A Variational-Hemivariational Inequality in Contact Mechanics

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Abstract This chapter deals with a new mathematical model for the frictional contact between an elastic body and a rigid foundation covered by a deformable layer made of soft material. We study the model in the form of a variational-hemivariational inequality for the displacement field. We review a unique solvability result of the problem under certain assumptions on the data. Then we turn to the numerical solution of the problem, based on the finite element method. We derive an optimal order error estimate for the linear finite element solution. Finally, we present numerical simulation results in the study of a two-dimentional academic example. The theoretically predicted optimal convergence order is observed numerically. Moreover, we provide mechanical interpretations of the numerical results for our contact model.

1 Introduction

Phenomena of contact involving deformable bodies abound in industry and daily life. Due to their inherent complexity, they lead to mathematical models expressed in terms of nonlinear boundary value problems which, in variational formulation, give rise to challenging inequality problems. Analysis of these problems is based on arguments of nonlinear functional analysis through the theory of variational and hemivariational inequalities.

The theory of variational inequalities started in early sixties and has gone through substantial development since then, see for instance [1, 5, 6, 14] and the references therein. It was built on arguments of monotonicity and convexity, including properties

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F. dell'Isola et al. (eds.), *Mathematical Modelling in Solid Mechanics*,
Advanced Structured Materials 69, DOI 10.1007/978-981-10-3764-1_16

of the subdifferential of a convex function. In contrast, the theory of hemivariational inequalities is based on properties of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions which may be nonconvex. Analysis of hemivariational inequalities, including existence and uniqueness results, can be found in [12, 17, 20, 23]. Both variational and hemivariational inequalities have been extensively used in the study of various problems in Mechanics, Physics and Engineering Sciences and, in particular, in Contact Mechanics. References on this matter include [4, 7, 8, 13, 15, 17, 22–24, 26], among others. Variational-hemivariational inequalities are inequality problems where both convex and nonconvex functions are involved. They have been introduced in the pioneering work [21] and were further studied in [20, 23].

Recently, a new variational-hemivariational inequality is studied in [9]. The inequality involves two nonlinear operators and two nondifferentiable functionals, of which at least one is convex. There, solution existence, uniqueness and data continuous dependence are shown. Moreover, the finite element method is studied for solving the inequality problem. For the first time in the literature, an optimal order error estimate is derived for the linear element solution of a hemivariational inequality under appropriate solution regularity assumptions. A more general variational-hemivariational inequality is analyzed in [19]. Solution existence and uniqueness are proved, together with a result on the continuous dependence of the solution on the data. This study was continuated in [10, 11] where numerical analysis of variational-hemivariational inequalities was performed.

The purpose of this chapter is to illustrate the use of variational-hemivariational inequalities in the analysis and numerical approximations of an elastic contact problem. We use an abstract result to prove the unique solvability of the problem. For the finite element method of the problem, we derive error estimates, which are of optimal order for the linear elements. We provide numerical simulation results to illustrate the performance of the numerical method, including numerical convergence order.

The rest of the chapter is organized as follows. In Sect. 2 we introduce the contact problem in which the material's behavior is modeled with a nonlinear elastic constitutive law and the contact conditions are in a subdifferential form and are associated with unilateral constraints. In Sect. 3, we list the assumptions on the data and state a unique solvability result on the problem. The proof of the unique solvability statement is based on a recent abstract result obtained in [19]. In Sect. 4, we provide numerical analysis of the contact model, including convergence and error estimation results. Finally, in Sect. 5, we report numerical simulation results which provide numerical evidence of our optimal order error estimate and give rise to interesting mechanical interpretations.

2 The Contact Model

Let Ω be the reference configuration of the elastic body, assumed to be an open, bounded and connected set in $\mathbb{R}^d$ $(d = 2, 3)$. The boundary $\Gamma = \partial\Omega$ is assumed Lipschitz continuous and is partitioned into three disjoint and measurable parts Γ_1,

Γ_2 and Γ_3 such that meas $(\Gamma_1) > 0$. The body is in equilibrium under the action of a body force of density $\mathbf{f}_0$ in Ω and a surface traction of density $\mathbf{f}_2$ on Γ_2, is fixed on Γ_1, and is in frictional contact on Γ_3 with a foundation. We use $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$. Also, "$\cdot$" and "$\|\cdot\|$" will represent the canonical inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$. We denote by $\mathbf{u}\colon \Omega \to \mathbb{R}^d$ and $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$ the displacement field and the stress field, respectively. In addition, we use $\boldsymbol{\varepsilon}(\mathbf{u})$ to denote the linearized strain tensor. Let ν be the unit outward normal vector, defined a.e. on Γ. For a vector field $\mathbf{v}$, we use $v_\nu := \mathbf{v}\cdot\nu$ and $\mathbf{v}_\tau := \mathbf{v} - v_\nu\nu$ for the normal and tangential components of $\mathbf{v}$ on Γ. Similarly, for the stress field $\boldsymbol{\sigma}$, its normal and tangential components on the boundary are defined as $\sigma_\nu := (\boldsymbol{\sigma}\nu)\cdot\nu$ and $\boldsymbol{\sigma}_\tau := \boldsymbol{\sigma}\nu - \sigma_\nu\nu$, respectively.

With the above notation, the contact model to be studied is the following.

