Chapter 5
Degradation-Based Reliability Modeling of Complex Systems in Dynamic Environments

Weiwen Peng, Lanqing Hong, and Zhisheng Ye

Abstract Benefiting from the intimate link and the sufficient information conveyed by the degradation-threshold failure mechanism, degradation analysis has gradually become a hot topic in reliability engineering, which has been investigated extensively in the recent two decades. Various degradation models have been introduced to facilitate the reliability modeling and assessment of modern products, especially for highly reliable products. As the continual evolving of these models, there is a growing trend of investigation of reliability modeling and assessment based on degradation analysis. However, modern complex systems are characterized as multi-functional and subject to dynamic environments. Two aspects are indispensable for the investigation of degradation based reliability modeling and assessment of modern complex systems: (1) how to deal with complex systems with more than one degradation indicators, and (2) how to incorporate the effects of dynamic environments. To advance the research on degradation modeling and analysis of complex systems, this paper presents a summary of the state of the arts on the research of reliability modeling of complex systems by taking account of these two aspects. In this paper, the review is delivered in two progressive stages: multiple degradation processes under static environments, and multiple degradation processes under dynamic environments. Some discussion on further research topics from both theoretical and practical perspectives are presented to conclude the paper.

Keywords Stochastic process models • Inverse Gaussian process • Reliability modelling • Reliability assessment • Data-driven methods

W. Peng
School of Mechatronics Engineering, University of Electronic Science and Technology of China, Chengdu, China

L. Hong • Z. Ye (✉)
Department of Industrial Systems Engineering and Management, National University of Singapore, Singapore, Singapore
e-mail: yez@nus.edu.sg

D.-G. (Din) Chen et al. (eds.), *Statistical Modeling for Degradation Data*, ICSA Book Series in Statistics, DOI 10.1007/978-981-10-5194-4_5

5.1 Introduction

Continually evolving paces of technology advances and ever-increasing requirements of system performance have made modern systems become more and more complex. Reliability engineers are encountered with paradoxical situations of reliability modelling of complex systems, for which the situation introduced by limited failure time data is an inevitable challenge [1], however this situation is contradicted by the growing reality that reliability data are stepping into big data situation [2], which is enhanced by the gradual availability of the system performance data, system operating profile, and working environment information. To facilitate the reliability analysis of complex system with limited failures under big data situation, degradation based reliability modeling and assessment has gradually become a hot topic, benefiting from the intimate link and the sufficient information conveyed by the degradation-threshold failure mechanism [3]. For the degradation based reliability analysis, the failure process of a complex system is reflected by degradation processes of some performance indicators, where the system fails when these degradation processes reach predefined thresholds. Accordingly, the failure time distribution can be determined through the degradation analysis of system performance data together with the system operating profile and working environment information, which provides a promising solution to the difficulty introduced by limited failure time data.

In recent two decades, degradation based reliability modelling has been investigated extensively. Classical examples include, but not limited to the works presented recently by Pan and Balakrishnan [4], Wang and Pham [5], Kharoufeh et al. [6], Liao and Tian [7], Wang et al. [8], Bian and Gebraeel [9], Si et al. [10], Ye and Xie [11], Hong et al. [12], Peng et al. [13], and Xu et al. [14]. Generally, two critical assumptions are used in the degradation based reliability analysis, which include the assumptions of (1) single degradation indicator, and (2) constant external factors. For complex systems under dynamic environments, these two assumptions are challenged greatly. These challenges are raised from the growing awareness that complex systems are composed of multiple components with multiple functionalities, and these systems are subject to various operating profiles under dynamic working conditions. The failure process of a complex system generally relates to degradation processes of multiple performance indicators, which are often presented with various characteristics. Moreover, the degradation processes are affected by the operating profiles and working conditions, which lead to the indicator-to-indicator dependency within the multiple degradation processes, and the unit-to-unit variability among the system population group. A classic example is the multivariate dependent degradation analysis of one type of heavy duty lathes presented in Peng et al. [15]. Within this study, the heavy duty lathes are founded subject to two types of gradually-evolving failures: losing of machining accuracy, and accumulation of lubrication debris, which are critical to the reliability of these lathes. In addition, by summarizing operating and maintenance records, these types of failures are found to vary from factories to factories, which is caused by differences of loading profiles and environmental conditions experienced by the heavy duty lathes. Accordingly, two critical aspects arise for the degradation based reliability modelling of complex systems under dynamic environments: (1) how to deal with complex systems with more than one degradation indicators, and (2) how to incorporate the effects of dynamic operating profile and time-varying working environments.

To facilitate the research on degradation based reliability analysis, this paper devotes to the review of degradation based reliability modelling of complex systems under dynamic environments. There are excellent review papers on degradation based reliability modelling over the past few decades, such as the reviews by Singpurwalla [16], Lee and Whitmore [17], Bae et al. [18], Si et al. [19], and Ye and Xie [11]. These review papers have covered various aspects of degradation based reliability modelling. However, the challenges introduced by multiple degradation processes and dynamic environments are still underdeveloped. Therefore, this paper is to highlight the state of arts on multiple degradation analysis under dynamic environments through two progressive categories: multiple degradation processes under static environments, and multiple degradation processes under dynamic environments.

The remainder of this paper is organized as follows. Section 5.2 describes the characteristics and modelling of dynamic environments encountered by complex systems. Section 5.3 reviews the degradation based reliability modelling of complex systems with multiple degradation process models under static environments. Section 5.4 reviews the degradation based reliability modelling through multiple degradation processes under dynamic environments. A short discussion on further research topics is presented in Sect. 5.5 to conclude the paper.

5.2 Dynamic Environments

Degradation processes of a complex system are closely related to the system's operation profile and working environments. This is because failure processes of a complex system are often driven or affected by the factors and stresses originated from the dynamic environments experienced by the system. The rates and the modes by which the system degrades are closely related to what type of missions the system is fulfilling, and which kind of environments is experienced. Take a manufacturing system as an example, the system can fulfill the missions of turning, drilling, and milling with different working speeds and depths on various types of materials. These parameters consist of the basic operational profile of the manufacturing system, for which a specific mission with particular working loads on one specific working piece determine the basic degradation rate and mode of the manufacturing systems. In addition, the temperature, humidity, and vibration condition experienced by the manufacturing system consist of the basic working environments, which further modify or change the degradation rate or even degradation mode of the system.

The dynamic environments experienced by a system is generally composed by its operational profile and working environment. The operating profile is composed of the variables that can be modified or determined by the operators or users to fulfill different missions. The working environment mainly refers to the factors that can hardly be controlled and are mainly determined by natural forces. The differentiation of these two groups is aimed to highlight the effect introduced by different types factors, and to deliver the notion that some variables can be well modeled and incorporated into the degradation modelling yet some factors can hardly be well characterized or integrated. As a result, two aspects are critical for the handling of dynamic environments within the degradation based reliability analysis: (1) how to characterize the dynamic environments, and (2) how to incorporate them in degradation modelling.

5.2.1 Characterization of Dynamic Environments

The characterization of dynamic environments is implemented based on the availability of environmental information and the capability of mathematical models [16, 20]. General methods for the characterization of dynamic environments include the time-varying deterministic function [22], probability distribution model [23], stochastic processes model [24, 29], time series models [2, 12], and so on.

Time-varying deterministic function is generally used for the situation where environmental variables are well controlled and their dynamic behavior follows specific patterns, such as accelerated degradation tests [25, 26]. The probability distribution model is used to model the environmental variables which are presented as shocks with random strengths, such as vibration shocks for mechanical systems [27, 28]. The stochastic processes model is often adopted for the situation where the environmental variables are evolved stochastically with temporal variability and epistemic uncertainty, such as different operating speeds for a rotational machine [24, 29]. The time series model is used for the situation where the environmental variables have highly variable behaviors with periodicity and autocorrelations, such as the solar radiation and temperature for organic coatings [12, 30].

These methods for the characterization of dynamic environments are mainly used for the environmental variables that can be well identified and collected, where system operating/environmental data highlighted by Meeker and Hong [2] can be obtained for the corresponding model derivation and parameter estimation. There are situations that environmental variables cannot be identified or collected, such as the micro-shock for a micro-electromechanical system. This kind of environmental variables and their effect are lumped together and incorporated into the degradation modelling through a random effect or frailty term [31–34].

