Finite Elasto-Plastic Models for Lattice Defects in Crystalline Materials

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Abstract Elasto-plastic models for continuous distributed defects are provided for materials endowed with Cartan-Riemannian differential geometric structures. The geometrical measure of defects, dislocations and disclinations, are related to the incompatibilities of the so-called plastic distortion and plastic connection, respectively. The coupling between defects is described through the non-local evolution equations which are compatible with the free energy imbalance principle.

1 Introduction

The continuum elasto-plastic constitutive models proposed in this paper describe the behaviour of crystalline material with microstructural defects in terms of three configurations:

Let k be a fixed reference configuration of the body $\mathscr{B}$, $k(\mathscr{B}) \subset E$, and $\mathscr{B}$ will be identified with $k(\mathscr{B})$;
$\chi(\cdot, t)$ the deformed configuration at time t, for any motion of the body $\mathscr{B}$, $\chi : \mathscr{B} \times R \longrightarrow E$, and $\mathbf{F}(\mathbf{X}, t) = \nabla \chi((\mathbf{X}, t)$ denotes the deformation gradient;
there exists $\mathscr{K}$, a time dependent anholonomic configuration (so- called configuration with torsion), defined by the pair $(\mathbf{F}^p, \overset{(p)}{\boldsymbol{\Gamma}})$, $\mathbf{F}^p$-plastic distortion and $\overset{(p)}{\boldsymbol{\Gamma}}$-plastic connection.

The reference and deformed (actual) configurations, which are global configurations of the elasto-plastic body, characterize the material within Riemannian geometry, while the local configurations, attached to the material points of the body, are

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anholonomic configuration, and the Riemann-Cartan geometry describes geometrically the measures of defects, see Yavari and Goriely [19]. The geometrical measure of defects, dislocations and disclinations, are related to the incompatibilities of the so-called plastic distortion and plastic connection, respectively. The incompatibility of the plastic distortion, i.e. $curl\,\mathbf{F}^p \neq 0$, which means the presence of dislocations, and the incompatibility of the so-called plastic connection, i.e. $\overset{(p)}{\boldsymbol{\Gamma}} \neq (\mathbf{F}^p)^{-1}\nabla\mathbf{F}^p$, which means the presence of the disclinations, see for instance de Wit [10], Cleja-Ţigoiu et al. [9], Fressengeas et al. [13]. The interplay between the defects such as dislocations and disclinations is described through the Cartan torsion attached to the plastic connection, denoted by $\mathbf{S}^p$. The dislocation density tensor (called also the geometrically necessary dislocation) and disclination density tensor characterize non-zero Burgers and Frank vector, defined at the end of this section. The physical motivations for the defects such as disclinations can be found in [18], see also [4, 13] In the present paper the plastic connection which is $\mathbf{C}^p$-metric compatible has been considered as in the previous papers [5, 6], apart from the paper by Clayton et al. [4], devoted to finite miscropolar elastoplasticity, where a connection defined by Minagawa [15] has been introduced. Let us remark here that when $\mathbf{Qu}$ for all vector $\mathbf{u}$ is a skew-symmetric second order tensor, the coefficients of this connection and our plastic connection coincide.

The energetic arguments are necessary to complete the description of the elasto-plastic models with defects, such as dislocations and disclinations. The balance equations for the micro forces have been revised in this paper, starting from the basic hypothesis concerning the expression of the internal dissipation power during the elasto-plastic material with microstructural defects, in conjunction with the virtual power assumptions. As the principle of the virtual power expending during the plastic and disclination mechanisms have been formulated in terms of the incompatible second order virtual rates, contrary to the virtual power principle considered by Fosdick [12] where the virtual second order velocity field $\widetilde{\mathbf{L}}$ is compatible, i.e. $\tilde{\mathbf{L}} = \nabla\tilde{\mathbf{v}}$, only one balance equation for appropriate micro forces has been provided. For the macro balance equations similar to those proposed by Fleck et al. [11], see also [8], have been adopted.

Clayton et al. [3, 4] introduced the balance equations for micro forces similar to Fleck et al. [11].

Fosdick [12] considered the following example of the *principle of virtual power*

$$\int_{\mathscr{P}} (\tilde{\mathbf{T}} \cdot \nabla\mathbf{u} + \mathscr{J} \cdot \nabla(\nabla\mathbf{u}) dV = \int_{\partial\mathscr{P}} (\mathbf{t} \cdot \mathbf{u} + \mathbf{J} \cdot \nabla\mathbf{u}) dA \quad \forall \mathscr{P} \subset \mathscr{B} \quad \forall \mathbf{u},$$

where the left hand side represents the internal virtual power and the right hand side is the external virtual power on $\mathscr{P}$, in which the body force term was neglected. $\mathbf{t}$ and $\mathbf{J}$ depend on the normal to the surface, $\mathbf{n}$, and have independent physical significance, being restricted to different, possible conditions. The principle is valid for any arbitrary part $\mathscr{P}$ of a body and for any arbitrary virtual velocity field $\mathbf{u}$. Fosdick analyzed all the consequences that can be drawn and remarked that the principle of virtual power has been applied for models by considering various length scales, because of "several major consequences," that follows from the assumption that the principle of the virtual work holds for "arbitrary parts of the body," "have not been clearly expressed" in certain papers.

The postulate of the free energy imbalance expresses the restriction on the elasto-plastic material to be satisfied in the configuration with torsion, $\mathscr{K}$, as an *imbalanced free energy condition*, see Cleja-Ţigoiu [5, 6], as well as Gurtin et al. [14], for the initial original ideas related to the free energy imbalance.

List of the Notations:

$\mathscr{V}$-the vector space of translations of the three dimensional Euclidean space $\mathscr{E}$;
Lin-the set of the linear mappings from $\mathscr{V}$ to $\mathscr{V}$, i.e. the set of second order tensors; $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\times\mathbf{v}$, $\mathbf{u}\otimes\mathbf{v}$ denote scalar, cross and tensorial products of vectors;
$(\mathbf{u},\mathbf{v},\mathbf{z}) := (\mathbf{u}\times\mathbf{v})\cdot\mathbf{z}$ is the mixt product of the vectors from $\mathscr{V}$;
$\mathbf{a}\otimes\mathbf{b}$ and $\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}$ are defined to be a second order tensor and a third order tensor by $(\mathbf{a}\otimes\mathbf{b})\mathbf{u} = \mathbf{a}(\mathbf{b}\cdot\mathbf{u})$, $(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c})\mathbf{u} = (\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\cdot\mathbf{u})$, for all vectors $\mathbf{u}$;
For any second order tensor $\mathbf{A}\in Lin$ we use the notations $\{\mathbf{A}\}^S$, $\{\mathbf{A}\}^a$ for its symmetric and skew-symmetric part;
$\mathbf{I}$ the identity tensor in Lin, $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}\in Lin$;
$\partial_{\mathbf{A}}\phi(x)$ denotes the partial differential of the function ϕ with respect to the field $\mathbf{A}$;
$\mathbf{A}\cdot\mathbf{B} := \mathrm{tr}(\mathbf{A}\mathbf{B}^T) = A_{ij}B_{ij}$ is the scalar product of $\mathbf{A},\mathbf{B}\in Lin$;

