Chapter 13
A Short Survey on Dislocated Metric Spaces via Fixed-Point Theory

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Abstract In this survey, we collect and combine basic notions and results for the fixed points of certain operators in the frame of dislocated metric (respectively, b-metric) spaces. By preparing a fundamental source, we shall aim to show that there are some rooms for researchers in this interesting and applicable research direction.

13.1 Introduction and Preliminaries

The notion of distance is as old as the history of humanity and it was axiomatically formulated by Fréchet [21] at the beginning of nineteen century. Indeed, after realizing the Euclidean distance between two points given by the absolute difference, Fréchet formulated and generalized the distance concept in an abstract form. It is an indispensable fact that the formulation of the metric notion opens a new age to mathematical analysis and hence the related sciences. The notion metric has been generalized, extended, and improved in different directions by a number of authors, due to the fundamental roles of it in analytic sciences and their applications. As a consequence of this trend, the notions fuzzy metric, symmetric, quasi-metric, partial metric, G-metric, D-metric, b-metric, 2-metric, ultra-metric, dislocated metric, modular metric, Hausdorff metric, and so on have been appeared in the literature. It is quite clear that the survey of this trend cannot be collected in a chapter. For this reason, we restrict ourselves on the merging of two interesting notions dislocated metric and b-metric. On the other hand, the letter $\mathbb{N}$ represents the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The real numbers will be denoted by $\mathbb{R}$ and $\mathbb{R}_0^+ = [0, \infty)$.

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Definition 13.1 [22] For a nonempty set M, a metric is a function $m : M \times M \to \mathbb{R}_0^+$ such that

(M_0) $m(x, y) \geq 0$ (nonnegativity),
(M_1) $x = y \Rightarrow m(x, y) = 0$ (self-distance),
(M_2) $m(x, y) = 0 \Rightarrow x = y$ (indistancy),
(M_3) $m(x, y) = m(y, x)$ (symmetry), and
(M_4) $m(x, y) \leq m(x, z) + m(z, y)$ (triangularity),

for all $x, y, z \in M$. Here, the ordered pair (M, m) is called a metric space.

One of the interesting extensions of metric space is given by Matthews [35, 36] who introduce the notion of a partial metric. Roughly speaking, apart from the notion of metric, the self-distance not necessarily be zero in partial metric.

Definition 13.2 [35, 36] For a nonempty set M, a partial metric is a function $p : M \times M \to \mathbb{R}_0^+$ such that

(PM_0) $p(x, y) \geq 0$ (nonnegativity),
(PM_1) $p(x, x) \leq p(x, y)$ (pseudo-self-distance),
(PM_2) $p(x, y) = p(x, x) = p(y, y) \Rightarrow x = y$ (pseudo-indistancy),
(PM_3) $p(x, y) = p(y, x)$ (symmetry), and
(PM_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ (pseudo-triangularity),

for all $x, y, z \in M$ and the pair (M, p) is called a partial metric space.

As an immediate example, we can consider the maximum of two numbers on the nonnegative real numbers with a maximum operator, that is, $M = \mathbb{R}_0^+$ and $d(x, y) = \max\{x, y\}$.

Although partial metric seems unnatural, it has an unexpectedly wide application potential in computer science, in particular domain theory. Mainly, the motivation of partial metric space comes from the question in the context of computer science: "How we can terminate the computer program in an '*economic way*'?". By dispose of the necessity of being self-distance zero, Matthews [35, 36] successfully get some results in this direction.

The notion of dislocated metric is defined by Hitzler [22]. It is another generalization of metric that is originated from the needs of computer science. The concept of dislocated metric is rediscovered by Amini-Harandi [5] as a "metric-like." Due to the historical development process, we prefer to use dislocated metric instead of metric-like.

Definition 13.3 [22] For a nonempty set M, a dislocated metric is a function $\rho : M \times M \to \mathbb{R}_0^+$ such that for all $x, y, z \in M$:

(ρ_0) $\rho(x, y) \geq 0$ (nonnegativity),
(ρ_1) $\rho(x, y) = 0 \Rightarrow x = y$. (pseudo-indistancy),
(ρ_2) $\rho(x, y) = \rho(y, x)$ (symmetry), and
(ρ_3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, (triangularity).

Moreover, the pair (M, ρ) is said to be dislocated metric space (DbMS).

It is clear that each partial metric forms a dislocated metric. But the converse is not true. Notice also that every metric necessarily forms a partial metric and hence a dislocated metric. For more details for dislocated metric space, we refer e.g., [1, 2, 20, 22, 23, 26, 51, 53, 55–57].

Example 13.1 Let $M = \{p, q\}$ where $p, q \in \mathbb{R}$ and $\{a, b\} \subset (0, \infty)$ with $2a < b$. Define $\rho(x, y) = b$ if $x = y = p$, and $\rho(x, y) = a$ otherwise. Then, the ordered pair (M, ρ) forms a dislocated metric space, but it is not a partial metric space since $\rho(p, p) \not\leq \rho(p, q)$

Another generalization of metric was introduced by Czerwik [9, 10] (for earlier considerations see e.g., Bourbaki [15], Bakhtin [8]).

Definition 13.4 ([8, 10]) Let M be a set and let $s \geq 1$ be a given real number. A function $b : M \times M \to \mathbb{R}_0^+$ is said to be a b-metric if the following conditions are satisfied:

(bM_o) $b(x, y) \geq 0$ (nonnegativity),
(bM_1) $x = y \Rightarrow b(x, y) = 0$ (self-distance),
(bM_2) $b(x, y) = 0 \Rightarrow x = y$ (indistancy),
(bM_3) $b(x, y) = b(y, x)$, (symmetry), and
(bM_4) $b(x, z) \leq s[b(x, y) + b(y, z)]$, (weakened triangularity),

for all $x, y, z \in M$. Furthermore, the ordered pair (M, b) is called a b-metric space.

It is expected that each b-metric forms a metric. On the other hand, the converse is not case. The followings are the standard examples of b-metric spaces, for more details, see e.g., [6, 7, 16–18, 29, 33, 48].

Example 13.2 Let $M = L^p[0,1]$ be the collections of all real functions $x(t)$ such that $\int_0^1 |x(t)|^p dt < \infty$, where $t \in$[0,1] and $0 < p < 1$. For the function $b : M \times M \to \mathbb{R}_0^+$ defined by

$$b(x, y) := (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}, \text{ for each } x, y \in L^p[0, 1],$$

the ordered pair (M, b) forms a b-metric space with $s = 2^{1/p}$.

Example 13.3 Let $M = l^p(\mathbb{R})$ be the collection of all real sequences such that

$$l^p(\mathbb{R}) := \{(x_n) \subset \mathbb{R}| \sum_{n=1}^{\infty} |x_n|^p < \infty\},$$

where $0 < p < 1$. For the function $b : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$ defined by

$$b(x, y) := (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p},$$

is a b-metric space with coefficient $s = 2^{1/p} > 1$, where $x = (x_n), y = (y_n) \in l^p(\mathbb{R})$. Notice that the above result holds for the general case $l^p(E)$ with $0 < p < 1$, where E is a Banach space.

By merging the notions of b-metric and dislocated metric, we obtain a more general form that can be called dislocated b-metric space.

Definition 13.5 Let M be a set and let $s \geq 1$ be a given real number. A function $d : M \times M \to \mathbb{R}_0^+$ is said to be a dislocated b-metric if the following conditions are satisfied:

(DbM_0) $d(x, y) \geq 0$ (nonnegativity),
(DbM_1) $d(x, y) = 0 \Rightarrow x = y$ (indistancy),
(DbM_2) $d(x, y) = b(y, x)$, (symmetry), and
(DbM_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$ (weakened triangularity),

for all $x, y, z \in M$. Furthermore, the ordered pair (M, d) is called a dislocated b-metric space, in short, DbMS.

Example 13.4 Let $M = \mathbb{R}_0^+$ and $d(x, y) = |x - y|^2 + \max\{x, y\}$. It is clear that (M, d) forms a dislocated b-metric space with $s = 2$.

