchestrations.

Several studies examined how teachers use technological tools in their classrooms. Using instrumental orchestration, researchers investigated how teachers support students' instrumentation of technological tools in classroom (Alqahtani & Powell, 2015b; Drijvers et al., 2010; Erfjord, 2011). Others analyzed teachers' pedagogical interventions in classrooms to support students' mathematical understanding while working with digital technologies (Biza, 2011; Dove & Hollenbrands, 2014; Laborde, 2007; Sutherland, Olivero, & Weeden, 2004). Teachers support students' learning by making available appropriate tools and materials and by engaging students with tasks that enhance their mathematical understanding.

In our review of the literature, we found that studies investigated how DGEs shape mathematical understanding and how teachers organize learning environments to support students' use of digital technologies. Learners' interactions with DGEs influenced their mathematical understanding. Teachers' different instructional configurations supported students' learning with digital technologies. Among these studies, we found only one study that investigated how teachers support students' learning with collaborative DGE (Alqahtani & Powell, 2015b). This suggests a need to further understand how teachers' orchestration of mathematics classrooms that use synchronous, collaborative digital technologies.

4.3 Theoretical Framework

To understand how teachers use collaborative technologies to support students' mathematics learning with these technologies, we draw on several theoretical foundations for our design and analysis. We employ a cultural historical perspective that encourages learners to collaborate with each other and communicate their ideas. Using Vygotsky's ideas about tool-mediated activity and the role of signs and tools in human development, we view learners' interactions in VMTwG as mediated activity through which students develop their understanding of mathematics and the VMTwG environment. We explain how learners develop their understanding of VMTwG and its different functions using Rabardel and Beguin's (2005) notion of instrumental genesis. It allows us to describe how users appropriate tools and use them as instruments to solve mathematical problems. Finally, we use the construct of instrumental orchestration (Trouche, 2004) to describe how teachers organize and support students' learning of mathematics while using technological tools.

To understand how learners use technological tools to collaborate in solving mathematics problems, we draw on Vygotsky's perspective on the role in human development of cultural signs and tools. He believed that material tools, which are developed historically in cultures, influence human's cognitive behavior and development. In addition to tools, he included signs (e.g., written and spoken language, number systems) in human activity. The "alloy of speech and action has a very specific function in the history of the child's development" (Vygotsky, 1978, p. 30). This perspective on the role of signs and tools informs our conceptual view of how learners use technological tools (online collaborative environments and dynamic mathematics software) and cultural signs (natural language and symbols) to construct together geometric figures and solve jointly geometrical problems. While performing mathematical activities, learners interact with each other to work on shared tasks using available environmental tools and signs. The mediational role of the tools and signs supports learners' cognitive development through a process of internalization. In it, learners transform external activities that are linked to tools into internal activities that are linked to signs (Mariotti, 2000; Vygotsky, 1978). The external actions learners perform with technological tools transform into signs that learners use to think about and communicate mathematical ideas.

The link between external actions and signs in the internalization process indicates the significance of human interactions. A major implication of Vygotsky's theory is the importance of social interactions in learning and human development. During an activity that is directed toward an object, learners employ "tools, speech directed toward the person conducting the experiment or speech that simply accompanies the action" to achieve their goal (Vygotsky, 1978, p. 30). Learners' engagement with these actions supports the internalization process that transforms social phenomena into psychological phenomena (mental functions) (Wertsch, 1985). This perspective emphasizes the importance of collaboration among students during mathematical activities. Collaboration with others gives learners opportunities to reflect on their own thinking and on thinking of others (Daniels, 2001) as well as improves mathematics achievement (Springer, Stanne, & Donovan, 1999).

Building on Vygotsky's work, researchers have developed other constructs to explain how learners build knowledge while interacting with others through technological tools. Rabardel and Beguin introduced the notion of instrumental genesis (Lonchamp, 2012; Rabardel & Beguin, 2005), which theorizes how learners interact with tools that mediate their activity. Learners transform tools into instrument by developing their own knowledge of how to use them. The instrument then mediates activities between learners and a task. In the activities, learners perform actions upon an object (matter, reality, object of work, etc.) in order to achieve a goal using a tool (technical or material component). Rabardel and Beguin (2005) emphasize that the instrument is not just the tool or the artifact, the material device or semiotic construct, it "is a composite entity made up of an artifact component and a scheme component." (p. 442). To transform the tool into an instrument (appropriation), learners develop their own utilization schemes through two important dialectical processes that account for potential changes in the instrument and in learners, called instrumentalization and instrumentation. Instrumentalization is "the process in which the learner enriches the artifact properties" (Rabardel & Beguin, 2005, p. 444). Instrumentation is about the development of the learner side of the instrument; the learner assimilates an artifact to a scheme or adapts utilization schemes. When engaging students with different technological tools in mathematics classrooms, instrumentation plays a significant role in how students build their knowledge about using the tools and how these tools support and shape students' mathematical knowledge.