$curl\mathbf{A}$, a second-order tensor field, is defined by

$(curl\mathbf{A})(\mathbf{u}\times\mathbf{v}) := (\nabla\mathbf{A}(\mathbf{u}))\mathbf{v} - (\nabla\mathbf{A}(\mathbf{v}))\mathbf{u} \quad \forall\mathbf{u},\mathbf{v}\in\mathscr{V}$ and
$(curl\mathbf{A})_{pi} = \in_{ijk}\dfrac{\partial A_{pk}}{\partial X^j}$, are the component of $curl\mathbf{A}$ given in a Cartesian basis;

$\in$ and $\in_{ijk}$ denote Ricci permutation tensor and its components, respectively;
$\nabla\mathbf{A}$ the derivative (or the gradient) of the field $\mathbf{A}$ in a coordinate system $\{\mathbf{x}^a\}$ (with respect to the reference configuration), $\nabla\mathbf{A} = \dfrac{\partial A_{ij}}{\partial X^k}\mathbf{e}^i\otimes\mathbf{e}^j\otimes\mathbf{e}^k$;
$\nabla_\chi\mathbf{L} \equiv \dfrac{\partial}{\partial x^k}(\dfrac{\partial v_i}{\partial x^j})\mathbf{e}^i\otimes\mathbf{e}^j\otimes\mathbf{e}^k$, where the dual basis $\mathbf{e}^a$, is defined by the inner product $\mathbf{e}^b\cdot\mathbf{e}_a = \delta^b{}_a$.

We also denote:

$Lin(\mathscr{V}, Lin) = \{\mathbf{N} : \mathscr{V}\longrightarrow Lin, \quad \text{linear}\}$ - the space of all third-order tensors; an element of this pspace is given by $\mathbf{N} = N_{ijk}\mathbf{i}^i\otimes\mathbf{i}^j\otimes\mathbf{i}^k$;

$\mathbf{N}\cdot\mathbf{M} = N_{ijk}M_{ijk}$ - the scalar product of third-order tensors expressed in a Cartesian basis.

The differential of any smooth tensor field $\bar{\mathbf{F}}$, defined on $k(\mathscr{B})$, with respect to the configuration with torsion $\mathscr{K}$ is given by

$$(\nabla_{\mathscr{K}}\bar{\mathbf{F}})\tilde{\mathbf{u}} = (\nabla\bar{\mathbf{F}})(\mathbf{F}^p)^{-1}\tilde{\mathbf{u}}, \quad \forall, \tilde{\mathbf{u}}\in\mathbf{F}^{\mathbf{p}}\mathscr{V}.$$

Moreover, $\boldsymbol{\Lambda}_1 \times \boldsymbol{\Lambda}_2$ denotes the third-order tensor generated by the second order (covariant) tensors $\boldsymbol{\Lambda}_j$ j = 1, 2 defined by

$$((\boldsymbol{\Lambda}_1 \times \boldsymbol{\Lambda}_2)\mathbf{u})\mathbf{v} = \boldsymbol{\Lambda}_1\mathbf{u} \times \boldsymbol{\Lambda}_2\mathbf{v}, \quad \forall\, \mathbf{u}, \mathbf{v}$$

and $Skw\mathscr{N}$ is the third-order tensor associated with $\mathscr{N}$, defined by

$$((Skw\mathscr{N})\mathbf{u})\mathbf{v} = (\mathscr{N}\mathbf{u})\mathbf{v} - (\mathscr{N}\mathbf{v})\mathbf{u},\ \forall\, \mathbf{u}, \mathbf{v},\ \text{i.e.}((Skw\mathscr{N})\mathbf{u})\mathbf{v} = -((Skw\mathscr{N})\mathbf{v})\mathbf{u}.$$

Finally, the Frank vector is defined in terms of the disclination tensor (following the definitions introduced by Cleja-Ţigoiu [7]) by

$$\boldsymbol{\omega}_{\mathscr{K}} = \int_{C_0} \widetilde{\boldsymbol{\Lambda}}\mathbf{F}^p\, d\mathbf{X} = \int_{\mathscr{A}_0} \mathrm{curl}(\widetilde{\boldsymbol{\Lambda}}\mathbf{F}^p)\mathbf{N} dA.$$

In the model we consider here, *the disclination tensor* $\widetilde{\boldsymbol{\Lambda}}$ is introduced as independent measure of certain defects, and we consider the expression of the disclination density in terms of the disclination tensor in the anholonomic configuration, i.e.

$$\boldsymbol{\alpha}^{\Lambda}_{\mathscr{K}} = \frac{1}{\det\mathbf{F}^p}\mathrm{curl}(\widetilde{\boldsymbol{\Lambda}}\mathbf{F}^p)(\mathbf{F}^p)^T.$$

The Burgers vector associated with the circuit C_0 is defined (following Cleja-Ţigoiu [5]) by

$$\mathbf{b}_{\mathscr{K}} = \int_{C_0} \mathbf{F}^p\, d\mathbf{X} = \int_{\mathscr{A}_0} (\mathrm{curl}\mathbf{F}^p)\mathbf{N} dA,$$

see also Acharya [1], Bilby [2] and Fressengeas et al. [13].

The *dislocation density tensor* $\boldsymbol{\alpha}_{\mathscr{K}}$ is expressed by

$$\boldsymbol{\alpha}_{\mathscr{K}} := \frac{1}{\det\mathbf{F}^p}(\mathrm{curl}\ \mathbf{F}^p)(\mathbf{F}^p)^T,$$

in a configuration with torsion, is called *Noll's dislocation density*, and was introduced by Noll [17].

In what follows, the anholonomic basis vectors are related to *the crystal* and is defined by $\mathbf{e}_j = \mathbf{F}^p\mathbf{G}_j$, where $\{\mathbf{G}_j\}_{j=1,2,3}$ is a basis in the reference configuration and the Christoffel symbols are represented by $\overset{(p)}{\boldsymbol{\Gamma}}(\mathbf{G}_j, \mathbf{G}_k) = \overset{(p)}{\Gamma}{}^i_{jk}\, \mathbf{G}_i$.

