Mapping Properties of One Class of Quasielliptic Operators

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Abstract. The paper is devoted to the theory of quasielliptic operators. We consider scalar and homogeneous quasielliptic operators $\mathcal{L}(D_x)$ with lower terms in the whole space $\mathbb{R}^n$. Our aim is to study mapping properties of these operators in weighted Sobolev spaces. We introduce a special scale of weighted Sobolev spaces $W^l_{p,q,\sigma}(\mathbb{R}^n)$. These spaces coincide with known spaces of Sobolev type for some parameters l, q, σ. We investigate mapping properties of the operators $\mathcal{L}(D_x)$ in the spaces $W^l_{p,q,\sigma}(\mathbb{R}^n)$. We indicate conditions for unique solvability of quasielliptic equations and systems in these spaces, obtain estimates for solutions and formulate an isomorphism theorem for quasielliptic operators. To prove our results we construct special regularizers for quasielliptic operators.

Keywords: Quasielliptic operators · Weighted Sobolev spaces · Isomorphism

1 Introduction

In the paper a class of quasielliptic operators $\mathcal{L}(D_x)$ is considered in the whole space $\mathbb{R}^n$. This class belongs to the classes of quasielliptic operators introduced by S.M. Nikol'skii [1] and L.R. Volevich [2]. Our aim is to study mapping properties of the operators $\mathcal{L}(D_x)$ in special weighted Sobolev spaces $W^l_{p,q,\sigma}(\mathbb{R}^n)$ and to establish isomorphism theorems.

The first isomorphism theorems for scalar elliptic operators were proved by L.A. Bagirov and V.A. Kondratiev [3], M. Cantor [4,5], R.C. McOwen [6,7]. Isomorphism theorems for matrix homogeneous elliptic operators were proved by Y. Choquet-Bruhat and D. Christodoulou [8], R.B. Lockhart and R.C. McOwen [9].

As a rule, isomorphism theorems for elliptic operators are not trivial. For example, consider the elliptic operator

$$\triangle - \varepsilon I : W^2_p(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n), \qquad 1 < p < \infty,$$

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where $\triangle$ is the Laplace operator. If $\varepsilon > 0$ then the mapping is an isomorphism. However, the mapping

$$\triangle : W_p^2(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n)$$

is not an isomorphism. Taking into account results of L.D. Kudryavtsev [10], L.A. Bagirov and V.A. Kondratyev [3], L. Nirenberg and H.F. Walker [11], M. Cantor [4,5], it is necessary to use weighted Sobolev spaces for proving isomorphism theorems for the Laplace operator. The first isomorphism theorems for the Laplace operator were proved by M. Cantor [4] and R.C. McOwen [6]. They used special weighted Sobolev spaces.

It should be noted that isomorphism theorems for matrix elliptic operators can be more complicated. For example, consider the Stokes operator

$$\begin{pmatrix} -\triangle & 0 & 0 & D_{x_1} \\ 0 & -\triangle & 0 & D_{x_2} \\ 0 & 0 & -\triangle & D_{x_3} \\ D_{x_1} & D_{x_2} & D_{x_3} & 0 \end{pmatrix}, \qquad x \in \mathbb{R}^3.$$

This operator is elliptic in the Douglis–Nirenberg sense. One can prove an isomorphism theorem for the Stokes operator (see [12]). However, it is necessary to use a product of special weighted Sobolev spaces with different components of smoothness vectors and different weights.

The first isomorphism theorems for matrix homogeneous quasielliptic operators were proved by G.V. Demidenko [13,14]. The investigations [13,14] were continued by G.N. Hile [15]. In the present paper we consider a more general class of quasielliptic operators $\mathcal{L}(D_x)$ in $\mathbb{R}^n$.

2 Quasielliptic Operators

First we consider the following scalar differential operator

$$L(D_x) = \sum_{\beta} a_\beta D_x^\beta,$$

where the coefficients a_β are constants. Suppose that its symbol $L(i\xi)$, $\xi \in \mathbb{R}^n$, satisfies the following conditions.

Condition 1. The symbol $L(i\xi)$ is homogeneous with respect to a vector $\alpha = (\alpha_1, \dots, \alpha_n)$, $1/\alpha_j \in \mathbb{N}$, $j = 1, \dots, n$; i.e.,

$$L(c^\alpha i\xi) = cL(i\xi), \qquad c > 0.$$

Condition 2. The equality

$$L(i\xi) = 0, \qquad \xi \in \mathbb{R}^n,$$

holds if and only if $\xi = 0$.