Example 13.5 Let $M = \{a, b, c\}$ and $d(x, y) = \begin{cases} 3 & \text{if } x = y = a, \\ 1 & \text{otherwise.} \end{cases}$

It is easy to see that (M, d) forms a dislocated b-metric space with $s = 2$.

Example 13.6 Let $M = \{p, q\}$ where $p, q \in \mathbb{R}$. Define $d(x, y) = 3$ if $x = y = p$, and $d(x, y) = 1$ otherwise. Then, the ordered pair (M, d) forms a dislocated b-metric space with $s = \frac{3}{2}$. It is clear that it is neither metric (fails in triangle inequality property) nor b-metric (fails in the self-distance property). Notice also that it is not a partial metric since $d(p, p) \nleq d(p, q)$.

Example 13.7 Let (M, ρ) be a dislocated metric space. Define a function $d : M \times M \to \mathbb{R}_0^+$ such that $d(x, y) = (\rho(x, y))^p$, where $p > 1$. Then, (M, d) forms a dislocated b-metric space with $s = 2^{p-1}$.

Remark 13.1 One can easily derive the "quasi" form of the notions above by omitting the property "symmetry" in definitions above. In this short survey, we skip this case to avoid to increase the number of pages so much. We should also mention that G-metric was proposed by Mustafa and Sims [38] to correct the notion of D-metric and to cover the inconsistency. Recently, it was realized that G-metric coincides with quasi-metric and almost all fixed-point results in G-metric can be derived from the existence results in the context of metric space in the literature. For more details, see e.g., [25, 50].

The topology of dislocated b-metric spaces as well as the basic topological properties (convergence, completeness, etc.) can be obtained by regarding the analogy of the standard metric space topology. Let us recall some essential notions together with the basic observations. Each dislocated b-metric d on a nonempty set M has a topology τ_d that was generated by the family of open balls

$$B_d(x,\varepsilon)=\{y\in M:|d(x,y)-d(x,x)|<\varepsilon\},\ \text{ for all } x\in M \text{ and } \varepsilon>0.$$

In the frame of the dislocated b-metric (M,d), a given sequence $\{x_n\}$ converges to a point $x\in M$ if the following limit exists (and finite):

$$\lim_{n\to\infty} d(x_n,x)=d(x,x).$$

As it is expected, a sequence $\{x_n\}$ is said to be Cauchy if the following limit

$$L=\lim_{n\to\infty} d(x_n,x_m), \tag{13.1}$$

exists and is finite. Additionally, if $L=0$ in (13.1), then we say that $\{x_n\}$ is a $0-$Cauchy sequence. Furthermore, a pair (M,d) is called complete DbMS if for each Cauchy sequence $\{x_n\}$, there is some $x\in M$ such that

$$M=\lim_{n\to\infty} d(x_n,x)=d(x,x)=\lim_{n\to\infty} d(x_n,x_m). \tag{13.2}$$

Moreover, a pair (M,d) is said to be $0-$complete DbMS if for each $0-$Cauchy sequence $\{x_n\}$, converges to a point $x\in M$ so that $M=0$ in (13.2). Remark that every 0-Cauchy sequence in (M,d) is a Cauchy sequence in (M,d), and that every complete dislocated b-metric space is 0-complete (see e.g., [44, 45]). On the other hand, the converse is not the case.

Let (M,d_1) and (K,d_2) be DbMSs. A mapping $T:M\to K$ is called continuous if

$$\lim_{n\to\infty} d_1(x_n,x)=d(x,x)=\lim_{n,m\to\infty} d_1(x_n,x_m),$$

then we have

$$\lim_{n\to\infty} d_2(Tx_n,Tx)=d_2(Tx,Tx)=\lim_{n,m\to\infty} d_2(Tx_n,Tx_m).$$

Definition 13.6 Let (M,d) be a DbMS and S be a subset of M. We say S is open subset of M, if for all $x\in M$ there exists $r>0$ such that $B_d(x,r)\subseteq S$. Also, $F\subseteq X$ is a closed subset of M if $(M\backslash F)$ is a open subset of M.

The proofs of the assertions in the following are straightforward, and hence we omit them.

Lemma 13.1 *For a DbMS (M,ρ), we have the following observations:*

(A) If $d(x,y)=0$ then $d(x,x)=d(y,y)=0$.
(B) For a sequence $\{x_n\}$ with $\lim_{n\to\infty} d(x_n,x_{n+1})=0$, we have

$$\lim_{n\to\infty} d(x_n, x_n) = \lim_{n\to\infty} d(x_{n+1}, x_{n+1}) = 0.$$

(C) *If* $x \neq y$ *then* $d(x, y) > 0$.
(D) *Let* V *be a closed subset of* M *and* $\{x_n\}$ *be a sequence in* V. *If* $x_n \to x$ *as* $n \to \infty$, *then* $x \in V$.

13.1.1 (c)-Comparison Functions

A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0, \infty)$. We denote by Φ, the class of the comparison function $\varphi : [0, \infty) \to [0, \infty)$. For more details and examples, see e.g., [12, 47]. Among them, we recall the following essential result.

Lemma 13.2 ([12, 47]) *If* $\varphi : [0, \infty) \to [0, \infty)$ *is a comparison function, then*

(1) *each iterate* φ^k *of* φ, $k \geq 1$, *is also a comparison function;*
(2) φ *is continuous at* 0*; and*
(3) $\varphi(t) < t$, *for any* $t > 0$.

Later, Berinde [12] introduced the concept of (c)-*comparison function* in the following way.

Definition 13.7 ([12]) A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a (c)-comparison function if

(c_1) φ is increasing,
(c_2) there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

The collection of all (c)-comparison functions will be denoted by Ψ.

13.1.2 (b)-Comparison Functions

Definition 13.8 [14] Let $s \geq 1$ be a real number. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a (b)-comparison function if the following conditions are fulfilled

(1) φ is monotone increasing;
(2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

We denote by Ψ_b for the class of (b)-comparison function $\varphi : [0, \infty) \to [0, \infty)$.

Lemma 13.3 [13] *Let $s \geq 1$ be a real number. If $\varphi : [0, \infty) \to [0, \infty)$ is a (b)-comparison function, then we have the following:*

(1) the series $\sum_{k=0}^{\infty} s^k \varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;

(2) the function $S_b : [0, \infty) \to [0, \infty)$ defined by $S_b(t) = \sum_{k=0}^{\infty} s^k \varphi^k(t),\ t \in [0, \infty)$, is increasing and continuous at 0.

Lemma 13.4 [39] *We note that any (b)-comparison function is a comparison function.*

13.1.3 Admissible Mappings

Samet et al. [49] proposed the following auxiliary function:

Definition 13.9 [49] Let M be a nonempty set and $\alpha : M \times M \to [0, \infty)$ be mapping. A self-mapping $T : M \to M$ is called an α-admissible if the following implication holds:

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1 \text{ for all } x, y \in M. \tag{13.3}$$

Definition 13.10 [28] An α-admissible T is said to be a triangular-α-admissible if

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, z) \Longrightarrow \alpha(x, z) \geq 1, \text{ for all } x, y, z \in M. \tag{13.4}$$

These notions are refined by Popescu [40] who introduce the concepts of α-orbital admissible mappings and triangular α-orbital admissible mappings:

Definition 13.11 [40] Let $T : M \to M$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is an α-orbital admissible if

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1.$$

Furthermore, T is called a triangular α-orbital admissible if T is α-orbital admissible and

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1.$$

It is clear that each α-admissible (respectively, triangular α-admissible) mapping is an α-orbital admissible (respectively, triangular α-orbital admissible) mapping. For more details and distinctive examples, see e.g., [3, 27, 30, 34, 40, 48].

