Chapter 11
On the Qualitative Behaviors of Nonlinear Functional Differential Systems of Third Order

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Abstract In this paper, the author gives new sufficient conditions for the boundedness and globally asymptotically stability of solutions to certain nonlinear delay functional differential systems of third order. The technique of proof involves defining an appropriate Lyapunov–Krasovskii functional and applying LaSalle's invariance principle. The obtained results include and improve the results in literature.

11.1 Introduction

Ordinary and functional differential equations are frequently encountered as mathematical models arisen from a variety of applications including control systems, electrodynamics, mixing liquids, medicine, biomathematics, economics, atomic energy, information theory, neutron transportation and population models, etc. In addition, it is well known that ordinary and functional differential equations of third order play extremely important and useful roles in many scientific areas such as atomic energy, biology, chemistry, control theory, economy, engineering, information theory, biomathematics, mechanics, medicine, physics, etc. For example, the readers can find applications such as nonlinear oscillations in Afuwape et al. [8], Andres [11], Fridedrichs [19], physical applications in Animalu and Ezeilo [12], nonresonant oscillations in Ezeilo and Onyia [17], prototypical examples of complex dynamical systems in a high-dimensional phase space, displacement in a mechanical system, velocity, acceleration in Chlouverakis and Sprott [14], Eichhorn et al. [16] and Linz [25], the biological model and other models in Cronin- Scanlon [15], electronic theory in Rauch [32], problems in biomathematics in Chlouverakis and Sprott [14] and Smith [36], etc.

Qualitative properties of solutions of ordinary and functional equations of third order such as stability, instability, oscillation, boundedness, and periodicity of solutions have been studied by many authors; in this regard, we refer the reader to the monograph by Reissig et al. [33], and the papers of Adams et al. [1], Ademola and

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Arawomo [2–5], Ademola et al. [6], Afuwape and Castellanos [7], Afuwape and Omeike [9], Ahmad and Rao [10], Bai and Guo [13], Ezeilo and Tejumola [18], Graef et al. [20, 21], Graef and Tunç [22], Kormaz and Tunç [24], Mahmoud and Tunç [26], Ogundare [27], Ogundare et al. [28], Olutimo [29], Omeike [30], Qian [31], Remili and Oudjedi [34], Sadek [35], Swick [37], Tejumola and Tchegnani [38], Tunç [39]–[57], Tunç and Ates [58], Tunç and Gozen [59], Tunç and Mohammed [60], Tunç and Tunç [61], Tunç [62, 63], Zhang and Yu [65], Zhu [66], and theirs references.

However, to the best of our knowledge from the literature, by this time, a little attention was given to the investigation of the stability/boundedness/ultimately boundedness in functional differential systems of third order (see Mahmoud and Tunç [26], Omeike [30], Tunç [56], Tunç and Mohammed [59]).

Recently, Tunç and Mohammed [60], Mahmoud and Tunç [26], and Tunç [56] discussed the stability and boundedness in nonlinear vector delay differential equation of third order, respectively:

$$X''' + \Psi(X')X'' + BX'(t-\tau_1) + cX(t-\tau_1) = P(t), \tag{11.1}$$

$$X''' + AX'' + G(X') + H(X(t-\tau)) = P(t), \tag{11.2}$$

and

$$X''' + H(X')X'' + G(X'(t-\tau)) + cX(t-\tau) = F(t, X, X', X''). \tag{11.3}$$

In addition, very recently Omeike [30] investigated the stability and boundedness in a nonlinear differential system of third order with variable delay, $\tau(t)$:

$$X''' + AX'' + BX' + H(X(t-\tau(t))) = P(t). \tag{11.4}$$

In this paper, instead of these delay differential equations, we consider vector delay differential equation of third order

$$X''' + H(X')X'' + G(X'(t-\tau)) + \Phi(X(t-\tau)) = E(t, X, X', X''), \tag{11.5}$$

where $\tau > 0$ is the fixed constant delay, $G : \Re^n \to \Re^n$ and $\Phi : \Re^n \to \Re^n$ are continuous differentiable functions with $G(0) = \Phi(0) = 0$ and H is an $n \times n-$ continuous differentiable symmetric matrix function. In addition, throughout this paper, we assume that the Jacobian matrices $J_H(X')$, $J_G(X')$, and $J_\Phi(X)$ exist and are symmetric and continuous, that is,

$$J_H(X') = \left(\frac{\partial h_{ik}}{\partial x'_j}\right), J_G(X') = \left(\frac{\partial g_i}{\partial x'_j}\right), J_\Phi(X) = \left(\frac{\partial \phi_i}{\partial x_j}\right), (i, j, k = 1, 2, ..., n),$$

where $(x_1, x_2, ..., x_n)$, $(x'_1, x'_2, ..., x'_n)$, (h_{ik}), (g_i), and (ϕ_i) are components of X, X', H, G, and Φ, respectively; $E : \Re^+ \times \Re^n \times \Re^n \times \Re^n \to \Re^n$ is a continuous function, $\Re^+ = [0, \infty)$, and the primes in Eq. (11.5) indicate differentiation with respect to t, $t \geq t_0 \geq 0$.

The continuity of the functions H, G, Φ, and E is a sufficient condition for existence of the solutions of Eq. (11.5). In addition, we assume that the functions H, G, Φ and E satisfy a Lipschitz condition on their respective arguments, like X, X', and X''. In this case, the uniqueness of solutions of Eq. (11.5) is guaranteed.

We can write equation as the system

$$\begin{aligned} X'_1 &= X_2, \\ X'_2 &= X_3, \\ X'_3 &= -H(X_2)X_3 - G(X_2) - \Phi(X_1) + \int_{t-\tau}^{t} J_G(X_2(s))X_3(s)ds \\ &\quad + \int_{t-\tau}^{t} J_\Phi(X_1(s))X_2(s)ds + E(t, X_1, X_2, X_3), \end{aligned} \tag{11.6}$$

which were obtained by setting $X = X_1$, $X' = X_2$, $X'' = X_3$ from Eq. (11.5).