5.2.2 Incorporation of Dynamic Environments

The incorporation of dynamic environments into degradation modelling is carried out based on the availability of influence mechanism and the flexibility of mathematical models. General methods for the incorporation of dynamic environments include the covariate method [20–23] and cumulative damage method [6, 24, 35].

The covariate method has been used extensively to incorporate the effect of environmental factors into degradation modelling. The environmental factors are represented as explanatory variables, and their effects are modelled through a covariate-effect function. The covariate-effect function is further used to modify the model parameters, which are related to the degradation rate, shape, mode or diffusion of a degradation process model. The covariate-effect function is generally determined based on the physical, chemical, and engineering knowledge, such as Arrhenius relationship, Eyring relationship, power low relationship, linear relationship, inverse-logit relationship and exponential relationship [3]. Classical examples of covariate method for incorporation of dynamic environments are the works presented by Ye et al. [11] and Lawless and Crowder [23] separately for Wiener process and gamma process models.

The cumulative damage method is introduced for the situation that the degradation rate of the degradation process is dominated by random environmental factors. The environmental factors are characterized as stochastic processes, such as the Wiener process [16], continuous-time Markov chain process [6], and Semi-Markov process [24]. The degradation rate of the degradation process is modelled through a functional relationship of the environmental factors. The degradation at a specific point is the accumulation result of the degradation increments throughout the time interval from the beginning to this specific point, where the degradation increments within an infinitesimal time interval is dominated by the specific environmental factors within that interval. This method is generally used for the situation that the degradation process of a system cannot be directly observed, however the environmental factors that dominate the degradation process can be observed and their relationship is well understood. Classical example of cumulative damage method for incorporation of dynamic environments can refer to the method introduced by Bian et al. [29] and Flory et al. [36].

5.3 Multiple Degradation Processes Under Static Environments

The premise of degradation based reliability assessment of complex system is to choose a proper model to characterize the degradation process of the complex system, based on the its performance indicators and dynamic environments. Various types of multiple degradation processes models have been introduced in the recent decade, which can be categorized into two types, that is, the models for multiple degradation process models under static environments and the models for multiple degradation process models under dynamic environments. This section devotes to the first type under static environments, including the multivariate Gaussian distribution based model, the bivariate Birnbaum-Saunders distribution based model, the degradation rate interaction model, and the copula based multivariate degradation process model.

5.3.1 Multivariate Gaussian Distribution Based Model

The multivariate Gaussian distribution based model is introduced out of the consideration that a system may have multiple degradation paths and the distribution of the degradation observations of these paths at a specific time point can be described by a joint multivariate Gaussian distribution [37]. Within this model, each degradation path is described by a marginal Gaussian distribution from the joint multivariate Gaussian distribution. The dependency among the degradation paths is characterized by the variance-covariance of the joint multivariate Gaussian distribution.

Suppose a product has L performance indicators. Let $Y_l(t)$ with $l = 1, \ldots, L$ denote the degradation process of the lth performance indicator. The joint distribution of the L performance indicators, $\mathbf{Y}(t) = (Y_1(t), \ldots, Y_L(t))^{\mathrm{T}}$, at a specific time point t is given as follow.

$$f(y_1(t), \ldots, y_L(t)) = \frac{1}{\sqrt{(2\pi)^L |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}(\mathbf{y}(t) - \boldsymbol{\mu}(t))^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{y}(t) - \boldsymbol{\mu}(t))\right) \tag{5.1}$$

where $\mathbf{y}(t) = (y_1(t), \ldots, y_L(t))^{\mathrm{T}}$, $\boldsymbol{\mu}(t) = (\mu_1(t), \ldots, \mu_L(t))^{\mathrm{T}}$ with $\mu_l(t)$ denoting degradation mean function of the lth performance indicator, and $\boldsymbol{\Sigma}$ is the covariance matrix and $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$.

The degradation mean function is a description of the average degradation observation of the performance indicator, such as $\mu_l(t) = at$ indicating linear degradation path of the lth performance indicator. The covariance matrix is a description of the variance and correlation among the degradation processes. Let $\mathrm{Var}(Y_l(t))$ denote the variance of $Y_l(t)$, and $\mathrm{Cov}(Y_{l-1}(t), Y_l(t))$ denote the covariance of $Y_{l-1}(t)$ and $Y_l(t)$. The general form of the covariance matrix $\boldsymbol{\Sigma}$ is given as

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{Var}(Y_1(t)) & \mathrm{Cov}(Y_1(t), Y_2(t)) & \cdots & \mathrm{Cov}(Y_1(t), Y_L(t)) \\ \mathrm{Cov}(Y_2(t), Y_1(t)) & \mathrm{Var}(Y_2(t)) & \cdots & \mathrm{Cov}(Y_2(t), Y_L(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(Y_L(t), Y_1(t)) & \mathrm{Cov}(Y_L(t), Y_2(t)) & \cdots & \mathrm{Var}(Y_n(t)) \end{bmatrix} \tag{5.2}$$

Generally, it's difficult to specify the functional forms of Var($Y_l(t)$) and Cov($Y_{l-1}(t), Y_l(t)$). The assumption of time-invariant variance and covariance is adopted for the application of this model [37]. Under the time-invariant assumption, the covariance matrix $\boldsymbol{\Sigma}$ is constant over time, where the variance and covariance are given as $\mathrm{Var}\left(Y_l(t)\right) = \sigma_l^2$ and $\mathrm{Cov}(Y_{l-1}(t), Y_l(t)) = \rho_{l-1,l}\sigma_{l-1}\sigma_l$ with $\rho_{l-1,l}$ indicating the relevance between $Y_{l-1}(t)$ and $Y_l(t)$. The parameters within the covariance matrix $\boldsymbol{\Sigma}$ are greatly reduced under the time-invariant assumption. In addition, the marginal distribution of the degradation path for the lth performance indicator is given as $Y_l(t) \sim \mathrm{N}\left(\mu_l(t), \sigma_l^2\right)$, and its degradation increments is $\Delta Y_l(t) \sim \mathrm{N}\left(\Delta\mu_l(t), \sigma_l^2\right)$. The model is simplified to a great extent. The temporal variability within the degradation process is missing under the time-invariant assumption due to the time invariant variance $\mathrm{Var}\left(\Delta Y_l(t)\right) = \sigma_l^2$.

The failure time T of the product is defined as the first time point that any of the L degradation processes reaches its degradation threshold D_l with $l = 1, \ldots, L$. The reliability function of the product is then given as

$$\begin{aligned} R(t) &= \Pr\left(Y_1(t) \le D_1, \ldots, \mathrm{Y}_L(t) \le D_L\right) \\ &= \int_0^{D_1} \cdots \int_0^{D_L} f\left(y_1(t), \ldots, y_L(t)\right) dy_1(t) \cdots dy_L(t) \\ &= \Phi_L\left(D_1, \ldots, D_L; \boldsymbol{\mu}(t), \boldsymbol{\Sigma}\right) \end{aligned} \tag{5.3}$$

where $\Phi_L(\bullet; \boldsymbol{\mu}(t), \boldsymbol{\Sigma})$ is the cumulative distribution function (CDF) of L dimension multivariate Gaussian distribution with mean $\boldsymbol{\mu}(t)$ and covariance matrix $\boldsymbol{\Sigma}$.

The multivariate Gaussian distribution based model does not receive wide application in degradation data analysis. However, this idea has roused the investigation on degradation analysis with multiple performance indicators, such as the works presented by Pan and Balakrishnan [4], Bian and Gebraeel [9], Sari et al. [38] and Peng et al. [15], which are serving as key stones for the following developed multiple degradation process models.

5.3.2 *Multivariate Birnbaum-Saunders Distribution Based Model*

The multivariate Birnbaum-Saunders distribution based model is introduced by taking the idea that the failure time distribution derived for the one-dimensional degradation can be approximated closely by Birnbaum-Saunders distribution [39]. The multivariate Birnbaum-Saunders distribution and its marginal distributions are then adopted to construct a multivariate degradation process model by Pan and Balakrishnan [4] and Pan et al. [40]. Within this model, the marginal degradation processes are modelled by gamma processes. The dependency between these degradation processes is constructed by assuming that the degradation increments of the multivariate degradation process within the same time interval are dependent, where a time-invariant correlation coefficient is used to characterize this dependency.

