Chapter 2
Diophantine approximation

2.1 Diophantine approximation on the line

In this short chapter we present without proof classical material about Diophantine approximation. More details and complete proofs can be found for instance in [50], [51], [8]. We are primarily interested in the rational approximation to algebraic numbers; more precisely, we are interested in estimating the accuracy in the approximation to such numbers with respect to the denominator of the approximant. The following theorem gives the best possible result for an arbitrary irrational number.

Theorem 2.1.1 (Dirichlet). *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be a real irrational number. There exist infinitely many rational numbers a/b (a, b coprime integers, $b > 0$) such that*

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^2}.$$

For instance, one can take for a/b the truncated continued fraction expansion of α.

Some irrational numbers can be approximated to a higher degree; for instance, Liouville's number $\alpha := \sum_{n=1}^{\infty} 10^{-n!}$ has the property that for every positive μ there exist infinitely many rationals a/b (a, b coprime integers, $b > 0$) such that

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^\mu}.$$

Such numbers are never algebraic; actually, a theorem of Liouville states that:

Theorem 2.1.2 (Liouville). *Let α be a real irrational algebraic number of degree d over $\mathbb{Q}$. There exists a positive number $c(\alpha)$ such that for all rational numbers a/b*

$$\left|\alpha - \frac{a}{b}\right| \geq \frac{c(\alpha)}{b^d}.$$

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A deeper theorem, due to Roth (1955) [47], improves on the exponent d:

Theorem 2.1.3 (Roth's Theorem). *Let α be a real algebraic number, $\epsilon > 0$. For all but finitely many rational numbers a/b, the following inequality holds:*

$$\left|\alpha - \frac{a}{b}\right| > \frac{1}{b^{2+\epsilon}}. \tag{2.1.4}$$

In an other formulation: if α is algebraic irrational, there exists a positive real number $c(\alpha, \epsilon)$ such that *for all* rational numbers a/b,

$$\left|\alpha - \frac{a}{b}\right| > \frac{c(\alpha, \epsilon)}{b^{2+\epsilon}}. \tag{2.1.5}$$

Roth's proof is ineffective, in the sense that it does not provide any means of finding the finitely many rational numbers a/b which violate the inequality (2.1.4). Looking at its second formulation, by the ineffective nature of Roth's proof it is not possible to calculate the function $c(\alpha, \epsilon)$.

Roth's theorem is best possible as far as the exponent is concerned in view of the mentioned result of Dirichlet (Theorem 2.1.1). However, one can try to improve on Roth's exponent after restricting the approximations to suitable classes of rational numbers. For instance, one can consider the set of rational numbers which, once written in base ten, have only finitely many digits. These numbers form the ring of S-integers $\mathbb{Z}[\frac{1}{10}] = \mathbb{Z}[\frac{1}{2}, \frac{1}{5}]$.

In that case, Ridout [45] improved Roth's bound by proving that: for every irrational algebraic number α and every positive real $\epsilon > 0$, there are only finitely many pairs of integers $(a, n) \in \mathbb{Z} \times \mathbb{N}$ such that $|\alpha - \frac{a}{10^n}| < 10^{-(1+\epsilon)n}$.

A similar result holds whenever the numerators of the approximations are supposed to be of special type, e.g. products of powers of primes from a fixed finite set. When both numerators and denominators are subject to lie in a finitely generated multiplicative semi-group, then the exponent can be lowered to "ϵ" (see Corollary 2.1.10).

In another direction, one can try to replace the rational number field $\mathbb{Q}$ by an arbitrary number field $\kappa \subset \mathbb{C}$. Of course, the expected exponent should change; for instance, if $\kappa \subset \mathbb{R}$ and has degree $d = [\kappa : \mathbb{Q}]$ over the rational, a variation of Dirichlet's theorem asserts that each real number $\alpha \in \mathbb{R} \setminus \kappa$ can be approximated to a degree $-2d$ with respect to the "height" of the approximant (see below for the precise definition of height).

Still another generalization concerns p-adic approximation: one can fix a p-adic algebraic number $\alpha \in \mathbb{Q}_p$ and study its approximations by rational numbers.

In order to find the most appropriate generalization, containing Roth's and Ridout's theorem and their natural extensions to number fields, we introduce the language of heights and revise the theory of absolute values in a number field.

Let κ be a number field. For every place ν of κ, the corresponding absolute values differ logarithmically by a positive constant: namely, if $|\cdot|_\nu$ and $\|\cdot\|_\nu$

are two equivalent absolute values of κ there exists a positive real number δ such that for every $x \in \kappa$, $|x|_\nu = \|x\|_\nu^\delta$. We are looking for a canonical normalization, which will simplify the notation in the formulation of results from Diophantine approximation. One natural choice would be simply to choose the ν-adic absolute values extending the natural ones already defined in the rational number field $\mathbb{Q}$. However, there is another possibility, which is less canonical since it depends on the number field κ, but has the advantage that by adopting this new convention, the generalization and extensions of Roth's theorem will be easier to state. We proceed to define this second normalization.

