Some Problems in Second Order Evolution Inclusions with Boundary Condition: A Variational Approach

Charles Castaing, Truong Le Xuan, Paul Raynaud de Fitte and Anna Salvadori

Abstract We prove, under appropriate assumptions, the existence of solutions for a second order evolution inclusion with boundary conditions via a variational approach.

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C. Castaing (✉)
Département de Mathématiques, Université Montpellier II, Case 051, 34095 Montpellier Cedex 5, France
e-mail: castaing.charles@gmail.com

T. Le Xuan
Department of Mathematics and Statistics, University of Economics of Ho Chi Minh City, 59C Nguyen Dinh Chieu Str. Dist. 3, Ho Chi Minh City, Vietnam
e-mail: lxuantruong@gmail.com

P. Raynaud de Fitte
Laboratoire Raphaël Salem, UMR CNRS 6085, Normandie Université, Rouen, France
e-mail: prf@univ-rouen.fr

A. Salvadori
Dipartimento di Matematica, Università‘di Perugia, via Vanvitelli 1, 06123 Perugia, Italy
e-mail: mateas@unipg.it

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1 Introduction

In the present paper, we prove, under appropriate assumptions, the existence of solutions for a second order evolution inclusion with boundary conditions governed by subdifferential operators of the form

$$f(t) \in \ddot{u}(t) + M\dot{u}(t) + \partial\varphi(u(t)), t \in [0, T]. \tag{I}$$

Here, M is positive, φ is a lower semicontinuous convex proper function defined on $\mathbf{R}^d$ and $\partial\varphi(u(t))$ is the subdifferential of the function φ at the point $u(t)$ and the perturbation f belongs to $L^2_{\mathbf{R}^d}([0, T])$. It is well known that this problem is difficult and needs a specific treatment via the Moreau-Yosida approximation or epiconvergence approach. See Attouch–Cabot–Redon [4] and Schatzmann [24] for a deep study of these problems, Castaing–Raynaud de Fitte–Salvadori [11], Castaing–Le Xuan Truong [8] dealing with second order evolution with m-point boundary conditions via the epiconvergence approach. These considerations lead us to consider the variational limits of a fairly general approximating problem

$$f^n(t) \in \ddot{u}^n(t) + M\dot{u}^n(t) + \partial\varphi_n(u^n(t)), t \in [0, T] \tag{II}$$

where u^n is a $W^{2,1}_{\mathbf{R}^d}([0, T])$-solution, f^n weakly converging in $L^2_{\mathbf{R}^d}([0, T])$ to f^∞, φ_n is a convex Lipschitz function which epiconverges to a lower semicontinuous convex proper function φ_∞. This approximating problem covers various type of problems of practical interest in several dynamic systems, evolution inclusion, control theory etc. Here we focus on several variational limits of solutions via the Biting Lemma and Young measures and other tools occurring in this approach by showing under suitable limit assumption on the boundary conditions that $(\ddot{u}^n)$ is $L^1_{\mathbf{R}^d}([0, T])$-bounded. This main fact allows to study the variational limit of solutions in this problem, in particular, the traditional estimated energy for the variational limit solutions is conserved almost everywhere. The applicability of our abstract framework given therein (Proposition 3.3) will be exemplified in considering the existence of solution for second order differential inclusions

$$f(t) \in \ddot{u}(t) + M\dot{u}(t) + \partial\varphi(u(t)), t \in [0, T]$$

under m-point boundary condition or anti-periodic conditions and further related second order evolution inclusions in the literature. This will be done by applying our abstract result to the single valued approximating problem

$$f^n(t) = \ddot{u}^n(t) + M\dot{u}^n(t) + \nabla\varphi_n(u^n(t)), t \in [0, T] \tag{III}$$

where $\nabla\varphi_n$ is the gradient of the C^1, Lipschitz, convex function φ_n that epi-converges to a proper convex lower semicontinuous function φ_∞ and f^n weakly converges in $L^2_{\mathbf{R}^d}([0, T])$ to f^∞ so that the variational limit solutions u^∞ to (III) are *generalized solutions* to the inclusion

$$f^\infty(t) \in \ddot{u}^\infty(t) + M\dot{u}^\infty(t) + \partial\varphi_\infty(u^\infty(t)), t \in [0, T]$$

with appropriate properties, namely, the solution limit u^∞ is $W^{1,1}_{BV}([0, T])$, that is, u^∞ is continuous and its derivative $\dot{u}^\infty$ is bounded variation (BV for short) and the estimated energy holds almost everywhere

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u_0) + \frac{1}{2}||\dot{u}_0)||^2 - M\int_0^t ||\dot{u}^\infty(s)||^2 ds + \int_0^t \langle f^\infty(s), \dot{u}^\infty(s)\rangle ds$$

with further related variational inclusion, in particular,

$$f^\infty(t) \in \zeta^\infty(t) + Mu^\infty(t) + \partial\varphi_\infty(u^\infty(t)), t \in [0, T]$$

almost everywhere, ζ^∞ being the biting limit of the $L^1_{\mathbf{R}^d}([0, T])$-bounded sequence $(\ddot{u}^n)$. Section 3 is devoted to second order evolution inclusion with boundary conditions. We present the variational limits of the general approximating problem (II) and the applications of variational limits of the approximating problem (III) to the existence problem of second order evolution inclusion (I) involving variational techniques, the Biting Lemma, the characterization of the second dual of $L^1_{\mathbf{R}^d}$ and Young measures. It is worth to mention that the approximation (III) occurs in practical cases of second order evolution inclusion governed by subdifferential operators. For instance, Attouch–Cabot–Redon [4] considered the approximating problem

$$0 = \ddot{u}^n(t) + \gamma\dot{u}^n(t) + \nabla\varphi_n(u^n(t)), t \in [0, T]$$

$$u^n(0) = u^n_0, \dot{u}^n(0) = \dot{u}^n_1$$

where γ is positive, $\nabla\varphi_n$ is the gradient of a C^1, smooth function. Schatzmann [24] considered the approximating problem

$$f(t) = \ddot{u}_\lambda(t) + \partial\varphi_\lambda(u_\lambda(t)), t \in [0, T]$$

$$u_\lambda(0) = u_0, \dot{u}_\lambda(0) = u_1$$

where $f \in L^2_{\mathbf{R}^d}([0, T])$ and $\partial\varphi_\lambda$ is the Moreau-Yosida approximation to the lower semicontinuous convex proper function φ. M. Mabrouk [19] continued the work of M. Schatzmann [24] by considering the approximating problem

$$f_\lambda(t) = \ddot{u}_\lambda(t) + \nabla\varphi_\lambda(u_\lambda(t)), t \in [0, T]$$
$$u_\lambda(0) = u_0, \dot{u}_\lambda(0) = u_1,$$

with $f_\lambda \in L^1_{\mathbf{R}^d}([0,T])$. In Sect. 4, we apply our techniques to the study of both first order and second order evolution equations with anti-periodic boundary condition using the approximating problem

$$\begin{gathered} f^n(t) = \ddot{u}^n(t) + M\dot{u}^n(t) + \nabla\varphi_n(u^n(t)), t \in [0,T] \\ u^n(0) = -u^n(T), \end{gathered}$$

where $u^n \in W^{2,2}_{\mathbf{R}^d}([0,T])$ and $f^n \in L^2_{\mathbf{R}^d}([0,T])$, see H. Okochi [22], A. Haraux [17], Aftabizadeh, Aizicovici and Pavel [1, 2], Aizicovici and Pavel [3] and the references therein.

A general analysis of some related problems in Hilbert space is available, c.f K. Maruo [19] and M. Schatzmann [24].

2 Some Existence Theorems in Second Order Evolution Inclusions with *m*-Point Boundary Condition

We will use the following definitions and notations and summarize some basic results.

- Let E be a separable Banach space, $\overline{B}_E(0,1)$ is the closed unit ball of E.
- $c(E)$ (resp. $cc(E)$) (resp. $ck(E)$)(resp. $cwk(E)$) is the collection of nonempty closed (resp. closed convex) (resp. compact convex) (resp. weakly compact convex) subsets of E.
- If A is a subset of E, $\delta^*(.,A)$ is the support function of A.
- $\mathcal{L}([0,T])$ is the σ-algebra of Lebesgue measurable subsets of $[0,T]$.
- If X is a topological space, $\mathcal{B}(X)$ is the Borel tribe of X.
- $L^1_E([0,T],dt)$ (shortly $L^1_E([0,T])$) is the Banach space of Lebesgue–Bochner integrable functions $f:[0,T]\to E$.
- A mapping $u:[0,T]\to E$ is *absolutely continuous* if there is a function $\dot{u} \in L^1_E([0,T])$ such that $u(t) = u(0) + \int_0^t \dot{u}(s)\,ds,\ \forall t \in [0,T]$.
- If X is a topological space, $\mathcal{C}_E(X)$ is the space of continuous mappings $u:X\to E$ equipped with the norm of uniform convergence.
- A set-valued mapping $F:[0,T]\rightrightarrows E$ is measurable if its graph belongs to $\mathcal{L}([0,T])\otimes\mathcal{B}(E)$.
- A convex weakly compact valued mapping $F:X\to ck(E)$ defined on a topological space X is scalarly upper semicontinuous if for every $x^*\in E^*$, the scalar function $\delta^*(x^*,F(.))$ is upper semicontinuous on X.

We refer to [13] for measurable multifunctions and Convex Analysis.

For the sake of completeness, we recall and summarize some results developed in [9]. By $W^{2,1}_E([0,T])$ we denote the set of all continuous functions in $\mathcal{C}_E([0,T])$ such that their first derivatives are continuous and their second derivatives belong to $L^1_E([0,T])$.

Lemma 2.1 *Assume that E is a separable Banach space. Let* $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\gamma > 0$, $m > 3$ *be an integer number, and* $\alpha_i \in \mathbf{R}$ $(i = 1, \ldots, m-2)$ *satisfying the condition*

$$\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma\eta_i)) \neq 0.$$

Let $G : [0, 1] \times [0, 1] \to \mathbf{R}$ *be the function defined by*

$$G(t, s) = \begin{cases} \frac{1}{\gamma}(1 - \exp(-\gamma(t - s))), & 0 \le s \le t \le 1 \\ 0, & t < s \le 1 \end{cases} + \frac{A}{\gamma}(1 - \exp(-\gamma t))\,\phi(s), \tag{2.1}$$

where

$$\phi(s) = \begin{cases} 1 - \exp(-\gamma(1 - s)) - \sum_{i=1}^{m-2} \alpha_i (1 - \exp(-\gamma(\eta_i - s))), & 0 \le s < \eta_1, \\ 1 - \exp(-\gamma(1 - s)) - \sum_{i=2}^{m-2} \alpha_i (1 - \exp(-\gamma(\eta_i - s))), & \eta_1 \le s \le \eta_2, \\ \ldots\ldots & \\ 1 - \exp(-\gamma(1 - s)), & \eta_{m-2} \le s \le 1, \end{cases} \tag{2.2}$$

and

$$A = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma\eta_i) \right)^{-1}. \tag{2.3}$$

Then the following assertions hold

(i) For every fixed $s \in [0, 1]$, *the function* $G(., s)$ *is right derivable on* $[0, 1[$ *and left derivable on* $]0, 1]$. *Its derivative is given by*

$$\left(\frac{\partial G}{\partial t}\right)_+ (t, s) = \begin{cases} \exp(-\gamma(t - s)), & 0 \le s \le t < 1 \\ 0, & 0 \le t < s < 1 \end{cases} + A \exp(-\gamma t)\phi(s), \tag{2.4}$$

$$\left(\frac{\partial G_\tau}{\partial t}\right)_- (t, s) = \begin{cases} \exp(-\gamma(t - s)), & 0 \le s < t \le 1 \\ 0, & 0 < t \le s \le 1 \end{cases} + A \exp(-\gamma t)\phi(s). \tag{2.5}$$

(ii) $G(\cdot, \cdot)$ *and* $\frac{\partial G}{\partial t}(\cdot, \cdot)$ *satisfies*

$$|G(t, s)| \le M_G \quad \text{and} \quad \left|\frac{\partial G}{\partial t}(t, s)\right| \le M_G \quad \forall (t, s) \in [0, 1] \times [0, 1],$$

where

$$M_G = \max\{\gamma^{-1}, 1\} \left[1 + |A| \left(1 + \sum_{i=1}^{m-2} |\alpha_i| \right) \right].$$

(iii) *If* $u \in W_E^{2,1}([0,1])$ *with* $u(0) = x$ *and* $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$, *then*

$$u(t) = e_x(t) + \int_0^1 G(t,s)(\ddot{u}(s) + \gamma \dot{u}(s))ds, \quad \forall t \in [0,1],$$

where

$$e_x(t) = x + A\left(1 - \sum_{i=1}^{m-2} \alpha_i\right)(1 - \exp(-\gamma t))x.$$

(iv) *Let* $f \in L_E^1([0,1])$ *and let* $u_f : [0,1] \to E$ *be the function defined by*

$$u_f(t) = e_x(t) + \int_0^1 G(t,s) f(s)ds \quad \forall t \in [0,1].$$

Then we have

$$u_f(0) = x \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i).$$

Further the function u_f *is weakly derivable on* $[0,1]$ *and its weak derivative* $\dot{u}_f$ *is defined by*

$$\dot{u}_f(t) = \lim_{h \to 0} \frac{u_f(t+h) - u_f(t)}{h} = \dot{e}_x(t) + \int_\tau^1 \frac{\partial G}{\partial t}(t,s) f(s)ds,$$

with

$$\dot{e}_x(t) = \gamma A\left(1 - \sum_{i=1}^{m-2} \alpha_i\right)\exp(-\gamma t)x.$$

(v) *If* $f \in L_E^1([0,1])$, *the function* $\dot{u}_f$ *is weakly derivable, and its weak derivative* $\ddot{u}_f$ *satisfies*

$$\ddot{u}_f(t) + \gamma \dot{u}_f(t) = f(t) \quad \text{a.e. } t \in [0,1].$$

The following is a direct consequence of Lemma 2.1.

Proposition 2.1 *Let* $f \in L_E^1([0,1])$. *The m-point boundary problem*

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = f(t), \ t \in [0,1] \\ u(0) = x, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{cases}$$

has a unique $W_E^{2,1}([0,1])$*-solution* u_f, *with integral representation formulas*

$$\begin{cases} u_f(t) = e_x(t) + \int_0^1 G(t,s) f(s)ds, \ t \in [0,1] \\ \dot{u}_f(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s)ds, \ t \in [0,1]. \end{cases}$$

where

$$\begin{cases} e_x(t) = x + A(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x \\ \dot{e}_x(t) = \gamma A \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma t)x \\ A \quad = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\right)^{-1}. \end{cases}$$

The following result and its notation will be used in the next section.

Proposition 2.2 *With the hypotheses and notations of Proposition 2.1, let E be a separable Banach space and let $X : [0, 1] \rightrightarrows E$ be a measurable convex weakly compact valued and integrably bounded mapping. Then the solution set of $W_E^{2,1}([0, 1])$-solutions to*

$$\begin{cases} \ddot{u}_f(t) + \gamma \dot{u}_f(t) = f(t), \ f \in S_X^1 \\ u_f(0) = x, \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i) \end{cases}$$

is bounded, convex, equicontinuous and sequentially weakly compact in $\mathcal{C}_E([0, 1])$.