13.1.4 Simulation Functions

Definition 13.12 (See [31]) A function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is said to be *simulation* if it satisfies the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{13.5}$$

The family of all simulation functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ will be denoted by $\mathscr{Z}$. On account of (ζ_2), we observe that

$$\zeta(t, t) < 0 \;\text{ for all }\; t > 0,\; \zeta \in \mathscr{Z}. \tag{13.6}$$

Example 13.8 (See e.g., [4, 31, 37, 43]) Let $\zeta_i : [0, \infty) \times [0, \infty) \to \mathbb{R}$, $i \in \{1, 2, 3\}$, be mappings defined by

(i) $\zeta_1(t, s) = \psi(s) - \phi(t)$ for all $t, s \in [0, \infty)$, where $\phi, \psi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if, and only if, $t = 0$, and $\psi(t) < t \leq \phi(t)$ for all $t > 0$.
(ii) $\zeta_2(t, s) = s - \dfrac{f(t, s)}{g(t, s)} t$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty) \to (0, \infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.
(iii) $\zeta_3(t, s) = s - \varphi(s) - t$ for all $t, s \in [0, \infty)$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if, and only if, $t = 0$.
(iv) If $\varphi : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$, and we define

$$\zeta_T(t, s) = s\varphi(s) - t \qquad \text{for all } s, t \in [0, \infty),$$

then ζ_T is a simulation function.
(v) If $\eta : [0, \infty) \to [0, \infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, and we define

$$\zeta_{BW}(t, s) = \eta(s) - t \qquad \text{for all } s, t \in [0, \infty),$$

then ζ_{BW} is a simulation function.

(vi) If $\phi : [0, \infty) \to [0, \infty)$ is a function such that $\int_0^\varepsilon \phi(u)du$ exists and $\int_0^\varepsilon \phi(u) du > \varepsilon$, for each $\varepsilon > 0$, and we define

$$\zeta_K(t, s) = s - \int_0^t \phi(u)du \qquad \text{for all } s, t \in [0, \infty),$$

then ζ_K is a simulation function.

13.2 Fixed Point of α-ψ Contractive Mapping on Dislocated b-Metric Spaces

One can find more interesting examples of simulation functions in [4, 31, 43].

Definition 13.13 (cf. [31]) Suppose (M, d) is an either dislocated b-metric space or dislocated b-metric space. Suppose also that T is a self-mapping on M and $\zeta \in \mathscr{Z}$. A mapping T is a *$\mathscr{Z}_b$-contraction with respect to* ζ if there exists $\psi \in \Psi_b$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\zeta(\alpha(x, y)d(Tx, Ty), \psi(d(x, y))) \geq 0 \qquad \text{for all } x, y \in M.$$

Since (ζ_2) holds, we have the following inequality:

$$x \neq y \implies d(Tx, Ty) \neq d(x, y).$$

Thus, we conclude that T cannot be an isometry whenever T is a $\mathscr{Z}$-contraction. In other words, if a $\mathscr{Z}$-contraction T in a metric space has a fixed point, then it is necessarily unique.

We can now state the main result of this paper.

Theorem 13.1 *Let (M, d) be a dislocated b-complete metric space and let $T : M \to M$ be an α-admissible $\mathscr{Z}_b$-*contraction *with respect to ζ. Suppose that*

(i) T is α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.

Then there exists $u \in M$ such that $Tu = u$.

Proof On account of (ii), we have $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$. Starting from this point $x_0 \in M$ we shall construct an iterative sequence $\{x_n\}$ in M by letting $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}_0$. Throughout the proof, we shall assume that $d(x_n, x_{n+1}) > 0$ and hence $x_n \neq x_{n+1}$ for all n. Indeed, if there exists an n_0 such that $d(x_{n_0}, x_{n_0+1}) = 0$, then by Lemma 13.1 (A), we find $u = x_{n_0} = x_{n_0+1} = Tx_{n_0} = Tu$. Hence, x_{n_0} becomes a fixed point of T that terminate the proof.

As a result, we have

$$d(x_n, x_{n+1}) > 0, \text{ for all } n \in \mathbb{N}_0. \tag{13.7}$$

Taking the fact that T is α-admissible into account, we obtain that

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Recursively, one can conclude that

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}_0. \tag{13.8}$$

Combining (13.17) and (13.8), we derive that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}), \psi(d(x_n, x_{n-1}))) \\ &= \zeta(\alpha(x_n, x_{n-1})d(x_{n+1}, x_n), \psi(d(x_n, x_{n-1}))) \\ &< \psi(d(x_n, x_{n-1})) - \alpha(x_n, x_{n-1})d(x_{n+1}, x_n), \end{aligned} \tag{13.9}$$

for all $n \geq 1$. Accordingly, we find that

$$d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1})d(x_n, x_{n+1}) < \psi(d(x_n, x_{n-1})) \text{ for all } n \in \mathbb{N}. \tag{13.10}$$

Inductively, we derive that

$$d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1)), \text{ for all } n \in \mathbb{N}_0. \tag{13.11}$$

The modified triangle inequality together with the inequality (13.11) yield, for all $p \geq 1$, that

$$\begin{aligned} d(x_n, x_{n+p}) &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{p-2} d(x_{n+p-3}, x_{n+p-2}) \\ &\quad + s^{p-1} d(x_{n+p-2}, x_{n+p-1}) + s^{p-1} d(x_{n+p-1}, x_{n+p}) \\ &\leq s\psi^n(d(x_0, x_1)) + s^2\psi^{n+1}(d(x_0, x_1)) + \cdots + s^{p-2}\psi^{n+p-3}(d(x_0, x_1)) \\ &\quad + s^{p-1}\psi^{n+p-2}(d(x_0, x_1)) + s^{p-1}\psi^{n+p-1}(d(x_0, x_1)) \\ &= \frac{1}{s^{n-1}}[s^n\psi^n(d(x_0, x_1)) + \cdots + s^{n+p-2}\psi^{n+p-2}(d(x_0, x_1)) \\ &\quad + s^{n+p-1}\psi^{n+p-1}(d(x_0, x_1))]. \end{aligned}$$

Denoting $L_n = \sum_{k=0}^{n} s^k \psi^k(d(x_0, x_1))$, $n \geq 1$ we obtain

$$d(x_n, x_{n+p}) \leq \frac{1}{s^{n-1}}[L_{n+p-1} - L_{n-1}], \; n \geq 1, \; p \geq 1. \tag{13.12}$$

On the account of (13.7) together with Lemma 13.3, we deduce that the series $\sum_{k=0}^{n} s^k \psi^k(d(x_0, x_1))$ is convergent. So, there exists $L = \lim_{n\to\infty} L_n \in [0, \infty)$. Taking $s \geq 1$ into account, the estimation (13.12) yields that the sequence $\{x_n\}_{n\geq 0}$ is 0-Cauchy in dislocated b-metric space (M, d).

$$\lim_{n\to\infty} d(x_n, x_m) = 0. \tag{13.13}$$

Since (M, d) is complete, there exists $x^* \in M$ such that $x_n \to x^*$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} d(x_n, x^*) = 0 = \lim_{n\to\infty} d(x_n, x_m). \tag{13.14}$$

From the continuity of f, it follows that $x_{n+1} = T(x_n) \to T(x^*)$ as $n \to \infty$:

$$\lim_{n\to\infty} d(x_{n+1}, Tx^*) = \lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n,m\to\infty} d(Tx_n, Tx_m) = \lim_{n\to\infty} d(x_{n+1}, x_{m+1}) = 0.$$

By the uniqueness of the limit, we get $x^* = T(x^*)$, that is, x^* is a fixed point of T. □

Theorem 13.2 *Let (M, d) be a complete dislocated b-metric space and let $T : M \to M$ be an α-admissible $\mathscr{Z}$*-contraction *with respect to ζ. Suppose that*

(i) T is triangular α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then there exists $u \in M$ such that $Tu = u$.