Definition 1. *The differential operator $L(D_x)$ is called quasielliptic, if its symbol satisfies Conditions 1, 2.*

This class of operators belongs to the class of differential operators introduced by S.M. Nikol'skii [1].

Quasielliptic operators $L(D_x)$ whose symbols $L(i\xi)$ are homogeneous with respect to a vector α are usually called *quasielliptic operators without lower terms*. Such operators can be written in the form

$$L(D_x) = \sum_{\beta\alpha=1} a_\beta D_x^\beta. \tag{1}$$

Examples of such operators are elliptic operators, $2b$-parabolic operators without lower terms, etc.

Note that the symbol of the quasielliptic operator (1) satisfies the following estimate

$$c_1\langle\xi\rangle \leq |L(i\xi)| \leq c_2\langle\xi\rangle, \qquad \langle\xi\rangle^2 = \sum_{j=1}^n \xi_j^{2/\alpha_j}, \qquad \xi \in \mathbb{R}^n,$$

where c_1, $c_2 > 0$ are constants.

We now consider the differential operators

$$\mathcal{L}(D_x) = L(D_x) + \sum_{\beta\alpha<1} a_\beta D_x^\beta, \tag{2}$$

where $L(D_x)$ is the quasielliptic operator (1). We will call operators of the form (2) *quasielliptic operators with lower terms*. Denote the differential operator corresponding to the lower terms by

$$L'(D_x) = \sum_{\beta\alpha<1} a_\beta D_x^\beta.$$

Condition 3. Suppose that the symbol of the differential operator (2) satisfies the estimate

$$c_3\left(\langle\xi\rangle + \langle\xi\rangle^q\right) \leq |L(i\xi) + L'(i\xi)| \leq c_4\left(\langle\xi\rangle + \langle\xi\rangle^q\right), \qquad \xi \in \mathbb{R}^n, \tag{3}$$

where $0 \leq q < 1$, c_3, $c_4 > 0$ are constants.

Example 1. Consider the differential operator

$$\mathcal{L}(D_x) = \triangle^m + \varepsilon(-1)^{m-k}\triangle^k, \qquad m > k, \quad \varepsilon > 0. \tag{4}$$

We have

$$L(D_x) = \triangle^m, \qquad L'(D_x) = \varepsilon(-1)^{m-k}\triangle^k, \qquad \alpha_1 = \ldots = \alpha_n = 1/(2m).$$

Obviously, Conditions 1–3 are fulfilled for $q = k/m$.

We now consider the matrix differential operator

$$L(D_x) = \sum_{\beta\alpha=1} A_\beta D_x^\beta, \tag{5}$$

where the coefficients A_β are constant $(m \times m)$-matrices with real or complex entries. Suppose that its symbol $L(i\xi)$ satisfies the following condition.

Condition 4. The equality

$$\det L(i\xi) = 0, \qquad \xi \in \mathbb{R}^n,$$

holds if and only if $\xi = 0$.

Definition 2. *The matrix differential operator (5) is called homogeneous quasi-elliptic operator if its symbol satisfies Condition 4.*

This class of operators belongs to the class of differential operators introduced by L.R. Volevich [2]. Examples of such operators are homogeneous elliptic operators, $2b$-parabolic operators without lower terms, parabolic operators with 'opposite times directions', etc.

We now consider matrix differential operators of the form

$$\mathcal{L}(D_x) = L(D_x) + \sum_{\beta\alpha<1} A'_\beta D_x^\beta, \tag{6}$$

where $L(D_x)$ is the matrix quasielliptic operator of the form (5), the coefficients A'_β are constant $(m \times m)$-matrices.

We will call operators of the form (6) *homogeneous quasielliptic operator with lower terms.* Suppose that its symbol $\mathcal{L}(i\xi)$ satisfies the following condition.

Condition 5. Suppose that the symbol of the differential operator (6) satisfies the estimate

$$c_5 \left(\langle\xi\rangle + \langle\xi\rangle^q\right)^m \le |\det \mathcal{L}(i\xi)| \le c_6 \left(\langle\xi\rangle + \langle\xi\rangle^q\right)^m, \qquad \xi \in \mathbb{R}^n, \tag{7}$$

where $0 \le q < 1$, c_5, $c_6 > 0$ are constants.