Proof Let us set

$$\mathcal{X} := \left\{ u_f \in \mathcal{C}_E([0, 1]) : u_f(t) = e_x(t) + \int_0^1 G(t, s) f(s) ds, \ t \in [0, 1], \ f \in S_X^1 \right\}$$

with

$$\begin{cases} e_x(t) = x + A(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x, \ t \in [0, 1] \\ \dot{e}_x(t) = \gamma A \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma t)x, \ t \in [0, 1] \\ A \quad = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\right)^{-1}. \end{cases}$$

Taking account of the properties of G in Lemma 2.1, it is not difficult to show that $\mathcal{X}$ is bounded, convex, equicontinuous and relatively weakly compact in $\mathcal{C}_E([0, 1])$ because for each $t \in [0, T]$, $\int_0^1 G(t, s)X(s)ds$ is convex and weakly compact, see e.g. [11]. We only need to check the compactness property since other properties are obvious. Indeed, let $u_{f_n} \in \mathcal{X}$. As S_X^1 is $\sigma(L_E^1, L_{E_s^*}^\infty)$ sequentially compact, we may assume that (f_n) $\sigma(L_E^1, L_{E_s^*}^\infty)$ converges to $f_\infty \in S_X^1$. Then we have for each $t \in [0, 1]$,

$$\begin{aligned} \text{w-}\lim_n u_{f_n}(t) &= e_x(t) + \text{w} - \lim_n \int_0^1 G(t, s) f_n(s) ds \\ &= e_x(t) + \int_0^1 G(t, s) f_\infty(s) ds := u_{f_\infty}(t). \end{aligned}$$

This means that $u_{f_n}(t)$ converges to $u_{f_\infty}(t)$ in E_σ for every $t \in [0, 1]$. Hence u_{f_n} converges weakly in $\mathcal{C}_E([0, 1])$ to $u_{f_\infty} \in \mathcal{X}$. Similarly using the properties of $\frac{\partial G}{\partial t}$ in Lemma 2.1,

$$\mathcal{Y} := \left\{ \dot{u}_f \in \mathcal{C}_E([0, 1]) : \dot{u}_f(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G}{\partial t}(t, s) f(s) ds, \ t \in [0, 1], \ f \in S^1_X \right\}$$

is bounded, convex, equicontinuous and sequentially weakly compact in $\mathcal{C}_E([0, 1])$ with

$$\begin{cases} \dot{e}_x(t) = \gamma A \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma t) x, \ t \in [0, 1] \\ A = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\right)^{-1}, \end{cases}$$

and we have

$$\begin{aligned} \mathrm{w} - \lim_n \dot{u}_{f_n}(t) &= \dot{e}_x(t) + \mathrm{w} - \lim_n \int_0^1 \frac{\partial G}{\partial t}(t, s) f_n(s) ds \\ &= \dot{e}_x(t) + \int_0^1 \frac{\partial G}{\partial t}(t, s) f_\infty(s) ds := u_{f_\infty}(t). \end{aligned}$$

This means that $\dot{u}_{f_n}(t)$ converges to $\dot{u}_{f_\infty}(t)$ in E_σ for every $t \in [0, 1]$. ■

Remark In the context of Control Theory, we have stated in the proof of Proposition 2.2, the dependence of the solution with respect to the control $f \in S^1_X$. Namely, if u_{f_n} is the $W^{2,1}_E([0, 1])$-solution to

$$\begin{cases} \ddot{u}_{f_n}(t) + \gamma \dot{u}_{f_n}(t) = f_n(t), \quad t \in [0, 1] \\ u_{f_n}(0) = x, \quad u_{f_n}(1) = \sum_{i=1}^{m-2} \alpha_i u_{f_n}(\eta_i) \end{cases}$$

and if (f_n) converges $\sigma(L^1_E, L^\infty_{E^*_s})$ to $f_\infty \in S^1_X$, then $(u_{f_n}(t))$ converges to $u_{f_\infty}(t)$ and $(\dot{u}_{f_n}(t))$ converges to $\dot{u}_{f_\infty}(t)$, in E_σ for every $t \in [0, 1]$ where u_{f_∞} is the $W^{2,1}_E([0, 1])$-solution to

$$\begin{cases} \ddot{u}_{f_\infty}(t) + \gamma \dot{u}_{f_\infty}(t) = f_\infty(t), \quad t \in [0, 1] \\ u_{f_\infty}(0) = x, \quad u_{f_\infty}(1) = \sum_{i=1}^{m-2} \alpha_i u_{f_\infty}(\eta_i). \end{cases}$$

The above remark is of importance since it allows to prove further results. Here is an application to the existence of $W^{2,1}_E([0, 1])$-solution to a second order differential inclusion with m-point boundary condition.

Proposition 2.3 *Let* $X : [0, 1] \rightrightarrows E$ *be a convex weakly compact valued measurable and integrably bounded mapping,* $F : [0, 1] \times E \times E \rightrightarrows E$ *be a convex weakly compact valued mapping satisfying*

(1) For each $x^* \in E^*$, the scalar function $\delta^*(x^*, F(., ., .))$ is $\mathcal{L}_\lambda([0,1]) \otimes \mathcal{B}(E_\sigma) \otimes \mathcal{B}(E_\sigma)$-measurable,[1]

(2) For each $x^* \in E^*$ and for each $t \in [0,1]$, the scalar function $\delta^*(x^*, F(t, ., .))$ is sequentially weakly upper semicontinuous, i.e., for any sequence (x_n) in E weakly converging to $x \in E$, for any sequence (y_n) in E weakly converging to $y \in E$, $\limsup_n \delta^*(x^*, F(t, x_n, y_n)) \le \delta^*(x^*, F(t, x, y))$,

(3) $F(t, x, y) \in X(t)$ for all $(t, x, y) \in [0,1] \times E \times E$.
Then the $W_E^{2,1}([0,1])$-solutions set to

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) \in F(t, u(t), \dot{u}(t))), \ t \in [0,1] \\ u(0) = x, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{cases}$$

is non empty and weakly compact in the space $C_E([0,1])$.

Proof The sets

$$\mathcal{X} := \left\{ u_f \in \mathcal{C}_E([0,1]) : u_f(t) = e_x(t) + \int_0^1 G(t,s) f(s) ds, \ \ f \in S_X^1, \ \ t \in [0,1] \right\} \tag{2.3.1}$$

and

$$\mathcal{Y} := \left\{ \dot{u}_f \in \mathcal{C}_E([0,1]) : \dot{u}_f(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s) ds, \ t \in [0,1], \ \ f \in S_X^1 \right\} \tag{2.3.2}$$

are bounded, convex, equicontinuous and weakly compact in $C_E([0,1])$. By condition (3), it is clear that

$$F(t, u_f(t), \dot{u}_f(t)) \subset X(t) \tag{2.3.4}$$

for all $t \in [0,1]$ and for all $f \in S_X^1$. Further, recall that S_X^1 is $\sigma(L_E^1, L_{E^*}^\infty)$-compact (see e.g. [10]). Using (1)–(3), for each $f \in S_X^1$, let us consider the convex $\sigma(L_E^1, L_{E^*}^\infty)$-compact valued mapping $\Psi : S_X^1 \rightrightarrows S_X^1$ defined by

$$\Psi(f) := \{g \in S_X^1 : g(t) \in F(t, u_f(t), \dot{u}_f(t)), \text{a.e. } t \in [0,1]\}.$$

Now we are going to show that Ψ is upper semi continuous on the convex $\sigma(L_E^1, L_{E^*}^\infty)$-compact set S_X^1. We need to check that the graph of Ψ is $\sigma(L_E^1, L_{E^*}^\infty)$-closed in $S_X^1 \times S_X^1$. Let $g_n \in \Psi(f_n)$ such that f_n, $\sigma(L_E^1, L_{E^*}^\infty)$-converges to $f \in S_X^1$ and g_n $\sigma(L_E^1, L_{E^*}^\infty)$-converges to $g \in S_X^1$. By compactness of $\mathcal{X}$ and $\mathcal{Y}$, it follows that $u_{f_n}(t) \to u_f(t)$ in E_σ and $\dot{u}_{f_n}(t) \to \dot{u}_f(t)$ in E_σ for every $t \in [0,1]$. From the inclusion $g_n \in \Psi(f_n)$, we have, for each $x^* \in E^*$ and for each $A \in \mathcal{L}_\lambda([0,1])$

[1] Actually $\mathcal{B}(E_\sigma) = \mathcal{B}(E)$ since E is separable.

$$\langle 1_A(t)x^*, g_n(t)\rangle \le 1_A(t)\delta^*(x^*, F(t, u_{f_n}(t), \dot{u}_{f_n}(t))),$$

so that, by integration,

$$\int_A \langle x^*, g_n(t)\rangle dt \le \int_A \langle x^*, F(t, u_{f_n}(t), \dot{u}_{f_n}(t))\rangle dt.$$

We thus have

$$\begin{aligned}\int_A \langle x^*, g(t)\rangle dt &= \lim_n \int_A \langle x^*, g_n(t)\rangle dt \\ &\le \limsup_n \int_A \delta^*(x^*, F(t, u_{f_n}(t), \dot{u}_{f_n}(t))dt \\ &\le \int_A \delta^*(x^*, F(t, u_f(t), \dot{u}_f(t))\rangle dt.\end{aligned}$$

Whence we get

$$\int_A \langle x^*, g(t)\rangle dt \le \int_A \delta^*(x^*, F(t, u_f(t), \dot{u}_f(t))dt$$

for every $A \in \mathcal{L}_\lambda([0, 1])$. Consequently

$$\langle x^*, g(t)\rangle \le \delta^*(x^*, F(t, u_f(t), \dot{u}_f(t)) \text{ a.e.}$$

Taking a dense sequence (e_k^*) in E^* with respect to the Mackey topology $\tau(E^*, E)$, we get

$$\langle e_k^*, g(t)\rangle \le \delta^*(e_k^*, F(t, u_f(t), \dot{u}_f(t)) \text{ a.e.}$$

for all $k \in \mathbf{N}$. By [13, Proposition III.35], we get finally

$$g(t) \in F(t, u_f(t), \dot{u}_f(t))) \text{ a.e.}$$

which proves that $g\ in \Psi(f)$. Whence by Kakutani-Ky Fan fixed point theorem Ψ admits a fixed point $f \in S^1_X$. This is a solution to the second order differential inclusion under consideration. Using Lemma 2.1, such a fixed point f verifies

$$\begin{cases} \ddot{u}_f(t) + \gamma \dot{u}_f(t) \in F(t, u_f(t), \dot{u}_f(t)), \text{ a.e. } t \in [0, 1] \\ u_f(0) = x, \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \end{cases}$$

The compactness of the solution set follows from the compactness of $\mathcal{X}$. ■

Second Order Evolution Inclusions Governed by Subdifferential Operators

We need to recall and summarize some notions on the subdifferential mapping of local Lipchitz functions developed by L. Thibault [25]. Let E be a separable Banach space. Let $f : E \to \mathbf{R}$ be a locally Lipschitz function. By Christensen [14, Theorem 7.5], there is a set D_f such that its complementary is Haar-nul (hence D_f is dense in E) such that for all $x \in D_f$ and for all $v \in E$

$$r_f(x, v) = \lim_{\delta \to 0} \frac{f(x + \delta v) - f(x)}{\delta}$$

exists and $v \mapsto r_f(x, v)$ is linear and continuous. Let us set $\nabla f(x) = r_f(x, .) \in E^*$. Then $r_f(x, v) = \langle \nabla f(x), v \rangle$, $\nabla f(x)$ is the gradient of f at the point x. Let us set

$$\mathcal{L}_f(x) = \{\lim_{j \to \infty} \nabla f(x_j) | x_j \in D_f, x_j \to x\}.$$

By definition, the subdifferential $\partial f(x)$ in the sense of Clarke [15] at the point $x \in E$ is defined by

$$\partial f(x) = \overline{co}\, \mathcal{L}_f(x).$$

The generalized directional derivative of f at a point $x \in E$ in the direction $v \in E$ is denoted by

$$f^{\cdot}(x, v) = \limsup_{h \to 0, \delta \to 0} \frac{f(x + h + \delta v) - f(x + h)}{\delta}.$$

Proposition 2.4 *Let $f : E \to \mathbf{R}$ be a locally Lipchitz function. Then the subdifferential $\partial f(x)$ at the point $x \in E$ is convex weak star compact and*

$$f^{\cdot}(x, v) = \sup\{\langle \zeta^*, v \rangle | \zeta^* \in \partial f(x)\} \quad \forall v \in E$$

that is, $f^{\cdot}(x, .)$ is the support function of $\partial f(x)$.

Proof See Thibault [25, Proposition I.12]. ■

Here are some useful properties of the subdifferential mapping.

Proposition 2.5 *Let $f : E \to \mathbf{R}$ be a locally Lipchitz function. Then the convex weak star compact valued subdifferential mapping ∂f is upper semicontinuous with respect to the weak star topology.*

Proof See [25, Proposition I.17]. Indeed we have

$$\delta^*(v, \partial f(x)) = f^{\cdot}(x; v) = \limsup_{h \to 0, \delta \to 0} \frac{[f(x + h + \delta v) - f(x + h)]}{\delta}.$$

As $f^{\cdot}(.; v)$ is upper semicontinuous and ∂f is convex compact valued in E^*_s, by [13], ∂f is upper semicontinuous in E^*_s. ■

Proposition 2.6 *Let $(T, \mathcal{T})$ a measurable space, and let $f : T \times E \to \mathbf{R}$ such that $f(., \zeta)$ is $\mathcal{T}$-measurable, for every $\zeta \in E$.*

$f(t,.)$ is locally Lipschitz for every $t \in T$.
Let $f_t^{\cdot}(x; v)$ be the directional derivative of $f(t,.) := f_t$ in the direction v for every fixed $t \in T$. Let x and v be two $\mathcal{T}$-measurable mappings from T to E. Then the following hold:

(a) the mapping $t \mapsto f_t^{\cdot}(x(t); v(t))$ is $\mathcal{T}$-measurable.
(b) the mapping $t \mapsto \partial f_t(x(t))$ is graph measurable, that is, its graph belongs to $\mathcal{T} \otimes \mathcal{B}(E_s^*)$.

Proof See Thibault [25, Proposition I.20 and Corollary I.21]. Note that the convex weak star compact valued mapping $t \mapsto \partial f_t(x(t))$ is scalarly $\mathcal{T}$-measurable, and so enjoys good measurability properties because E_s^* is a locally convex Lusin space. ■

We begin with a second order differential inclusion involving the subdifferential operator.

Proposition 2.7 *Assume that $E = \mathbf{R}^d$, and $h : [0,1] \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ be a bounded Carathéodory mapping, that is, h is separately Lebesque-measurable on $[0,1]$, separately continuous on $\mathbf{R}^d \times \mathbf{R}^d$, $||h(t,x,y)|| \le \alpha(t)$, $\forall (t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{R}^d$ where α is positive Lebesque-integrable. Let $f : [0,1] \times E \to \mathbf{R}$ be a mapping such that*

(1) $\forall x \in E$, $f(.,x)$ is Lebesgue-measurable,
(2) There exists $\beta \in L^1_{\mathbf{R}^+}([0,1])$ such that, for all $t \in [0,1]$, for all $x, y \in E$,

$$||f(t,x) - f(t,y)|| \le \beta(t)||x-y||.$$

Then the following hold

(a) $\partial f_t(x) \subset \beta(t)\overline{B}_E$, for all $(t,x) \in [0,1] \times E$,
(b) The $W_E^{2,1}([0,1])$-solution set to

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) \in \partial f_t(u(t)) + h(t,u(t),\dot{u}(t)), \text{ a.e. } t \in [0,1] \\ u(0) = x, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{cases}$$

is compact in the space $\mathcal{C}_E([0,T])$.

Proof The proof is immediate by applying Proposition 2.3 to the convex compact valued mapping $(t,x,y) \mapsto \partial f_t(x) + h(t,x,y)$, taking account of the properties of the subdifferential mapping and its measurable properties given in Proposition 2.6. ■

We finish this section with a variant which has some importance in the study of epiconvergence problem for the approximating system

$$\ddot{u}(t) + \gamma\dot{u}(t) = h(t,u(t),\dot{u}(t)) - \nabla\varphi(u(t))$$

where φ is C^1 and Lipschitz.

Proposition 2.8 *Assume that* $E = \mathbf{R}^d$, $\varphi : E \to \mathbf{R}$ *is* C^1, *Lipschitz, and that* $h : [0, 1] \times \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ *is a bounded Carathéodory mapping, that is, h is separately Lebesque-measurable on* $[0, 1]$, *separately continuous on* $\mathbf{R}^d \times \mathbf{R}^d$, $||h(t, x, y)|| \leq \alpha(t)$, $\forall(t, x, y) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^d$ *where* α *is positive Lebesque-integrable.Then the* $W_E^{2,1}([0, 1])$*-solution set to*

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = h(t, u(t), \dot{u}(t)) - \nabla\varphi(u(t)) \text{ a.e. } t \in [0, 1] \\ u(0) = x, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{cases}$$

is compact in the space $\mathcal{C}_E([0, T])$.

Proof The proof is immediate by applying Proposition 2.3 with $F(t, x, y) = h(t, x, y) - \nabla\varphi(x), \forall(t, x, y) \in [0, 1] \times E \times E$ and by observing that the subdifferential $x \mapsto \partial\varphi(x) = \nabla\varphi(x)$ is bounded and continuous. ■

3 Applications. Towards the Variational Convergence in Second Order Evolution Inclusions

Let us recall a useful Gronwall type lemma [12].

Lemma 3.1 (A Gronwall-like inequality) *Let* $p, q, r : [0, T] \to [0, \infty[$ *be three nonnegative Lebesgue integrable functions such that for almost all* $t \in [0, T]$

$$r(t) \leq p(t) + q(t) \int_0^t r(s)\, ds.$$

Then

$$r(t) \leq p(t) + q(t) \int_0^t [p(s) \exp(\int_s^t q(\tau)\, d\tau)]\, ds$$

for all $t \in [0, T]$.