2 Geometry and Kinematics of Elasto-Plastic Body

The behaviour of the elasto-plastic material with microstructural defects will be described based on three configurations: the initial, k, and the deformed configurations, χ which are global configurations, and so-called configuration with torsion associated with any material particle, X, of the body, formally denoted by $\mathscr{K}$.

We assume the multiplicative decomposition

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{1}$$

As a consequence of the multiplicative decomposition of the deformation gradient (1), we obtain

$$\begin{aligned} &\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1}, \quad \text{where} \\ &\mathbf{L} = \dot{\mathbf{F}}(\mathbf{F})^{-1}, \quad \mathbf{L}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}, \quad \mathbf{L}^e = \dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1}, \end{aligned} \tag{2}$$

L is the velocity gradient in the actual configuration.

The geometrical structure of the configuration $\mathscr{K}$ is characterized by the pair of plastic distortion, $\mathbf{F}^p$, and the so-called plastic connection $\overset{(p)}{\boldsymbol{\Gamma}}$, which is $\mathbf{C}^p$-metric connection. Following [5] the plastic connection with metric property is represented under the form

$$\begin{aligned} &\overset{(p)}{\boldsymbol{\Gamma}} = \overset{(p)}{\mathscr{A}} + (\mathbf{C}^p)^{-1}(\boldsymbol{\Lambda} \times \mathbf{I}), \quad \text{with} \\ &\mathbf{C}^p = (\mathbf{F}^p)^T\mathbf{F}^p, \quad \overset{(p)}{\mathscr{A}} = (\mathbf{F}^p)^{-1}\,\nabla\mathbf{F}^p, \end{aligned} \tag{3}$$

where the second order tensor $\boldsymbol{\Lambda}$ is called the *disclination* tensor and $\overset{(p)}{\mathscr{A}}$ is a Bilby type plastic connection, see [2].

As a direct consequence of the multiplicative decomposition of the deformation gradient (1), the material connection $\boldsymbol{\Gamma}$ is represented in terms of the Bilby type elastic and plastic connection by

$$\begin{aligned} &\boldsymbol{\Gamma} = (\mathbf{F}^p)^{-1}\,\overset{(e)}{\mathscr{A}}_{\mathscr{K}}\,[\mathbf{F}^p, \mathbf{F}^p] + \overset{(p)}{\mathscr{A}}, \quad \text{with} \\ &\overset{(e)}{\mathscr{A}}_{\mathscr{K}} = (\mathbf{F}^e)^{-1}\nabla_{\mathscr{K}}\,\mathbf{F}^e, \end{aligned} \tag{4}$$

where $\overset{(e)}{\mathscr{A}}_{\mathscr{K}}$ represent the Bilby type elastic connection with respect to the configuration with torsion.

We introduce the notation $\mathbf{S}^p$ for the third order Cartan torsion, associated to the plastic connection $\overset{(p)}{\boldsymbol{\Gamma}}$, which can be expressed as a consequence of (3) by

$$\mathbf{S}^p = Skw\overset{(p)}{\mathscr{A}} + Skw((\mathbf{C}^p)^{-1}(\boldsymbol{\Lambda} \times \mathbf{I})), \quad \text{where} \quad \overset{(p)}{\mathscr{A}} = (\mathbf{F}^p)^{-1} \nabla \mathbf{F}^p. \tag{5}$$

Let us introduce the notation

$$\mathscr{S}^p_{\mathscr{K}} = -\mathbf{F}^p\mathbf{S}^p[(\mathbf{F}^p)^{-1}, (\mathbf{F}^p)^{-1}], \tag{6}$$

and write the formula

$$\mathscr{S}^p_{\mathscr{K}} = Skw\overset{(p)}{\mathscr{A}}_{\mathscr{K}} + Skw(\widetilde{\boldsymbol{\Lambda}} \times \mathbf{I}), \quad \text{where} \quad \overset{(p)}{\mathscr{A}}_{\mathscr{K}} = \mathbf{F}^p \nabla_{\mathscr{K}}(\mathbf{F}^p)^{-1}, \tag{7}$$

which holds for $\boldsymbol{\Lambda}$ and $\widetilde{\boldsymbol{\Lambda}}$, related by

$$\widetilde{\boldsymbol{\Lambda}} = \frac{1}{\det\mathbf{F}^p}\, \mathbf{F}^p \boldsymbol{\Lambda}(\mathbf{F}^p)^{-1}, \quad \widetilde{\rho}\det\mathbf{F}^p = \rho_0. \tag{8}$$

Remark $Skw\overset{(p)}{\mathscr{A}}_{\mathscr{K}}$ can be viewed as a measure of dislocation, motivated by the relationships between the fields referred to the configurations $\mathscr{K}$ and k, which is expressed under the form

$$((Skw\ \overset{(p)}{\mathscr{A}}_{\mathscr{K}})\tilde{\mathbf{u}})\tilde{\mathbf{v}} = (\overset{(p)}{\mathscr{A}}_{\mathscr{K}}\ \tilde{\mathbf{u}})\tilde{\mathbf{v}} - (\overset{(p)}{\mathscr{A}}_{\mathscr{K}}\ \tilde{\mathbf{v}})\tilde{\mathbf{u}} = (curl\ \mathbf{F}^p)(\mathbf{u} \times \mathbf{v}), \tag{9}$$

$\forall\ \tilde{\mathbf{u}} = \mathbf{F}^p\mathbf{u}, \quad \tilde{\mathbf{v}} = \mathbf{F}^p\mathbf{v}.$

We put into evidence the rates of the above geometrical fields

$$\begin{aligned}
\frac{d}{dt}(\mathbf{S}^p_{\mathscr{K}}) &= Skw\left\{\frac{d}{dt}(\overset{(p)}{\mathscr{A}}_{\mathscr{K}})\right\} + Skw\left\{\left(\frac{d}{dt}(\widetilde{\boldsymbol{\Lambda}}) \times \mathbf{I}\right)\right\},\\
\frac{d}{dt}\big(\overset{(e)}{\mathscr{A}}_{\mathscr{K}}\big) &= (\mathbf{F}^p)^{-1}\nabla_{\mathscr{K}}\mathbf{L}^p[\mathbf{F}^p, \mathbf{F}^p] - \nabla_{\mathscr{K}}\mathbf{L}^p +\\
&\quad +\mathbf{L}^p\ \overset{(e)}{\mathscr{A}} - \overset{(e)}{\mathscr{A}}_{\mathscr{K}}\ \mathbf{L}^p - \overset{(e)}{\mathscr{A}}_{\mathscr{K}}\ [\mathbf{I}, \mathbf{L}^p],\\
\frac{d}{dt}\big(\overset{(p)}{\mathscr{A}}_{\mathscr{K}}\big) &= -\nabla_{\mathscr{K}}\mathbf{L}^p + \mathbf{L}^p\ \overset{(p)}{\mathscr{A}} - \overset{(p)}{\mathscr{A}}_{\mathscr{K}}\ \mathbf{L}^p - \overset{(p)}{\mathscr{A}}_{\mathscr{K}}\ [\mathbf{I}, \mathbf{L}^p].
\end{aligned} \tag{10}$$