It should be noted any investigation of the stability and boundedness in vector functional differential equations of third order, using the Lyapunov–Krasovskii functional method, first requires the definition or construction of a suitable Lyapunov–Krasovskii functional, which gives meaningful results. In reality, this case can be an arduous task. The situation becomes more difficult when we replace an ordinary differential equation with a functional vector differential equation. However, once a viable Lyapunov–Krasovskii functional has been defined or constructed, researchers may end up with working with it for a long time, deriving more information about stability. To arrive at the objective of this paper, we define a new suitable Lyapunov–Krasovskii functional.

The motivation of this paper is inspired by the results established in Graef and Tunç [22], Omeike [30], Mahmoud and Tunç [26], Tunç [56], Tunç and Mohammed [60], Zhang and Yu [65], Zhu [66], the mentioned papers and theirs references. The aim of this paper is to obtain some new globally asymptotically stability/boundedness/ultimately boundedness results in Eq. (11.5). In verification of our main results the Lyapunov–Krasovskii functional approach is used. By this paper, we will extend and improve the results of Graef and Tunç [22], Omeike [30], Mahmoud and Tunç [26], Tunç [56], Tunç and Mohammed [60], Zhang and Yu [65], and Zhu [66]. It is clear that Eq. (11.5) includes Eqs. (11.1), (11.2), (11.3), and (11.4) when $\tau(t) = \tau$ (constant). In addition, this paper may be useful for researchers working on the qualitative properties of solutions of functional differential equations. These cases show the novelty and originality of the present paper.

One tool to be used here is the LaSalle's invariance principle.

Consider delay differential system

$$\dot{x} = F(x_t), x_t = x(t+\theta), \quad -r \leq \theta \leq 0, t \geq 0.$$

We take $C = C([-r, 0], \Re^n)$ to be the space of continuous function from $[-r, 0]$ into $\Re^n$ and ask that $F : C \to \Re^n$ be continuous. We say that $V : C \to \Re$ is a Lyapunov function on a set $G \subset C$ relative to F if V is continuous on $\bar{G}$, the closure of G, $V \geq 0$, V is positive definite, $\dot{V}$ is defined on G, and $\dot{V} \leq 0$ on G.

The following form of the LaSalle's invariance principle can be found reference in Smith [36].

Theorem 11.1 *If V is a Lyapunov function on G and $x_t(\phi)$ is a bounded solution such that $x_t(\phi) \in G$ for $t \geq 0$, then $\omega(\phi) \neq 0$ is contained in the largest invariant subset of $E \equiv \{\psi \in \bar{G} : \dot{V}(\psi) = 0\}$, ω denotes the omega limit set of a solution.*

The following lemmas are needed in the proofs of main results.

Lemma 11.1 (Hale [23]) *Suppose $F(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, nonnegative and continuous for $0 \leq s < \infty$, $u(s) \to \infty$ as $u \to \infty$ with $u(0) = 0$. If for all $\phi \in C$, $u(|\phi(0)|) \leq V(\phi)$, $V(\phi) \geq 0$, $\dot{V}(\phi) \leq 0$, then the zero solution of $\dot{x} = F(x_t)$ is stable.*

If we define $Z = \{\phi \in C_H : \dot{V}(\phi) = 0\}$, then the zero solution of $\dot{x} = F(x_t)$ is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

Lemma 11.2 *Let A be a real symmetric $n \times n$-matrix. Then for any $X_1 \in \Re^n$*

$$\delta_a \|X_1\|^2 \leq \langle AX_1, X_1 \rangle \leq \Delta_a \|X_1\|^2 ,$$

where δ_a and Δ_a are, respectively, the least and greatest eigenvalues of the matrix A.

11.2 Stability

Our first result is for the case where $E(.) \equiv 0$.

Assume that there are positive constants ε, α, a_0, a_1, b_0, b_1, c_0, and c such that for all $X_1, X_2 \in \Re^n$ the following conditions hold:

(C1) $a_0 b_0 c - c_0^2 > 0$, $1 - \alpha a_0 > 0$, $G(0) = 0$, $n \times n$-symmetric matrices J_G and H commute with each other, and

$$b_0 \leq \lambda_i(J_G(X_2)) \leq b_1, \quad 2a_0 + \varepsilon \leq \lambda_i(H(X_2)) \leq a_1;$$

(C2) $\Phi(0) = 0$, $c \leq \lambda_i(J_\Phi(X_1)) \leq c_0$.
Let

$$\ell_5 = 2(a_0 b_0 - c_0) - \alpha a_0 b_0 [a_0 + c^{-1}(b_1 - b_0)^2] > 0$$

and

$$\ell_6 = 2\varepsilon[1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2] > 0.$$

Theorem 11.2 *Assume that* $E(.) \equiv 0$ *and conditions* $(C1)$ *and* $(C2)$ *hold. If*

$$\tau < \min\left\{\frac{\alpha a_0 b_0 c}{\Delta_1}, \frac{\ell_5}{\Delta_2}, \frac{2\ell_6}{\Delta_3}\right\},$$

then all solutions of Eq. (11.5) are bounded and the zero solution of Eq. (11.5) is globally asymptotically stable, where Δ_1*,* Δ_2*, and* Δ_3 *are some positive constants to be determined later in the proof.*