We recall below some notations and summarize some results which describe the limiting behavior of a bounded sequence in $L_H^1([0, T])$. See [10, Proposition 6.5.17].

Proposition 3.1 *Let H be a separable Hilbert space. Let* (ζ_n) *be a bounded sequence in* $L_H^1([0, T])$. *Then the following hold:*

(1) (ζ_n) *(up to an extracted subsequence) stably converges to a Young measure* ν *that is, there exist a subsequence* (ζ_n') *of* (ζ_n) *and a Young measure* ν *belonging to the space of Young measure* $\mathcal{Y}([0, T]; H_\sigma)$ *with* $t \mapsto bar(\nu_t) \in L_H^1([0, T])$ *(here* $bar(\nu_t)$ *denotes the barycenter of* ν_t*) such that*

$$\lim_{n\to\infty} \int_0^T h(t, \zeta_n'(t)))\, dt) = \int_0^T \left[\int_H h(t, x)\, \nu_t(dx)\right] dt$$

for all bounded Carathéodory integrands $h : [0, T] \times H_\sigma \to \mathbf{R}$,

(2) (ζ_n) *(up to an extracted subsequence) weakly biting converges to an integrable function* $f \in L^1_H([0, T])$, *which means that there is a subsequence* (ζ'_m) *of* (ζ_n) *and an increasing sequence of Lebesgue-measurable sets* (A_p) *with* $\lim_p \lambda(A_p) = 1$ *and* $f \in L^1_H([0, T])$ *such that, for each* p,

$$\lim_{m\to\infty} \int_{A_p} \langle h(t), \zeta'_m(t)\rangle\, dt = \int_{A_p} \langle h(t), f(t)\rangle\, dt$$

for all $h \in L^\infty_H([0, T])$,

(3) (ζ_n) *(up to an extracted subsequence) Komlós converges to an integrable function* $g \in L^1_H([0, T])$, *which means that there is a subsequence* $(\zeta_{\beta(m)})$ *and an integrable function* $g \in L^1_H([0, T])$, *such that*

$$\lim_{n\to\infty} \frac{1}{n} \Sigma^n_{j=1} \zeta_{\gamma(j)}(t) = g(t), \text{ a.e. } \in [0, T],$$

for every subsequence $(f_{\gamma(n)})$ *of* $(f_{\beta(n)})$.

(4) There is a filter $\mathcal{U}$ *finer than the Fréchet filter such that* $\mathcal{U} - \lim_n \zeta_n = l \in (L^\infty_H)'_{weak}$ *where* $(L^\infty_H)'_{weak}$ *is the second dual of* $L^1_H([0, T])$.
Let $w_{l_a} \in L^1_H([0, T])$ *be the density of the absolutely continuous part* l_a *of* l *in the decomposition* $l = l_a + l_s$ *in absolutely continuous part* l_a *and singular part* l_s.
If we have considered the same extracted subsequence in (1)–(4), then one has

$$f(t) = g(t) = bar(\nu_t) = w_{l_a}(t) \text{ a.e. } t \in [0, T].$$

By $W^{2,1}_{\mathbf{R}^d}([0, T])$ (resp. $W^{2,2}_{\mathbf{R}^d}([0, T])$ we denote the set of all continuous functions in $\mathcal{C}_{\mathbf{R}^d}([0, T])$ such that their first derivatives are continuous and their second derivatives belong to $L^1_{\mathbf{R}^d}([0, T])$ (resp. $L^2_{\mathbf{R}^d}([0, T])$) and by $W^{1,1}_{BV}([0, T])$ we denote the set of all continuous functions in $\mathcal{C}_{\mathbf{R}^d}([0, T])$ such that their first derivatives are of bounded variation (BV for short).

We begin with a preliminary result which shows the limiting properties of $W^{2,1}_{\mathbf{R}^d}([0, 1])$-solutions for a second order ordinary differential equation with m-point boundary conditions.

Proposition 3.2 *Let* $E = \mathbf{R}^d$. *Let* $(f_n)_{n\in\mathbf{N}}$ *be a bounded sequence in* $L^1_E([0, 1])$. *For each* $n \in \mathbf{N}$, *let us consider the* $W^{2,1}_E([0, 1])$*-solution* $u_n : [0, 1] \to E$ *of the equation*

$$\ddot{u}_n(t) + \gamma\dot{u}_n(t) = f_n(t),\ t \in [0, 1]; \quad u_n(0) = x, \quad u_n(1) = \sum_{i=1}^{m-2} \alpha_i u_n(\eta_i).$$

Then there exist a subsequence of (u_n) *still denoted by* (u_n), *a* $W^{1,1}_{BV}([0, 1])$*-function* $u : [0, 1] \to E$ *and a Young measure* $\nu \in \mathcal{Y}([0, 1]; E)$ *such that* $t \mapsto bar(\nu_t)$ *belongs to* $L^1_E([0, 1])$ *which satisfy the following conditions:*

(a) $(u_n(.))$ *converges in* $\mathcal{C}_E([0,1])$ *to* $u(.)$ *with* $u(0) = x, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$.
(b) $(\dot{u}_n(.))$ *converges in* $L^1_E([0,1])$ *to* $\dot{u}(.)$.
(c) $(\delta_{\ddot{u}_n})$ *stably converges in* $\mathcal{Y}([0,1], E)$ *to* ν.
(d) *Assume further that the negative parts* $\langle u_n, \ddot{u}_n\rangle^-$ *of the functions* $\langle u_n, \ddot{u}_n\rangle$ *are uniformly integrable in* $L^1_{\mathbf{R}}([0,1])$.
Then

$$\liminf_{n\to\infty} \int_0^1 \langle u_n(t), \ddot{u}_n(t)\rangle\, dt \geq \int_0^1 \langle u(t), bar(\nu_t)\rangle\, dt = \int_0^1 \left[\int_E \langle u(t), x\rangle\, \nu_t(dx)\right] dt.$$

Proof Existence and uniqueness of a $W^{2,1}_E([0,1])$-solution for the equation

$$\ddot{u}_n(t) + \gamma \dot{u}_n(t) = f_n(t),\ \ t \in [0,1];\ \ u(0) = x, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i).$$

are ensured by Proposition 2.1 with integral representation formulas

$$\begin{cases} u_n(t) = e_x(t) + \int_0^1 G(t,s) f_n(s) ds,\ t \in [0,1] \\ \dot{u}_n(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G}{\partial t}(t,s) f_n(s) ds,\ t \in [0,1] \end{cases}$$

where

$$\begin{cases} e_x(t) = x + A(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x \\ \dot{e}_x(t) = \gamma A \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma t) x \\ A \quad = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\right)^{-1}. \end{cases}$$

Since $(f_n(.))$ is bounded in $L^1_E([0,1])$ by assumption, $(\dot{u}_n(.))$ is uniformly bounded by using the integral formula for $\dot{u}_n$ and the boundedness of the Green function G given in Lemma 3.1. So $(\dot{u}_n(.))$ is uniformly bounded and bounded in variation. In view of the Helly–Banach theorem (see e.g. [20, p. 11]), we may, by extracting a subsequence, assume that $(\dot{u}_n(.))$ pointwise converges to a BV function $v(.)$. Let us set $u(t) = \int_0^t v(s)\, ds$ for all $t \in [0,1]$. Then $u \in W^{1,1}_{BV}([0,1])$ with $\dot{u}(t) = v(t)$ for almost every $t \in [0,1]$. Then $(\dot{u}_n(.))$ is uniformly bounded and pointwise converges to $v(.)$. By Lebesgue's theorem, we conclude that $(\dot{u}_n(.))$ converges in $L^1_E([0,1])$ to $\dot{u}(.)$. Hence $(u_n(.))$ converges uniformly to $u(.)$ with $u(0) = x, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$. It remains to check (c) and (d). Since $(\ddot{u}_n(.))$ is bounded, in view of Proposition 3.1, we may assume that the sequence $(\delta_{\ddot{u}_n})$ of associated Young measures stably converges in $\mathcal{Y}([0,1], E)$ to a Young measure ν such that $t \mapsto \mathrm{bar}(\nu_t)$ belongs to $L^1_E([0,1])$. Let us prove the last Fatou property (d). We may suppose that

$$a := \lim_{n\to\infty} \int_0^1 \langle u_n(t), \ddot{u}_n(t)\rangle\, dt \in \mathbf{R}.$$

Furthermore, since $(\ddot{u}_n(.))$ is bounded in $L^1_E([0,1])$, in view of Proposition 3.1 we may suppose that $(\ddot{u}_n(.))$ weakly biting converges to a function $f \in L^1_E([0,1])$, that is, there exist a subsequence (still denoted by $(\ddot{u}_n(.))$) of $(\ddot{u}_n(.))$ and an increasing sequence of measurable sets (A_p) in $[0,1]$ such that $\lim_{p\to\infty} \lambda(A_p) = 1$, and such that, for each p and for each $g \in L^\infty_E(A_p, A_p \cap \mathcal{L}([0,1]), \lambda|_{A_p})$, the following holds:

$$\lim_{n\to\infty} \int_{A_p} \langle \ddot{u}_n(t), g(t)\rangle\, dt = \int_{A_p} \langle f(t), g(t)\rangle\, dt.$$

Let $\varepsilon > 0$ be given. Pick $N \in \mathbf{N}$ such that

$$\int_{A_N} \langle u(t), f(t)\rangle\, dt \geq \int_{[0,1]} \langle u(t), f(t)\rangle\, dt - \varepsilon,$$

and that

$$\limsup_{n\to\infty} \int_{[0,1]\setminus A_N} \langle u_n(t), \ddot{u}_n(t)\rangle^-\, dt \leq \varepsilon$$

(this is possible because $(\langle u_n, \ddot{u}_n\rangle^-)_n$ is uniformly integrable by hypothesis). As $||u_n(.) - u(.)|| \to 0$ uniformly, it is easy to see that

$$\lim_{n\to\infty} \int_{A_N} ||u_n(t) - u(t)||\; ||\ddot{u}_n(t)||\, dt = 0.$$

See [6, 16] for a more general case. Whence

$$\lim_{n\to\infty} \Big[\int_{A_N} \langle u_n(t), \ddot{u}_n(t)\rangle\, dt - \int_{A_N} \langle u(t), \ddot{u}_n(t)\rangle\, dt \Big] = 0.$$

An easy computation gives

$$\begin{aligned} a &\geq \lim_{n\to\infty} \int_{A_N} \langle u_n(t), \ddot{u}_n(t)\rangle - \limsup_{n\to\infty} \int_{[0,1]\setminus A_N} \langle u_n(t), \ddot{u}_n(t)\rangle^-\, dt \\ &\geq \lim_{n\to\infty} \int_{A_N} \langle u_n(t), \ddot{u}_n(t)\rangle\, dt - \varepsilon. \end{aligned}$$

Finally we get

$$\begin{aligned} a &\geq \lim_{n\to\infty} \int_{A_N} \langle u_n(t), \ddot{u}_n(t)\rangle\, dt - \varepsilon \\ &= \lim_{n\to\infty} \int_{A_N} \langle u(t), \ddot{u}_n(t)\rangle\, dt - \varepsilon \\ &= \int_{A_N} \langle u(t), f(t)\rangle\, dt - \varepsilon \end{aligned}$$

$$\geq \int_{[0,1]} \langle u(t), f(t) \rangle\, dt - 2\varepsilon.$$

By virtue of Proposition 3.1 $f(t) = \text{bar}(\nu_t)$ a.e. The proof is therefore complete because

$$\int_0^1 \langle u(t), \text{bar}(\nu_t) \rangle\, dt = \int_0^1 \left[\int_E \langle u(t), x \rangle\, \nu_t(dx) \right] dt.$$ ∎

The above techniques can be used to prove the existence of a solution for second order evolution inclusion with boundary conditions governed by subdifferential operators of the form

$$f(t) \in \ddot{u}(t) + Mu(t) + \partial\varphi(u(t)), t \in [0, T] \tag{I}$$

where M is positive, φ is a proper convex proper lower semicontinuous function defined on $\mathbf{R}^d$, and $\partial\varphi(u(t))$ is the subdifferential of the function φ at the point $u(t)$ and the perturbation f belongs to $L^2_{\mathbf{R}^d}([0, T])$. Similar results in this direction are obtained by [1–4, 11].

Now we present a fairly general result for the approximating problem via the epiconvergence approach in a second order evolution problem. The applicability of our abstract results will be exemplified below.

Proposition 3.3 *Assume that* $M > 0$, $\beta \in L^2_{\mathbf{R}^+}([0, T])$. *For each* $n \in \mathbf{N}$, *let* $\varphi_n : \mathbf{R}^d \to \mathbf{R}^+$ *be a convex, Lipschitz function and let* φ_∞ *be a nonnegative l.s.c proper function defined on* $\mathbf{R}^d$ *such that* $\varphi_n(x) \leq \varphi_\infty(x)$ *for all* $n \in \mathbf{N}$ *and for all* $x \in \mathbf{R}^d$. *Let* $f^n \in L^2_{\mathbf{R}^d}([0, T])$ *such that* $||f_n(t)|| \leq \beta(t)$, $\forall n \in \mathbf{N}$, $\forall t \in [0, T]$. *For each* $n \in \mathbf{N}$, *let* u^n *be a* $W^{2,1}_{\mathbf{R}^d}([0, T])$*-solution to the problem*

$$\begin{cases} f^n(t) \in \ddot{u}^n(t) + M\dot{u}^n(t) + \partial\varphi_n(u^n(t)), t \in [0, T] \\ u^n(0) = u^n_0;\ \dot{u}^n(0) = \dot{u}^n_0. \end{cases}$$

Assume that

(i) f^n $\sigma(L^2_{\mathbf{R}^d}, L^2_{\mathbf{R}^d})$*-converges to* $f^\infty \in L^2_{\mathbf{R}^d}([0, T])$,
(ii) φ_n *epi-converges to* φ_∞,
(iii) $\lim_n u^n(0) = u^\infty_0 \in dom\ \varphi_\infty$, $\lim_n \varphi_n(u^n(0)) = \varphi_\infty(u^\infty_0)$, *and* $\lim_n \dot{u}^n(0) = \dot{u}^\infty_0$,
(iv) *There exist* $r_0 > 0$ *and* $x_0 \in \mathbf{R}^d$ *such that*

$$\sup_{n\in\mathbf{N}} \sup_{v\in\overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}} \int_0^T \varphi_\infty(x_0 + r_0 v(t)) < +\infty$$

where $\overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}$ *is the closed unit ball in* $L^\infty_{\mathbf{R}^d}([0, T])$.

(a) *Then up to extracted subsequences,* (u^n) *converges uniformly to a* $W^{1,1}_{BV}([0,T])$*-function* u^∞ *and* $(\dot u^n)$ *pointwisely converges to a BV function* v^∞ *with* $v^\infty = \dot u^\infty$*, and* $(\ddot u^n)$ *biting converges to a function* $\zeta^\infty \in L^1_{\mathbf{R}^d}([0,T])$ *so that the limit function* u^∞, $\dot u^\infty$ *and the biting limit* ζ^∞ *satisfy the variational inclusion*

$$f^\infty \in \zeta^\infty + M\dot u^\infty + \partial I_{\varphi_\infty}(u^\infty)$$

where $\partial I_{\varphi_\infty}$ *denotes the subdifferential of the convex lower semicontinuous integral functional* I_{φ_∞} *defined on* $L^\infty_{\mathbf{R}^d}([0,T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt, \;\; \forall u \in L^\infty_{\mathbf{R}^d}([0,T]).$$

Furthermore $\lim_n \varphi_n(u^n(t)) = \varphi_\infty(u^\infty(t)) < \infty$ *a.e. and* $\lim_n \int_0^T \varphi_n(u^n(t))\,dt = \int_0^T \varphi_\infty(u^\infty(t))dt$*. Subsequently, the energy estimate holds true almost everywhere* $t \in [0,T]$*,*

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot u^\infty(t)||^2 = \varphi_\infty(u^\infty_0)) + \frac{1}{2}||\dot u^\infty_0||^2 - \int_0^t \langle M\dot u^\infty(s), \dot u^\infty(s)\rangle ds + \int_0^t \langle \dot u^\infty(s), f^\infty(s)\rangle ds.$$

Further $(\ddot u^n)$ *weakly converges to the vector measure* $m \in \mathcal{M}^b_{\mathbf{R}^d}([0,T])$ *so that the limit functions* $u^\infty(.)$ *and the limit measure* m *satisfy the following variational inequality:*

$$\int_0^T \varphi_\infty(v(t))\,dt \ge \int_0^1 \varphi_\infty(u^\infty(t))\,dt + \int_0^1 \langle -M\dot u^\infty(t) + f^\infty(t), v(t) - u^\infty(t)\rangle\,dt + \langle -m, v - u^\infty\rangle_{(\mathcal{M}^b_{\mathbf{R}^d}([0,T]), \mathcal{C}_{\mathbf{R}^d}([0,T]))}.$$

In other words, the vector measure $-m + [-M\dot u^\infty + f^\infty]\,dt$ *belongs to the subdifferential* $\partial J_{\varphi_\infty}(u^\infty)$ *of the convex functional integral* J_{φ_∞} *defined on* $\mathcal{C}_{\mathbf{R}^d}([0,T])$ *by* $J_{\varphi_\infty}(v) = \int_0^1 \varphi_\infty(t, v(t))\,dt$, $\forall v \in \mathcal{C}_{\mathbf{R}^d}([0,T])$.