For a product with L performance indicators, the degradation process of the lth performance indicator is modelled as a gamma process, $Y_l(t) \sim \text{Ga}(\nu_l t, \gamma_l)$ with $l = 1, \ldots, L$. The degradation process $Y_l(t)$ has independent and gamma-distributed increments as $\Delta Y_l(t) \sim \text{Ga}(\nu_l \Delta t, \gamma_l)$, where $\Delta Y_l(t) = Y_l(t + \Delta t) - Y_l(t)$. The probability density function (PDF) of the degradation increment $\Delta Y_l(t)$ is given as

$$f\left(\Delta y_l(t) | \nu_l \Delta t, \gamma_l\right) = \frac{1}{\Gamma\left(\nu_l \Delta t\right) \gamma_l^{\nu_l \Delta t}} \left(\Delta y_l(t)\right)^{\nu_l \Delta t - 1} \exp\left(-\frac{\Delta y_l(t)}{\gamma_l}\right) \tag{5.4}$$

The degradation increments of different performance indicators at the same time interval are dependent, which is further described by time-invariant correlation coefficient as $\text{Corr}(Y_{l-1}(t), Y_l(t)) = \rho_{l-1,l}$. However, the degradation increments at different time intervals are assumed independent. To further describe the dependence among the performance indicators, a random variable is introduced by normalizing the degradation increments as $X_l(t) = (\Delta Y_l(t) - \nu_l \Delta t \gamma_l) / \left(\sqrt{\nu_l \Delta t} \gamma_l\right)$, and all the observation intervals are assumed having the same length . In addition, the dependency among the performance indicator is further described as $\text{Corr}(X_{l-1}(t), \text{X}_l(t)) = \rho_{l-1,l}$. Given the degradation threshold of all the performance indicators as D_l with $l = 1, \ldots, L$, the reliability function of the product is given as follows by utilizing the central limit theorem [40].

$$\begin{aligned}
R(t) &= \Pr\left(Y_1(t) \le D_1, \ldots, \text{Y}_L(t) \le D_L\right) \\
&= \Pr\left(\sum_{j=1}^{t/\Delta t} \Delta Y_{1j}\left(t_j\right) \le D_1, \ldots, \sum_{j=1}^{t/\Delta t} \Delta Y_{Lj}\left(t_j\right) \le D_L\right) \\
&= \Pr\left(\sqrt{\frac{\Delta t}{t}} \sum_{j=1}^{t/\Delta t} \frac{\Delta Y_{1j}\left(t_j\right) - \nu_1 \Delta t \gamma_1}{\sqrt{\nu_1 \Delta t} \gamma_1} \le \frac{D_1 - \nu_1 t \gamma_1}{\sqrt{\nu_1 t} \gamma_1}, \ldots, \right. \\
&\qquad \left. \sqrt{\frac{\Delta t}{t}} \sum_{j=1}^{t/\Delta t} \frac{\Delta Y_{Lj}\left(t_j\right) - \nu_L \Delta t \gamma_L}{\sqrt{\nu_L \Delta t} \gamma_L} \le \frac{D_L - \nu_L t \gamma_L}{\sqrt{\nu_L t} \gamma_L}\right) \\
&= \Pr\left(\sum_{j=1}^{t/\Delta t} X_{1j}\left(t_j\right) \le \frac{D_1 - \nu_1 t \gamma_1}{\sqrt{\nu_1 t} \gamma_1}, \ldots, \sum_{j=1}^{t/\Delta t} X_{Lj}\left(t_j\right) \le \frac{D_L - \nu_L t \gamma_L}{\sqrt{\nu_L t} \gamma_L}\right) \\
&\approx \Phi_L\left(\frac{D_1 - \nu_1 t \gamma_1}{\sqrt{\nu_1 t} \gamma_1}, \ldots, \frac{D_L - \nu_L t \gamma_L}{\sqrt{\nu_L t} \gamma_L}; 0, \boldsymbol{\Sigma}\right)
\end{aligned} \tag{5.5}$$

where $\boldsymbol{\Sigma}$ the covariance matrix with the correlation coefficient between different performance indicators includes, which is given as

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1L} \\ \rho_{21} & 1 & \cdots & \rho_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{L1} & \rho_{L2} & \cdots & 1 \end{bmatrix} \tag{5.6}$$

By extending the bivariate Birnbaum-Saunders distribution introduced by Kundu et al. [41], Pan et al. [40] further derived that the lifetime time distribution of the product, $F(t) = 1 - R(t)$ can be expressed by the multivariate Birnbaum-Saunders distribution of a L-dimensional vector and all its marginal distributions.

Compared with the multivariate Gaussian distribution based model, the multivariate Birnbaum-Saunders distribution based model is derived from the perspective of degradation increments of performance indicators, where the distribution and dependence of degradation increments are highlighted, and the degradation process of performance indicators are monotonic. The temporal variability within each performance indicator is also characterized by the gamma process. However, due to the assumptions made in the derivation, the multivariate Birnbaum-Saunders distribution model is limited to the situation that the shape function of the gamma process should be linear as $\nu(t) = \nu_l t$. Parameter estimation method for this model is investigated by Pan et al. [40]. The application of this model for bivariate degradation analysis has also been presented by Pan and Balakrishnan [4] and Pan et al. [42].

5.3.3 Degradation Rate Interaction Model

The degradation rate interaction model is introduced from the perspective of a multi-component product by leveraging the idea of degradation rate modelling [9]. The product is composed of multiple components, and these components are associated with multiple dependent performance indicators. The dependence is represented through the consideration that the degradation process of one component can be affected by the deterioration of other components. This effect is characterized through the modelling of the degradation rate function, which consists of two separated parts of the inherent degradation rate of the component and the inductive degradation rate by contributing components. A multivariate degradation process model is constructed by separately integrating the degradation rate function and associating a stationary Brownian motion noise for each component.

Suppose a product is composed of L components, and each component has a performance indicator described by the degradation process $Y_l(t)$ with $l = 1, \ldots, L$. Let $r_l(t)$ denote the degradation rate associated with the degradation process $Y_l(t)$. To incorporate the dependence among the degradation processes, a general form of degradation rate $r_l(t)$ is given as $r_l(t) = r_l(t; \kappa(t), h(\mathbf{Y}(t)))$ [9]. The degradation rate is defined on t and further determined by the functional relationship of $\kappa(t)$ and $h(\mathbf{Y}(t))$, for which $\kappa(t)$ is the inherent degradation rate of the component without the interaction of other components, and $h(\mathbf{Y}(t))$ is the inductive degradation rate by taking account of the effect from other components. A simplified form of the degradation rate is given as follows [43].

$$r_l(t) = \kappa_l + \sum_{l \neq k} \delta_{lk} y_k(t) \tag{5.7}$$

where a time invariant inherent degradation rate κ_l, and a linear combination of inductive degradation rates $\delta_{lk} y_k(t)$ with constant coefficient δ_{lk} is used.

Based on the degradation rate function, the degradation process $Y_l(t)$ can be given as follows.

$$Y_l(t) = \int_0^t r_l\left(u; \kappa(u), h\left(\mathbf{Y}(u)\right)\right) du + \varepsilon_l(t) \tag{5.8}$$

where $\varepsilon_l(t)$ is an error item for capturing the measurement noise and unidentified uncertainty, which is generally given as a white noise process or a stationary Brownian motion process.

It is difficult to derive the failure time distribution of the product under the degradation process model given in Eq. 5.8. By assuming a simplified form of the degradation rate as given in Eq. 5.7 and a stationary Brownian error item, a approximated failure time distribution is obtained by Bian & Gebraeel [43], where the degradation processes of the product are reconstructed as

$$\mathrm{d}\mathbf{Y}(t) = (\boldsymbol{\kappa} + \boldsymbol{\delta} \times \mathbf{Y}(t))\,\mathrm{d}t + \mathrm{d}\boldsymbol{\varepsilon} \tag{5.9}$$

where $\boldsymbol{\kappa} = (\kappa_1, \ \ldots, \kappa_L)$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ \ldots, \varepsilon_L)$ which follows a multivariate normal distribution $\mathrm{MVN}(0, \boldsymbol{\sigma}^2 t)$ with $\boldsymbol{\sigma}^2 = \mathrm{diag}\left(\sigma_1^2, \ldots, \sigma_L^2\right)$, and $\boldsymbol{\delta} \in \mathbb{R}^{L \times L}$ with non-diagonal entries δ_{lk} and diagonal entries $\delta_{ll} = 0$.

