Chapter 7
Hydrodynamic Formulation: Mass and Salt Conservation Equations

When hydrographic properties and motions in an estuary have spatial and temporal variation, they are termed as non-uniform and unsteady, as opposed to uniform and in steady-state. In the previous chapter, salinity in the estuary was, by hypothesis, in steady-state conditions in longitudinal segments during complete tidal cycles and at high and low tidal conditions. However, estuaries are dynamic systems, and salinity and current velocity vary in time and space from almost at rest (slack water) to speeds of up to several meters per second, in estuaries forced by macro-tides. In observational data analysis, it is usual to simulate steady-state conditions of hydrographic properties and circulation by calculating mean values during a time interval of one or more tidal cycles, under the assumption that the river discharge remains constant during this time. Tidal co-oscillation is the main driving force of the non-steady-state condition, however, in some situations other forces may also be important, such as the abnormal storm surge due to wind shear stress acting on the continental shelf.

The mathematical development of this chapter starts with the equation of mass conservation, also named the continuity equation, which complements the equation of motion, which will be studied in the next chapter. In practical applications it will be necessary to assume as given the estuary geometry, the river discharge and the initial and boundary conditions. For investigation of any hydrographic property, it will also be necessary to use the corresponding conservation equation; in the case of non-conservative properties, sources and sinks must also be specified.

For the application of the fundamental principles of Fluid Mechanics, it is necessary to consider an infinitesimally small fluid sample. Usually this sample is referred as a *material element,* or more often as a *volume element*. Another assumption is that all variables which will represent physical properties (scalar or vectorial, such as hydrographic properties or current velocity, respectively) are continuous functions of space and time; in this way the mathematical rules of the differential and integral calculus can be applied.

7.1 State of a Volume Element

When the fluid is in motion, its properties (scalar or vector) are functions of space (x, y, z) and time (t). Then, any property of the fluid, generically indicated by P, is expressed by the function:

$$P = P(x, y, z, t) = P(\overrightarrow{r}, t), \qquad (7.1)$$

where $\overrightarrow{r} = \overrightarrow{r}(x, y, z)$ indicating the position vector of a small volume, δV, and x, y and z being its coordinates in space.

Conceptually, this volume of fluid presents the characteristics of properties associated with this elementary volume within a given water flow. According to Symon (1957) and Gill (1982), if this element is at the position $\overrightarrow{r}$ at the instant of time t, its position in the space may be generically indicated by the vector position $\overrightarrow{r} = \overrightarrow{r}[x(t), y(t), z(t)] = \overrightarrow{r}(t)$. Then, a generic property of the volume element is expressed by the following functional relationship:

$$P = P[x(t), y(t), z(t), t] = P[\overrightarrow{r}(t), t]. \qquad (7.2)$$

From this expression it follows that the total rate of variation (dP/dt) of the property is,

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt}, \qquad (7.3)$$

or

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \frac{d\overrightarrow{r}}{dt} \bullet \nabla P, \qquad (7.4)$$

where the symbol ∇ is the *nabla* operator, $\nabla = (\frac{\partial}{\partial x})\overrightarrow{i} + (\frac{\partial}{\partial y})\overrightarrow{j} + (\frac{\partial}{\partial z})\overrightarrow{k}$, and the dot (•) indicates the scalar product. Equation (7.4) indicates that: (i) the rate at which the property, P, is changing with time at a fixed point in space is the partial derivative with respect to time ($\partial P/\partial t$), which is itself a function of x, y, z, and; (ii) the rate at which the property, P, is changing with respect to a point moving along with the fluid. Another component $(\frac{d\overrightarrow{r}}{dt})$ is the variation of the volume element's position in space,

$$\frac{d\overrightarrow{r}}{dt} = \overrightarrow{v} = \overrightarrow{v}(u, v, w) = \overrightarrow{v}(x, y, z, t), \qquad (7.5)$$

where u = u(x, y, z, t), v = v(x, y, z, t) and w = w(x, y, z, t) are the velocity components of the velocity vector in the coordinate axes Ox, Oy and Oz,

respectively, and are functions of time (t). Then, the equation that defines the state of a volume element, δV, of fluid is given by,

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + \vec{v} \bullet \nabla P. \tag{7.6}$$

Thus, the total time variation of the property, P, is composed of the local variation ($\partial P/\partial t$) and the variation due to the advection, which depends on the fluid velocity and the property gradient ($\vec{v} \bullet \nabla P$). When the local variation is zero ($\partial P/\partial t = 0$), the spatial property variation is considered to be in *steady-state*, and when it has no spatial variation it is *uniform*.

To simplify the notation of equations, it is useful to define the total derivative operator, d/dt, as:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \bullet \nabla. \tag{7.7}$$

This definition is very convenient, as may be seen considering the salt conservation equation of seawater, simply equating P = S. In fact, if the molecular and turbulent diffusion are neglected, the volume element in motion will retain the same concentration of its dissociate components, and its mass will remain constant during the motion. Mathematically,

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \vec{v} \bullet \nabla S = 0. \tag{7.8a}$$

This equation indicates that during the motion there is an equilibrium between the local ($\partial S/\partial t$) and advective ($\vec{v} \bullet \nabla S$) variation. In the steady-state of the salinity field $\vec{v} \bullet \nabla S = 0$, or,

$$u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} = 0. \tag{7.8b}$$

7.2   Mass and Salt Conservation Equations

Let us consider the fluid motion in a laminar flow regime, which usually holds for slow motions. Even if the volume element has a constant mass, its volume may vary due to the pressure acting on its surface during the motion. As density is defined by the ratio of mass by volume ($\rho = m/V$, $[\rho] = [ML^{-3}]$), it follows that density, being a dependant property, may vary with element volume changes. The equation relating the fluid's density with its motion (velocity) may be defined by the mass conservation principle, which is traditionally named continuity equation since it

follows the conservation laws, and it is related to the density and velocity of a continuous medium.

The continuity equation is of fundamental importance to studies related to fluid motion, and its deduction may be obtained with different theoretical developments (Symon 1957; Brand 1959; Kinsman 1965; Neumann and Pierson 1966; Gill 1982, and others). A straightforward Eulerian formulation may be made using Gauss' divergence theorem, equating the local density time rate $(\partial\rho/\partial t)$, integrated in a differential volume element, V, enclosed by its area A,

$$\int_V (\frac{\partial\rho}{\partial t})dV, \tag{7.9a}$$

with the mass transport into the volume, V, through the closed surface area, A, which is expressed by:

$$-\int_A \rho\overrightarrow{v}\bullet\overrightarrow{n}\,dA = -\int_V (\nabla\bullet\rho\overrightarrow{v})dV, \tag{7.9b}$$

where $\overrightarrow{n}$ is the unity vector orthogonal to the closed surface, oriented outward of the volume V. Hence, equating the mass transport $[MT^{-1}]$ expressed by Eqs. 7.9a with the corresponding mass transport on the right-hand-side of Eq. 7.9b,

$$\int_V (\frac{\partial\rho}{\partial t})dV = -\int_V (\nabla\bullet\rho\overrightarrow{v})dV, \tag{7.9c}$$

it follows the mass conservation property inside the control volume V, or the continuity equation,

$$\frac{\partial\rho}{\partial t} + \nabla\bullet\rho\overrightarrow{v} = 0. \tag{7.10}$$

This equation indicates the following physical principle: the local density variation inside a volume element is due only to the divergence operator of the mass flux $(\rho\overrightarrow{v})$, $([\rho\overrightarrow{v}] = [ML^{-2}T^{-1}])$ through a closed surface of the fluid element. Using the divergent operator, the second term of this equation may be written as:

$$\nabla\bullet\rho\overrightarrow{v} = \overrightarrow{v}\bullet\nabla\rho + \rho\nabla\bullet\overrightarrow{v}, \tag{7.11}$$

combining expressions (7.11) and (7.10),

$$\frac{\partial\rho}{\partial t} + \overrightarrow{v}\bullet\nabla\rho + \rho\nabla\bullet\overrightarrow{v} = 0, \tag{7.12}$$

and, taking into account the total derivative (7.7), the continuity equation is reduced to the following expression:

$$\frac{d\rho}{dt} + \rho \nabla \bullet \vec{v} = 0, \tag{7.13a}$$

or

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla \bullet \vec{v} = 0, \tag{7.13b}$$

which expresses the relationship between the time variation of fluid density and its velocity.

Equations (7.10) and (7.13b) are different mathematical expressions of the continuity equation. The first term of Eq. (7.13b) is the relative change of the total density variation, and the second is the divergent operator of the velocity field. At this point we should remember that the divergent operator may be positive, negative or zero, indicating the divergent, convergent or non-divergent fields, respectively.

The continuity equation in the differential form (7.13b) is valid for fluids in laminar motion with only one component, such as pure water. In the field of Physical Oceanography, seawater is considered a solution with two components (pure water + salt) and the mass of a volume element may vary due to salt diffusion through its geometric boundaries. Thus, when the continuity equation is applied to seawater, unless this diffusion process is negligible, it must be compensated by the introduction of a parcel which takes this into account to preserve the mass conservation principle.

To demonstrate that the salt diffusion may be disregarded in coastal and estuarine water masses which have non-constant ionic composition, Csanady (1982) presented the following development to the quantitative determination of the relative density time rate and the divergence of the velocity field of Eq. (7.13b). For a typical summer day, it is estimated that the time taken to heat the surface layer of the estuarine water mass by 1.0 °C is three hours. Thus, the temperature time rate increase is: $dT/dt = 1.0 \times 10^{-4}$ °C s^{-1}. Adopting a typical value of 1.0×10^{-4} °C^{-1} to the thermal expansion coefficient, the relative rate at which the density is changing with time at a fixed point in space (first term of Eq. (7.13b) is estimated to be 1.0×10^{-8} s^{-1}.

To estimate the molecular salt diffusion on the local density time rate, a value of 1.0×10^{-9} m^2 s^{-1} was adopted for the kinematic salt molecular diffusion coefficient (D). And, as the salt molecular diffusion obeys the Fickian law,

$$\frac{dS}{dt} = D\nabla^2 S, \tag{7.14}$$

the total salt variation (dS/dt) may be calculated, for the most unfavorable condition ($\nabla^2 S = 1$), as equal to $1.0 10^{-9}$ s^{-1}. Using a mean value for the saline contraction

coefficient (β) of $7.5 \times 10^{-4} \approx 10^{-3}$, the estimated value of the kinematic molecular salt diffusion coefficient is 1.0×10^{-12} m^2 s^{-1}. These results indicate that the influences of the local heating and salt diffusion on the relative local density variation are by a magnitude of less than or equal to 1.0×10^{-8} s. For an estuary with a length of 10 km (1.0×10^4 m), a longitudinal density variation of 10 kg m^{-3} between its mouth and its head, and a velocity variation of 1.0 m s^{-1}, it follows that the relative density time variation (first term of the Eq. 7.13b) is less than or equal to 1.0×10^{-6} s^{-1}.

Let us now estimate the order of magnitude of the second parcel of Eq. (7.13b), representing the divergence of the velocity field. Observational data of estuaries indicate that the u-velocity component may vary from 0 to 1.0 m s^{-1} over distances of up to 1.0×10^4 m, and its divergence value is estimated in 1.0×10^{-4} s^{-1}. Comparing this value with the estimated value with the estimated value for the first parcel (1.0×10^{-6} s^{-1}), the conclusion is that the influence of the velocity divergence is predominant, even in the extreme conditions of the above example. Then, for practicality, the continuity equation is reduced to the simple expression in the Cartesian coordinate system:

$$\nabla \bullet \overrightarrow{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{7.15}$$

This mass conservation Eq. (7.15) may also be obtained from Eq. (7.13b) under the hypothesis that the fluid density is a constant (ρ = const.) or its relative value doesn't change during motion ($\frac{1}{\rho}\frac{\partial \rho}{\partial t} = 0$), which corresponds to the behavior of incompressible fluids. Thus, for practicality, estuarine water mass is considered to be an incompressible fluid. In some texts of Hydrodynamics, Eq. (7.13a, 7.13b) are named conservation of mass, and the expression continuity equation is usually used for Eq. (7.15).

As the motion regime in an estuary is transitional, changing from laminar to turbulent, the continuity equation must be adapted to take into account the turbulent flow. This may be accomplished by eliminating the random (or turbulent) small scale velocity fluctuations, dividing the velocity into two terms which are uncorrelated with one another: a mean time ($\langle \overrightarrow{v} \rangle$), and a turbulence velocity value ($\langle \overrightarrow{v'} \rangle$). The mean value $\langle \overrightarrow{v} \rangle$, is calculated from a time interval Δt which is long enough (generally a few minutes) to eliminate the turbulent fluctuations $\overrightarrow{v'}$, but short enough that the larger-scale variations do not affect the mean value. That is, the average value of the turbulent fluctuations should equal zero ($\langle \overrightarrow{v'} \rangle = 0$). Substituting this instantaneous value into Eq. (7.15), gives,

$$\nabla \bullet (\langle \overrightarrow{v} \rangle + \overrightarrow{v'}) = 0, \tag{7.16}$$

and calculating its mean time value for the time interval (Δt),

$$\frac{1}{\Delta t} \int_0^{\Delta t} \nabla \bullet (\langle \overrightarrow{v} \rangle + \langle \overrightarrow{v}' \rangle) dt = 0. \tag{7.17}$$

As the divergence is calculated as spatial derivatives of vector velocity, which is assumed to be a continuous function, according to the Schwartz's theorem it is possible to change the order of the derivative and integration operations. Taking into account that $\langle \langle \overrightarrow{v} \rangle \rangle = \langle \overrightarrow{v} \rangle$ and $\langle \overrightarrow{v}' \rangle = 0$, it follows that the expression of the continuity equation for a turbulent fluid flow is,

$$\nabla \bullet (\langle \overrightarrow{v} \rangle) = \nabla \bullet \overrightarrow{v} = 0, \tag{7.18}$$

where, to simplify the notation, the time mean value $(\langle \overrightarrow{v} \rangle)$ is substituted by $\overrightarrow{v}$ $(\langle \overrightarrow{v} \rangle = \overrightarrow{v} = u\overrightarrow{i} + v\overrightarrow{j} + w\overrightarrow{k})$. This vector now has u, v and w components, which are time mean values of a relatively short time interval (Δt). Then, the continuity equation, for a transitional or turbulent flow in the Cartesian frame of reference (Oxyz), is formally expressed by a similar equation which holds for laminar fluid flow (Eq. 7.15),

$$\nabla \bullet \overrightarrow{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{7.19}$$

When this expression of the continuity equation is integrated with respect to a geometric volume, such as for an estuary, at the free surface and bottom layers there will be sources and sinks of mass (evaporation-precipitation balance, snow, condensation on the surface, and bottom spring water), which must be adequately specified.

Now, let us apply the principle of mass conservation to other properties that are used to characterize the state of a water mass, such as its salt content. In practice, the principle of continuity is most often used together with the principle of conservation of salt to study the flow of relatively enclosed bodies of water, such as estuaries. By conservative properties we mean concentrations, such as salinity, that are altered locally, except at the boundaries, by diffusion and advection only.