(b) *There are a filter* $\mathcal{U}$ *finer than the Fréchet filter,* $l \in L^\infty_{\mathbf{R}^d}([0,T])'$ *such that*

$$\mathcal{U} - \lim_n [f^n - \ddot u^n - M\dot u^n] = l \in L^\infty_{\mathbf{R}^d}([0,T])'_{weak}$$

where $L^\infty_{\mathbf{R}^d}([0,T])'_{weak}$ *is the second dual of* $L^1_{\mathbf{R}^d}([0,T])$ *endowed with the topology* $\sigma(L^\infty_{\mathbf{R}^d}([0,T])', L^\infty_{\mathbf{R}^d}([0,T]))$ *and* $\mathbf{n} \in \mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ *such that*

$$\lim_n [f^n - \ddot u^n - M\dot u^n] = \mathbf{n} \in \mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$$

where $\mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ denotes the space $\mathcal{C}_{\mathbf{R}^d}([0,T])'$ endowed with the weak topology $\sigma(\mathcal{C}_{\mathbf{R}^d}([0,T])', \mathcal{C}_{\mathbf{R}^d}([0,T]))$. Let l_a be the density of the absolutely continuous part l_a of l in the decomposition $l = l_a + l_s$ in absolutely continuous part l_a and singular part l_s. Then

$$l_a(h) = \int_0^T \langle h(t), f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t)\rangle dt$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$ so that

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(f^\infty - \zeta^\infty - M\dot{u}^\infty) + \delta^*(l_s, dom\, I_{\varphi_\infty})$$

where φ^*_∞ is the conjugate of φ_∞, $I_{\varphi^*_\infty}$ the integral functional defined on $L^1_{\mathbf{R}^d}([0,T])$ associated with φ^*_∞, $I^*_{\varphi_\infty}$ the conjugate of the integral functional I_{φ_∞}, $dom\, I_{\varphi_\infty} := \{u \in L^\infty_{\mathbf{R}^d}([0,T]) : I_{\varphi_\infty}(u) < \infty\}$ and

$$\langle \mathbf{n}, h\rangle = \int_0^T \langle f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt + \langle \mathbf{n}_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

with $\langle \mathbf{n}_s, h\rangle = l_s(h)$, $\forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$. Further $\mathbf{n}$ belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex lower semicontinuous integral functional J_{φ_∞} defined on $\mathcal{C}_{\mathbf{R}^d}([0,T])$

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\, dt, \ \forall u \in \mathcal{C}_{\mathbf{R}^d}([0,T]).$$

(c) *Consequently the density $f^\infty - \zeta^\infty - M\dot{u}^\infty$ of the absolutely continuous part n_a*

$$\mathbf{n}_a(h) := \int_0^T \langle f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,T])$$

satisfies the inclusion

$$f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e.}$$

and for any nonnegative measure θ on $[0,T]$ with respect to which $\mathbf{n}_s$ is absolutely continuous

$$\int_0^T h_{\varphi^*_\infty}(\frac{d\mathbf{n}_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t)$$

here $h_{\varphi^*_\infty}$ denotes the recession function of φ^*_∞.

Proof Step 1 $||\dot{u}^n(.)||$ *and* $\varphi_n(u_n(.))$ *are uniformly bounded.*

Multiplying scalarly the inclusion

$$f^n(t) - \ddot{u}^n(t) - M\dot{u}^n(t) \in \partial\varphi_n(u^n(t))$$

by $\dot{u}^n(t)$ and applying the chain rule theorem [21, Theorem 2] yields

$$\langle \dot{u}^n(t), f^n(t)\rangle - \langle \dot{u}^n(t), \ddot{u}^n(t)\rangle - \langle \dot{u}^n(t), M\dot{u}_n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))]$$

that is,

$$-\langle M\dot{u}^n(t), \dot{u}^n(t)\rangle + \langle \dot{u}^n(t), f^n(t)\rangle = \frac{d}{dt}\left[\varphi_n(u_n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2\right]. \quad (3.3.1)$$

Integrating this equality on $[0, t]$, we get

$$\begin{aligned}
\varphi_n(u^n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2
&= \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 \\
&\quad - \int_0^t \langle M\dot{u}^n(s), \dot{u}^n(s)\rangle ds + \int_0^t \langle \dot{u}^n(s), f^n(s)\rangle ds \\
&\le \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 \\
&\quad + M\int_0^t ||\dot{u}^n(s)||^2 ds + ||f^n||_{L^2_{\mathbf{R}^d}([0,T])}\left(\int_0^t ||\dot{u}^n(s)||^2 ds\right)^{\frac{1}{2}} \\
&\le \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 \\
&\quad + M\int_0^t ||\dot{u}^n(s)||^2 ds + \frac{1}{2}||f^n||_{L^2_{\mathbf{R}^d}([0,T])}\left(1 + \int_0^t ||\dot{u}^n(s)||^2 ds\right) \\
&\le \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 \\
&\quad + M\int_0^t ||\dot{u}^n(s)||^2 ds + \frac{1}{2}||\beta||_{L^2_{\mathbf{R}}([0,T])}\left(1 + \int_0^t ||\dot{u}^n(s)||^2 ds\right).
\end{aligned}$$

Then, from (iii), the preceding estimate and the Gronwall like inequality (Lemma 3.1), it is immediate that

$$\sup_{n\ge 1}\sup_{t\in[0,T]} ||\dot{u}^n(t)|| < +\infty \quad \text{and} \quad \sup_{n\ge 1}\sup_{t\in[0,T]} \varphi_n(u^n(t)) < +\infty. \quad (3.3.2)$$

Step 2 Estimation of $||\ddot{u}^n(.)||$. As

$$z^n(t) := f^n(t) - \ddot{u}^n(t) - M\dot{u}^n(t) \in \partial\varphi_n(u^n(t))$$

by the subdifferential inequality for convex lower semi continuous functions we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbf{R}^d$. Now let $v \in \overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}$, the closed unit ball of $L^\infty_{\mathbf{R}^d}[0, T])$. Taking $x = w(t) := x_0 + r_0 v(t)$ in the preceding inequality we get

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle w(t) - u^n(t), z^n(t)\rangle.$$

Integrating the preceding inequality gives

$$\begin{aligned}\int_0^T \langle x_0 + r_0 v(t) - u^n(t), z^n(t)\rangle dt &\\ &= \int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt + r_0 \int_0^T \langle v(t), z^n(t)\rangle dt\\ &\leq \int_0^T \varphi_n(x_0 + r_0 v(t))dt - \int_0^T \varphi_n(u^n(t))dt.\end{aligned}$$

Whence follows

$$\begin{aligned}r_0 \int_0^T \langle v(t), z^n(t)\rangle dt \leq \int_0^T \varphi_n(x_0 + r_0 v(t))dt &\\ - \int_0^T \varphi_n(u^n(t))dt - \int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt.\end{aligned} \tag{3.3.3}$$

We compute the last integral in the preceding inequality. For simplicity, let us set $v^n(t) = u^n(t) - x_0$ for all $t \in [0, T]$. By integration by parts and taking into account (3.3.2), we have

$$\begin{aligned}&-\int_0^T \langle x_0 - u^n(t), z^n(t)\rangle dt = -\int_0^T \langle v^n(t), \ddot{v}^n(t) + M\dot{v}^n(t)\rangle - f^n(t)\rangle dt \qquad (3.3.4)\\ &= -[\langle v^n(t), \dot{v}^n(t) + Mv^n(t)]_0^T + \int_0^T \langle \dot{v}^n(t), \dot{v}^n(t) + Mv^n(t)\rangle dt + \int_0^T \langle v^n(t), f^n(t)\rangle dt\\ &\leq -\langle v^n(T), \dot{v}^n(T)\rangle + \langle v^n(0), \dot{v}^n(0)\rangle - \langle Mv^n(T), v^n(T)\rangle\\ &\quad + \langle Mv^n(0), v^n(0)\rangle + \int_0^T ||\dot{v}^n(t)||^2 dt + \int_0^T \langle \dot{v}^n(t), Mv^n(t)\rangle dt + \int_0^T \langle v^n(t), f^n(t)\rangle dt.\end{aligned}$$

By (3.3.2)–(3.3.4), we get

$$r_0 \int_0^T \langle v(t), z^n(t)\rangle dt \leq \int_0^T \varphi_\infty(x_0 + r_0 v(t))dt + L \tag{3.3.5}$$

for all $v \in \overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}$, where L is a generic positive constant independent of $n \in \mathbf{N}$. By (iv) and (3.3.5) we conclude that $(z^n = f^n - \ddot{u}^n - M\dot{u}^n)$ is bounded in $L^1_{\mathbf{R}^d}([0,T])$, then so is $(\ddot{u}^n)$. It turns out that the sequence $(\dot{u}^n)$ is uniformly bounded by using (3.3.2) and is bounded in variation. By Helly theorem, we may assume that $(\dot{u}^n)$ pointwisely converges to a BV function $v^\infty : [0,T] \to \mathbf{R}^d$ and the sequence (u^n) converges uniformly to an absolutely continuous function u^∞ with $\dot{u}^\infty = v^\infty$ a.e. At this point, it is clear that $(\dot{u}_n)$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to v^∞, using (3.3.2) and the dominated convergence theorem. Hence $(M\dot{u}^n(.))$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to $Mv^\infty(.)$.

Step 3 Young measure limit and biting limit of $\ddot{u}_n$. As $(\ddot{u}_n)$ is bounded in $L^1_{\mathbf{R}^d}([0,T])$, we may assume that $(\ddot{u}^n)$ stably converges to a Young measure $\nu \in \mathcal{Y}([0,T]); \mathbf{R}^d)$ with $\mathrm{bar}(\nu) : t \mapsto \mathrm{bar}(\nu_t) \in L^1_{\mathbf{R}^d}([0,T])$ (here $\mathrm{bar}(\nu_t)$ denotes the barycenter of ν_t). Further by Proposition 3.1, we may assume that $(\ddot{u}^n)$ biting converges to a function $\zeta^\infty : t \mapsto \mathrm{bar}(\nu_t)$ that is, there exists a decreasing sequence of Lebesgue-measurable sets (B_p) with $\lim_p \lambda(B_p) = 0$ such that the restriction of $(\ddot{u}_n)$ on each B_p^c converges weakly in $L^1_{\mathbf{R}^d}([0,T])$ to ζ^∞. Note that $(M\dot{u}^n)$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to Mv^∞. It follows that the restriction of $(z^n = f^n - \ddot{u}^n - M\dot{u}^n)$ to each B_p^c weakly converges in $L^1_{\mathbf{R}^d}([0,T])$ to $z^\infty := f^\infty - \zeta^\infty - Mv^\infty$, because (f^n) weakly converges in $L^1_{\mathbf{R}^d}([0,T])$ to f^∞, $(M\dot{u}^n)$ converges in $L^1_{\mathbf{R}^d}([0,T])$ to Mv^∞ and $(\ddot{u}^n)$ biting converges to $\zeta^\infty \in L^1_{\mathbf{R}^d}([0,T])$. It follows that

$$\lim_n \int_B \langle -\ddot{u}^n - W^n(t), w(t) - u^n(t)\rangle = \int_B \langle -\mathrm{bar}(\nu_t) - W(t), w(t) - u(t)\rangle dt \tag{3.3.6}$$

for every $B \in B_p^c \cap \mathcal{L}([0,T])$, and for every $w \in L^\infty_{\mathbf{R}^d}([0,T])$, where $W^n(t) = M\dot{u}^n(t) - f^n(t)$ and $W(t) = M\dot{u}^\infty(t) - f^\infty(t)$. Indeed, we note that $(w(t) - u^n(t))$ is a bounded sequence in $L^\infty_{\mathbf{R}^d}([0,1])$ which pointwisely converges to $w(t) - u^\infty(t)$, it converges uniformly on every uniformly integrable subset of $L^1_{\mathbf{R}^d}([0,T])$ by virtue of a Grothendieck Lemma [16], recalling here that the restriction of $-\ddot{u}^n - W^n$ on each B_p^c is uniformly integrable. Now, since φ_n lower epiconverges to φ_∞, for every Lebesgue-measurable set A in $[0,T]$, by virtue of Corollary 4.7 in [11], we have

$$+\infty > \liminf_n \int_A \varphi_n(u^n(t))dt \geq \int_A \varphi_\infty(u^\infty(t))dt. \tag{3.3.7}$$

Combining (3.3.2)–(3.3.5)–(3.3.6)–(3.3.7) and using the subdifferential inequality

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle -\ddot{u}^n(t) - W^n(t), w(t) - u^n(t)\rangle$$

gives

$$\int_B \varphi_\infty(w(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt + \int_B \langle -\mathrm{bar}(\nu_t) - W(t), w(t) - u^\infty(t)\rangle\,dt.$$

This shows that $t \mapsto -\operatorname{bar}(\nu_t) - W(t)$ is a subgradient at the point u^∞ of the convex integral functional I_{φ_∞} restricted to $L^\infty_{\mathbf{R}^d}(B^c_p)$, consequently,

$$-\operatorname{bar}(\nu_t) - W(t) \in \partial\varphi_\infty(u^\infty(t)), \text{ a.e. on} B^c_p.$$

As this inclusion is true on each B^c_p and $B^c_p \uparrow [0, T]$, we conclude that

$$-\operatorname{bar}(\nu_t) - W(t) \in \partial\varphi_\infty(u^\infty(t)), \text{ a.e. on}[0, T].$$

Step 4 Limit measure in $\mathcal{M}^b_{\mathbf{R}^d}([0, T])$ *of* $\ddot{u}^n$. As $(\ddot{u}_n)$ is bounded in $L^1_{\mathbf{R}^d}([0, T])$, we may assume that $(\ddot{u}^n)$ weakly converges to the vector measure $m \in \mathcal{M}^b_{\mathbf{R}^d}([0, T])$ so that the limit functions $u^\infty(.)$ and the limit measure m satisfy the following variational inequality:

$$\int_0^T \varphi_\infty(v(t))\, dt \geq \int_0^1 \varphi_\infty(u^\infty(t))\, dt + \int_0^1 \langle -M\dot{u}^\infty(t) + f^\infty(t), v(t) - u^\infty(t)\rangle\, dt$$
$$+ \langle -m, v - u^\infty\rangle_{(\mathcal{M}^b_E([0,T]),\mathcal{C}_{\mathbf{R}^d}([0,T]))}.$$

In other words, the vector measure $-m + [-M\dot{u}^\infty + f^\infty]\, dt = -m - W.dt$ belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex functional integral J_{f_∞} defined on $\mathcal{C}_{\mathbf{R}^d}([0, T])$ by $J_{\varphi_\infty}(v) = \int_0^1 \varphi_\infty(v(t))\ dt,\ \forall v \in \mathcal{C}_{\mathbf{R}^d}([0, T])$. Indeed, let $w \in \mathcal{C}_{\mathbf{R}^d}([0, T])$. Integrating the subdifferential inequality

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle -\ddot{u}^n(t) - W^n(t), w(t) - u^n(t)\rangle$$

and noting that $\varphi_\infty(w(t)) \geq \varphi_n(w(t))$ gives immediately

$$\int_0^T \varphi_\infty(w(t))dt \geq \int_0^T \varphi_n(w(t))dt$$
$$\geq \int_0^T \varphi_n(u^n(t))dt + \langle -\ddot{u}^n(t) - W^n(t), w(t) - u^n(t)\rangle dt.$$

We note that

$$\lim_n \int_0^T \langle -W^n(t), w(t) - u^n(t)\rangle dt = \int_0^T \langle -W(t), w(t) - u^\infty(t)\rangle dt$$

because $(W^n := M\dot{u}^n - f^n)$ is uniformly integrable, and weakly converges to $W := M\dot{u}^\infty - f^\infty$ and the bounded sequence in $w(t) - u^n(t)$ pointwise converges to $w - u^\infty$ so that it converges uniformly on uniformly integrable subsets by virtue of Grothendieck lemma. Whence follows

$$\int_0^T \varphi_\infty(w(t))dt \geq \int_0^T \varphi_\infty(u^\infty(t))dt + \int_0^T \langle -W(t), w(t) - u^\infty(t)\rangle dt$$
$$+ \langle -m, w - u^\infty\rangle_{(\mathcal{M}^b_{\mathbf{R}^d}([0,T]), \mathcal{C}_{\mathbf{R}^d}([0,T]))},$$

which shows that the vector measure $-m - W.dt$ is a subgradient at the point u^∞ of the of the convex integral functional J_{φ_∞} defined on $\mathcal{C}_{\mathbf{R}^d}([0, T]))$ by $J_{\varphi_\infty}(v) := \int_0^T \varphi_\infty(v(t))dt, \forall v \in \mathcal{C}_{\mathbf{R}^d}([0, T])$.