The stochastic differential equations given in Eq. 5.9 with initial condition $\mathbf{Y}(0) = \mathbf{Y}_0$ has been demonstrated to have a closed-form as $\mathbf{Y}(t)\mathrm{|}(\boldsymbol{\kappa}, \boldsymbol{\sigma}^2)$, which follows a multivariate normal distribution with mean vector $\boldsymbol{\mu}_0(t)\mathrm{|}(\boldsymbol{\kappa}, \boldsymbol{\sigma}^2)$ and covariance matrix $\boldsymbol{\Sigma}_0(t)\mathrm{|}(\boldsymbol{\kappa}, \boldsymbol{\sigma}^2)$ as follow [43]

$$\boldsymbol{\mu}_0(t) \mid \left(\boldsymbol{\kappa}, \boldsymbol{\sigma}^2\right) = \exp\left(t\boldsymbol{\delta}\right) \times \mathbf{Y}_0 + \int_0^t \exp\left(\left(t-u\right)\boldsymbol{\delta}\right) \times \boldsymbol{\kappa}\,\mathrm{d}u \tag{5.10}$$

$$\boldsymbol{\Sigma}_0(t) \mid \left(\boldsymbol{\kappa}, \boldsymbol{\sigma}^2\right) = \int_0^t \exp\left(\left(t-u\right)\boldsymbol{\delta}\right) \times \boldsymbol{\sigma}^2 \times \exp\left(\left(t-u\right)\boldsymbol{\delta}\right)^{\mathrm{T}} \mathrm{d}u \tag{5.11}$$

Given the degradation thresholds of the L performance indicators, the reliability function of the product can be approximated as follows.

$$R(t) = \Pr\left(Y_1(t) \le D_1, \ldots, Y_L(t) \le D_L\right)$$
$$\approx \Phi_L\left(D_1, \ldots, D_L; \boldsymbol{\mu}_0(t) | \left(\boldsymbol{\kappa}, \boldsymbol{\sigma}^2\right), \boldsymbol{\Sigma}_0(t) | \left(\boldsymbol{\kappa}, \boldsymbol{\sigma}^2\right)\right) \tag{5.12}$$

The calculation of the reliability function depends on the solution of the multidimensional integrals given in Eqs. 5.10 and 5.11, which can be obtained using mathematical software. Analytical solutions of Eq. 5.12 have been obtained by Bian and Gebraeel [43] for two special cases, that are the case with $\boldsymbol{\delta}$ being a diagonal matrix, and the case with $\boldsymbol{\delta}$ being a diagonalizable matrix.

Compared with the models introduced above, the degradation interaction model is introduced from the perspective of a multi-component system, where the dependence is originated from the mutual influences of components' deteriorations. A degradation interaction function and a linear system of stochastic differential equations are used to construct the multivariate degradation process model. If the characteristics of the systems, such as the structure, functionality, operating conditions, failure mechanism, and interaction patterns of components are well studied, the degradation rate interaction model is more suitable for degradation modelling and reliability analysis than the models presented above. However, limited by the availability of the product characteristic for determining the degradation interaction parameter $\boldsymbol{\delta}$, the degradation rate interaction has not received wide application in degradation based reliability modelling of the modern system.

5.3.4 Copula Based Multivariate Degradation Process Model

The copula based multivariate degradation process model is introduced by taking the copula based multivariate degradation process model is introduced by taking the advantage of the copula theory [44] for constructing multivariate probability distributions. By adopting a copula function, the dependence structure of random variables can be characterized separately from their marginal distribution functions. Within the copula based multivariate degradation process model, the degradation processes of performance indicators are separately modelled by marginal degradation processes with independent degradation increments, such as Wiener process, gamma process, and inverse Gaussian process. Their dependency is characterized by a copula function by assuming that their degradation increments of the multiple performance indicators at the same time interval are dependent. As a result, the characteristics of each performance indicator are described by its marginal degradation process. And their dependency is modelled by a copula function. This kind of multivariate degradation process model can facilitate the degradation modelling, parameter estimation, and reliability assessment, which has been widely investigated, such as the works presented by Sari et al. [38], Pan et al. [45], Wang and Pham [5], Wang et al. [46], and Peng et al. [13].

For a product with L performance indicators, let $Y_l(t)$ denote the degradation process of the lth performance indicator, which is modelled by a stochastic process. Let $\Delta Y_l(t_j) = Y_l(t_j) - Y_l(t_{j-1})$ denote the degradation increment within the time interval $[t_{j-1}, t_j]$, which follows a probability distribution with CDF $F_l(\Delta y_l(t_j))$ under the specific stochastic process chosen for the marginal degradation process. The dependence among the degradation processes is characterized through their degradation increments. It is assumed that the degradation increments $\boldsymbol{\Delta Y}_l(t_j) = (\Delta Y_1(t_j), \ldots, \Delta Y_L(t_j))$ for the L performance indicators within the same time interval $[t_{j-1}, t_j]$ are dependent. The degradation increments in disjoint time intervals are independent, e.g., $Y_l(t_j) - Y_l(t_{j-1})$ and $Y_l(t_{j-1}) - Y_l(t_{j-2})$ separately in $[t_{j-1}, t_j]$ and $[t_{j-2}, t_{j-1}]$ are s-independent. The joint probability distribution of $\boldsymbol{\Delta Y}_l(t_j)$ within the same time interval is characterized by leveraging a multivariate copula function as follow.

$$F\left(\boldsymbol{\Delta y}\left(t_j\right)\right) = \mathrm{C}\left(F_1\left(\Delta y_1\left(t_j\right)\right), \ldots, F_L\left(\Delta y_L\left(t_j\right)\right); \boldsymbol{\theta}^{\mathrm{Cop}}\right) \tag{5.13}$$

where $\boldsymbol{\Delta y}(t_j) = (\Delta y_1(t_j), \ldots, \Delta y_L(t_j))$ and $\mathrm{C}(u_1, \ldots, u_L; \boldsymbol{\theta}^{\mathrm{Cop}})$ is a L-dimensional multivariate copula function with parameters $\boldsymbol{\theta}^{\mathrm{Cop}}$ and $u_l \sim \mathrm{Uniform}(0, 1)$.

Under the dependency structure given in Eq. 5.13, the CDF of each degradation increment $F_l(\Delta y_l(t_j))$ is the marginal distribution of the joint CDF $F(\boldsymbol{\Delta y}(t_j))$. This characteristic makes the modeling of dependence among performance indicators separated from the modeling of marginal degradation process of each performance indicator. Such separation makes the construction of multivariate degradation model with different marginal degradation models feasible. In general, under the multiple degradation processes model presented in Sects. 5.3.1, 5.3.2, and 5.3.3, the marginal degradation models for the performance indicators are from the same stochastic model family, such as the normal distribution, Wiener process and gamma process. However, the performance indicators of a product may have different deterioration characteristics, which need different types of stochastic model families [15]. This practical requirement can be fulfilled through the copula based multivariate degradation process model. For instance, the performance indicators can be separately modelled by Wiener processes, gamma processes, inverse Gaussian process and so on, and their dependence under different stochastic processes can be constructed through Eq. 5.13.

Since the performance indicator is assumed following a stochastic process with independent degradation increments, the multivariate degradation process model based on copula function is constructed as follows.

$$Y_l\left(t_j\right) = \sum_{k=2}^{j} \Delta Y_l\left(t_k\right), l = 1, \ldots, L, j = 2, \ldots, +\infty,$$
$$\begin{cases} \Delta Y_l\left(t_k\right) \sim \mathrm{PRO}\left(\Delta y_l\left(t_k\right); \boldsymbol{\theta}_l^{\mathrm{Mar}}\right) \\ F\left(\boldsymbol{\Delta y}\left(t_j\right)\right) = \mathrm{C}\left(F_1\left(\Delta y_1\left(t_j\right)\right), \ldots, F_L\left(\Delta y_L\left(t_j\right)\right); \boldsymbol{\theta}^{\mathrm{Cop}}\right) \end{cases} \tag{5.14}$$

where $\mathrm{PRO}\left(\Delta y_1\left(t_k\right);\boldsymbol{\theta}_l^{\mathrm{Mar}}\right)$ is the distribution of degradation increment of the lth performance indicator, which is determined based on the stochastic process chosen for the modelling of this performance indicator. The stochastic processes generally used for degradation modelling include the Wiener process, gamma process and inverse Gaussian process [3], which separately giving rise to the $\mathrm{PRO}\left(\Delta y_1\left(t_k\right);\boldsymbol{\theta}_l^{\mathrm{Mar}}\right)$ presenting as the normal distribution, gamma distribution, and inverse Gaussian distribution.