The time-derivative of $\nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}}$ is expressed by

$$\frac{d}{dt}\big(\nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}}\big) = \nabla_{\mathscr{K}}\left(\frac{d}{dt}\widetilde{\boldsymbol{\Lambda}}\right) - (\nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}})\mathbf{L}^p. \tag{11}$$

3 Energetic Assumptions

3.1 Micro Balance Equations

Concerning the micro forces we assume that they generate internal power and satisfy their own balance equations. In order to put into evidence their appropriate balance equations, we start from the expression of the internal power, expended during the elasto-plastic process and expressed with respect to the configuration $\mathscr{K}$, at which level the presence of the defects can be emphasized.

Ax.internal power The internal power in the configuration with torsion is postulated to be given by the expression

$$(\mathscr{P}_{int})_{\mathscr{K}} = \frac{1}{\rho}\mathbf{T}\cdot\mathbf{L}^e + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}\cdot((\mathbf{F}^e)^{-1}(\nabla_\chi\mathbf{L})[\mathbf{F}^e,\mathbf{F}^e] - \nabla_{\mathscr{K}}\mathbf{L}^p) + \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p\cdot\mathbf{L}^p + \\ + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p\cdot\nabla_{\mathscr{K}}\mathbf{L}^p + \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^\lambda\cdot\frac{d}{dt}\widetilde{\boldsymbol{\Lambda}} + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^\lambda\cdot\nabla_{\mathscr{K}}\frac{d}{dt}\widetilde{\boldsymbol{\Lambda}}, \tag{12}$$

- $\boldsymbol{\Upsilon}^p$ and $\boldsymbol{\mu}^p$ denote the plastic micro stress and micro stress momentum, with respect to the configuration with torsion $\mathscr{K}$, which are power conjugated with $\mathbf{L}^p$ and its gradient $\nabla_{\mathscr{K}}\mathbf{L}^p$, respectively,
- $\boldsymbol{\Upsilon}^\lambda$ and $\boldsymbol{\mu}^\lambda$ represent the micro stress and micro stress momentum, which are related with the disclination mechanism and power conjugated with the rate $\frac{d}{dt}\widetilde{\boldsymbol{\Lambda}}$ and the gradient of the appropriate rate $\nabla_{\mathscr{K}}\frac{d}{dt}\widetilde{\boldsymbol{\Lambda}}$, respectively.

Micro balance equations will be derived from the formulated principle of virtual power relative to the disclination and plastic mechanisms, as independent. Here a basic role is played by the supposition, (12), concerning the virtual internal power postulated within the constitutive framework.

Ax.disclination virtual power We assume that the virtual internal power related to the disclination mechanism is equal to the external virtual power, produced by the virtual variation associated with the disclination

$$\int_{\mathscr{K}(\mathscr{B}^p,t)} \{\boldsymbol{\Upsilon}^\lambda\cdot\delta\widetilde{\boldsymbol{\Lambda}} + \boldsymbol{\mu}^\lambda\cdot\nabla_{\mathscr{K}}\delta\widetilde{\boldsymbol{\Lambda}}\}dV_{\mathscr{K}} = \\ = \int_{\partial\mathscr{K}(\mathscr{B}^p,t)} \mathbf{M}^\lambda(\mathbf{n}_{\mathscr{K}})\cdot\delta\widetilde{\boldsymbol{\Lambda}}\,dA_{\mathscr{K}} + \int_{\mathscr{K}(\mathscr{B}^p,t)} \tilde{\rho}\mathbf{B}^\lambda\cdot\delta\widetilde{\boldsymbol{\Lambda}}\,dV_{\mathscr{K}}, \tag{13}$$

holds for any virtual rate associated with the disclination, $\delta\widetilde{\boldsymbol{\Lambda}}$, which is an *incompatible second order field*. Here $\mathbf{n}_{\mathscr{K}}$ the unit vector of the normal. The integral is taken over arbitrary part of the plastically deformed domain $\mathscr{K}(\mathscr{B}^p, t)$.

We can apply "the common tetrahedron argument," and the linearity of the mapping $\mathbf{n} \in \mathscr{V} \longrightarrow \mathbf{M}^{\lambda}(\mathbf{n}) \in Lin$ follows, namely there exists $\widetilde{\boldsymbol{\mu}}$ such that $\widetilde{\boldsymbol{\mu}}\mathbf{n}_{\mathscr{K}} = \mathbf{M}^{\lambda}(\mathbf{n}_{\mathscr{K}})$. Using Green's formula first the equality $\widetilde{\boldsymbol{\mu}} = \boldsymbol{\mu}^{\lambda}$ yields and the following result can be proved:

Proposition 4.1 *The micro balance equation for micro forces associated with the disclination is written in the configuration with torsion $\mathscr{K}$, under the form*

$$\begin{aligned} \boldsymbol{\Upsilon}^{\lambda} &= div_{\mathscr{K}}\ \boldsymbol{\mu}^{\lambda} + \tilde{\rho}\mathbf{B}^{\lambda}, \quad or \\ \frac{1}{\widetilde{\widetilde{\rho}}}\boldsymbol{\Upsilon}^{\lambda} &= div\left(\frac{1}{\widetilde{\widetilde{\rho}}}\boldsymbol{\mu}^{\lambda}\ (\mathbf{F}^{p})^{-1}\right) + \mathbf{B}^{\lambda}. \end{aligned} \tag{14}$$

Here $\tilde{\rho}\mathbf{B}^{\lambda}$ is mass density of the couple body force, $\boldsymbol{\Upsilon}^{\lambda}$ is micro stress and $\boldsymbol{\mu}^{\lambda}$ is micro momentum associated with the disclinations.