Proof We define a Lyapunov–Krasovskii functional $V_0 = V_0(t) = V_0(X_1(t), X_2(t), X_3(t))$ given by

$$\begin{aligned}
2V_0 = {} & 2a_0\int_0^1 \langle \Phi(\sigma X_1), X_1\rangle\, d\sigma + 2a_0\int_0^1 \langle \sigma H(\sigma X_2)X_2, X_2\rangle\, d\sigma \\
& +\alpha a_0 b_0^2\langle X_1, X_1\rangle + 2\int_0^1 \langle G(\sigma X_2), X_2\rangle\, d\sigma \\
& +\langle X_3, X_3\rangle + 2\alpha a_0^2 b_0\langle X_1, X_2\rangle + 2\alpha a_0 b_0\langle X_1, X_3\rangle \\
& +2a_0\langle X_2, X_3\rangle + 2\langle \Phi(X_1), X_2\rangle - \alpha a_0 b_0\langle X_2, X_2\rangle \\
& +2\lambda\int_{-\tau}^0\int_{t+s}^t \|X_2(\theta)\|^2\, d\theta ds + 2\eta\int_{-\tau}^0\int_{t+s}^t \|X_3(\theta)\|^2\, d\theta ds, \quad (11.7)
\end{aligned}$$

where

$$0 < \alpha < \min\left\{\frac{1}{a_0}, \frac{a_0}{b_0}, \frac{a_0 b_0 - c_0}{a_0 b_0[a_0 + c^{-1}(b_1 - b_0)^2]}, \frac{c}{a_0 b_0 (a_1 - a_0)^2}\right\},$$

$a_1 > a_0$, $b_1 \neq b_0$, and λ and η are positive constants that will be determined later in the proof.

It is clear that

$$V_0(0, 0, 0) = 0.$$

From

$$\Phi(0) = 0, \quad \frac{\partial}{\partial\sigma}\Phi(\sigma X_1) = J_\Phi(\sigma X_1)X_1,$$

$$G(0) = 0, \quad \frac{\partial}{\partial\sigma}G(\sigma X_2) = J_G(\sigma X_2)X_2,$$

and (C2), it follows that

$$\begin{aligned}
2a_0\int_0^1 \langle \Phi(\sigma X_1), X_1\rangle\, d\sigma &= 2\int_0^1\int_0^1 \sigma_1\, \langle J_\Phi(\sigma_1\sigma_2 X_1)X_1, X_1\rangle\, d\sigma_1 d\sigma_2 \\
&\geq a_0 c\, \|X_1\|^2
\end{aligned}$$

and

$$\int_0^1 \langle G(\sigma X_2), X_2\rangle \, d\sigma = \int_0^1 \int_0^1 \sigma_1 \langle J_G(\sigma_1\sigma_2 X_2)X_2, X_2\rangle \, d\sigma_1 d\sigma_2.$$

Then, from (11.7), we obtain

$$\begin{aligned} 2V_0 \geq{}& a_0 b_0 \left\| a_0^{-\frac{1}{2}} X_2 + a_0^{-\frac{1}{2}} b_0^{-1} \Phi(X_1) \right\|^2 + \|X_3 + a_0 X_2 + \alpha a_0 b_0 X_1\|^2 \\ &+ 2a_0 \int_0^1 \langle \sigma H(\sigma X_2)X_2, X_2\rangle \, d\sigma - 2a_0^2 \|X_2\|^2 + a_0(a_0 - \alpha b_0) \|X_2\|^2 \\ &+ 2\int_0^1 \int_0^1 \sigma_1 \langle J_G(\sigma_1\sigma_2 X_2)X_2, X_2\rangle \, d\sigma_1 d\sigma_2 - b_0 \|X_2\|^2 \\ &+ \alpha a_0 b_0^2 (1 - \alpha a_0) \|X_1\|^2 + a_0 c \langle X_1, X_1\rangle - b_0^{-1} \langle \Phi(X_1), \Phi(X_1)\rangle \\ &+ 2\lambda \int_{-\tau}^0 \int_{t+s}^t \|X_2(\theta)\|^2 \, d\theta ds + 2\eta \int_{-\tau}^0 \int_{t+s}^t \|X_3(\theta)\|^2 \, d\theta ds. \end{aligned} \tag{11.8}$$

From

$$\Phi(0) = 0, \quad \frac{\partial}{\partial \sigma_1} \Phi(\sigma_1 X_1) = J_\Phi(\sigma_1 X_1) X_1,$$

it follows that

$$\frac{\partial}{\partial \sigma_1} \langle \Phi(\sigma_1 X_1), \Phi(\sigma_1 X_1)\rangle = 2 \langle J_\Phi(\sigma_1 X_1)X_1, \Phi(\sigma_1 X_1)\rangle .$$

Integrations of the last two estimates, from $\sigma_1 = 0$ to $\sigma_1 = 1$, respectively, imply

$$\Phi(X_1) = \int_0^1 J_\Phi(\sigma_1 X_1) X_1 d\sigma_1$$

and

$$\langle \Phi(X_1), \Phi(X_1)\rangle = 2 \int_0^1 \langle J_\Phi(\sigma_1 X_1)X_1, \Phi(\sigma_1 X_1)\rangle \, d\sigma_1.$$

Further, it is clear that

$$\frac{\partial}{\partial \sigma_2} \langle \Phi(\sigma_1\sigma_2 X_1), J_\Phi(\sigma_1 X_1)X_1\rangle = \langle \sigma_1 J_\Phi(\sigma_1 X_1)X_1, J_\Phi(\sigma_1 X_1)X_1\rangle .$$

Integration of the both sides of the last equality, from $\sigma_2 = 0$ to $\sigma_2 = 1$, implies

$$\langle \Phi(\sigma_1 X_1), J_\Phi(\sigma_1 X_1)X_1\rangle = \int_0^1 \langle \sigma_1 J_\Phi(\sigma_1 X_1)X_1, J_\Phi(\sigma_1 X_1)X_1\rangle \, d\sigma_2.$$

