Chapter 1
Hesitant Fuzzy Set and Its Extensions

As uncertainty takes place almost everywhere in our daily life, many different tools have been developed to recognize, represent, manipulate, and tackle such uncertainty. Among the most popular theories to handle uncertainty include the probability theory and the fuzzy set theory, which are proposed to interpret statistical uncertainty and fuzzy uncertainty, respectively. These two types of models possess philosophically different kinds of information: the probability theory conveys information about relative frequencies, while the fuzzy set theory represents similarities of objects to the imprecisely defined properties (Bezdek 1993). Since it was originally introduced by Zadeh (1965), the fuzzy set has turned out to be one of the most efficient decision aid techniques providing the ability to deal with uncertainty and vagueness. After the pioneering work of Zadeh (1965), the fuzzy set theory has been extended in a number of directions, the most impressive one of which relates to the representation of the membership grades of the underlying fuzzy set (Yager 2014). Recently, on the basis of the extensional forms of fuzzy set, Torra (2010) proposed a new generalized type of fuzzy set called hesitant fuzzy set (HFS), which opens new perspectives for further research on decision making under hesitant environments.

HFS shows many advantages over traditional fuzzy set and its other extensions, especially in group decision making with anonymity. The HFS has attracted many scholars' attentions. Torra (2010) firstly gave the concept of HFS, and defined the complement, union and intersection of HFSs. Furthermore, Torra and Narukawa (2009) presented an extension principle permitting to generalize the existing operations on fuzzy sets to HFSs, and described the application of this new type of set in the framework of decision making. Xu and Xia (2011a, b) originally gave the mathematical expressions of HFS, and investigated the distance, similarity and correlation measures for HFSs. Torra (2010) also established the relationship between HFS and intuitionistic fuzzy set (IFS), based on which, Xia and Xu (2011a) gave some operational laws for HFSs, such as the addition and multiplication operations. Afterwards, Liao and Xu (2014a) introduced the subtraction and division operations over HFSs.

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In this chapter, we first introduce the HFS and its operations, and then give the subtraction and division operations over HFSs. The motivation of introducing these operations for HFSs is based on the relationship between HFS and IFS: HFS encompasses IFS as a particular case and the envelope of a HFS is an IFS (Torra 2010). Several operational laws of these two operations over HFSs are given. The relationship between IFS and HFS is further verified in terms of these two operations. In addition, the relationships between these two operations are established. We also discuss the comparison laws for HFSs. HFS has been extended into different forms, such as the interval-valued hesitant fuzzy set (IVHFS) (Chen et al. 2013b), the dual hesitant fuzzy set (DHFS) (Zhu et al. 2012) and the hesitant fuzzy linguistic term set (Rodríguez et al. 2012). In this chapter, we also introduce the definitions, the operational laws and the comparison laws of these extended HFSs.

1.1 Hesitant Fuzzy Set

1.1.1 Introduction to Hesitant Fuzzy Set

Zadeh (1965) introduced the concept of fuzzy set, which leads to a completely new and very active research area today named as fuzzy logic.

Definition 1.1 (*Zadeh* 1965). An ordinary fuzzy set F in a set X is characterized by a membership function μ_F which takes the values in the interval $[0,1]$, i.e., $\mu_F : X \to [0,1]$. The value of μ_F at x, $\mu_F(x)$, named fuzzy number, represents the grade of membership (grade, for short) of x in F and is a point in $[0,1]$.

For example, we can use the fuzzy set

$$\begin{aligned} F &= \mu_F(x_1)/x_1 + \mu_F(x_2)/x_2 + + \mu_F(x_3)/x_3 + \mu_F(x_4)/x_4 \\ &= 1/0 + 0.9/0.1 + 0.7/0.2 + 0.4/0.3 \end{aligned} \tag{1.1}$$

to represent the linguistic term “low”, where the operation “ + ” stands for logical sum (or).

As the membership grades in a fuzzy set are expressed as precise values drawn from the unit interval $[0,1]$, the fuzzy set cannot capture the human ability in expressing imprecise and vague membership grades of a fuzzy set. On the one hand, in realistic decision making, imprecision may arise due to the unquantifiable information, incomplete information, unobtainable information, partial ignorance, and so forth. To cope with imperfect and imprecise information that two or more sources of vagueness appear simultaneously, the traditional fuzzy set shows some limitations. It uses a crisp number in unit interval [0,1] as a membership degree of an element to a set; however, very often, such a crisp number is difficult to be determined by a decision maker (or an expert). On the other hand, if a group of decision makers (or experts) are asked to evaluate the candidate alternatives, they often find some disagreements among themselves. Since the decision makers (or experts) may have different opinions over the alternatives and they cannot persuade each other easily, a consensus result is hard to be obtained but a set of possible values. In such a case, the traditional fuzzy set cannot be used to depict the group's opinions. Hence, the classical fuzzy set has been extended into several different forms, such as the IFS (Atanassov 1986), the interval-valued IFS (Atanassov and Gargov 1989), the type 2 fuzzy set (Mizumoto and Tanaka 1976), the type n fuzzy set (Dubois and Prade 1980), and the fuzzy multisets (also named the fuzzy bags) (Yager 1986). All these extensions are based on the same rationale that it is not clear to assign the membership degree of an element to a fixed set.

The IFS, which assigns to each element a membership degree, a non-membership degree and a hesitancy degree, is more powerful than fuzzy set in dealing with vagueness and uncertainty.

Definition 1.2 (*Atanassov* 1983, 2012). Let a crisp set X be fixed and let $A \subset X$ be a fixed set. An IFS A^* on X is an object of the following form:

$$A^* = \{ <x, \mu_A(x), v_A(x)> | x \in X \} \tag{1.2}$$

where the functions $\mu_A : A \to [0, 1]$ and $v_A : A \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ to the set A, respectively, and for every $x \in X$

$$0 \le \mu_A + v_A \le 1 \tag{1.3}$$

Obviously, every ordinary fuzzy set has the form:

$$A^* = \{ <x, \mu_A(x), 1 - \mu_A(x)> | x \in X \} \tag{1.4}$$

That is to say, the ordinary fuzzy set is a special case of IFS.

For each IFS A^* on X

$$\pi_A(x) = 1 - \mu_A(x) - v_A(x) \tag{1.5}$$

is called the degree of non-determinacy (uncertainty) of the membership of the element $x \in X$ to the set A. In the case of ordinary fuzzy sets, $\pi_A(x) = 0$ for every $x \in X$.

However, when giving the membership degree of an element to a set, the difficulty of establishing the membership degree is not because we have some possibility distribution (as in type 2 fuzzy set), or a margin of error (as in interval fuzzy set and IFS), but because we have a set of possible values. In such cases, HFS, as a generalization of fuzzy set, permits the membership degree of an element to a set presented by several possible values between 0 and 1. It can better describe the situations where people have hesitancy in providing their preferences over objects in the process of decision making. The HFS was originally proposed by Torra (2010).

Definition 1.3 (*Torra* 2010). Let X be a fixed set, a HFS on X is in terms of a function h that when applied to X returns a subset of $[0, 1]$.

To be easily understood, Xia and Xu (2011a) represented the HFS in terms of the following mathematical symbol:

$$H = \{ <x, \ h_A(x) > | x \in X\} \tag{1.6}$$

where $h_A(x)$ is a set of values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set $A \subset X$. For convenience, Xia and Xu (2011b) called $h_A(x)$ a hesitant fuzzy element (HFE), which denotes a basic component of the HFS.

Since the possible values of the membership degree in a HFS are random, the HFS is, to some extent, more natural in representing the fuzziness and vagueness than all the other extensional forms of fuzzy set. On the one hand, it is very close to human's cognitive process by using HFS. It is noted that modeling fuzzy information by other extended forms of fuzzy set is based on the elicitation of single or interval values that should encompass and express the information provided by the decision makers (or experts) when determining the membership of an element to a given set. Nevertheless, in some cases, the decision makers (or experts) involved in the problem may have a set of possible values, and thus cannot provide a single or an interval value to express their preferences or assessments because they are thinking of several possible values at the same time. In such a case, the HFS, whose membership degree is represented by a set of possible values, can solve this problem perfectly, while the other extensions of fuzzy set are invalid.

On the other hand, due to the increasing complexity of socio-economic environments, it is less and less possible for single decision maker (or expert) to consider all relevant aspects of a problem when evaluating the considered objects. Hence, in order to get a more reasonable decision result, a decision organization, such as the board of directors of a company, which contains a collection of decision makers (or experts), is set up explicitly or implicitly to assess the alternatives. As pointed by Yu (1973), "*when a group of individuals intend to form a corporation with themselves as the shareholders or form a union to increase their total bargaining power, they usually find some disagreements among themselves. The disagreements come from the difference in their subjective evaluations of the decision making problems which arise*." Since the decision makers (or experts) may have different opinions over the alternatives due to their different knowledge backgrounds or benefits and they cannot persuade each other easily, a consensus evaluation result is sometimes hard to obtain but several possible evaluation values. Then the HFS is suitable to handle this issue, and it is more powerful than all the other extended fuzzy sets. For example, suppose that a decision organization is asked to provide the degree to which an alternative is superior to another, and the decision makers prefer to use the values between 0 and 1 to express their preferences. Some decision makers in the organization provide 0.2, some provide 0.6, and the others provide 0.8. These three parts cannot persuade each other, and thus, the degree to which the alternative is superior to the other can be represented by the hesitant fuzzy element (HFE) $\{0.2, 0.6, 0.8\}$. Note that the HFE $\{0.2, 0.6, 0.8\}$ can describe the above situation more objectively than the crisp number 0.2 (or 0.6 or 0.8), or the interval-valued fuzzy number $[0.2, 0.8]$, or the intuitionistic fuzzy number $(0.2, 0.8)$, because the degrees to which an alternative is superior to another are not the convex combination of 0.2 and 0.8, or the interval between 0.2 and 0.8, but just three possible values 0.2, 0.6 and 0.8. If we use any of the extended fuzzy sets to represent the assessments given by these three parts of the decision organization, much useful information may be lost and this may lead to an unreasonable decision. Therefore, it is more suitable and powerful to describe the uncertain evaluation information by HFS.

The HFS encompasses IFS as a particular case, and it is a particular case of type 2 fuzzy set. The typical HFS is the one where $h(x)$ is finite. Torra (2010) gave some special HFEs for x in X:

(1) Empty set: $h(x) = \{0\}$, denoted as O^* for simplicity.
(2) Full set: $h(x) = \{1\}$, denoted as E^*.
(3) Complete ignorance (all is possible): $h(x) = [0, 1]$, denoted as U^*.
(4) Nonsense set: $h(x) = \emptyset^*$.

Liao and Xu (2014a) made some deep clarifications on these special HFEs from the view points of the definition of HFS and also from the practical decision making process. As presented in the definition, the HFS on a reference set X is in terms of a function h that when applied to X returns a subset of $[0, 1]$. Hence, if the HFS h returns no value, it is adequate for us to assert that h is a nonsense set. Analogously, if it returns the set $[0, 1]$, which means all values between 0 and 1 are possible, we call it complete ignorance. Particularly, if it returns only one value $\gamma \in [0, 1]$, this certainly makes sense because single value $\gamma \in [0, 1]$ can also be seen as a subset of $[0, 1]$, i.e., we can take γ as $[\gamma, \gamma]$. When $\gamma = 0$, which means the membership degree is zero, then we call it the empty set; if $\gamma = 1$, then we call it the full set. Note that we shall not take the empty set as the set that there is no any value in it, and we also should not take the full set as the set of all possible values. This is the difference between the HFS and the traditional set. The interpretation of these four special HFEs in decision making process is obvious. Consider that an organization with several experts from different areas evaluates an alternative using HFS. The empty set depicts that all experts oppose the alternative. The full set means that all experts agree with it. The complete ignorance represents that all experts have no idea on the alternative, and the nonsense set implies nonsense.

Given an intuitionistic fuzzy number (IFN) (Xu 2007b)$(x, \mu_A(x), v_A(x))$, its corresponding HFE is straightforward: $h(x) = [\mu_A(x), 1 - v_A(x)]$ if $\mu_A(x) \neq 1 - v_A(x)$. But, the construction of IFN from HFE is not so easy when the HFE contains more than one value for each $x \in X$. As for this issue, Torra (2010) pointed out that the envelope of a HFE is an IFN, expressed in the following definition:

Definition 1.4 (*Torra* 2010). Given a HFE h, the IFN $A_{env}(h)$ is defined as the envelope of h, where $A_{env}(h)$ can be represented as $(h^-, 1 - h^+)$, with $h^- = \min\{\gamma | \gamma \in h\}$ and $h^+ = \max\{\gamma | \gamma \in h\}$.

Definition 1.5 (*Liao et al.* 2015b). For a reference set X, let $h(x) = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a HFE with γ_k $(k = 1, 2, \ldots, l)$ being the possible membership grades of $x \in X$ to a given set and l being the number of values in $h(x)$. The mean of the HFE $h(x)$ is defined as:

$$\bar{h}(x) = \frac{1}{l}\sum\nolimits_{k=1}^{l} \gamma_k \tag{1.7}$$

Definition 1.6 (*Liao et al.* 2015b). For a reference set X, let $h(x) = \{\gamma_1, \gamma_2, \ldots, \gamma_l\}$ be a HFE with γ_k $(k = 1, 2, \ldots, l)$ being the possible membership grades of $x \in X$ to a given set and l being the number of values in $h(x)$. The hesitant degree of the HFE $h(x)$ is defined as:

$$\varphi_{h(x)} = \sqrt{\frac{1}{l}\sum\nolimits_{k=1}^{l} \left[\gamma_k - (\bar{h}(x))\right]^2} = \sqrt{\frac{1}{l}\sum\nolimits_{k=1}^{l} \left[\gamma_k - \left(\frac{1}{l}\sum\nolimits_{k=1}^{l} \gamma_k\right)\right]^2} \tag{1.8}$$

Example 1.1 (Liao et al. 2015b). For two HFEs $h_1 = \{0.1, 0.3, 0.5\}$ and $h_2 = \{0.1, 0.3, 0.8\}$, based on Eqs. (1.7) and (1.8), we have $\bar{h}_1 = 0.45$, $\bar{h}_2 = 0.6$, $\varphi_{h_1} = 0.2217$, and $\varphi_{h_2} = 0.3786$. Therefore, the HFE h_2 is more hesitant than the HFE h_1.

1.1.2 Operational Laws of Hesitant Fuzzy Elements

Torra (2010) defined some operations such as complement, union and intersection for HFEs:

Definition 1.7 (*Torra* 2010). For three HFEs h, h_1 and h_2, the following operations are defined:

(1) Lower bound: $h^-(x) = \min h(x)$.
(2) Upper bound: $h^+(x) = \max h(x)$.
(3) $h^c = \cup_{\gamma \in h}\{1 - \gamma\}$.
(4) $h_1 \cup h_2 = \left\{h \in h_1 \cup h_2 | h \geq \max(h_1^-, h_2^-)\right\}$.
(5) $h_1 \cap h_2 = \left\{h \in h_1 \cup h_2 | h \geq \min(h_1^+, h_2^+)\right\}$.