Ax.(plastic virtual power) We assume that the virtual internal power related to the plastic mechanism is equal to the external virtual power, produced by the virtual variation of the rate of plastic distortion

$$\begin{aligned} &\int_{\mathscr{K}(\mathscr{B}^{p},t)} \{(\boldsymbol{\Upsilon}^{p} - \Sigma_{\mathscr{K}}) \cdot \widetilde{\mathbf{L}}^{p} + (\boldsymbol{\mu}^{p} - \boldsymbol{\mu}_{\mathscr{K}}) \cdot \nabla_{\mathscr{K}}\widetilde{\mathbf{L}}^{p}\} dV_{\mathscr{K}} = \\ &= \int_{\partial\mathscr{K}(\mathscr{B}^{p},t)} \mathbf{M}^{p}(\mathbf{n}_{\mathscr{K}}) \cdot \widetilde{\mathbf{L}}^{p}\ dA_{\mathscr{K}} + \int_{\mathscr{K}(\mathscr{B}^{p},t)} \tilde{\rho}\mathbf{B}^{p} \cdot \widetilde{\mathbf{L}}^{p}\ dV_{\mathscr{K}}, \\ &\text{where } \Sigma_{\mathscr{K}} = (\det \mathbf{F}^{e})(\mathbf{F}^{e})^{T}\mathbf{T}(\mathbf{F}^{e})^{-T}, \end{aligned} \tag{15}$$

holds for any virtual rate of plastic distortion, $\widetilde{\mathbf{L}}^{p}$, which is an *incompatible second order field*. The integral is taken over arbitrary part of the plastically deformed domain, generically denoted here by $\mathscr{K}(\mathscr{B}^{p}, t)$.

Remark In the formula (15) the Mandel type stress tensor associated with the Cauchy stress tensor has been introduced, as a direct consequence of the power expression.

By a similar argument the local form of the balance equation for micro forces associated with plastic mechanism can be proved.

Proposition 4.2 *The micro balance equation for micro forces associated with the plastic mechanism is written in the configuration with torsion $\mathscr{K}$, under the form*

$$\begin{aligned} \boldsymbol{\Upsilon}^{p} - \boldsymbol{\Sigma}_{\mathscr{K}} &= div_{\mathscr{K}}\ (\boldsymbol{\mu}^{p} - \boldsymbol{\mu}_{\mathscr{K}}) + \tilde{\rho}\mathbf{B}^{p}, \quad or \\ \frac{1}{\widetilde{\widetilde{\rho}}}\left(\boldsymbol{\Upsilon}^{p} - \boldsymbol{\Sigma}_{\mathscr{K}}\right) &= div\ \left(\frac{1}{\widetilde{\widetilde{\rho}}}(\boldsymbol{\mu}^{p} - \boldsymbol{\mu}_{\mathscr{K}})(\mathbf{F}^{p})^{-1}\right) + \mathbf{B}^{p}. \end{aligned} \tag{16}$$

Here $\tilde{\rho}\mathbf{B}^p$ is mass density of the couple body force, $\boldsymbol{\Upsilon}^p$ is micro stress and $\boldsymbol{\mu}^p$ is micro momentum associated with the plastic mechanism, while $\boldsymbol{\mu}_{\mathscr{K}}$ is the macro stress momentum. The appropriate boundary condition for micro stress momentum has to be associated.

Remark The virtual principles as they were formulated above, namely (13) and (15), have been applied for any virtual rates $\tilde{\mathbf{L}}^p$ and $\delta\boldsymbol{\Lambda}$, respectively, which are described by *incompatible* second order tensors. Consequently only one balance equation characterizes the peculiar *microstructural mechanism*, namely one balance equation for micro forces associated with disclination mechanism and another one associated with the plastic mechanism.

We now pass to the reference configuration and we derive the transformed balance equation in terms of the micro forces in reference configuration.

Remark We proved here the appropriate micro balance equations for disclination mechanism. For micro balance equations related with plastic behaviour we make references to Cleja-Ţigoiu [5], or to Cleja-Ţigoiu and Ţigoiu [8] in a paper concerning a strain gradient finite elasto-plastic model. The micro balance equation (13) contains only the micro forces, a similar point of view appears in Clayton et al. [4], apart from the micro balance equation (15) which contain the difference between macro and micro forces (like in the models developed by Gurtin [14], Cleja-Ţigoiu and Ţigoiu [8]).

An alternative formulation of the internal power in $\mathscr{K}$ has been postulated, see [7, 9], as the free energy has been postulated through an appropriate expression in the reference configuration, k. In the aforementioned papers the variation in time of the disclination with respect to the configuration with torsion and its gradient, respectively, were introduced in terms of the rate of disclination tensor with respect to the reference configuration, and its gradient, respectively, pushed away to the configuration with torsion.

3.2 Free Energy Density Function

We introduce now the expression of the free energy density postulated with respect to the configuration with torsion. We assume that

Ax.1: The *free energy density* is postulated to be dependent on the second order elastic deformation, in terms of $(\mathbf{C}^e, \overset{(e)}{\mathscr{A}}_{\mathscr{K}})$, and on the defects through $(\mathbf{S}^p_{\mathscr{K}}, \widetilde{\boldsymbol{\Lambda}}, \nabla_{\mathscr{K}}\, \widetilde{\boldsymbol{\Lambda}})$, as

$$\psi_{\mathscr{K}} = \psi(\mathbf{C}^e, \overset{(e)}{\mathscr{A}}_{\mathscr{K}}, \mathbf{S}^p_{\mathscr{K}}, \widetilde{\boldsymbol{\Lambda}}, \nabla_{\mathscr{K}}\, \widetilde{\boldsymbol{\Lambda}}), \quad \mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e. \tag{17}$$

The elements which enter the free energy density function in $\mathscr{K}$ have been represented in terms of the appropriate expressions.

Remark In Cleja-Ţigoiu [7] the free energy density was postulated to be dependent on the second order elastic deformation, in terms of $(\mathbf{C}^e, \overset{(e)}{\mathscr{A}}_{\mathscr{K}})$, and being dependent on the second order plastic deformation through $(\mathbf{S}^e_{\mathscr{K}}, \widetilde{\boldsymbol{\Lambda}})$,

$$\psi = \psi_{\mathscr{K}}(\mathbf{C}^e, \overset{(e)}{\mathscr{A}}_{\mathscr{K}}, \mathbf{S}^e_{\mathscr{K}}, \widetilde{\boldsymbol{\Lambda}}), \quad \mathbf{C}^e = (\mathbf{F}^e)^T \mathbf{F}^e, \tag{18}$$

while in [6] the free energy density has been postulated as

$$\psi = \psi_{\mathscr{K}}(\mathbf{C}^e, \overset{(e)}{\mathscr{A}}_{\mathscr{K}}, (\mathbf{F}^p)^{-1}, \overset{(p)}{\mathscr{A}}_{\mathscr{K}}, \widetilde{\boldsymbol{\Lambda}}, \nabla_{\mathscr{K}} \widetilde{\boldsymbol{\Lambda}}), \quad \mathbf{C}^e = (\mathbf{F}^e)^T \mathbf{F}^e, \tag{19}$$

where

$$\overset{(p)}{\mathscr{A}}_{\mathscr{K}} \equiv -\mathbf{F}^p \overset{(p)}{\mathscr{A}} [(\mathbf{F}^p)^{-1}, (\mathbf{F}^p)^{-1}]. \tag{20}$$