From these estimates and assumptions (C1) and (C2), we have

$$\langle \Phi(X_1), \Phi(X_1)\rangle = 2\int_0^1\int_0^1 \langle \sigma_1 J_\Phi(\sigma_1 X_1)X_1, J_\Phi(\sigma_1 X_1)X_1\rangle\, d\sigma_1 d\sigma_2 \le c_0^2 \|X_1\|^2,$$

$$a_0 c\,\langle X_1, X_1\rangle - b_0^{-1}\langle \Phi(X_1), \Phi(X_1)\rangle \ge (a_0 c - b_0^{-1}c_0^2)\|X_1\|^2 \ge 0,$$

$$2\int_0^1 \langle G(\sigma X_2), X_2\rangle\, d\sigma = 2\int_0^1\int_0^1 \sigma_1\langle J_G(\sigma_1\sigma_2 X_2)X_2, X_2\rangle\, d\sigma_1 d\sigma_2 \ge \delta_b \|X_2\|^2,$$

$$\begin{aligned}&2a_0\int_0^1 \langle \sigma H(\sigma X_2)X_2, X_2\rangle\, d\sigma - 2a_0^2\|X_2\|^2\\ &= 2a_0\int_0^1\int_0^1 \sigma_1\langle J_H(\sigma_1\sigma_2 X_2)X_2, X_2\rangle\, d\sigma_1 d\sigma_2 - 2a_0^2\|X_2\|^2 \ge \varepsilon a_0\|X_2\|^2,\end{aligned}$$

$$2\int_0^1\int_0^1 \sigma_1\langle J_G(\sigma_1\sigma_2 X_2)X_2, X_2\rangle\, d\sigma_1 d\sigma_2 - b_0\|X_2\|^2 \ge 0,$$

$$\alpha a_0 b_0^2(1-\alpha a_0)\|X_1\|^2 = \mu_1\|X_1\|^2, \quad \mu_1 = \alpha a_0 b_0^2(1-\alpha a_0) > 0,$$

$$(a_0 c - b_0^{-1}c_0^2)\|X_1\|^2 = \quad \mu_2\|X_1\|^2, \quad \mu_2 = (a_0 c - b_0^{-1}c_0^2) > 0,$$

$$a_0(a_0 - \alpha b_0)\|X_2\|^2 = \mu_3\|X_2\|^2, \quad \mu_3 = a_0(a_0 - \alpha b_0) > 0.$$

Combining these estimates into (11.8), it follows that

$$\begin{aligned}V_0 \ge\ & \frac{1}{2}a_0 b_0\left\|a_0^{-\frac{1}{2}}X_2 + a_0^{-\frac{1}{2}}b_0^{-1}\Phi(X_1)\right\|^2\\ &+\frac{1}{2}\|X_3 + a_0X_2 + \alpha a_0 b_0 X_1\|^2\\ &+\frac{1}{2}(\mu_1+\mu_2)\|X_1\|^2 + \frac{1}{2}(a_0\varepsilon + \mu_3)\|X_2\|^2\\ &+2\lambda\int_{-\tau}^0\int_{t+s}^t \|X_2(\theta)\|^2\, d\theta ds + 2\eta\int_{-\tau}^0\int_{t+s}^t \|X_3(\theta)\|^2\, d\theta ds.\end{aligned} \tag{11.9}$$

It can be obtained from the first four terms of (11.9) that there exist sufficiently small positive constants ℓ_i, $(i = 1, 2, 3)$, such that

$$V_0 \ge \ell_1\|X_1\|^2 + \ell_2\|X_2\|^2 + \ell_3\|X_3\|^2.$$

Let

$$\ell_4 = \min\{\ell_1, \ell_2, \ell_3\}.$$

Then

$$V_0 \geq \ell_4(\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2).$$

Therefore, we can conclude that the Lyapunov–Krasovskii functional V_0 is positive definite.

Differentiating the Lyapunov–Krasovskii functional $V_0(t)$ along any solution $(X_1(t), X_2(t), X_3(t))$ of (11.6), it follows from (11.7) and (11.6) that

$$\begin{aligned}
\dot{V}_0(t) = & -\alpha a_0 b_0 \langle \Phi(X_1), X_1\rangle - a_0 \langle G(X_2), X_2\rangle + \langle J_\Phi(X_1)X_2, X_2\rangle \\
& +\alpha a_0^2 b_0 \|X_2\|^2 - \alpha a_0 b_0 \langle X_1, H(X_2)X_3\rangle + \alpha a_0^2 b_0 \langle X_1, X_3\rangle \\
& - \langle H(X_2)X_3, X_3\rangle + a_0 \|X_3\|^2 - \alpha a_0 b_0 \langle X_1, G(X_2)\rangle \\
& +\alpha a_0 b_0^2 \langle X_1, X_2\rangle + \left\langle X_3, \int_{t-\tau}^{t} J_G(X_2(s))X_3(s)ds \right\rangle \\
& + \left\langle X_3, \int_{t-\tau}^{t} J_\Phi(X_1(s))X_2(s)ds \right\rangle + \alpha a_0 b_0 \left\langle X_1, \int_{t-\tau}^{t} J_G(X_2(s)X_3(s)ds \right\rangle \\
& +\alpha a_0 b_0 \left\langle X_1, \int_{t-\tau}^{t} J_\Phi(X_1(s))X_2(s)ds \right\rangle + a_0 \left\langle X_2, \int_{t-\tau}^{t} J_G(X_2(s))X_3(s)ds \right\rangle \\
& +a_0 \left\langle X_2, \int_{t-\tau}^{t} J_\Phi(X_1(s))X_2(s)ds \right\rangle + \lambda\tau \|X_2\|^2 + \eta\tau \|X_3\|^2 \\
& -\lambda \int_{t-\tau}^{t} \|X_2(\theta)\|^2 d\theta - \eta \int_{t-\tau}^{t} \|X_3(\theta)\|^2 d\theta.
\end{aligned}$$