In mathematics classroom settings, how teachers support learners' instrumental genesis is multifaceted. To understand how to support students' instrumentation of technological tools, Trouche (2004, 2005) introduces "instrumental orchestration" to describe how teachers plan and implement mathematics lessons that integrate technological tools. The instrumentation process is an important multidimensional process, including individual as well as social dimensions (Trouche, 2005). Since instrumental genesis mainly accounts for the individual dimension, instrumental orchestration accounts for the social dimension of the instrumentation process. It describes the arrangements of artifacts in the environment, *didactical configurations*, and teacher and student move within these configurations, *exploitation modes*

(Trouche, 2004, 2005). A third component was added by Drijvers et al. (2010), *didactical performance*, to describe teachers' instructional decisions responding to circumstances during mathematical lesson. There are different combinations of didactical configurations, exploitation modes, and didactical performance that teachers use to support their students' instrumentation. The didactical configurations concern the "layout of the artifacts available in the environment" (Trouche, 2004, p. 296) for the students to interact with during the mathematical lesson. The exploitation modes represent actions that teachers choose for students to perform based on their lesson's objects. The didactical configurations and the exploitation modes work together to support students' achievement of lesson's objects. Combinations of didactical configurations and the exploitation modes act on three levels: artifacts, instruments, and students' relationship with the instruments (Trouche, 2005). Within these levels, teachers attend to the tools as artifacts (before instrumentation) and after students appropriate the tools. After students appropriate the tools, teachers' decisions involve how to guide students' interactions with the instrument and support their mathematical understanding.

4.4 Online Environment for Collaborative Learning

The online environment, VMTwG, is an interactional, synchronous space. It contains support for chat rooms with collaborative tools for mathematical explorations, including a multiuser version of GeoGebra, where team members can construct dynamic objects and drag elements to visually explore relationships (see Fig. 4.1). VMTwG records users' chat postings and GeoGebra actions, which teachers can review and even replay at various speeds. The research team designed dynamic geometry tasks to encourage participants to discuss and collaboratively manipulate and construct dynamic geometry objects, notice relations and dependencies among the objects, make conjectures, and build justifications.

The data for this study come from the second course and concern the work of a high school mathematics teacher, Mr. S. He engaged his class in VMTwG in small teams of three to four students each. The class worked in a computer lab, and Mr. S. encouraged students to communicate only through VMTwG. To understand his instrumental orchestration, we analyze qualitatively four sources of data: (1) the tasks he used with his students, (2) the modifications he made to the tasks after reviewing teams' work, (3) the logged VMTwG interactions of two teams of his students on the tasks, and (4) his reflections on their work, which he wrote after each class session. We chose to analyze two teams, team 1 and team 6, since Mr. S. considered those teams to be most collaborative.

On each of the four data sources, we performed conventional and directed content analysis (Hsieh & Shannon, 2005). We were particularly interested in coding and categorizing both Mr. S.'s pedagogical interventions and the deductive justifications of two teams of his students. The data drives our analysis, and we interpret them using the theories of instrumental genesis and orchestration whenever there are links between the data and the theories.