The multivariate copula function in Eq. 5.14 is a multivariate distribution with uniformly distributed marginal distributions on $[0,1]$. A group of CDFs of degradation increments $(F_1(\Delta y_1(t_j)), \ldots, F_L(\Delta y_L(t_j)))$ is a sample from the multivariate copula function. Accordingly, when the CDFs of degradation increments are available, the choice of copula function to model their dependency can be implemented through the methods for multivariate probability distribution selection. Qualitative method such as the scatter plots presented in Wu [47], and quantitative method such as Bayesian model selection introduced by Huard and Evin [48], both can be adopted to choose the right copula function for multivariate degradation modelling. In bivariate degradation processes, there are various types of copula functions can be used for degradation modelling, which are listed as follows.

1. Gaussian copula

$$\mathrm{C}\left(u_1,u_2\right) = \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)}\frac{1}{2\pi\sqrt{1-\alpha^2}}\exp\left(-\frac{u_1^2-2u_1u_2+u_2^2}{2\left(1-\alpha^2\right)}\right)\mathrm{d}u_1\mathrm{d}u_2,\alpha\in[-1,1] \tag{5.15}$$

2. Frank copula

$$\mathrm{C}\left(u_1,u_2\right) = -\frac{1}{\alpha}\ln\left(1+\frac{\left(\mathrm{e}^{-\alpha u_1}-1\right)\left(\mathrm{e}^{-\alpha u_2}-1\right)}{\mathrm{e}^{-\alpha}-1}\right),\quad \alpha\neq 0 \tag{5.16}$$

3. Gumbel copula

$$\mathrm{C}\left(u_1,u_2\right) = \exp\left(-((-\ln u_1)^{\alpha}+(-\ln u_2)^{\alpha})^{1/\alpha}\right),\quad \alpha\in[1,\infty) \tag{5.17}$$

4. Clayton copula

$$\mathrm{C}\left(u_1,u_2\right) = \max\left(\left(u_1^{-\alpha}+u_2^{-\alpha}-1\right)^{-1/\alpha},0\right),\quad \alpha\in[-1,\infty)\setminus 0 \tag{5.18}$$

In the multivariate degradation process, the multivariate Gaussian copula [49], the multivariate t-copula [50], and the vine copula [51] are commonly used in various literature.

Given the degradation thresholds of the L performance indicators, the reliability function of the product is given as follows.

$$R(t) = \Pr\left\{\sup_{s\le t} Y_1(s) < D_1, \ldots, \sup_{s\le t} Y_l(s) < D_l, \ldots, \sup_{s\le t} Y_L(s) < D_L\right\} \tag{5.19}$$

It's often difficult to calculate the reliability function due to the unavailability of the analytical solution. Simulation method is generally used to obtain the reliability of the product based on the model in Eq. 5.14 [13]. When all the parameters for the copula function and the marginal degradation processes are available, a group of random samples, $[\tilde{u}_1, \ldots, \tilde{u}_L]$, are firstly generated from the copula function, $C(u_1, \ldots, u_L; \boldsymbol{\theta}^{\mathrm{Cop}})$. By calculating the inverse CDF of degradation increments based on the generated samples, the degradation increments for marginal degradation processes at a specific time interval are obtained as $\Delta\tilde{y}_l(t_k) = F_l^{-1}\left(\tilde{u}_l; \boldsymbol{\theta}_l^{\mathrm{Mar}}\right)$. The marginal degradation process at a specific time point is obtained based the degradation increments of the passed time intervals as $\tilde{y}_l\left(t_j\right) = \sum_{k=2}^{j} \Delta\tilde{y}_l(t_k)$. The failure time point of the product can then be obtained by comparing the simulated observations of the marginal degradation processes with their respective degradation thresholds. By repeating the simulation process above, a group of samples of failure time points is obtained, and the reliability function can be statistically summarized from these generated failure time samples.

The copula based model has been investigated extensively for bivariate degradation processes modeling, which can be summarized in the following three groups according to the stochastic processes incorporated, i.e., bivariate Wiener process model, bivariate gamma process model, and bivariate inverse Gaussian process model. Bivariate degradation processes model based on Wiener process and copula function has been studied by Pan et al. [45], Wang et al. [52], and Jin and Matthews [53]. In detail, Pan et al. [45] presented a Bayesian method based on Markov chain Monte Carlo method to facilitate parameter estimation and reliability assessment. Wang et al. [52] further introduced a Bayesian method for residual life estimation under this kind of bivariate degradation process model. Jin and Mathews [53] introduced a method for degradation test planning and measurement plan optimization for products modelled by the bivariate Wiener process and copula function.

Bivariate degradation process model based on gamma process and copula function has been investigated by Pan and Sun [54], Hong et al. [55], and Wang et al. [46]. Wang et al. [46] introduced a two-stage method to estimate the parameter of a bivariate non-stationary gamma degradation process, for which the residual life estimation can be implemented in an adaptive manner. Pan and Sun [54] presented a method for step-stress accelerated degradation test planning under the bivariate degradation model based on gamma process and copula function. Hong et al. [55] investigated the condition-based maintenance optimization for products with dependent components deteriorations, where the dependent deteriorations were characterized by gamma process and copula function.

Bivariate degradation process models based on inverse Gaussian process and copula function have been studied by Liu et al. [56] and Peng et al. [13]. Within these studies, Liu et al. [56] incorporated time scale transformation and random drift into the bivariate inverse Gaussian process model to account for the nonlinear of degradation process and heterogeneity within a product population. Peng et al. [13] introduced a two-stage Bayesian parameter estimation and reliability assessment method to deal with the incomplete degradation observations for products modelled by the bivariate inverse Gaussian process and copula function.

According to the literature review presented above, it can be found that most of the models are introduced for bivariate degradation processes. Limited exceptions are the works presented by Peng et al. [15], Pan and Sun [54] and Wang et al. [57]. Although the copula based model given in Eq. 5.14 is flexible for constructing multivariate degradation process models, the study on copula based multivariate degradation process model is still limited and deserves more investigation on both the utilization of multivariate copula function and the applications. There is also a strong imperative to advance investigation on residual life assessment, degradation test planning, optimal maintenance decision based on the copula based multivariate degradation process models.

5.4 Multiple Degradation Processes Under Dynamic Environments

The model for the degradation process under dynamic environments is aimed to deal with the two critical aspects highlighted above for degradation modelling of complex systems under dynamic environments. Various methods and models have been summarized above to deal with the two critical aspects, which include the modelling of multiple degradation process and the incorporation of dynamic environmental effect. However, the research on multiple degradation modelling for complex system under dynamic environments has not been well studied. There are generally two types of models having been introduced, which are the multiple degradation process and random shock models [5, 58, 59], and the multiple degradation process and dynamic covariate model [15].

5.4.1 Multiple Degradation Process and Random Shock Models

The multiple degradation process and random shock models have been introduced for the situation where a system is subject to degradation processes and random shocks [5]. The degradation processes are associated with the inherent failure mechanism of the system, and the random shocks are related to the exterior environmental effects. The dependence among degradation processes can be either from the inherent dependent failure mechanism of the system, or from the exterior effect of the random shocks, or both. The inherent dependent failure mechanism of the system is characterized through the copula function [5], which is similar to the copula based multivariate degradation process model. The random shocks introduced dependence is characterized through instantaneous degradation increments [58, 59] or degradation rate acceleration or both [5], where the effect of random shocks is incorporated.