The vector which characterizes the advective salt flux $(\overrightarrow{S})$, expressed by mass of salt per area and time $([\overrightarrow{S}] = [ML^{-2}T^{-1}])$, generated by a laminar motion, $\overrightarrow{v}$, is expressed by $\overrightarrow{S} = \rho S \overrightarrow{v}$. Substituting in Eq. (7.10) the density, ρ, with the scalar quantity, ρS, which has the same dimension as density $([\rho S] = [ML^{-3}])$, but physically represents the concentration of mass of salt dissociated in seawater, it follows that:

$$\frac{\partial(\rho S)}{\partial t} + \nabla \bullet \rho S \overrightarrow{v} = 0, \tag{7.20}$$

or

$$\rho(\frac{\partial S}{\partial t} + \overrightarrow{v} \bullet \nabla S) + S(\frac{\partial \rho}{\partial t} + \nabla \bullet \rho \overrightarrow{v}) = 0. \tag{7.21}$$

These equations are the analytical expressions of the principle of conservation of salt, only due to the advection. As the expression between the parentheses of the second parcel of Eq. (7.21) is the continuity Eq. (7.10), and is equal to zero, then

$$\frac{\partial S}{\partial t} + \overrightarrow{v} \bullet \nabla S = 0, \tag{7.22}$$

or, in the scalar notation

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} = \frac{dS}{dt} = 0. \tag{7.23}$$

As stated previously, $\frac{dS}{dt}$ is the total time variation of the salinity. This differential equation is the principle of the conservation of salt, under the action of advection for a small volume of seawater, with the assumption that the molecular diffusion has been disregarded. However, as estuaries usually have a turbulent flow regime, the salt flux due to the turbulent diffusion is much higher. Thus, the influence of turbulent motion on the salt balance of estuarine waters must also be taken into account. Consider a cubic volume with surface area units normal to the coordinate axis, for an estuarine water mass without free surface. The salt conservation equation in the differential form is rigorously written as (Sverdrup et al. 1942; Pritchard 1958; Cameron and Pritchard 1963, and others):

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} = \frac{\partial}{\partial x}(K_x\frac{\partial S}{\partial x}) + \frac{\partial}{\partial y}(K_y\frac{\partial S}{\partial y}) + \frac{\partial}{\partial z}(K_z\frac{\partial S}{\partial z}). \tag{7.24}$$

In this equation, K_x, K_y and K_z are the kinematic[1] coefficients of turbulent diffusion of salt in the horizontal (Ox and Oy) and vertical (Oz) axes, respectively, which in general are functions of the spatial and temporal scales of the estuarine processes, with dimensions $[L^2T^{-1}]$. Equation (7.24) indicates that the local salinity variation $(\frac{\partial S}{\partial t})$ is dependent on the advection (velocity components u, v and w in the

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[1]The dynamic coefficients of eddy diffusion (dispersion), A_x, A_y and A_z, which have dimensions of $[ML^{-1}T^{-1}]$, is obtained from the product of density, ρ, by the corresponding kinematic coefficient; $A_x = \rho K_x$, $A_y = \rho K_y$ and $A_z = \rho K_z$.

left-hand-side terms), and turbulent diffusion (terms in the right-hand-side) simulated by the Fickian law.

Another expression for the salt conservation equation may be obtained when combined with the continuity Eq. (7.19):

$$\frac{\partial S}{\partial t} + \frac{\partial (uS)}{\partial x} + \frac{\partial (vS)}{\partial y} + \frac{\partial (wS)}{\partial z} = \frac{\partial}{\partial x}\left(K_x \frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial S}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial S}{\partial z}\right).$$

$$(7.25)$$

It is implicit in this equation that $S = S(x, y, z, t)$ represents the average salinity obtained from a time interval that is long enough to eliminate the turbulent variations (S'), but short enough for this mean value not to be affected by long- term variations. In the same way, the velocity components $u = u(x, y, z, t)$, $v = v(x, y, z, t)$ and $w = w(x, y, z, t)$ in the Eqs. (7.24) and (7.25) also represent mean values. If advection alone is responsible for the mixing process in steady-state conditions, these equations are reduce to their vector formulation $\vec{v} \bullet \nabla S = 0$, or $\nabla \bullet S\vec{v} = 0$.

The kinematics eddy diffusion coefficients K_x, K_y and K_z, with dimension $[L^2 T^{-1}]$, are parameterized by cross correlations of the velocity turbulent fluctuations (u', v', w') and S', with expressions similar to those of the turbulent or eddy kinematic viscosity coefficients developed by Osborne Reynolds in 1894 (Pritchard 1954; Bowden 1963; Lacombe 1965, and others):

$$K_x = -\frac{\langle u'S' \rangle}{\frac{\partial S}{\partial x}}; K_y = -\frac{\langle v'S' \rangle}{\frac{\partial S}{\partial y}}; K_z = -\frac{\langle w'S' \rangle}{\frac{\partial S}{\partial z}}. \qquad (7.26)$$

It should be noted that the numerators of Eq. (7.26), multiplied by the density ($\rho\langle u'S'\rangle$, $\rho\langle v'S'\rangle$, $\rho\langle w'S'\rangle$), have dimensions of the salt fluxes generated by turbulent or eddy diffusion. Salinity and current velocity measurements in the James River estuary (Virginia, USA) taken over several tidal cycles in a cross section, gave the following results (Pritchard 1954):

- The horizontal advective (ρSu) and the vertical non-advective ($\rho\langle w'S'\rangle$) fluxes of salt were the most important factors in maintaining the salt balance.
- The mean vertical advective (ρSw) and the horizontal non-advective ($\rho\langle u'S'\rangle$) fluxes were of secondary importance, but still significant and small, respectively.
- In addition, the vertical non-advective flux ($\rho\langle w'S'\rangle$) of salt is partly related to the magnitude of the oscillatory tidal currents, and is dependent on the vertical salinity stratification.

Pritchard's work confirmed the hypothesis that the mixing process in an estuary is mainly related to the tidal forcing, and suggested the possibility of calculating the turbulent diffusion terms using a modified version of Eq. (7.26) for a laterally homogeneous estuary, taking into account its width variation.

A conservation equation, similar to (7.24), may also be used in the mathematical simulation of a conservative concentration of a property dissociated in an estuary. If $C = C(x, y, z, t)$ denotes the property's concentration, $[C] = [ML^{-3}]$, the conservation equation is:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = \frac{\partial}{\partial x}\left(K_{xC}\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{yC}\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_{zC}\frac{\partial C}{\partial z}\right),$$

(7.27)

where K_{xC}, K_{yC} and K_{zC} are the kinematic eddy diffusion coefficients of the property, whose theoretical determinations are given by similar expressions as presented in Eqs. (7.26).

The quantity C may also represent the concentration of suspended cohesive or non-cohesive sediments, transported by velocities along the bottom which usually are very low in comparison to the velocities in the upper layer. For non-conservative substances, such as nutrients, dissolved oxygen, domestic effluents and radioactive substances, an additional term must be included in Eq. (7.27) to analytically represent sources and/or sinks. If the property has a first order exponential decay, its mathematical simulation is given by $\frac{\partial C}{\partial t} = -k_C C$, where k_C is a proportionality coefficient with dimension $[T^{-1}]$. In any case, it is important to remember that the velocity components of the advective terms (u, v, w) and the solution of Eq. (7.27), $C = C(x, y, z, t)$, represent average values for a time interval Δt, which must be long enough to eliminate the turbulent fluctuations.

The partial differential Eqs. (7.24) or (7.25) and (7.27), which have the salinity, $S = S(x, y, z, t)$, and concentration, $C = C(x, y, z, t)$ fields as unknowns, are named as Eulerian formulations. Mathematically, the solutions may be obtained if the turbulent diffusion coefficients of these properties and the velocity field $(\overrightarrow{v})$ are known quantities. The solutions of these equations are also dependent on the initial and boundary conditions and the estuary geometry.

7.3 Integral Formulas: Mass and Salt Conservation Equations

7.3.1 Volume Integration

When the solution to an estuarine physics problem for a water body doesn't require detailed knowledge of the interior domain, a simple solution may be obtained by applying the continuity and salt conservation equations (or any other conservative property) integrated with respect to the volume domain.

To start, let us integrate the differential expression of the continuity Eq. (7.19) with respect to a small volume element (ΔV) limited by a closed continuous surface (ΔA). Under the assumption that all regularity conditions necessary for the

application of Gauss's divergence theorem are met, the volume integral may be transformed into an integral in the area,

$$\iiint_{\Delta V} (\nabla \bullet \overrightarrow{v})dV = \iint_{\Delta A} \overrightarrow{(v \bullet n)}dA = 0. \tag{7.28}$$

In this equation, the unitary vector $\overrightarrow{n}$ ($|\overrightarrow{n}| = 1$) is normal to the area ΔA and it is oriented from the interior of the volume ΔV to the exterior of the closed surface. This equation may be generalized to a finite volume, V, of the estuarine water mass, limited by an area A, then

$$\iiint_{V} (\nabla \bullet \overrightarrow{v})dV = \iint_{A} \overrightarrow{v} \bullet dA = 0. \tag{7.29}$$

The integral over the area in Eq. (7.29) is volume transport [L^3T^{-1}], through the geometric limits of volume V, enclosed by the area, A, and according to these conservation equations, are equal to zero. In the SI system of units this transport is calculated in $m^3 \ s^{-1}$.

To obtain the integrated form of the salt conservation Eq. (7.24), for a differential water volume forced only by the advective process, the salinity and density fields in the volume, V, of the estuarine water mass must be, by hypothesis, stationary fields representing mean values during complete tidal cycles, and the salt conservation equation is reduced to the simplest differential expression:

$$\nabla \bullet \rho S \overrightarrow{v} = 0. \tag{7.30}$$

Integrating this equation in the geometric volume, V, of the estuarine mixing zone (MZ), and applying the Gauss theorem, it follows that:

$$\iiint_{V} (\nabla \bullet \rho S \overrightarrow{v})dV = \iint_{A} \rho S \overrightarrow{v} \bullet \overrightarrow{n}dA = 0. \tag{7.31}$$

The surface integral in the second member of this equation physically represents the advective salt transport [MT^{-1}] (kg s^{-1} in the SI system of units) through the surface area A, enclosed by the geometric volume of the estuarine water mass. As this salt transport is equal to zero, the mass of salt entering the volume, V, is counterbalanced by an equal value exiting, according to the principle of conservation of salt.

Examples of the practical application of the conservation principles of mass and salt (Eqs. 7.19 and 7.24) applied on a relatively small scale are presented according to Officer (1976) and Team course (2001). Let us take a water volume, V, bounded by two vertical transverse sections, where areas A_1 and A_2 have uniform mean salinities S_1 and S_2, respectively. Water enters the channel through A_1 and exits through A_2, with mean velocities $\overrightarrow{v_1} = u_1 \overrightarrow{n}$ and $\overrightarrow{v_2} = u_2 \overrightarrow{n}$, respectively (Fig. 7.1). Two sources of input or output water will be considered: the fresh river discharge
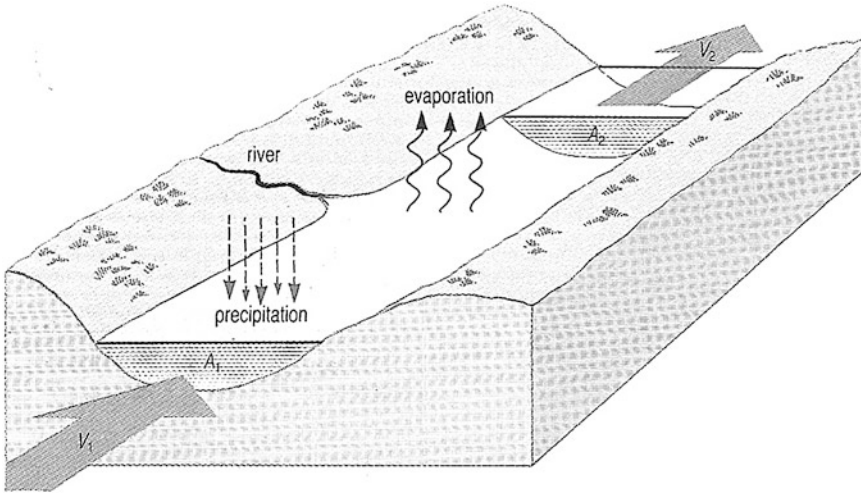
Fig. 7.1 Schematic diagram of a stationary estuarine water body bounded by vertical transverse sections, A_1 and A_2, and by the free surface (A_{su}) and the bottom. S_1, S_2, V_1 and V_2 are mean salinities and velocities, respectively (according the Team Course 2001)

(Q_f) and the input or outflow of fresh water through its free surface (A_{su}) by precipitation (P) and evaporation (E_v). Sources of bottom spring water and run-off will be disregarded.

Denoting $\bar{r}$ as the mean value of the difference evaporation to precipitation (E-P) per unit of time, the product $\vec{r}A_{su}$ is the volume transport of fresh water through the surface layer. Then, if $E_v > P$ or $E_v < P$, it follows that $\bar{r} > 0$ or $\bar{r} < 0$, respectively, and the volume transport across the surface area A_{su} is exiting or entering the system, respectively; when there is a counterbalance of precipitation and evaporation, the transport across the free surface is null ($\bar{r}A_{su} = 0$). However, it should be noted that usually $Q_f \gg \bar{r}A_{su}$, with the exception of estuaries in dry regions where hypersaline (or negative) estuaries are formed.

Under the assumption of steady-state conditions for the inflow and outflow of fresh water, we may apply the integrated continuity Eq. (7.29). Taking into account the particular geometry (Fig. 7.1) and the kinematic boundary condition ($\vec{v} \bullet \vec{n} = 0$), the scalar product $\vec{v} \bullet \vec{n} = 0$ is only different from zero on the transverse sections A_1 and A_2. Thus, applying the integrated formulation of continuity equation, the total volume of water entering this portion of the channel may be equal to the total volume leaving, resulting in the following expression:

$$-V_1 A_1 + V_2 A_2 + (Q_f + \bar{r}A_{su}) = 0. \qquad (7.32)$$

Taking into account the geometric characteristics, the velocity and the salinity fields, applying Eq. (7.31) will give the following balance of the salt transport through the closed surface:

$$-\rho_1 S_1 V_1 A_1 + \rho_2 S_2 V_2 A_2 = 0, \tag{7.33}$$

because the salt transport through the bottom and the free surface are equal to zero. Under the assumption that the density variation may be disregarded in the salt balance ($\rho_1 \approx \rho_2$) this equation may be rewritten as:

$$-S_1 V_1 A_1 + S_2 V_2 A_2 = 0. \tag{7.34}$$

This approximation isn't restrictive, because salinity is a parameter that can be measured, and the density may be calculated with the equation of state of seawater. Using SI units for velocity, area, and $\bar{r}$, and psu units ($S \times 10^{-3}$) for salinity, the parcels of Eqs. (7.32) and (7.34) for the volume and salt transports are calculated in $m^3 \, s^{-1}$ and $kg \, s^{-1}$, respectively.