From (C1) and (C2), we find

$$\begin{aligned}
-\alpha a_0 b_0 \langle \Phi(X_1), X_1\rangle &= -\alpha a_0 b_0 \int_0^1 \langle J_\Phi(\sigma_1 X_1)X_1, X_1\rangle d\sigma_1 \\
&\leq -\alpha a_0 b_0 c \|X_1\|^2
\end{aligned}$$

and

$$\langle J_\Phi(X_1)X_2, X_2\rangle \leq c_0 \|X_2\|^2 .$$

Then

$$\begin{aligned}
\dot{V}_0(t) \leq & -\frac{1}{2}\alpha a_0 b_0 c \left\|X_1\right\|^2 - \langle a_0 G(X_2), X_2\rangle \\
& + \left\langle (c_0 I + \alpha a_0^2 b_0 I) X_2, X_2 \right\rangle - \langle (H(X_2) - a_0 I) X_3, X_3\rangle \\
& - \frac{1}{4}\alpha a_0 b_0 \left\| c^{\frac{1}{2}} X_1 + 2c^{-\frac{1}{2}} (H(X_2) - a_0 I) X_3 \right\|^2 \\
& + \frac{1}{4}\alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}} (H(X_2) - a_0 I) X_3 \right\|^2 \\
& - \frac{1}{4}\alpha a_0 b_0 \left\| c^{\frac{1}{2}} X_1 + 2c^{-\frac{1}{2}} (G(X_2) X_2 - b_0 X_2) \right\|^2 \\
& + \frac{1}{4}\alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}} (G(X_2) X_2 - b_0 X_2) \right\|^2 \\
& + \left\langle X_3, \int_{t-\tau}^{t} J_G(X_2(s)) X_3(s) ds \right\rangle + \left\langle X_3, \int_{t-\tau}^{t} J_\Phi(X_1(s)) X_2(s) ds \right\rangle \\
& + \alpha a_0 b_0 \left\langle X_1, \int_{t-\tau}^{t} J_G(X_2(s) X_3(s) ds \right\rangle \\
& + \alpha a_0 b_0 \left\langle X_1, \int_{t-\tau}^{t} J_\Phi(X_1(s)) X_2(s) ds \right\rangle \\
& + a_0 \left\langle X_2, \int_{t-\tau}^{t} J_G(X_2(s)) X_3(s) ds \right\rangle \\
& + a_0 \left\langle X_2, \int_{t-\tau}^{t} J_\Phi(X_1(s)) X_2(s) ds \right\rangle + \lambda\tau \left\|X_2\right\|^2 + \eta\tau \left\|X_3\right\|^2 \\
& - \lambda \int_{t-\tau}^{t} \left\|X_2(\theta)\right\|^2 d\theta - \eta \int_{t-\tau}^{t} \left\|X_3(\theta)\right\|^2 d\theta.
\end{aligned} \tag{11.10}$$

Assumptions (C1) and (C2), imply that

$$\begin{aligned}
\langle a_0 G(X_2), X_2\rangle &= \int_0^1 \langle a_0 J_G(\sigma X_2) X_2, X_2\rangle \, d\sigma \\
&\geq a_0 b_0 \left\|X_2\right\|^2,
\end{aligned}$$

$$\begin{aligned}
\langle a_0 G(X_2), X_2\rangle & - \left\langle (c_0 I + \alpha a_0^2 b_0 I) X_2, X_2 \right\rangle \\
& \geq (a_0 b_0 - c_0 - \alpha a_0^2 b_0) \left\|X_2\right\|^2,
\end{aligned}$$

$$\left\langle X_3, \int_{t-\tau}^{t} J_G(X_2(s))X_3(s)ds \right\rangle \le \|X_3\| \int_{t-\tau}^{t} \|J_G(X_2(s))\| \, \|X_3(s)\| \, ds$$
$$\le \sqrt{n}b_1 \|X_3\| \int_{t-\tau}^{t} \|X_3(s)\| \, ds$$
$$\le \frac{1}{2}\sqrt{n}b_1 \int_{t-\tau}^{t} \{\|X_3(t)\|^2 + \|X_3(s)\|^2\} ds$$
$$= \frac{1}{2}\sqrt{n}b_1\tau \|X_3\|^2 + \frac{1}{2}\sqrt{n}b_1 \int_{t-\tau}^{t} \|X_3(s)\|^2 \, ds,$$

$$\left\langle X_3, \int_{t-\tau}^{t} J_\Phi(X_1(s))X_2(s)ds \right\rangle \le \|X_3\| \int_{t-\tau}^{t} J_\Phi(X_1(s)) \, \|X_2(s)\| \, ds$$
$$\le \sqrt{n}c_0 \|X_3\| \int_{t-\tau}^{t} \|X_2(s)\| \, ds$$
$$\le \frac{1}{2}\sqrt{n}c_0\tau \|X_3\|^2 + \frac{1}{2}\sqrt{n}c_0 \int_{t-\tau}^{t} \|X_2(s)\|^2 \, ds,$$

$$\alpha a_0 b_0 \left\langle X_1, \int_{t-\tau}^{t} J_G(X_2(s)X_3(s)ds \right\rangle \le \alpha a_0 b_0 \|X_1\| \int_{t-\tau}^{t} \|J_G(X_2(s))\| \, \|X_3(s)\| \, ds$$
$$\le \frac{1}{2}\alpha a_0 b_0 b_1 \sqrt{n} \int_{t-\tau}^{t} \{\|X_1(t)\|^2 + \|X_3(s)\|^2\} ds$$
$$= \frac{1}{2}\alpha a_0 b_0 b_1 \tau \sqrt{n} \|X_1\|^2$$
$$+ \frac{1}{2}\alpha a_0 b_0 b_1 \sqrt{n} \int_{t-\tau}^{t} \|X_3(s)\|^2 \, ds,$$

$$\alpha a_0 b_0 \left\langle X_1, \int_{t-\tau}^{t} J_\Phi(X_1(s))X_2(s)ds \right\rangle \le \alpha a_0 b_0 c_0 \sqrt{n} \|X_1\| \int_{t-\tau}^{t} \|X_2(s)\| \, ds$$
$$\le \frac{1}{2}\alpha a_0 b_0 c_0 \tau \sqrt{n} \|X_1\|^2$$
$$+ \frac{1}{2}\alpha a_0 b_0 c_0 \sqrt{n} \int_{t-\tau}^{t} \|X_2(s)\|^2 \, ds,$$