Afterwards, Xia and Xu (2011a) gave other forms of (4) and (5) as follows:

(6) $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$.
(7) $h_1 \cap h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$.

Torra (2010) further studied the relationships between HFEs and IFNs:

Proposition 1.1 (Torra 2010). *Let h, h_1 and h_2 be three HFEs. Then,*

(1) $A_{env}(h^c) = (A_{env}(h))^c$.
(2) $A_{env}(h_1 \cup h_2) = A_{env}(h_1) \cup A_{env}(h_2)$.
(3) $A_{env}(h_1 \cap h_2) = A_{env}(h_1) \cap A_{env}(h_2)$.

Proposition 1.2 (Torra 2010). *Let h_1 and h_2 be two HFEs with $h(x)$ being a nonempty convex set for all x in X, i.e., h_1 and h_2 are IFNs. Then,*

(1) *h_1^c is equivalent to IFS complement.*
(2) *$h_1 \cap h_2$ is equivalent to IFS intersection.*
(3) *$h_1 \cup h_2$ is equivalent to IFS union.*

Proposition 1.2 reveals that the operations defined for HFEs are consistent with the ones for IFNs. Based on the relationships between HFEs and IFNs, Xia and Xu (2011a) gave some operational laws for HFEs.

Definition 1.8 (*Xia and Xu* 2011a). Let h, h_1 and h_2 be three HFEs, and λ be a positive real number, then

(1) $h^{\lambda} = \cup_{\gamma \in h}\{\gamma^{\lambda}\}$.
(2) $\lambda h = \cup_{\gamma \in h}\left\{1-(1-\gamma)^{\lambda}\right\}$.
(3) $h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$.
(4) $h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1\gamma_2\}$.

Let $h_j (j = 1, 2, \ldots, n)$ be a collection of HFEs, Liao et al. (2014a) generalized (3) and (4) in Definition 1.8 to the following forms:

(5) $\overset{n}{\underset{j=1}{\oplus}} h_j = \cup_{\gamma_j \in h_j}\left\{1-\prod_{j=1}^{n}(1-\gamma_j)\right\}$.
(6) $\overset{n}{\underset{j=1}{\otimes}} h_j = \cup_{\gamma_j \in h_j}\left\{\prod_{j=1}^{n}\gamma_j\right\}$.

It is noted that the number of values in different HFEs may be different. Let l_{h_j} be the number of the HFE h_j. Based on the above operational laws, the following theorem holds:

Theorem 1.1 (Liao et al. 2014a). *Suppose h_1 and h_2 are two HFEs, then*

$$l_{h_1 \oplus h_2} = l_{h_1} \times l_{h_2},\ l_{h_1 \otimes h_2} = l_{h_1} \times l_{h_2} \tag{1.9}$$

Similarly, it also holds when there are n different HFEs, i.e.,

$$l_{\overset{n}{\underset{j=1}{\oplus}} h_j} = \prod_{j=1}^{n} l_{h_j},\ l_{\overset{n}{\underset{j=1}{\otimes}} h_j} = \prod_{j=1}^{n} l_{h_j} \tag{1.10}$$

Example 1.2 (Liao and Xu 2013). Let $h_1 = (0.1, 0.2, 0.7)$ and $h_2 = (0.2, 0.4)$ be two HFEs, then by the operational laws of HFSs given in Definition 1.8, we have

$$\begin{aligned} h_1 \oplus h_2 &= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\} \\ &= \{0.1+0.2-0.1*0.2, 0.1+0.4-0.1*0.4, 0.2+0.2-0.2*0.2, 0.2+0.4-0.2*0.4, \\ &\quad 0.7+0.2-0.7*0.2, 0.7+0.4-0.7*0.4\} = \{0.28, 0.36, 0.46, 0.52, 0.76, 0.82\} \end{aligned}$$

$$h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\} = \{0.1 * 0.2, 0.1 * 0.4, 0.2 * 0.2, 0.2 * 0.4, 0.7 * 0.2, 0.7 * 0.4\}$$
$$= \{0.02, 0.04, 0.04, 0.08, 0.14, 0.28\}$$

Thus, $l_{h_1 \oplus h_2} = 6 = 3 \times 2 = l_{h_1} \times l_{h_2}$, $l_{h_1 \otimes h_2} = 6 = 3 \times 2 = l_{h_1} \times l_{h_2}$.

Theorem 1.1 and Example 1.2 reveal that the dimension of the derived HFE may increase as the addition or multiplication operations are done, which may increase the complexity of calculation. In order not to increase the dimension of the derived HFE in the process of calculation, Liao et al. (2014a) adjusted the operational laws of HFEs into the following forms:

Definition 1.9 (*Liao et al.* 2014a). Let $h_j (j = 1, 2, \ldots, n)$ be a collection of HFEs, and λ be a positive real number, then

(1) $h^\lambda = \{(h^{\sigma(t)})^\lambda, t = 1, 2, \ldots, l\}$.
(2) $\lambda h = \{1 - (1 - h^{\sigma(t)})^\lambda, t = 1, 2, \ldots, l\}$.
(3) $h_1 \oplus h_2 = \{h_1^{\sigma(t)} + h_2^{\sigma(t)} - h_1^{\sigma(t)} h_2^{\sigma(t)}, t = 1, 2, \ldots, l\}$.
(4) $h_1 \otimes h_2 = \{h_1^{\sigma(t)} h_2^{\sigma(t)}, t = 1, 2, \ldots, l\}$.
(5) $\overset{n}{\underset{j=1}{\oplus}} h_j = \{1 - \prod_{j=1}^n (1 - h_j^{\sigma(t)}), t = 1, 2, \ldots, l\}$.
(6) $\overset{n}{\underset{j=1}{\otimes}} h_j = \{\prod_{j=1}^n h_j^{\sigma(t)}, t = 1, 2, \ldots, l\}$.

where $h_j^{\sigma(t)}$ is the tth smallest value in h_j.

Example 1.3 (Liao and Xu 2013). Let $h_1 = \{0.2, 0.3, 0.5, 0.8\}$ and $h_2 = \{0.4, 0.6, 0.8\}$ be two HFEs respectively. Taking addition and multiplication operations as an example, by using Definition 1.9, we have

$$h_1 \oplus h_2 = \left\{ h_1^{\sigma(t)} + h_2^{\sigma(t)} - h_1^{\sigma(t)} h_2^{\sigma(t)} \middle| t = 1, 2, 3, 4 \right\}$$
$$= \{0.2 + 0.4 - 0.2 \times 0.4, 0.3 + 0.5 - 0.3 \times 0.5, 0.5 + 0.6 - 0.5 \times 0.6, 0.8 + 0.8 - 0.8 \times 0.8\}$$
$$= \{0.52, 0.65, 0.8, 0.96\}$$

$$h_1 \otimes h_2 = \left\{ h_1^{\sigma(t)} h_2^{\sigma(t)} \middle| t = 1, 2, \ldots, l \right\} = \{0.2 \times 0.4, 0.3 \times 0.5, 0.5 \times 0.6, 0.8 \times 0.8\}$$
$$= \{0.08, 0.15, 0.3, 0.64\}$$

It is noted that neither Torra (2010) nor Xia and Xu (2011a) paid any attention to the subtraction and division operations over HFEs. The subtraction and division operations are significantly important in forming the integral theoretical framework of HFS. Meanwhile, it is also an indispensable foundation in developing some well-known decision making method such as PROMETHEE with hesitant fuzzy information. Hence, in the following, we introduce these basic operations over HFEs.

Considering the relationships between IFS and HFS, to start our investigation, let us first review the subtraction and division operations over IFSs. The subtraction and division operations over IFSs were firstly proposed by Atanassov and Riečan (2006).

Later, Chen (2007) also introduced these operations for IFSs, which were derived from the deconvolution for equations using addition and multiplication operations of IFSs, and the forms of these two operations they proposed were similar to those of Atanassov and Riečan (2006). Based on the different versions of the operation “negation”, Atanassov (2009) further developed a family of different kinds of subtraction operations for IFSs. Among all these different subtraction operations, Atanassov (2012) finally chose the following forms as the standard definitions for subtraction and division operations over IFSs in his recent published book:

Definition 1.10 (*Atanassov* 2012). For two given IFSs A and B, the subtraction and division operations have the forms:

$$A \ominus B = \left\{ \left(x, \mu_{A \ominus B}(x), v_{A \ominus B}(x) \right) | x \in X \right\} \tag{1.11}$$

where

$$\mu_{A \ominus B}(x) = \begin{cases} \frac{\mu_A(x) - \mu_B(x)}{1 - \mu_B(x)} & \begin{array}{l} \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } v_A(x) \leq v_B(x) \\ \text{and } v_B(x) > 0 \\ \text{and } v_A(x)\pi_B \leq \pi_A(x) v_B(x) \end{array} \\ 0, & \text{otherwise} \end{cases} \tag{1.12}$$

and

$$v_{A \ominus B}(x) = \begin{cases} \frac{v_A(x)}{v_B(x)}, & \begin{array}{l} \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } v_A(x) \leq v_B(x) \\ \text{and } v_B(x) > 0 \\ \text{and } v_A(x)\pi_B(x) \leq \pi_A(x) v_B(x) \end{array} \\ 1, & \text{otherwise} \end{cases} \tag{1.13}$$

and

$$A \oslash B = \left\{ \left(x, \mu_{A \oslash B}(x), v_{A \oslash B}(x) \right) | x \in X \right\} \tag{1.14}$$

where

$$\mu_{A \oslash B}(x) = \begin{cases} \frac{\mu_A(x)}{\mu_B(x)}, & \begin{array}{l} \text{if } \mu_A(x) \leq \mu_B(x) \text{ and } v_A(x) \geq v_B(x) \\ \text{and } \mu_B(x) > 0 \\ \text{and } \mu_A(x)\pi_B(x) \leq \pi_A(x) \mu_B(x) \end{array} \\ 0, & \text{otherwise} \end{cases} \tag{1.15}$$

and

$$v_{A \oslash B}(x) = \begin{cases} \frac{v_A(x) - v_B(x)}{1 - v_B(x)} & \begin{array}{l} \text{if } \mu_A(x) \leq \mu_B(x) \text{ and } v_A(x) \geq v_B(x) \\ \text{and } \mu_B(x) > 0 \\ \text{and } \mu_A(x)\pi_B(x) \leq \pi_A(x) \mu_B(x) \end{array} \\ 1 & \text{otherwise} \end{cases} \tag{1.16}$$

Inspired by Definition 1.10 and based on the relationships between IFSs and HFSs, the definitions of subtraction and division operations over HFEs can be introduced:

Definition 1.11 (*Liao and Xu* 2014a). Let h, h_1 and h_2 be three HFEs, then

(1) $h_1 \oslash h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{t\}$, where

$$t = \begin{cases} \frac{\gamma_1 - \gamma_2}{1 - \gamma_2}, & \textit{if } \gamma_1 \geq \gamma_2 \textit{ and } \gamma_2 \neq 1 \\ 0, & \textit{otherwise} \end{cases}$$

(2) $h_1 \oslash h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{t\}$, where

$$t = \begin{cases} \frac{\gamma_1}{\gamma_2}, & \textit{if } \gamma_1 \leq \gamma_2 \textit{ and } \gamma_2 \neq 0 \\ 1, & \textit{otherwise} \end{cases}$$

To make it more adequate, let $h \oslash U^* = O^*, h \oslash U^* = O^*$. According to Definition 1.11, it is obvious that for any HFE h, the following equations hold:

- $h \ominus h = O^*$; $h \ominus O^* = h$; $h \ominus E^* = O^*$.
- $h \oslash h = E^*$; $h \oslash E^* = h$; $h \oslash O^* = E^*$.

In addition, it follows from the above equations that some special cases hold:

- $E^* \ominus E^* = O^*$; $U^* \ominus E^* = O^*$; $O^* \ominus E^* = O^*$.
- $E^* \ominus U^* = O^*$; $U^* \ominus U^* = O^*$; $O^* \ominus U^* = O^*$.
- $E^* \ominus O^* = E^*$; $U^* \ominus O^* = U^*$; $O^* \ominus O^* = O^*$.
- $E^* \oslash E^* = E^*$; $U^* \oslash E^* = U^*$; $O^* \oslash E^* = O^*$.
- $E^* \oslash U^* = O^*$; $U^* \oslash U^* = O^*$; $O^* \oslash U^* = O^*$.
- $E^* \oslash O^* = E^*$; $U^* \oslash O^* = E^*$; $O^* \oslash O^* = E^*$.

For the brevity of presentation, in the process of theoretical derivation thereafter, we shall not consider the particular case where $t = 0$ in subtraction operation and $t = 1$ in division operation. It is noted that the HFS encompasses the IFS as a particular case; thus, the subtraction and division operations over HFEs should be equivalent to the subtraction and division operations over IFNs when not considering the nonmembership degree of each IFN. Comparing Definitions 1.10 and 1.11, we can see that this requirement is met. The following theorems show that the subtraction and division operations over HFEs in Definition 1.11 are convincing and they satisfy some basic properties:

Theorem 1.2 (Liao and Xu 2014a). *Let h_1 and h_2 be two HFEs, then*

(1) $(h_1 \ominus h_2) \oplus h_2 = h_1$, if $\gamma_1 \geq \gamma_2$, $\gamma_2 \neq 1$.
(2) $(h_1 \oslash h_2) \otimes h_2 = h_1$, if $\gamma_1 \leq \gamma_2$, $\gamma_2 \neq 0$.

Theorem 1.3 (Liao and Xu 2014a). *Let h_1 and h_2 be two HFEs*, $\lambda > 0$, *then*

(1) $\lambda(h_1 \ominus h_2) = \lambda h_1 \ominus \lambda h_2$, if $\gamma_1 \geq \gamma_2, \gamma_2 \neq 1$.
(2) $(h_1 \oslash h_2)^{\lambda} = h_1^{\lambda} \oslash h_2^{\lambda}$, if $\gamma_1 \leq \gamma_2, \gamma_2 \neq 0$.

Theorem 1.4 (Liao and Xu 2014a). *Let* $h = \cup_{\gamma \in h}\{\gamma\}$ *be a HFE, and* $\lambda_1 \geq \lambda_2 > 0$, *then*

(1) $\lambda_1 h \ominus \lambda_2 h = (\lambda_1 - \lambda_2)h$, if $\gamma \neq 1$.
(2) $h^{\lambda_1} \oslash h^{\lambda_2} = h^{(\lambda_1 - \lambda_2)}$, if $\gamma \neq 0$.

Theorem 1.5 (Liao and Xu 2014a). *For three HFEs* h_1, h_2, *and* h_3, *the following conclusions are valid:*

(1) $h_1 \ominus h_2 \ominus h_3 = h_1 \ominus h_3 \ominus h_2$, if
$\gamma_1 \geq \gamma_2, \gamma_1 \geq \gamma_3, \gamma_2 \neq 1, \gamma_3 \neq 1, \gamma_1 - \gamma_2 - \gamma_3 + \gamma_2\gamma_3 \geq 0$.
(2) $h_1 \oslash h_2 \oslash h_3 = h_1 \oslash h_3 \oslash h_2$, if $\gamma_1 \leq \gamma_2\gamma_3$, $\gamma_2 \neq 0, \gamma_3 \neq 0$.