Problem P. *Find a displacement field* $\mathbf{u}\colon \Omega \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$ *and an interface force* $\xi_\nu\colon \Gamma_3 \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathscr{F}(\boldsymbol{\varepsilon}(\mathbf{u})) \qquad \text{in } \Omega, \tag{1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0} \qquad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_1, \tag{3}$$

$$\boldsymbol{\sigma}\nu = \mathbf{f}_2 \qquad \text{on } \Gamma_2, \tag{4}$$

$$u_\nu \le g,\ \sigma_\nu + \xi_\nu \le 0,\ (u_\nu - g)(\sigma_\nu + \xi_\nu) = 0,\ \xi_\nu \in \partial j_\nu(u_\nu) \qquad \text{on } \Gamma_3, \tag{5}$$

$$\|\boldsymbol{\sigma}_\tau\| \le F_b(u_\nu), \quad -\boldsymbol{\sigma}_\tau = F_b(u_\nu)\frac{\mathbf{u}_\tau}{\|\mathbf{u}_\tau\|} \ \text{ if } \ \mathbf{u}_\tau \ne \mathbf{0} \qquad \text{on } \Gamma_3. \tag{6}$$

In (1)–(6) and sometimes below, we do not indicate explicitly the dependence of various functions on the spatial variable $\mathbf{x} \in \Omega \cup \Gamma$. We now present a short description of the equations and conditions in Problem P and we refer the reader to the books [17, 26] for more details on the modelling of contact problems. First, Eq. (1) is the constitutive law for elastic materials in which $\mathscr{F}$ represents the elasticity operator, allowed to be nonlinear. Equation (2) is the equilibrium equation and is used here since the process is assumed to be static. Condition (3) represents the displacement condition and condition (4) is the traction condition. Relations (5) and (6) represent the contact condition and the friction law, respectively. Here $g \ge 0$, ∂j_ν denotes the Clarke subdifferential of the given function j_ν, and F_b denotes a positive function, the friction bound.

Note that condition (5) models the contact with a foundation made of a rigid body covered by a layer of soft material, say asperities. It is obtained through the following considerations:

(a) The penetration is restricted by the rigid body, i.e.

$$u_\nu \le g, \tag{7}$$

where $g \geq 0$ represents the thickness of the soft layer. We consider the non-homogeneous case, i.e., g is allowed to be a function of the spatial variable $\mathbf{x} \in \Gamma_3$.

(b) The normal stress has an additive decomposition of the form

$$\sigma_\nu = \sigma_\nu^D + \sigma_\nu^R, \tag{8}$$

where the term σ_ν^D describes the reaction of the soft layer and σ_ν^R describes the reaction of the rigid body.

(c) The component σ_ν^D satisfies a multivalued normal compliance condition of the form

$$-\sigma_\nu^D \in \partial j_\nu(u_\nu). \tag{9}$$

Examples of contact conditions of the form (9) can be found in [17], for instance.

(d) The component σ_ν^R satisfies the Signorini unilateral condition in a form with the gap g, i.e.

$$\sigma_\nu^R \leq 0, \qquad \sigma_\nu^R(u_\nu - g) = 0. \tag{10}$$

Comments and mechanical interpretation on the contact condition (10) can be found in [24] and the references therein.

Denote $-\sigma_\nu^D = \xi_\nu$. Then, it is easy to see that the contact condition (5) is a direct consequence of relations (7)–(9).

The friction law (6) was used in [25], associated with a multivalued normal compliance contact condition without unilateral constraint. Here the friction bound F_b may depend on the normal displacement u_ν, which is reasonable from the physical point of view, as explained in [25].

Note that, due to the strong nolinearities involved, in general Problem P does not have classical solution. Therefore, as usual in Contact Mechanics, its study is made by using a weak formulation, the so-called variational formulation. The formulation will allow one to prove the unique solvability of the problem and to construct numerical schemes for the approximation of the weak solution.

3 Variational Analysis

In the study of Problem P we use standard notation for Lebesgue and Sobolev spaces. For the stress and strain fields, we use the space $Q = L^2(\Omega; \mathbb{S}^d)$, which is a Hilbert space with the canonical inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q := \int_\Omega \sigma_{ij}(\mathbf{x})\, \tau_{ij}(\mathbf{x})\, dx, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in Q$$

and the associated norm $\|\cdot\|_Q$. The displacement fields will be sought in a subset of the space

$$V = \left\{\mathbf{v} = (v_i) \in H^1(\Omega;\mathbb{R}^d) \mid \mathbf{v} = \mathbf{0} \ \text{ on } \Gamma_1\right\}.$$

Since meas $(\Gamma_1) > 0$, it is known that V is a Hilbert space with the inner product

$$(\mathbf{u},\mathbf{v})_V := \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u})\cdot\boldsymbol{\varepsilon}(\mathbf{v})\,dx, \quad \mathbf{u},\mathbf{v}\in V$$

and the associated norm $\|\cdot\|_V$. We denote by V^* the topological dual of V, and by $\langle\cdot,\cdot\rangle_{V^*\times V}$ the duality pairing of V and V^*. When no confusion may arise, we simply write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_{V^*\times V}$. For $\mathbf{v}\in H^1(\Omega;\mathbb{R}^d)$ we use the same symbol $\mathbf{v}$ for the trace of $\mathbf{v}$ on Γ. By the Sobolev trace theorem we have

$$\|\mathbf{v}\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le \|\Gamma\|\,\|\mathbf{v}\|_V \quad \forall\,\mathbf{v}\in V, \tag{11}$$

$\|\gamma\|$ being the norm of the trace operator $\gamma\colon V\to L^2(\Gamma_3;\mathbb{R}^d)$.