As salinities S_1 and S_2 at the transverse sections A_1 and A_2, respectively, and the volume transports are known, Eqs. (7.32) and (7.34) may be solved for the velocities averages velocities in the transverse sections (V_1 and V_2), and for the volume transports through the cross sections A_1 and A_2 ($V_1 A_1$ and $V_2 A_2$), and the results are

$$V_1 = \frac{\overline{(rA_{su} + Q_f)}S_2}{A_1(S_2 - S_1)}, \quad \text{and} \quad T_{V1} = V_1 A_1 = \frac{\overline{(rA_{su} + Q_f)}S_2}{(S_2 - S_1)}, \tag{7.35}$$

and

$$V_2 = \frac{\overline{(rA_{su} + Q_f)}S_1}{A_2(S_2 - S_1)}, \quad \text{and} \quad T_{V2} = V_2 A_2 = \frac{\overline{(rA_{su} + Q_f)}S_1}{(S_2 - S_1)}, \tag{7.36}$$

With all quantities in the second member of these equations in units of the SI system, the velocity components (V_1 and V2) and the volume transports (T_{V1} and T_{V2}) are expressed in $m \, s^{-1}$ and $m^3 \, s^{-1}$, respectively.

In extreme conditions where $Q_f = 0$, with evaporation is greater than precipitation ($E_v > P$, $\bar{r} > 0$) and $S_2 > S_1$, analysis of the solutions (7.35 and 7.36) indicates that the velocity directions ($V_1 > 0$ and $V_2 > 0$) are in agreement with those indicated in Fig. (7.1). For this ideal system, the flow is from the regions of low salinity towards the high salinity regions, in agreement with the salinity gradient direction.

Let us now consider the opposite process, that is, the precipitation rate exceeds the evaporation ($\bar{r} < 0$), which corresponds with $P > E_v$, and seawater is diluted by fresh water. Also, with $S_2 > S_1$ it follows from Eqs. (7.35) and (7.36) that the flow is in the opposite direction to the former condition ($V_1 < 0$ and $V_2 < 0$) and opposite to the salinity gradient.

Finally, with $Q_f \rightarrow 0$ and $\bar{r} \rightarrow 0$, the residual flow and volume transport through sections A_1 and A_2 are equal to zero. Hence, the difference of $P - E_v$ determines the driving motions in water bodies at coastlines, such as choked and hyper-saline coastal lagoons.

With H_0 denoting the mean depth of the water column of the closed water body shown in Fig. 7.1, the time interval (Δt) required for its interior volume of water to be completely removed from a choked coastal lagoon may be estimated by:

$$\Delta t = \frac{H_0 A_{su}}{V_2 A_2} = \frac{H_0 (S_2 - S_1)}{\bar{r} S_1}. \tag{7.37}$$

With the variables in this equation expressed in the SI units, the time interval Δt is calculated in seconds, and usually this quantity is converted in hours or days.

Consider now a similar problem, but for a salt wedge estuary in steady-state condition. The dynamics of this estuary is dominated by the river discharge, and the vertical salt distribution is generated by the entrainment. The continuity and the salt conservation equations integrated with respect to the volume (Eqs. 7.29 and 7.31) may be applied. Because the mean flow is one-dimensional in these equations, it will be considered along the longitudinal axis (Ox), oriented down-estuary (Fig. 7.2). This figure indicates the upper and lower salt-wedge transverse sections A_2 and A_1, and their mean velocity values are indicated by u_s and u_i, respectively. The mean salinities in these upper and lower sections are also considered as known, and are indicated by S_s and S_i, respectively. In Chap. 3 (Sect. 3.2) we have seen that for this estuarine type, the following inequality holds: $S_i \gg S_s$.

Hence, the integrated equations of continuity (Eq. 7.29) and the corresponding principle of salt conservation (Eq. 7.31) may be applied in the calculation of the intensity of the velocities u_s and u_i and the associated volume transports. Taking into account the MZ geometry and the kinematic boundary condition ($\vec{v} \bullet \vec{n} \neq 0$
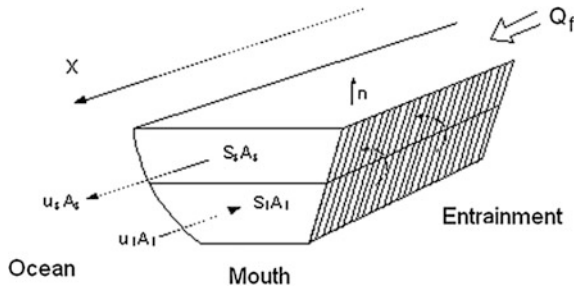

Fig. 7.2 Schematic diagram of a bidirectional motion through a vertical section localized at the mouth of a salt wedge estuary. The index of the quantities $A_{s,i}$, $u_{s,i}$ and $S_{s,i}$ indicate the areas of the *upper* (s) and *lower* (i) sections, and the corresponding mean velocity and salinity values, respectively. The unit vector, $\vec{n}$, (not shown) is normal to the closed surface oriented positively outward of the volume

only in the transverse sections A_s and A_i), from the conservation equations we have the following relationships:

$$\iint\limits_{A} \overrightarrow{v} \bullet \overrightarrow{n}.dA = u_s A_s - u_i A_i - Q_f = 0, \tag{7.38}$$

and

$$\iint\limits_{A} \rho S \overrightarrow{v} \bullet \overrightarrow{n}.dA = \rho_s S_s u_s A_s - \rho_i S_i u_i A_i = 0. \tag{7.39}$$

Disregarding the density variations ($\rho_s \approx \rho_i$) the equation system (7.38 and 7.39) is reduced to:

$$u_s A_s - u_i A_i = Q_f, \tag{7.40}$$

and

$$S_s u_s A_s = S_i u_i A_i. \tag{7.41}$$

If the mean salinity values (S_s and S_i), the area of the vertical sections (A_s and A_i), and the river discharges are all known, this equation system has only two unknowns, u_s and u_i, and the solutions are:

$$u_s = \frac{S_i Q_f}{A_s(S_i - S_s)} = \frac{Q_f}{A_s(1 - \frac{S_s}{S_i})}, \tag{7.42}$$

and

$$u_1 = \frac{S_s Q_f}{A_i(S_i - S_s)} = \frac{Q_f}{A_i(\frac{S_i}{S_s} - 1)}. \tag{7.43}$$

With these results, it is also possible to calculate the transport of volumes ($u_s A_s$ and $u_i A_i$) and salt ($u_s S_s A_s$ and $u_i S_i\ A_i$) in the upper and lower layers, respectively.

This practical application of the principles of continuity and conservation of salt integrated with respect to the volume exemplify how it is possible to calculate the mean velocities in transverse sections and the corresponding values of the volume and salt transports of a salt wedge estuary, when its geometry and scalar properties (salinity and river discharge) are known. In relation to Eqs. (7.42 and 7.43), which are used to calculate the velocities u_s and u_i, it is possible to observe that, even if A_s and A_i have the same areas, the velocity of the upper layer is always higher than the lower layer velocity ($u_s > u_i$) because $S_s \ll S_i$. Hence, this result is in agreement with the salt wedge estuary dynamics.

A numerical application of Eqs. (7.42 and 7.43) is presented to theoretically estimate the vertical velocity profiles at the mouth of the Fraser River estuary

(British Columbia, Canada). This estuary is classified as salt wedge (or type 4, according the Stratification-circulation diagram). The Fraser river estuary is a typical example of salt wedge estuary in a region of meso-tides. The following data were estimated from the article of Geyer and Farmer (1989): mean river discharge $Q_f = 3000$ m^3 s^{-1}, geometry at the upper and lower sections $A_s = 3750$ m^2 and $A_i = 4500$ m^2, and salinities $S_s = 14.0^o/_{oo}$ and $S_i = 30.0^o/_{oo}$, respectively, representing mean values at the upper and lower sections of the halocline, respectively. The estimated vertical profile of salinity and the theoretical simulations of the vertical velocity profile are presented in Fig. 7.3; the mean velocities at the upper and lower vertical sections are $u_s \approx 1.5$ m s^{-1} and $u_i \approx -0.6$ m s^{-1}, respectively. The discontinuity of the vertical salinity profile at depth $z = 5$ m, generated similar characteristics in the velocity profile, because theoretical equations don't include dissipative forces due to the internal friction and at the bottom.

From the results of the velocity, the volume transport was calculated and its landward and seaward values were $Q_s = 5525$ m^3 s^{-1} and $Q_i = -2525$ m^3 s^{-1}, respectively. Hence, the volume transport is in balance with the river discharge. The increase in the volume transport seaward, in comparison with the river discharge (Q_f), clearly indicates the influence on the upper transport, forced by the entrainment of seawater into the layer above the halocline.

We leave it to the reader to demonstrate the following dot marks:

- Solutions (7.42) and (7.43) identically satisfy the principles of mass and salt conservations;
- The mean speed at the mouth transverse section is calculated by: $\dfrac{Q_f}{(A_s + A_i)} = u_f$;
- The salt transports may be determined by $\rho_i u_i S_i A_i$ and $\rho_s u_s S_s A_s$, landward and seaward, respectively.

Fig. 7.3 Vertical theoretical velocity profiles of a salt wedge estuary (a). Estimates of salinity profile from experimental data (b), obtained near the estuary mouth
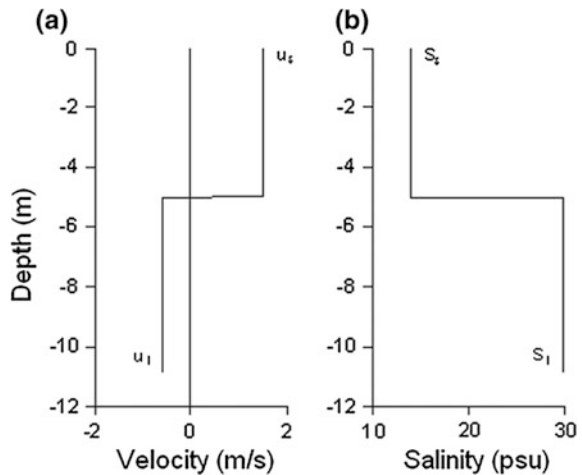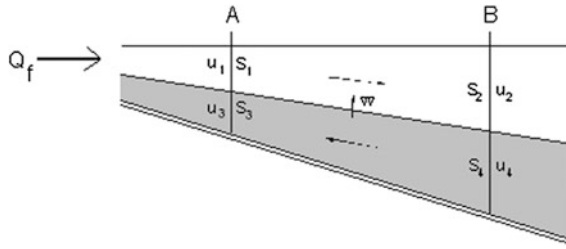
Fig. 7.4 Schematic diagram of bidirectional motion and salt transport through vertical sections, A and B, of a highly stratified estuary. The indexes 1–2 and 3–4 indicate physical properties in the *upper* and *lower* layers, respectively, bounded by the halocline (adapted from Defant 1961)

The classical Knudsen hydrographic theorem was presented at the beginning of the 19th century, stating relationships between a known salinity field and the velocity under stationary conditions. Let us assume that the estuary is highly stratified and its geometry and salinity are known. Under these conditions, the mean longitudinal motion, in relation to the Ox axis, is bidirectional in two layers separated by a sharp halocline; seaward and landward motions are in the surface and lower layers, respectively (Fig. 7.4).

Applying the continuity and salt conservation Eqs. (7.29 and 7.31) to the volume between the transverse sections (A, B), taking into account the channel geometry and the areas of the upper $(A_1 + A_3)$ and lower $(A_2 + A_4)$ layers, and knowing the salinities S_1 and S_3 (at A_1 and A_3), and S_2 and S_4 (at A_2 and A_4), the following volume and salt transport balances may be written:

$$-u_1 A_1 + u_3 A_3 = -Q_f, \tag{7.44}$$

$$u_2 A_2 - u_4 A_4 = Q_f, \tag{7.45}$$

$$-u_1 A_1 + u_3 A_3 + u_2 A_2 - u_4 A_4 = 0, \tag{7.46}$$

and

$$-S_1 u_1 A_1 + S_3 u_3 A_3 + S_2 u_2 A_2 - S_4 u_4 A_4 = 0, \tag{7.47}$$

with the approximation $\rho_1 \approx \rho_2 \approx \rho_3 \approx \rho_4$.

As the net salt transport across the transversal section A (sub-sections A_1 and A_3) and section B (sub-sections A_2 and A_4) must be equal to zero, the following equalities may be written from Eq. (7.47):

$$S_1 u_1 A_1 = S_3 u_3 A_3, \tag{7.48a}$$

and

$$S_2 u_2 A_2 = S_4 u_4 A_4. \tag{7.48b}$$

Equations (7.44), (7.45), (7.48a and 7.48b) form a system of four equations with four unknowns u_1, u_2, u_3 and u_4. Calculating these unknowns and multiplying by areas we then obtain the volume transports:

$$u_1 = \frac{S_3 Q_f}{A_1(S_3 - S_1)}, \rightarrow u_1 A_1 = \frac{S_3 Q_f}{(S_3 - S_1)}, \tag{7.49a}$$

$$u_3 = \frac{S_1 Q_f}{A_3(S_3 - S_1)}, \rightarrow u_3 A_3 = \frac{S_1 Q_f}{(S_3 - S_1)}, \tag{7.49b}$$

$$u_2 = \frac{S_4 Q_f}{A_2(S_4 - S_2)}, \rightarrow u_2 A_2 = \frac{S_4 Q_f}{(S_4 - S_2)}, \tag{7.49c}$$

and

$$u_4 = \frac{S_2 Q_f}{A_4(S_4 - S_2)}, \rightarrow u_4 A_4 = \frac{S_2 Q_f}{(S_4 - S_2)}. \tag{7.49d}$$

Then, with knowledge hydrologic and hydrographic data it is possible to calculate the velocity components (u_i, i = 1, 2, 3, 4), the volume ($u_i A_i$, i = 1, 2, 3, 4), and salt transports ($u_i A_i S_i$, i = 1, 2, 3, 4) across the upper and lower sections shown in Fig. 7.4. As $S_3 > S_1$ and $S_4 > S_2$, it follows from these equations that the velocity and volume transport modules are positive. As the flow direction has already been taken into account in the water column stratification ($S_1 < S_3$), from Eqs. (7.49a and 7.49b) it follows that $u_1 > u_3$, and the velocity in the upper layer is higher than the lower layer velocity. Also, if $A_2 = A_3$, from Eqs. (7.49c and 7.49d) it follows that $u_2 > u_4$. These theoretical inequalities, between the mean speeds in the upper and lower layers separated by the halocline, may be verified experimentally.

Although the Knudsen hydrographic theorem only takes into account the advective process, it is a good approximation for highly stratified and salt wedge estuaries, because vertical mixing due to turbulent diffusion is suppressed by the entrainment. This theorem has been applied by Scandinavian oceanographers in studies of the circulation in fjord type estuaries, and some examples may be found in Defant (1961) and Dyer (1973). To estimate the areas A_i (i = 1, 3) and A_j(j = 2, 4) usually the interface between the upper and lower layers is taken as the mean depth of the halocline.