Step 5 **Claim** $\lim_n \varphi_n(u^n(t)) = \varphi_\infty(u^\infty(t)) < \infty$ a.e. and $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt < \infty$, and subsequently, the energy estimate holds for a.e. $t \in [0, T]$:

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u^\infty(0)) + \frac{1}{2}||\dot{u}^\infty(0)||^2$$
$$- \int_0^t \langle M\dot{u}^\infty(s), \dot{u}^\infty(s)\rangle ds + \int_0^t \langle \dot{u}^\infty(s), f^\infty(s)\rangle ds.$$

With the above results and notations, applying the subdifferential inequality

$$\varphi_n(w(t)) \geq \varphi_n(u^n(t)) + \langle -\ddot{u}^n(t) - W^n(t), w(t) - u^n(t)\rangle$$

with $w = u^\infty$, integrating on $[0, T]$, and passing to the limit when n goes to ∞, gives the inequalities

$$\int_B \varphi_\infty(u^\infty(t))dt \geq \liminf_n \int_B \varphi_n(u^n(t))dt$$
$$\geq \int_B \varphi_\infty(u^\infty(t))dt \geq \limsup_n \int_B \varphi_n(u^n(t))dt$$

on $B \in B^c_p \cap \mathcal{L}([0, T])$ so that

$$\lim_n \int_B \varphi_n(u^n(t))dt = \int_B \varphi_\infty(u^\infty(t))dt \tag{3.3.8}$$

on $B \in B^c_p \cap \mathcal{L}([0, T])$. Now, from the chain rule theorem given in Step 1, recall that

$$\langle \dot{u}^n(t), f^n(t)\rangle - \langle \dot{u}^n(t), \ddot{u}^n(t) - M\dot{u}_n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))],$$

that is,

$$\langle \dot{u}^n(t), z^n(t)\rangle = \frac{d}{dt}[\varphi_n(u_n(t))].$$

By the estimate (3.3.2) and the boundedness in $L^1_{\mathbf{R}^d}([0, T])$ of (z^n), it is immediate that $(\frac{d}{dt}[\varphi_n(u_n(t))])$ is bounded in $L^1_{\mathbf{R}}([0, T])$ so that $(\varphi_n(u_n(.))$ is bounded in variation. By Helly theorem, we may assume that $(\varphi_n(u_n(.))$ pointwisely converges to a BV function ψ. By (3.3.2), $(\varphi_n(u_n(.))$ converges in $L^1_{\mathbf{R}}([0,T])$ to ψ. In particular, for every $k \in L^\infty_{\mathbf{R}^+}([0,T])$ we have

$$\lim_{n\to\infty}\int_0^T k(t)\varphi_n(u_n(t))dt = \int_0^T k(t)\psi(t)dt. \tag{3.3.9}$$

Combining (3.3.8) and (3.3.9) yields

$$\int_B \psi(t)\,dt = \lim_{n\to\infty}\int_B \varphi_n(u^n(t))\,dt = \int_B \varphi_\infty(u^\infty(t))\,dt$$

for all $\in B^c_p \cap \mathcal{L}([0,T])$. As this inclusion is true on each B^c_p and $B^c_p \uparrow [0,T]$, we conclude that

$$\psi(t) = \lim_n \varphi_n(u_n(t)) = \varphi_\infty(u^\infty(t)) \text{ a.e.}$$

Hence we get $\lim_n \varphi_n(u_n(t)) = \varphi_\infty(u^\infty(t))$ a.e. Subsequently, using (iii) the passage to the limit when n goes to ∞ in the equation

$$\varphi_n(u^n(t)) + \frac{1}{2}||\dot{u}^n(t)||^2 = \varphi_n(u^n(0)) + \frac{1}{2}||\dot{u}^n(0)||^2 - \int_0^t \langle M\dot{u}^n(s), \dot{u}^n(s)\rangle ds + \int_0^t \langle \dot{u}^n(s), f^n(s)\rangle ds$$

yields for a.e. $t \in [0,T]$

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u^\infty_0) + \frac{1}{2}||\dot{u}^\infty_0||^2 - \int_0^t \langle M\dot{u}^\infty(s), \dot{u}^\infty(s)\rangle ds + \int_0^t \langle \dot{u}^\infty(s), f^\infty(s)\rangle ds.$$

Noting that (f^n) is uniformly integrable and $\dot{u}^n$ is uniformly bounded and pointwise converges to $\dot{u}^\infty$, by virtue of Grothendieck lemma [16], it converges uniformly on uniformly integrable (=relatively weakly compact) subsets of $L^1_{\mathbf{R}^d}([0,T])$, so that

$$\lim_n \int_0^t \langle \dot{u}^n(s), f^n(s)\rangle ds = \int_0^t \langle \dot{u}^\infty(s), f^\infty(s)\rangle ds.$$

Step 6 Localization of further limits and final step.
As $(z^n = f^n - \ddot{u}^n - M\dot{u}^n)$ is bounded in $L^1_{\mathbf{R}^d}([0,T])$, in view of Step 3, it is relatively compact in the second dual $L^\infty_{\mathbf{R}^d}([0,T])'$ of $L^1_{\mathbf{R}^d}([0,T])$ endowed with the weak topology $\sigma(L^\infty_{\mathbf{R}^d}([0,T])', L^\infty_{\mathbf{R}^d}([0,T]))$. Furthermore, (z^n) can be viewed as a bounded sequence in $\mathcal{C}_{\mathbf{R}^d}([0,T])'$. Hence there are a filter $\mathcal{U}$ finer than the Fréchet filter, $l \in L^\infty_{\mathbf{R}^d}([0,T])'$ and $\mathbf{n} \in \mathcal{C}_{\mathbf{R}^d}([0,T])'$ such that

$$\mathcal{U} - \lim_n z^n = l \in L^\infty_{\mathbf{R}^d}([0,T])'_{weak} \tag{3.3.10}$$

and

$$\lim_n z^n = \mathbf{n} \in \mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak} \tag{3.3.11}$$

where $L^\infty_{\mathbf{R}^d}([0,T])'_{weak}$ is the second dual of $L^1_{\mathbf{R}^d}([0,T])$ endowed with the topology $\sigma(L^\infty_{\mathbf{R}^d}([0,T])', L^\infty_{\mathbf{R}^d}([0,T]))$ and $\mathcal{C}_{\mathbf{R}^d}([0,T])'_{weak}$ denotes the space $\mathcal{C}_{\mathbf{R}^d}([0,T])'$ endowed with the weak topology $\sigma(\mathcal{C}_{\mathbf{R}^d}([0,T])', \mathcal{C}_{\mathbf{R}^d}([0,T]))$, because $\mathcal{C}_{\mathbf{R}^d}([0,T])$ is a separable Banach space for the norm sup, so that we may assume by extracting subsequences that (z^n) weakly converges to $\mathbf{n} \in \mathcal{C}_{\mathbf{R}^d}([0,T])'$. Using Step 4, we note that $\mathbf{n} = -m - W.dt = -m - (M\dot{u}^\infty - f^\infty).dt$. Let l_a be the density of the absolutely continuous part l_a of l in the decomposition $l = l_a + l_s$ in absolutely continuous part l_a and singular part l_s, in the sense there is an decreasing sequence (A_n) of Lebesgue measurable sets in $[0,T]$ with $A_n \downarrow \emptyset$ such that $l_s(h) = l_s(1_{A_n}h)$ for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$ and for all $n \geq 1$. As $(z^n = f^n - \ddot{u}^n - M\dot{u}^n)$ biting converges to $z^\infty = f^\infty - \zeta^\infty - M\dot{u}^\infty$ in Step 4, it is already seen (cf. Proposition 3.1) that

$$l_a(h) = \int_0^T \langle h(t), f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t)\rangle dt$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,T])$, shortly $z^\infty = f^\infty - \zeta^\infty - M\dot{u}^\infty$ coincides a.e. with the density of the absolutely continuous part l_a. By [13, 23], we have

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(f^\infty - \zeta^\infty - M\dot{u}^\infty) + \delta^*(l_s, \text{dom}\, I_{\varphi_\infty}),$$

where φ^*_∞ is the conjugate of φ_∞, $I_{\varphi^*_\infty}$ is the integral functional defined on $L^1_{\mathbf{R}^d}([0,T])$ associated with φ^*_∞, $I^*_{\varphi_\infty}$ is the conjugate of the integral functional I_{φ_∞} and

$$\text{dom}\, I_{\varphi_\infty} := \{u \in L^\infty_{\mathbf{R}^d}([0,T]) : I_{\varphi_\infty}(u) < \infty\}.$$

Using the inclusion

$$z^\infty = f^\infty - \zeta^\infty - M\dot{u}^\infty \in \partial I_{\varphi_\infty}(u^\infty),$$

that is,

$$I_{\varphi^*_\infty}(f^\infty - \zeta^\infty - M\dot{u}^\infty) = \langle f^\infty - \zeta^\infty - M\dot{u}^\infty, u^\infty\rangle - I_{\varphi_\infty}(u^\infty),$$

we see that

$$I^*_{\varphi_\infty}(l) = \langle f^\infty - \zeta^\infty - M\dot{u}^\infty, u^\infty\rangle - I_{\varphi_\infty}(u^\infty) + \delta^*(l_s, \text{dom}\, I_{\varphi_\infty}).$$

Coming back to the inclusion $z^n(t) \in \partial\varphi_n(u^n(t))$, we have

$$\varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbf{R}^d$. By substituting x by $h(t)$ in this inequality, where $h \in L^\infty_{\mathbf{R}^d}([0, T])$, and by integrating

$$\int_0^T \varphi_n(h(t))\, dt \geq \int_0^T \varphi_n(u^n(t))\, dt + \int_0^T \langle h(t) - u^n(t), z^n(t)\rangle\, dt.$$

Arguing as in Step 4 by passing to the limit in the preceding inequality, involving the epiliminf property for integral functionals $\int_0^T \varphi_n(h(t))dt$ defined on $L^\infty_{\mathbf{R}^d}([0, T])$, it is easy to see that

$$\int_0^T \varphi_\infty(h(t))\, dt \geq \int_0^T \varphi_\infty(u^\infty(t))\, dt + \langle h - u^\infty, \mathbf{n}\rangle.$$

Since this holds, in particular, when $h \in \mathcal{C}_{\mathbf{R}^d}([0, T])$, we conclude that $\mathbf{n}$ belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex lower semicontinuous integral functional J_{φ_∞} defined on $\mathcal{C}_{\mathbf{R}^d}([0, T])$

$$J_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\, dt, \ \ \forall u \in \mathcal{C}_{\mathbf{R}^d}([0, T]).$$

Now, let $B : \mathcal{C}_{\mathbf{R}^d}([0, T]) \to L^\infty_{\mathbf{R}^d}([0, T])$ be the continuous injection, and let $B^* : L^\infty_{\mathbf{R}^d}([0, T])' \to \mathcal{C}_{\mathbf{R}^d}([0, T])'$ be the adjoint of B given by

$$\langle B^* l, h\rangle = \langle l, Bh\rangle = \langle l, h\rangle, \quad \forall l \in L^\infty_{\mathbf{R}^d}([0, T])', \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0, T]).$$

Then we have $B^* l = B^* l_a + B^* l_s$, $l \in L^\infty_{\mathbf{R}^d}([0, T])'$ being the limit of $(z_n = f^n - \ddot{u}^n - M\dot{u}^n)$ under the filter $\mathcal{U}$ given in Sect. 4 and $l = l_a + l_s$ being the decomposition of l in absolutely continuous part l_a and singular part l_s. It follows that

$$\langle B^* l, h\rangle = \langle B^* l_a, h\rangle + \langle B^* l_s, h\rangle = \langle l_a, h\rangle + \langle l_s, h\rangle$$

for all $h \in \mathcal{C}_{\mathbf{R}^d}([0, T])$. But it is already seen that

$$\begin{aligned}\langle l_a, h\rangle &= \langle f^\infty - \zeta^\infty - M\dot{u}^\infty, h\rangle \\ &= \int_0^T \langle f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in L^\infty_{\mathbf{R}^d}([0, T])\end{aligned}$$

so that the measure $B^* l_a$ is absolutely continuous

$$\langle B^* l_a, h\rangle = \int_0^T \langle f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0, T])$$

and its density $f^\infty - \zeta^\infty - M\dot{u}^\infty$ satisfies the inclusion

$$f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e.}$$

and the singular part B^*l_s satisfies the equation

$$\langle B^*l_s, h\rangle = \langle l_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0, T]).$$

As $B^*l = \mathbf{n}$, using (3.3.10) and (3.3.11), it turns out that $\mathbf{n}$ is the sum of the absolutely continuous measure $\mathbf{n}_a$ with

$$\langle \mathbf{n}_a, h\rangle = \int_0^T \langle f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0, T])$$

and the singular part $\mathbf{n}_s$ given by

$$\langle \mathbf{n}_s, h\rangle = \langle l_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0, T]),$$

which satisfies the property: for any nonnegative measure θ on $[0, T]$ with respect to which $\mathbf{n}_s$ is absolutely continuous,

$$\int_0^T h_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_s}{d\theta}(t)\right) d\theta(t) = \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t),$$

where $h_{\varphi_\infty^*}$ denotes the recession function of φ_∞^*. Indeed, as $\mathbf{n}$ belongs to $\partial J_{\varphi_\infty}(u^\infty)$ by applying Theorem 5 in [23] we have

$$J_{\varphi_\infty}^*(n) = I_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_a}{dt}\right) + \int_0^T h_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_s}{d\theta}(t)\right) d\theta(t) \tag{3.3.12}$$

with

$$I_{\varphi_\infty^*}(v) := \int_0^T \varphi_\infty^*(v(t))dt, \forall v \in L^1_{\mathbf{R}^d}([0, T]).$$

Recall that

$$\frac{d\mathbf{n}_a}{dt} = f^\infty - \zeta^\infty - M\dot{u}^\infty \in \partial I_{\varphi_\infty}(u^\infty),$$

that is,

$$I_{\varphi_\infty^*}\left(\frac{d\mathbf{n}_a}{dt}\right) = \langle f^\infty - \zeta^\infty - M\dot{u}^\infty, u^\infty\rangle_{\langle L^1_{\mathbf{R}^d}([0,T]), L^\infty_{\mathbf{R}^d}([0,T])\rangle} - I_{\varphi_\infty}(u^\infty). \tag{3.3.13}$$

From (3.3.13), we deduce

$$
\begin{aligned}
J^*_{\varphi_\infty}(n) &= \langle u^\infty, \mathbf{n}\rangle_{\langle \mathcal{C}_{\mathbf{R}^d}([0,T]), \mathcal{C}_{\mathbf{R}^d}([0,T])'\rangle} - J_{\varphi_\infty}(u^\infty) \\
&= \langle u^\infty, \mathbf{n}\rangle_{\langle \mathcal{C}_{\mathbf{R}^d}([0,T]), \mathcal{C}_{\mathbf{R}^d}([0,T])'\rangle} - I_{\varphi_\infty}(u^\infty) \\
&= \int_0^T \langle u^\infty(t), f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t)\rangle dt \\
&\quad + \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t) - I_{\varphi_\infty}(u^\infty) \\
&= I_{\varphi^*_\infty}\left(\frac{d\mathbf{n}_a}{dt}\right) + \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t)).
\end{aligned}
$$

Coming back to (3.3.12) we get the equality

$$
\int_0^T h_{\varphi^*_\infty}\left(\frac{d\mathbf{n}_s}{d\theta}(t)\right) d\theta(t) = \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t)). \qquad \blacksquare
$$

Actually, Proposition 3.3 completes Proposition 4.6 in [7], which is a precursor of some results we present here.