We now turn to the assumptions on the data. First, the elasticity operator $\mathscr{F}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d$ and the potential function $j_\nu\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}$, are assumed to have the following properties:

$$\left\{\begin{array}{l}
\text{(a) there exists } L_{\mathscr{F}}>0 \text{ such that for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \text{ a.e. } \mathbf{x}\in\Omega,\\
\qquad \|\mathscr{F}(\mathbf{x},\boldsymbol{\varepsilon}_1)-\mathscr{F}(\mathbf{x},\boldsymbol{\varepsilon}_2)\| \le L_{\mathscr{F}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|;\\
\text{(b) there exists } m_{\mathscr{F}}>0 \text{ such that for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \text{ a.e. } \mathbf{x}\in\Omega,\\
\qquad (\mathscr{F}(\mathbf{x},\boldsymbol{\varepsilon}_1)-\mathscr{F}(\mathbf{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathscr{F}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2;\\
\text{(c) } \mathscr{F}(\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d;\\
\text{(d) } \mathscr{F}(\mathbf{x},\mathbf{0})=\mathbf{0} \text{ for a.e. } \mathbf{x}\in\Omega.
\end{array}\right. \tag{12}$$

$$\left\{\begin{array}{l}
\text{(a) } j_\nu(\cdot,r) \text{ is measurable on } \Gamma_3 \text{ for all } r\in\mathbb{R} \text{ and there}\\
\qquad \text{exists } \bar e\in L^2(\Gamma_3) \text{ such that } j_\nu(\cdot,\bar e(\cdot))\in L^1(\Gamma_3);\\
\text{(b) } j_\nu(\mathbf{x},\cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } \mathbf{x}\in\Gamma_3;\\
\text{(c) } |\partial j_\nu(\mathbf{x},r)| \le \bar c_0+\bar c_1|r| \text{ for a.e. } \mathbf{x}\in\Gamma_3,\\
\qquad \text{for all } r\in\mathbb{R} \text{ with } \bar c_0,\ \bar c_1\ge 0;\\
\text{(d) } j_\nu^0(\mathbf{x},r_1;r_2-r_1)+j_\nu^0(\mathbf{x},r_2;r_1-r_2)\le\alpha_{j_\nu}|r_1-r_2|^2\\
\qquad \text{for a.e. } \mathbf{x}\in\Gamma_3,\ \text{all } r_1,r_2\in\mathbb{R} \text{ with } \alpha_{j_\nu}\ge 0.
\end{array}\right. \tag{13}$$

On the penetration bound $g\colon \Gamma_3\to\mathbb{R}$ and the friction bound $F_b\colon \Gamma_3\times\mathbb{R}\to\mathbb{R}_+$, we assume

$$g\in L^2(\Gamma_3), \quad g(\mathbf{x})\ge 0 \ \text{ a.e. on } \Gamma_3, \tag{14}$$

$$\left\{\begin{array}{l}
\text{(a) there exists } L_{F_b}>0 \text{ such that}\\
\qquad |F_b(\mathbf{x},r_1)-F_b(\mathbf{x},r_2)|\le L_{F_b}|r_1-r_2| \quad \forall\,r_1,r_2\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Gamma_3;\\
\text{(b) } F_b(\cdot,r) \text{ is measurable on } \Gamma_3,\ \text{for all } r\in\mathbb{R};\\
\text{(c) } F_b(\mathbf{x},r)=0 \text{ for } r\le 0,\ F_b(\mathbf{x},r)\ge 0 \text{ for } r\ge 0,\ \text{a.e. } \mathbf{x}\in\Gamma_3.
\end{array}\right. \tag{15}$$

Finally, on the densities of the body force and the surface traction, we assume

$$\mathbf{f}_0 \in L^2(\Omega;\mathbb{R}^d), \quad \mathbf{f}_2 \in L^2(\Gamma_2;\mathbb{R}^d). \tag{16}$$

Define $\mathbf{f} \in V^*$ by

$$\langle \mathbf{f}, \mathbf{v}\rangle_{V^*\times V} = (\mathbf{f}_0, \mathbf{v})_{L^2(\Omega;\mathbb{R}^d)} + (\mathbf{f}_2, \mathbf{v})_{L^2(\Gamma_2;\mathbb{R}^d)} \quad \forall\, \mathbf{v} \in V. \tag{17}$$

Corresponding to the constraint $u_\nu \le g$ on Γ_3 in (5), we introduce the following subset of the space V:

$$U := \{\mathbf{v} \in V \mid v_\nu \le g \text{ on } \Gamma_3\}. \tag{18}$$

Also, we use the notation $j_\nu^0(u,v)$ for the generalized directional derivative of j_ν at $u \in \mathbb{R}$ in the direction $v \in \mathbb{R}$, defined by

$$j_\nu^0(u;v) := \limsup_{y\to u,\,\lambda\downarrow 0} \frac{j_\nu(y+\lambda v) - j_\nu(y)}{\lambda}.$$

Then, from the definition of Clarke subdifferential the following implication holds:

$$\xi_\nu \in \partial j_\nu(u_\nu) \ \text{ a.e. on } \Gamma_3 \quad \Longrightarrow \quad j_\nu^0(u_\nu; v_\nu) \ge \xi_\nu v_\nu \ \text{ a.e. on } \Gamma_3,\ \forall\, \mathbf{v} \in V. \tag{19}$$