According to Geyer (2010), let us make as an exercise the following simplification of the original Knudsen hydrographic theorem, displacing the cross-section areas at the positions A and B (Fig. 7.4) towards the estuary head and mouth, respectively, thus, at the new section A position the velocity and salinity have the following values: (i) $u_1 = u_f$ and $S_1 = S_f = 0$, and there will no more the quantities S_3 and u_3; ii) at the section B, now located at the estuary mouth, its properties above and below the halocline will remains with the same previous notations. Thus,

applying the volume and salt transport conservations equations the following expressions of two equations with the unknowns Q_2 (or u_2) and Q_4 (or u_4) are written as:

$$Q_2 = Q_f + Q_4, \quad \text{and} \quad u_2 A_2 S_2 = u_4 A_a S_4, \rightarrow Q_2 S_2 = Q_4 S_4, \tag{7.50a}$$

with the simplification $\rho_2 \approx \rho_4$. Solving this system of equations we find the following expressions to calculate volume transports and velocities and at the estuary head (A) and at position B:

$$Q_2 = u_2 A_2 = \frac{S_4}{S_4 - S_2} Q_f, \quad \text{and} \quad , Q_4 = u_4 A_4 = \frac{S_2}{S_4 - S_2} Q_f, \tag{7.50b}$$

or

$$u_2 = \frac{1}{A_2} \frac{S_4}{(S_4 - S_2)} Q_f, \quad \text{and} \quad , u_4 = \frac{1}{A_4} \frac{S_2}{(S_4 - S_2)} Q_f. \tag{7.50c}$$

We leave to the reader to demonstrate that these solutions satisfy the volume and salt transport conservation.

To establish the horizontal continuity of the flow, as indicated in Fig. (7.4), an upward mean velocity $(\overline{w})$, generated by entrainment, is necessary across the halocline. Thus, if A_h indicates the horizontal area of the halocline, the associate entrained volume transport is calculated by $(\overline{w}A_h)$, which is generate by the volume transport convergence on the lower layer, and may be calculated by: $\overline{w}A_h = u_4 A_4 - u_3 A_3$. Then, from the volume transports calculated by Eqs. (7.52) and (7.54a, 7.54b), it follows that:

$$\overline{w}A_h = Q_f \left(\frac{S_2}{S_4 - S_2} - \frac{S_1}{S_3 - S_1} \right), \tag{7.51a}$$

and

$$\overline{w} = \frac{Q_f}{A_h} \left(\frac{S_2}{S_4 - S_2} - \frac{S_1}{S_3 - S_1} \right). \tag{7.51b}$$

In the presented theory, only the principle of mass conservation (continuity) and salt conservation in its integrated formulation were used, enabling the solutions for velocity field and transports at the boundaries of the estuary only. The driving forces were the river discharge input and the evaporation-precipitation rate, but the dissipative force (friction) and the turbulent diffusion were taken as negligible. Obtaining a solution for the inner circulation and property distributions in natural estuaries requires a complete set of differential equations, including the equations of motion, which will be presented in Chap. 8.

7.3.2   Bi-Dimensional Formulation: Vertical Integration

Estuaries are transitional water bodies with free surface and morphologic charac-
teristics which may vary from a simply geometry, such as a channel, to complex
system with a net of interconnected channels. The tridimensional equations of
continuity and salt conservation have already been presented (Eqs. 7.19 and 7.24).
Under the assumptions that the turbulent coefficients of salt diffusion and the
velocity components are known, Eq. (7.24) may be solved to calculate the salinity
field, $S = S(x, y, z, t)$. However, its analytical solution is extremely difficult, per-
haps even impossible, particularly for complex geometries.

Coastal plain estuaries which have a longitudinal channel geometry, low river
discharge and high tidal amplitude are practically well-mixed (type 1 or C), and
variations in velocity and property concentrations mainly occur in the transverse
sections (plane Oxy). However, when estuaries are forced by moderate or high river
discharge, variations in property concentrations may occur mainly in the Oxz plane,
such as in partially mixed and salt wedge estuaries (types 2 and 4, or A and B).
With these particular geometries and driving forces, the conservation equations may
be simplified to two dimensions.

Let us now present the deduction of the two-dimensional continuity equation
from its three-dimensional formulation (Eq. 7.19), which is often used in problems
related to well-mixed estuaries. Properties variations in these estuaries are mainly in
the Ox and Oy directions, oriented according to the reference system in Fig. 7.5.

To eliminate variations in the Oz direction, it is sufficient to integrate the con-
tinuity equation using the local depth $z = -H_0(x, y)$ and the ordinate of the free
surface $z = \eta(x, y, t)$ as limits, disregarding the large-scale temporal depth varia-
tions due to erosion and sedimentation,

$$\int_{-H_0}^{\eta} \left(\frac{\partial u}{\partial x}\right)dz + \int_{-H_0}^{\eta} \left(\frac{\partial v}{\partial y}\right)dz + w|_\eta - w|_{-H_0} = 0. \qquad (7.52)$$

In this equation, $w|_\eta = w(x, y, \eta, t)$ and $w|_{-H0} = w(x, y, -H_0, t)$ are values of the
vertical velocity component at the surface and on the bottom, respectively. As its
integration limits are functions of x, y and t, it is necessary simplify the equation to
a more convenient expression for practical applications, using the Leibnitz rule of
an integral derivation[2] (Severi 1956, p. 354):

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[2]When the estuary bottom is plane (H_0 = const.), and due to the very long tidal wave, the tidal
elevation may be considered uniform along the estuary, $\eta = \eta(t)$, and it is possible to change the
order of the integral operator and the derivative. In these conditions $w|_{-H0} = 0$ and $w|_\eta = d\eta/dt = \partial\eta/\partial t$ are the kinematic boundary conditions.
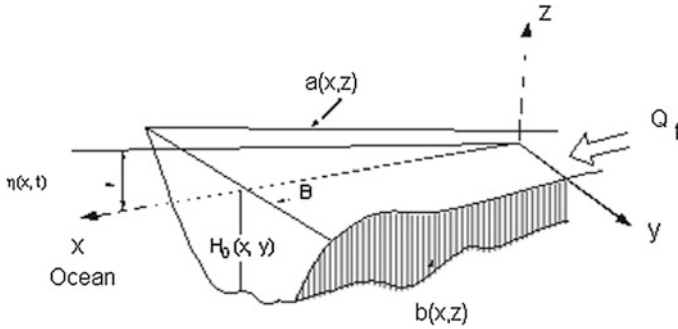
Fig. 7.5 Geometric limits of an estuary. The coordinates on the surface and bottom are indicated by z = η(x, y, t) and z = H₀(x, y), and from the right to *left*, a(x, z) and b(x, z) are lateral boundaries

$$\int_{-H_0}^{\eta} \left(\frac{\partial u}{\partial x}\right) dz = \frac{\partial}{\partial x} \left(\int_{-H_0}^{\eta} u dz \right) + u|_{-H_0} \frac{\partial(-H_0)}{\partial x} - u|_{\eta} \frac{\partial \eta}{\partial x}, \qquad (7.53a)$$

and

$$\int_{-H_0}^{\eta} \left(\frac{\partial v}{\partial y}\right) dz = \frac{\partial}{\partial y} \left(\int_{-H_0}^{\eta} v dz \right) + v|_{-H_0} \frac{\partial(-H_0)}{\partial y} - v|_{\eta} \frac{\partial \eta}{\partial y}, \qquad (7.53b)$$

where $u|_{\eta} = u(x, y, \eta, t)$, $v|_{\eta} = v(x, y, \eta, t)$, $u|_{-H_0} = u(x, y, -H_0, t)$ and $v|_{-H0} = v(x, y, -H_0, t)$ are values of velocity horizontal components in the free surface ($z = \eta$) and on the bottom ($z = -H_0$), respectively.

By substituting expressions (7.53a, 7.53b) into Eq. (7.52), and taking into account the vertical velocity components generated by the bottom topography and the sea-surface, because $H_0 = H_o(x, y)$ and $\eta = \eta(x, y, t)$, the following kinematic boundary conditions must be imposed:

$$w|_{-H_0} = u|_{-H_0} \frac{\partial(-H_0)}{\partial x} + v|_{-H_0} \frac{\partial(-H_0)}{\partial y}, \qquad (7.54a)$$

and

$$w|_{\eta} = u|_{\eta} \frac{\partial \eta}{\partial x} + v|_{\eta} \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial t}. \qquad (7.54b)$$

The final result is the expression,

$$\frac{\partial}{\partial x}\left(\int_{-H_0}^{\eta} u\,dz\right) + \frac{\partial}{\partial y}\left(\int_{-H_0}^{\eta} v\,dz\right) + \frac{\partial \eta}{\partial t} = 0. \tag{7.55}$$

In this equation, the integrands u and v are, by hypothesis, independent of the depth. Finalizing the integration, the continuity equation in two dimensions for an estuary may be given by[3]:

$$\frac{\partial(uh)}{\partial x} + \frac{\partial(vh)}{\partial y} + \frac{\partial \eta}{\partial t} = 0, \text{ or } \nabla_H \bullet h\,\vec{v} = -\frac{\partial \eta}{\partial t}, \tag{7.56a}$$

or, when the longitudinal depth variation (∂h/∂x) may be disregarded,

$$h\frac{\partial u}{\partial x} + \frac{\partial(vh)}{\partial y} + \frac{\partial \eta}{\partial t} = 0. \tag{7.56b}$$

where,

$$\int_{-H_0}^{\eta} dz = \eta + H_0 = h(x, y, t). \tag{7.56c}$$

The quantity h(x, y, t) = η(x, y, t) + H_0(x, y) is the thickness of the water column and holds the identity ∂h/∂t = ∂η/∂t.

The bi-dimensional continuity Eq. (7.56a) has the following physical interpretation: the divergence ($\nabla_H \bullet h\,\vec{v} > 0$) or the convergence ($\nabla_H \bullet h\,\vec{v} < 0$) must be compensated by a decrease (∂h/∂t < 0) or an increase (∂h/∂t > 0) of the thickness of water layer, respectively, where the vector $h\,\vec{v}$ is the volume transport vertically integrated by width unity. As a result of the vertical integration, each term of the continuity equation has dimension of velocity [LT^{-1}], and the coordinate z was substituted by a geometric characteristic of the estuary (the depth, h).

By analogy, the vertically integrated deduction of the salt conservation equation results from the integration of Eq. (7.25) in the following limits: z = −H_0(x, y) and z = η(x, y, t) at the bottom and the free surface, respectively. Then, we have the following expression:

$$\int_{-H_0}^{\eta} \left(\frac{dS}{dt}\right)dz = \int_{-H_0}^{\eta} \left[\frac{\partial}{\partial x}\left(K_x\frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial S}{\partial y}\right)\right]dz + \int_{-H_0}^{\eta} \frac{\partial}{\partial z}\left(K_z\frac{\partial S}{\partial z}\right)dz, \tag{7.57}$$

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[3]This operation is equivalent to the *Mean Value Theorem.*

and the integrand of the term on the left-hand-side is the total derivative of the salinity (local + advective variations). By completing the integration of the last term on the right-hand-side of the equation, we have:

$$\int_{-H_0}^{\eta} \frac{\partial}{\partial z}(K_z \frac{\partial S}{\partial z})dz = K_z \frac{\partial S}{\partial z}\Big|_{\eta} - K_z \frac{\partial S}{\partial z}\Big|_{-H_0}. \tag{7.58}$$

As K_z is the kinematic turbulent diffusion coefficient of salt $[K_z] = [L^2 T^{-1}]$, the terms of this equation have dimensions of velocity $[LT^{-1}]$, which may be interpreted physically as salt flux per density unit, through the surface $(z = \eta)$ and the bottom $(z = -H_0)$. As these salt fluxes must be zero within the estuary's geometric boundaries, the last parcel in the right-hand-side of Eq. (7.57) is equal to zero.

The vertical integration of the total derivative in the left-hand-side of the Eq. (7.57) is given by:

$$\int_{-H_0}^{\eta} \frac{dS}{dt}dz = \int_{-H_0}^{\eta} \frac{\partial S}{\partial t}dz + \int_{-H_0}^{\eta} \frac{\partial(uS)}{\partial x}dz + \int_{-H_0}^{\eta} \frac{\partial(vS)}{\partial y}dz + \int_{-H_0}^{\eta} \frac{\partial(wS)}{\partial z}dz. \tag{7.59}$$

Let us again apply the Leibnitz integration rule to the first three terms of the right-hand-side of this equation:

$$\int_{-H_0}^{\eta} \frac{\partial S}{\partial t}dz = \frac{\partial}{\partial t}(\int_{-H_0}^{\eta} Sdz) - S\big|_{\eta} \frac{\partial \eta}{\partial t}, \tag{7.60a}$$

$$\int_{-H_0}^{\eta} \frac{\partial(uS)}{\partial x}dz = \frac{\partial}{\partial x}(\int_{-H_0}^{\eta} uSdz) - uS\big|_{\eta} \frac{\partial \eta}{\partial x} + uS\big|_{-H_0} \frac{\partial(-H_0)}{\partial x}, \tag{7.60b}$$

and

$$\int_{-H_0}^{\eta} \frac{\partial(vS)}{\partial y}dz = \frac{\partial}{\partial y}(\int_{-H_0}^{\eta} vSdz) - vS\big|_{\eta} \frac{\partial \eta}{\partial y} + vS\big|_{-H_0} \frac{\partial(-H_0)}{\partial y}. \tag{7.60c}$$

Finally, integrating the last term of the right-hand-side of Eq. (7.59) gives,

$$\int_{-H_0}^{\eta} \frac{\partial(wS)}{\partial z}dz = wS\big|_{\eta} - wS\big|_{-H_0}. \tag{7.61}$$

In these equations, the quantities $S|\eta = S(x, y, \eta, t)$ and $S|_{-H0} = S(x, y, -H_0, t)$ are the salinity values at the surface and bottom, respectively.