Example 2. Consider the parabolic operator with 'opposite times directions'

$$\mathcal{L}(D_x) = \begin{pmatrix} D_{x_n} - \triangle' & \alpha \\ \beta & D_{x_n} + \triangle' \end{pmatrix},$$

where $\triangle'$ is the Laplace operator in $\mathbb{R}^{n-1}$ and $\alpha\beta > 0$. Obviously,

$$\mathcal{L}(D_x) = \begin{pmatrix} D_{x_n} - \triangle' & 0 \\ 0 & D_{x_n} + \triangle' \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}.$$

Consequently, Conditions 4, 5 are fulfilled for

$$m = 2, \qquad \alpha = \left(\frac{1}{2}, \dots, \frac{1}{2}, 1\right), \qquad q = 0.$$

Isomorphism theorems for quasielliptic operators of the forms (1), (5) without lower terms were proved in [13–15]. In our paper we study mapping properties of quasielliptic operators of the forms (2), (6). Particularly, we formulate isomorphism theorems for these operators.

3 Weighted Sobolev Spaces

We introduce the *weighted Sobolev spaces* $W^l_{p,q,\sigma}(\mathbb{R}^n)$. Using these spaces, one can solve the problem on isomorphism for quasielliptic operators $\mathcal{L}(D_x)$ of the form (2) or (6).

Definition 3. *Let* $l = (1/\alpha_1, \ldots, 1/\alpha_n)$, $1/\alpha_j \in \mathbb{N}$, $j = 1, \ldots, n$, $1 < p < \infty$, $0 \le q \le 1$, $\sigma \ge 0$. *Denote by* $W^l_{p,q,\sigma}(\mathbb{R}^n)$ *the weighted Sobolev space of functions* $u \in L_{loc}(\mathbb{R}^n)$ *having the weak derivatives* $D^\nu_x u$, $\nu\alpha \le 1$, *such that*

$$D^\nu_x u \in L_p(\mathbb{R}^n) \quad for \quad q \le \nu\alpha \le 1,$$

$$\|(1 + \langle x \rangle)^{-\sigma(q-\nu\alpha)} D^\nu_x u(x),\, L_p(\mathbb{R}^n)\| < \infty \quad for \quad 0 \le \nu\alpha < q.$$

Here $\langle x \rangle^2 = \sum\limits_{j=1}^{n} x_j^{2/\alpha_j}$.

Introduce the norm

$$\|u,\, W^l_{p,q,\sigma}(\mathbb{R}^n)\| = \sum_{q \le \nu\alpha \le 1} \|D^\nu_x u(x),\, L_p(\mathbb{R}^n)\| + \sum_{0 \le \nu\alpha < q} \|(1 + \langle x \rangle)^{-\sigma(q-\nu\alpha)} D^\nu_x u(x),\, L_p(\mathbb{R}^n)\|. \tag{8}$$

The weighted Sobolev spaces $W^l_{p,q,\sigma}(\mathbb{R}^n)$ coincide with well-known spaces for some parameters l, q, σ. We consider several examples.

Example 3. Obviously, the space $W^l_{p,q,0}(\mathbb{R}^n) = W^l_{p,0,\sigma}(\mathbb{R}^n)$ is the Sobolev space $W^l_p(\mathbb{R}^n)$.

Example 4. The space $W^l_{p,1,\sigma}(\mathbb{R}^n)$ coincides with the space $W^l_{p,\sigma}(\mathbb{R}^n)$ introduced in [16]. Indeed, by definition [16],

$$\|u,\, W^l_{p,\sigma}(\mathbb{R}^n)\| = \sum_{0 \le \nu\alpha \le 1} \|(1 + \langle x \rangle)^{-\sigma(1-\nu\alpha)} D^\nu_x u(x),\, L_p(\mathbb{R}^n)\|.$$

Example 5. In the isotropic case $1/\alpha_1 = \ldots = 1/\alpha_n = \bar{l}$ the norm (8) for $q = \sigma = 1$ is equivalent to the norm

$$\sum_{0 \le |\beta| \le \bar{l}} \|(1 + |x|)^{-(\bar{l}-|\beta|)} D^\beta_x u(x),\, L_p(\mathbb{R}^n)\|. \tag{9}$$

Then, from the work [10] of L.D. Kudryavtsev it follows that the space $W^l_{p,1,1}(\mathbb{R}^n)$ for $p > n$ coincides with the Sobolev space

$$W^{\bar{l}}_{p,\Box}(\mathbb{R}^n), \quad \Box = \{x \in \mathbb{R}^n : \ |x_j| < 1, \ j = 1, \dots, n\},$$

where

$$\|u, W^{\bar{l}}_{p,\Box}(\mathbb{R}^n)\| = \int\limits_{\Box} |u(x)|dx + \sum_{|\beta|=\bar{l}} \|D^\beta_x u(x), L_p(\mathbb{R}^n)\|.$$

Example 6. Consider the Nirenberg–Walker–Cantor space $M^p_{\ell,k}(\mathbb{R}^n)$ [4,11] whose norm is defined as

$$\|u, M^p_{\ell,k}(\mathbb{R}^n)\| = \sum_{|\beta|\le\ell} \|(1+|x|)^{k+|\beta|} D^\beta_x u(x), L_p(\mathbb{R}^n)\|.$$

Clearly, by (9) the space $W^l_{p,1,1}(\mathbb{R}^n)$ coincides with the space $M^p_{\bar{l},-\bar{l}}(\mathbb{R}^n)$ in the isotropic case $1/\alpha_1 = \ldots = 1/\alpha_n = \bar{l}$ for $q = \sigma = 1$, $p > 1$.