By a standard approach, based on integration by parts and the inequality (19), the following weak formulation of the contact problem P can be derived.
PROBLEM P_V. *Find a displacement field* $\mathbf{u} \in U$ *such that*

$$(\mathscr{F}(\boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u}))_Q + \int_{\Gamma_3} F_b(u_\nu)\,(\|\mathbf{v}_\tau\| - \|\mathbf{u}_\tau\|)\,d\Gamma$$
$$+ \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\,d\Gamma \ge \langle \mathbf{f}, \mathbf{v}-\mathbf{u}\rangle_{V^*\times V} \quad \forall\, \mathbf{v} \in U. \tag{20}$$

Note that the inequality (20) has both a convex and nonconvex structure. Its convex structure is given by the subset of the admisible displacement fields U, which is convex, and the function

$$\mathbf{v} \mapsto \int_{\Gamma_3} F_b(u_\nu)\|\mathbf{v}_\tau\|\,d\Gamma,$$

which is a convex function on V. The nonconvex structure of the inequality (20) follows from the term

$$\int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu - u_\nu)\,d\Gamma$$

which involves a possibly nonconvex locally Lipschitz functions j_ν. We conclude from here that the inequality (20) represents a variational-hemivariational inequality.

The analysis of inequalities of the form (20) has been carried out in [11, 19], in an abstract functional framework. There, a general existence and uniqueness result for inequalities with pseudomonotone operators was provided, under a smallness assumption on the data. The use of this abstract result in the study of (20) is straighforward and, therefore, we skip it. The main point is the use of smallness assumption, that we describe in what follows.

Let $\lambda_{1,V} > 0$ be the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx = \lambda \int_{\Gamma_3} \mathbf{u} \cdot \mathbf{v}\, d\Gamma \quad \forall\, \mathbf{v} \in V,$$

and let $\lambda_{1\nu,V} > 0$ be the smallest eigenvalue of the eigenvalue problem

$$\mathbf{u} \in V, \quad \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx = \lambda \int_{\Gamma_3} u_\nu v_\nu d\Gamma \quad \forall\, \mathbf{v} \in V.$$

Assume also that

$$L_{F_b} \lambda_{1,V}^{-1} + \alpha_{j_\nu} \lambda_{1\nu,V}^{-1} < m_{\mathscr{F}}, \tag{21}$$

Then, using the abstract result in [11] it follows that, under the assumptions (12), (12)–(16) and (21), Problem P_V has a unique solution $\mathbf{u} \in U$.

Let $\mathbf{u} \in U$ be the solution of Problem P_V and denote by $\boldsymbol{\sigma} \in Q$ the function given by $\boldsymbol{\sigma} = \mathscr{F} \boldsymbol{\varepsilon}(\nu)$. The couple $(\mathbf{u}, \boldsymbol{\sigma})$ is called a weak solution to the contact problem P. We conclude from the above discussion that the latter has a unique weak solution.

4 Numerical Analysis

We now consider the finite element method of solving Problem P_V. For simplicity, assume Ω is a polygonal/polyhedral domain and express the three parts of the boundary, Γ_k, $1 \le k \le 3$, as unions of closed flat components with disjoint interiors:

$$\overline{\Gamma_k} = \cup_{i=1}^{i_k} \Gamma_{k,i}, \quad 1 \le k \le 3.$$

Let $\{\mathscr{T}^h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles/tetrahedrons that are compatible with the partition of the boundary $\partial\Omega$ into $\Gamma_{k,i}$, $1 \le i \le i_k$, $1 \le k \le 3$, in the sense that if the intersection of one side/face of an element with one set $\Gamma_{k,i}$ has a positive measure with respect to $\Gamma_{k,i}$, then the side/face lies entirely in $\Gamma_{k,i}$. Construct the linear element space corresponding to $\mathscr{T}^h$:

$$V^h = \left\{ \mathbf{v}^h \in C(\overline{\Omega})^d \mid \mathbf{v}^h|_T \in \mathbb{P}_1(T)^d,\ T \in \mathscr{T}^h,\ \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_1 \right\},$$

and the related finite element subset $U^h = V^h \cap U$. Assume g is a concave function. Then

$$U^h = \left\{\mathbf{v}^h \in V^h \mid v_\nu^h \le g \text{ at node points on } \Gamma_3\right\}.$$

Note that $\mathbf{0} \in U^h$. Define the following numerical method for Problem P_V.

PROBLEM P_V^h. *Find a displacement field* $\mathbf{u}^h \in U^h$ *such that*

$$(\mathscr{F}(\boldsymbol{\varepsilon}(\mathbf{u}^h)), \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{u}^h))_Q + \int_{\Gamma_3} F_b(u_\nu^h)\left(\|\mathbf{v}_\tau^h\| - \|\mathbf{u}_\tau^h\|\right) d\Gamma + \int_{\Gamma_3} j_\nu^0(u_\nu^h; v_\nu^h - u_\nu^h)\, d\Gamma \ge \langle \mathbf{f}, \mathbf{v}^h - \mathbf{u}^h\rangle_{V^*\times V} \quad \forall\, \mathbf{v}^h \in U^h. \tag{22}$$

For an error analysis, we assume

$$\mathbf{u} \in H^2(\Omega)^d, \quad \boldsymbol{\sigma}\nu \in L^2(\Gamma_3)^d. \tag{23}$$

Note that for many application problems, $\boldsymbol{\sigma}\nu \in L^2(\Gamma_3)^d$ follows from $\mathbf{u} \in H^2(\Omega)^d$; e.g., this is the case where the material is linearized elastic with suitably smooth coefficients, or where the elasticity operator $\mathscr{F}$ depends on $\mathbf{x}$ smoothly.