At this stage the first two terms of the right-hand-side of Eq. (7.57), relating to the lateral influence of the turbulent diffusion, are still missing from the vertical integration of Eq. (7.57). Applying the Leibnitz rule to these terms, and they may be rewritten as:

$$\int_{-H_0}^{\eta} \frac{\partial}{\partial x}(K_x \frac{\partial S}{\partial x})dz = \frac{\partial}{\partial x}(\int_{-H_0}^{\eta} K_x \frac{\partial S}{\partial x}dz) - K_x \frac{\partial S}{\partial x}|_\eta \frac{\partial \eta}{\partial x} + K_x \frac{\partial S}{\partial x}|_{-H_0} \frac{\partial(-H_0)}{\partial x},$$

(7.62a)

and

$$\int_{-H_0}^{\eta} \frac{\partial}{\partial y}(K_y \frac{\partial S}{\partial y})dz = \frac{\partial}{\partial y}(\int_{-H_0}^{\eta} K_y \frac{\partial S}{\partial y}dz) - K_y \frac{\partial S}{\partial y}|_\eta \frac{\partial \eta}{\partial y} + K_y \frac{\partial S}{\partial y}|_{-H_0} \frac{\partial(-H_0)}{\partial y}.$$

(7.62b)

Finally, substituting Eqs. (7.60a, b, c) and (7.62a, b) into Eq. (7.57), and taking into account that:

- The diffusive salt flux in the estuary boundaries are zero;

$$K_x \frac{\partial S}{\partial x}|_{-H_0\eta} = K_y \frac{\partial S}{\partial y}|_{-H_0\eta} = K_z \frac{\partial S}{\partial z}|_{-H_0\eta} = 0,$$

(7.63a)

- The kinematic boundary conditions indicated in Eqs. (7.54a, b) are valid when multiplied by $S|_{-Ho}$ and $S|\eta$:

$$wS|_{-H_0} = S|_{-H_0}[u|_{-H_0} \frac{\partial(-H_0)}{\partial x} + v|_{-H_0} \frac{\partial(-H_0)}{\partial y}],$$

(7.63b)

and

$$wS|_\eta = S|_\eta[u|_\eta \frac{\partial \eta}{\partial x} + v|_\eta \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial t}].$$

(7.63c)

Then, the following vertically integrated formulation of the continuity equation has been obtained:

$$\frac{\partial}{\partial t}\left(\int_{-H_0}^{\eta} S dz\right) + \frac{\partial}{\partial x}\left[\int_{-H_0}^{\eta} (uS)dz\right] + \frac{\partial}{\partial y}\left[\int_{-H_0}^{\eta} (vS)dz\right]$$

$$= \frac{\partial}{\partial x}\left[\int_{-H_0}^{\eta} \left(K_x \frac{\partial S}{\partial x}\right)dz\right] + \frac{\partial}{\partial y}\left[\int_{-H_0}^{\eta} \left(K_y \frac{\partial S}{\partial y}\right)dz\right]. \tag{7.64}$$

Imposing the conditions that the estuary is well-mixed (vertically homogeneous), the quantities u, S, uS, vS, $K_x \frac{\partial S}{\partial x}$ and $K_y \frac{\partial S}{\partial y}$ are independent of depth, and, taking into account that the integral of the differential dz in the limits $z = -H_0$ and $z = \eta$ is equal to the local depth, $h = H_0 + \eta$, the integration of Eq. (7.64), yields the expression for the salt conservation equation:

$$\frac{\partial (Sh)}{\partial t} + \frac{\partial (uSh)}{\partial x} + \frac{\partial (vSh)}{\partial y} = \frac{\partial}{\partial x}\left(K_x h \frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y h \frac{\partial S}{\partial y}\right). \tag{7.65}$$

Some simplifications may be made in this equation if, for instance, the tidal oscillation is much less than the estuary depth ($\eta \ll H_0$), h(x, y) = H_0(x, y). Then the continuity Eq. (7.56a) and the salt conservation (7.65) may be rewritten as:

$$\frac{\partial (uH_0)}{\partial x} + \frac{\partial (vH_0)}{\partial y} + \frac{\partial \eta}{\partial t} = 0, \tag{7.66}$$

and

$$\frac{\partial (SH_0)}{\partial t} + \frac{\partial (uSH_0)}{\partial x} + \frac{\partial (vSH_0)}{\partial y} = \frac{\partial}{\partial x}\left(K_x H_0 \frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y H_0 \frac{\partial S}{\partial y}\right). \tag{7.67}$$

Analysis of Eq. (7.66) indicates that when there is divergence ($\nabla_H \bullet H_0 \vec{v} > 0$) or convergence ($\nabla_H \bullet H_0 \vec{v} < 0$) of the volume transport along the depth axis, they must be compensated by negative or positive of the vertical velocity component on the surface ($\partial \eta / \partial t < 0$ or $\partial \eta / \partial t > 0$, respectively).

A simple salt conservation equation may be obtained from Eq. (7.67) by separating derived variables, such as S × h, S × uh and S × vh, and combining these variables with the continuity Eq. (7.56a) to give,

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} = \frac{1}{h}\left[\frac{\partial}{\partial x}\left(K_x h \frac{\partial S}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y h \frac{\partial S}{\partial y}\right)\right] + \frac{\partial S}{\partial z}\Big|_{z=\eta}(P - E_V). \tag{7.68a}$$

This equation may be further simplified when h = const.,

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} = [\frac{\partial}{\partial x}(K_x\frac{\partial S}{\partial x}) + \frac{\partial}{\partial y}(K_y\frac{\partial S}{\partial y})] + \frac{\partial S}{\partial z}|_{z=\eta}(P - E_V). \quad (7.68b)$$

In Eqs. (7.68a, b), the last term of the right-hand-side was introduced to simulate the salinity time variation due to fresh water exchanges at the free surface through precipitation (P) and evaporation (E_v) rates, $[P] = [E_v] = [LT^{-1}]$. When $P > E_V$ or $P < E_v$, there will be a fresh water source or sink at the surface ($z = \eta$), respectively; when $P = E_v$, there will be no fresh water interchanges at the free surface.

Pritchard (1954) used Eq. (7.68b) to study the salt balance in the James river coastal plain estuary (Virginia, USA) under steady-state conditions ($\partial S/\partial t = 0$) and with $P = E_v$. Based on a time series over several tidal cycles of salinity and current velocity, it was observed that the horizontal flux due to advection and the vertical non-advective salt flux were the most important factors in maintaining a simplified salt balance equation, such as:

$$u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} = \frac{\partial}{\partial y}(K_y\frac{\partial S}{\partial y}). \quad (7.69)$$

Equations (7.65) and (7.68a, 7.68b) are the physical-mathematical formulation of the Eulerian description of the bi-dimensional salinity field variation $S = S(x, y, t)$ in the water column. If the estuary geometry, the velocity field, the kinematic coefficients K_x and K_y, and the initial and boundary conditions are known, these equations may be integrated to calculate the mean salinity in the water column.

7.3.3 Bi-Dimensional Formulation: Lateral Integration

Let us now consider a second bi-dimensional model, under the assumption that the estuarine water mass is laterally homogeneous, which is generally the case of narrow partially-mixed estuaries; its circulation, salinity and others properties are independent on the Oy axis. In these conditions, the conservation equations of mass (7.19) and salt (7.24 or 7.25) must be integrated along the lateral direction, from y = a(x, z) to y = b(x, z) coordinates, and their mean values calculated. The difference being b(x, z) − a(x, z) = B(x, z) will indicate the estuary width (Fig. 7.5). The dependence on the Oz direction will take into account the time variation of the lateral coordinates, because z is dependent on the time variation of the free surface, $\eta = \eta(x, t)$.

By analogy with the mathematical development used for the vertically homogeneous estuary, it is necessary to laterally integrate the tri-dimensional equations, calculate their means and then reduce them, applying the Leibnitz rule, with consideration that the integration limits of the estuary boundaries, a and b are functions of x and z, a = a(x, z) and b = b(x, z). Following this procedure, it is necessary to apply the boundary conditions in order to eliminate influences that are topographically generated by the bottom and margins of the estuarine channel, and

impose the condition that salt advection and diffusion through its boundaries must be zero.

Beginning with the continuity Eq. (7.19), it follows that:

$$\int_a^b \frac{\partial u}{\partial x}dy + \int_a^b \frac{\partial v}{\partial y}dy + \int_a^b \frac{\partial w}{\partial z}dy = 0. \tag{7.70}$$

Because the integration of the second parcel is immediate, applying the Leibnitz rule, this equation is rewritten as:

$$\frac{\partial}{\partial x}(\int_a^b udy) - u|_b \frac{\partial b}{\partial x} + u|_a \frac{\partial a}{\partial x} + v|_b - v|_a + \frac{\partial}{\partial z}(\int_a^b wdy) - w|_b \frac{\partial b}{\partial z} + w|_a \frac{\partial a}{\partial z} = 0, \tag{7.71}$$

where $u|_b = u(x, b, z, t)$, $v|_b = v(x, b, z, t)$, $u|_a = u(x, a, z, t)$ and $v|_a = v(x, a, z, t)$.

Due to variations in the margin geometry of the estuarine channel, the coordinates $a = a(x, z)$ and $b = b(x, z)$ may induce, due to topographic influences, transversal components of the velocity, which is necessary the imposition of the following boundary conditions:

$$v|_a = u|_a \frac{\partial a}{\partial x} + w|_a \frac{\partial a}{\partial z}, \tag{7.72a}$$

and

$$v|_b = u|_b \frac{\partial b}{\partial x} + w|_b \frac{\partial b}{\partial z}. \tag{7.72b}$$

For an estuarine channel with a uniform rectangular transversal section, the adherence principle states that at the margins $v|_a = v|_b = 0$.

Finally, with the hypothesis of lateral uniformity, or imposing the conditions that the velocity components are independent of the variable, y, it follows that the expression of the laterally integrated continuity equation is:

$$\frac{\partial(uB)}{\partial x} + \frac{\partial(wB)}{\partial z} = 0, \tag{7.73a}$$

and

$$\int_a^b dy = b(x, z, t) - a(x, z, t) = B(x, z, t). \tag{7.73b}$$

The continuity Eq. (7.73a) for a laterally homogeneous estuary indicates that the vector $\overrightarrow{v} = uB\,\overrightarrow{i} + wB\,\overrightarrow{k}$ is non-divergent $(\nabla_V \bullet B\,\overrightarrow{v} = 0)$, and the associated current function, $\Psi = \Psi(x, z)$, is defined by,

$$\frac{\partial\psi(x, z)}{\partial x} = w(x, z)B \text{ and } \frac{\partial\psi(x, z)}{\partial z} = -u(x, z)B, \tag{7.73c}$$

and satisfy identically the continuity equation; its dimension is equivalent to the volume transport $[\Psi(x, z)] = [L^3 T^{-1}]$.

Applying an analogous procedure, the salt conservation Eq. (7.25) will be laterally integrated from $a = a(x, z)$ to $b = b(x, z)$,

$$\int_a^b \frac{dS}{dt} = \int_a^b [\frac{\partial}{\partial x}(K_x\frac{\partial S}{\partial x}) + \frac{\partial}{\partial y}(K_y\frac{\partial S}{\partial y})]dy + \int_a^b (\frac{\partial}{\partial z}(K_z\frac{\partial S}{\partial z})dy, \tag{7.74}$$

applying the Leibnitz rule and the following boundary conditions:

$$vS|_a = S|_a[u|_a\frac{\partial a}{\partial x} + w|_a\frac{\partial a}{\partial z}], \tag{7.75a}$$

and

$$vS|_b = S|_b[u|_a\frac{\partial b}{\partial x} + w|_a\frac{\partial b}{\partial z}]. \tag{7.75b}$$

Imposing the lateral homogeneity condition and that the salt flux, per density unity, at its geometric boundaries is zero,

$$K_x\frac{\partial S}{\partial x}|_{a,b} = K_z\frac{\partial S}{\partial z}|_{a,b} = 0, \tag{7.75c}$$

it follows that the expression of the salt conservation equation for laterally homogeneous estuaries is:

$$\frac{\partial(BS)}{\partial t} + \frac{\partial(uBS)}{\partial x} + \frac{\partial(wBS)}{\partial z} = \frac{\partial}{\partial x}(BK_x\frac{\partial S}{\partial x}) + \frac{\partial}{\partial z}(BK_z\frac{\partial S}{\partial z}). \tag{7.76}$$

Combining this result with the continuity Eq. (7.73a), it follows that the most usual salt conservation equation is:

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + w\frac{\partial S}{\partial z} = \frac{1}{B}[\frac{\partial}{\partial x}(BK_x\frac{\partial S}{\partial x}) + \frac{\partial}{\partial z}(BK_z\frac{\partial S}{\partial z})]. \tag{7.77}$$

This equation is the Eulerian formulation of the bi-dimensional salinity variation $S = S(x, z, t)$ of a laterally homogeneous and partially-mixed estuary (type 2 or B). If the estuary geometry, the velocity field, the turbulent diffusion coefficients and the initial and boundary conditions are known, this equation may be solved for the salinity field distribution.

Equation (7.77) may be further simplified according to the estuary characteristics; for a steady-state well-mixed estuary (type 1 or C) with a constant cross-sectional area A, a width B, and a constant kinematic turbulent diffusion coefficient (K_x), forced by the river discharge, the Eq. (7.77) is simplified to:

$$u_f \frac{dS}{dx} = K_x \left(\frac{d^2 S}{dx^2} \right). \tag{7.78a}$$

Imposing the following boundary conditions: (i) $S(x)|_{x=0} = S_0$, and; (ii) $S|_{x \to \infty} = 0$, which indicate the salinity at the estuary mouth and head, respectively, the solution to this equation is:

$$S(x) = S_0 \exp\left(-\frac{u_f}{K_x} x \right), \tag{7.78b}$$

and the salinity decreases exponentially from the head down its mouth.

Another example of simplifying the Eq. (7.77) is presented for a highly stratified estuary, such as a salt wedge (type 4 or A). In this estuary, the dominant mixing process is advection, and the salinity increase across the halocline is due to entrainment. Then, the salt conservation equation is simplified to:

$$u \frac{\partial S}{\partial x} + w \frac{\partial S}{\partial z} = 0. \tag{7.79a}$$

However, with the exception of salt wedge estuaries, in the layer over the halocline, the diffusion term may still be important and holds the expression,

$$\frac{1}{B} \frac{\partial}{\partial z} \left(BK_z \frac{\partial S}{\partial z} \right), \tag{7.79b}$$

which may be further simplified if the width (B) is constant.

In coastal plain estuaries forced by macro or hyper-tides and with moderate river discharge, there will be random velocity fluctuations generated by internal turbulence and friction at the estuary boundaries. The vertical mixing of the upper and lower layers is enhanced, and the halocline is partially eroded. Thus, the salinity increase in the upper layer, increasing the seaward salt transport, while the salinity in the lower layer decreases landward, and the estuary becomes partially mixed (type 2 or B). If the estuary is laterally homogeneous and in steady-state (according to mean values over tidal cycles), the most important terms of the salt conservation Eq. (7.77) are the horizontal and vertical advection and the vertical diffusion. Then, the principle of salt conservation is reduced to:

$$u \frac{\partial S}{\partial x} + w \frac{\partial S}{\partial z} = \frac{1}{B} \frac{\partial}{\partial z} \left(BK_z \frac{\partial S}{\partial z} \right), \tag{7.80}$$

because the non-advective horizontal term is small and may be disregarded (Pritchard 1954, 1955).

Although the two-dimensional equations in the planes (Ox, y) or (Ox, z) are simplifications of the tridimensional equation, their steady-state solutions for natural estuarine systems, from analytical and time dependent numerical methods, have some complexity. However, analytical solutions were obtained for simple geometries and steady-state conditions in the classical articles of Pritchard and Kent (1956), Rattray and Hansen (1962), Hansen and Rattray (1965), Fisher et al. (1972), Officer (1977), among others. A non-steady-state numerical solution using the natural geometry of the Potomac river estuary may be found in Blumberg (1975).