Definition 4. *Denote by* $\overset{\circ}{W}{}^l_{p,q,\sigma}(\mathbb{R}^n)$ *the completion of* $C^\infty_0(\mathbb{R}^n)$ *with respect to the norm (8).*

From Definitions 3 and 4 it follows that the space $\overset{\circ}{W}{}^l_{p,q,\sigma}(\mathbb{R}^n)$ is embedded in the space $W^l_{p,q,\sigma}(\mathbb{R}^n)$. One can show that the strict embedding holds

$$\overset{\circ}{W}{}^l_{p,q,\sigma}(\mathbb{R}^n) \subset W^l_{p,q,\sigma}(\mathbb{R}^n)$$

for sufficiently large $\sigma > 1$.

In the next theorem we indicate the condition when these spaces coincide. Note that theorems of such type are very important in the theory of differential operators.

Theorem 1. *If* $0 \le \sigma \le 1$ *then* $\overset{\circ}{W}{}^l_{p,q,\sigma}(\mathbb{R}^n) = W^l_{p,q,\sigma}(\mathbb{R}^n)$.

Definition 5. *Denote by*

$$L_{p,\gamma}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad \gamma \in R,$$

the space of integrable functions with the norm

$$\|u, L_{p,\gamma}(\mathbb{R}^n)\| = \|(1+\langle x\rangle)^{-\gamma} u(x), L_p(\mathbb{R}^n)\|.$$

Thereafter we will say that a vector-function

$$U(x) = (u_1(x), \dots, u_m(x))^T, \qquad m \ge 1$$

belongs to the weighted Sobolev space $W^l_{p,q,\sigma}(\mathbb{R}^n)$, if every its component u_j belongs to $W^l_{p,q,\sigma}(\mathbb{R}^n)$. By definition,

$$\|U, W^l_{p,q,\sigma}(\mathbb{R}^n)\| = \sum_{j=1}^{m} \|u_j, W^l_{p,q,\sigma}(\mathbb{R}^n)\|.$$

Analogously, a vector-function

$$F(x) = (f_1(x), \dots, f_m(x))^T, \qquad m \geq 1$$

belongs to the weighted space $L_{p,\gamma}(\mathbb{R}^n)$, if every its component f_j belongs to $L_{p,\gamma}(\mathbb{R}^n)$ and

$$\|F, L_{p,\gamma}(\mathbb{R}^n)\| = \sum_{j=1}^{m} \|f_j, L_{p,\gamma}(\mathbb{R}^n)\|.$$

4 Mapping Properties of the Operators (2), (6)

Consider the quasielliptic operator $\mathcal{L}(D_x)$ defined by (2) or (6). Introduce the notation $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$.

The following theorems hold.

Theorem 2. *Let $\beta = (\beta_1, \dots, \beta_n)$, $1 \geq \beta\alpha \geq q$. Then the following estimate is satisfied for every $U \in C_0^\infty(\mathbb{R}^n)$*

$$\|D_x^\beta U(x), L_p(\mathbb{R}^n)\| \leq c_\beta \|\mathcal{L}(D_x)U(x), L_p(\mathbb{R}^n)\|,$$

where the constant $c_\beta > 0$ does not depend on U.

Theorem 3. *Let $\beta = (\beta_1, \dots, \beta_n)$, $q > \beta\alpha \geq 0$ and*

$$\frac{|\alpha|}{p} > \sigma(q - \beta\alpha) > q - \beta\alpha - \frac{|\alpha|}{p'}, \quad 1 \geq \sigma \geq 0, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Then the following estimate is satisfied for every $U \in C_0^\infty(\mathbb{R}^n)$

$$\|\langle x\rangle^{-\sigma(q-\beta\alpha)} D_x^\beta U(x), L_p(\mathbb{R}^n)\| \leq c\|\langle x\rangle^{(1-\sigma)(q-\beta\alpha)} \mathcal{L}(D_x)U(x), L_p(\mathbb{R}^n)\|,$$

where the constant $c_\beta > 0$ does not depend on U.