Theorem 1.6 (Liao and Xu 2014a). *For three HFEs* h_1, h_2, *and* h_3, *the following conclusions are valid:*

(1) $h_1 \ominus h_2 \ominus h_3 = h_1 \ominus (h_2 \oplus h_3)$, if
$\gamma_1 \geq \gamma_2, \gamma_1 \geq \gamma_3, \gamma_2 \neq 1, \gamma_3 \neq 1, \gamma_1 - \gamma_2 - \gamma_3 + \gamma_2\gamma_3 \geq 0$.
(2) $h_1 \oslash h_2 \oslash h_3 = h_1 \oslash (h_2 \otimes h_3)$, if $\gamma_1 \leq \gamma_2\gamma_3, \gamma_2 \neq 0, \gamma_3 \neq 0$.

It should be noted that in the above theorems, the equations hold only under the given precondition. Moreover, the relationship between IFNs and HFEs can be further verified in terms of these two operations:

Theorem 1.7 (Liao and Xu 2014a). *Let* h_1 *and* h_2 *be two HFEs, then*

(1) $A_{env}(h_1 \ominus h_2) = A_{env}(h_1) \ominus A_{env}(h_2)$.
(2) $A_{env}(h_1 \oslash h_2) = A_{env}(h_1) \oslash A_{env}(h_2)$.

Theorem 1.7 further reveals that the subtraction and division operations defined for HFEs are consistent with the ones for IFNs. The following theorem reveals the relationship between these two operations:

Theorem 1.8 (Liao and Xu 2014a). *For two HFEs* h_1 *and* h_2, *the following conclusions are valid:*

(1) $h_1^c \ominus h_2^c = (h_1 \oslash h_2)^c$.
(2) $h_1^c \oslash h_2^c = (h_1 \ominus h_2)^c$.

Example 1.4 (Liao and Xu 2014a). Consider two HFEs $h_1 = \{0.3, 0.2\}$ and $h_2 = \{0.1, 0.2\}$. According to Definition 1.11, we have

$$h_1 \ominus h_2 = \left\{\frac{0.3 - 0.1}{1 - 0.1}, \frac{0.3 - 0.2}{1 - 0.2}, \frac{0.2 - 0.1}{1 - 0.1}, \frac{0.2 - 0.2}{1 - 0.2}\right\} = \left\{\frac{2}{9}, \frac{1}{8}, \frac{1}{9}, 0\right\}$$

In addition, as $h_1^c = \{0.7, 0.8\}$, and $h_2^c = \{0.9, 0.8\}$, by Definition 1.11, we obtain

$$h_1^c \oslash h_2^c = \left\{ \frac{0.7}{0.9}, \frac{0.8}{0.9}, \frac{0.7}{0.8}, \frac{0.8}{0.8} \right\} = \left\{ \frac{7}{9}, \frac{8}{9}, \frac{7}{8}, 1 \right\}$$

Since

$$(h_1 \ominus h_2)^c = \left\{ 1 - \frac{2}{9}, 1 - \frac{1}{8}, 1 - \frac{1}{9}, 1 - 0 \right\} = \left\{ \frac{7}{9}, \frac{8}{9}, \frac{7}{8}, 1 \right\}$$

Then, $(h_1 \ominus h_2)^c = h_1^c \oslash h_2^c$, which verifies (2) of Theorem 1.8. In analogous, (1) of Theorem 1.8 can also be verified.

The subtraction and division operations are significantly important in forming the integral theoretical framework of HFS. Meanwhile, it is also critical in developing some well-known decision making method such as PROMETHEE (Behzadian et al. 2010) with hesitant fuzzy information. The operations of HFEs can be immediately extended into interval-valued HFEs and dual HFEs.

1.1.3 Comparison Laws of Hesitant Fuzzy Elements

It is noted that the number of values in different HFEs may be different. Let l_{h_j} be the number of values in h_j. For two HFEs h_1 and h_2, let $l = \max\{l_{h_1}, l_{h_2}\}$. To operate correctly, Xu and Xia (2011a) gave the following regulation, which is based on the assumption that all the decision makers are pessimistic: If $l_{h_1} < l_{h_2}$, then h_1 should be extended by adding the minimum value in it until it has the same length with h_2; If $l_{h_1} > l_{h_2}$, then h_2 should be extended by adding the minimum value in it until it has the same length with h_1. For example, let $h_1 = \{0.1, 0.2, 0.3\}$, $h_2 = \{0.4, 0.5\}$. To operate correctly, we should extend h_2 until it has the same length with h_1. The pessimist may extend it as $h_2 = \{0.4, 0.4, 0.5\}$, and the optimist may extend h_2 as $h_2 = \{0.4, 0.5, 0.5\}$ which adds the maximum value instead. The results may be different if we extend the shorter one by adding different values. It is reasonable because the decision makers' risk preferences can directly influence the final decision. As to the situation where the decision makers are neither pessimistic nor optimistic, then the added value should be the mean value of the shorter HFE. We can also extend the shorter one by adding the value of 0.5 in it. In such a case, we assume that the decision makers have uncertain information.

Xia and Xu (2011a) defined the score function of a HFE:

Definition 1.12 (*Xia and Xu* 2011a). For a HFE h,

$$s(h) = \frac{1}{l_h} \sum_{\gamma \in h} \gamma \tag{1.17}$$

is called the score function of h, where l_h is the number of values in h. For two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

However, in some special cases, this comparison law cannot be used to distinguish two HFEs:

Example 1.5 (Liao et al. 2014a). Let $h_1 = (0.1, 0.2, 0.6)$ and $h_2 = (0.2, 0.4)$ be two HFEs, then by (1.17), we have

$$s(h_1) = \frac{0.1 + 0.2 + 06}{3} = 0.3, \quad s(h_2) = \frac{0.2 + 0.4}{2} = 0.3.$$

Since $s(h_1) = s(h_2)$, we cannot tell the difference between h_1 and h_2 by only using Definition 1.12. Actually, such a case is common in practice. Hence, Liao et al. (2014a) introduced the variance function of HFE.

Definition 1.13 (*Liao et al.* 2014a). For a HFE h,

$$v_1(h) = \frac{1}{l_h}\sqrt{\sum\nolimits_{\gamma_i,\gamma_j \in h} (\gamma_i - \gamma_j)^2} \tag{1.18}$$

is called the variance function of h, where l_h is the number of values in h, and $v_1(h)$ is called the variance degree of h. For two HFEs h_1 and h_2, if $v_1(h_1) > v_1(h_2)$, then $h_1 < h_2$; if $v_1(h_1) = v_1(h_2)$, then $h_1 = h_2$.

Example 1.6 (Liao et al. 2014a). According to Eq. (1.18), in Example 1.5, we have

$$v_1(h_1) = \frac{\sqrt{0.2^2 + 0.4^2 + 0.5^2}}{3} = 0.2160, v_1(h_2) = \frac{\sqrt{0.2^2}}{2} = 0.1$$

Then, $v_1(h_1) > v_1(h_2)$, i.e., the variance degree of h_1 is higher than that of h_2. Thus, $h_1 < h_2$.

From the above analysis, we can see that the relationship between the score function and the variance function is similar to the relationship between mean and variance in statistics. It is noted that recently, Liao and Xu (2015c) modified the variance function into the following form:

$$v_2(h) = \frac{2}{l_h(l_h - 1)}\sqrt{\sum\nolimits_{\gamma_i,\gamma_j \in h} (\gamma_i - \gamma_j)^2} \tag{1.19}$$

where l_h in the coefficient of Eq. (1.18) is replaced by $C_{l_h}^2 = \frac{l_h(l_h-1)}{2}$.

In addition, Chen et al. (2015) introduced the deviation function of a HFE:

Definition 1.14 (*Chen et al.* 2015). For a HFE h, we define the deviation degree $v_3(h)$ of h as:

$$v_3(h) = \sqrt{\frac{1}{l_h}\sum\nolimits_{\gamma \in h} (\gamma - s(h))^2} \tag{1.20}$$

As it can be seen that $v_3(h)$ is just conventional standard variance in statistics, which reflects the deviation degree between all values in the HFE h and their mean value.

Based on the score function $s(h)$ and the variance function $v_q(h)(q = 1, 2, 3)$, a comparison scheme can be developed to rank any HFEs (Liao et al. 2014a):

- If $s(h_1) < s(h_2)$, then $h_1 < h_2$;
- If $s(h_1) = s(h_2)$, then
  - If $v_q(h_1) < v_q(h_2)$, then $h_1 > h_2$.
  - If $v_q(h_1) = v_q(h_2)$, then $h_1 = h_2$.

Note that we cannot claim that "For two HFEs h_1 and h_2, if $v(h_1) > v(h_2)$, then $h_1 < h_2$; If $v(h_1) = v(h_2)$, then $h_1 = h_2$" due to the fact that sometimes variance is bad, while sometimes variance is good. This assentation holds only under the precondition that $s(h_1) = s(h_2)$. It is well known that an efficient estimator is a measure of the variance of an estimate's sampling distribution in statistics. Hence, under the condition that the score values are equal, which implies that the average values are the same in statistics, it is appropriate to stipulate that the smaller the variance, the more stable the HFE, and thus, the greater the HFE. Similar schemes can be seen in the process of comparing two vague sets (Hong and Choi 2000), and also the comparison between two IFNs (Xu and Yager 2006).

1.2 Extensions of Hesitant Fuzzy Set

1.2.1 Interval-Valued Hesitant Fuzzy Set

In many decision making problems, due to the insufficiency of available information, it may be difficult for decision makers (or experts) to exactly quantify the membership degrees of an element to a set by crisp numbers but by interval-valued numbers within [0, 1]. Consequently, it is necessary to introduce the concept of interval-valued hesitant fuzzy set (IVHFS), which permits the membership degree of an element to a given set to have a few different interval values. The situation is similar to that encounters in intuitionistic fuzzy environment where the concept of IFS has been extended to interval-valued IFS (Atanassov and Gargov 1989).

Definition 1.15 (*Chen et al.* 2013b). Let X be a reference set, and $D[0, 1]$ be the set of all closed subintervals of [0, 1]. An IVHFS on X is

$$\tilde{H} = \{ < x, \tilde{h}_A(x) > | x \in X \} \tag{1.21}$$

where $\tilde{h}_A(x) : X \to D[0,1]$ denotes all possible interval-valued membership degrees of the element $x \in X$ to the set $A \subset X$. For convenience, we call $\tilde{h}_A(x)$ an interval-valued hesitant fuzzy element (IVHFE), which reads

$$\tilde{h}_A(x) = \left\{ \tilde{\gamma} \middle| \tilde{\gamma} \in \tilde{h}_A(x) \right\} \tag{1.22}$$

Here $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$ is an interval-valued number. $\tilde{\gamma}^L = \inf \tilde{\gamma}$ and $\tilde{\gamma}^U = \sup \tilde{\gamma}$ represent the lower and upper limits of $\tilde{\gamma}$, respectively.

The IVHFE is the basic unit of the IVHFS. It can be considered as a special case of the IVHFS. The relationship between IVHFE and IVHFS is similar to that between the interval-valued fuzzy number and interval-valued fuzzy set (Zadeh 1975).

Example 1.7 (Chen et al. 2013b). Let $X = \{x_1, x_2\}$ be a reference set, and the IVHFEs $h_A(x_1) = \{[0.1,\ 0.3], [\ 0.4, 0.5]\}$ and $h_A(x_2) = \{[0.1, 0.2], [0.3, 0.5], [0.7, 0.9]\}$ denote the membership degrees of $x_i (i = 1, 2)$ to a set $A \subset X$ respectively. We call $\tilde{H}$ an IVHFS, where

$$\tilde{H} = \{ < x_1, \{[0.1,\ 0.3],\ [0.4, 0.5]\} > ,\ < x_2, \{[0.1, 0.2], [0.3, 0.5], [0.7, 0.9]\}\}$$

When a decision making problem needs to be characterized by interval-valued numbers rather than crisp numbers, the IVHFS is a preferable choice because it has a great ability in handling imprecise and ambiguous information. For example, supposing two decision makers (or experts) discuss the membership degree of an element x to a set A, one wants to assign [0.3, 0.5] and the other wants to assign [0.6, 0.7]. They cannot reach consensus. In such a circumstance, the degree can be represented by an IVHFE {[0.3, 0.5], [0.6, 0.7]}. Furthermore, in a usual interval-valued fuzzy logic, it is common to average these interval membership degrees or take the smallest interval that contains all these interval degrees. However, the IVHFE can keep all interval values proposed by the decision makers (or experts). That is to say, potentially, it keeps more information about the decision makers' (or experts') opinions, the information that is normally dismissed. It therefore can give a better result in information aggregation.

It should be noted that when the upper and lower bounds of the interval values are identical, the IVHFS becomes the HFS, indicating that the HFS is a special case of the IVHFS. Moreover, when the membership degree of each element belonging to a given set only has an interval value, the IVHFE reduces to the interval-valued fuzzy number and the IVHFS becomes the interval-valued fuzzy set. We can introduce some special IVHFEs, such as:

(1) Empty set: $\tilde{O}^* = \left\{ < x,\ \tilde{h}^\circ(x) > | x \in X \right\}$, where $\tilde{h}^\circ(x) = \{[0, 0]\}$, $\forall x \in X$.
(2) Full set: $\tilde{E}^* = \left\{ < x,\ \tilde{h}^*(x) > | x \in X \right\}$, where $\tilde{h}^*(x) = \{[1, 1]\}$, $\forall x \in X$.
(3) Complete ignorance (all is possible): $\tilde{U}^* = \left\{ < x,\ \tilde{h}(x) > | x \in X \right\}$, where $\tilde{h}(x) = \{[0, 1]\}$, $\forall x \in X$.
(4) Nonsense set: $\tilde{\oslash}^* = \left\{ < x,\ \tilde{h}(x) > | x \in X \right\}$, where $\tilde{h}(x) = \oslash$, $\forall x \in X$.

Chen et al. (2013b) defined some operations on IVHFEs through the connection between IVHFEs and HFEs.

Definition 1.16 (*Chen et al.* 2013b). Let $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ be three IVHFEs, then

(1) $\tilde{h}^c = \{[1-\tilde{\gamma}^U, 1-\tilde{\gamma}^L] | \tilde{\gamma} \in \tilde{h}\}$.
(2) $\tilde{h}_1 \cup \tilde{h}_2 = \{[\max(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \max(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] | \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$.
(3) $\tilde{h}_1 \cap \tilde{h}_2 = \{[\min(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \min(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] | \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$.
(4) $\tilde{h}^\lambda = \left\{[(\tilde{\gamma}^L)^\lambda, (\tilde{\gamma}^U)^\lambda] | \tilde{\gamma} \in \tilde{h}\right\}$, $\lambda > 0$.
(5) $\lambda\tilde{h} = \left\{[1-(1-\tilde{\gamma}^L)^\lambda, 1-(1-\tilde{\gamma}^U)^\lambda] | \tilde{\gamma} \in \tilde{h}\right\}$, $\lambda > 0$.
(6) $\tilde{h}_1 \oplus \tilde{h}_2 = \{[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L,\ \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] | \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$.
(7) $\tilde{h}_1 \otimes \tilde{h}_2 = \{[\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L,\ \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] | \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}$.