The starting point for obtaining error estimates is the inequality

$$\|\mathbf{u} - \mathbf{u}^h\|_V^2 \le c\left[\|\mathbf{u} - \mathbf{v}^h\|_V^2 + \|\mathbf{u} - \mathbf{v}^h\|_{L^2(\Gamma_3)^d} + R(\mathbf{v}^h)\right] \quad \forall\, \mathbf{v}^h \in U^h. \tag{24}$$

This inequality is based on the properties of the operators $\mathscr{F}$, the function F_b, the potential j_ν and the trace inequality (11). Its proof follows from an abstract error estimation result in the study of elliptic variational-hemivariational inequalities which can be found in [11]. In (24) and below, c represents a positive constant which does not depend on h and whose value may change from line to line and $R(\mathbf{v}^h)$ is a residual term defined by

$$R(\mathbf{v}^h) = (\mathscr{F}(\boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\mathbf{v}^h - \mathbf{u}))_Q + \int_{\Gamma_3} F_b(u_\nu)\left(\|\mathbf{v}_\tau^h\| - \|\mathbf{u}_\tau\|\right) d\Gamma + \int_{\Gamma_3} j_\nu^0(u_\nu; v_\nu^h - u_\nu)\, d\Gamma - \langle \mathbf{f}, \mathbf{v}^h - \mathbf{u}\rangle_{V^*\times V}.$$

We now derive a bound for this residual term and, to this end, we follow the procedure found in [7]. Take $\mathbf{v} = \mathbf{u} \pm \mathbf{w}$ with $\mathbf{w}$ in the subset $\tilde{U}$ of U defined by

$$\tilde{U} := \left\{\mathbf{w} \in C^\infty(\overline{\Omega})^d \mid \mathbf{w} = \mathbf{0} \text{ on } \Gamma_1 \cup \Gamma_3\right\},$$

and derive from (20) that

$$(\mathscr{F}(\boldsymbol{\varepsilon}(\mathbf{u})), \boldsymbol{\varepsilon}(\mathbf{w}))_Q = \langle \mathbf{f}, \mathbf{w}\rangle_{V^*\times V} \quad \forall\, \mathbf{w} \in \tilde{U}.$$

Therefore,

$$\mathrm{Div}\mathscr{F}(\boldsymbol{\varepsilon}(\mathbf{u})) + \mathbf{f}_0 = \mathbf{0} \quad \text{in } \Omega, \tag{25}$$

$$\boldsymbol{\sigma}\nu = \mathbf{f}_2 \quad \text{on } \Gamma_2. \tag{26}$$

Then multiply (25) by $\mathbf{v} - \mathbf{u}$ with $\mathbf{v} \in U$, integrate over Ω, and integrate by parts,

$$\int_{\partial\Omega} \boldsymbol{\sigma}\nu\cdot(\mathbf{v}-\mathbf{u})\,d\Gamma - \int_{\Omega} \mathscr{F}(\boldsymbol{\varepsilon}(\mathbf{u}))\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\,dx + \int_{\Omega} \mathbf{f}_0\cdot(\mathbf{v}-\mathbf{u})\,dx = 0,$$

i.e.,

$$\int_{\Omega} \mathscr{F}(\boldsymbol{\varepsilon}(\mathbf{u}))\cdot\boldsymbol{\varepsilon}(\mathbf{v}-\mathbf{u})\,dx = \langle \mathbf{f}, \mathbf{v}-\mathbf{u}\rangle_{V^*\times V} + \int_{\Gamma_3} \boldsymbol{\sigma}\nu\cdot(\mathbf{v}-\mathbf{u})\,d\Gamma. \tag{27}$$

Thus,

$$R(\mathbf{v}^h) = \int_{\Gamma_3} \left[\boldsymbol{\sigma}\nu\cdot(\mathbf{v}^h-\mathbf{u}) + F_b(u_\nu)\left(\|\mathbf{v}_\tau^h\| - \|\mathbf{u}_\tau\|\right) + j_\nu^0(u_\nu; v_\nu^h - u_\nu)\right] d\Gamma,$$

and then,

$$\left|R(\mathbf{v}^h)\right| \le c\,\|\mathbf{u}-\mathbf{v}^h\|_{L^2(\Gamma_3)^d}. \tag{28}$$

Finally, from (24), we derive the inequality

$$\|\mathbf{u}-\mathbf{u}^h\|_V^2 \le c\left(\|\mathbf{u}-\mathbf{v}^h\|_V^2 + \|\mathbf{u}-\mathbf{v}^h\|_{L^2(\Gamma_3)^d}\right) \quad \forall\,\mathbf{v}^h \in U^h. \tag{29}$$

Under additional solution regularity assumption

$$\mathbf{u}|_{\Gamma_{3,i}} \in H^2(\Gamma_{3,i};\mathbb{R}^d), \quad 1 \le i \le i_3, \tag{30}$$

we have the optimal order error bound

$$\|\mathbf{u}-\mathbf{u}^h\|_V \le c\,h. \tag{31}$$

We comment that similar results hold for the frictionless version of the model, i.e., where the friction condition (6) is replaced by

$$\boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_3.$$

Then the problem is to solve the inequality (20) without the term

$$\int_{\Gamma_3} F_b(u_\nu)\left(\|\mathbf{v}_\tau\| - \|\mathbf{u}_\tau\|\right) d\Gamma.$$

The condition (21) reduces to $\alpha_{j_\nu}\lambda_{1\nu,V}^{-1} < m_{\mathscr{F}}$. The inequality (29) and the error bound (31) still hold for the linear finite element solution.