$$
\begin{aligned}
& a_0 \left\langle X_2, \int_{t-\tau}^{t} J_G(X_2(s))X_3(s)ds \right\rangle \\
& \leq a_0 b_1 \|X_2\| \int_{t-\tau}^{t} \|J_G(X_2(s))\| \, \|X_3(s)\| \, ds \\
& \leq \frac{1}{2} a_0 b_1 \tau \sqrt{n} \|X_2\|^2 + \frac{1}{2} a_0 b_1 \sqrt{n} \int_{t-\tau}^{t} \|X_3(s)\|^2 \, ds,
\end{aligned}
$$

$$
\begin{aligned}
a_0 \left\langle X_2, \int_{t-\tau}^{t} J_\Phi(X_1(s))X_2(s)ds \right\rangle & \leq a_0 c_0 \sqrt{n} \|X_2\| \int_{t-\tau}^{t} \|X_2(s)\| \, ds \\
& \leq \frac{1}{2} a_0 c_0 \sqrt{n} \int_{t-\tau}^{t} \{\|X_2(t)\|^2 + \|X_2(s)\|^2\} ds \\
& = \frac{1}{2} a_0 c_0 \tau \sqrt{n} \|X_2\|^2 + \frac{1}{2} a_0 c_0 \sqrt{n} \int_{t-\tau}^{t} \|X_2(s)\|^2 \, ds.
\end{aligned}
$$

Gathering all these estimates into (11.10) and rearranging we deduce that

$$
\begin{aligned}
\dot{V}_0(t) \leq & -\frac{1}{2} \alpha a_0 b_0 c \|X_1\|^2 - (a_0 b_0 - c_0 - \alpha a_0^2 b_0) \|X_2\|^2 \\
& - \langle (H(X_2) - a_0 I) X_3, X_3 \rangle \\
& - \frac{1}{4} \alpha a_0 b_0 \left\| c^{\frac{1}{2}} X_1 + 2c^{-\frac{1}{2}} (H(X_2) - a_0 I) X_3 \right\|^2 \\
& + \frac{1}{4} \alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}} (H(X_2) - a_0 I) X_3 \right\|^2 \\
& - \frac{1}{4} \alpha a_0 b_0 \left\| c^{\frac{1}{2}} X_1 + 2c^{-\frac{1}{2}} (G(X_2) - b_0 I) X_2 \right\|^2 \\
& + \frac{1}{4} \alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}} (G(X_2) - b_0 I) X_2 \right\|^2 \\
& + \frac{1}{2} \alpha a_0 b_0 b_1 \tau \sqrt{n} \|X_1\|^2 + \frac{1}{2} \alpha a_0 b_0 c_0 \tau \sqrt{n} \|X_1\|^2 \\
& + \frac{1}{2} a_0 b_1 \tau \sqrt{n} \|X_2\|^2 + \frac{1}{2} a_0 c_0 \tau \sqrt{n} \|X_2\|^2 \\
& + \frac{1}{2} b_1 \tau \sqrt{n} \|X_3\|^2 + \frac{1}{2} c_0 \tau \sqrt{n} \|X_3\|^2 + \lambda \tau \|X_2\|^2 + \eta \tau \|X_3\|^2 \\
& - \{\lambda - \frac{1}{2} (a_0 + \alpha a_0 b_0 + 1) c_0 \sqrt{n}\} \int_{t-\tau}^{t} \|X_2(s)\|^2 \, ds \\
& - \{\eta - (1 + a_0 + \frac{1}{2} \alpha a_0 b_0) b_1 \sqrt{n}\} \int_{t-\tau}^{t} \|X_3(s)\|^2 \, ds.
\end{aligned}
$$

Let

$$
\lambda = \frac{1}{2} (a_0 + \alpha a_0 b_0 + 1) c_0 \sqrt{n} \text{ and } \eta = (1 + a_0 + \frac{1}{2} \alpha a_0 b_0) b_1 \sqrt{n}.
$$

Hence, we obtain

$$\begin{aligned}\dot{V}_0(t) \le & -\frac{1}{2}\alpha a_0 b_0 c \|X_1\|^2 - (a_0 b_0 - c_0 - \alpha a_0^2 b_0)\|X_2\|^2 \\ & - \langle (H(X_2) - a_0 I)X_3, X_3\rangle \\ & + \frac{1}{4}\alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}}(H(X_2) - a_0 I)X_3 \right\|^2 \\ & + \frac{1}{4}\alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}}(G(X_2) - b_0 I)X_2 \right\|^2 \\ & + \frac{1}{2}(\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c_0)\tau\sqrt{n}\|X_1\|^2 \\ & + \frac{1}{2}(a_0 b_1 + a_0 c_0)\tau\sqrt{n}\|X_2\|^2 \\ & + \frac{1}{2}(a_0 + \alpha a_0 b_0 + 1)c_0\sqrt{n}\tau\|X_2\|^2 \\ & + \frac{1}{2}b_1\tau\sqrt{n}\|X_3\|^2 + \frac{1}{2}c_0\tau\sqrt{n}\|X_3\|^2 \\ & + (1 + a_0 + \frac{1}{2}\alpha a_0 b_0)b_1\sqrt{n}\tau\|X_3\|^2 .\end{aligned}$$