Example 1.8 (Chen and Xu 2014). Suppose there are three IVHFEs $\tilde{h}_1 = \{[0.4, 0.6]\}$, $\tilde{h}_2 = \{[0.2,\ 0.3],\ [0.5, 0.7], [0.6, 0.8]\}$, $\tilde{h}_3 = \{[0.3,\ 0.4],\ [0.7, 0.8]\}$. Let $\lambda = 2$, then we have

(1)
$$\begin{aligned}\tilde{h}_3^c &= \{[1-\tilde{\gamma}_3^U, 1-\tilde{\gamma}_3^L] | \tilde{\gamma}_3 \in \tilde{h}_3\}\\ &= \{[1-0.8, 1-0.7], [1-0.4, 1-0.3]\} = \{[0.2, 0.3], [0.6, 0.7]\}.\end{aligned}$$

(2)
$$\begin{aligned}\tilde{h}_1 \cup \tilde{h}_2 &= \{[\max(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \max(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] | \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}\\ &= \{[\max(0.4, 0.2), \max(0.6, 0.3)], [\max(0.4, 0.5), \max(0.6, 0.7)],\\ &\qquad [\max(0.4, 0.6), \max(0.6, 0.8)]\}\\ &= \{[0.4, 0.6], [0.5, 0.7], [0.6, 0.8]\}.\end{aligned}$$

(3)
$$\begin{aligned}\tilde{h}_1 \cap \tilde{h}_2 &= \{[\min(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L), \min(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U)] | \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}\\ &= \{[\min(0.4, 0.2), \min(0.6, 0.3)], [\min(0.4, 0.5), \min(0.6, 0.7)],\\ &\qquad [\min(0.4, 0.6), \min(0.6, 0.8)]\\ &= \{[0.2, 0.3], [0.4, 0.6]\}\end{aligned}$$

Noted that the symbol "{}" means the set of interval-valued numbers. Considering that any two elements in a set must be different, the repeated elements are thus deleted.

(4)
$$\begin{aligned}\tilde{h}_1 \oplus \tilde{h}_2 &= \{[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L,\ \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] | \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\}\\ &= \{[0.4 + 0.2 - 0.4 \cdot 0.2,\ 0.6 + 0.3 - 0.6 \cdot 0.3], [0.4 + 0.5 - 0.4 \cdot 0.5,\ 0.6 + 0.7 - 0.6 \cdot 0.7],\\ &\qquad [0.4 + 0.6 - 0.4 \cdot 0.6,\ 0.6 + 0.8 - 0.6 \cdot 0.8]\}\\ &= \{[0.52, 0.72], [0.7, 0.88], [0.76, 0.92]\}.\end{aligned}$$

(5) $$\tilde{h}_1 \otimes \tilde{h}_2 = \left\{ \left[\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L ,\ \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U \right] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2 \right\}$$
$$= \{[0.4 \cdot 0.2 ,\ 0.6 \cdot 0.3], [0.4 \cdot 0.5 ,\ 0.6 \cdot 0.7], [0.4 \cdot 0.6 ,\ 0.6 \cdot 0.8]\}$$
$$= \{[0.08,\ 0.18], [0.2,\ 0.42], [0.24,\ 0.48]\}.$$

(6) $$\lambda\tilde{h}_3 = \left\{ \left[1 - (1 - \tilde{\gamma}_3^L)^2, 1 - (1 - \tilde{\gamma}_3^U)^2 \right] \middle| \tilde{\gamma}_3 \in \tilde{h}_3 \right\}$$
$$= \left\{ \left[1 - (1 - 0.3)^2, 1 - (1 - 0.4)^2 \right], \left[1 - (1 - 0.7)^2, 1 - (1 - 0.8)^2 \right] \right\}$$
$$= \{[0.51, 0.64], [0.91, 0.96]\}.$$

(7) $$\tilde{h}_3^2 = \left\{ \left[(\tilde{\gamma}_3^L)^2, (\tilde{\gamma}_3^U)^2 \right] \middle| \tilde{\gamma}_3 \in \tilde{h}_3 \right\}$$
$$= \left\{ \left[(0.3)^2, (0.4)^2 \right], \left[(0.7)^2, (0.8)^2 \right] \right\} = \{[0.09, 0.16], [0.49, 0.64]\}.$$

It is pointed out that if $\tilde{\gamma}^L = \tilde{\gamma}^U$, then the operations in Definition 1.16 reduce to those of HFEs.

Theorem 1.9 (Chen and Xu 2014). *Let* $\tilde{h}$ *be an IVHFE and* $\lambda, \lambda_1, \lambda_2 > 0$, *then*

(1) $\tilde{h} \cup \tilde{h} = \tilde{h}$, $\tilde{h} \cap \tilde{h} = \tilde{h}$.
(2) $\tilde{h} \cup \tilde{h}^\circ = \tilde{h}$, $\tilde{h} \cap \tilde{h}^\circ = \tilde{h}^\circ$.
(3) $\tilde{h} \cup \tilde{h}^* = \tilde{h}^*$, $\tilde{h} \cap \tilde{h}^* = \tilde{h}$.
(4) $\tilde{h} \oplus \tilde{h}^\circ = \tilde{h}$, $\tilde{h} \otimes \tilde{h}^\circ = \tilde{h}^\circ$.
(5) $\tilde{h} \oplus \tilde{h}^* = \tilde{h}^*$, $\tilde{h} \otimes \tilde{h}^* = \tilde{h}$.
(6) $\lambda\tilde{h}^\circ = \tilde{h}^\circ$, $\lambda\tilde{h}^* = \tilde{h}^*$.
(7) $\left(\tilde{h}^\circ\right)^\lambda = \tilde{h}^\circ$, $\left(\tilde{h}^*\right)^\lambda = \tilde{h}^*$.
(8) $\left(\tilde{h}^{\lambda_1}\right)^{\lambda_2} = \left(\tilde{h}^{\lambda_2}\right)^{\lambda_1} = \tilde{h}^{\lambda_1\lambda_2}$, $\lambda_2\left(\lambda_1\tilde{h}\right) = \lambda_1\left(\lambda_2\tilde{h}\right) = (\lambda_1\lambda_2)\tilde{h}$.

Theorem 1.10 (Chen and Xu 2014). *Let* $\tilde{h}_1$, $\tilde{h}_2$ *and* $\tilde{h}_3$ *be three IVHFEs, then*

(1) $\tilde{h}_1 \cup \tilde{h}_2 = \tilde{h}_2 \cup \tilde{h}_1$.
(2) $\tilde{h}_1 \cap \tilde{h}_2 = \tilde{h}_2 \cap \tilde{h}_1$.
(3) $\tilde{h}_1 \cap (\tilde{h}_2 \cap \tilde{h}_3) = (\tilde{h}_1 \cap \tilde{h}_2) \cap \tilde{h}_3$.
(4) $\tilde{h}_1 \cup (\tilde{h}_2 \cup \tilde{h}_3) = (\tilde{h}_1 \cup \tilde{h}_2) \cup \tilde{h}_3$.
(5) $\tilde{h}_1 \oplus (\tilde{h}_2 \oplus \tilde{h}_3) = (\tilde{h}_1 \oplus \tilde{h}_2) \oplus \tilde{h}_3$.
(6) $\tilde{h}_1 \otimes (\tilde{h}_2 \otimes \tilde{h}_3) = (\tilde{h}_1 \otimes \tilde{h}_2) \otimes \tilde{h}_3$.
(7) $\tilde{h}_1 \cap (\tilde{h}_2 \cup \tilde{h}_3) = \left(\tilde{h}_1 \cap \tilde{h}_2\right) \cup \left(\tilde{h}_1 \cap \tilde{h}_3\right)$.
(8) $\tilde{h}_1 \cup (\tilde{h}_2 \cap \tilde{h}_3) = \left(\tilde{h}_1 \cup \tilde{h}_2\right) \cap \left(\tilde{h}_1 \cup \tilde{h}_3\right)$.

Theorem 1.11 (Chen and Xu 2014). *Let* $\tilde{h}_1$ *and* $\tilde{h}_2$ *be two IVHFEs, then*

(1) $\tilde{h}_1 \cap (\tilde{h}_1 \cup \tilde{h}_2) = \tilde{h}_1$.
(2) $\tilde{h}_1 \cup (\tilde{h}_1 \cap \tilde{h}_2) = \tilde{h}_1$.

Theorem 1.12 (Chen and Xu 2014). *Let* $\tilde{h}_1$ *and* $\tilde{h}_2$ *be two IVHFEs and* $\lambda > 0$, *then*

(1) $\lambda(\tilde{h}_1 \cup \tilde{h}_2) = \lambda\tilde{h}_1 \cup \lambda\tilde{h}_2$.
(2) $\lambda(\tilde{h}_1 \cap \tilde{h}_2) = \lambda\tilde{h}_1 \cap \lambda\tilde{h}_2$.
(3) $(\tilde{h}_1 \cup \tilde{h}_2)^\lambda = \tilde{h}_1^\lambda \cup \tilde{h}_2^\lambda$.
(4) $(\tilde{h}_1 \cap \tilde{h}_2)^\lambda = \tilde{h}_1^\lambda \cap \tilde{h}_2^\lambda$.

Theorem 1.13 (Chen et al. 2013b). *Let* $\tilde{h}$, $\tilde{h}_1$ *and* $\tilde{h}_2$ *be three IVHFEs, we have*

(1) $\tilde{h}_1 \oplus \tilde{h}_2 = \tilde{h}_2 \oplus \tilde{h}_1$.
(2) $\tilde{h}_1 \otimes \tilde{h}_2 = \tilde{h}_2 \otimes \tilde{h}_1$.
(3) $\lambda(\tilde{h}_1 \oplus \tilde{h}_2) = \lambda\tilde{h}_1 \oplus \lambda\tilde{h}_2$, $\lambda > 0$.
(4) $(\tilde{h}_1 \otimes \tilde{h}_2)^\lambda = \tilde{h}_1^\lambda \otimes \tilde{h}_2^\lambda$, $\lambda > 0$.
(5) $\lambda_1\tilde{h} \oplus \lambda_2\tilde{h} = (\lambda_1 + \lambda_2)\tilde{h}$, $\lambda_1, \lambda_2 > 0$.
(6) $\tilde{h}^{\lambda_1} \otimes \tilde{h}^{\lambda_2} = \tilde{h}^{(\lambda_1 + \lambda_2)}$, $\lambda_1, \lambda_2 > 0$.

Proof For three IVHFEs $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$, we have

(1)
$$\begin{aligned}\tilde{h}_1 \oplus \tilde{h}_2 &= \left\{[\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L,\ \tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\right\} \cdot \\ &= \left\{[\tilde{\gamma}_2^L + \tilde{\gamma}_1^L - \tilde{\gamma}_2^L \cdot \tilde{\gamma}_1^L,\ \tilde{\gamma}_2^U + \tilde{\gamma}_1^U - \tilde{\gamma}_2^U \cdot \tilde{\gamma}_1^U] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\right\} \\ &= \tilde{h}_2 \oplus \tilde{h}_1\end{aligned}$$

(2)
$$\begin{aligned}\tilde{h}_1 \otimes \tilde{h}_2 &= \left\{[\tilde{\gamma}_1^L \cdot \tilde{\gamma}_1^L,\ \tilde{\gamma}_1^U \cdot \tilde{\gamma}_1^U] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\right\} \\ &= \left\{[\tilde{\gamma}_2^L \cdot \tilde{\gamma}_1^L,\ \tilde{\gamma}_2^U \cdot \tilde{\gamma}_2^U] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\right\} = \tilde{h}_2 \otimes \tilde{h}_1.\end{aligned}$$

(3)
$$\begin{aligned}\lambda(\tilde{h}_1 \oplus \tilde{h}_2) &= \left\{[1 - (1 - (\tilde{\gamma}_1^L + \tilde{\gamma}_2^L - \tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L))^\lambda, 1 - (1 - (\tilde{\gamma}_1^U + \tilde{\gamma}_2^U - \tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U))^\lambda] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\right\} \\ &= \left\{[1 - (1 - \tilde{\gamma}_1^L)^\lambda (1 - \tilde{\gamma}_2^L)^\lambda, 1 - (1 - \tilde{\gamma}_1^U)^\lambda (1 - \tilde{\gamma}_2^U)^\lambda] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\right\} \\ &= \left\{[1 - (1 - \tilde{\gamma}_1^L)^\lambda + 1 - (1 - \tilde{\gamma}_2^L)^\lambda - (1 - (1 - \tilde{\gamma}_1^L)^\lambda)(1 - (1 - \tilde{\gamma}_2^L)^\lambda),\right. \\ &\quad \left. 1 - (1 - \tilde{\gamma}_1^U)^\lambda + 1 - (1 - \tilde{\gamma}_2^U)^\lambda - (1 - (1 - \tilde{\gamma}_1^U)^\lambda)(1 - (1 - \tilde{\gamma}_2^U)^\lambda)] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\right\} \\ &= \lambda\tilde{h}_1 \oplus \lambda\tilde{h}_2.\end{aligned}$$

(4) $$(\tilde{h}_1 \otimes \tilde{h}_2)^{\lambda} = \left\{[(\tilde{\gamma}_1^L \cdot \tilde{\gamma}_2^L)^{\lambda}, (\tilde{\gamma}_1^U \cdot \tilde{\gamma}_2^U)^{\lambda}] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\right\}$$
$$= \left\{[(\tilde{\gamma}_1^L)^{\lambda} \cdot (\tilde{\gamma}_2^L)^{\lambda}, (\tilde{\gamma}_1^U)^{\lambda} \cdot (\tilde{\gamma}_2^U)^{\lambda}] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2\right\}$$
$$= \tilde{h}_1^{\lambda} \otimes \tilde{h}_2^{\lambda}.$$

(5) $$\lambda_1\tilde{h} \oplus \lambda_2\tilde{h} = \Big\{[1-(1-\tilde{\gamma}^L)^{\lambda_1} + 1-(1-\tilde{\gamma}^L)^{\lambda_2} - (1-(1-\tilde{\gamma}^L)^{\lambda_1})(1-(1-\tilde{\gamma}^L)^{\lambda_2}),$$
$$1-(1-\tilde{\gamma}^U)^{\lambda_1} + 1-(1-\tilde{\gamma}^U)^{\lambda_2} - (1-(1-\tilde{\gamma}^U)^{\lambda_1})(1-(1-\tilde{\gamma}^U)^{\lambda_2})] \big| \tilde{\gamma} \in \tilde{h}\Big\}$$
$$= \left\{[1-(1-\tilde{\gamma}^L)^{\lambda_1}(1-\tilde{\gamma}^L)^{\lambda_2}, 1-(1-\tilde{\gamma}^U)^{\lambda_1}(1-\tilde{\gamma}^U)^{\lambda_2}] \middle| \tilde{\gamma} \in \tilde{h}\right\}$$
$$= \left\{[1-(1-\tilde{\gamma}^L)^{\lambda_1+\lambda_2}, 1-(1-\tilde{\gamma}^U)^{\lambda_1+\lambda_2}] \middle| \tilde{\gamma} \in \tilde{h}\right\}$$
$$= (\lambda_1+\lambda_2)\tilde{h}.$$

(6) $$\tilde{h}^{\lambda_1} \otimes \tilde{h}^{\lambda_2} = \left\{[(\tilde{\gamma}^L)^{\lambda_1} \cdot (\tilde{\gamma}^L)^{\lambda_2}, (\tilde{\gamma}^U)^{\lambda_1} \cdot (\tilde{\gamma}^U)^{\lambda_2}] \middle| \tilde{\gamma} \in \tilde{h}\right\}$$
$$= \left\{[(\tilde{\gamma}^L)^{\lambda_1+\lambda_2}, (\tilde{\gamma}^U)^{\lambda_1+\lambda_2}] \middle| \tilde{\gamma} \in \tilde{h}\right\} = \tilde{h}^{(\lambda_1+\lambda_2)}. \square$$

This completes the proof.