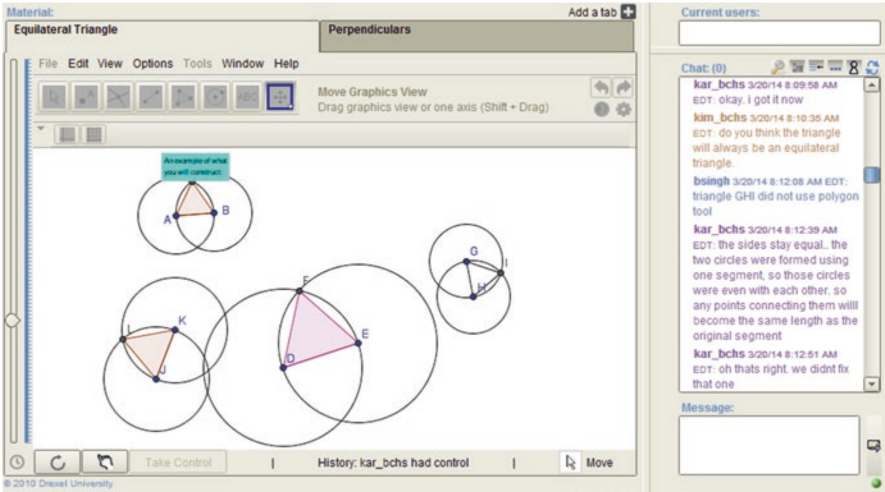

Fig. 4.1 Screenshot of VMTwG environment with the work of Mr. S.'s students

4.5 Pedagogical Setting: Teachers Learning

The work of the high school mathematics teacher who engaged his students in an online collaborative environment, Virtual Math Team with GeoGebra (VMTwG), to extend their geometric knowledge was informed by a particular pedagogical setting. The setting is a professional development project, "Computer-Supported Math Discourse among Teachers and Students," that involves middle and high school teachers in two 15-weeklong, technology-focused online courses. The first course engages teachers, working synchronously in small groups, in interactive, discursive learning of dynamic geometry through collaborating in VMTwG to solve 55 tasks that involve constructing geometric figures and solving open-ended geometric problems. In addition, the teachers realize and, in writing, reflect on their mathematical and collaborative practices; read and discuss synchronously and asynchronously articles about technology and pedagogy (Battista, 2002; Mishra & Koehler, 2006; Stahl, 2009a), lesson types with technology (McGraw & Grant, 2005), collaboration and discourse (Mercer & Sams, 2006; Michaels, O'Connor, & Resnick, 2007; Resnick, Michaels, & O'Connor, 2010), and mathematical practices (Common Core State Standards Initiative, 2010, pp. 6–8); and collaboratively plan the content and means to implement what they learn in the course in lessons with their students.

The Implementation Plan is a major component of the 15-weeklong professional development course. Completed over 10 of the 15-weeklong professional development course, it is divided into the following five phases:

Phases of the Implementation Plan
Phase 1: Develop a plan for garnering school and district support and for technology availability.
Phase 2: Select a focus for VMTwG lessons that is of interest to you individually as a teacher. Discuss your focus during your team's second synchronous session, and post your focus to the Blackboard Discussion forum. Do other teachers within your team or in other teams share the same focus?
Phase 3: Define your focus statement including a clear set of pedagogical goals. Collaborate within your VMT Team, and also exchange ideas with teachers from other teams through Blackboard Discussion forums. It is acceptable if several teachers share a focus statement, but it is not required.
Phase 4: Develop an activity list of collaborative dynamic mathematics activities to foster a developmental trajectory aligned with your focus statement. It is acceptable if several teachers share activities or all or part of a developmental trajectory, but it is not required.
Phase 5: Develop a coherent set of scripted VMTwG sessions, and decide how you might implement this curriculum next term. It is acceptable if several teachers share some or all scripted VMTwG sessions, but it is not required.

Each phase is further explained in an implementation document that the teachers receive. Over the course of the 10 weeks, each teacher uploads to the discussion board space of an online course management system (Blackboard™) their response to each phase of the plan and receives feedback from other teachers in the course as well as the course facilitators. This collaborative development of the teachers' plan is a substantial way in which the course supports each teacher's implementation efforts with their students.

A second major support for teachers' classroom implementation is the second course, which is a reflective practicum. In the course, teachers post their planned lessons to Blackboard, receive constructive feedback and exchange ideas, post a reflection about each lesson that includes information about their students' learning and about challenges and triumphs and to these receive feedback. In the teachers' reflection on each of their lessons they write about the goals of the lesson, how whether the students achieved the goals, what worked and did not work, and what support the teacher provided suggest changes for further revisions to improvement; from the discursive and inscriptive, data highlight the collaborative and mathematical practices; comment on each teachers' reflection.

4.6 Pedagogical Setting: Teachers Supporting Student Learning

The teachers engage their students in at least 10 h of class sessions to learn dynamic geometry through the use of VMTwG to work on construction and problem-solving tasks. This study examines a teacher's and his students' initial engagement with the VMTwG program in an urban high school in southern New Jersey. The high school has a diverse student body, where 52% of the students identify as Black, 35% as White, 10% as Hispanic, and 3% as Asian. Twenty-one percent of the students have a classified disability, while 57% live in economically disadvantaged households.