5 Numerical Simulations

This section is devoted to some numerical simulation results in order to illustrate the solution of the frictional contact Problem P_V^h and to provide a numerical evidence of the theoretical error bound obtained in Sect. 4. We comment that the solution of Problem P_V^h is based on numerical methods presented in detail in [2, 3]. Numerous standard numerical methods for contact mechanics can be found for instance in [16, 27].

Numerical example. The physical setting of the numerical example related to Problem P_V^h is depicted in Fig. 1. There, the unit square body $\Omega = (0, 1) \times (0, 1) \subset \mathbb{R}^2$ is considered and

$$\Gamma_1 = [0, 1] \times \{1\},\ \ \Gamma_2 = (\{0\} \times (0, 1)) \cup (\{1\} \times (0, 1)),\ \ \Gamma_3 = [0, 1] \times \{0\}.$$

The domain Ω represents the cross section of a three-dimensional linearly elastic body subjected to the action of tractions in such a way that a plane stress hypothesis is assumed. On the part Γ_1 the body is clamped and, therefore, the displacement field vanishes there. Horizontal compressions act on the part $(\{0\} \times [0.5, 1)) \cup (\{1\} \times [0.5, 1))$ of the boundary Γ_2 and the part $(\{0\} \times (0, 0.5)) \cup (\{1\} \times (0, 0.5))$ is traction free. Constant vertical body forces are assumed to act on the elastic body. We consider that the deformable body is in frictional contact with an obstacle on the subset $\Gamma_3 = [0, 1] \times \{0\}$ of its boundary.

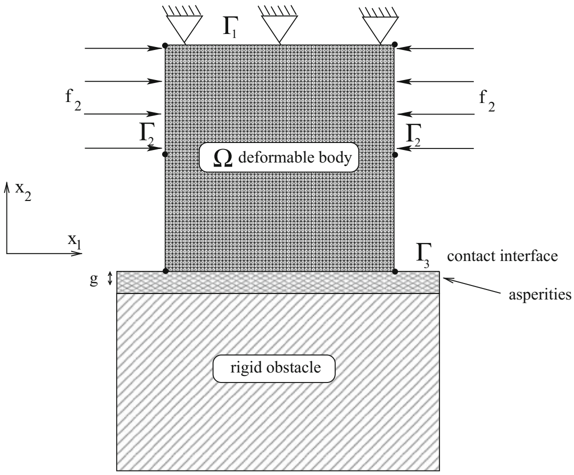

Fig. 1 Reference configuration of the two-dimensional example

Let $0 < r_\nu^1 < r_\nu^2$ be given, and let $p_\nu : \mathbb{R} \to \mathbb{R}$, $j_\nu : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$p_\nu(r) = \begin{cases} 0 \quad \text{if} \quad r \le 0, \\ c_\nu^1 r \quad \text{if} \quad r \in (0, r_\nu^1], \\ c_\nu^1 r_\nu^1 + c_\nu^2 (r - r_\nu^1) \quad \text{if} \quad r \in (r_\nu^1, r_\nu^2), \\ c_\nu^1 r_\nu^1 + c_\nu^2 (r_\nu^2 - r_\nu^1) + c_\nu^3 (r - r_\nu^2) \quad \text{if} \quad r \ge r_\nu^2, \end{cases} \tag{32}$$

$$j_\nu(r) = \int_0^r p_\nu(s)\, ds \quad \forall\, s \in \mathbb{R}. \tag{33}$$

In the numerical example, we consder the frictional contact conditions (5) and (6) in which the function j_ν is given by (32), (33) and

$$F_b(r) = \mu p_\nu(r) \quad \forall\, r \in \mathbb{R} \tag{34}$$

where $\mu \ge 0$ represents a given coefficient of friction. Note that the fuction p_ν is continuous but is not monotone and, therefore, j_ν is a locally Lipschitz nonconvex function. With this choice, the frictional contact condition we use on Γ_3 takes the following form:

$$u_\nu \le g, \quad \sigma_\nu + \xi_\nu \le 0, \quad (u_\nu - g)(\sigma_\nu + \xi_\nu) = 0,$$
$$\xi_\nu = \begin{cases} 0 \quad \text{if} \quad u_\nu \le 0, \\ c_\nu^1 u_\nu \quad \text{if} \quad u_\nu \in (0, r_\nu^1], \\ c_\nu^1 r_\nu^1 + c_\nu^2 (u_\nu - r_\nu^1) \quad \text{if} \quad u_\nu \in (r_\nu^1, r_\nu^2), \\ c_\nu^1 r_\nu^1 + c_\nu^2 (r_\nu^2 - r_\nu^1) + c_\nu^3 (u_\nu - r_\nu^2) \quad \text{if} \quad u_\nu \ge r_\nu^2, \end{cases}$$
$$\|\boldsymbol{\sigma}_\tau\| \le \mu \xi_\nu, \qquad -\boldsymbol{\sigma}_\tau = \mu \xi_\nu \frac{\mathbf{u}_\tau}{\|\mathbf{u}_\tau\|} \quad \text{if} \quad \mathbf{u}_\tau \ne \mathbf{0}.$$