As a practical example, under steady-state conditions and with a constant width (B = const.), Eq. (7.80) will be used to calculate the salinity profile S = S(z) of a partially mixed estuary, with the following quantities known: the vertical velocity profile $u = u(z)$ or $u = u(Z)$, the mean longitudinal salinity gradient $(\partial S/\partial X) \approx \overline{S}_x$), and w = 0. From some algebraic rearrangement, the equation is reduced to:

$$\frac{\partial^2 S}{\partial z^2} = \frac{u(z)}{K_z} \overline{S}_x, \text{ or } \frac{d^2 S}{dZ^2} = \frac{h^2}{K_z} u(Z) \overline{S}_x,$$

where $Z = z/h$ ($0 \leq Z \leq 1$), and with the non-dimensional depth (Z) used in the latter expression. The general solution of this second order ordinary differential equation is,

$$S(Z) = \frac{u(Z)h^2}{2K_z} \overline{S}_x Z^2 + C_1 Z + C_2.$$

The non-dimensional constants C_1 and C_2 are calculated using the following boundary conditions: at the surface $S(0) = S_s$, and at the bottom $S(1) = S_b$. In the general solution, they are given by:

$$C_1 = S_s \text{ and } C_2 = (S_b - S_s) - \frac{u(Z)h^2}{2K_z} \overline{S}_x,$$

where the difference $S_b - S_s$ for partially mixed estuaries may vary from just a few to values to over twenty psu, for weakly and high stratified conditions, respectively.

Substituting the constants C_1 and C_2 into the general solution, the final vertical salinity profile is:

$$S(Z) = [\frac{u(Z)h^2}{2K_z}\overline{S}_x]Z^2 + [(S_b - S_s) - \frac{u(Z)h^2}{2K_z}\overline{S}_x]Z + S_s,$$

which satisfies the boundary conditions, and may be reduced to a final solution if the vertical velocity profile u = u(Z) has previously been calculated.

7.3.4 One-Dimensional Formulation: Integration in an Area

Estuaries that are long, narrow and shallow, with accentuated tidal forcing, and are vertically non-stratified and laterally homogeneous, can be studied as one-dimensional system, and the continuity and salt conservation equations may be simplified to be applied to these well-mixed estuaries (type 1, C). In these estuaries, the longitudinal variation is prevalent and the velocity, salinity and the concentration of properties may be taken as functions of the longitudinal distance and time, i.e., u = u(x, t), S = S(x, t) and C = C(x, t), respectively. The one-dimensional deduction of these equations are not trivial, because the integrated mean values of Eqs. (7.19), (7.25) and (7.27) must be along the transverse plane (Oyz), orthogonal to the axis Ox, as illustrated in Fig. 7.6.

Beginning with the integration of the continuity equation,[4] let us indicate by A = A(x, t) the area of the transverse section limited by the closed line c (Fig. 7.6), which varies both along the estuary and with time

$$\iint_A \frac{\partial u}{\partial x}dydz + \iint_A \frac{\partial v}{\partial y}dydz + \iint_A \frac{\partial w}{\partial z}dydz = 0, \qquad (7.81)$$

where dydz = dA is a small elementary area.

The first term of Eq. (7.81) must be transformed by applying the Leibnitz derivation rule of a double integral, resulting in (Pritchard 1958; Okubo 1964):

$$\frac{\partial}{\partial x,t}\left(\iint_A udydz\right) = \iint_A \frac{\partial u}{\partial x,t}dydz + \frac{1}{c}\left(\frac{\partial A}{\partial x,t}\right)\oint_c udl. \qquad (7.82)$$

In this equation, c is the length of the continuous curve limiting the area, A, depicted by the closed line going through in the positive sense of direction (Fig. 7.6), and dl is the differential arch element.

The remaining terms (second and third) of Eq. (7.82) may be adequately reduce using the Green's formula (Severi 1956, p. 369), which transform the surface integral into a line integral,

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[4]This type of estuary is usually shallow and the influence of the gravitational circulation may be disregarded and the baroclinic bumping landward is negligible.
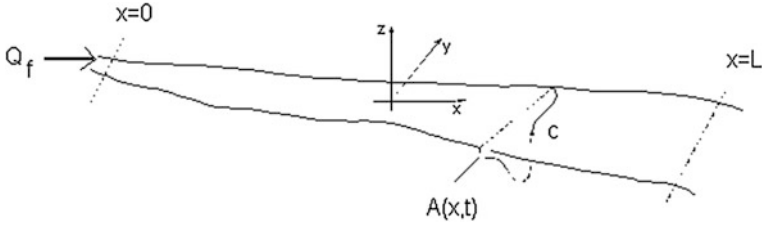
Fig. 7.6 Estuary approximated by one-dimensional model with uniform property distributions in the transversal section A = A(x, t). The boundary of the area, A, is the closed line circulating in the positive orientation (area is located at left)

$$\iint_A \frac{\partial v}{\partial y} dy dz = \oint_c v dz,$$
(7.83)

and

$$\iint_A \frac{\partial w}{\partial z} dy dz = - \oint_c w dy,$$
(7.84)

where, dy and dz are differential elements of the contour line (c).

Substituting Eqs. (7.82), (7.83) and (7.84) into the Eq. (7.81) we have the following expression:

$$\frac{\partial}{\partial x} \left(\iint_A u dy dz \right) - \frac{1}{c} \left(\frac{\partial A}{\partial x} \right) \oint_c u d\ell + \oint_c v dz - \oint_c w dy = 0.$$
(7.85)

As the u-velocity component is uniform in the transversal section A, this equation may be rewritten as:

$$\frac{\partial (uA)}{\partial x} - \frac{\partial A}{\partial x} u|_A + \oint_c v dz - \oint_c w dy = 0,$$
(7.86)

where the following identities have been taken into account: $\int_A dy dz = A$, and $\frac{1}{c} \oint_c d\ell = 1$, and $u|_A = u$, is the velocity mean value in the area A.

Now, for the physical interpretation of the sum of the two last terms of Eq. (7.86), we may use a *consequence* the Green theorem, which states that (Severi 1956, p. 369):

$$A = \frac{1}{2}(\oint_c y\,dz - \oint_c z\,dy). \tag{7.87}$$

As $A = A(x, t)$, the local variation $\partial A/\partial t$ is formulated by:

$$\frac{\partial A}{\partial t} = \frac{1}{2}(\oint_c v\,dz - \oint_c w\,dy),\ \text{or}\ 2\frac{\partial A}{\partial t} = (\oint_c v\,dz - \oint_c w\,dy), \tag{7.88}$$

and the functions being integrated, v and w are given by

$$v = \frac{dy}{dt}\,;\, w = \frac{dz}{dt}. \tag{7.89}$$

The second term of Eq. (7.86) may be rewritten as:

$$\frac{\partial A}{\partial x}u\Big|_A = \frac{\partial A}{\partial x}\frac{dx}{dt} = \frac{\partial A}{\partial t}, \tag{7.90}$$

and substituting Eqs. (7.88) and (7.90) into the Eq. (7.86), the result is,

$$\frac{\partial(uA)}{\partial x} - \frac{\partial A}{\partial t} + 2\frac{\partial A}{\partial t} = 0, \tag{7.91}$$

and the analytical expression of the one-dimensional principle of continuity is reduced to (Pritchard 1958):

$$\frac{\partial(uA)}{\partial x} + \frac{\partial A}{\partial t} = 0. \tag{7.92a}$$

For a wide shallow estuary ($B \gg H_0$), $A = B(H_0 + \eta) \approx B\eta$, this equation is reduced to

$$\frac{1}{B}\frac{\partial(uA)}{\partial x} + \frac{\partial \eta}{\partial t} = 0. \tag{7.92b}$$

As the area, A, is a known geometric property in this equation, it may be solved using the mean velocity at the transverse section $u = u(x, t)$, as well as to the volume transport, $u(x, t)A = T_V(x, t)$; for convenience, this volume transport may be also denoted by Q ($uA = T_V = Q$). Then, in order to satisfy the mass conservation principle, if in a longitudinal location, x, the volume transport increases ($\partial T_V/\partial t > 0$) or decreases ($\partial T_V/\partial t < 0$), this must be compensated by a time decrease ($\partial A/\partial t < 0$) or increase ($\partial A/\partial t > 0$) of the cross-section area, respectively. However, in steady-state condition, the continuity equation is reduced to $T_V = Q_f = uA = \text{const}$, or $u = u_f = Q_f/A$.

Similar to the mass continuity, the one-dimensional salt conservation equation is obtained by multiplying the tridimensional expression (7.25) by the differential area dydz, developing the integral terms using the Leibnitz rule and the Green theorem, it follows that:

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\iint_{A} S dy dz\right) &- \frac{\partial A}{\partial t} S|_{A} + \frac{\partial}{\partial x}\left(\iint_{A} u S dy dz\right) - \frac{\partial A}{\partial x} u S|_{A} \\
&+ \oint_{c} v S dz - \oint_{c} w S dy = \frac{\partial}{\partial x}\left[\iint_{A}\left(K_{x} \frac{\partial S}{\partial x}\right) dy dz\right] \\
&- \frac{1}{c} \frac{\partial A}{\partial x} \oint_{c}\left(K_{x} \frac{\partial S}{\partial x}\right) dl + \oint_{c}\left(K_{y} \frac{\partial S}{\partial y}\right) dz - \oint_{c}\left(K_{z} \frac{\partial S}{\partial z}\right) dy.
\end{aligned}
\tag{7.93}
$$

Taking into account Eqs. (7.88) and (7.90), and that the salinity, S, and the u-velocity component are uniform in the area, A, the terms on the left-hand-side of Eq. (7.93) simplifies to:

$$
\frac{\partial(AS)}{\partial t} + \frac{\partial(uAS)}{\partial x},
\tag{7.94}
$$

because $\int_{A} dy dz = A$, and the sum of the remaining terms are equal to zero,

$$
-\frac{\partial A}{\partial t} S|_{A} - \frac{\partial A}{\partial x} u S|_{A} + S|_{A}\left(\oint_{c} v dz - \oint_{c} w dy\right) = 0.
\tag{7.95}
$$

In the first term on right-hand-side of the Eq. (7.93), salinity and the longitudinal salt diffusive term, $K_{x}(\partial S/\partial x)$, are uniform in the area A. As the line integrals are equal to zero, these terms are reduced to:

$$
\frac{\partial}{\partial x}\left(AK_{x} \frac{\partial S}{\partial x}\right),
\tag{7.96}
$$

and it follows that the one-dimensional salt conservation equation (Pritchard 1958) is:

$$
\frac{\partial(AS)}{\partial t} + \frac{\partial(uAS)}{\partial x} = \frac{\partial}{\partial x}\left(AK_{x} \frac{\partial S}{\partial x}\right).
\tag{7.97}
$$

Finally, developing the products indicate in the left-hand-side of the differential equation, and combining the result with the one-dimensional continuity Eq. (7.92a, 7.92b), the salt conservation equation may be simplified to,

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = \frac{1}{A}[\frac{\partial}{\partial x}(AK_x \frac{\partial S}{\partial x})].\tag{7.98}$$

Under steady-state conditions and for A = const., this equation may be further reduced to $u_f S = K_x(\frac{dS}{dx})$ or $\rho(u_f S) = \rho[K_x(\frac{dS}{dx})]$, which states that the downstream advective salt flux driven by the river velocity ($\rho u_f S$) will counteract the upstream diffusive flux driven by all other mechanism, $\rho K_x(\partial S/\partial x)$. This equation has been used to estimate the kinematic (or dynamic) eddy diffusion coefficient, which may be approximated by $K_x = u_f S/(\partial S/\partial x)$. However, coefficients calculated with this equation and used in analytical and numerical models are subjected to interpretation and must be validated with observational results of velocity and salinity profiles.

We have seen that when the geometry of a transverse section (A), the uniform longitudinal velocity component (u), the kinematic eddy diffusion coefficient (K_x) and the initial and boundary conditions are known, it is possible to solve the differential Eq. (7.98) for S = S(x, t) or S = S(x). The following statements, according to Okubo (1964) and Fischer et al. (1979), should be noted: (i) Time derivative describes the change of water property per tidal cycle, and A may indicate the cross-sectional area at mean time interval of the tidal cycle; (ii) Because of the cross-sectional area divergence, the effective mean flow velocity tends to decrease towards the sea; (iii) On the other hand, the local fresh water inflow through the sides of the estuarine may counteract, to some extent, the cross-section divergence of the mean flow.

Similar expressions to Eqs. (7.97) and (7.98) with a few adaptations, may be transformed into one-dimensional conservation equations for analysis of the one dimensional concentration, C = C(x, t), diffusion of pollutants, based on the time and space averaged equations formulated by:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \frac{1}{A}[\frac{\partial}{\partial x}(AK_{xC}\frac{\partial C}{\partial x})] \pm (\text{sources or sink terms}),\tag{7.99a}$$

or

$$A\frac{\partial C}{\partial t} + Q_f\frac{\partial C}{\partial x} = [\frac{\partial}{\partial x}(AK_{xC}\frac{\partial C}{\partial x})] \pm (\text{sources or sink terms}).\tag{7.99b}$$

In these equations, C, is a non-dimensional concentration [C] = [MM^{-1}] and K_{xC} [K_{xC}] = [L^{2}T^{-1}] is the kinematic eddy-diffusion coefficient.

For the Eqs. (7.99a, 7.99b) the same restrictions hold as those indicated for Eq. (7.98). A rigorous deduction of this equation, describing the space and time variation of the pollutant diffusion in a one-dimensional estuary was developed by Okubo (1964). The conditions for which it is appropriate to use for pollutant diffusion in a one-dimensional estuary under steady-state condition are: (i) after a critical initial time period; (ii) when, due to tidal fluctuations of cross-sectional areas, the property concentration and density are sufficiently small compared with the respective mean value; (iii) when tidal mean velocity weighted by

cross-sectional area is used for velocity (u), instead a simple tidal mean velocity, and; (iv) when an effective eddy diffusivity is defined, including non-homogeneity within the cross-section, and the river discharge into the estuary is small.