Suppose a product has L degradation processes $Y_l(t)$ with $l = 1, \ldots, L$, and the arrival of random shocks experienced by the product follows a Poisson process $N(t)$. To facilitate the derivation, a simple model for the degradation process $Y_l(t)$ is often used, such as a multiplicative path function as $Y_l(t) = X_l\eta_l(t)$ with X_l being a random variable [5]. The effect of the random shocks on the degradation processes is characterized into two types, i.e., (1) the cumulative degradation increments as $S_l(t) = \sum_{k=1}^{N(t)}\omega_{lk}$ with ω_{lk} denoting the instantaneous degradation increment introduced by the kth random shock, and (2) the degradation rate acceleration, which is incorporated into $Y_l(t)$ through the idea of accelerated degradation modelling, such as the scaling of t into $te^{G_l(t)}$ with $G_l(t) = \lambda_1 N(t) + \lambda_2\sum_{k=1}^{N(t)}\omega_k$ [5]. By considering both effects of the random shocks, the degradation process model of the lth degradation process is presented as

$$M_l(t) = Y_l\left(t; G_l(t)\right) + S_l(t) \tag{5.20}$$

The marginal distribution of the degradation process $M_l(t)$ can be derived based on the model for the degradation process $Y_l(t)$, the model for random shocks $N(t)$, and the model for the instantaneous degradation increment ω_{lk}. For instance, assume $Y_l(t) = X_l\eta_l(t)$ with $F_{X_l}(x_l)$ being the distribution of random parameter X_l, $N(t)$ follows a homogeneous Poisson process with occurring rate λ, and ω_{lk} follows an exponential distribution with mean μ_l, the distribution of $M_l(t)$ can be derived as

$$\begin{aligned} F_l\left(m_l(t)\right) &= \Pr\left(M_l(t) < m_l(t)\right) = \Pr\left(N(t) = 0\right)\Pr\left(Y_l(t) < m_l(t)\right) \\ &\quad + \sum_{n=1}^{+\infty} \Pr\left(N(t) = n\right)\Pr\left(Y_l\left(t; G_l(t)\right) + S_l(t) < m_l(t) | N(t) = n\right) \\ &= e^{-\lambda t}F_{X_l}\left(\frac{m_l(t)}{\eta_l(t)}\right) + \sum_{n=1}^{+\infty}\frac{e^{-\lambda t}(-\lambda t)^n}{n!}\int_{u=0}^{m_l(t)} F_{X_i}\left(\frac{m_l(t) - u}{\eta_l\left(te^{G_l(t)}\right)}\right)\frac{u^{n-1}e^{-\frac{u}{\mu_l}}}{\Gamma(n)\mu_l^n}du \end{aligned} \tag{5.21}$$

The dependency among $\mathbf{M}(t) = (M_1(t), \ldots, M_L(t))$ mainly originates from two parts, which are the dependency introduced by the dependence of the inherent degradation processes $\mathbf{Y}(t) = (Y_1(t), \ldots, Y_L(t))$, and the dependency originated from the series of random shocks. Assume the instantaneous degradation increments ω_{lk} with $l = 1, \ldots, L$ and $k = 1, \ldots, N(t)$ are independent, the dependence among $\mathbf{M}(t)$ can be characterized through the joint distribution of $\mathbf{M}(t)$ as

$$\begin{aligned} F\left(m_1(t), \ldots, m_L(t)\right) = \Pr\left(M_1(t) < m_1(t), \ldots, M_L(t) < m_L(t)\right) \\ = \sum_{n=0}^{+\infty} \mathbf{C}\left(F_1\left(m_1(t)|N(t) = n\right), \ldots, F_L\left(m_L(t)|N(t) = n\right)\right) \Pr\left(N(t) = n\right) \end{aligned} \tag{5.22}$$

where $\mathbf{C}(F(m_1(t)|\, N(t) = n), \ldots, F(m_L(t)|\, N(t) = n))$ is a copula function used to model the dependence originated from the inherent dependence of degradation processes $\mathbf{Y}(t)$, and $F_l(m_l(t)|\, N(t) = n)$ is the marginal distribution of degradation process $M_l(t)$ condition on $N(t) = n$, which can be derived similarly to Eq. 5.22.

Given the degradation thresholds of the degradation processes, the reliability of the product is given as

$$R(t) = \Pr\left\{M_1(t) < D_1, \ldots, M_L(t) < D_L\right\} = F\left(D_1(t), \ldots, D_L(t)\right) \tag{5.23}$$

By substituting the degradation thresholds into Eq. 5.22, the reliability function of the product can be obtained. However, it is often difficult to obtain an analytical solution for the reliability function. Wang and Pham [5] derive the reliability bounds for the reliability of product with bivariate degradation processes and random shocks. A two-stage parameter inference method and a comparison of the reliability under the model introduced above with constant copulas and time-varying copulas have also been studied by Wang and Pham [5]. Song et al. [58] investigated the reliability of multi-component systems with multiple degradation processes and random shocks, where the model presented above is simplified without considering the effect of degradation rate acceleration by the random shocks. The maintenance modeling and optimization under the multiple degradation process and random shock model has been studied by Song et al. [58]. Song et al. [59] further extended the model introduced above into a more advanced model, where the dependency of transmitted shock sizes to hard failure process, and shock damages to specific degradation processes (soft failure processes) for all components have been studied.

Compared with the degradation process model under static environments, the multiple degradation process and random shock model introduced in this section successfully incorporates the effect of dynamic environments through the modeling of the random shocks and the dependency caused by the effect of random shocks. However, more assumptions need to be assumed to derive the model, which requires a deep understanding of the failure mechanism of the degradation processes, the arrival of random shocks, and their influences on the degradation processes as well. A limit on the models used for the marginal degradation processes is necessary to facilitate the calculation of the reliability as given in Eq. 5.23, where a multiplicative path function or a degradation path model is generally applicable. In addition, methods for parameter estimation, degradation analysis and residual life prediction with this kind of multiple degradation process and random shock model has not been sufficiently studied.

5.4.2 Multiple Degradation Process and Dynamic Covariate Models

The multiple degradation process with dynamic covariate model is introduced by utilizing the idea of dynamic covariate and copula function. The dynamic covariate is used to incorporate the dynamic environments into degradation process models. The copula function is used to model the dependency among the degradation processes. This kind model is similar to the copula based multivariate degradation process models introduced in Sect. 5.3.4. Major difference is that the marginal degradation processes used in Sect. 5.3.4 are substituted with marginal degradation processes with dynamic covariates.

Suppose a product has L degradation processes $Y_l(t)$ with $l=1,\ldots,L$, and the dynamic environments experienced by the product are summarized into the external factors $\mathbf{X}^E$. Under the effect of the external factors, the marginal degradation processes $Y_l(t)$ are modelled using stochastic process models with dynamic covariates. Following the methods and models summarized in Sect. 5.2, the baseline stochastic process models $Y_l(t)$ are modified into $Y_l(t;\mathbf{X}^E)$, such as the Wiener process model $Y_l(t)=\eta_l(t)+\sigma_l\mathsf{B}(t)$ with $\mathsf{B}(t)$ being a standard Brownian motion and the inverse Gaussian process model $Y_l(t)\sim \mathrm{IG}\left(\Lambda_l(t),\lambda_l\Lambda_l^2(t)\right)$ with $\Lambda_l(t)$ being a nonnegative and monotonically increasing function are separately modified into $Y_l(t;\mathbf{X}^E)=\eta_l(t;\mathbf{X}^E)+\sigma_l\mathsf{B}(t)$ and $Y_l\left(t;\mathbf{X}^E\right)\sim \mathrm{IG}\left(\Lambda_l\left(t;\mathbf{X}^E\right),\lambda_l\Lambda_l^2\left(t;\mathbf{X}^E\right)\right)$.