The relationships between the defined operations on IVHFEs are given in Theorem 1.14.

Theorem 1.14 (Chen et al. 2013b). *For three IVHFEs* $\tilde{h}$, $\tilde{h}_1$ *and* $\tilde{h}_2$, *we have*

(1) $\tilde{h}_1^c \cup \tilde{h}_2^c = \left(\tilde{h}_1 \cap \tilde{h}_2\right)^c$.
(2) $\tilde{h}_1^c \cap \tilde{h}_2^c = \left(\tilde{h}_1 \cup \tilde{h}_2\right)^c$.
(3) $(\tilde{h}^c)^{\lambda} = (\lambda\tilde{h})^c$.
(4) $\lambda(\tilde{h}^c) = (\tilde{h}^{\lambda})^c$.
(5) $\tilde{h}_1^c \oplus \tilde{h}_2^c = \left(\tilde{h}_1 \otimes \tilde{h}_2\right)^c$.
(6) $\tilde{h}_1^c \otimes \tilde{h}_2^c = \left(\tilde{h}_1 \oplus \tilde{h}_2\right)^c$.

Proof For three IVHFEs $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$, we have

(1) $$\tilde{h}_1^c \cup \tilde{h}_2^c = \left\{\left[\max(1-\tilde{\gamma}_1^U, 1-\tilde{\gamma}_2^U), \max(1-\tilde{\gamma}_1^L, 1-\tilde{\gamma}_2^L)\right] | \tilde{\gamma}_1 \in h_1, \tilde{\gamma}_2 \in h_2\right\}$$
$$= \left\{\left[1-\min(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U), 1-\min(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L)\right] | \tilde{\gamma}_1 \in h_1, \tilde{\gamma}_2 \in h_2\right\} = (\tilde{h}_1 \cap \tilde{h}_2)^c.$$

(2) $$\tilde{h}_1^c \cap \tilde{h}_2^c = \left\{\left[\min(1-\tilde{\gamma}_1^U, 1-\tilde{\gamma}_2^U), \min(1-\tilde{\gamma}_1^L, 1-\tilde{\gamma}_2^L)\right] | \tilde{\gamma}_1 \in h_1, \tilde{\gamma}_2 \in h_2\right\}$$
$$= \left\{\left[1-\max(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U), 1-\max(\tilde{\gamma}_1^L, \tilde{\gamma}_1^L)\right] | \tilde{\gamma}_1 \in h_1, \tilde{\gamma}_2 \in h_2\right\} = (\tilde{h}_1 \cup \tilde{h}_2)^c.$$

(3) $$(\tilde{h}^c)^{\lambda} = \left\{\left[(1-\tilde{\gamma}^U)^{\lambda}, (1-\tilde{\gamma}^L)^{\lambda}\right] \middle| \tilde{\gamma} \in \tilde{h}\right\}$$
$$= \left\{\left[1-(1-\tilde{\gamma}^L)^{\lambda}, 1-(1-\tilde{\gamma}^U)^{\lambda}\right] \middle| \tilde{\gamma} \in \tilde{h}\right\}^C = (\lambda\tilde{h})^c.$$

(4) $$\lambda\tilde{h}^c = \lambda\{[1-\tilde{\gamma}^U, 1-\tilde{\gamma}^L]|\tilde{\gamma}\in\tilde{h}\} = \left\{\left[1-(1-(1-\tilde{\gamma}^U))^{\lambda}, 1-(1-(1-\tilde{\gamma}^L))^{\lambda}\right]|\tilde{\gamma}\in\tilde{h}\right\}$$
$$= \left\{\left[1-(\tilde{\gamma}^U)^{\lambda}, 1-(\tilde{\gamma}^L)^{\lambda}\right]|\tilde{\gamma}\in\tilde{h}\right\} = (\tilde{h}^{\lambda})^c.$$

(5) $$\tilde{h}_1^c \oplus \tilde{h}_2^c = \{[(1-\tilde{\gamma}_1^U)+(1-\tilde{\gamma}_2^U)-(1-\tilde{\gamma}_1^U)(1-\tilde{\gamma}_2^U),$$
$$(1-\tilde{\gamma}_1^L)+(1-\tilde{\gamma}_2^L)-(1-\tilde{\gamma}_1^L)(1-\tilde{\gamma}_2^L)]|\tilde{\gamma}_1\in\tilde{h}_1, \tilde{\gamma}_2\in\tilde{h}_2\}$$
$$= \{[1-\tilde{\gamma}_1^U\cdot\tilde{\gamma}_2^U, 1-\tilde{\gamma}_1^L\cdot\tilde{\gamma}_2^L]|\tilde{\gamma}_1\in\tilde{h}_1, \tilde{\gamma}_2\in\tilde{h}_2\} = \left(\tilde{h}_1\otimes\tilde{h}_2\right)^c.$$

(6) $$\tilde{h}_1^c \otimes \tilde{h}_2^c = \{[(1-\tilde{\gamma}_1^U)(1-\tilde{\gamma}_2^U), (1-\tilde{\gamma}_1^L)(1-\tilde{\gamma}_2^L)]|\tilde{\gamma}_1\in\tilde{h}_1, \tilde{\gamma}_2\in\tilde{h}_2\}$$
$$= \{[1-(\tilde{\gamma}_1^U+\tilde{\gamma}_2^U-\tilde{\gamma}_1^U\cdot\tilde{\gamma}_2^U), 1-(\tilde{\gamma}_1^L+\tilde{\gamma}_2^L-\tilde{\gamma}_1^L\cdot\tilde{\gamma}_2^L)]|\tilde{\gamma}_1\in\tilde{h}_1, \tilde{\gamma}_2\in\tilde{h}_2\} = \left(\tilde{h}_1\oplus\tilde{h}_2\right)^c.\square$$

This completes the proof.

Since the number of interval values for different IVHFEs could be different and the interval values are usually out of order, we arrange them in any order using Eq. (1.23). To facilitate the calculation between two IVHFEs, we let $l = \max\{l_{\tilde{\alpha}}, l_{\tilde{\beta}}\}$ with $l_{\tilde{\alpha}}$ and $l_{\tilde{\beta}}$ being the number of intervals in the IVHFEs $\tilde{\alpha}$ and $\tilde{\beta}$. To operate correctly, we give the following regulation: when $l_{\tilde{\alpha}} \neq l_{\tilde{\beta}}$, we can make them equivalent through adding elements to the IVHFE that has a less number of elements. In terms of pessimistic principles, the smallest element can be added while the opposite case will be adopted following optimistic principles. In this study we adopt the latter. Specifically, if $l_{\tilde{\alpha}} < l_{\tilde{\beta}}$, then $\tilde{\alpha}$ should be extended by adding the maximum value in it until it has the same length as $\tilde{\beta}$; if $l_{\tilde{\alpha}} > l_{\tilde{\beta}}$, then $\tilde{\beta}$ should be extended by adding the maximum value in it until it has the same length as $\tilde{\alpha}$.

Definition 1.17 (*Xu and Da* 2002). Let $\tilde{a} = [\tilde{a}^L, \tilde{a}^U]$ and $\tilde{b} = [\tilde{b}^L, \tilde{b}^U]$ be two interval numbers, and $\lambda \geq 0$, then

(1) $\tilde{a} = \tilde{b} \Leftrightarrow \tilde{a}^L = \tilde{b}^L$ and $\tilde{a}^U = \tilde{b}^U$.
(2) $\tilde{a} + \tilde{b} = [\tilde{a}^L + \tilde{b}^L, \tilde{a}^U + \tilde{b}^U]$.
(3) $\lambda\tilde{a} = [\lambda\tilde{a}^L, \lambda\tilde{a}^U]$, especially, $\lambda\tilde{a} = 0$, if $\lambda = 0$.

The possibility degree is proposed to compare two interval numbers:

Definition 1.18 (*Xu and Da* 2002). Let $\tilde{a} = [\tilde{a}^L, \tilde{a}^U]$ and $\tilde{b} = [\tilde{b}^L, \tilde{b}^U]$, and let $l_{\tilde{a}} = \tilde{a}^U - \tilde{a}^L$ and $l_{\tilde{b}} = \tilde{b}^U - \tilde{b}^L$. Then the degree of possibility of $\tilde{a} \geq \tilde{b}$ is formulated by

$$p(\tilde{a} \geq \tilde{b}) = \max\left\{1 - \max\left(\frac{\tilde{b}^U - \tilde{a}^L}{l_{\tilde{a}} + l_{\tilde{b}}}, 0\right), 0\right\} \tag{1.23}$$

The score function for IVHFEs is defined as follows:

Definition 1.19 (*Chen et al.* 2013b). For an IVHFE $\tilde{h}$,

$$s(\tilde{h}) = \frac{1}{l_{\tilde{h}}} \sum\nolimits_{\tilde{\gamma} \in \tilde{h}} \tilde{\gamma} \tag{1.24}$$

is called the score function of $\tilde{h}$ with $l_{\tilde{h}}$ being the number of interval values in $\tilde{h}$, and $s(\tilde{h})$ is an interval value belonging to [0, 1]. For two IVHFEs $\tilde{h}_1$ and $\tilde{h}_2$, if $s(\tilde{h}_1) \geq s(\tilde{h}_2)$, then $\tilde{h}_1 \geq \tilde{h}_2$.

Note that we can compare two scores using Eq. (1.23). Moreover, with Definition 1.19, we can compare two IVHFEs.

1.2.2 Dual Hesitant Fuzzy Set

Zhu et al. (2012) defined the dual hesitant fuzzy set in terms of two functions that return two sets of membership values and nonmembership values respectively for each element in the domain:

Definition 1.20 (*Zhu et al.* 2012). Let X be a fixed set, then a dual hesitant fuzzy set (DHFS) D on X is described as:

$$D = \{ <x,\ h_A(x), g_A(x) > | x \in X \} \tag{1.25}$$

in which $h_A(x)$ and $g_A(x)$ are two sets of some values in $[0, 1]$, denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to the set $A \subset X$ respectively, with the conditions:

$$0 \leq \gamma, \eta \leq 1,\ 0 \leq \gamma^+ + \eta^+ \leq 1, \tag{1.26}$$

where $\gamma \in h_A(x)$, $\eta \in g_A(x)$, $\gamma^+ \in h^+(x) = \cup_{\gamma \in h_A(x)} \max\{\gamma\}$, and $\eta^+ \in g^+(x) = \cup_{\eta \in g_A(x)} \max\{\eta\}$ for all $x \in X$. For convenience, the pair $d_A(x) = (h_A(x), g_A(x))$ is called a dual hesitant fuzzy element (DHFE) denoted by $d = (h, g)$, with the conditions: $\gamma \in h$, $\eta \in g$, $\gamma^+ \in h^+ = \cup_{\gamma \in h} \max\{\gamma\}$, $\eta^+ \in g^+ = \cup_{\eta \in g} \max\{\eta\}$, $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$.

There are some special DHFEs:

(1) Complete uncertainty: $d = (\{0\}, \{1\})$.
(2) Complete certainty: $d = (\{1\}, \{0\})$.
(3) Complete ill-known (all is possible): $d = [0, 1]$.
(4) Nonsense element: $d = \oslash$, i.e., $h = \oslash, g = \oslash$.

For a given $d \neq \oslash$, if h and g have only one value γ and η respectively, and $\gamma + \eta < 1$, then the DHFS reduces to an IFS. If h and g have only one value γ and η respectively, and $\gamma + \eta = 1$, or h owns one value, and $g = \oslash$, then the DHFS reduces to a fuzzy set. If $g = \oslash$ and $h \neq \oslash$, then the DHFS reduces to a HFS.

Hence, the DHFS encompasses the fuzzy set, the IFS, and the HFS as special cases. DHFS consists of two parts, i.e., the membership hesitancy function and the non-membership hesitancy function, which confront several different possible values indicating the cognitive degrees whether certainty or uncertainty. As we all know, when the decision makers provide their judgments over the objects, the more the information they take into account, the more the values we will obtain from the decision makers. As the DHFS can reflect the original information given by the decision makers as much as possible, it can be regarded as a more comprehensive set supporting a more flexible approach.

For simplicity, let $\gamma^- \in h^- = \cup_{\gamma \in h(x)} \min\{\gamma\}$, $\eta^- \in g^- = \cup_{\eta \in g(x)} \min\{\eta\}$. γ^+ and η^+ are defined as above. For a typical DHFS, h and g can be represented by two intervals as:

$$h = [\gamma^-, \gamma^+],\ g = [\eta^-, \eta^+] \tag{1.27}$$

Based on Definition 1.4, there is a transformation between IFN and HFE, we can also transform g to the second HFE $h^2(x) = [1-\eta^+, 1-\eta^-]$ denoting the possible membership degrees of the element $x \in X$ to the set $A \subset X$. In this way, both h and h^2 indicate the membership degrees. As such, we can use a "nested interval" to represent $d(x)$ as:

$$d = [[\gamma^-, \gamma^+], [1-\eta^+, 1-\eta^-]] \tag{1.28}$$

The common ground of these sets is to reflect fuzzy degrees to an object, according to either fuzzy numbers or interval-valued fuzzy numbers. Therefore, we use nonempty closed interval as a uniform framework to represent a DHFE d, which is divided into different cases as follows:

$$d = \begin{cases} \oslash,\ \textit{if}\ g = \oslash \text{ and } h = \oslash \\ \left\langle \begin{array}{l} (\gamma) \begin{cases} \textit{if}\ g = \oslash \text{ and } h \neq \oslash,\ \gamma^- = \gamma^+ = \gamma \\ \textit{if}\ g \neq \oslash \text{ and } h \neq \oslash,\ \gamma^- = \gamma^+ = \gamma = 1-\eta^- = 1-\eta^+ = 1-\eta \end{cases} \\ (1-\eta),\ \textit{if}\ g \neq \oslash \text{ and } h = \oslash,\ \eta^- = \eta^+ = \eta \end{array} \right. \\ \left\langle \begin{array}{l} [\gamma^-, \gamma^+],\ \textit{if}\ g = \oslash \text{ and } h \neq \oslash,\ \gamma^- \neq \gamma^+ \\ [1-\eta^+, 1-\eta^-],\ \textit{if}\ g \neq \oslash \text{ and } h = \oslash,\ \eta^- \neq \eta^+ \end{array} \right. \\ \left\langle \begin{array}{l} [\gamma, [1-\eta^+, 1-\eta^-]],\ \textit{if}\ g \neq \oslash \text{ and } h \neq \oslash,\ \eta^- \neq \eta^+, \gamma^- = \gamma^+ = \gamma \\ [[\gamma^-, \gamma^+], \eta],\ \textit{if}\ g \neq \oslash \text{ and } h \neq \oslash,\ \gamma^- \neq \gamma^+,\ \eta^- = \eta^+ = \eta \end{array} \right. \\ [[\gamma^-, \gamma^+], [1-\eta^+, 1-\eta^-]],\ \textit{if}\ g \neq \oslash \text{ and } h \neq \oslash,\ \eta^- \neq \eta^+, \gamma^- \neq \gamma^+ \end{cases} \tag{1.29}$$

Equation (1.29) reflects the connections between DHFS and other types of fuzzy set extensions. The merit of DHFS is more flexible to be valued in multifold ways according to the practical demands than the existing sets, taking into account much more information given by decision makers.