7.4 Simplifyed Forms of the Continuity Equation

In Chap. 2, a simplified one-dimensional continuity equation was presented for an estuary with width (B) and depth (H_0) constants (Eq. 2.17, Chap. 2). Let us derive another equation in which the condition that the width B $\neq$ const., starting with the bi-dimensional expression of the continuity (7.56a), performing its lateral integration from y = a(x, z) to y = b(x, z), the result is:

$$\int_a^b \frac{\partial(uh)}{\partial x}dy + \int_a^b \frac{\partial(vh)}{\partial y}dy + \int_a^b \frac{\partial h}{\partial t}dy = 0. \qquad (7.100)$$

As h = H_0 + η(x, t), in this equation it has been taken into account that $\partial\eta/\partial t = \partial h/\partial t$. Applying the Leibnitz rule to the first term, calculating the integral in the second term, and with the boundary conditions $vh|_{x=a} = vh|_{x=b} = 0$, it follows that:

$$\frac{\partial}{\partial x}[\int_a^b (uh)dy] + \int_a^b (\frac{\partial h}{\partial t})dy = 0. \qquad (7.101)$$

Because an estuary's length is generally much less than one quarter of the co-oscillating tidal wave length, the local depth (h) may be considered as independent on x and y, and the Eq. (7.101) may be rewritten as:

$$\frac{\partial}{\partial x}[\int_a^b (uh)dy] + \frac{\partial h}{\partial t}\int_a^b dy = 0, \qquad (7.102a)$$

and

$$\frac{\partial}{\partial x}[\int_a^b (uh)dy] + B\frac{\partial h}{\partial t} = 0. \qquad (7.102b)$$

As the u-velocity component is independent of the depth (z) and the water column height (h), in the first term it may be substituted by an integral (as in Eq. 7.56c); then, this equation may be re-written as:

$$\frac{\partial}{\partial x}[u \int_a^b (\int_{-H_0}^{\eta} dz)dy] + B(\frac{\partial h}{\partial t}) = 0, \tag{7.103a}$$

or

$$\frac{\partial Q}{\partial x} + B(\frac{\partial \eta}{\partial t}) = 0, \tag{7.103b}$$

where the volume transport is denoted by $Q = Q(x, t)$ and is calculated by

$$u[\int_a^b (\int_{-H_0}^{\eta} u dz)dy] = u(x, t)A = Q(x, t) \tag{7.103c}$$

By the longitudinal integration of equation of Eq. (7.103b), from the estuary head $(x = 0)$ down to the mouth $(x = L)$, it follows that,

$$Q_f - Q_L(t) = \frac{\partial \eta}{\partial t}(\int_0^L B dx) = (\frac{\partial \eta}{\partial t})A_{su}, \tag{7.104}$$

where Q_f and $Q_L(t)$ are the river discharge and the time variation of the volume transport at the estuary mouth, respectively. The integration of the estuary width (B) along its longitudinal length (L) may be identified as the surface area of the estuary (A_{su}), which is dependent on the tidal height, $A_{su} = A_{su}(\eta)$. If the river discharge (Q_f) is disregarded, this solution may be simplified as:

$$Q_L(t) = -(\frac{\partial \eta}{\partial t})A_{su}. \tag{7.105}$$

This equation may be used to estimate the volume transport through the mouth of a hyper-saline estuary or coastal lagoon in arid regions, and indicates that the temporal variation of the volume transport is proportional to the surface area and the tidal oscillation. As this surface area may be determined by the hypsometric characteristics of the coastal system (Fig. 7.7), with leveling techniques and aerial pictures, and the tide may be predicted, it is possible to calculate approximately the volume transport, $Q_L = Q_L(t)$.

It should be noted that during the tidal flood $(\partial \eta/\partial t > 0)$ and ebb $(\partial \eta/\partial t < 0)$ tides, the volume transport is negative $Q_L < 0$ and positive $Q_L > 0$, respectively. Then, the volume transport balance, $T_C = CQ_L(t)$, associated with any known conservative property concentration (C), may be used to estimate the importation $(T_C < 0)$ or exportation $(T_C > 0)$ of substances. With the concentration in units of property per volume $[prL^{-3}]$, the property transport is calculated in $[prT^{-1}]$.

Fig. 7.7 Schematic diagram of a coastal lagoon showing the inflow and outflow of volume transport (Q_L) and the associate maximum and minimum flooding areas, which may be calculate by hypsometric techniques
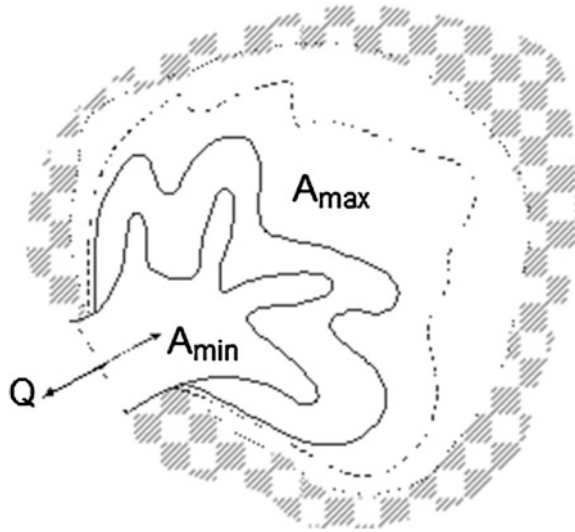
7.5   Application of the One-Dimensional Continuity Equation

To exemplify an application of the one-dimensional continuity Eq. (7.92a, b), let us investigate its analytical solution for a classical estuarine channel. Under the assumption that in this estuary the velocity is uniform in the cross-sectional area, the equation may be integrated from a section located at the estuary head ($x = 0$) down to a longitudinal position x (Fig. 7.6):

$$\int_0^x \frac{\partial (uA)}{\partial x} dx + \int_0^x \frac{\partial A}{\partial t} dx = 0, \tag{7.106a}$$

or

$$(uA)|_x - (uA)|_{x=0} + \int_o^x \frac{\partial A}{\partial t} dx = 0, \tag{7.106b}$$

where $(uA)|_{x=0} = Q_f$. As the integration limits are independent on the time,

$$(uA)|_x + \frac{\partial}{\partial t} (\int_0^x A dx) = Q_f, \tag{7.107a}$$

or

$$(uA)|_x + \frac{dV}{dt} = Q_f. \tag{7.107b}$$

In this equation, V is the volume of the estuarine water mass between the cross-sections located from x = 0 and the arbitrary position x, and dV/dt is its time variation. Solving this equation for the cross-section mean velocity, u = u(x, t), gives,

$$u(x,t) = \frac{1}{A}\left(Q_f - \frac{dV}{dt}\right), \tag{7.108}$$

where u = u(x, t) is the non-steady state u-velocity component generated by the river discharge and the barotropic gradient pressure force due to tidal oscillation, and its influence in the volume time variation (dV/dt). From this result, it is possible to compare the current intensities during the ebb (u_E) and flood (u_F) tides. In effect, because the river discharge may be taken as constant during the tidal period $u_f = Q_f/$ A, and the ebb velocity (u_E) usually is higher than the flood (u_F), because the volume decrease (dV/dt < 0) of the mixing zone (MZ). During the flood, dV/dt > 0, and its intensity $u_F < u_E$. Under steady-state condition dV/dt = 0 then $u_f = Q_f/A$.

Taking into account that the mean free surface $\overline{A}_{su}$ is occupied by the MZ, between the estuary head (x = 0) and the estuary mouth at longitudinal position x, the geometric volume of the estuarine water mass is calculated by $V(t) = \overline{A}_{su}\eta(x,t)$. Thus, the Eq. (7.108) may be rewritten as:

$$u(x,t) = \frac{1}{A}\left(Q_f - \overline{A}_{su}\frac{\partial\eta}{\partial t}\right), \tag{7.109}$$

which confirms that this velocity is generated by the river discharge and the barotropic gradient pressure force. With this equation, the volume transport Q, $[Q] = [L^3T^{-1}]$ may also be calculated by,

$$Q(x,t) = u(x,t)A = Q_f - \overline{A}_{su}\frac{\partial\eta}{\partial t} = Q_f - \frac{dV}{dt}, \tag{7.110}$$

which has similarities with Eq. (7.104).

In estuaries, the time-rate of the volume stored in the space between a cross-section and its head may be approximate by sine functions of time (t) and the angular frequency (ω). Then, the integral of the Eq. (7.107b) during a complete tidal cycle is given by (Pritchard 1958):

$$\int_0^T (uA)dt = \int_0^T Q_f dt - \int_0^T \frac{dV}{dt}dt. \tag{7.111}$$

As the river discharge Q_f is constant, and the last integral on the right-hand-side is equal to zero, as it is supposed that during a complete tidal cycle there will be no time variation in the stored volume ($\frac{dV}{dt} = 0$) in the mixing zone (MZ), then,

$$\int_0^T (uA)dt = Q_f T = R, \tag{7.112}$$

This result confirms the fresh water volume conservation during this time interval, and that the net volume transport is equal to R.

7.6 Application of the One-Dimensional Salt Conservation Equation

The tidal prism models where semi-empirical developed without the formalism of the principles of mass and salt conservation presented in this chapter. Additionally, the basic ideas of Ketchum's methods were described, taking into account the physics of the continuum in the classical article of Arons and Stommel (1951), which will be presented in Chap. 10.

In this topic, the mass and salt conservation equations will be applied with the same hypothesis as the discrete models: one-dimensional, steady-state and well-mixed estuaries. Let us consider a simple geometry (A = const.), with the kinematic effective (eddy) diffusion coefficient (K_H) in the MZ taken as constant. In these conditions, because $\partial A/\partial t = 0$, the continuity Eq. (7.92a, b) is simplified to ($\frac{\partial uA}{\partial x} = 0$), and $uA = Q_f$ is independent of the longitudinal position. In turn, the salt conservation Eq. (7.98) is simplified to:

$$u_f \frac{\partial S}{\partial x} = K_H \left(\frac{\partial^2 S}{\partial x^2} \right). \tag{7.113}$$

As the river discharge and the estuary geometry are known, the solution S = S(x) of Eq. (7.113) is dependent only on the boundary conditions and a known K_H coefficient. As the salinity field is uniform in the cross section area, the problem has been reduced to one dimension, and the differential Eq. (7.113) is an ordinary second order differential equation with constants coefficients. Rearranging its terms the equation to be solved is:

$$\frac{d^2 S}{dx^2} - \frac{u_f}{K_H} \frac{dS}{dx} = 0, \tag{7.114a}$$

or, reducing its order with a first integration from the river tidal zone (x = 0, where S = 0), and the longitudinal position x in the mixing zone (MZ), gives

$$u_f S = K_H \frac{dS}{dx} + A_1, \qquad (7.114b)$$

where A_1 is an integration constant. Let us solve this equation in relation to a local coordinate system with Ox oriented positively seaward from $x = -L$ (boundary TRZ/MZ) to $x = 0$ (estuary mouth) as shown in Fig. 7.8; with this orientation of the u-velocity component generated by the river discharge, (u_f) is positive.

At $x = -L$, the salinity and the product $K_H \frac{dS}{dx}$ are equal to zero and the integration constant $A_1 = 0$. Multiplying both members of the remaining equation by the density, it follows that the seaward advective salt flux counter balances the eddy diffusive salt flux due to the tidal forcing.

The simplifying hypothesis of this problem may limit its practical application because some influences have not been taken in consideration (mainly bottom and lateral friction, baroclinic forcing and vertical mixing). The solution, however, is very interesting because it demonstrates the physical nature of the estuary and its relationship with the main concepts presented in Chap. 6.

Returning to Eq. (7.114b), its solution with $A_1 = 0$ simplifies to

$$S(x) = S_0 \exp(\frac{u_f}{K_H} x). \qquad (7.115)$$

This solution was determined with the following boundary conditions: the salinity at the mouth ($x = 0$) is equal to the salinity at coastal region $S(0) = S_0$ (the only salt source), and the estuary is long enough that for $L \rightarrow -\infty$ the salinity decreases to zero $S(-\infty) = 0$. As $x < 0$ and $u_f > 0$, it is easy to show that this solution identically satisfies these boundary conditions.

The equation for the longitudinal distribution of a solute discharged into a steady-state well-mixed estuary forced by tides and river discharge, with transverse sections that vary in the longitudinal direction $A = A(x)$, was obtained in the classical work of Maximon and Morgan (1955). Their solution, when particularized for a constant transversal section ($A = $ const.), reduces identically to the solution (7.115).

With this analytical solution for $S = S(x)$, it is possible to apply the Eq. (6.11) (Chap. 6) to calculate the longitudinal variation of the fresh water fraction, $f = f(x)$, which is necessary to theoretically calculate the fresh water volume stored in the estuary. Thus, this expression becomes,

$$f(x) = 1 - \exp(\frac{u_f}{K_H} x), \qquad (7.116)$$

and for $x \rightarrow -\infty$, this solution converges to one $f(-\infty) \rightarrow 1$.

In these solutions (7.115 and 7.116), the longitudinal kinematic eddy diffusion coefficient (K_H) still remains an unknown physical quantity, which may be theoretically calculated by correlation of the turbulent fluctuations of the u-velocity and salinity (Eq. 7.26). However, its order of magnitude may be estimated if the
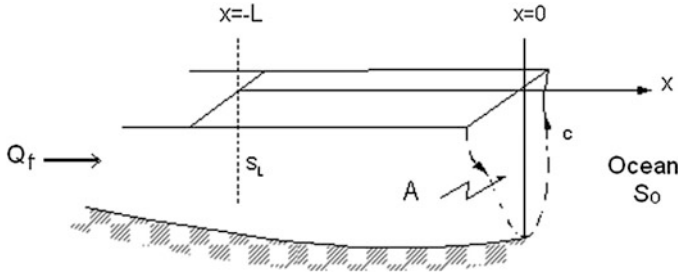
Fig. 7.8 One-dimensional model indicating the landward $(x = -L)$ and seaward $(x = 0)$ longitudinal coordinates of the mixing zone (MZ) between its head and mouth, respectively

longitudinal distance between the estuary mouth and head (mixing zone length) is known, along with the physical quantities such as Q_f, A and S_0. For example, if $Q_f = 100$ m^3 s^{-1}, A = 10^3 m^2, $S_0 = 35^o/_{oo}$ and $S(-L) = 1$, Eq. (7.116) may easily be solved for the coefficient K_H, and some results are presented in Table 7.1.

Although this table only shows the coefficient dependence on the mixing zone length, it is also directly dependent on the river discharge, fresh water fraction, and the estuary geometry (transverse cross-section area and length). As the transverse cross-section area usually varies along the estuarine channel, even for a constant river discharge, this coefficient will be dependent on the longitudinal distance, $K_H = K_H(x)$. This dependence was first investigated by Stommel (1953b), solving the Eq. (7.115) by finite differences around a longitudinal position, x,

$$K_H(x) = \frac{u_f S(x)}{\Delta S} \Delta x = \frac{Q_f S(x)}{A(x)\Delta S} \Delta x = \frac{Q_f[f(x) - 1]}{A(x)\Delta f} \Delta x. \qquad (7.117)$$

In the last term, the salinity $S = S(x)$ was changed by the fresh water fraction, as $S(x) = S_0[1-f(x)]$ and $\Delta S(x) = -S_0 \Delta f(x)$. In this equation, the salinity and the fresh water fraction are mean values at the cross-sections separated by a distance Δx, and ΔS and Δf are finite intervals of salinity and the fresh water fraction, respectively.

Equation (7.117) is a simple and direct method to determine the K_H coefficient using known quantities that may be obtained experimentally. With all variables in the SI system of units, the kinematic coefficient, K_H, is calculated in m^2 s^{-1}. For well-mixed estuaries with mean dimensions and moderate river discharge, we may,

Table 7.1 Estimates of the longitudinal kinematic eddy diffusion coefficient (K_H), based on the mixing zone length ($-L$) and the following data: $Q_f = 100$ m^3 s^{-1}, A = 10^3 m^2 and $S_0 = 35^o/_{oo}$

Mixing zone length L (m)	Coefficient K_H (m^2 s^{-1})
36	1
360	10
3600	100
36,000	1000

as a first approximation, adopt K_H values between 3.0×10^2 m^2 s^{-1} and 5.0×10^2 m^2 s^{-1}. Results obtained by various scientists who applied this method are compared by Officer (1978).