Theorem 4. *Let*

$$|\alpha| > q, \quad |\alpha|/p > \sigma q > |\alpha|/p - (|\alpha| - q).$$

Then for every $F \in L_{p,(\sigma-1)q}(\mathbb{R}^n)$ there exists a unique $U \in W^l_{p,q,\sigma}(\mathbb{R}^n)$ such that

$$\mathcal{L}(D_x)U(x) = F(x), \qquad x \in \mathbb{R}^n.$$

Moreover, the estimate holds

$$\|U, W^l_{p,q,\sigma}(\mathbb{R}^n)\| \leq c\|F, L_{p,(\sigma-1)q}(\mathbb{R}^n)\|$$

with a constant $c > 0$ independent of F.

Theorem 5. *Let $|\alpha|/p > q$. Then the mapping*

$$\mathcal{L}(D_x):\ W^l_{p,q,1}(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n), \quad 1 < p < \infty$$

is an isomorphism.

Remark 1. Theorems 4, 5 are analogs of some theorems in [13–15] for quasielliptic operators without lower terms.

We illustrate Theorem 5 by using the differential operator (4):

$$\mathcal{L}(D_x) = \triangle^m + \varepsilon(-1)^{m-k}\triangle^k, \quad m \geq k, \quad \varepsilon > 0.$$

Taking into account Example 1, we have

$$\alpha_1 = \ldots = \alpha_n = 1/(2m), \qquad q = k/m.$$

Consequently, by Theorem 5 the mapping

$$\mathcal{L}(D_x):\ W^l_{p,q,1}(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n), \quad l = (2m, \ldots, 2m), \tag{10}$$

is an isomorphism for $p \in (1, \frac{n}{2k})$, $n > 2k$.

Consider the critical cases in (4): $k = 0$ and $k = m$.

In the first case $k = 0$ we have $\triangle^0 = I$, $q = 0$ and $W^l_{p,0,1}(\mathbb{R}^n) = W^l_p(\mathbb{R}^n)$. Then (10) is rewritten in the form

$$\triangle^m + \varepsilon(-1)^m I:\ W^l_p(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n).$$

Therefore the isomorphism theorem gives the classical result.

In the second case $k = m$, we have $q = 1$ and $W^l_{p,1,1}(\mathbb{R}^n) = W^l_{p,1}(\mathbb{R}^n)$. Then (10) is rewritten in the form

$$(1 + \varepsilon)\triangle^m:\ W^l_{p,1}(\mathbb{R}^n) \longrightarrow L_p(\mathbb{R}^n).$$

The isomorphism theorem for $p \in \left(1, \frac{n}{2m}\right)$, $n > 2m$ follows from [7].

5 Elements of Used Technique

To prove of the above results we use a technique of integral representations for regularizers of differential operators. Our technique is based on the special representation by S.V. Uspenskii [17] for integrable functions:

$$\varphi(x) = \lim_{h \to 0} (2\pi)^{-n} \int_h^{h^{-1}} v^{-|\alpha|-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(i \frac{x-y}{v^\alpha} \xi\right) G(\xi)\varphi(y) d\xi dy dv, \tag{11}$$

where

$$G(\xi) = 2M\langle\xi\rangle^{2M} \exp(-\langle\xi\rangle^{2M}), \quad \langle\xi\rangle^2 = \sum_{i=1}^n \xi_i^{2/\alpha_i}, \quad M,\ 1/\alpha_i \in \mathbb{N}.$$

Applying the integral representation (10), we construct the following integral operators

$$P_{j,h}F(x) = (2\pi)^{-n} \int\limits_{h}^{h^{-1}} v^{-|\alpha|} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \exp\left(i\frac{x-y}{v^{\alpha}}\xi\right)$$

$$\times G(\xi)\left(\sum_{k=1}^{m} l^{j,k}(i\xi)F_k(y)\right) d\xi dy dv, \quad j=1,\ldots,m, \quad h>0,$$

where $l^{j,k}(i\xi)$ are entries of the inverse matrix $(\mathcal{L}(i\xi))^{-1}$. In the case of $m=1$ we write $(\mathcal{L}(i\xi))^{-1}F(y)$ instead of the sum

$$\sum_{k=1}^{m} l^{j,k}(i\xi)F_k(y).$$

In the present paper we use the operators $P_{j,h}$ for $h \ll 1$ in order to construct regularizers of the quasielliptic operators (2), (6). Using these regularizers, we indicate the conditions for unique solvability of the quasielliptic equations and systems in the weighted Sobolev spaces, obtain the estimates for the solutions and formulate the isomorphism theorem for the quasielliptic operators.

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