For the 2012–2013 school year, the high school's suspension rate was 48%, and its 2014 graduation rate was 82%.

This mathematics teacher, Mr. S., has taught at this high school since the beginning of his teaching career and, at the time of this study, had taught there for 6 years. During regular class time, from two different 10th and 11th grade classes, he engaged a total of 31 students in tasks in VMTwG. Nineteen of his students were females and 12 were males. Academically, they had performed below or at the average on statewide standardized assessments. Specifically, statewide assessment data for 25 of the 31 students were available, and of them, 19 passed the 7th grade assessment and still fewer, 17, passed it in the 8th grade. At the time of this study, eight of the 31 students were concurrently enrolled in a separate mathematics remediation course. All 31 students neither had prior experience with dynamic geometry nor previously worked in a computer-supported collaborative learning environment.

For working in VMTwG, the students worked in a computer lab and divided themselves into teams. Teams were formed based on students' already established social groupings since students chose their teammates according to with whom they normally socialized during class. They formed eight teams, seven teams with four students in each and one team of three. The teacher did not have students in regularly assigned seats. In the computer lab, the computers were arranged on pentagonal-shaped tables. There was a teacher station that was connected ceiling projector and two large wall-mounted whiteboards. The teacher provided his students with log-ins and assigned each student team to a VMTwG chat room. Students were not able to enter other teams' chat rooms.

4.6.1 Teacher's Instrumental Orchestration

Based on our analyses of Mr. S.'s implementation of the project design, his instrumental orchestration was directed at supporting three categories of students' actions: collaborative practices, mathematical reasoning, and the use of technology. In addition, the analysis reveals that Mr. S. followed a trajectory of pedagogical interventions focused on his students' discursive interactions and their emerging knowledge of dynamic geometry. In his reflections on his students' work, Mr. S. expresses an overall goal that, within their teams, students manipulate and construct dynamic geometric objects and notice and discuss relations among them, particularly relations of dependency. To achieve this goal, Mr. S.'s didactical configurations had students work in small groups in a computer lab and communicate online through VMTwG. His pedagogical interventions focused on how the teams of students collaborate. Having given his students a task designed to promote collaboration, Mr. S. expressed concern in his weekly reflection that the teams did not collaborate successfully. He reported that to ensure successful collaborative sessions, he subsequently discussed with his class features of successful collaborations and presented examples of what he considered good collaborative moves. To underscore his advice, he distributed a list of behaviors that he judged could help to ensure successful collaboration and called it "The Pledge." It contained statements of behaviors such as "Include everyone's ideas" and "Ask what my team members think and what their reasons are."

These pedagogical interventions and ones that we present below focused on collaboration. They reveal that Mr. S. choose exploitation modes (instructional decisions) that encourage students to be reflective of their work within their teams. His pedagogical interventions are mostly focused on the second and third level of his instrumental orchestration. Those levels are concerned, respectively, with the instrument and the students' relations with the instrument. Mr. S. used collaboration as a vehicle to orchestrate his students' appropriation of VMTwG artifacts and movement toward deductive justifications. In his weekly reflections, he assessed his students' reasoning by tracking their collaborative practices and their use of mathematical language.

Closely following Mr. S.'s interventions concerning his students' collaborative practices, he then focused on aspects of their use of the technological environment. This focus is at the first level—artifact level—of his instrumental orchestration. In his weekly reflections, he reported that during his students' engagement in VMTwG, he "monitored progress and resolved some tech issues." He helped students gain insights into the use of particular GeoGebra commands by modifying tasks and directing his students to view specific YouTube GeoGebra clips.

As Mr. S.'s teams of students increased their effective collaborative interactions, he shifted his pedagogical interventions more explicitly toward supporting their mathematical reasoning. He discussed with his class the concept of dependency in dynamic geometry to contrast it with dependency in other mathematical domains and modified the tasks to explicate particular mathematical ideas. He posed detailed questions to foreground mathematical discourse. For example, he found that the tasks' original questions were not specific enough to elicit mathematical reasoning, so he included the following questions in one of the tasks, "constructing an equilateral triangle":

1. What kinds of triangles can you find here?
2. Drag the points. Do any of the triangles change kind? Discuss this in the chat.
3. Are there some kinds [of triangles] you are not sure about?
4. Why are you sure about some relationships?
5. Does everyone in the team agree?