In [7] thermomechanic restrictions imposed on the elastic type constitutive functions show that the macro stress momentum is not vanishing, since $\frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}} = \partial_{\mathscr{A}^e}\psi + \partial_{\mathbf{S}^e}\psi$. We conclude that the macro stress momentum is involved in the constitutive models if, for instance, $\mathscr{A}^e$ is involved in the free energy density function. The evolution equation for plastic distortion, i.e. $\mathbf{L}^p$, and for the disclination tensor, i.e. $\dot{\widetilde{\boldsymbol{\Lambda}}}$ were defined to be compatible with the reduced dissipation inequality. As the micro momentum related to plastic mechanism is vanishing, $\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p = 0$, then the microforce $\boldsymbol{\Upsilon}^p$ can be identified with the Mandel stress measure, which is power conjugate with the rate of plastic strain $\mathbf{L}^p$, namely $\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p = \frac{1}{\tilde{\rho}}\boldsymbol{\Sigma}^p = \frac{1}{\rho}(\mathbf{F}^e)^T\{\mathbf{T}\}^S(\mathbf{F}^e)^{-T}$. The micro stress associated with the disclination mechanism remains undefined in the model performed in Cleja-Ţigoiu [7], $\widetilde{\boldsymbol{\Lambda}}$ can be viewed as internal variable. Consequently, it is possible to take $\boldsymbol{\Upsilon}^\lambda = \tilde{\rho}\partial_{\tilde{\boldsymbol{\Lambda}}}\psi$, see for instance the discussion concerning this issue in [16]. To avoid the above mentioned disadvantage the free energy density function has been reconsidered to be given by (17).

4 The Postulate of the Free Energy Imbalance

Within the constitutive framework developed herein, see [5, 6], the second law of thermodynamics is formulated as a postulate of the free energy imbalance on isothermal processes.

The postulate of the free energy imbalance expresses the restriction on the elasto-plastic material to be satisfied in the configuration with torsion, $\mathscr{K}$, under the form: the internal power has to be grater or equal to the rate of the free density energy

$$-\dot{\psi}_{\mathscr{K}} + (\mathscr{P}_{int})_{\mathscr{K}} \geq 0, \tag{21}$$

for an appropriate definition for the internal power $(\mathscr{P}_{int})_{\mathscr{K}}$ and for any virtual (isothermal) processes, when free energy density, $\psi_{\mathscr{K}}$, is given.

In order to investigate the consequences of the free energy imbalance, we proceed as follows

i. We emphasize a set of independent kinematic variables and their gradients, namely $\mathbf{L}$, $\mathbf{L}^p$, $\dot{\Lambda}$ and $\nabla_\chi \mathbf{L}$, $\nabla_{\mathscr{K}} \mathbf{L}^p$, $\nabla \dot{\boldsymbol{\Lambda}}$, respectively;
ii. Under the assumption that no evolution of the plastic distortion and of the disclination mechanism occurs, which means that $\mathbf{L}^p = 0$ and $\dot{\boldsymbol{\Lambda}} = 0$, the elastic type constitutive equations for macro forces are derived;
iii. The reduced dissipation inequality is derived;
iv. The possible consequences on the evolution for plastic distortion and disclination tensor and for their derivative are provided and analyzed to ensure their compatibility with the reduced dissipation inequality.

The time derivative of the free energy density function (17) is expressed through

$$\begin{aligned}\dot{\psi}_{\mathscr{K}} &= \partial_{\mathbf{C}^e}\psi \cdot \frac{d}{dt}(\mathbf{C}^e) + \partial_{\mathscr{A}^e_{\mathscr{K}}}\psi \cdot \frac{d}{dt}(\overset{(e)}{\mathscr{A}}_{\mathscr{K}}) + \partial_{\mathbf{S}^p_{\mathscr{K}}}\psi \cdot \left(Skw \frac{d}{dt} \overset{(p)}{\mathscr{A}}_{\mathscr{K}} + \right. \\ &\left. + Skw\left(\frac{d}{dt}\widetilde{\boldsymbol{\Lambda}}\right) \times \mathbf{I} \right) + \partial_{\widetilde{\boldsymbol{\Lambda}}}\psi \cdot \frac{d}{dt}(\widetilde{\boldsymbol{\Lambda}}) + \partial_{\nabla\widetilde{\boldsymbol{\Lambda}}}\psi \cdot \frac{d}{dt}(\nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}}).\end{aligned} \tag{22}$$

In (22) the rate of the appropriate fields have to be replaced by the formulae (10), and the formula

$$\frac{d}{dt}(\mathbf{C}^e) = 2(\mathbf{F}^e)^T \mathbf{D}^e \mathbf{F}^e, \quad \text{where} \quad \mathbf{D}^e = \{\mathbf{L}^e\}^S. \tag{23}$$

Proposition 4.3 *The free energy imbalance is satisfied for any virtual process, if the following inequality holds*

$$\begin{aligned}&\left\{\frac{1}{\rho}\{\mathbf{T}\}^S - 2\mathbf{F}\partial_{\bar{\mathbf{C}}^e}\psi\mathbf{F}^T\right\} \cdot \mathbf{D}^e + \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p \cdot \mathbf{L}^p + \left(\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p + \partial_{\mathbf{S}^p_{\mathscr{K}}}\psi\right) \cdot \nabla_{\mathscr{K}}\mathbf{L}^p + \\ &+ \left(\frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}} - \partial_{\mathscr{A}^e_{\mathscr{K}}}\psi\right) \cdot ((\mathbf{F}^e)^{-1}(\nabla_\chi \mathbf{L})[\mathbf{F}^e, \mathbf{F}^e] - \nabla_{\mathscr{K}}\mathbf{L}^p) + \\ &+ \left(\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^\lambda - \partial_{\widetilde{\boldsymbol{\Lambda}}}\psi\right) \cdot \dot{\widetilde{\boldsymbol{\Lambda}}} - \partial_{\mathbf{S}^p_{\mathscr{K}}}\psi \cdot Skw\{\dot{\widetilde{\boldsymbol{\Lambda}}} \times \mathbf{I}\} + \left(\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^\lambda - \partial_{\nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}}}\psi\right) \cdot \nabla_{\mathscr{K}}\dot{\widetilde{\boldsymbol{\Lambda}}} + \\ &+ \partial_{\nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}}}\psi \cdot (\nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}})\mathbf{L}^p - \partial_{\mathscr{A}^e_{\mathscr{K}}}\psi \cdot \left(\mathbf{L}^p \overset{(e)}{\mathscr{A}}_{\mathscr{K}} - \overset{(e)}{\mathscr{A}}_{\mathscr{K}} \mathbf{L}^p - \overset{(e)}{\mathscr{A}}_{\mathscr{K}} [\mathbf{I}, \mathbf{L}^p]\right) - \\ &- \partial_{\mathbf{S}^p_{\mathscr{K}}}\psi \cdot \left(\mathbf{L}^p \overset{(p)}{\mathscr{A}}_{\mathscr{K}} - \overset{(p)}{\mathscr{A}}_{\mathscr{K}} \mathbf{L}^p - \overset{(p)}{\mathscr{A}}_{\mathscr{K}} [\mathbf{I}, \mathbf{L}^p]\right) \geq 0.\end{aligned} \tag{24}$$