Proof Following the lines in the proof of Theorem 13.1, we find that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in M$. From (13.8) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k. Applying (13.17), for all k, we get that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n(k)}, u)d(Tx_{n(k)}, Tu), \psi(d(x_{n(k)}, u))) \\ &= \zeta(\alpha(x_{n(k)}, u)d(x_{n(k)+1}, Tu), \psi(d(x_{n(k)}, u))) \\ &< \psi(d(x_{n(k)}, u)) - \alpha(x_{n(k)}, u)d(x_{n(k)+1}, Tu), \end{aligned} \tag{13.15}$$

which is equivalent to

$$d(x_{n(k)+1}, Tu) = d(Tx_{n(k)}, Tu) \leq \alpha(x_{n(k)}, u)d(Tx_{n(k)}, Tu) < \psi(d(x_{n(k)}, u)). \tag{13.16}$$

By keeping Lemmas 13.2 and 13.4 in the mind, we derive that by letting $k \to \infty$ in the above equality. Hence, we get that $u = Tu$. □

For the uniqueness of a fixed point of a α-admissible $\mathscr{Z}$*-contraction* with respect to ζ, we shall suggest the following hypothesis.

(U_1) For all $x, y \in \text{Fix}(T)$, we have $\alpha(x, y) \geq 1$.

Here, $\text{Fix}(T)$ denotes the set of fixed points of T.

Theorem 13.3 *Adding condition (U_1) to the hypotheses of Theorem 13.1* (resp. Theorem 13.2)*, we obtain that u is the unique fixed point of T.*

We skip the proof of Theorem 13.3 which is a direct consequence of the property (ζ_2).

13.3 Consequences

In this section, we give a short list of consequences of the main results in the previous section. By regarding the condition (ζ_2) and combining Theorems 13.1 and 13.2, we find the first corollary.

Corollary 13.1 (See [18]) *Let (M, d) be a dislocated b-complete metric space and let $T : M \to M$ satisfy*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \text{ for all } x, y \in M,$$

where $\psi \in \Psi_b$. Suppose that

(i) T is α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) either, T is continuous, or,
(iii)' if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then there exists $u \in M$ such that $Tu = u$.

On account of the condition (ζ_2) and taking $\alpha(x, y) = 1$ in Theorem 13.3, we get the following result:

Corollary 13.2 *Let (M, d) be a dislocated b-complete metric space and let $T : M \to M$ satisfy*

$$d(Tx, Ty) \leq \psi(d(x, y)) \text{ for all } x, y \in M,$$

where $\psi \in \Psi_b$. Then there exists unique $u \in M$ such that $Tu = u$.

In particular, by taking $\psi(t) = kt$, $k \in [0, 1)$ in the corollary above, we derive the following consequence:

Corollary 13.3 *Let* (M, d) *be a dislocated b-complete metric space and let* $T : M \to M$ *satisfy*

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in M,$$

where $k \in [0, 1)$. *Then there exists unique* $u \in M$ *such that* $Tu = u$.

13.3.1 Fixed-Point Theorems on Dislocated b-Metric Spaces Endowed with a Partial Order

The research topic "existence of fixed point on metric spaces endowed with partial orders" was initiated by Turinici [54] and continued by Ran and Reurings in [42] with many others.

Definition 13.14 Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that T is nondecreasing with respect to $\preceq$ if

$$x, y \in X, \ x \preceq y \Longrightarrow Tx \preceq Ty.$$

Definition 13.15 Let $(X, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all n.

Definition 13.16 Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preceq, d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

We have the following result.

Corollary 13.4 *Let* $(X, \preceq)$ *be a partially ordered set and d be a dislocated b-metric on X such that* (X, d) *is complete. Let* $T : X \to X$ *be a nondecreasing mapping with respect to* $\preceq$. *Suppose that there exists a function* $\psi \in \Psi_b$ *such that*

$$d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$ *with* $x \succeq y$. *Suppose also that the following conditions hold:*

(i) there exists $x_0 \in X$ *such that* $x_0 \preceq Tx_0$*;*
(ii) T *is continuous or* $(X, \preceq, d)$ *is regular.*

Then T *has a fixed point. Moreover, if for all* $x, y \in X$ *there exists* $z \in X$ *such that* $x \preceq z$ *and* $y \preceq z$*, we have uniqueness of the fixed point.*

Proof Let $\alpha : X \times X \to [0, \infty)$ be defined as

$$\alpha(x, y) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } x \succeq y, \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, T is a generalized $\alpha - \psi$ contractive mapping, that is,

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)),$$

for all $x, y \in X$. From condition (i), we have $\alpha(x_0, Tx_0) \geq 1$. Moreover, for all $x, y \in X$, from the monotone property of T, we have

$$\alpha(x, y) \geq 1 \Longrightarrow x \succeq y \text{ or } x \preceq y \Longrightarrow Tx \succeq Ty \text{ or } Tx \preceq Ty \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Thus T is α-admissible. Now, if T is continuous, the existence of a fixed point follows from Theorem 13.1. Suppose now that $(X, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. From the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. This implies from the definition of α that $\alpha(x_{n(k)}, x) \geq 1$ for all k. In this case, the existence of a fixed point follows from Theorem 13.2. The uniqueness follows from Theorem 13.3. □

13.3.2 Fixed-Point Theorems for Cyclic Contractive Mappings

An interesting concept, cyclic contraction, was introduced by Kirk, Srinivasan and Veeramani [32]. After then, this notion has been studied by several authors. In this subsection, we shall prove our setup and able to get several fixed-point theorems for cyclic contractive mappings.

Corollary 13.5 *Let $\{A_i\}_{i=1}^2$ be nonempty, closed subsets of a complete dislocated b-metric space (M, d) and $T : Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$. Suppose that the following conditions hold:*

(I) $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$;
(II) there exists a function $\psi \in \Psi_b$ such that

$$d(Tx, Ty) \leq \psi(d(x, y)), \textit{ for all } (x, y) \in A_1 \times A_2.$$

Then T has a unique fixed point that belongs to $A_1 \cap A_2$.

Proof Since A_1 and A_2 are closed subsets of the complete dislocated b-metric space (M, d), then (Y, d) is complete. Define the mapping $\alpha : Y \times Y \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } (x,y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 \text{ otherwise.} \end{cases}$$

From (II) and the definition of α, we can write

$$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y)),$$

for all $x, y \in Y$. Thus T is a generalized $\alpha - \psi$ contractive mapping.

Let $(x,y) \in Y \times Y$ such that $\alpha(x,y) \geq 1$. If $(x,y) \in A_1 \times A_2$, from (I), $(Tx,Ty) \in A_2 \times A_1$, which implies that $\alpha(Tx,Ty) \geq 1$. If $(x,y) \in A_2 \times A_1$, from (I), $(Tx,Ty) \in A_1 \times A_2$, which implies that $\alpha(Tx,Ty) \geq 1$. Thus in all cases, we have $\alpha(Tx,Ty) \geq 1$. This implies that T is α-admissible.

Also, from (I), for any $a \in A_1$, we have $(a, Ta) \in A_1 \times A_2$, which implies that $\alpha(a, Ta) \geq 1$.

Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. This implies from the definition of α that

$$(x_n, x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \text{ for all } n.$$

Since $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set with respect to the Euclidean metric, we get that

$$(x,x) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which implies that $x \in A_1 \cap A_2$. Thus we get immediately from the definition of α that $\alpha(x_n, x) \geq 1$ for all n.

Now, all the hypotheses of Corollary 13.1 are satisfied. Consequently, we conclude that T has a unique fixed point that belongs to $A_1 \cap A_2$ (from (I)). □

13.3.3 Consequences on Standard b-Metric Spaces

Corollary 13.6 *Let (M, b) be a b-complete metric space and let $T : M \to M$ be an α-admissible $\mathscr{Z}_b$-*contraction *with respect to ζ. Suppose that*

(i) T is α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) either, T is continuous, or
(iii)' if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then there exists $u \in M$ such that $Tu = u$.