These questions prompted his students to attend to particular objects and relations in the construction and to discuss the behavior of these objects and relations.

4.6.2 Students' Work in VMTwG

Mr. S.'s instrumental orchestration and his other pedagogical interventions contributed to his teams of students' instrumentation and movement toward greater collaboration and deductive justifications. For example, according to Mr. S. and our analyses, a team of three students (team 6) improved their collaboration, explorations, and mathematical reasoning. In their third session, the task asked them to construct an equilateral triangle, find the relationships among objects in their construction, and justify their claims. The students first dragged a preconstructed figure of an equilateral triangle (see triangle ABC in Fig. 4.1 above) to explore elements of the construction and their behavior. Afterward, they each constructed a similar figure (see Fig. 4.1) and dragged their construction vigorously to validate and justify their construction. Below, an excerpt[3] of their discussion shows how a team of students articulated a valid justification of why their constructions were of equilateral triangles.

18	kar_bchs:	looks like we both got it [both successfully construct and drag the figures vigorously]
19	kim_bchs:	yay, it seems like for a second one of the circles appeared much larger. but that was my imagination.
20	kar_bchs:	oh. lol. why is the third point dependent on the distance between the first two points? (number 7)
21	kar_bchs:	it just connects the points and the circles. making them all one piece
22	kim_bchs:	as the segments change sides so does the radius of the circle. However, the triangle remains an equilateral triangle
23	bsingh:	[the teacher] be sure to read directions, ALL, and make the pledge
24	kim_bchs:	triangle
25	kar_bchs:	yea. even though the sizes of the sides change, the fact that it is an equilateral triangle doesn't
26	kar_bchs:	each side has the same distance in between it. even when you move the points
27	kim_bchs:	i notice that point d and e are on the circumference of one circle. while point f is an intersetion of both circle. making it dependent on both points.
28	kar_bchs:	if you try and move the intersected point (F and I), it won't move. but yea you're right, the intersecting point depends on the segment that was made
29	kim_bchs:	*point f is an intersect of both circles
30	bsingh:	[the teacher] there is something missing, are you reading the directions
31	bsingh:	[the teacher] we are only doing tab 1 today
32	kar_bchs:	i didnt use the polygon tool.. that's missing in mine
33	kim_bchs:	i just notice that.
34	kar_bchs:	can i try?
35	kar_bchs:	okay. i got it now
36	kim_bchs:	do you think the triangle will always be an equilateral triangle.
37	kar_bchs:	the sides stay equal.. the two circles were formed using one segment, so those circles were even with each other. so any points connecting them will become the same length as the original segment
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[3]This and the next excerpt in this chapter are from students engaged in chat communication in VMTwG. In this setting, at the same time that they write informally, without focusing on conventions of academic writing, students direct their attention to communicating quickly their mathematical ideas to themselves and their teammates. For this reason, we have chosen not to correct their orthography or any other aspect of their writing. We feel that it is important to honor and understand their authentic expressions.

50	kim_bchs:	the radius of a circle is the same distance. segment AB is Sure. the radii of both circles and Segment AC and BC are also radii of both circles. hence, the triangle should be equilateral.
51	kar_bchs:	the circles are equal. making the circumference of each equal to one another

This team of students noticed that the equilateral triangle depended on the relationship between the two circles that they created. They discussed their constructions and the relationships they noticed (lines: 18–29). Both students noticed that the construction maintains the triangle equilateral as vertices are dragged (lines 22 and 25). They tried to explain how the intersection points of the circles are dependent on the centers of the circles (lines 27–29). In line 36, kim_bchs asks whether the triangle is always equilateral. In response, kar_bchs states that the sides of the triangle are equal and mentions that the two circles are "even" or congruent. In line 50, it seems that kim_bchs builds on kar_bchs's observation and notes that the radii of both circles are equal and that imply that the triangle is equilateral and, in line 51, that the circumferences of the two circles are equal. The students successfully build on each other's ideas and justify why their constructions yield equilateral triangles and justify other equivalences that they notice. They also note that the congruence of their circles depends on the segment that they share (line 37: "the two circles were formed using one segment, so those circles were even with each other") and that two sides of the given triangle are dependent on segment AB (line 50: "the radius of a circle is the same distance. segment AB is Sure. the radii of both circles and Segment AC and BC are also radii of both circles. hence, the triangle should be equilateral."). This provides further evidence that these students are justifying mathematical relations, moving themselves toward deductive justification. This also indicates that this student team transformed artifacts of the technological environment such as chat, dragging, and tools involved in constructing equilateral triangles into instruments.