We introduce three types of second order tensors that can be associated with any pair of third order tensors, $\mathscr{A}$, $\mathscr{B}$, following the rules written below

$$
\begin{aligned}
&(\mathscr{A} \odot \mathscr{B}) \cdot \mathbf{L} = \mathscr{A}[\mathbf{I}, \mathbf{L}] \cdot \mathscr{B} = \mathscr{A}_{isk} L_{sn} \mathscr{B}_{ink} \\
&(\mathscr{A}\ _r \odot \mathscr{B}) \cdot \mathbf{L} = \mathscr{A} \cdot (\mathbf{L}\mathscr{B}) = \mathscr{A}_{ijk} L_{in} \mathscr{B}_{njk} \\
&(\mathscr{A} \odot_l \mathscr{B}) \cdot \mathbf{L} = \mathscr{A} \cdot (\mathscr{B}\mathbf{L}) = \mathscr{A}_{ijk} \mathscr{B}_{ijn} L_{kn}.
\end{aligned}
\tag{25}
$$

for all $\mathbf{L} \in Lin$, in order to put into evidence the linear dependence on $\mathbf{L}^p$ in the dissipation inequality (24).

We use the above representation and we get

$$
\partial_{\mathbf{S}^p_{\mathscr{K}}} \psi \cdot Skw(\dot{\tilde{\boldsymbol{\Lambda}}} \times \mathbf{I}) = -2(\in \odot_l \partial_{\mathbf{S}^p_{\mathscr{K}}} \psi) \cdot \dot{\tilde{\boldsymbol{\Lambda}}}. \tag{26}
$$

If we suppose that no evolution of the plastic distortion and of the disclination mechanism occurs, which means that $\mathbf{L}^p = 0$ and $\dot{\tilde{\boldsymbol{\Lambda}}} = 0$, then $\mathbf{L}^e = \mathbf{L}$, and we get from the free energy imbalance, (24), that the following inequality holds

$$
\left\{ \frac{1}{\rho} \{\mathbf{T}\}^S - 2\mathbf{F} \partial_{\mathbf{C}^e} \psi \mathbf{F}^T \right\} \cdot \ \mathbf{L} + \left(\frac{1}{\tilde{\rho}} \boldsymbol{\mu}_{\mathscr{K}} - \partial_{\mathscr{A}^e_{\mathscr{K}}} \psi \right) \cdot (\mathbf{F}^e)^{-1} (\nabla_\chi \mathbf{L})[\mathbf{F}^e, \mathbf{F}^e] \geq 0 \tag{27}
$$

for any virtual process, i.e. $\forall\ \mathbf{L}$, $\nabla_\chi \mathbf{L}$.

Theorem 4.1 *1. The thermomechanic restrictions imposed on the elastic type constitutive functions are*

$$
\begin{aligned}
&\frac{1}{\rho} \{\mathbf{T}\}^S = 2\mathbf{F} \partial_{\bar{\mathbf{C}}} \psi \mathbf{F}^T \\
&\frac{1}{\tilde{\rho}} \boldsymbol{\mu}_{\mathscr{K}} = \partial_{\mathscr{A}^e_{\mathscr{K}}} \psi.
\end{aligned}
\tag{28}
$$

2. The dissipative inequality (24) is reduced to the following inequality

$$
\begin{aligned}
&\frac{1}{\tilde{\rho}} \boldsymbol{\Upsilon}^p \cdot \mathbf{L}^p + \left(\frac{1}{\tilde{\rho}} \boldsymbol{\mu}^p + \partial_{\mathbf{S}^p_{\mathscr{K}}} \psi \right) \cdot \nabla_{\mathscr{K}} \mathbf{L}^p + \\
&+ \left(\frac{1}{\tilde{\rho}} \boldsymbol{\Upsilon}^\lambda - \partial_{\tilde{\boldsymbol{\Lambda}}} \psi + 2(\in \odot_l \partial_{\mathbf{S}^p_{\mathscr{K}}} \psi) \right) \cdot \dot{\tilde{\boldsymbol{\Lambda}}} + \left(\frac{1}{\tilde{\rho}} \boldsymbol{\mu}^\lambda - \partial_{\nabla_{\mathscr{K}} \tilde{\boldsymbol{\Lambda}}} \psi \right) \cdot \nabla_{\mathscr{K}} \dot{\tilde{\boldsymbol{\Lambda}}} + \\
&+ \left(\nabla_{\mathscr{K}} \tilde{\boldsymbol{\Lambda}} \odot_l \partial_{\nabla_{\mathscr{K}} \tilde{\boldsymbol{\Lambda}}} \psi \right) \cdot \mathbf{L}^p - \frac{1}{\tilde{\rho}} \boldsymbol{\mu}_{\mathscr{K}} \cdot \left(\mathbf{L}^p \overset{(e)}{\mathscr{A}}_{\mathscr{K}} - \overset{(e)}{\mathscr{A}}_{\mathscr{K}} \mathbf{L}^p - \overset{(e)}{\mathscr{A}}_{\mathscr{K}} [\mathbf{I}, \mathbf{L}^p] \right) - \\
&- \partial_{\mathbf{S}^p_{\mathscr{K}}} \psi \cdot \left(\mathbf{L}^p \overset{(p)}{\mathscr{A}}_{\mathscr{K}} - \overset{(p)}{\mathscr{A}}_{\mathscr{K}} \mathbf{L}^p - \overset{(p)}{\mathscr{A}}_{\mathscr{K}} [\mathbf{I}, \mathbf{L}^p] \right) \geq 0.
\end{aligned}
\tag{29}
$$