Corollary 13.7 *Adding condition (U_1) to the hypotheses of Corollary 13.6, we obtain that u is the unique fixed point of T.*

By taking (ζ_2) into consideration in Corollary 13.7, we conclude that

Corollary 13.8 *Let (M, b) be a b-complete metric space and let $T : M \to M$ satisfy*

$$\alpha(x, y)b(Tx, Ty) \leq \psi(b(x, y)) \text{ for all } x, y \in M,$$

where $\psi \in \Psi_b$. Suppose that

- *(i) T is α-orbital admissible;*
- *(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- *(iii) either, T is continuous, or,*
- *(iii)' if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*
- *(iv) the condition (U_1) is fulfilled.*

Then there exists a unique $u \in M$ such that $Tu = u$.

By letting $\alpha(x, y) = 1$ for all $x, y \in M$ in Corollary 13.8, we get that

Corollary 13.9 *Let (M, b) be a b-complete metric space and let $T : M \to M$ satisfy*

$$b(Tx, Ty) \leq \psi(b(x, y)) \text{ for all } x, y \in M,$$

where $\psi \in \Psi_b$. Then there exists unique $u \in M$ such that $Tu = u$.

In particular, by taking $\psi(t) = kt$, $k \in [0, 1)$ in the corollary above, we have

Corollary 13.10 *Let (M, b) be a b-complete metric space and let $T : M \to M$ satisfy*

$$b(Tx, Ty) \leq kb(x, y) \text{ for all } x, y \in M,$$

where $k \in [0, 1)$. Then there exists unique $u \in M$ such that $Tu = u$.

It easy to get some more consequences by repeating the similar arguments in Sects. 3.1 and 3.2.

13.3.4 Consequences on Standard Metric Spaces

It is clear that all results in the previous section can be repeated in the context of standard metric by letting $s = 1$. Regarding the analogy, we skip the details. On the other hand, we should underline that the analog of Corollary 13.10 is nothing but well-known Banach Contraction Mapping principle [11]. Moreover, the techniques used in Sect. 3.1 imply the famous results of Ran and Reuring [42]. On the other hand, Sect. 3.2 yields the initial fixed-point results in cyclic mapping due to Kirk et al. [32].

13.3.5 Consequences on Standard Dislocated Metric Spaces

Corollary 13.11 *Let (M, ρ) be a complete dislocated metric space and let $T : M \to M$ be an α-admissible $\mathscr{Z}_b$-*contraction *with respect to ζ. Suppose that*

(i) T is α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) either, T is continuous, or
(iii)' if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then there exists $u \in M$ such that $Tu = u$.

Corollary 13.12 *Adding condition (U_1) to the hypotheses of Corollary 13.11, we obtain that u is the unique fixed point of T.*

By taking (ζ_2) into consideration in Corollary 13.12, we conclude that

Corollary 13.13 *Let (M, ρ) be a complete dislocated metric space and let $T : M \to M$ satisfy*

$$\alpha(x, y)\rho(Tx, Ty) \leq \psi(\rho(x, y)) \text{ for all } x, y \in M,$$

where $\psi \in \Psi_b$. Suppose that

(i) T is α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) either, T is continuous, or,
(iii)' if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.
(iv) the condition (U_1) is fulfilled.

Then there exists a unique $u \in M$ such that $Tu = u$.

By letting $\alpha(x, y) = 1$ for all $x, y \in M$ in Corollary 13.13, we get that

Corollary 13.14 *Let (M, ρ) be a complete dislocated metric space and let $T : M \to M$ satisfy*

$$\rho(Tx, Ty) \leq \psi(\rho(x, y) \text{ for all } x, y \in M,$$

where $\psi \in \Psi_b$. Then there exists unique $u \in M$ such that $Tu = u$.

In particular, by taking $\psi(t) = kt$, $k \in [0, 1)$ in the corollary above, we have

Corollary 13.15 *Let (M, ρ) be a complete dislocated metric space and let $T : M \to M$ satisfy*

$$\rho(Tx, Ty) \leq k\rho(x, y) \text{ for all } x, y \in M,$$

where $k \in [0, 1)$. Then there exists unique $u \in M$ such that $Tu = u$.

As it is expected, it is possible to list more consequences. For example, all theorems in this subsection can be re-built in the frames of "partially ordered metric spaces" or "cyclic contractions" as in Sects. 3.1 and 3.2.

Remark 13.2 Since each partial metric space is dislocated spaces, all results in Sect. 3.5 can be reformulated in the context of partial metric spaces. Notice also that the analog of Corollary 13.19 in the frame of partial metric space yields the fixed-point result of Matthews [35, 36].

Corollary 13.16 [35, 36] *Let (M, p) be a complete partial metric space and let $T : M \to M$ satisfy*

$$p(Tx, Ty) \leq kp(x, y) \text{ for all } x, y \in M,$$

where $k \in [0, 1)$. Then there exists unique $u \in M$ such that $Tu = u$.

13.4 Generalized α-Admissible $\mathscr{Z}$-Contraction

In this section, we shall prove fixed-point theorems in the setting of dislocated metric space. Not surprisingly, the topology of dislocated space was produced by the family of open balls

$$B_\rho(x, \varepsilon) = \{y \in M : |\rho(x, y) - \rho(x, x)| < \varepsilon\}, \quad \text{for all } x \in M \text{ and } \varepsilon > 0.$$

Furthermore, the basic topological tools (convergence, completeness, etc.) can be observed in a similar way in standard metric theory. Here, we collect some important properties of this space.

Lemma 13.5 [26] *For a DMS (M, ρ), we have the following observations:*

(A) If $\rho(x, y) = 0$ then $\rho(x, x) = \rho(y, y) = 0$.
(B) For a sequence $\{x_n\}$ with $\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0$, we have

$$\lim_{n\to\infty} \rho(x_n, x_n) = \lim_{n\to\infty} \rho(x_{n+1}, x_{n+1}) = 0.$$

(C) If $x \neq y$ then $\rho(x, y) > 0$.
(D) $\rho(x, x) \leq \frac{2}{n} \sum_{i=1}^{i=n} \rho(x, x_i)$ holds for all $x_i, x \in M$ where $1 \leq i \leq n$.
(E) Let V be a closed subset of M and $\{x_n\}$ be a sequence in V. If $x_n \to x$ as $n \to \infty$, then $x \in V$.

(F) *For a sequence* $\{x_n\}$ *in* M *such that* $x_n \to x$ *as* $n \to \infty$ *with* $\rho(x, x) = 0$, *then* $\lim_{n\to\infty} \rho(x_n, y) = \rho(x, y)$ *for all* $y \in M$.

Definition 13.17 Let T be a self-mapping defined on a dislocated metric space (M, ρ). Suppose that there exist functions $\zeta \in \mathscr{Z}$, $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that

$$\zeta(\alpha(x, y)\rho(Tx, Ty), \psi(P(x, y))) \geq 0 \qquad \text{for all } x, y \in M, \tag{13.17}$$

where $P(x, y) = \max\left\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \dfrac{\rho(x, Ty) + \rho(y, Tx)}{4}\right\}$. Then we say that T is a *generalized* α*-admissible* $\mathscr{Z}$*-contraction of type* (I) *with respect to* ζ.

In what follows we recall the following lemma for determining whether the given sequence is Cauchy.

Lemma 13.6 (cf. [41]) *Let* (M, ρ) *be a dislocated metric space and let* $\{x_n\}$ *be a sequence in* M *such that* $d(x_{n+1}, x_n)$ *is nonincreasing and that* $\lim_{n\to\infty} \rho(x_{n+1}, x_n) = 0$. *If* $\{x_n\}$ *is not a Cauchy sequence, then there exist an* $\varepsilon > 0$ *and two sequences* $\{m_k\}$ *and* $\{n_k\}$ *of positive integers such that the following four sequences tend to* ε *when* $k \to \infty$*:*

$$\rho(x_{m_k}, x_{n_k}), \rho(x_{m_k+1}, x_{n_k+1}), \rho(x_{m_k-1}, x_{n_k}), \rho(x_{m_k}, x_{n_k-1})$$

We skip the proof of the lemma above since it is the verbatim of the proof of the corresponding lemma in [41]. Now, we shall state the main results of this chapter.