In view the facts

$$\begin{aligned}& \frac{1}{4}\alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}}(G(X_2) - b_0 I)X_2 \right\|^2 \\ & = \alpha a_0 b_0 \left\langle c^{-1}(G(X_2) - b_0 I)X_2, (G(X_2) - b_0 I)X_2 \right\rangle\end{aligned}$$

and

$$\begin{aligned}& \frac{1}{4}\alpha a_0 b_0 \left\| 2c^{-\frac{1}{2}}(H(X_2) - a_0 I)X_3 \right\|^2 \\ & = \alpha a_0 b_0 \left\langle c^{-1}(H(X_2) - a_0 I)X_3, (H(X_2) - a_0 I)X_3 \right\rangle ,\end{aligned}$$

it follows that

$$\begin{aligned}\dot{V}_0(t) \le & -\frac{1}{2}\alpha a_0 b_0 c \|X_1\|^2 - (a_0 b_0 - c_0 - \alpha a_0^2 b_0)\|X_2\|^2 \\ & - \langle (H(X_2) - a_0 I)X_3, X_3\rangle \\ & + \alpha a_0 b_0 \left\langle c^{-1}(H(X_2) - a_0 I)X_3, (H(X_2) - a_0 I)X_3 \right\rangle \\ & + \alpha a_0 b_0 \left\langle c^{-1}(G(X_2) - b_0 I)X_2, (G(X_2) - b_0 I)X_2 \right\rangle \\ & + \frac{1}{2}(\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c_0)\sqrt{n}\tau\|X_1\|^2 \\ & + \frac{1}{2}(a_0 b_1 + a_0 c_0)\sqrt{n}\tau\|X_2\|^2\end{aligned}$$

$$+\frac{1}{2}(a_0 + \alpha a_0 b_0 + 1)c_0\sqrt{n}\tau \,\|X_2\|^2$$
$$+(\frac{3}{2} + a_0 + \frac{c_0}{2b_1} + \frac{1}{2}\alpha a_0 b_0)b_1\sqrt{n}\tau \,\|X_3\|^2 .$$

By Lemma 11.2 and (C1) and (C2), we can obtain

$$\begin{aligned}
\dot{V}_0(t) \le & -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c_0)\sqrt{n}\tau\} \,\|X_1\|^2 \\
& -\big\langle\{(a_0 b_0 - c_0) - \alpha a_0 b_0[a_0 I + c^{-1}(G(X_2) - b_0 I)^2]\}X_2, X_2\big\rangle \\
& +\frac{1}{2}(a_0 b_1 + 2a_0 c_0 + \alpha a_0 b_0 c_0 + c_0)\sqrt{n}\tau \,\|X_2\|^2 \\
& -\big\langle\{(H(X_2) - a_0 I)[I - \alpha a_0 b_0 c^{-1}(H(X_2) - a_0 I)]\}X_3, X_3\big\rangle \\
& +(\frac{3}{2} + a_0 + \frac{c_0}{2b_1} + \frac{1}{2}\alpha a_0 b_0)b_1\sqrt{n}\tau \,\|X_3\|^2 \\
\le & -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c_0)\sqrt{n}\tau\} \,\|X_1\|^2 \\
& -\{(a_0 b_0 - c_0) - \alpha a_0 b_0[a_0 + c^{-1}(b_1 - b_0)^2]\} \,\|X_2\|^2 \\
& +\frac{1}{2}(a_0 b_1 + 2a_0 c_0 + \alpha a_0 b_0 c_0 + c_0)\sqrt{n}\tau \,\|X_2\|^2 \\
& -\varepsilon[1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2] \,\|X_3\|^2 \\
& +(\frac{3}{2} + a_0 + \frac{c_0}{2b_1} + \frac{1}{2}\alpha a_0 b_0)b_1\sqrt{n}\tau \,\|X_3\|^2 .
\end{aligned}$$

Let

$$\ell_5 = 2(a_0 b_0 - c_0) - \alpha a_0 b_0[a_0 + c^{-1}(b_1 - b_0)^2] > 0$$

and

$$\ell_6 = 2\varepsilon[1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)^2] > 0.$$

Hence,

$$\begin{aligned}
\dot{V}_0(t) \le & -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c_0)\sqrt{n}\tau\} \,\|X_1\|^2 \\
& -\frac{1}{2}\{\ell_5 - [((a_0 b_1 + 2a_0 c_0 + \alpha a_0 b_0 c_0 + c_0)]\sqrt{n}\tau\} \,\|X_2\|^2 \\
& -\frac{1}{2}\{\ell_6 - (\frac{3}{2} + a_0 + \frac{c_0}{2b_1} + \frac{1}{2}\alpha a_0 b_0)b_1\sqrt{n}\tau\} \,\|X_3\|^2 .
\end{aligned}$$

If

$$\tau < \min\left\{\frac{\alpha a_0 b_0 c}{\Delta_1}, \frac{\ell_5}{\Delta_2}, \frac{2\ell_6}{\Delta_3}\right\},$$

then, for some positive constants ℓ_7, ℓ_8, and ℓ_9,

$$\dot{V}_0(t) \leq -\ell_7 \|X_1\|^2 - \ell_8 \|X_2\|^2 - \ell_9 \|X_3\|^2 \leq 0,$$

where

$$\Delta_1 = \alpha a_0 b_0 (b_1 + c_0)\sqrt{n}, \quad \Delta_2 = (a_0 b_1 + 2a_0 c_0 + \alpha a_0 b_0 c_0 + c_0)]\sqrt{n},$$

$$\Delta_3 = (3b_1 + 2a_0 b_1 + c_0 + \alpha a_0 b_0 b_1)\sqrt{n}.$$

In addition, we can conclude that

$$V_0(X_1, X_2, X_3) \to \infty \text{ as } \|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \to \infty.$$

Consider the set defined by

$$\Omega \equiv \{(X_1, X_2, X_3) : \dot{V}_0(X_1, X_2, X_3) = 0\}.$$

If we apply the LaSalle's invariance principle, then $(X_1, X_2, X_3) \in \Omega$ implies that $X_1 = X_2 = X_3 = 0$. Clearly, this result implies that the largest invariant set contained in Ω is $(0, 0, 0) \in \Omega$. By Lemma 11.2, we conclude that the zero solution of (11.6) is globally asymptotically stable. Hence, all solutions of Eq. (11.5) are bounded and the zero solution of Eq. (11.5) is globally asymptotically stable. This proves Theorem 11.2. □

11.3 Boundedness

Our second result is for the case where $E(.) \neq 0$.

Assume that the following condition holds:

(C3) $\|E(t, X_1, X_2, X_3)\| \leq e(t)$ for all $t \geq 0$, $\max e(t) < \infty$ and $e \in L^1(0, \infty)$,

where $L^1(0, \infty)$ denotes the space of Lebesgue integrable functions.