Let $\Delta Y_l(t_j;\mathbf{X}^E)=Y_l(t_j;\mathbf{X}^E)-Y_l(t_{j-1};\mathbf{X}^E)$ denote the degradation increment within the time interval $[t_{j-1},t_j]$, which follows a probability distribution with CDF $F_l(\Delta y_l(t_j);\mathbf{X}^E)$ under the specific stochastic process chosen for the marginal degradation process. The dependence among the degradation processes is characterized through their degradation increments. It is assumed that the degradation increments $\mathbf{\Delta Y}_l(t_j)=(\Delta Y_1(t_j),\ldots,\Delta Y_L(t_j))$ for the L performance indicators within the same time interval $[t_{j-1},t_j]$ are dependent. Similar to the copula based multivariate process model presented in Sect. 5.3.4, the joint probability distribution of $\mathbf{\Delta Y}_l(t_j;\mathbf{X}^E)$ within the same time interval is characterized by leveraging a multivariate copula function as

$$F\left(\mathbf{\Delta y}\left(t_j\right);\mathbf{X}^E\right)=\mathrm{C}\left(F_1\left(\Delta y_1\left(t_j\right);\mathbf{X}^E\right),\ldots,F_L\left(\Delta y_L\left(t_j\right);\mathbf{X}^E\right);\boldsymbol{\theta}^{\mathrm{Cop}}\right) \tag{5.24}$$

where $\mathbf{\Delta y}(t_j)=(\Delta y_1(t_j),\ldots,\Delta y_L(t_j))$ and $\mathrm{C}(u_1,\ldots,u_L;\boldsymbol{\theta}^{\mathrm{Cop}})$ is a L-dimensional multivariate copula function with parameters $\boldsymbol{\theta}^{\mathrm{Cop}}$ and $u_l\sim\mathrm{Uniform}(0,1)$.

The multiple degradation process and dynamic covariate model, which is based on the modified marginal degradation process and copula function is constructed as

$$\begin{cases} Y_l\left(t_j;\mathbf{X}^E\right) = \sum_{k=2}^{j} \Delta Y_l\left(t_k;\mathbf{X}^E\right), l = 1,\ldots,L, j = 2,\ldots,+\infty, \\ \Delta Y_l\left(t_k;\mathbf{X}^E\right) \sim \mathrm{PRO}\left(\Delta y_l\left(t_k\right);\mathbf{X}^E,\boldsymbol{\theta}_l^{\mathrm{Mar}}\right) \\ F\left(\boldsymbol{\Delta}\mathbf{y}\left(t_j\right);\mathbf{X}^E\right) = \mathrm{C}\left(F_1\left(\Delta y_1\left(t_j\right);\mathbf{X}^E\right),\ldots,F_L\left(\Delta y_L\left(t_j\right);\mathbf{X}^E\right);\boldsymbol{\theta}^{\mathrm{Cop}}\right) \end{cases} \tag{5.25}$$

where $\mathrm{PRO}\left(\Delta y_1\left(t_k\right);\mathbf{X}^E,\boldsymbol{\theta}_l^{\mathrm{Mar}}\right)$ is the distribution of degradation increment of the lth performance indicator with the external factors incorporated.

Given the degradation thresholds of the L performance indicators, the reliability function of the product is given as

$$R(t) = \Pr\left\{\sup_{s\le t} Y_1\left(s;\mathbf{X}^E\right) < D_1,\ldots,\sup_{s\le t} Y_l\left(s;\mathbf{X}^E\right) < D_l,\ldots,\sup_{s\le t} Y_L\left(s;\mathbf{X}^E\right) < D_L\right\} \tag{5.26}$$

It is generally to obtain the analytical solution to the reliability function, simulation based method is needed to implement the calculation. To facilitate the degradation analysis of complex systems under a dynamic environment with the multiple degradation process and dynamic covariate model, Peng et al. [15] introduced a Bayesian parameter estimation method and a simulation based degradation inference, reliability assessment and residual life prediction method. However, their method is limited to the situation that the external factors $\mathbf{X}^E$ are assumed available during the degradation analysis, where no probabilistic model is constructed for these external factors. A more general model with external factors modelled as random variables or stochastic processes deserves further investigation.

Compared with the multiple degradation process and random shocks model, the model based on dynamic covariates and copula function is more flexible for incorporating external factors and characterizing degradation processes. There are various types of stochastic models, covariate models and copula functions available for constructing the multivariate degradation process model as given in Eq. 5.25. The Bayesian parameter estimation method and degradation analysis method presented by Peng et al. [15] can also be extended to the model with more general assumptions. More advanced methods for parameter estimation and residual life predication still deserve further investigation. In addition, the study on maintenance modeling and optimization and system health management under this kind of multivariate degradation process model has not been presented yet.

5.5 Conclusions

In this paper, a summary of the state of arts on the researches of reliability modelling of complex system under dynamic environments is presented by highlighting two critical aspects, i.e., (1) modelling multiple degradation processes, and (2) characterizing dynamic environments effects. We mainly focused on the various types of multivariate degradation process models because these models are critical for reliability modelling of modern complex system with multiple performance indicators or composed by multiple components. In addition, the characterization of dynamic environments and the methods for incorporating dynamic environments into reliability models has been discussed due to the consideration that the effects of environments are often simplified or omitted in general degradation modelling. Through these two aspects, the paper is organized into two progressive sessions, i.e., multiple degradation processes under static dynamic environments, and multiple degradation processes under dynamic environments.

For the multiple degradation processes under static environments, the multivariate degradation process models based on multivariate Gaussian distribution and multivariate Birnbaum-Saunders distribution, the degradation rate interaction model, and the models based on copula function and stochastic processes are reviewed. Among these models, the copula based multivariate degradation process model has the capability of modelling various types of degradation processes while keeping the model simple enough for model construction and parameter estimation. However, limited by the utilization of multivariate copula function in reliability engineering, most of the models are bivariate degradation process model. More research is needed to extend the research on multiple degradation processes under dynamic environments.

For the multiple degradation processes under dynamic environments, the multiple degradation process and random shocks models and the multiple degradation processes and dynamic covariates models are summarized. There is not too much research published on this topic, partially due to the limitation of real case examples, the unavailability of physical mechanism, and unjustifiability of complex models. Within the proposed model, the multiple degradation process with random shocks model is basically an extension of the degradation-threshold-shock model, which has been studied for one-dimensional degradation process. The multiple degradation process with dynamic covariate model is a combination of the copula based multivariate degradation model and dynamic covariate models, which has been investigated extensively for one-dimensional degradation process and failure time models. There is a strong imperative for more investigation on multivariate degradation modelling with dynamic environments incorporated. In addition, methods for parameter estimation, residual life predication, degradation test planning, maintenance strategy optimization, and model comparison and selection, are major open area deserving extensive investigations under the multivariate degradation modelling with dynamic environments highlighted.

Acknowledgements This project is supported by National Natural Science Foundation of China (Grant No. 51605081).

References

1. Bennett R, Booker JM, Keller-McNulty S, Singpurwalla ND (2003) Testing the ustestable: reliability in the 21st century. IEEE T Reliab 52:118–124
2. Meeker WQ, Hong Y (2014) Reliability meets big data: opportunities and challenges. Qual Eng 26:102–116. doi:10.1080/08982112.2014.846119
3. Ye ZS, Xie M (2014) Stochastic modelling and analysis of degradation for highly reliable products. Appl Stoch Model Bus Ind 31:16–32. doi:10.1002/asmb.2063
4. Pan Z, Balakrishnan N (2011) Reliability modeling of degradation of products with multiple performance characteristics based on gamma processes. Reliab Eng Syst Saf 96:949–957. doi:10.1016/j.ress.2011.03.014
5. Wang Y, Pham H (2012) Modeling the dependent competing risks with multiple degradation processes and random shock using time-varying copulas. IEEE T Reliab 61:13–22. doi:10.1109/TR.2011.2170253
6. Kharoufeh JP, Cox SM, Oxley ME (2013) Reliability of manufacturing equipment in complex environments. Ann Oper Res 209:231–254. doi:10.1007/s10479-011-0839-x
7. Liao H, Tian Z (2013) A framework for predicting the remaining useful life of a single unit under time-varying operating conditions. IIE Trans 45:964–980. doi:10.1080/0740817X.2012.705451
8. Wang D, Miao Q, Pecht M (2013) Prognostics of lithium-ion batteries based on relevance vectors and a conditional three-parameter capacity degradation model. J Power Sources 239:253–264. doi:10.1016/j.jpowsour.2013.03.129
9. Bian L, Gebraeel N (2014) Stochastic modeling and real-time prognostics for multi-component systems with degradation rate interactions. IIE Trans 46:470–482. doi:10.1080/0740817X.2013.812269
10. Si XS, Hu CH, Kong X, Zhou DH (2014) A residual storage life prediction approach for systems with operation state switches. IEEE T Ind Electron 61:6304–6315. doi:10.1109/TIE.2014.2308135
11. Ye ZS, Chen N, Shen Y (2015) A new class of Wiener process models for degradation analysis. Reliab Eng Syst Saf 139:58–67. doi:10.1016/j.ress.2015.02.005
12. Hong Y, Duan Y, Meeker WQ (2015) Statistical methods for degradation data with dynamic covariates information and an application to outdoor weathering data. Technometrics 57:180–193. doi:10.1080/00401706.2014.915891
13. Peng W, Li YF, Yang YJ, Zhu SP, Huang HZ (2016) Bivariate analysis of incomplete degradation observations based on inverse Gaussian processes and copulas. IEEE T Reliab 65:624–639. doi:10.1109/TR.2015.2513038
14. Xu X, Li Z, Chen N (2016) A hierarchical model for lithium-ion battery degradation prediction. IEEE T Reliab 65:310–325. doi:10.1109/TR.2015.2451074
15. Peng W, Li YF, Mi J, Yu L, Huang HZ (2016) Reliability of complex systems under dynamic conditions: a Bayesian multivariate degradation perspective. Reliab Eng Syst Saf 153:75–87. doi:10.1016/j.ress.2016.04.005
16. Singpurwalla ND (1995) Survival in dynamic environments. Stat Sci 10:86–103
17. Lee Ting ML, Whitmore GA (2006) Threshold regression for survival analysis: modelling event times by a stochastic process reaching a boundary. Stat Sci 21:501–513
18. Bae SJ, Kuo W, Kvam PH (2007) Degradation models and implied lifetime distributions. Reliab Eng Syst Saf 92:601–608
19. Si XS, Wang W, Hu CH, Zhou DH (2011) Remaining useful life estimation – a review on the statistical data driven approaches. Eur J Oper Res 213:1–14
20. Yashin AI, Manton KG (1997) Effects of unobserved and partially observed covariate processes on system failure: a review of models and estimation strategies. Stat Sci 12:20–34
21. Bagdonavičius V, Nikulin M (2000) Estimation in degradation models with explanatory variables. Lifetime Data Anal 7:85–103