Theorem 13.4 *Let* (M, ρ) *be a complete dislocated metric space and let* $T : M \to M$ *be generalized* α*-admissible* $\mathscr{Z}$*-contraction of type (I) with respect to* ζ. *Suppose that*

(i) T *is triangular* α*-orbital admissible;*
(ii) *there exists* $x_0 \in M$ *such that* $\alpha(x_0, Tx_0) \geq 1$*; and*
(iii) T *is continuous.*

Then there exists $z \in M$ *such that* $Tz = z$.

Proof On account of the assumption (ii), we have $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$. By taking initial values as x_0, we shall construct an iterative sequence $\{x_n\}$ in M where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

Notice that if $\rho(x_{n_0}, x_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, then $u = x_{n_0}$ turns to be a fixed point of T. Consequently, we shall assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$, thus

$$\rho(x_n, x_{n+1}) > 0, \text{ for all } n \in \mathbb{N}_0.$$

On the other hand, α-admissibility of the mapping T yields that

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

By repeating the observation above, we find that

$$\alpha(x_n, x_{n+1}) \geq 1, \ n \in \mathbb{N}_0, \tag{13.18}$$

Notice also that triangular α-orbital admissibility of the mapping T implies that

$$\alpha(x_n, x_m) \geq 1, \ n, m \in \mathbb{N}_0, n \neq m. \tag{13.19}$$

From (13.17) and (13.18), it follows that for all $n \geq 1$, we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x_{n-1})\rho(Tx_n, Tx_{n-1}), \psi(P(x_n, x_{n-1}))) \\ &= \zeta(\alpha(x_n, x_{n-1})\rho(x_{n+1}, x_n), \psi(P(x_n, x_{n-1}))) \\ &< \psi(P(x_n, x_{n-1})) - \alpha(x_n, x_{n-1})\rho(x_{n+1}, x_n). \end{aligned} \tag{13.20}$$

By combining the obtained inequality above together with Lemmas 13.2 and 13.4, we derive that

$$d(x_n, x_{n+1}) \leq \alpha(x_{n-1}, x_n)\rho(x_n, x_{n+1}) < \psi(P(x_n, x_{n-1})) < P(x_{n-1}, x_n), \ n \in \mathbb{N}. \tag{13.21}$$

Let us analyze the terms of $P(x, y)$:

$$\begin{aligned} P(x_{n-1}, x_n) &= \max\left\{\rho(x_{n-1}, x_n), \rho(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{\rho(x_{n-1}, x_{n+1}) + \rho(x_n, x_n)}{4}\right\} \\ &= \max\left\{\rho(x_{n-1}, x_n), \frac{\rho(x_{n-1}, x_n) + \rho(x_n, x_{n+1})}{2}\right\}, \end{aligned}$$

Under the observation above with the inequality with (13.21), we deduce that $P(x_{n-1}, x_n) = \rho(x_{n-1}, x_n)$. Hence, the sequence $\{\rho(x_n, x_{n+1})\}$ is monotonically decreasing and bounded below by zero. Thus, it is convergent, that is, there is a $L \geq 0$ such that $\lim_{n\to\infty} \rho(x_n, x_{n+1}) = L$. Notice that, from (13.21), $\lim_{n\to\infty} \alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) = L$. We aim to show that $L = 0$. Suppose, on the contrary, that $L > 0$. Then, due to (ζ_3), we have

$$\limsup \zeta(\alpha(x_{n-1}, x_n)\rho(x_n, x_{n+1}), \rho(x_n, x_{n+1})) < 0,$$

a contradiction since we have the condition (13.17). Consequently, we derive that

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0.$$

Next phase is to prove that the sequence $\{x_n\}$ is Cauchy. Suppose, on the contrary, that the constructed sequence $\{x_n\}$ is not Cauchy. Accordingly, there exist $\varepsilon > 0$ such that, for any $k \in \mathbb{N}$, there exist $m_k > n_k > k$ and $d(x_{n_k}, x_{m_k}) \geq \varepsilon$ with an additional condition that m_k is the smallest possible.

Due to Lemma 13.6, we have $\lim_{n\to\infty} \rho(x_{n_k}, x_{m_k+1}) = \lim_{n\to\infty} \rho(x_{n_k+1}, x_{m_k}) = \varepsilon$.

$$\lim_{n\to\infty} \rho(x_{n_k+1}, x_{m_k+1}) = \lim_{n\to\infty} \alpha(x_{n_k}, x_{m_k})\rho(x_{n_k+1}, x_{m_k+1}) = \varepsilon.$$

Taking the observations above into account together with (13.5), we find

$$\limsup_{n\to\infty} \zeta(\alpha(x_{n_k}, x_{m_k})d(x_{n_k+1}, x_{m_k+1}), d(x_{n_k}, x_{m_k})) < 0,$$

which contradicts the condition (13.17). By *reductio ad absurdum*, we conclude that $\{x_n\}$ is a 0-Cauchy sequence.

Since (X, d) is a complete dislocated metric space, there exist $x^* \in M$ so that the sequence $\{x_n\}$ converges to x^*. The continuity of T implies $Tx^* = x^*$. □

Theorem 13.5 *Let (M, ρ) be a complete dislocated metric space and let $T : M \to M$ be a generalized α-admissible $\mathscr{Z}$-contraction of type (I) with respect to ζ. Suppose that*

(i) T is triangular α-orbital admissible;
(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Then there exists $x^* \in M$ such that $Tx^* = x^*$.

Proof By following the lines in the proof of Theorem 13.1, we find a sequence $\{x_n\}$ converges to some $x^* \in M$, that is, $\rho(x_{n_k}, x^*) = 0$. From (13.8) and condition (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$, $k \in \mathbb{N}$. Applying (13.17), for all $k \in \mathbb{N}$, we get that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n_k}, x^*)\rho(Tx_{n_k}, Tx^*), \psi(P(x_{n_k}, x^*))) \\ &= \zeta(\alpha(x_{n_k}, x^*)d(x_{n_k+1}, Tz), \psi(P(x_{n_k}, x^*))) \\ &< \psi(P(x_{n_k}, x^*)) - \alpha(x_{n_k}, x^*)\rho(x_{n_k+1}, Tx^*), \end{aligned}$$

for

$$P(x_{n_k}, x^*) = \max\left\{\rho(x_{n_k}, x^*), \rho(x_{n_k}, x_{n_k+1}), \rho(x^*, Tx^*), \frac{d(x_{n_k}, Tz) + \rho(x_{n_k+1}, z)}{4}\right\}.$$

Hence, we have

$$0 \le \rho(x_{n_k+1}, Tx^*) \le \alpha(x_{n_k}, x^*)\rho(x_{n_k+1}, Tx^*) < \psi(P(x_{n_k}, x^*)) < P(x_{n_k}, x^*).$$

Letting $k \to \infty$, we have $\rho(x^*, Tx^*) = 0$, i.e., $Tx^* = x^*$. □

The uniqueness for the fixed point determined in Theorems 13.4 and 13.5, the condition (U_1) is not sufficient. For this reason, we prefer to revise the contraction condition as follows.

Definition 13.18 Let T be a self-mapping defined on a dislocated metric space (M, d). If there exist $\zeta \in \mathscr{Z}$ and $\alpha : M \times M \to [0, \infty)$ such that

$$\zeta(\alpha(x, y)\rho(Tx, Ty), \psi(Q(x, y))) \ge 0 \qquad \text{for all } x, y \in X, \tag{13.22}$$

where $\psi \in \Psi$ and

$$Q(x, y) = \max\left\{\rho(x, y), \frac{\rho(x, Tx) + \rho(y, Ty)}{4}, \frac{\rho(x, Ty) + \rho(y, Tx)}{4}\right\}. \tag{13.23}$$

Then, we say that T is a *generalized α-admissible $\mathscr{Z}$-contraction of type* **(II)** *with respect to ζ.*

Theorem 13.6 *Let (M, ρ) be a complete dislocated metric space and let $T : M \to M$ be a generalized α-admissible $\mathscr{Z}$-contraction of type* **(II)** *with respect to ζ. Suppose that*

- *(i) T is triangular α-orbital admissible;*
- *(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \ge 1$;*
- *(iii) either T is continuous, or*
- *(iii)' if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \ge 1$ for all k.*

Then there exists $x^* \in M$ such that $Tx^* = x^*$.