With the theoretical solutions of salinity, $S = S(x)$, and the fresh water fraction $f = f(x)$, it is possible to calculate the following fundamental physical quantities, defined in Chap. 6: fresh water volume V_f stored in the MZ, flushing time (t_q) and flushing rate (F). To exemplify this procedure, let us take $L = -10^4$ m, and the same values as we used previously: $Q_f = 100$ m^3 s^{-1}, $A = 10^3$ m^2, and $S_0 = 35.0°/_{oo}$. With these values, it is easy to show that:

$$S(x) = 35 \exp(3.6 \times 10^{-4}x), \tag{7.118a}$$

$$f(x) = 1 - \frac{S(x)}{35} = 1 - \exp(3.6 \times 10^{-4}x), \tag{7.118b}$$

$$V_f = A \int_{-L}^{0} f(x)dx = A\{L + 2.8 \times 10^3 [\exp(-3.6 \times 10^{-4}L) - 1]\}, \tag{7.118c}$$

and

$$F = \frac{S_0}{S_0 - \overline{S}} Q_f = \frac{35}{35 - 9.5} Q_f. \tag{7.118d}$$

Calculating these analytical solutions with the numerical values given above gives the following results: $V_f = 7.28 \times 10^6$ m^3, $t_q = 20.2$ h and F = 137 m^2 s^{-1}. As this theory is applied to well-mixed estuaries, the flux rate (F) will be almost constant due to the dominance of the diffusion in the mixing process. The longitudinal variations of salinity and the fresh water fraction in this steady-state and one-dimensional model are presented in Figs. 7.9a, b, respectively.

Applying the conservation Eq. (7.114a) to a conservative property, with the same simplifying conditions as this example, and with its source located at the estuary mouth, it is easy to verify that the solution is similar to the Eq. (7.115), with the following substitutions: at the estuary mouth the salinity S_0 by the maximum concentration C_0, and K_H substituted by the corresponding eddy diffusion coefficient of the property K_{HC}. Then, the longitudinal distribution of the property concentration, $C = C(x)$, decreases exponentially landward, following the exponential salinity decrease. This result has already been used qualitatively in Chap. 6, in a semi-empirical development (Eq. 6.83).

As a complementary exercise at the end of this topic, let us present another solution of the salt conservation Eq. (7.113), but in terms of the fresh water fraction (f). After integration of this equation, and considering the integration constant equal to zero, it may be rewritten as:

Fig. 7.9 Simulation of
longitudinal variations of
salinity and fresh water
fraction of a one dimensional
and steady-state well-mixed
estuary, with length $L = -10^4$
m, and kinematic coefficient
of eddy diffusion,
$K_H = 280$ m^2 s^{-1}
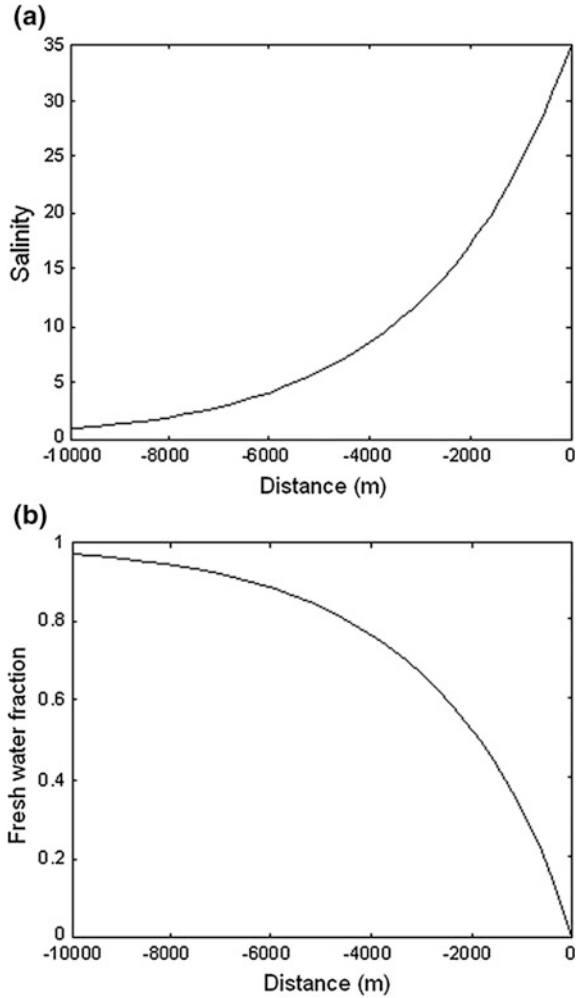
$$u_f S - K_x \frac{dS}{dx} = 0, \qquad (7.119)$$

and its first term may be associated with the mass conservation equation, for
$u_f = Q_f/A$. Taking into account that the fresh water fraction $f = f(x)$ defined in the
Eq. 6.10 (Chap. 6), it follow that the salinity may be expressed in terms of the fresh
water fraction $S = S_0(1 - f)$, and

$$\frac{dS}{dx} = -S_0 \frac{df}{dx}, \qquad (7.120)$$

where $0 \leq f \leq 1$. Combining Eqs. (7.119) and (7.120) and taking into account the mass conservation equation, it follows:

$$\left(\frac{Q_f}{A}\right)S + S_0 K_H\left(\frac{df}{dx}\right) = 0. \tag{7.121}$$

Dividing both members of this equation by S_0, and knowing that the ratio S/S_0 is equal to $(1 - f)$, the salt conservation equation in terms of the fresh water fraction is rewritten as (Officer 1978):

$$Q_f = fQ_f - K_H A\left(\frac{df}{dx}\right). \tag{7.122}$$

Thus, the fresh water fraction, $f = f(x)$, may be calculated solving the following differential equation:

$$\frac{df}{dx} = (f - 1)\frac{Q_f}{K_H A}, \tag{7.123a}$$

or

$$\frac{df}{dx} = (f - 1)\frac{u_f}{K_H}. \tag{7.123b}$$

This equation satisfies identically the longitudinal variation of the fresh water fraction: at the estuary mouth ($f \rightarrow 0$) and $\frac{df}{dx} < 0$, thus $f = f(x)$ decreases seaward. Otherwise, at the estuary head $\frac{df}{dx} = 0$, and $f = f(x)|_{x=1} \rightarrow f = \text{const.} = 1$. An exponential solution of $f = f(x)$ has already been presented in Eq. (7.118b).

7.7   Steady-State Concentration Distribution of a Non-conservative Substance

A study of the longitudinal distribution of the concentration of a non-conservative property (tracer), $C = C(x)$, in a well-mixed estuary under steady-state conditions is now presented as a particular solution of Eq. (7.99a). By hypothesis, the concentration is almost invariable from one tidal cycle to the next, and the river and effluent discharges are also constant during this time interval. As the concentration of a non-conservative substance will be considered (for conservative, the longitudinal variation follows the salinity very closely), its concentration may decrease with time even without the influence of tidal diffusion, and it is necessary to add a term into the equation to take that influence into account. In this case, the decrease with time may be expressed according to the first order exponential decay,

$$C(t) = c_0 \exp(-kt), \tag{7.124}$$

where c_0 is the initial concentration, $C(0) = c_0$, and k, $[k] = [T^{-1}]$ is a coefficient of proportionality, and $t = k^{-1}$ is the time required for the concentration to decay from the initial value c_0 to c_0/e.

With these definitions, the conservation equation for calculation of the steady-state longitudinal distribution, $C = C(x)$, of a non-conservative substance, undergoing a first order concentration decay, in a one-dimensional estuarine channel is reduced to the following second order differential equation:

$$\frac{d^2 C}{dx^2} - \frac{u_f}{K_{HC}} \frac{dC}{dx} + \frac{k}{K_{HC}} C = 0, \tag{7.125}$$

relative to the same referential system presented in Fig. 7.8. The non-dimensional coefficient K_{HC} is a tracer's kinematic eddy (effective) horizontal diffusion. Even though this particular example has limited potential to be directly applied to an estuary, its solution is of interest in showing the approximate forms of solutions that can be expected, and it is appropriate to view this equation as a postulate (or empirical) model, subject to verification (Fischer et al. 1979). This equation has as general solution $C(x) = e^{mx}$, and the characteristic second grade root of this equation has two solutions in the real numeric field:

$$C(x) = c_0 \exp[\frac{u_f x}{2K_{HC}} (1 \pm \sqrt{1 + \Psi})], \tag{7.126}$$

where $\Psi = 4 K_{HC} k / u_f^2$ is a non-dimensional quantity, and the x variable is negative, according to the reference system used. Because the velocity generated by the river discharge ($u_f > 0$) in well-mixed estuaries is usually low, and the quantity Ψ is inversely proportional to u_f, the value of Ψ may be very large.

In order to exemplify the application of the solution presented in Eq. (7.126), let us consider the input of a tracer in an estuary with a mass transport W, in kg s^{-1}, in the longitudinal position $x = x_w$ ($x_w < 0$). Then, the initial condition $C(0) = c_0$ is satisfied, and the equation's solution must be separated into two terms (Stommel 1953b, Fischer et al. 1979):

- Landward from the effluent throw position x_w ($x \leq -x_w$)

$$C_{ac}(x) = c_0 \exp[\frac{u_f}{2K_{HC}} (1 + \sqrt{1 + \psi})(x - x_w)]. \tag{7.127a}$$

- Seaward from the effluent throw position ($x \leq -x_w$)

$$C_{ac}(x) = c_0 \exp[\frac{u_f}{2K_{HC}} (1 + \sqrt{1 + \psi})(x - x_w)]. \tag{7.127b}$$

In these solutions, it is easy to see that for x = x_w the concentration at this point, $C(x_w)$, is equal to c_0, $C(x_w) = c_0$ and, according to the simplifying hypothesis, the initial concentration is determined by the equation,

$$c_0 = \frac{W}{Q_f} f_0, \tag{7.128}$$

where f_0 is the fresh water fraction at the effluent's launching position, and W/Q_f is the initial concentration due to dilution by the river discharge (Eq. 6.78, Chap. 6), without the influence of diffusion or concentration decay.

Let us exemplify numeric results using the above equations under the assumption that the tracer input is made into the estuary whose longitudinal salinity distribution, $S = S(x)$, and fresh water fraction, $f = f(x)$, were theoretically obtained and presented in the Eqs. (7.118a, b), which were calculated with A = 10^3 m^2 and Q_f = 100 m^3 s^{-1}. For the tracer discharge point, we adopt the following positions: x_W = −3000 m and −5000 m, where, according to the Eq. (7.118b), $f(x)|_{x=0}$ = f_0 = 0.65 and f_0 = 0.83, respectively. Assuming of a tracer transport of 20.0 kg s^{-1}, K_{xC} = 60 m^2 s^{-1} and k = 2.3 × 10^{-4} s^{-1}, the longitudinal landward (C_{ac}) and seaward (C_{ab}) distributions from the discharge positions are given by:

$$C_{ac}(x) = 0.13. \exp[2.94 \times 10^{-3}(x + 3 \times 10^3)], \tag{7.129a}$$

$$C_{ab}(x) = 0.13. \exp[-1.55 \times 10^{-3}(x + 3 \times 10^3)], \tag{7.129b}$$

and

$$C_{ac}(x) = 0.17. \exp[2.94 \times 10^{-3}(x + 5 \times 10^3)], \tag{7.130a}$$

$$C_{ab}(x) = 0.17. \exp[-1.55 \times 10^{-3}(x + 5 \times 10^3)]. \tag{7.130b}$$

In these solutions, the initial concentrations $C(x_w) = c_0 = 0.13$ kg m^{-3} and 0.17 kg m^{-3}, were calculated with Eq. (7.128); however, the primary phase of the tracer decay has not been taken into account. As may be observed, due to the seaward decrease of the fresh water fraction, the displacement of the tracer input in this direction reduces the initial tracer concentration (c_0).

Distributions of concentrations, C = C(x), for the discharge positions considered in this exercise, are presented in Fig. 7.10, along with the corresponding salinity S = S(x) and fresh water fraction f = f(x) variations, representing the longitudinal behavior of a conservative property above and below the input position, respectively.

The analytical solution of this problem of a conservative substance is related to the semi-empirical solution of Ketchum (1955), shown in Fig. (6.14, Chap. 6), and the analytical solution of Stommel (1953a) may be similarly interpreted. If salinity
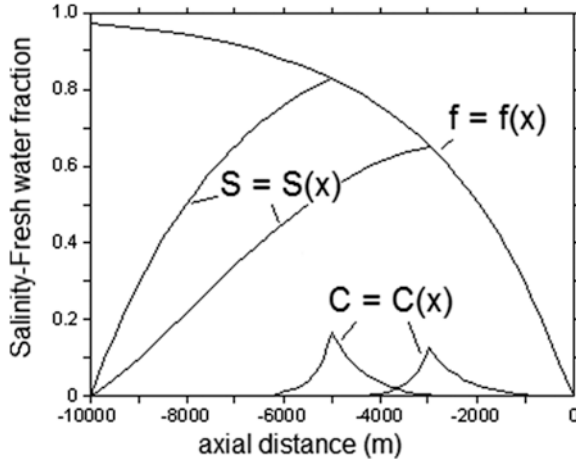
Fig. 7.10 Longitudinal distributions of the concentration, $C = C(x)$, of a non-conservative tracer undergoing a first order decay, with the fresh water fraction, $f = f(x)$ and salinity $S = S(x)$, simulating conservatives properties. The one dimensional solutions were obtained landward and seaward of the tracer input position at -3000 m and -5000 m. (adapted from Stommel 1953a, b). The estuary mouth is located at $x = 0$

is introduced at the locations $x = -6$ km and -3 km, upstream of the outfall they will follow the $S = S(x)$ curves, as shown in Fig. 7.10; however, the remainder of the solution downstream of the outfall their concentrations will follow the fresh water concentration curves, $f = f(x)$. In this simple case it is clear that no matter what the location of the outfall, the concentration is every-where reduced upstream of the outfall if the location of the outfall is moved towards the sea. This solution also indicates that, up to the intersection of the curve $S = S(x)$ with fresh water fraction curve, $f = f(x)$ indicates the maximum input concentration of the effluent.

Solutions for the concentration distribution, $C = C(x)$, for a non-conservative tracer discharged into the estuary by outfalls located at $x = -3.0 \times 10^3$ m and $x = -5.0 \times 10^3$ m, are also shown in the Fig. 7.10. A remarkable feature of these distributions is that the peak concentration, even at the outfalls, are very much reduced in comparison to the conservative tracer. As shown in the figure, the tracer concentration extends both upstream and downstream of the outfall and, unlike a conservative tracer, its concentration can be reduced at a point bellow by an upstream displacement of the outfall (Stommel 1953b; Ketchum, *unpublished report*). These theoretical and practical results indicate the following influence of the river discharge on the tracer longitudinal distribution: an increase in river discharge causes an increase in the fresh water fraction in the mixing zone, reducing the landward displacement of the tracer, and its initial concentration at the discharge position, as well as the seaward concentration of a conservative substance.

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