The complement of the DHFS can be defined regarding to different situations.

Definition 1.21 (*Zhu et al.* 2012). Given a DHFE represented by the function d, and $d \neq \oslash$, its complement is defined as:

$$d^c = \begin{cases} (\cup_{\eta \in g}\{\eta\}, \cup_{\gamma \in h}\{\gamma\}), \text{ if } g \neq \oslash \text{ and } h \neq \oslash \\ (\cup_{\gamma \in h}\{1-\gamma\}, \{\oslash\}), \text{ if } g = \oslash \text{ and } h \neq \oslash \\ (\{\oslash\}, \cup_{\eta \in g}\{1-\eta\}), \text{ if } h = \oslash \text{ and } g \neq \oslash \end{cases} \tag{1.30}$$

Apparently, the complement can be correspondingly represented as $(d^c)^c = d$.

For two DHFSs d_1 and d_2, the corresponding lower and upper bounds to h and g are h^-, h^+, g^- and g^+, respectively, where $h^- = \cup_{\gamma \in h}\min\{\gamma\}$, $h^+ = \cup_{\gamma \in h}\max\{\gamma\}$, $g^- = \cup_{\eta \in g}\min\{\eta\}$, and $g^+ = \cup_{\eta \in g}\max\{\eta\}$. Then the union and intersection of DHFSs can be defined as follows:

Definition 1.22 (*Zhu et al.* 2012). Let d_1 and d_2 be two DHFEs. Then,

(1) $d_1 \cup d_2 = (\{h \in (h_1 \cup h_2) | h \geq \max(h_1^-, h_2^-)\}, \{g \in (g_1 \cap g_2) | g \leq \min(g_1^+, g_2^+)\})$.
(2) $d_1 \cap d_2 = (\{h \in (h_1 \cap h_2) | h \leq \min(h_1^+, h_2^+)\}, \{g \in (g_1 \cup g_2) | g \geq \max(g_1^-, g_2^-)\})$.

Example 1.9 (Zhu et al. 2012). Let $d_1 = (\{0.1, 0.3, 0.4\}, \{0.3, 0.5\})$ and $d_2 = (\{0.2, 0.5\}, \{0.1, 0.2, 0.4\})$ be two DHFEs, then we have

(1) Complement: $d_1^c = (\{0.3, 0.5\}, \{0.1, 0.3, 0.4\})$.
(2) Union: $d_1 \cup d_2 = (\{0, 2, 0.3, 0.4, 0.5\}, \{0.1, 0.2, 0.3, 0.4\})$.
(3) Intersection: $d_1 \cap d_2 = (\{0, 1, 0.2, 0.3, 0.4\}, \{0.3, 0.4, 0.5\})$.

Definition 1.23 (*Zhu et al.* 2012). For two DHFEs d_1 and d_2, let n be a positive integer, then the following operations are valid:

(1)
$$\begin{aligned} d_1 \oplus d_2 &= (h_{d_1} \oplus h_{d_2}, g_{d_1} \otimes g_{d_2}) \\ &= (\cup_{\gamma_{d_1} \in h_{d_1}, \gamma_{d_2} \in h_{d_2}}\{\gamma_{d_1} + \gamma_{d_2} - \gamma_{d_1}\gamma_{d_2}\}, \cup_{\eta_{d_1} \in g_{d_1}, \eta_{d_2} \in g_{d_2}}\{\eta_{d_1}\eta_{d_2}\}). \end{aligned}$$

(2)
$$\begin{aligned} d_1 \otimes d_2 &= (h_{d_1} \otimes h_{d_2}, g_{d_1} \oplus g_{d_2}) \\ &= (\cup_{\gamma_{d_1} \in h_{d_1}, \gamma_{d_2} \in h_{d_2}}\{\gamma_{d_1}\gamma_{d_2}\}, \cup_{\eta_{d_1} \in g_{d_1}, \eta_{d_2} \in g_{d_2}}\{\eta_{d_1} + \eta_{d_2} - \eta_{d_1}\eta_{d_2}\}). \end{aligned}$$

(3) $nd = (\cup_{\gamma_d \in h_d}\{1 - (1 - \gamma_d)^n\}, \cup_{\eta_d \in g_d}\{(\eta_d)^n\})$.
(4) $d^n = (\cup_{\gamma_d \in h_d}\{(\gamma_d)^n\}, \cup_{\eta_d \in g_d}\{1 - (1 - \eta_d)^n\})$.

Theorem 1.15 (Zhu et al. 2012). *Let d, d_1 and d_2 be any three DHFEs, $\lambda \geq 0$, then*

(1) $d_1 \oplus d_2 = d_2 \oplus d_1$.
(2) $d_1 \otimes d_2 = d_2 \otimes d_1$.
(3) $\lambda(d_1 \otimes d_2) = \lambda d_1 \otimes \lambda d_2$.
(4) $(d_1 \otimes d_2)^\lambda = d_1^\lambda \otimes d_2^\lambda$.

It is noted that the above operations for DHFEs are based on the Algebraic t-conorm and t-norm. In fact, there are various types of t-conorm and t-norm. If we replace the Algebraic t-conorm and t-norm in the above operations for DHFEs with other forms of t-conorm and t-norm, we shall get more operational methods for DHFEs. For example, the Einstein t-conorm and t-norm are given as:

$$S^E(x,y) = \frac{x+y}{1+xy},\ T^E(x,y) = \frac{xy}{1+(1-x)(1-y)} \tag{1.31}$$

Based on the Einstein t-conorm and t-norm, Zhao et al. (2015) defined the Einstein sum and the Einstein product of DHFEs as follows:

Definition 1.24 (*Zhao et al.* 2016a). For any two DHFEs $d_1 = (h_1, g_1)$ and $d_2 = (h_2, g_2)$, we have

(1) $$d_1 \dot{\oplus} d_2 = (\cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\frac{\gamma_1 + \gamma_2}{1+\gamma_1\gamma_2}\}, \cup_{\eta_1 \in g_1, \eta_2 \in g_2}\{\frac{\eta_1\eta_2}{1+(1-\eta_1)(1-\eta_2)}\}).$$

(2) $$d_1 \dot{\otimes} d_2 = (\cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{\frac{\gamma_1\gamma_2}{1+(1-\gamma_1)(1-\gamma_2)}\}, \cup_{\eta_1 \in g_1, \eta_2 \in g_2}\{\frac{\eta_1 + \eta_2}{1+\eta_1\eta_2}\}).$$

To get the Einstein scalar multiplication and the Einstein power for DHFEs, the following theorems are introduced:

Theorem 1.16 (Zhao et al. 2016a). *Let $d = (h, g)$ be a DHFE, and n be any positive real number, then*

$$nd = (\cup_{\gamma \in h}\{\frac{(1+\gamma)^n - (1-\gamma)^n}{(1+\gamma)^n + (1-\gamma)^n}\},\ \cup_{\eta \in g}\{\frac{2\eta^n}{(2-\eta)^n + \eta^n}\}) \tag{1.32}$$

where $nd = \overbrace{d \dot{\oplus} d \dot{\oplus} \cdots \dot{\oplus} d}^{n}$. *Moreover, nd is a DHFE.*

Proof We use mathematical induction to prove that Eq. (1.32) holds for the positive integer n.

(1) For $n = 1$, it is obvious that Eq. (1.32) holds.
(2) Assume Eq. (1.32) holds for $n = k$. Then for $n = k + 1$, we have

$$\begin{aligned}(k+1)d = kd \dot{\oplus} d &= (\cup_{\gamma \in h}\{\frac{(1+\gamma)^k - (1-\gamma)^k}{(1+\gamma)^k + (1-\gamma)^k}\}, \cup_{\eta \in g}\{\frac{2\eta^k}{(2-\eta)^k + \eta^k}\}) \dot{\oplus} (\cup_{\gamma \in h}\{\gamma\}, \cup_{\eta \in g}\{\eta\}) \\ &= (\cup_{\gamma \in h}\{\frac{\frac{(1+\gamma)^k - (1-\gamma)^k}{(1+\gamma)^k + (1-\gamma)^k} + \gamma}{1 + \frac{(1+\gamma)^k - (1-\gamma)^k}{(1+\gamma)^k + (1-\gamma)^k}\gamma}\}, \cup_{\eta \in g}\{\frac{\frac{2\eta^k}{(2-\eta)^k + \eta^k}\eta}{1 + (1 - \frac{2\eta^k}{(2-\eta)^k + \eta^k})(1-\eta)}\}) \\ &= (\cup_{\gamma \in h}\{\frac{(1+\gamma)^{k+1} - (1-\gamma)^{k+1}}{(1+\gamma)^{k+1} + (1-\gamma)^{k+1}}\}, \cup_{\eta \in g}\{\frac{2\eta^{k+1}}{(2-\eta)^{k+1} + \eta^{k+1}}\})\end{aligned}$$

Thus, Eq. (1.32) holds for $n = k + 1$.

In the following, we prove that Theorem 1.16 holds when n is a positive real number.

Since $0 \le \gamma \le 1$, $0 \le \eta \le 1$, $1 \le 2-\eta \le 2$, and $1-\gamma \ge \eta \ge 0$, $1-\eta \ge \gamma \ge 0$, obviously, we have

$$0 \le \frac{(1+\gamma)^n - (1-\gamma)^n}{(1+\gamma)^n + (1-\gamma)^n} \le 1 \tag{1.33}$$

$$0 \le \frac{2\eta^n}{(2-\eta)^n + \eta^n} \le 1 \tag{1.34}$$

$$0 \le \frac{(1+\gamma)^n - (1-\gamma)^n}{(1+\gamma)^n + (1-\gamma)^n} \le \frac{(1+\gamma)^n - (1-\gamma)^n}{(1+\gamma)^n + \eta^n} \le \frac{(1+\gamma)^n - \eta^n}{(1+\gamma)^n + \eta^n} \tag{1.35}$$

$$0 \le \frac{2\eta^n}{(2-\eta)^n + \eta^n} = \frac{2\eta^n}{(1+(1-\eta))^n + \eta^n} \le \frac{2\eta^n}{(1+(1-\eta))^n + \eta^n} \le \frac{2\eta^n}{(1+\gamma)^n + \eta^n} \tag{1.36}$$

From Eqs. (1.35) and (1.36), we have

$$0 \le \frac{(1+\gamma)^n - (1-\gamma)^n}{(1+\gamma)^n + (1-\gamma)^n} + \frac{2\eta^n}{(2-\eta)^n + \eta^n} \le 1 \tag{1.37}$$

Combining Eqs. (1.33), (1.34) and (1.37), we know that the DHFE nd is a DHFE for any positive real number n. This completes the proof of Theorem 1.16.

Theorem 1.17 (Zhao et al. 2016a). *Let* $d = (h, g)$ *be a DHFE, and* n *be any positive real number, then*

$$d^n = \left(\cup_{\gamma \in h} \left\{ \frac{2\gamma^n}{(2-\gamma)^n + \gamma^n} \right\},\ \cup_{\eta \in g} \left\{ \frac{(1+\eta)^n - (1-\eta)^n}{(1+\eta)^n + (1-\eta)^n} \right\} \right) \tag{1.38}$$

where $d^n = \overbrace{d \dot{\otimes} d \dot{\otimes} \cdots \dot{\otimes} d}^{n}$, and d^n is a DHFE.

Based on Theorems 1.16 and 1.17, the Einstein scalar multiplication and the Einstein power of DHFE can be defined:

Definition 1.25 (*Zhao et al.* 2016a). Let $d = (h, g)$ be a DHFE, $\lambda > 0$, then

(1) $\lambda d = \left(\cup_{\gamma \in h} \left\{ \frac{(1+\gamma)^\lambda - (1-\gamma)^\lambda}{(1+\gamma)^\lambda + (1-\gamma)^\lambda} \right\},\ \cup_{\eta \in g} \left\{ \frac{2\eta^\lambda}{(2-\eta)^\lambda + \eta^\lambda} \right\} \right)$.

(2) $d^\lambda = \left(\cup_{\gamma \in h} \left\{ \frac{2\gamma^\lambda}{(2-\gamma)^\lambda + \gamma^\lambda} \right\},\ \cup_{\eta \in g} \left\{ \frac{(1+\eta)^\lambda - (1-\eta)^\lambda}{(1+\eta)^\lambda + (1-\eta)^\lambda} \right\} \right)$.

Theorem 1.18 (Zhao et al. 2016a). *Let* d, d_1 *and* d_2 *be any three DHFEs*, $\lambda > 0$, *then*

(1) $d_1 \dot{\oplus} d_2 = d_2 \dot{\oplus} d_1$.
(2) $d_1 \dot{\otimes} d_2 = d_2 \dot{\otimes} d_1$.
(3) $\lambda(d_1 \dot{\oplus} d_2) = \lambda d_1 \dot{\oplus} \lambda d_2$.
(4) $(d_1 \dot{\otimes} d_2)^{\lambda} = d_1^{\lambda} \dot{\otimes} d_2^{\lambda}$.
(5) $\lambda_1 d \dot{\oplus} \lambda_2 d = (\lambda_1 \dot{\oplus} \lambda_2) d$.
(6) $d^{\lambda_1} \dot{\otimes} d^{\lambda_2} = d^{\lambda_1 + \lambda_2}$.

Proof (1) and (2) are obvious. We prove (3) and (5), while (4) and (6) can be proven similarly.