We begin with a second order evolution equation with m-point boundary condition

Proposition 3.4 *Assume that* $E = \mathbf{R}^d$, $M > 0$, $\beta \in L^2_{\mathbf{R}^+}([0, T])$. *For each* $n \in \mathbf{N}$, *let* $\varphi_n : \mathbf{R}^d \to \mathbf{R}^+$ *be a* C^1, *convex, Lipschitz function and let* φ_∞ *be a nonnegative l.s.c proper function defined on* $\mathbf{R}^d$ *such that* $\varphi_n(x) \le \varphi_\infty(x)$ *for all* $n \in \mathbf{N}$ *and for all* $x \in \mathbf{R}^d$. *Let* $f : [0, T] \times E \times E \to E$ *satisfying*

(1) *For each* $(x, y) \in E \times E$ *the scalar function* $t \mapsto f(t, x, y)\rangle$ *is Lebesgue measurable,*
(2) *For each* $t \in [0, 1]$, *function* $f(t, ., .)$ *is continuous on* $E \times E$,
(3) $||f(t, x, y)|| \le \beta(t)$ *for all* $(t, x, y) \in [0, 1] \times E \times E$.
For each $n \in \mathbf{N}$, *let* u^n *be a* $W^{2,1}_{\mathbf{R}^d}([0, 1])$-*solution to the approximating problem*

$$
(\mathcal{P}_n) \begin{cases} f(t, u^n(t), \dot{u}^n(t)) = \ddot{u}^n(t) + M\dot{u}^n(t) + \nabla\varphi_n(u^n(t)), t \in [0, 1] \\ u^n(0) = x \in dom\ \varphi_\infty, \quad u_n(1) = \sum_{i=1}^{m-2} \alpha_i u_n(\eta_i) \end{cases}
$$

Assume that

(i) φ_n *epi-converges to* φ_∞,
(ii) $\lim_n \dot{u}^n(0) = \dot{u}^\infty_0$,
(iii) *There exist* $r_0 > 0$ *and* $x_0 \in \mathbf{R}^d$ *such that*

$$
\sup_{v \in \overline{B}_{L^\infty_{\mathbf{R}^d}([0,1])}} \int_0^T \varphi_\infty(x_0 + r_0 v(t)) < +\infty
$$

where $\overline{B}_{L^\infty_{\mathbf{R}^d}([0,1])}$ *is the closed unit ball in* $L^\infty_{\mathbf{R}^d}([0, 1])$.

(a) *Then, up to extracted subsequences,* (u^n) *converges uniformly to a* $W^{1,1}_{BV}([0,1])$*-function* u^∞ *with* $u^\infty(0) = x \in dom\,\varphi_\infty$, $u^\infty(1) = \sum_{i=1}^{m-2} \alpha_i u^\infty(\eta_i)$ *and* $(\dot u^n)$ *pointwisely converges to a BV function* v^∞ *with* $v^\infty = \dot u^\infty$, *and* $(\ddot u^n)$ *biting converges to a function* $\zeta^\infty \in L^1_{\mathbf{R}^d}([0,1])$ *so that the limit function* u^∞, $\dot u^\infty$ *and the biting limit* ζ^∞ *satisfy the variational inclusion*

$$(\mathcal{P}_\infty) \qquad f^\infty \in \zeta^\infty + M\dot u^\infty + \partial I_{\varphi_\infty}(u^\infty)$$

where $f^\infty(t) := f(t, u^\infty(t), \dot u^\infty(t), \forall t \in [0,1]$, $\partial I_{\varphi_\infty}$ *denotes the subdifferential of the convex lower semicontinuous integral functional* I_{φ_∞} *defined on* $L^\infty_{\mathbf{R}^d}([0,1])$ *by*

$$I_{\varphi_\infty}(u) := \int_0^1 \varphi_\infty(u(t))\,dt, \ \ \forall u \in L^\infty_{\mathbf{R}^d}([0,1]).$$

(b) $(\ddot u^n)$ *weakly converges to the vector measure* $m \in \mathcal{M}^b_E([0,1])$ *so that the limit functions* $u^\infty(.)$ *and the limit measure* m *satisfy the following variational inequality:*

$$\int_0^1 \varphi_\infty(v(t))\,dt \geq \int_0^1 \varphi_\infty(u^\infty(t))\,dt + \int_0^1 \langle -M\dot u^\infty(t) + f^\infty(t), v(t) - u^\infty(t)\rangle\,dt + \langle -m, v - u^\infty\rangle_{(\mathcal{M}^b_{\mathbf{R}^d}([0,1]), \mathcal{C}_E([0,1]))}.$$

(c) *Furthermore* $\lim_n \int_0^1 \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt.$ *Subsequently the energy estimate*

$$\varphi_\infty\left(u^\infty(t)) + \frac{1}{2}||\dot u^\infty(t)||^2 \leq \varphi_\infty(x) + \frac{1}{2}||\dot u^\infty_0\right)||^2 - \int_0^t \langle M\dot u^\infty(s), \dot u^\infty(s)\rangle ds + \int_0^t \langle \dot u^\infty(s), f^\infty(s)\rangle ds$$

holds a.e.

(d) *There are a filter* $\mathcal{U}$ *finer than the Fréchet filter,* $l \in L^\infty_{\mathbf{R}^d}([0,1])'$ *such that*

$$\mathcal{U} - \lim_n [f^n - \ddot u^n - M\dot u^n] = l \in L^\infty_{\mathbf{R}^d}([0,1])'_{weak}$$

where $L^\infty_{\mathbf{R}^d}([0,1])'_{weak}$ *is the second dual of* $L^1_{\mathbf{R}^d}([0,1])$ *endowed with the topology* $\sigma(L^\infty_{\mathbf{R}^d}([0,1])', L^\infty_{\mathbf{R}^d}([0,1]))$ *and* $\mathbf{n} \in \mathcal{C}_{\mathbf{R}^d}([0,1])'_{weak}$ *such that*

$$\lim_n [f^n - \ddot u^n - M\dot u^n] = \mathbf{n} \in \mathcal{C}_{\mathbf{R}^d}([0,1])'_{weak}$$

where $\mathcal{C}_{\mathbf{R}^d}([0,1])'_{weak}$ *denotes the space* $\mathcal{C}_{\mathbf{R}^d}([0,1])'$ *endowed with the weak topology* $\sigma(\mathcal{C}_{\mathbf{R}^d}([0,1])', \mathcal{C}_{\mathbf{R}^d}([0,1]))$ *so that* $\mathbf{n} = -m - (M\dot u^\infty - f^\infty)dt$.

Let l_a be the density of the absolutely continuous part l_a of l in the decomposition $l = l_a + l_s$ in absolutely continuous part l_a and singular part l_s. Then

$$l_a(h) = \int_0^T \langle h(t), f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t)\rangle dt$$

for all $h \in L^\infty_{\mathbf{R}^d}([0,1])$ so that

$$I^*_{\varphi_\infty}(l) = I_{\varphi^*_\infty}(f^\infty - \zeta^\infty - M\dot{u}^\infty) + \delta^*(l_s, dom\, I_{\varphi_\infty})$$

where φ^*_∞ is the conjugate of φ_∞, $I_{\varphi^*_\infty}$ the integral functional defined on $L^1_{\mathbf{R}^d}([0,1])$ associated with φ^*_∞, $I^*_{\varphi_\infty}$ the conjugate of the integral functional I_{φ_∞}, $dom\, I_{\varphi_\infty} := \{u \in L^\infty_{\mathbf{R}^d}([0,1]) : I_{\varphi_\infty}(u) < \infty\}$ and

$$\langle \mathbf{n}, h\rangle = \int_0^1 \langle f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt + \langle \mathbf{n}_s, h\rangle, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,1])$$

with $\langle \mathbf{n}_s, h\rangle = l_s(h)$, $\forall h \in \mathcal{C}_{\mathbf{R}^d}([0,1])$. Further $\mathbf{n}$ belongs to the subdifferential $\partial J_{\varphi_\infty}(u^\infty)$ of the convex lower semicontinuous integral functional J_{φ_∞} defined on $\mathcal{C}_{\mathbf{R}^d}([0,1])$

$$J_{\varphi_\infty}(u) := \int_0^1 \varphi_\infty(u(t))\, dt, \ \ \forall u \in \mathcal{C}_{\mathbf{R}^d}([0,1]).$$

(c) Consequently the density $f^\infty - \zeta^\infty - M\dot{u}^\infty$ of the absolutely continuous part $\mathbf{n}_a$

$$\mathbf{n}_a(h) := \int_0^1 \langle f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t), h(t)\rangle dt, \quad \forall h \in \mathcal{C}_{\mathbf{R}^d}([0,1])$$

satisfies the inclusion

$$f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t) \in \partial\varphi_\infty(u^\infty(t)), \quad \text{a.e.}$$

and for any nonnegative measure θ on $[0,T]$ with respect to which $\mathbf{n}_s$ is absolutely continuous

$$\int_0^1 h_{\varphi^*_\infty}(\frac{d\mathbf{n}_s}{d\theta}(t))d\theta(t) = \int_0^T \langle u^\infty(t), \frac{d\mathbf{n}_s}{d\theta}(t)\rangle d\theta(t)$$

where $h_{\varphi^*_\infty}$ denotes the recession function of φ^*_∞.

Proof Existence of a $W^{2,1}_{\mathbf{R}^d}([0,1])$-solution for the approximating equation

$$\begin{cases} \ddot{u}_n(t) + M\dot{u}_n(t) + \nabla\varphi_n(u^n(t) = f(t, u^n(t), \dot{u}^n(t)), \ \text{a.e. } t \in [0,1] \\ u_n(0) = x, \quad u_n(1) = \sum_{i=1}^{m-2} \alpha_i u_n(\eta_i) \end{cases}$$

is ensured by Proposition 2.8 with integral representation formulas

$$\begin{cases} u_n(t) = e_x(t) + \int_0^1 G(t,s)[\ddot{u}_n(t) + M\dot{u}_n(s)]ds, \ t \in [0,1] \\ \dot{u}_n(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G}{\partial t}(t,s)[\ddot{u}_n(t) + M\dot{u}_n(s)]ds, \ t \in [0,1] \end{cases}$$

$$\begin{cases} e_x(t) = x + A(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x \\ \dot{e}_x(t) = \gamma A \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma t)x \\ A \quad = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\right)^{-1} \end{cases}$$

where G is the Green function given by Lemma 2.1. Then $u^n(0) = x$ and $u_n(1) = \sum_{i=1}^{m-2} \alpha_i u_n(\eta_i)$.

The rest of the proof follows the same lines as that of Proposition 3.3. ■

The following is a new variant on the existence of solutions for the second order evolution inclusion with m-point boundary condition.

Proposition 3.5 *Let $(\partial\varphi_n)$ $(n \in \mathbf{N} \cup \{\infty\})$ be a sequence of subdifferential operators associated with a sequence of nonnegative normal convex integrands (φ_n) $(n \in \mathbf{N} \cup \{\infty\})$. Assume that the following conditions are satisfied:*

(1) For each $n \in \mathbf{N}$, $|\varphi_n(t,x) - \varphi_n(t,y)| \leq \beta_n(t)||x - y||$ for all $t \in [0,1]$ and for all $x, y \in \mathbf{R}^d$, where β_n is a nonnegative integrable functions.

(2) For each Lebesgue-measurable set $A \in [0,1]$, for each $w \in L^\infty_{\mathbf{R}^d}([0,1])$,

$$\limsup_n \int_A \varphi_n(t, w(t))\, dt \leq \int_A \varphi_\infty(t, w(t))\, dt.$$

(3) For each $t \in [0,1]$, $\varphi_n(t,.)$ lower epiconverges to $\varphi_\infty(t,.)$, that is, for each fixed $t \in [0,1]$, for each (x_n) in $\mathbf{R}^d$, converging to $x \in \mathbf{R}^d$, $\liminf \varphi_n(t,x_n) \geq \varphi_\infty(t,x)$.
For each $n \in \mathbf{N}$, let $u^n : [0,1] \to \mathbf{R}^d$ be a $W^{2,1}_{\mathbf{R}^d}([0,1])$-solution to

$$\begin{cases} \ddot{u}^n(t) + \gamma\dot{u}^n(t) \in \partial\varphi_n(t, u^n(t)), \text{ a.e. } t \in [0,1] \\ u^n(0) = x, \quad u^n(1) = \sum_{i=1}^{m-2} \alpha_i u^n(\eta_i). \end{cases}$$

(4) Assume further that

$$\sup_{n\in\mathbf{N}} \int_0^1 \varphi_n(t, u_n(t))dt < +\infty$$

and

$$\sup_{n\in\mathbf{N}} \int_0^1 |\partial\varphi_n(t, u^n(t))|dt < +\infty.$$

Then the following hold:

(a) *Up to extracted subsequences,* (u^n) *converges uniformly to a* $W^{1,1}_{BV}([0,1])$ *function* u^∞ *with* $u^\infty(0) = x$, $u^\infty(1) = \sum_{i=1}^{m-2} \alpha_i u^\infty(\eta_i)$ *and* $(\dot u^n)$ *pointwisely converges to the BV function* $\dot u^\infty$*, and* $(\ddot u^n)$ *stably converges to a Young measure* $\nu^\infty \in \mathcal{Y}([0,1]; \mathbf{R}^d)$ *with* $t \mapsto bar(\nu^\infty_t) \in L^1_{\mathbf{R}^d}([0,1])$ *(here* $bar(\nu^\infty_t)$ *denotes the barycenter of* ν^∞_t*) such that the limit functions* $u^\infty(.)$*,* $\dot u^\infty(.)$ *and the Young limit measure* ν^∞ *satisfy*

$$\int_0^1 \varphi_\infty(t, u^\infty(t))dt \le \liminf_n \int_0^1 \varphi_n(t, u^n(t))dt$$

consequently

$$\lim_n \int_0^1 \varphi_n(t, u^n(t))dt = \int_0^1 \varphi_\infty(t, u^\infty(t))dt < \infty$$

and

$$bar(\nu^\infty_t) + \gamma \dot u^\infty(t) \in \partial\varphi_\infty(t, u^\infty(t)), \text{ a.e.}$$

equivalently the function $t \mapsto bar(\nu^\infty_t) + \gamma\dot u^\infty(t)$ *belongs to the subdifferential* $\partial I_{\varphi_\infty}(u^\infty)$ *of the convex lower semicontinuous integral functional* I_{φ_∞} *defined on* $L^\infty_{\mathbf{R}^d}([0,T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(t, u(t))\, dt, \ \ \forall u \in L^\infty_{\mathbf{R}^d}([0,T]).$$

(b) *Up to extracted subsequences,* (u^n) *converges uniformly to a* $W^{1,1}_{BV}([0,1])$ *function* u^∞ *with* $u^\infty(0) = x$, $u^\infty(1) = \sum_{i=1}^{m-2} \alpha_i u^\infty(\eta_i)$ *and* $(\dot u^n)$ *pointwisely converges to the BV function* $\dot u^\infty$*,* $(\ddot u^n)$ *weakly converges to* $m^\infty \in \mathcal{M}^b_{\mathbf{R}^d}([0,1])$ *so that the limit functions* $u^\infty(.)$ *and the limit measure* m^∞ *satisfy the variational inequality: for every* $v \in \mathcal{C}_{\mathbf{R}^d}([0,1])$*,*

$$\int_0^1 \varphi_\infty(t, v(t))\, dt \ge \int_0^1 \varphi_\infty(t, u^\infty(t))\, dt + \int_0^1 \langle \gamma \dot u^\infty(t)), v(t) - u^\infty(t)\rangle\, dt \\ + \langle m^\infty, v - u^\infty\rangle_{(\mathcal{M}^b_{\mathbf{R}^d}([0,1]), \mathcal{C}_{\mathbf{R}^d}([0,1]))}.$$

In other words, the vector measure $m^\infty + \gamma\dot u^\infty\, dt$ *belongs to the subdifferential* $\partial I_{\varphi_\infty}(u)$ *of the convex functional integral* I_{φ_∞} *defined on* $\mathcal{C}_{\mathbf{R}^d}([0,1])$ *by* $I_{\varphi_\infty}(v) = \int_0^1 \varphi_\infty(t, v(t))\, dt$*,* $\forall v \in \mathcal{C}_{\mathbf{R}^d}([0,1])$*.*