The compressible material response, considered here, is governed by a linear elastic constitutive law defined by the elasticity tensor $\mathscr{F}$ given by

$$(\mathscr{F}\boldsymbol{\tau})_{\alpha\beta} = \frac{E\kappa}{1-\kappa^2}(\tau_{11} + \tau_{22})\delta_{\alpha\beta} + \frac{E}{1+\kappa}\tau_{\alpha\beta}, \quad 1 \le \alpha, \beta \le 2, \ \forall \boldsymbol{\tau} \in \mathbb{S}^2,$$

where E and κ are Young's modulus and Poisson's ratio of the material and $\delta_{\alpha\beta}$ denotes the Kronecker symbol.

For the numerical simulations, the following data are used:

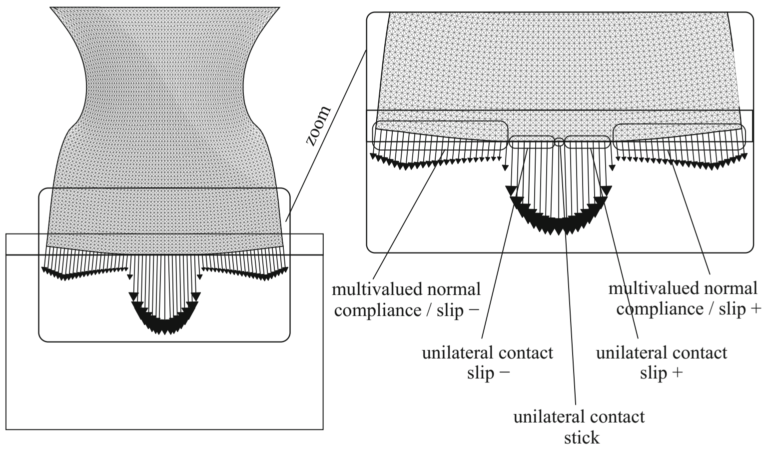

Fig. 2 Deformed meshes and interface forces on Γ_3 corresponding to the Problem P_V^h

$$
\begin{aligned}
&E = 2000 N/m^2, \quad \kappa = 0.4,\\
&\mathbf{f}_0 = (0, -0.5 \times 10^{-3}) N/m^2,\\
&\mathbf{f}_2 = \begin{cases} \;(8 \times 10^{-3}, 0)\, N/m & \text{on } \{0\} \times [0.5, 1),\\ (-8 \times 10^{-3}, 0)\, N/m & \text{on } \{1\} \times [0.5, 1), \end{cases}\\
&c_\nu^1 = 100, \quad c_\nu^2 = -100, \quad c_\nu^3 = 400, \quad r_\nu^1 = 0.1\, m, \quad r_\nu^2 = 0.15\, m,\\
&g = 0.15\, m \quad \mu = 0.2.
\end{aligned}
$$

In Fig. 2, we plotted the deformed mesh and the interface forces on Γ_3. We observe that the contact nodes on the extremities of the boundary Γ_3 are in multivalued normal compliance with either backward slip (slip-) or forward slip (slip+); there, the normal displacement u_ν does not reach the penetration bound, that is $u_\nu < g$. All the remaining nodes of Γ_3 are in unilateral contact; there, the penetration bound is reached, that is $u_\nu = g$. Most of these nodes are in the slip status, except the node in the center of the boundary Γ_3 which is in stick status.

Numerical convergence orders. The aim of this part is to illustrate the convergence of the discrete solutions and to provide numerical evidence of the optimal error estimate obtained in Sect. 4. To this end, we computed a sequence of numerical solutions by using uniform discretization of the Problem P_V^h according to the spatial discretization parameter h. For instance, for $h = 1/64$, we obtained the deformed configurations and the interface forces plotted in Fig. 2.

The numerical errors $\|\mathbf{u} - \mathbf{u}^h\|_E$ are computed by using the energy norm $\|\cdot\|_E$ for several discretization parameters of h. The energy norm $\|\cdot\|_E$ is equivalent to the canonical norm $\|\cdot\|_V$. Since it is not possible to calculate the exact solution $\mathbf{u}$ in an analytical way, we consider a "reference" solution $\mathbf{u}_{\text{ref}}$ corresponding to a fine discretization of Ω, instead of the exact solution. Here, each line segment component of the boundary Γ of Ω is divided into $1/h$ equal parts. We start with $h = 1/4$ which is successively halved. The numerical solution $\mathbf{u}_{\text{ref}}$ corresponding to

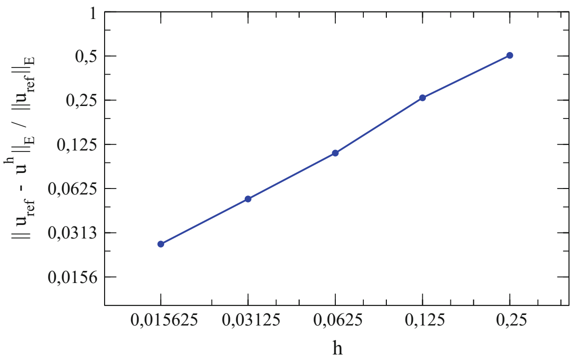

Fig. 3 Relative numerical errors in the energy norm for Problem P_V^h

$h = 1/256$ was taken as the "reference" solution. This fine discretization corresponds to a problem with 132612 degrees of freedom and 131329 finite elements. The numerical results are presented in Fig. 3 where the dependence of the relative error $\|\mathbf{u}_{\text{ref}} - \mathbf{u}^h\|_E / \|\mathbf{u}_{\text{ref}}\|_E$ with respect to h is plotted for the Problem P_V^h. Note that these results provide a numerical evidence of the theoretically predicted optimal order estimate obtained in Sect. 4 and highlight the linear asymptotic convergence of the numerical solutions.