5 Viscoplastic Type Evolution Equations

Based on the dissipation inequality written in (29), we formulate the *constitutive hypotheses* in plastically deformed configuration:

Ax.5 The plastic micro stress momentum and micro stress momentum related with the disclination mechanism are represented through certain energetic relationships

$$\begin{aligned}\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p &= -\partial_{\mathbf{S}^p_{\mathscr{K}}}\psi \\ \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^\lambda &= \partial_{\nabla_{\mathscr{K}}\tilde{\boldsymbol{\Lambda}}}\psi.\end{aligned} \tag{30}$$

Using the definition of the operators introduced by (25) the linear terms in $\mathbf{L}^p$ can be grouped together as follows

$$\begin{aligned}&\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p\cdot\mathbf{L}^p+\left(-\frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}\ {}_r\odot\overset{(e)}{\mathscr{A}}_{\mathscr{K}}+\overset{(e)}{\mathscr{A}}_{\mathscr{K}}\odot_l\frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}+\frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}\odot\overset{(e)}{\mathscr{A}}_{\mathscr{K}}\right)\cdot\mathbf{L}^p+\\ &+\left(\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p\ {}_r\odot\overset{(p)}{\mathscr{A}}_{\mathscr{K}}-\overset{(p)}{\mathscr{A}}_{\mathscr{K}}\odot_l\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p-\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p\odot\overset{(p)}{\mathscr{A}}_{\mathscr{K}}\right)\cdot\mathbf{L}^p+\\ &+\left(\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^\lambda-\partial_{\tilde{\boldsymbol{\Lambda}}}\psi+2(\in\odot_l\partial_{\mathbf{S}^p_{\mathscr{K}}}\psi)\right)\cdot\dot{\tilde{\boldsymbol{\Lambda}}}+\left(\nabla_{\mathscr{K}}\tilde{\boldsymbol{\Lambda}}\odot_l\partial_{\nabla_{\mathscr{K}}\tilde{\boldsymbol{\Lambda}}}\psi\right)\cdot\mathbf{L}^p\geq 0.\end{aligned} \tag{31}$$

We introduce now the evolution equation for the plastic distortion, $\mathbf{F}^p$, and for the disclination tensor, $\tilde{\boldsymbol{\Lambda}}$, in the configuration with torsion.
Ax.6 The rate of plastic distorsion in the configuration with torsion is characterized in terms of micro and macro forces by

$$\begin{aligned}\xi_1\mathbf{L}^p &= \frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p-\frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}\ {}_r\odot\overset{(e)}{\mathscr{A}}_{\mathscr{K}}+\overset{(e)}{\mathscr{A}}_{\mathscr{K}}\odot_l\frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}+\frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}\odot\overset{(e)}{\mathscr{A}}_{\mathscr{K}}+\\ &+\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p\ {}_r\odot\overset{(p)}{\mathscr{A}}_{\mathscr{K}}-\overset{(p)}{\mathscr{A}}_{\mathscr{K}}\odot_l\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p-\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p\odot\overset{(p)}{\mathscr{A}}_{\mathscr{K}}+\nabla_{\mathscr{K}}\tilde{\boldsymbol{\Lambda}}\odot_l\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^\lambda\end{aligned} \tag{32}$$

Ax.7 The variation in time of the disclination tensor, $\tilde{\boldsymbol{\Lambda}}$, is characterized by the micro forces as

$$\xi_2\dot{\tilde{\boldsymbol{\Lambda}}}=\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^\lambda-\partial_{\tilde{\boldsymbol{\Lambda}}}\psi-2\left(\in\odot_l\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p\right) \tag{33}$$

in terms of the micro forces.

Remark If the viscous parameters ξ_1 and ξ_2 are scalar positive functions, then the reduced dissipation inequality (31) is satisfied

$$\xi_2 \dot{\widetilde{\boldsymbol{\Lambda}}} \cdot \dot{\widetilde{\boldsymbol{\Lambda}}} + \xi_1 \mathbf{L}^p \cdot \mathbf{L}^p \geq 0. \tag{34}$$

6 Concluding Remarks

We end our work with the following concluding remarks.

1. The rate of plastic distortion described by (32) is strongly dependent on the macro stress momentum $\boldsymbol{\mu}_{\mathscr{K}}$ and the micro force related to the plastic mechanism, $(\frac{1}{\tilde{\rho}}\boldsymbol{\Upsilon}^p, \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p)$, as well as on the micro stress momentum related to the disclination mechanism $\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^\lambda$.

2. A coupling term in the evolution equation shows that the disclination mechanism is influenced by the plastic mechanism.

3. By eliminating the micro stresses from the evolution Eqs. (32) and (33), via the balance equation for micro forces (14) and (16) in the absence of the mass densities of the couple body forces, the differential type evolution equations for $\mathbf{F}^p$ and $\widetilde{\boldsymbol{\Lambda}}$ become

$$\begin{aligned}\xi_1 \mathbf{L}^p = {} & \boldsymbol{\Sigma}_{\mathscr{K}} + \operatorname{div}\left(\frac{1}{\tilde{\rho}}(\boldsymbol{\mu}^p - \boldsymbol{\mu}_{\mathscr{K}})(\mathbf{F}^p)^{-1}\right) + \\ & -\frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}}\ {}_r\!\odot \overset{(e)}{\mathscr{A}}_{\mathscr{K}} + \overset{(e)}{\mathscr{A}}_{\mathscr{K}} \odot_l \frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}} + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}_{\mathscr{K}} \odot \overset{(e)}{\mathscr{A}}_{\mathscr{K}} + \\ & + \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p\ {}_r\!\odot \overset{(p)}{\mathscr{A}}_{\mathscr{K}} - \overset{(p)}{\mathscr{A}}_{\mathscr{K}} \odot_l \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p - \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p \odot \overset{(p)}{\mathscr{A}}_{\mathscr{K}} + \nabla_{\mathscr{K}}\widetilde{\boldsymbol{\Lambda}} \odot_l \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^\lambda\end{aligned} \tag{35}$$

and

$$\xi_2 \dot{\widetilde{\boldsymbol{\Lambda}}} = \operatorname{div}\left(\frac{1}{\tilde{\rho}}\boldsymbol{\mu}^\lambda\ (\mathbf{F}^p)^{-1}\right) - \partial_{\widetilde{\boldsymbol{\Lambda}}}\psi - 2\left(\in \odot_l \frac{1}{\tilde{\rho}}\boldsymbol{\mu}^p\right) \tag{36}$$

3. The evolution equation for the disclination tensor $\widetilde{\boldsymbol{\Lambda}}$ is characterized by the micro stress momentum only, i.e. no direct influence of the macro forces has been emphasized in the proposed model.

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