22. Bagdonavičius V, Nikulin M (2009) Statistical models to analyze failure, wear, fatigue, and degradation data with explanatory variables. Commun Stat – Theory M 38:3031–3047
23. Lawless J, Crowder M (2004) Covariates and random effects in a gamma process model with application to degradation and failure. Lifetime Data Anal 10:213–227
24. Kharoufeh JP, Solo CJ, Ulukus MY (2010) Semi-markov models for degradation-based reliability. IIE Trans 42:599–612
25. Meeker WQ, Escobar LA, Lu J (1998) Accelerated degradation tests: modeling and analysis. Technometrics 40:89–99. doi:10.1080/00401706.1998.10485191
26. Park C, Padgett WJ (2006) Stochastic degradation models with several accelerating variables. IEEE Trans Reliab 55:379–390
27. Bae SJ, Kvam PH (2004) A nonlinear random-coefficients model for degradation testing. Technometrics 46:460–469
28. Liao HT, Elsayed EA (2006) Reliability inference for field conditions from accelerated degradation testing. Nav Res Logist 53:576–587
29. Bian L, Gebraeel N, Kharoufeh JP (2015) Degradation modeling for real-time estimation of residual lifetimes in dynamic environments. IIE Trans 47:471–486. doi:10.1080/0740817X.2014.955153
30. Chan V, Meeker WQ (2001) Estimation of degradation-based reliability in outdoor environments. Stat Prepr 25:1–31
31. Ye ZS, Chen N (2014) The inverse Gaussian process as a degradation model. Technometrics 56:302–311. doi:10.1080/00401706.2013.830074
32. Peng W, Li YF, Yang YJ, Huang HZ, Zuo MJ (2014) Inverse Gaussian process models for degradation analysis: a Bayesian perspective. Reliab Eng Syst Saf 130:175–189. doi:10.1016/j.ress.2014.06.005
33. Wang L, Pan R, Li X, Jiang T (2013) A Bayesian reliability evaluation method with integrated accelerated degradation testing and field information. Reliab Eng Syst Saf 112:38–47. doi:10.1016/j.ress.2012.09.015
34. Ye ZS, Hong Y, Xie Y (2013) How do heterogeneities in operating environments affect field failure predictions and test planning? Ann Appl Stat 7:2249–2271. doi:10.1214/13-AOAS666
35. Kharoufeh JP, Cox SM (2005) Stochastic models for degradation-based reliability. IIE Trans 37:533–542
36. Flory JA, Kharoufeh JP, Gebraeel NZ (2014) A switching diffusion model for lifetime estimation in randomly varying environments. IIE Trans 46:1227–1241
37. Wang P, Coit DW (2004) Reliability prediction based on degradation modeling for systems with multiple degradation measures. Reliab Maintainab Annu Symp – RAMS 2004:302
38. Sari JK, Newby MJ, Brombacher AC, Tang LC (2009) Bivariate constant stress degradation model: LED lighting system reliability estimation with two-stage modelling. Qual Reliab Eng Int 25:1067–1084
39. Park C, Padgett WJ (2005) Accelerated degradation models for failure based on geometric Brownian motion and Gamma processes. Lifetime Data Anal 11:511–527
40. Pan Z, Feng J, Sun Q (2016) Lifetime distribution and associated inference of systems with multiple degradation measurements based on Gamma processes. Eksploat Niezawodn 18: 307–313
41. Kundu D, Balakrishnan N, Jamalizadeh A (2010) Bivariate Birnbaum-Sanders distribution and associated inference. J Multivar Anal 101:113–125
42. Pan Z, Sun Q, Feng J (2016) Reliability modeling of systems with two dependent degrading components based on gamma processes. Commun Stat A-Theor 45:1923–1938
43. Bian L, Gebraeel N (2014) Stochastic framework for partially degradation systems with continuous component degradation-rate-interactions. Nav Res Logist 61:286–303
44. Nelsen RB (2006) An introduction to copulas. Springer, New York
45. Pan Z, Balakrishnan N, Sun Q, Zhou J (2013) Bivariate degradation analysis of products based on Wiener processes and copulas. J Stat Comput Simul 83:1316–1329
46. Wang X, Balakrishnan N, Guo B, Jiang P (2015) Residual life estimation based on bivariate non-stationary gamma degradation process. J Stat Comput Simul 85(2):405–421

47. Wu S (2014) Construction of asymmetric copulas and its application in two-dimensional reliability modelling. Eur J Oper Res 238:476–485
48. Huard D, Evin G, Favre AC (2006) Bayesian copula selection. Comput Stat Data Anal 51: 809–822
49. Jaworski PJ, Durante F, Härdle WK, Rychlik T (2010) Copula theory and its applications. Springer, Heidelberg
50. Frahm G, Junker M, Szimayer A (2003) Elliptical copulas: applicability and limitations. Stat Probab Lett 63:275–286
51. Kurowicka D, Joe H (2011) Dependence modeling: vine copula handbook. World Scientific Publishing, Singapore
52. Wang X, Guo B, Cheng ZJ (2014) Residual life estimation based on bivariate Wiener degradation process with time-scale transformations. J Stat Comput Simul 84:545–563
53. Jin G, Matthews D (2014) Measurement plan optimization for degradation test design based on the bivariate Wiener process. Qual Reliab Eng Int 30:1215–1231
54. Pan Z, Sun Q (2014) Optimal design for step-stress accelerated degradation test with multiple performance characteristics based on gamma processes. Commun Stat B-Simul 43:298–314
55. Hong HP, Zhou W, Zhang S, Ye W (2014) Optimal condition-based maintenance decisions for systems with dependent stochastic degradation of components. Reliab Eng Syst Saf 121: 276–288
56. Liu Z, Ma X, Yang J, Zhao Y (2014) Reliability modeling for systems with multiple degradation processes using inverse Gaussian process and copulas. Math Probl Eng 2014:829597-1–10
57. Wang X, Balakrishnan N, Guo B (2015) Residual life estimation based on nonlinear-multivariate Wiener processes. J Stat Comput Simul 85:1742–1764
58. Song S, Coit DW, Feng Q, Peng H (2014) Reliability analysis for multi-component systems subject to multiple dependent competing failure processes. IEEE Trans Reliab 63:331–345
59. Song S, Coit DW, Feng Q (2016) Reliability analysis of multiple-component series systems subject to hard and soft failures with dependent shock effects. IIE Trans 48:720–735