(3)
$$\lambda(d_1 \dot{\oplus} d_2) = \lambda\left(\cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\left\{\frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2}\right\}, \cup_{\eta_1 \in g_1, \eta_2 \in g_2}\left\{\frac{\eta_1 \eta_2}{1 + (1 - \eta_1)(1 - \eta_2)}\right\}\right)$$
$$= \left(\cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\left\{\frac{(1 + \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2})^{\lambda} - (1 - \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2})^{\lambda}}{(1 + \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2})^{\lambda} + (1 - \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2})^{\lambda}}\right\},\right.$$
$$\left.\cup_{\eta_1 \in g_1, \eta_2 \in g_2}\left\{\frac{2(\frac{\eta_1 \eta_2}{1 + (1 - \eta_1)(1 - \eta_2)})^{\lambda}}{(2 - \frac{\eta_1 \eta_2}{1 + (1 - \eta_1)(1 - \eta_2)})^{\lambda} + (\frac{\eta_1 \eta_2}{1 + (1 - \eta_1)(1 - \eta_2)})^{\lambda}}\right\}\right)$$
$$= \left(\cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\left\{\frac{(1 + \gamma_1)^{\lambda}(1 + \gamma_2)^{\lambda} - (1 - \gamma_1)^{\lambda}(1 - \gamma_2)^{\lambda}}{(1 + \gamma_1)^{\lambda}(1 + \gamma_2)^{\lambda} + (1 - \gamma_1)^{\lambda}(1 - \gamma_2)^{\lambda}}\right\},\right.$$
$$\left.\cup_{\eta_1 \in g_1, \eta_2 \in g_2}\left\{\frac{2(\eta_1 \eta_2)^{\lambda}}{(2 - \eta_1)^{\lambda}(2 - \eta_2)^{\lambda} + (\eta_1 \eta_2)^{\lambda}}\right\}\right)$$

$$\lambda d_1 \dot{\oplus} \lambda d_2 = \left(\cup_{\gamma_1 \in h_1}\left\{\frac{(1 + \gamma_1)^{\lambda} - (1 - \gamma_1)^{\lambda}}{(1 + \gamma_1)^{\lambda} + (1 - \gamma_1)^{\lambda}}\right\}, \cup_{\eta_1 \in g_1}\left\{\frac{2\eta_1^{\lambda}}{(2 - \eta_1)^{\lambda} + \eta_1^{\lambda}}\right\}\right)$$
$$\dot{\oplus}\left(\cup_{\gamma_2 \in h_2}\left\{\frac{(1 + \gamma_2)^{\lambda} - (1 - \gamma_2)^{\lambda}}{(1 + \gamma_2)^{\lambda} + (1 - \gamma_2)^{\lambda}}\right\}, \cup_{\eta_2 \in g_2}\left\{\frac{2\eta_2^{\lambda}}{(2 - \eta_2)^{\lambda} + \eta_2^{\lambda}}\right\}\right)$$
$$= \left(\cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\left\{\frac{\frac{(1 + \gamma_1)^{\lambda} - (1 - \gamma_1)^{\lambda}}{(1 + \gamma_1)^{\lambda} + (1 - \gamma_1)^{\lambda}} + \frac{(1 + \gamma_2)^{\lambda} - (1 - \gamma_2)^{\lambda}}{(1 + \gamma_2)^{\lambda} + (1 - \gamma_2)^{\lambda}}}{1 + \frac{(1 + \gamma_1)^{\lambda} - (1 - \gamma_1)^{\lambda}}{(1 + \gamma_1)^{\lambda} + (1 - \gamma_1)^{\lambda}} \frac{(1 + \gamma_2)^{\lambda} - (1 - \gamma_2)^{\lambda}}{(1 + \gamma_2)^{\lambda} + (1 - \gamma_2)^{\lambda}}}\right\},\right.$$
$$\left.\cup_{\eta_1 \in g_1, \eta_2 \in g_2}\left\{\frac{\frac{2\eta_1^{\lambda}}{(2 - \eta_1)^{\lambda} + \eta_1^{\lambda}} \frac{2\eta_2^{\lambda}}{(2 - \eta_2)^{\lambda} + \eta_2^{\lambda}}}{1 + (1 - \frac{2\eta_1^{\lambda}}{(2 - \eta_1)^{\lambda} + \eta_1^{\lambda}})(1 - \frac{2\eta_2^{\lambda}}{(2 - \eta_2)^{\lambda} + \eta_2^{\lambda}})}\right\}\right)$$
$$= \left(\cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\left\{\frac{(1 + \gamma_1)^{\lambda}(1 + \gamma_2)^{\lambda} - (1 - \gamma_1)^{\lambda}(1 - \gamma_2)^{\lambda}}{(1 + \gamma_1)^{\lambda}(1 + \gamma_2)^{\lambda} + (1 - \gamma_1)^{\lambda}(1 - \gamma_2)^{\lambda}}\right\},\right.$$
$$\left.\cup_{\eta_1 \in g_1, \eta_2 \in g_2}\left\{\frac{2(\eta_1 \eta_2)^{\lambda}}{(2 - \eta_1)^{\lambda}(2 - \eta_2)^{\lambda} + (\eta_1 \eta_2)^{\lambda}}\right\}\right).$$

Thus, $\lambda(d_1 \dot{\oplus} d_2) = \lambda d_1 \dot{\oplus} \lambda d_2$.

(5)
$$\lambda_1 d \dot{\oplus} \lambda_2 d = (\cup_{\gamma \in h}\{\frac{(1+\gamma)^{\lambda_1} - (1-\gamma)^{\lambda_1}}{(1+\gamma)^{\lambda_1} + (1-\gamma)^{\lambda_1}}\}, \cup_{\eta \in g}\{\frac{2\eta^{\lambda_1}}{(2-\eta)^{\lambda_1} + \eta^{\lambda_1}}\})$$
$$\dot{\oplus}(\cup_{\gamma \in h}\{\frac{(1+\gamma)^{\lambda_2} - (1-\gamma)^{\lambda_2}}{(1+\gamma)^{\lambda_2} + (1-\gamma)^{\lambda_2}}\}, \cup_{\eta \in g}\{\frac{2\eta^{\lambda_2}}{(2-\eta)^{\lambda_2} + \eta^{\lambda_2}}\})$$
$$= (\cup_{\gamma \in h}\{\frac{\frac{(1+\gamma)^{\lambda_1} - (1-\gamma)^{\lambda_1}}{(1+\gamma)^{\lambda_1} + (1-\gamma)^{\lambda_1}} + \frac{(1+\gamma)^{\lambda_2} - (1-\gamma)^{\lambda_2}}{(1+\gamma)^{\lambda_2} + (1-\gamma)^{\lambda_2}}}{1 + \frac{(1+\gamma)^{\lambda_1} - (1-\gamma)^{\lambda_1}}{(1+\gamma)^{\lambda_1} + (1-\gamma)^{\lambda_1}} \frac{(1+\gamma)^{\lambda_2} - (1-\gamma)^{\lambda_2}}{(1+\gamma)^{\lambda_2} + (1-\gamma)^{\lambda_2}}}\},$$
$$\cup_{\eta \in g}\{\frac{\frac{2\eta^{\lambda_1}}{(2-\eta)^{\lambda_1} + \eta^{\lambda_1}} \frac{2\eta^{\lambda_2}}{(2-\eta)^{\lambda_2} + \eta^{\lambda_2}}}{1 + (1 - \frac{2\eta^{\lambda_1}}{(2-\eta)^{\lambda_1} + \eta^{\lambda_1}})(1 - \frac{2\eta^{\lambda_2}}{(2-\eta)^{\lambda_2} + \eta^{\lambda_2}})}\})$$
$$= (\cup_{\gamma \in h}\{\frac{(1+\gamma)^{\lambda_1+\lambda_2} - (1-\gamma)^{\lambda_1+\lambda_2}}{(1+\gamma)^{\lambda_1+\lambda_2} + (1-\gamma)^{\lambda_1+\lambda_2}}\},$$
$$\cup_{\eta \in g}\{\frac{2\eta^{\lambda_1+\lambda_2}}{(2-\eta)^{\lambda_1+\lambda_2} + \eta^{\lambda_1+\lambda_2}}\}) = (\lambda_1 \dot{\oplus} \lambda_2)d.$$

Thus, $\lambda_1 d \dot{\oplus} \lambda_2 d = (\lambda_1 \dot{\oplus} \lambda_2)d$. This completes the proof.

To compare the DHFEs, inspired by the comparison method of HFEs, the following definition is given:

Definition 1.26 (*Zhu et al.* 2012). Let $d = \{h, g\}$ be any two DHFEs,

$$s(d) = \frac{1}{l_h}\sum_{\gamma \in h}\gamma - \frac{1}{l_g}\sum_{\eta \in g}\eta \tag{1.39}$$

is called the score function of d, and

$$p(d) = \frac{1}{l_h}\sum_{\gamma \in h}\gamma + \frac{1}{l_g}\sum_{\eta \in g}\eta \tag{1.40}$$

is called the accuracy function of d, where l_h and l_g are the numbers of the elements in h and g, respectively.

Based on the score function and accuracy function of DHFEs, the following scheme is proposed to compare any two DHFEs d_1 and d_2:

(1) If $s(d_1) > s(d_2)$, then d_1 is superior to d_2, denoted by $d_1 \succ d_2$.
(2) If $s(d_1) = s(d_2)$, then

   (a) if $p(d_1) = p(d_2)$, then d_1 is equivalent to d_2, denoted by $d_1 \sim d_2$.
   (b) If $p(d_1) > p(d_2)$, then d_1 is superior than d_2, denoted by $d_1 \succ d_2$.

Example 1.10 (Zhu et al. 2012). Let $d_1 = (\{0.1, 0.3\}, \{0.3, 0.5\})$ and $d_2 = (\{0.2, 0.4\}, \{0.4, 0.6\})$ be two DHFEs, then based on Definition 1.26, we obtain $s(d_1) = s(d_2) = 0$, $p(d_2)(= 0.8) > p(d_1)(= 0.6)$. Thus, $d_2 \succ d_1$.

1.2.3 Hesitant Fuzzy Linguistic Term Set

It is noted that the above mentioned different forms of fuzzy sets suit the problems that are defined as quantitative situations. However, in real world decision making problems, many aspects of different activities cannot be assessed in a quantitative form, but rather in a qualitative one. Using linguistic information to express experts' opinions is suitable and straightforward because it is very close to human's cognitive processes. A common approach to model linguistic information is the fuzzy linguistic approach proposed by Zadeh (1975), which represents qualitative information as linguistic variables. Although it is less precise than a number, the linguistic variable, defined as "*a variable whose values are not numbers but words or sentences in a natural or artificial language*", enhances the flexibility and reliability of decision making models and provides good results in different fields. Nevertheless, similar to fuzzy sets, the fuzzy linguistic approach has some limitations and thus different linguistic representation models have been introduced, such as the 2-tuple fuzzy linguistic representation model (Herrera and Martínez 2000), the linguistic model based on type-2 fuzzy set (Türkşen 2002), the virtual linguistic model (Xu 2004a), the proportional 2-tuple model (Wang and Hao 2006), and so on. However, all these extended models are still very limited due to the fact that they are based on the elicitation of single or simple terms that should encompass and describe the information provided by decision makers (or experts) regarding to a linguistic variable. When the experts hesitate among different linguistic terms and need to use a more complex linguistic term that is not usually defined in the linguistic term set to depict their assessments, the above mentioned fuzzy linguistic approaches are out of use. Thus, motivated by the HFS, Rodríguez et al. (2012) proposed the concept of hesitant fuzzy linguistic term set (HFLTS), which provides a different and great flexible form to represent the assessments of decision makers.

In fuzzy linguistic approach, the decision makers' opinions are taken as the values of a linguistic variable which is established by linguistic descriptors and their corresponding semantics (Herrera and Herrera-Viedma 2000b). Once the experts provide the linguistic evaluation information, the following step is to translate these linguistic inputs into a machine manipulative format in which the computation can be carried out. Such translation is conducted by some fuzzy tools. Meanwhile, the outputs of the computing with words (CWW) model should also be easy to be converted into the linguistic information. To do so, Xu (2005b) proposed the subscript-symmetric additive linguistic term set, shown as

$$S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\} \quad (1.41)$$

where the mid linguistic label s_0 represents an assessment of "indifference", and the rest of them are placed symmetrically around it. In particular, $s_{-\tau}$ and s_τ are the lower and upper bounds of the linguistic labels used by the decision makers in practical applications. τ is a positive integer, and S satisfies the following conditions:

(1) If $\alpha > \beta$, then $s_\alpha > s_\beta$;
(2) The negation operator is defined: neg $(s_\alpha) = s_{-\alpha}$, especially, neg $(s_0) = s_0$.

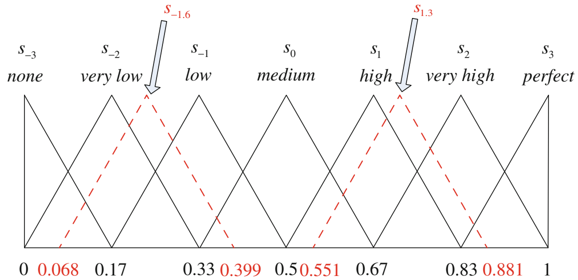
The linguistic term set S is a discrete linguistic term set and thus is not convenient for calculation and analysis. To preserve all given linguistic information, Xu (2005b) extended the discrete linguistic term set to a continuous linguistic term set $\bar{S} = \{s_\alpha | \alpha \in [-q, q]\}$, where $q(q > \tau)$ is a sufficiently large positive integer. In general, the linguistic term $s_\alpha (s_\alpha \in S)$ is determined by the decision makers, and the virtual linguistic term $\bar{s}_\alpha (\bar{s}_\alpha \in \bar{S})$ only appears in computation. The virtual linguistic term provides a tool to compute with the linguistic terms. The mapping between virtual linguistic terms and their corresponding semantics is easy to build, shown as Fig. 1.1 (Liao et al. 2014b).

As traditional fuzzy linguistic approach can only use single linguistic term, such as "*medium*", "*high*" or "*a little high*", to represent the value of a linguistic variable but cannot express complicated linguistic expressions such as "*between medium and high*", "*at least a little high*", Rodríguez et al. (2012) introduced the concept of HFLTS, which can be used to elicit several linguistic terms or linguistic expression for a linguistic variable.

Definition 1.27 (*Rodríguez et al.* 2012). Let $S = \{s_0, \ldots, s_\tau\}$ be a linguistic term set. A HFLTS, H_S, is an ordered finite subset of the consecutive linguistic terms of S.

Since Definition 1.27 does not give any mathematical form for HFLTS, Liao et al. (2015a) redefined the HFLTS mathematically as follows, which is much easier to be understood. Liao and Xu (2015c) also replaced the linguistic term set by the subscript-symmetric linguistic term set $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$.

Fig. 1.1 Semantics of virtual linguistic terms



Definition 1.28 (*Liao et al.* 2015a). Let $x \in X$, be fixed and $S = \{s_t | t = -\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ be a linguistic term set. A HFLTS in X, H_S, is in mathematical terms of

$$H_S = \{ <x, h_S(x)> | x \in X\} \tag{1.42}$$

where the function $h_S(x) : X \to S$ defines the possible membership grades of the element $x \in X$ to the set $A \subset X$ and for every $x \in X$, the value of $h_S(x)$ is represented by a set of some values in the linguistic term set S and can be expressed as $h_S(x) = \{ s_{\varphi_l}(x) | s_{\varphi_l}(x) \in S,\ l = 1, \ldots, L(x) \}$ with $\varphi_l \in \{-\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ being the subscript of a linguistic term $s_{\varphi_l}(x)$ and $L(x)$ being the number of linguistic terms in $h_S(x)$.