Proof Existence of a $W^{2,1}_{\mathbf{R}^d}([0,1])$-solution u^n to

$$\begin{cases} \ddot u^n(t) + \gamma \dot u^n(t) \in \partial\varphi_n(t, u^n(t)), \text{ a.e. } t \in [0,1] \\ u^n(0) = x, \quad u^n(1) = \sum_{i=1}^{m-2} \alpha_i u^n(\eta_i) \end{cases}$$

is ensured by Proposition 2.7 with integral representation formulas

$$\begin{cases} u^n(t) = e_x(t) + \int_0^1 G(t,s)[\ddot{u}^n(s) + \gamma \dot{u}^n(s)]ds, \ t \in [0,1] \\ \dot{u}^n(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G}{\partial t}(t,s)[\ddot{u}^n(s) + \gamma \dot{u}^n(s)]ds, \ t \in [0,1] \end{cases}$$

where

$$\begin{cases} e_x(t) = x + A(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x \\ \dot{e}_x(t) = \gamma A \left(1 - \sum_{i=1}^{m-2} \alpha_i\right) \exp(-\gamma t)x \\ A \quad = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\right)^{-1} \end{cases}$$

where G is the Green function given by Lemma 2.1.
Step 1 (a) As $\sup_n \int_0^1 |\partial\varphi_n(t, u^n(t))|dt < +\infty$, it follows that $(\ddot{u}^n + \gamma \dot{u}^n)$ is bounded in $L^1_{\mathbf{R}^d}([0,1])$, namely

$$\sup_n \int_0^1 ||(\ddot{u}^n(t) + \gamma \dot{u}^n(t)||dt < +\infty,$$

so that, by the representation formulas given above, it is immediate that (u^n) and $(\dot{u}^n)$ are uniformly bounded. Hence $(\ddot{u}^n)$ is bounded in $L^1_{\mathbf{R}^d}([0,1])$ and $(\dot{u}_n(.))$ is bounded in variation because $\sup_n \int_0^1 ||\ddot{u}_n(t)||\, dt < +\infty$. In view of the Helly–Banach theorem, we may, by extracting a subsequence, assume that $(\dot{u}^n(.))$ converges pointwisely to a BV function $v^\infty(.)$. Let us set $u^\infty(t) = \int_0^t v^\infty(s)\, ds$ for all $t \in [0,1]$. Then $u^\infty \in W^{1,1}_{BV}([0,1])$. As $(\dot{u}_n(.))$ is uniformly bounded and pointwise converges to $v^\infty(.)$, by Lebesgue's theorem, we conclude that $(\dot{u}^n(.))$ converges in $L^1_{\mathbf{R}^d}([0,1])$ to $\dot{u}^\infty(.)$. Hence $u^n(.)$ converges uniformly to $u^\infty(.)$ with $u^\infty(0) = x, \quad u^\infty(1) = \sum_{i=1}^{m-2} \alpha_i u^\infty(\eta_i)$. So (a) is proved. From the general compactness result for Young measures, [5, 10] one may assume that $\ddot{u}^n$ stably converge to an Young measure ν^∞. Further, by virtue of Proposition 3.1 we may assume that $(\ddot{u}^n)$ biting converges to the integrable function $\text{bar}(\nu^\infty) : t \mapsto \text{bar}(\nu^\infty_t)$, that is, there exists a decreasing sequence (B_p) of Lebesgue measurable sets with $\lambda(\cap B_p) = 0$ such that the restriction of $(\ddot{u}^n)$ on each B^c_p converges $\sigma(L^1, L^\infty)$ to $\text{bar}(\nu)$. It follows that

$$\lim_n \int_B \langle \ddot{u}^n + \gamma \dot{u}^n(t), w(t) - u^n(t)\rangle\, dt = \int_B \langle \text{bar}(\nu_t) + \gamma \dot{u}^\infty(t), w(t) - u^\infty(t)\rangle\, dt \tag{3.5.1}$$

for every $B \in B^c_p \cap \mathcal{L}([0,1])$, and for every $w \in L^\infty_E([0,1])$ because the sequence $(w - u^n)$ in $L^\infty_{\mathbf{R}^d}([0,1])$ is bounded and pointwise converges to $w - u^\infty$, so it converges uniformly on uniformly integrable subsets of $L^1_{\mathbf{R}^d}([0,1])$. Since (φ_n) lower epiconverges to φ_∞, by Corollary 4.7 in [11], we have

$$\liminf_n \int_A \varphi_n(t, u^n(t))\, dt \geq \int_A \varphi_\infty(t, u^\infty(t))\, dt \tag{3.5.2}$$

for every Lebesgue-measurable set A in $[0, 1]$. Combining (3.5.1), (3.5.2) and Assumption (2), and integrating the subdifferential inequality

$$\varphi_n(t, w(t)) \geq \varphi_n(t, u^n(t)) + \langle \ddot{u}^n(t) + \gamma \dot{u}^n(t), w(t) - u^n(t) \rangle \tag{3.5.3}$$

on each $B \in B_p^c \cap \mathcal{L}([0, 1])$ and for every $w \in L^\infty_{\mathbf{R}^d}([0, 1])$, we get

$$\int_B \varphi_\infty(t, w(t))\, dt \geq \int_B \varphi_\infty(t, u^\infty(t))\, dt + \int_B \langle \mathrm{bar}(\nu_t^\infty) + \gamma \dot{u}^\infty(t), w(t) - u^\infty(t) \rangle\, dt.$$

This shows that $t \mapsto \mathrm{bar}(\nu_t^\infty) + \gamma \dot{u}^\infty(t)$ is a subgradient at the point u^∞ of the convex integral functional I_{φ_∞} restricted to $L^\infty_E(B_p^c)$, consequently,

$$\mathrm{bar}(\nu_t) + \gamma \dot{u}^\infty(t) \in \partial \varphi_\infty(t, u^\infty(t)), \text{ a.e. on } B_p^c.$$

As this inclusion is true on each B_p^c and $B_p^c \uparrow [0, 1]$, we conclude that

$$\mathrm{bar}(\nu_t^\infty) + \gamma \dot{u}^\infty(t) \in \partial \varphi_\infty(t, u^\infty(t)), \text{ a.e. on } [0, 1].$$

Finally, applying the above subdifferential inequality, and putting $w = u^\infty$ in (3.5.3), we deduce

$$\begin{aligned}
&\int_B \varphi_\infty(t, u^\infty(t) dt \\
&\qquad \geq \limsup_n \int_B \varphi_n(t, u^\infty(t)) dt \\
&\qquad \geq \limsup_n \int_B [\varphi_n(t, u^n(t)) + \langle \ddot{u}^n(t) + \gamma \dot{u}^n(t), u^\infty(t) - u^n(t) \rangle] dt \\
&\qquad = \limsup_n \int_B \varphi_n(t, u^n(t)) dt \geq \liminf_n \int_B \varphi_n(t, u^n(t)) dt \\
&\qquad \geq \int_B \varphi_\infty(t, u^\infty(t)) dt
\end{aligned}$$

because

$$\lim_n \int_B \langle \ddot{u}^n(t) + \gamma \dot{u}^n(t), u^\infty(t) - u^n(t) \rangle] dt = 0$$

recalling that $1_B[\ddot{u}^n + \gamma \dot{u}^n]$ is uniformly integrable. Whence follows

$$\lim_n \int_B \varphi_n(t, u^n(t)) dt = \int_B \varphi_\infty(t, u^\infty(t)) dt.$$

As this inclusion is true on each B in B_p^c and $B_p^c \uparrow [0, 1]$, we conclude that

$$\lim_n \int_0^1 \varphi_n(t, u^n(t))dt = \int_0^1 \varphi_\infty(t, u^\infty(t))dt.$$

Step 2 (b) Repeating the results in Step 1, up to extracted subsequences, (u^n) converges uniformly to a $W_{BV}^{1,1}([0, 1])$ function u^∞ with $u^\infty(0) = x, u^\infty(1) = \sum_{i=1}^{m-2} \alpha_i u^\infty(\eta_i)$ and $(\dot{u}^n)$ pointwisely converges to the BV function $\dot{u}^\infty$. As $(\ddot{u}_n)$ is L^1-bounded we may assume that $(\ddot{u}_n)$ weakly converges to a vector measure $m^\infty \in \mathcal{M}^b_{\mathbf{R}^d}([0, 1])$ since the Banach space $\mathcal{C}_{\mathbf{R}^d}([0, 1])$ is separable and its topological dual is $\mathcal{M}^b_{\mathbf{R}^d}([0, 1])$. Let $w \in \mathcal{C}_{\mathbf{R}^d}(([0, 1])$. Integrating the subdifferential inequality

$$\varphi_n(t, w(t)) \geq \varphi_n(t, u^n(t)) + \langle \ddot{u}^n(t) + \gamma \dot{u}^n(.), w(t) - u^n(t)\rangle$$

and passing to the limit gives immediately

$$\int_0^1 \varphi_\infty(t, w(t))\, dt \geq \int_0^1 \varphi_\infty(t, u^\infty(t))\, dt + \int_0^1 \langle \gamma \dot{u}^\infty(t), w(t) - u^\infty(t)\rangle\, dt$$
$$+ \langle m^\infty, w - u\rangle_{(\mathcal{M}^b_{\mathbf{R}^d}([0,1]), \mathcal{C}_{\mathbf{R}^d}([0,1]))},$$

which shows that the vector measure $m^\infty + \gamma \dot{u}^\infty\, dt$ belongs to the subdifferential $\partial I_{\varphi_\infty}$ of the convex functional integral I_{φ_∞} defined on $\mathcal{C}_{\mathbf{R}^d}([0, 1])$ by $I_{\varphi_\infty}(v) := \int_0^1 \varphi_\infty(t, v(t))\, dt, \forall v \in \mathcal{C}_{\mathbf{R}^d}([0, 1])$. ■

4 Further Applications: Second Order Evolution Problems with Anti-periodic Boundary Condition

It is worth to focus on the main difference in discussing the various approximating problems.

$$f^n(t) = [\ddot{u}^n(t) + M\dot{u}^n(t)] + \nabla\varphi_n(u^n(t)), t \in [0, T] \tag{4.1}$$

$$f^n(t) \in [\ddot{u}^n(t) + M\dot{u}^n(t)] + \partial\varphi_n(u^n(t)), t \in [0, T] \tag{4.2}$$

$$f^n(t) = -[\ddot{u}^n(t) + M\dot{u}^n(t)] + \nabla\varphi_n(u^n(t)), t \in [0, T] \tag{4.3}$$

$$f^n(t) \in -[\ddot{u}^n(t) + M\dot{u}^n(t)] + \partial\varphi_n(u^n(t)), t \in [0, T]. \tag{4.4}$$

Equations (4.1) and (4.2) are usual in second order dynamical systems. We refer to Attouch et al. [4] and Schatzmann [24] for a deep study of such models. See also the results developed in Propositions 3.2–3.5. Here, according to a traditional vein, we prove the existence of generalized solution with the conservation of energy in (3.3) and (3.4). Meanwhile (4.3) and (4.4) appear in the problem of anti-periodic solution developed in Aizicovici et al. [1–3]. Here in Proposition 4.3 we present a first result of the existence of generalized solution for the problem

$$f(t) \in [\ddot{u}(t) + M\dot{u}(t)] + \partial\varphi(u(t))$$

using the approximating problem (4.2) with application (Proposition 3.4) to problem

$$f(t, u(t), \dot{u}(t)) \in \ddot{u}(t) + M\dot{u}(t) + \partial\varphi(u(t)), t \in [0, T]$$

with m-point boundary condition using the approximating problem

$$f(t, u^n(t), \dot{u}^n(t)) = \ddot{u}^n(t) + M\dot{u}^n(t)] + \nabla\varphi_n(u(t)), t \in [0, T]$$

with m-point boundary condition. Here one can see that the techniques employed in (4.1) and (4.2) cannot be used to develop similar results to (4.3) and (4.4), in particular, we cannot obtain the conservation of energy for the variational limits in (4.3) and (4.4) by contrast with (4.1) and (4.2). So it is worth to mention that our tools allow to study the approximating problem of anti-periodic solution in the framework of Haraux–Okochi with anti-periodic solution

$$f^n(t) = [\ddot{u}^n(t) + M\dot{u}^n(t)] + \nabla\varphi_n(u^n(t)), t \in [0, T],$$
$$u_n(0) = -u_n(T).$$

In our opinion, the general problem of the existence of energy conservation solution to second order evolution inclusion of the form

$$f(t) \in [\ddot{u}(t) + M\dot{u}(t)] + \partial\varphi(u(t)) \tag{4.5}$$

where φ is a lower semicontinuous convex proper function is a difficult problem when the perturbation $f \in L^1_H([0, T])$ and H is a separable Hilbert space.

Now, to finish the paper, we show that our abstract result in Proposition 3.3 and the tool developed therein can be applied to the first order of evolution equation and also to the second order evolution equation with anti-periodic boundary conditions. H. Okochi initiated the study for anti-periodic solutions to evolution equations in Hilbert spaces. Following Okochi's work, A. Haraux proved some existence and uniqueness theorems for anti-periodic solutions by using Brouwer's or Schauder fixed point theorems. Aftabizadeh, Aizicovici and Pavel have studied the anti-periodic solutions to second order evolution equation in Hilbert spaces and Banach spaces by using monotone and accretive operator theory for equations of type (4.3) and (4.4). Here we show the applicability of our abstract result to the study of evolution equations of type (4.1) and (4.2) with anti-periodic boundary condition. For notational convenience let us denote by $\mathcal{H}$ the set of of functions $f \in L^2_{loc}(\mathbf{R}, H)$ such that f is anti-periodic, that is, $f(t + T) = -f(t)$ for all $t \in \mathbf{R}$ and

$$\mathcal{H}_\beta([0,T]) := \{f \in \mathcal{H} : ||f(t)|| \le \beta(t),\ \beta \in L^2_{\mathbf{R}}([0,T]),\ t \in [0,T]\}.$$

We begin with some examples in the first order of evolution equation with anti-periodic condition.

Proposition 4.1 *Let $H = \mathbf{R}^d$. Assume that $\varphi_n : \mathbf{R}^d \to [0,+\infty[$ are even, convex, Lipschitz and $\varphi_\infty : \mathbf{R}^d \to [0,+\infty]$ is proper lower semicontinuous convex function such that $\varphi_n(x) \le \varphi_\infty(x)$ for all $n \in \mathbf{N}$ and for all $x \in \mathbf{R}^d$. Let f^n be sequence in $\mathcal{H}_\beta([0,T])$ and let u^n be a $W^{1,2}_{\mathbf{R}^d}([0,T])$-solution to the problem*

$$\begin{cases} f^n(t) \in \dot{u}^n(t) + \partial\varphi_n(u^n(t)) & t \in [0,T], \\ u_n(T) = -u_n(0) \end{cases}$$

Assume that the following conditions are satisfied:

(i) φ_n *epiconverges to* φ_∞,
(ii) $\lim_n u^n(0) = u_0^\infty \in$ *dom* φ_∞ *and* $\lim_n \varphi_n(u^n(0)) = \varphi_\infty(u_0^\infty)$.
(iii) f^n $\sigma(L^2_{\mathbf{R}^d}([0,T]), L^2_{\mathbf{R}^d}([0,T]))$*-converges to* $f^\infty \in L^2_{\mathbf{R}^d}([0,T])$.

Then the following hold

(a) Up to extracted subsequences, (u^n) converges pointwisely to an anti-periodic absolutely continuous mapping u^∞ with $u^\infty(T) = -u^\infty(0)$, $(\dot{u}^n)$ $\sigma(L^2_{\mathbf{R}^d}, L^2_{\mathbf{R}^d})$-converges to $\zeta^\infty \in L^2_{\mathbf{R}^d}([0,T])$ with $\zeta^\infty = \dot{u}^\infty$, $\lim_n \varphi_n(u^n(t)) = \varphi_\infty(u^\infty(t)) < +\infty$ a.e. and $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt < +\infty$.
(b) $f^\infty - \zeta^\infty \in \partial I_{\varphi_\infty}(u^\infty)$ where $\partial I_{\varphi_\infty}$ denotes the subdifferential of the convex lower semicontinuous integral functional I_{φ_∞} defined on $L^\infty_{\mathbf{R}^d}([0,T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt,\ \ \forall u \in L^\infty_{\mathbf{R}^d}([0,T]).$$

Proof Existence of $W^{1,2}_{\mathbf{R}^d}([0,T])$-solution u^n to the problem

$$\begin{cases} f^n(t) \in \dot{u}^n(t) + \partial\varphi_n(u^n(t)) & t \in [0,T], \\ u_n(T) = -u_n(0) \end{cases}$$

is ensured. See Haraux [17], Okochi [22].
Step 1 Estimation of u^n, $\dot{u}^n$, and $\varphi_n(u^n(.)$ Multiplying scalarly the inclusion

$$f^n(t) - \dot{u}^n(t) \in \partial\varphi_n(u^n(t)$$

by $\dot{u}^n(t)$ and applying the chain rule formula [21] for the Lipschitz function φ_n gives

$$\langle \dot{u}^n(t), f^n(t)\rangle - ||\dot{u}^n(t)||^2 = \frac{d}{dt}[\varphi_n(u^n(t))]. \tag{4.1.1}$$

Hence by integration of (4.1.1) on $[0, T]$ and anti-periodicity condition we get the estimate

$$||\dot{u}^n||_{L^2_H([0,T])} \leq ||f^n||_{L^2_H([0,T])}. \tag{4.1.2}$$

From the Poincaré inequality

$$||u^n(t)|| \leq \sqrt{T}\,||\dot{u}^n||_{L^2_H([0,T])}, \forall t \in [0, T]. \tag{4.1.3}$$

Integrating (4.1.1) on $[0, t]$ we get

$$\begin{aligned} 0 \leq \varphi_n(u^n(t)) = \varphi_n(u^n(0)) - \int_0^t ||\dot{u}^n(s)||^2 ds + \int_0^t \langle \dot{u}^n(s), f^n(s)\rangle ds \quad (4.1.4) \\ \leq \varphi_n(u^n(0)) + \int_0^t \langle \dot{u}^n(s), f^n(s)\rangle ds \end{aligned}$$

so that by using the above estimates (4.1.2)–(4.1.3)–(4.1.4), the weak convergence of f^n in $L^2_H([0, T])$ and (ii) we note that $\varphi_n(u^n(t))$ is uniformly bounded.