Acknowledgements The work of W.H. was partially supported by NSF under grant DMS-1521684.

References

1. Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities: Applications to Free-Boundary Problems. John Wiley, Chichester (1984)
2. Barboteu, M., Bartosz, K., Kalita, P.: An analytical and numerical approach to a bilateral contact problem with nonmonotone friction. Int. J. Appl. Math. Comput. Sci. **23**, 263–276 (2013)
3. Barboteu, M., Bartosz, K., Kalita, p, Ramadan, A.: Analysis of a contact problem with normal compliance, finite penetration and nonmonotone slip dependent friction. Commun. Contemp. Math. **15**, 1350016 (2013). doi:10.1142/S0219199713500168
4. Eck, C., Jarušek, J., Krbec, M.: Unilateral Contact Problems: Variational Methods and Existence Theorems, Pure and Applied Mathematics 270. Chapman/CRC Press, New York (2005)
5. Glowinski, R.: Numerical Methods for Nonlinear Variational Problems. Springer-Verlag, New York (1984)
6. Glowinski, R., Lions, J.-L., Trémolières, R.: Numerical Analysis of Variational Inequalities. North-Holland, Amsterdam (1981)
7. Han, W., Sofonea, M.: Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity, Studies in Advanced Mathematics, vol. 30. American Mathematical Society, Providence, RI-International Press, Somerville, MA (2002)
8. Han, W., Reddy, B.D.: Plasticity: Mathematical Theory and Numerical Analysis, 2nd edn. Springer-Verlag, New York (2013)
9. Han, W., Migórski, S., Sofonea, M.: A class of variational-hemivariational inequalities with applications to frictional contact problems. SIAM J. Math. Anal. **46**, 3891–3912 (2014)
10. Han, W., Sofonea, M., Barboteu, M.: Numerical Analysis of Elliptic Hemivariational Inequalities, to appear in SIAM J. Numer. Anal

11. Han, W., Sofonea, M., Danan, D.: Numerical Analysis of Stationary Variational-Hemivariational Inequalities (submitted)
12. Haslinger, J., Miettinen, M., Panagiotopoulos, P.D.: Finite Element Method for Hemivariational Inequalities. Theory, Methods and Applications. Kluwer Academic Publishers, Boston, Dordrecht, London (1999)
13. Hlaváček, I., Haslinger, J., Necǎs, J., Lovíšek, J.: Solution of Variational Inequalities in Mechanics. Springer-Verlag, New York (1988)
14. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and their Applications, Classics in Applied Mathematics, vol. 31. SIAM, Philadelphia (2000)
15. Kikuchi, N., Oden, J.T.: Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods. SIAM, Philadelphia (1988)
16. Laursen, T.: Computational Contact and Impact Mechanics. Springer, Berlin (2002)
17. Migórski, S., Ochal, A., Sofonea,: Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems. Advances in Mechanics and Mathematics, vol. 26. Springer, New York (2013)
18. Migórski, S., Ochal, A., Sofonea, M.: History-dependent variational-hemivariational inequalities in contact mechanics. Nonlinear Anal. Real World Appl. **22**, 604–618 (2015)
19. Migórski, S., Ochal, A., Sofonea, M.: A Class of Variational-Hemivariational Inequalities in Reflexive Banach Spaces, to appear in Jounral of Elasticity
20. Naniewicz, Z., Panagiotopoulos, P.D.: Mathematical Theory of Hemivariational Inequalities and Applications. Marcel Dekker Inc., New York, Basel, Hong Kong (1995)
21. Panagiotopoulos, P.D.: Nonconvex problems of semipermeable media and related topics. ZAMM Z. Angew. Math. Mech. **65**, 29–36 (1985)
22. Panagiotopoulos, P.D.: Inequality Problems in Mechanics and Applications. Birkhäuser, Boston (1985)
23. Panagiotopoulos, P.D.: Hemivariational Inequalities, Applications in Mechanics and Engineering. Springer-Verlag, Berlin (1993)
24. Shillor, M., Sofonea, M., Telega, J.J.: Models and Analysis of Quasistatic Contact. Lecture Notes in Physics, vol. 655. Springer, Berlin, Heidelberg (2004)
25. Sofonea, M., Han, W., Migórski, S.: Numerical analysis of history-dependent variational inequalities with applications to contact problems. Eur. J. Appl. Math. **26**, 427–452 (2015)
26. Sofonea, M., Matei, A.: Mathematical Models in Contact Mechanics, London Mathematical Society Lecture Note Series, vol. 398. Cambridge University Press, Cambridge (2012)
27. Wriggers, P.: Computational Contact Mechanics. Wiley, Chichester (2002)