For convenience, Liao et al. (2015a) called $h_S(x)$ the hesitant fuzzy linguistic element (HFLE) and let $\mathbb{H}_S$ be the set of all HFLEs on S. For simplicity, $h_S(x)$, $s_{\varphi_l}(x)$ and $L(x)$ can be written respectively as h_S, s_{φ_l} and L for short. There are several special HFLEs, such as:

(1) empty HFLE: $h_S = \{\}$.
(2) full HFLE: $h_S = S$.
(3) the complement of HFLE h_S: $h_S^c = S - h_S = \{ s_{\varphi_l} | s_{\varphi_l} \in S \textit{ and } s_{\varphi_l} \notin h_S \}$.

Although the HFLTS can be used to elicit several linguistic values for a linguistic variable, it is still not similar to the human way of thinking and reasoning. Thus, Rodríguez et al. (2012) further proposed a context-free grammar to generate simple but elaborated linguistic expressions *ll* that are more similar to the human expressions and can be easily represented by means of HFLTS. The grammar G_H is a 4-tuple (V_N, V_T, I, P) where V_N is a set of nonterminal symbols, V_T is the set of terminals' symbols, I is the starting symbols, and P is the production rules.

Definition 1.29 (*Rodríguez et al.* 2012). Let S be a linguistic term set, and G_H be a context-free grammar. The elements of $G_H = (V_N, V_T, I, P)$ are defined as:

$$\begin{aligned} V_N &= \{ <\textit{primary term}>, <\textit{composite term}>, \\ &\quad <\textit{unary relation}>, <\textit{binary relation}>, <\textit{conjunction}> \} \\ V_T &= \{\textit{lower than}, \textit{greater than}, \textit{at least}, \textit{at most}, \textit{between}, \\ I &\in V_N; \end{aligned}$$

$$\begin{aligned}
P = \{ I ::= & <primary\ term> | <composite\ term> \\
& <composite\ term> \ ::= <unary\ relation> \ <primary\ term> | \\
& \qquad <binary\ relation> \ <conjunction> \ <primary\ term> \\
& <primary\ term> \ ::= s_{-\tau} | \cdots | s_{-1} | s_0 | s_1 | \cdots | s_\tau \\
& <unary\ relation> \ ::= lower\ than | greater\ than \\
& <binary\ relation> \ ::= between \\
& <conjunction> \ ::= and \}.
\end{aligned}$$

Note: In the above definition, the brackets enclose optional elements and the symbol "|" indicates alternative elements.

The expressions ll generated by the context-free grammar G_H may be either single valued linguistic terms $s_t \in S$ or linguistic expressions. The transformation function E_{G_H} can be used to transform the expressions ll that are produced by G_H into HFLTS.

Definition 1.30 (*Rodríguez et al.* 2012). Let E_{G_H} be a function that transforms linguistic expressions $ll \in S_{ll}$, obtained by using G_H, into the HFLTS H_S. S is the linguistic term set used by G_H, and S_{ll} is the expression domain generated by G_H:

$$E_{G_H} : S_{ll} \to H_S \tag{1.43}$$

The linguistic expression generated by G_H using the production rules are converted into HFLTS by means of the following transformations:

- $E_{G_H}(s_t) = \{s_t | s_t \in S\}$;
- $E_{G_H}(at\ most\ s_m) = \{s_t | s_t \in S\ and\ s_t \le s_m\}$;
- $E_{G_H}(lower\ than\ s_m) = \{s_t | s_t \in S\ and\ s_t < s_m\}$;
- $E_{G_H}(at\ least\ s_m) = \{s_t | s_t \in S\ and\ s_t \ge s_m\}$;
- $E_{G_H}(great\ than\ s_m) = \{s_t | s_t \in S\ and\ s_t > s_m\}$;
- $E_{G_H}(between\ s_m\ and\ s_n) = \{s_t | s_t \in S\ and\ s_m \le s_t \le s_n\}$.

With the transformation function E_{G_H} defined as Definition 1.30, it is easy to transform the initial linguistic expressions into HFLTS. Liao et al. (2015a) used a figure (see Fig. 1.2) to show the relationships among the context-free grammar G_H, the linguistic expression ll and the HFLTS H_S.

Example 1.11 (Liao et al. 2015a). Quality management is more and more popular in our daily life. In the process of quality management, many aspects of certain products cannot be measured as crisp values but only qualitative values. Here we

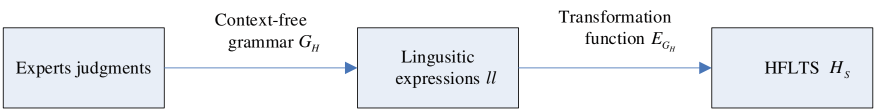


Fig. 1.2 The way to obtain a HFLTS

just consider a simple example that an expert evaluates the operational complexity of three automatic systems, represented as x_1, x_2 and x_3. Since this criterion is qualitative, it is impossible to give crisp values but only linguistic terms. The operational complexity of these automatic systems can be taken as a linguistic variable. The linguistic term set for the operational complexity can be set up as:

$$S = \{s_{-3} = \textit{very complex},\ s_{-2} = \textit{complex}, s_{-1} = \textit{a little complex},\ s_0 = \textit{medium},\\ s_1 = \textit{a little easy},\ s_2 = \textit{easy},\ s_3 = \textit{very easy}\}$$

With the linguistic term set and also the context-free grammar, the expert determines his/her judgments over these three automatic systems with linguistic expressions, which are $ll_1 = \textit{at least a little easy}$, $ll_2 = \textit{between complex and medium}$ and $ll_3 = \textit{great than easy}$. These linguistic expressions are similar to human way of thinking and they can reflect the expert's hesitant cognition intuitively. Using the transformation function E_{G_H}, a HFLTS can be yielded as $H_S(x) = \{<x_1, h_S(x_1)>, <x_2, h_S(x_2)>, <x_3, h_S(x_3)>\}$ with $h_S(x_1) = \{s_1, s_2, s_3\}$, $h_S(x_2) = \{s_{-2}, s_{-1}, s_0\}$, and $h_S(x_3) = \{s_3\}$ being three HFLEs.

Example 1.12 (Liao et al. 2015a). Consider a simple example that a Chief Information Officer (CIO) of a company evaluates the candidate ERP system in terms of three criteria, i.e., x_1 (potential cost), x_2 (function), and x_3 (operation complexity). Since the three criteria are qualitative, the CIO gives his evaluation values in linguistic expressions. Different criteria are associated with different linguistic term sets and different semantics. The linguistic term sets for these three criteria are set up as:

$$S_1 = \{s_{-3} = \textit{very expensive},\ s_{-2} = \textit{expensive},\ s_{-1} = \textit{a little expensive},\ s_0 = \textit{medium},\\ s_1 = \textit{a little cheap},\ s_2 = \textit{cheap},\ s_3 = \textit{very cheap}\}$$

$$S_2 = \{s_{-3} = \textit{none},\ s_{-2} = \textit{very low},\ s_{-1} = \textit{low}, s_0 = \textit{medium},\\ s_1 = \textit{high}, s_2 = \textit{very high},\ s_3 = \textit{perfect}\}$$

$$S_3 = \{s_{-3} = \textit{too complex},\ s_{-2} = \textit{complex}, s_{-1} = \textit{a little complex},\ s_0 = \textit{medium},\\ s_1 = \textit{a little easy},\ s_2 = \textit{easy},\ s_3 = \textit{every easy}\}$$

respectively. With these linguistic term sets and also the context-free grammar, the CIO provides his evaluation values in linguistic expressions for a ERP system as: $ll_1 = \textit{between cheap and very cheap}, ll_2 = \textit{at least high}, ll_3 = \textit{great than easy}$. Using the transformation function E_{G_H}, a HFLTS is obtained as $H(x) = \{<x_1, h_{S_1}(x_1)>, <x_2, h_{S_2}(x_2)>, <x_3, h_{S_3}(x_3)>\}$ with $h_{S_1}(x_1) = \{s_2, s_3 | s_2, s_3 \in S_1\}$, $h_{S_2}(x_2) = \{s_1, s_2, s_3 | s_1, s_2, s_3 \in S_2\}$ and $h_{S_3}(x_3) = \{s_3 | s_3 \in S_3\}$. Furthermore, if we ignore the influence of different semantics over different linguistic term sets on the criteria, i.e., let $S = \{s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, s_3\}$, then the HFLTS $H(x)$ can be rewritten as

$H_S(x) = \{<x_1, h_S(x_1)>, <x_2, h_S(x_2)>, <x_3, h_S(x_3)>\}$ with $h_S(x_1) = \{s_2, s_3\}$, $h_S(x_2) = \{s_1, s_2, s_3\}$ and $h_S(x_3) = \{s_3\}$.

From Examples 1.11 and 1.12, we can find that X, in Definition 1.28, could be either a set of objects on a linguistic variable or a set of linguistic variables of an object (in this case, the influence of different semantics over different linguistic term sets on different linguistic variables should be ignored).

Rodríguez et al. (2012) defined the complement, union and intersection of HFLTSs:

Definition 1.31 (*Rodríguez et al.* 2012). For three HFLEs h_S, h_S^1 and h_S^2, the following operations are defined:

(1) Lower bound: $h_S^- = \min(s_t) = s_k$, $s_t \in h_S$ and $s_t \geq s_k$, $\forall t$.
(2) Upper bound: $h_S^+ = \max(s_t) = s_k$, $s_t \in h_S$ and $s_t \leq s_k$, $\forall t$.
(3) $h_S^c = S - h_S = \{s_t | s_t \in S$ and $s_t \notin h_S\}$.
(4) $h_S^1 \cup h_S^2 = \{s_t | s_t \in h_S^1$ or $s_t \in h_S^2\}$.
(5) $h_S^1 \cap h_S^2 = \{s_t | s_t \in h_S^1$ and $s_t \in h_S^2\}$.

For a HFLE $h_S = \{s_{\varphi_l} | l = 1, 2, \ldots, L\}$, the linguistic terms in it might be out of order. To simplify the computation, we can arrange the linguistic terms s_{φ_l} $(l = 1, \ldots, L)$ in any of the following orders (Liao and Xu 2015c): ① ascending order $\delta : (1, 2, \ldots, n) \rightarrow (1, 2, \ldots, n)$ is a permutation satisfying $\delta_l \leq \delta_{l+1}$, $l = 1, \ldots, L$; ② descending order $\eta : (1, 2, \ldots, n) \rightarrow (1, 2, \ldots, n)$ is a permutation satisfying $\eta_l \geq \eta_{l+1}$, $l = 1, \ldots, L$. In addition, considering that different HFLEs may have different numbers of linguistic terms, we can extend the short HFLEs by adding some linguistic terms in it till they have same length. Liao et al. (2014b) introduced a method to add linguistic terms in a HFLE. For a HFLE $h_S = \{s_{\varphi_l} | l = 1, 2, \ldots, L\}$, let s^+ and s^- be the maximal and minimal linguistic terms in the HFLE h_S, defined as $s^+ = \max_{\varphi_l}\{s_{\varphi_l} | l = 1, 2, \ldots, L\}$ and $s^- = \min\limits_{\varphi_l}\{s_{\varphi_l} | l = 1, 2, \ldots, L\}$, respectively, and ξ $(0 \leq \xi \leq 1)$ be an optimized parameter, then we can add the linguistic term:

$$\bar{s} = \xi s^+ \oplus (1 - \xi) s^- \tag{1.44}$$

to the HFLE. The optimized parameter, which is used to reflect the decision makers' risk preferences, is provided by the decision makers.

Motivated by the score function and the variance function of HFS, Liao et al. (2015c) introduced the score function and the variance function for HFLE.

Definition 1.32 (*Liao et al.* 2015c). For a HFLE $h_S = \cup_{s_{\delta_l} \in h_S} \{s_{\delta_l} | l = 1, \ldots, L\}$ where L is the number of linguistic terms in h_S, $\rho(h_S) = \frac{1}{L}\sum_{s_{\delta_l} \in h_S} s_{\delta_l} = s_{\frac{1}{L}\sum_{l=1}^{L} \delta_l}$ is called the score function of h_S.

Definition 1.33 (*Liao et al.* 2015c). For a HFLE $h_S = \cup_{s_{\delta_l} \in h_S} \{s_{\delta_l} | l = 1, \ldots, L\}$ where L is the number of linguistic terms in h_S, $\sigma(h_S) = \frac{1}{L}\sqrt{\sum_{s_{\delta_l}, s_{\delta_k} \in h_S} (s_{\delta_l} - s_{\delta_k})^2} = s_{\frac{1}{L}\sqrt{\sum_{s_{\delta_l}, s_{\delta_k} \in h_S} (\delta_l - \delta_k)^2}}$ is called the variance function of h_S.

The relationship between the score function and the variance function of HFLE is similar to the relationship between mean and variance in statistics. Thus, for two HFLEs h_S^1 and h_S^2, the following approach can be used to compare any two HFLEs:

- If $\rho(h_S^1) > \rho(h_S^2)$, then $h_S^1 > h_S^2$.
- Else if $\rho(h_S^1) = \rho(h_S^2)$, then,
  - if $\sigma(h_S^1) < \sigma(h_S^2)$, then $h_S^1 > h_S^2$;
  - else if $\sigma(h_S^1) = \sigma(h_S^2)$, then $h_S^1 = h_S^2$.

Example 1.13 (Liao et al. 2015c). Let $S = \{s_{-3} = none,\ s_{-2} = very\ low,\ s_{-1} = low,\ s_0 = medium,\ s_1 = high,\ s_2 = very\ high,\ s_3 = perfect\}$ be a linguistic term set. The linguistic information obtained by means of the context-free grammar is $\phi_1 = high$, $\phi_2 = lower\ than\ medium$, $\phi_3 = greater\ than\ high$, and $\phi_4 = between\ medium\ and\ very\ high$. With the transformation function, the above linguistic information can be represented as $H_S = \{h_S^1, h_S^2, h_S^3, h_S^4\}$ with $h_S^1 = \{s_1\}$, $h_S^2 = \{s_{-3}, s_{-2}, s_{-1}, s_0\}$, $h_S^3 = \{s_1, s_2, s_3\}$ and $h_S^4 = \{s_0, s_1, s_2\}$. Then, we have $\rho(h_S^1) = s_1$, $\rho(h_S^2) = s_{-1.5}$, $\rho(h_S^3) = s_2$, $\rho(h_S^4) = s_1$. Since $\rho(h_S^3) > \rho(h_S^1) = \rho(h_S^4) > \rho(h_S^2)$, it yields that $MAX(H_S) = h_S^3$, $MIN(H_S) = h_S^2$.

Calculating the variance functions of h_S^1 and h_S^4, we have $\sigma(h_S^1) = s_0$, $\sigma(h_S^4) = s_{0.8165}$. Since $\sigma(h_S^1) < \sigma(h_S^4)$, then we get $h_S^1 > h_S^4$. Hence, the rank of these four HFLEs is $h_S^3 > h_S^1 > h_S^4 > h_S^2$.

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