Now, by adding the hypothesis (U_1), we can the uniqueness of the existing fixed point of T.

Theorem 13.7 *Adding condition (U_1) to the hypotheses of Theorem 13.6, we obtain that z is the unique fixed point of T.*

Proof Following the lines in the proof of Theorem 13.6, we guarantee the existence fixed point of T. We claim that the obtained fixed point of T in Theorem 13.6 is unique. Suppose, on the contrary, that both $y, z \in M$ are distinct fixed points of T:

$$d(z, y) \le \alpha(z, y)d(z, y) < \psi(Q(z, y)) = \psi(\max\{d(z, y)\}) < d(z, y),$$

a contradiction. Hence, the constructed fixed point of T in Theorem 13.6 is unique. □

13.4.1 More Consequences on Dislocated Metric Spaces

By taking (ζ_2) into consideration in Corollary 13.7, we conclude that

Corollary 13.17 *Let (M, ρ) be a complete dislocated metric space and let $T : M \to M$ satisfy*

$$\alpha(x, y)\rho(Tx, Ty) \leq \psi(Q(x, y)) \text{ for all } x, y \in M,$$

where $\psi \in \Psi_b$ and $Q(x, y)$ is defined as in (13.23). Suppose that

- *(i) T is α-orbital admissible;*
- *(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- *(iii) either, T is continuous, or,*
- *(iii)' if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*
- *(iv) the condition (U_1) is fulfilled.*

Then there exists a unique $u \in M$ such that $Tu = u$.

By letting $\alpha(x, y) = 1$ for all $x, y \in M$ in Corollary 13.17, we get that

Corollary 13.18 *Let (M, ρ) be a complete dislocated metric space and let $T : M \to M$ satisfy*

$$\rho(Tx, Ty) \leq \psi(Q(x, y)) \text{ for all } x, y \in M,$$

where $\psi \in \Psi_b$ and $Q(x, y)$ is defined as in (13.23). Then there exists unique $u \in M$ such that $Tu = u$.

In particular, by taking $\psi(t) = kt$, $k \in [0, 1)$ in the corollary above, we have

Corollary 13.19 *Let (M, ρ) be a complete dislocated metric space and let $T : M \to M$ satisfy*

$$\rho(Tx, Ty) \leq kQ(x, y) \text{ for all } x, y \in M,$$

where $k \in [0, 1)$ and $Q(x, y)$ is defined as in (13.23). Then there exists unique $u \in M$ such that $Tu = u$.

13.4.2 Consequences on Standard Partial Metric Spaces

Here, we list immediate consequences in the setting of partial metric spaces.

Corollary 13.20 *Let (M, p) be a complete partial metric space and $T : M \to M$ be a self-mapping. Suppose that there exist functions $\zeta \in \mathscr{Z}$, $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$\zeta(\alpha(x, y)p(Tx, Ty), \psi(P(x, y))) \geq 0 \quad for\ all\ x, y \in M, \tag{13.24}$$

where

$$P(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{4}\right\}. \tag{13.25}$$

Suppose that

- *(i) T is α-orbital admissible;*
- *(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;*
- *(iii) either, T is continuous, or,*
- *(iii)' if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then there exists $u \in M$ such that $Tu = u$.

Corollary 13.21 *Adding condition (U_1) to the hypotheses of Corollary 13.20, we obtain that u is the unique fixed point of T.*

Proof The existence fixed point of T is concluded from Corollary 13.20. We shall indicate that the existence fixed point of T in Corollary 13.20 is unique. Suppose, on the contrary, that both $y, z \in M$ are distinct fixed points of T. Since $\max\{p(z, z), p(y, y)\} \leq p(z, y)$, then we have

$$p(z, y) \leq \alpha(z, y)p(z, y) < \psi(P(z, y)) = \psi(\max\{p(z, y)\}) < p(z, y),$$

a contradiction. Hence, guaranteed fixed point of T in Corollary 13.20 is unique. □

By taking (ζ_2) into consideration in Corollary 13.21, we conclude that

Corollary 13.22 *Let (M, p) be a complete partial metric space and let $T : M \to M$ satisfy*

$$\alpha(x, y)p(Tx, Ty) \leq \psi(P(x, y)) \ for\ all\ x, y \in M,$$

where $\psi \in \Psi_b$ and $P(x, y)$ is defined as in (13.25). Suppose that

- *(i) T is α-orbital admissible;*
- *(ii) there exists $x_0 \in M$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *either, T is continuous, or,*
(iii)' *if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in M$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*
(iv) *the condition (U_1) is fulfilled.*

Then there exists a unique $u \in M$ such that $Tu = u$.

By letting $\alpha(x, y) = 1$ for all $x, y \in M$ in Corollary 13.22, we get that

Corollary 13.23 *Let (M, p) be a b-complete metric space and let $T : M \to M$ satisfy*

$$p(Tx, Ty) \leq \psi(P(x, y)) \text{ for all } x, y \in M,$$

where $\psi \in \Psi_b$ and $P(x, y)$ is defined as in (13.25). Then there exists unique $u \in M$ such that $Tu = u$.

In particular, by taking $\psi(t) = kt$, $k \in [0, 1)$ in the corollary above, we have

Corollary 13.24 *Let (M, p) be a complete partial metric space and let $T : M \to M$ satisfy*

$$p(Tx, Ty) \leq kP(x, y) \text{ for all } x, y \in M,$$

where $k \in [0, 1)$ and $P(x, y)$ is defined as in (13.25). Then there exists unique $u \in M$ such that $Tu = u$.

It is quite easy to extend the list of consequences of the given theorems. For instance, all results in this subsection can be reformulated in the setting of "cyclic contractions" or "partially ordered set" as in Sects. 3.1 and 3.2. Furthermore, taking Example 13.8 into account, we can deduce more consequences of the given theorems involving simulation function.

References

1. Aage, C.T., Salunke, J.N.: The results on fixed points in dislocated and dislocated quasi-metric space. Appl. Math. Sci. **2**(59), 2941–2948 (2008)
2. Aage, C.T., Salunke, J.N.: Some results of fixed point theorem in dislocated quasi-metric spaces. Bull. Marathadawa Math. Soc. **9**, 1–5 (2008)
3. Ali, M.U., Kamran, T., Karapınar, E.: On (α, ψ, ξ)-contractive multi-valued mappings. Fixed Point Theory Appl. **2014**, 7 (2014)
4. Alsulami, H.H., Karapınar, E., Khojasteh, F., Roldán-López-de-Hierro, A.F.: A proposal to the study of contractions in quasi-metric spaces. Discrete Dyn. Nat. Soc. **2014**, Article ID 269286 (2014)
5. Amini Harandi, A.: Metric-like spaces, partial metric spaces and fixed points. Fixed Point Theory Appl. **2012**, 204 (2012)
6. Aydi, H., Bota, M.F., Karapınar, E., Moradi, S.: A common fixed point for weak-phi-contractions on b-metric spaces. Fixed Point Theory **13**(2), 337–346 (2012)