4.7 Discussion

Mr. S's students' actions led to their transformation of technological artifacts of VMTwG into instruments of their knowledge building. In this process, they accomplished movement between visual and dragging explorations and discursive deductive justifications. Their movement toward deductive justifications was evidenced in their discursive, interaction motivated by their perception of mathematical properties and relations that they notice while manipulating mathematical objects to develop and communicate convincing arguments about the mathematical relations that satisfy their team members.

These student knowledge-building actions were supported by Mr. S.'s pedagogical orchestrations. As we defined it earlier, such orchestrations are instructional actions initiated by teachers that precede, invite, sustain, monitor, or reflect on students' activity. Initial actions in the trajectory of Mr. S.'s pedagogical orchestrations began with a focus on supporting teams of his students to have effective collaborative interactions. The extension of their collaborative practices evidence collaborative learning, as Jeong and Hmelo-Silver (2016) suggest: "a group of people engage[s] in activities toward a shared goal. They may divide the tasks in the process of working together, but the ultimate goal is to produce an outcome that collectively advances the knowledge of individuals as well as the collectives" (p. 248). Once Mr. S. was satisfied that, within teams, students were listening to each other and building on each other's ideas, he shifted to focus his instructional interventions around ideas of mathematical reasoning and justifications. Our analysis of his weekly reflections, his later analysis of his students' work, and our analysis of his students' work indicate that, in parallel with his trajectory, his students progressed toward more pointed justifications of geometric relations that they noticed, including, for dynamic geometry, mathematically significant relations of dependencies (Stahl, 2013; Talmon & Yerushalmy, 2004).

Mr. S.'s pedagogical orchestrations not only shaped his students' transformation of technological artifacts of VMTwG into instruments for knowledge building but also inform the theory of instrumental orchestration (Trouche, 2003, 2004, 2005). His instructional actions undergird a model of pedagogical orchestration, the purpose of which is to support students' instrumental genesis (Rabardel & Beguin, 2005) of collaborative mathematical environments such as VMTwG. The didactical configuration involves a technological environment specifically designed to support collaborative knowledge building among small teams of interlocutors, interacting in coordinated discursive (chat) and inscriptive (GeoGebra) spaces. Another aspect of the didactical configuration is the open-ended, collaborative, and discourse-provoking nature of the mathematical tasks that the teacher chose and modified. The choices of technological environment and tasks are instructional moves that shift students' focus in the classroom from their teacher to their peer collaborators.

In the exploitation mode, teacher and student moves vary significantly from Mr. S.'s students' previous mathematical experiences in school. His students' school experiences in mathematical classrooms neither include little to any work in collaborative teams nor with technological or dynamic geometry. They had no experience working on open-ended tasks in which they were expected to build their own geometrical knowledge. This expectation that students were to build their own geometrical knowledge as they collaborative resolved open-ended tasks and constructed geometric figures rather than being told what they were to learn served to decentralize further the teacher's role. As a consequence, the teacher's pedagogical orchestration meant that there were minimal opportunities for Mr. S. to intervene contemporaneously while he was interacting in VMTwG. After the students' sessions, he reviewed their interactional work and made decisions in the realm of didactical performance to respond to circumstances that had occurred in his students' online mathematical work. His whole-class discussion of collaborative moves, including "The Pledge," is how he addressed the challenge of supporting his students' development of productive collaborative practices.

The example of Mr. S. provides evidence of pedagogical orchestration that supports students' instrumentation. He models how teachers can support students' instrumentation of collaborative environments and the extension of their mathematical understanding. In this model, during students' mathematical activity, teachers progressively decentralize their role and, simultaneously, support students' development and performance of collaborative practices. This model augments the theory of instrumental orchestration (Trouche, 2003, 2004, 2005) by providing a pedagogical intervention trajectory that supports students' instrumental genesis of collaborative mathematical environments and shifts students' focus from their teacher to their peer collaborators. In general, this model contributes to an understudied area of DGEs, teacher practice (Sinclair et al., 2016).

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