Step 2 Using the results in Step 1, up to extracted subsequences (u^n) converges pointwisely to an anti-periodic absolutely continuous mapping u^∞ with $u^\infty(T) = -u^\infty(0)$, $(\dot{u}^n)$ $\sigma(L^2_{\mathbf{R}^d}, L^2_{\mathbf{R}^d})$-converges to $\zeta^\infty \in L^2_{\mathbf{R}^d}([0, T])$ with $\zeta^\infty = \dot{u}^\infty$. For simplicity set $z^n(t) := f^n(t) - \dot{u}^n(t)$. Since we have

$$\langle \dot{u}^n(t), z^n(t)\rangle = \frac{d}{dt}[\varphi_n(u^n(t))]$$

and $\langle \dot{u}^n(.), z^n(.)\rangle$ is bounded in $L^1_{\mathbf{R}}([0, T])$, $\varphi_n(u^n(t))$ is of bounded variation and uniformly bounded.

Claim $\lim_n \varphi_n(u_n(t)) = \varphi_\infty(u_\infty(t)) < \infty$ a.e and $\lim_n \int_0^T \varphi_n(u_n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt < \infty$.

From the above estimates and Helly theorem, we may assume that $(\varphi_n(u_n(.))$ pointwisely converges to a BV function θ so that $(\varphi_n(u_n(.))$ converges in $L^1_{\mathbf{R}}([0, T])$ to θ. In particular, for every $k \in L^\infty_{\mathbf{R}^+}([0, T])$, we have

$$\lim_{n\to\infty} \int_0^T k(t)\varphi_n(u_n(t))dt = \int_0^T k(t)\theta(t)dt.$$

Coming back to the inclusion $z^n(t) \in \partial\varphi_n(u^n(t))$, and using the fact that $\varphi_n(x) \leq \varphi_\infty(x)$, $\forall n \in \mathbf{N}$, $\forall x \in \mathbf{R}^d$, we have

$$\varphi_\infty(x) \geq \varphi_n(x) \geq \varphi_n(u^n(t)) + \langle x - u^n(t), z^n(t)\rangle$$

for all $x \in \mathbf{R}^d$. Let $h \in L^\infty_{\mathbf{R}^d}([0, T])$. Substituting x by $h(t)$ in this inequality and by integrating on each measurable set B gives

$$\int_B \varphi_\infty(h(t))\,dt \geq \int_B \varphi_n(h(t))\,dt \geq \int_B \varphi_n(u^n(t))\,dt + \int_B \langle h(t)-u^n(t), z^n(t)\rangle\,dt$$

and passing to the limit in the preceding inequality when n goes to $+\infty$, we get

$$\int_B \varphi_\infty(h(t))\,dt \geq \int_B \theta(t)\,dt + \int_B \langle h(t)-u^\infty(t), z^\infty(t)\rangle\,dt \tag{4.1.5}$$

with $z^\infty = f^\infty - \dot{u}^\infty$. In particular, by taking $h = u^\infty$ we get the estimate

$$\int_B \varphi_\infty(u^\infty(t))\,dt \geq \int_B \theta(t)\,dt$$

for all $B \in \mathcal{L}([0,T])$. By the epi-lower convergence result [11, Corollary 4.7], we have

$$\int_B \theta(t)\,dt = \lim_{n\to\infty} \int_B \varphi_n(u^n(t))\,dt \geq \liminf_{n\to\infty} \int_B \varphi_\infty(u^n(t))\,dt \geq \int_B \varphi_\infty(u^\infty(t))\,dt$$

for all $B \in \mathcal{L}([0,T])$. It turns out that $\varphi_\infty(u^\infty(t)) = \theta(t)$ a.e. and

$$\lim_{n\to\infty} \int_B \varphi_n(u^n(t))\,dt = \int_B \varphi_\infty(u^\infty(t))\,dt < \infty. \tag{4.1.6}$$

From (4.1.5) and (4.1.6) it follows that $f^\infty - \zeta^\infty \in \partial I_{\varphi_\infty}(u^\infty)$ where $\partial I_{\varphi_\infty}$ denotes the subdifferential of the convex lower semicontinuous integral functional I_{φ_∞} defined on $L^\infty_{\mathbf{R}^d}([0,T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt,\ \ \forall u \in L^\infty_{\mathbf{R}^d}([0,T]).$$ ■

Here is a variant of Proposition 4.1.

Proposition 4.2 *Let $H = \mathbf{R}^d$. Assume that $\gamma > 0$, $\varphi_n : \mathbf{R}^d \to [0,+\infty]$ is even, convex, Lipschitz, $\varphi_\infty : \mathbf{R}^d \to [0,+\infty]$ is proper lower semicontinuous convex function such that $\varphi_n(x) \leq \varphi_\infty(x)$ for all $n \in \mathbf{N}$ and for all $x \in \mathbf{R}^d$. Let (f^n) be an anti-periodic sequence in $\mathcal{H}_\beta([0,T])$. Let u^n be a $W^{1,2}_{\mathbf{R}^d}([0,T])$ anti-periodic solution to the problem*

$$\begin{cases} f^n(t) \in \dot{u}^n(t) + \partial\varphi_n(u^n(t)) - \gamma u^n(t),\ t \in [0,T] \\ u_n(T) = -u_n(0). \end{cases}$$

Assume that the following conditions are satisfied:

(i) *φ_n epiconverges to φ_∞,*
(ii) $\lim_n u^n(0) = u_0^\infty \in$ *dom* φ_∞ *and* $\lim_n \varphi(u^n(0)) = \varphi_\infty(u_0^\infty)$,
(iii) *f^n $\sigma(L^2_{\mathbf{R}^d}([0,T]), L^2_{\mathbf{R}^d}([0,T]))$-converges to $f^\infty \in L^2_{\mathbf{R}^d}([0,T])$.*

Then the following hold

(a) *Up to extracted subsequences,* (u^n) *converges pointwisely to an anti-periodic absolutely continuous mapping* u^∞ *with* $u^\infty(T) = -u^\infty(0)$*,* $(\dot u^n)$ $\sigma(L^2_{\mathbf{R}^d}, L^2_{\mathbf{R}^d})$*-converges to* $\zeta^\infty \in L^2_{\mathbf{R}^d}([0,T])$ *with* $\zeta^\infty = \dot u^\infty$*,* $\lim_n \varphi_n(u^n(t)) = \varphi_\infty(u^\infty(t)) < +\infty$ *a.e. and* $\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt < +\infty$.

(b) $f^\infty - \zeta^\infty \in \partial I_{\varphi_\infty}(u^\infty)$ *where* $\partial I_{\varphi_\infty}$ *denotes the subdifferential of the convex lower semicontinuous integral functional* I_{φ_∞} *defined on* $L^\infty_{\mathbf{R}^d}([0,T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt, \ \ \forall u \in L^\infty_{\mathbf{R}^d}([0,T]).$$

Proof Existence of u^n for the problem

$$\begin{cases} f^n(t) - \dot u^n(t) + \gamma u^n(t) \in \partial\varphi_n(u^n(t)) & t \in [0,T], \\ u_n(T) = -u_n(0), \end{cases}$$

is ensured. See Haraux [17], Okochi [22].

Step 1 Estimation of $\dot u^n$ *and* u^n. Multiplying scalarly the inclusion

$$f^n(t) - \dot u^n(t) + \gamma u^n(t) \in \partial\varphi_n(u^n(t)) \tag{4.2.1}$$

by $\dot u^n(t)$ and applying the chain rule formula [21] for the Lipschitz function φ_n gives

$$\langle \dot u^n(t), f^n(t)\rangle - ||\dot u^n(t)||^2 + \gamma\langle \dot u^n(t), u^n(t)\rangle = \frac{d}{dt}[\varphi(u^n(t))]. \tag{4.2.2}$$

Hence by integration in (4.2.1) and anti-periodicity conditions we get the estimate

$$||\dot u^n||_{L^2_H([0,T])} \le ||f^n||_{L^2_H([0,T])}. \tag{4.2.3}$$

From the Poincaré inequality,

$$||u^n(t)|| \le \sqrt{T}\,||\dot u^n||_{L^2_H([0,T])} \le \sqrt{T}\,||f^n||_{L^2_H([0,T])}. \tag{4.2.4}$$

Integrating (4.2.2), we get

$$0 \le \varphi_n(u^n(t)) = \varphi_n(u^n(0)) - \int_0^t ||\dot u^n(s)||^2 ds + \int_0^t \langle \dot u^n(s), f^n(s)\rangle ds + \gamma\int_0^t \langle \dot u^n(s), u^n(s)\rangle ds$$

We note that

$$\int_0^t \langle \dot{u}^n(s), f^n(s)\rangle ds \leq \frac{1}{2}||f^n||_{L^2_H([0,T])}(1+\int_0^t ||\dot{u}^n(s)||^2 ds) \leq \text{Const.}$$

$$\gamma \int_0^t \langle \dot{u}^n(s), u^n(s)\rangle ds \leq \text{Const.}||f^n||^2_{L^2_H([0,T])}$$

so that by using the above estimate, the $\sigma(L^2_{\mathbf{R}^d}([0,T]), L^2_{\mathbf{R}^d}([0,T]))$ convergence of f^n and (ii), we conclude that $\varphi_n(u^n(t))$ is uniformly bounded. Now the remainder of the proof is similar to that of Proposition 4.1. ■

We finish the paper with the approximating problem in second order evolution equation with anti-periodic condition

$$\begin{cases} f^n(t) &= \ddot{u}^n(t) + M\dot{u}^n(t) + \nabla\varphi_n(u^n(t)), \\ u^n(T) = -u^n(0). \end{cases}$$

where M is a positive constant, φ_n are convex Lipschitz, C^1, even, functions that epi-converges to a lower semicontinuous convex proper function φ_∞, (f_n) is a sequence in $L^2_H([0,T])$ which weakly converges to a function $f_\infty \in L^2_H([0,T])$. Existence of a $W^{2,2}_{\mathbf{R}^d}([0,T])$ anti-periodic -solution to this approximating problem is well known. See Haraux [17], Okochi [22].

Proposition 4.3 *Let $H = \mathbf{R}^d$, $M \in \mathbf{R}^+$. Assume that $\varphi_n : \mathbf{R}^d \to [0,+\infty[$ is $\mathcal{C}^1$, even, convex, Lipschitz and, $\varphi_\infty : \mathbf{R}^d \to [0,+\infty]$ is proper convex lower semicontinuous with $\varphi_n(x) \leq \varphi_\infty(x)$, $\forall x \in \mathbf{R}^d$. Let $f^n \in \mathcal{H}_\beta([0,T])$ Let u^n be a $W^{2,2}_{\mathbf{R}^d}([0,T])$ anti-periodic solution to the approximated problem*

$$\begin{cases} f^n(t) &= \ddot{u}^n(t) + M\dot{u}^n(t) + \nabla\varphi_n(u^n(t)), t \in [0,T], \\ u_n(T) = -u_n(0). \end{cases}$$

Assume that

(i) $f^n \sigma(L^2_H, L^2_H)$ *converges to* $f^\infty \in L^2_H([0,T])$.
(ii) $\lim_n u^n(0) = u_0^\infty \in$ *dom* φ_∞, $\lim_n \varphi_n(u^n(0)) = \varphi_\infty(u_0^\infty)$, *and* $\lim_n \dot{u}^n(0) = \dot{u}_0^\infty$,
(iii) φ_n *epi-converges to* φ_∞,
(iv) *There exist* $r_0 > 0$ *and* $x_0 \in \mathbf{R}^d$ *such that*

$$\sup_{v \in \overline{B}_{L^\infty_{\mathbf{R}^d}([0,T])}} \int_0^T \varphi_\infty(x_0 + r_0 v(t))) < +\infty$$

where $\overline{B}_{L^\infty_{\mathbf{R}^d}([0,1])}$ *is the closed unit ball in* $L^\infty_{\mathbf{R}^d}([0,T])$.

Then the following hold

(a) *Up to extracted subsequences,* (u^n) *converges uniformly to a* $W^{1,1}_{BV}([0,T])$ *anti-periodic function* u^∞ *with* $u^\infty(T) = -u^\infty(0)$, *and* $(\dot{u}^n)$ *pointwisely converges to*

the BV function $\dot{u}^\infty$, and $(\ddot{u}^n)$ biting converges to a function $\zeta^\infty \in L^1_{\mathbf{R}^d}([0, T])$ which satisfy the variational inclusion

$$f^\infty - \zeta^\infty - M\dot{u}^\infty \in \partial I_{\varphi_\infty}(u^\infty)$$

where $\partial I_{\varphi_\infty}$ denotes the subdifferential of the convex lower semicontinuous integral functional I_{φ_∞} defined on $L^\infty_{\mathbf{R}^d}([0, T])$

$$I_{\varphi_\infty}(u) := \int_0^T \varphi_\infty(u(t))\,dt, \ \ \forall u \in L^\infty_{\mathbf{R}^d}([0, T]).$$

Furthermore

$$\lim_n \varphi_n(u^n(t)) = \varphi_\infty(u^\infty(t)) < \infty \text{ a.e.}$$
$$\lim_n \int_0^T \varphi_n(u^n(t))dt = \int_0^T \varphi_\infty(u^\infty(t))dt < \infty.$$

Subsequently, the estimated energy holds almost everywhere

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u^\infty(0)) + \frac{1}{2}||\dot{u}^\infty(0)||^2 - M\int_0^t ||\dot{u}^\infty(s)||^2\,ds + \int_0^t \langle \dot{u}^\infty(s), f^\infty(s)\rangle ds.$$

Further $(\ddot{u}^n)$ weakly converges to the vector measure $m \in \mathcal{M}^b_H([0, T])$ so that the limit functions $u^\infty(.)$ and the limit measure m satisfy the following variational inequality:

$$\int_0^T \varphi_\infty(v(t))\,dt \geq \int_0^T \varphi_\infty(u^\infty(t))\,dt + \int_0^T \langle -M\dot{u}^\infty(t) + f^\infty(t), v(t) - u^\infty(t)\rangle\,dt + \langle -m, v - u^\infty\rangle_{(\mathcal{M}^b_E([0,T]),\mathcal{C}_E([0,T]))}.$$

In other words, the vector measure $-m + [-M\dot{u}^\infty + f^\infty]\,dt$ belongs to the subdifferential $\partial I_{f_\infty}(u)$ of the convex functional integral I_{f_∞} defined on $\mathcal{C}_H([0, T])$ by $I_{\varphi_\infty}(v) = \int_0^T \varphi_\infty(t, v(t))\,dt$, $\forall v \in \mathcal{C}_H([0, T])$.

Proof Existence of $W^{2,2}_{\mathbf{R}^d}([0, T])$-solution u^n for the approximated problem

$$\begin{cases} f^n(t) \ = \ddot{u}^n(t) + M\dot{u}^n(t) + \nabla\varphi_n(u^n(t)) \quad t \in [0, T], \\ u_n(T) = -u_n(0) \end{cases}$$

follows from Haraux [17]. Now we can finish the proof by repeating mutatis mutandis the machinery developed in Proposition 3.3. Therefore our $W^{1,1}_{BV}([0, T])$ anti-periodic limit u^∞ of (u^n) and biting limit ζ^∞ of $(\ddot{u}^n)$ satisfies the inclusion

$$f^\infty(t) - \zeta^\infty(t) - M\dot{u}^\infty(t) \in \partial\varphi_\infty(u^\infty(t))$$

and the energy estimate holds

$$\varphi_\infty(u^\infty(t)) + \frac{1}{2}||\dot{u}^\infty(t)||^2 = \varphi_\infty(u_0^\infty) + \frac{1}{2}||\dot{u}_0^\infty||^2 - M\int_0^t ||\dot{u}^\infty(s)||^2\,ds + \int_0^t \langle \dot{u}^\infty(s), f^\infty(s)\rangle ds$$

almost everywhere. ∎

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