Theorem 11.3 *Assume that $E(.) \neq 0$ and conditions $(C1)$, $(C2)$, and $(C3)$ hold. If*

$$\tau < \min\left\{\frac{\alpha a_0 b_0 c}{\Delta_1}, \frac{\ell_5}{\Delta_2}, \frac{2\ell_6}{\Delta_3}\right\},$$

then there exists a constant $K > 0$ such that any solution $(X_1(t), X_2(t), X_3(t))$ of (11.6) determined by

$$X_1(0) = X_{10}, \quad X_2(0) = X_{20}, \quad X_3(0) = X_{30}$$

satisfies

$$\|X_1(t)\| \le K, \quad \|X_2(t)\| \le K, \quad \|X_3(t)\| \le K$$

for all $t \in \Re^+$.

Proof Let $E(.) = E(t, X_1, X_2, X_3) \neq 0$. If assumptions (C1), (C2), and (C3) hold, then we can obtain

$$\begin{aligned}
\dot{V}_0(t) &\le -\frac{1}{2}\{\alpha a_0 b_0 c - (\alpha a_0 b_0 b_1 + \alpha a_0 b_0 c_0)\sqrt{n}\tau\} \|X_1\|^2 \\
&\quad -\frac{1}{2}\{\ell_5 - [((a_0 b_1 + 2a_0 c_0 + \alpha a_0 b_0 c_0 + c_0)]\sqrt{n}\tau\} \|X_2\|^2 \\
&\quad -\frac{1}{2}\{\ell_6 - (\frac{3}{2} + a_0 + \frac{c_0}{2b_1} + \frac{1}{2}\alpha a_0 b_0) b_1 \sqrt{n}\tau\} \|X_3\|^2 \\
&\quad + \langle X_3, E(.)\rangle + \alpha a_0 b_0 \langle X_1, E(.)\rangle + a_0 \langle X_2, E(.)\rangle \\
&\le -\ell_7 \|X_1\|^2 - \ell_8 \|X_2\|^2 - \ell_9 \|X_3\|^2 \\
&\quad + (\alpha a_0 b_0 \|X_1\| + a_0 \|X_2\| + \|X_3\|) \|E(.)\| \\
&\le \ell(\|X_1\| + \|X_2\| + \|X_3\|) \|E(.)\| \\
&\le \ell(3 + \|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2) e(t),
\end{aligned}$$

where

$$\ell = \max\{\alpha a_0 b_0, a_0, 1\}.$$

It is obvious that

$$\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \le \ell_4^{-1} V_0.$$

Then

$$\dot{V}_0(t) \le 3\ell e(t) + \ell \ell_4^{-1} V_0(t) e(t).$$

Integrating both sides of the last estimate from 0 to t $(t \ge 0)$, we have

$$V_0(t) \le V_0(0) + 3\ell \int_0^t e(s) ds + \ell \ell_4^{-1} \int_0^t V_0(s) e(s) ds.$$

Let

$$M = V_0(0) + 3\ell \int_0^\infty e(s) ds.$$

Then

$$V_0(t) \le M + \ell \ell_4^{-1} \int_0^\infty V_0(s) e(s) ds.$$

From the Gronwall-Bellman inequality, we can get

$$V_0(t) \leq M \exp(\ell \ell_4^{-1} \int_0^\infty e(s)ds).$$

In view of $\|X_1\|^2 + \|X_2\|^2 + \|X_3\|^2 \leq \ell_4^{-1} V_0$ and the assumption $e \in L^1(0, \infty)$, we can conclude that all solutions of (11.6) are bounded. The proof of Theorem 11.3 is complete. □

11.4 Ultimately Boundedness

Our last result is for the case where $E(.) \neq 0$.

Assume that the following condition holds:

(C4) $\|E(t, X_1, X_2, X_3)\| \leq \Delta$ for all $t \geq 0$, where Δ is a positive constant.

Theorem 11.4 *Assume that* $E(.) \neq 0$ *and conditions* $(C1)$*,* $(C2)$*, and* $(C4)$ *hold. If*

$$\tau < \min\left\{ \frac{\alpha a_0 b_0 c}{\Delta_1}, \frac{\ell_5}{\Delta_2}, \frac{2\ell_6}{\Delta_3} \right\},$$

then there exists a constant $K_1 > 0$ *such that any solution* $(X_1(t), X_2(t), X_3(t))$ *of (11.6) determined by*

$$X_1(0) = X_{10}, \quad X_2(0) = X_{20}, \quad X_3(0) = X_{30}$$

ultimately satisfies

$$\|X_1(t)\|^2 + \|X_2(t)\|^2 + \|X_3(t)\|^2 \leq K_1$$

for all $t \in \Re^+$.

Proof Let $E(.) = E(t, X_1, X_2, X_3) \neq 0$. If assumptions (C1), (C2), and (C4) hold, then we can arrive at

$$\begin{aligned}
\dot{V}_0(t) \leq & -\ell_7 \|X_1\|^2 - \ell_8 \|X_2\|^2 - \ell_9 \|X_3\|^2 \\
& + (\alpha a_0 b_0 \|X_1\| + a_0 \|X_2\| + \|X_3\|) \|E(.)\| \\
\leq & -\ell_7 \|X_1\|^2 - \ell_8 \|X_2\|^2 - \ell_9 \|X_3\|^2 \\
& + (\alpha a_0 b_0 \delta_0 \|X_1\| + a_0 \delta_0 \|X_2\| + \delta_0 \|X_3\|).
\end{aligned}$$

The remaining of the proof can be completed by following a similar procedure as shown in Omeike [30]. Therefore, we omit the details of the proof. □

Conclusion

A class of nonlinear vector functional differential equations of third order with a constant delay has been considered. Qualitative properties of solutions like globally asymptotically stability/boundedness/ultimately boundedness of solutions have been investigated. The technique of proofs involves defining an appropriate Lyapunov–Krasovskii functional. Our results include and improve some recent results in the literature.

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