7. Aydi, H., Bota, M.F., Mitrovic, S., Karapınar, E.: A fixed point theorem for set-valued quasi-contractions in b-metric spaces. Fixed Point Theory Appl. **2012**, 88 (2012)
8. Bakhtin, I.A.: The contraction mapping principle in quasimetric spaces. Funct. Anal. Unianowsk Gos. Ped. Inst. **30**, 26–37 (1989)
9. Czerwik, S.: Contraction meppings in b-metric spaces. Acta. Math. Inform. Univ. Ostraviensis. **1**, 5–11 (1993)
10. Czerwik, S.: Nonlinear set-valued contraction mappings in b-metric spaces. Atti Sem. Mat. Univ. Modena. **46**, 263–276 (1998)
11. Banach, S.: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. **3**, 133–181 (1922)
12. Berinde, V.: Contracţii generalizate şi aplicaţii, Editura Club Press 22, Baia Mare (1997)
13. Berinde, V.: Generalized contractions in quasimetric spaces. Seminar on Fixed Point Theory. Preprint **3**, 3–9 (1993)
14. Berinde, V.: Sequences of operators and fixed points in quasimetric spaces. Stud. Univ. Babeş-Bolyai, Math. **16**(4), 23–27 (1996)
15. Bourbaki, N.: Topologie Générale. Herman, Paris (1974)
16. Bota, M.F., Karapınar, E., Mlesnite, O.: Ulam-Hyers stability results for fixed point problems via $\alpha - \psi$-contractive mapping in b-metric space. Abstr. Appl. Anal. **2013**, Article ID 825293 (2013)
17. Bota, M.F., Karapınar, E.: A note on "Some results on multi-valued weakly Jungck mappings in b-metric space". Cent. Eur. J. Math. **11**(9), 1711–1712 (2013)
18. Bota, M.F., Chifu, C., Karapınar, E.: Fixed point theorems for generalized (α-ψ)-Ćirić-type contractive multivalued operators in b-metric spaces. J. Nonlinear Sci. Appl. **9**(3), 1165–1177 (2016)
19. Cvetkovic, M., Karapınar, E., Rakocevic, V.: Some fixed point results on quasi-b-metric-like spaces. J. Inequal. Appl. **2015**, 374 (2015)
20. Daheriya, R.D., Jain, R., Ughade, M.: Some fixed point theorem for expansive type mapping in dislocated metric space. ISRN Math. Anal. **2012**, Article ID 376832 (2012)
21. Fréchet, M.: Sur quelques points du calcul fonctionnel. Rendic. Circ. Mat. Palermo. **22**, 1–74 (1906)
22. Hitzler, P.: Generalized metrics and topology in logic programming semantics, Ph.D. Thesis, School of Mathematics, Applied Mathematics and Statistics, National University Ireland, University college Cork (2001)
23. Hitzler, P., Seda, A.K.: Dislocated topologies. J. Electr. Engin. **51**(12), 3–7 (2000)
24. Hussain, N., Roshan, J.R., Parvaneh, V., Abbas, M.: Common fixed point results for weak contractive mappings in ordered b-dislocated metric spaces with applications. J. Inequal. Appl. **2013**, 486 (2013)
25. Jleli, M., Samet, B.: Remarks on G-metric spaces and fixed point theorems. Fixed Point Theory Appl. **2012**, 210 (2012)
26. Karapınar, E., Salimi, P.: Dislocated metric spaces to metric spaces with some fixed point theorems. Fixed Point Theory Appl. **2013**, 222 (2013)
27. Karapınar, E., Samet, B.: Generalized $\alpha - \psi$-contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal. **2012**, Article ID 793486 (2012)
28. Karapınar, E., Kumam, P., Salimi, P.: On $\alpha - \psi$-Meri-Keeler contractive mappings. Fixed Point Theory Appl. **2013**, 94 (2013)
29. Karapınar, E., Piri, H., Alsulami, H.H.: Fixed points of generalized F-Suzuki type contraction in complete b-metric spaces. Discrete Dyn. Nat. Soc. **2015**, Article ID 969726 (2015)
30. Karapınar, E., Alsulami, H.H., Noorwali, M.: Some extensions for Geragthy type contractive mappings. J. Ineq. Appl. **2015**, 303 (2015)
31. Khojasteh, F., Shukla, S., Radenović, S.: A new approach to the study of fixed point theorems via simulation functions. FILOMAT **29**(6), 1189–1194 (2015)
32. Kirk, W.A., Srinivasan, P.S., Veeramani, P.: Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory **4**(1), 79–89 (2003)

33. Kutbi, M.A., Karapınar, E., Ahmed, J., Azam, A.: Some fixed point results for multi-valued mappings in b-metric spaces. J. Ineq. Appl. **2014**, 126 (2014)
34. Latif, A., Gordji, M.E., Karapınar, E., Sintunavarat, W.: Fixed point results for generalized (α, ψ)-Meir-Keeler contractive mappings and applications. J. Ineq. Appl. **2014**, 68 (2014)
35. Matthews, S.G.: Partial metric topology. Research Report 212. Department of Computer Science, University of Warwick (1992)
36. Matthews, S.G.: Partial metric topology. Ann. New York Acad. Sci. **728**, 183–197 (1994)
37. Mukhtar-Hassan, A., Karapınar, E., Alsulami, H. H.: Ulam-Hyers stability for MKC mappings via fixed point theory. J. Funct. Spaces. **2016**, Article ID 9623597 (2016)
38. Mustafa, Z., Sims, B.: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. **7**(2), 289–297 (2006)
39. Pacurar, M.: A fixed point result for φ-contractions on b-metric spaces without the boundedness assumption. Fasc. Math. **43**, 127–137 (2010)
40. Popescu, O.: Some new fixed point theorems for α-Geraghty-contraction type maps in metric spaces. Fixed Point Theory Appl. **2014**, 190 (2014)
41. Radenović, S., Kadelburg, Z., Jandrlić, D., Jandrlić, A.: Some results on weak contraction maps. Bull. Iranian Math. Soc. **38**(3), 625–645 (2012)
42. Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**, 1435–1443 (2003)
43. Roldán-López-de-Hierro, F., A., Karapınar, E., Roldán-López-de-Hierro, C., Martínez-Moreno, J.: Coincidence point theorems on metric spaces via simulation functions. J. Comput. Appl. Math. **275**, 345–355 (2015)
44. Romaguera, S.: A Kirk type characterization of completeness for partial metric spaces. Fixed Point Theory Appl. **2010**, Article ID 493298 (2010)
45. Romaguera, S.: Fixed point theorems for generalized contractions on partial metric spaces. Topol. Appl. **159**, 194–199 (2012)
46. Rus, I.A.: The theory of a metrical fixed point theorem: theoretical and applicative relevances. Fixed Point Theory **9**(2), 541–559 (2008)
47. Rus, I.A.: Generalized Contractions and Applications. Cluj University Press, Cluj-Napoca (2001)
48. Samet, B.: The class of $(\alpha - \psi)$-type contractions in b-metric spaces and fixed point theorems. Fixed Point Theory Appl. **2015**, 92 (2015)
49. Samet, B., Vetro, C., Vetro, P.: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal. **75**, 2154–2165 (2012)
50. Samet, B., Vetro, C., Vetro, F.: Remarks on G-metric spaces. Int. J. Anal. **2013**, Article ID 917158 (2013)
51. Sarma, I.R., Kumari, P.S.: On dislocated metric spaces. Int. J. Math. Arch. **3**(1), 72–77 (2012)
52. Shukla, S.: Partial b-metric spaces and fixed point theorems. Mediterr. J. Math. **11**, 703–711 (2014)
53. Shrivastava, R., Ansari, Z.K., Sharma, M.: Some results on fixed points in dislocated and dislocated quasi-metric spaces. J. Adv. Stud. Topol. **3**(1), 25–31 (2012)
54. Turinici, M.: Abstract comparison principles and multivariable Gronwall-Bellman inequalities. J. Math. Anal. Appl. **117**, 100–127 (1986)
55. Zeyada, F.M., Hassan, G.H., Ahmed, M.A.: A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces. Arab. J. Sci. Eng. Sect. A. **31**(1), 111–114 (2005)
56. Zoto, K., Hoxha, E., Isufati, A.: Some new results in dislocated and dislocated quasi-metric spaces. Appl. Math. Sci. **6**(71), 3519–3526 (2012)
57. Zoto, K., Hoxha, E.: Fixed point theorems for ψ-contractive type mappings in dislocated quasi-metric spaces. Int. Math. Forum. **7**(51), 2503–2508 (2012)