Let then ν be a place of κ; the completion κ_ν, which is independent of the chosen normalization for the absolute value, is a finite algebraic extension of the corresponding completion of $\mathbb{Q}$, which is either the real number field $\mathbb{R}$ or a field of p-adic numbers $\mathbb{Q}_p$. If ν is ultrametric, we let p be the characteristic of the residue field $\kappa(\nu)$ (so that κ_ν contains $\mathbb{Q}_p$) and normalize the absolute value $|\cdot|_\nu$ on κ so that on $\mathbb{Q}$ it becomes

$$|x|_\nu = |x|_p^{\frac{[\kappa_\nu:\mathbb{Q}_p]}{[\kappa:\mathbb{Q}]}} \qquad \forall x \in \mathbb{Q}.$$

Since the absolute value is determined by the place up to renormalization, the above relation defines the absolute value on the whole of κ. In other words, there is an embedding $j_\nu : \kappa \hookrightarrow \mathbb{C}_p$ such that for all $x \in \kappa$, $|x|_\nu = |j_\nu(x)|_p^{\frac{[\kappa_\nu:\mathbb{Q}_p]}{[\kappa:\mathbb{Q}]}}$.

If, on the contrary, ν is archimedean, then it corresponds to an embedding $j_\nu : \kappa \hookrightarrow \mathbb{C}$; we then normalize the absolute value $|\cdot|_\nu$ by putting

$$|x|_\nu = |j_\nu(x)|^{\frac{[\kappa_\nu:\mathbb{R}]}{[\kappa:\mathbb{Q}]}},$$

where the symbol $|\cdot|$ on the right-hand side stands for the usual complex absolute value.

With this choice of the normalizations the absolute logarithmic Weil height reads

$$h(x) = \sum_\nu \log^+ |x|_\nu \qquad \forall x \in \kappa,$$

where the sum runs over the places of κ and $\log^+ = \max(0, \log)$. We also put

$$H(x) = \exp(h(x))$$

and call it the height of the algebraic number x. It turns out to be independent of the number field κ containing x.

Also, the product formula can be written 'without weights', as

$$\prod_\nu |x|_\nu = 1 \qquad \forall x \in \kappa^*.$$

We can now formulate the first extension of Roth's theorem: we study the degree of approximation of algebraic numbers by elements of a given number field κ. The result is the following

Theorem 2.1.6. *Let κ be a number field, ν be a place of κ and $\alpha \in \kappa_\nu$ be an element of the topological closure of κ, algebraic over κ but not lying in κ. Let $\|\cdot\|_\nu$ denote the absolute value normalized with respect to κ and extended to κ_ν. Then for every positive real number $\epsilon > 0$ there exists a number $c(\alpha, \nu, \epsilon)$ such that for all $\beta \in \kappa$*

$$|\alpha - \beta|_\nu > c(\alpha, \nu, \epsilon) \cdot H(\beta)^{-2-\epsilon}.$$

Let us consider the particular case where ν is archimedean and $\kappa \subset \kappa_\nu = \mathbb{R}$. While generic real numbers can be approximated by a sequence of rationals with an error bounded by Dirchlet's Theorem, we expect that using as approximants elements of κ instead of only rational numbers the degree of approximability of any real number will increase. Since κ is a vector space of dimension $[\kappa : \mathbb{Q}]$ over $\mathbb{Q}$, it should be possible to make the error in the approximation as little as the height of the approximant to the power $-2[\kappa : \mathbb{Q}]$. Actually this is true, and can be proved via the classical pigeon-hole principle. However, in Theorem 2.1.6 above the usual exponent 2 appears; taking into consideration our normalization, the same inequality written with respect to the usual real absolute value would show precisely the exponent $-2[\kappa : \mathbb{Q}]$; so Theorem 2.1.6 states that for algebraic numbers no improvement on Dirichlet's exponent can be obtained.

The most general version of Roth's Theorem, encompassing both Ridout's theorem and the above Theorem 2.1.6, was formulated by Lang in [40]:

Theorem 2.1.7. *Let κ be a number field; let S be a finite set of places of κ. Let, for every $\nu \in S$, $|\cdot|_\nu$ be the extension of the ν-adic absolute value to κ_ν, normalized with respect to κ and let $\alpha_\nu \in \kappa_\nu$ be an algebraic number. For every $\epsilon > 0$ there exists a number $c = c(S, (\alpha_\nu)_{\nu \in S}, \epsilon)$ such that for all $\beta \in \kappa$ with $\beta \neq \alpha_\nu$ for every $\nu \in S$,*

$$\prod_{\nu \in S} |\alpha_\nu - \beta|_\nu > c \cdot H(\beta)^{-2-\epsilon}.$$

Notice that interesting cases arise when some, or even all, the α_ν lie in κ. Indeed, another equivalent formulation of the general Roth's Theorem 2.1.7 involves only κ-rational points. It appears e.g. in [10] and reads as follows:

Theorem 2.1.8. *Let κ be a number field, $d \geq 1$ an integer, $\alpha_1, \ldots, \alpha_d$ be pairwise distinct elements of κ. Let $S_1, \ldots, S_d$ be pairwise disjoint finite sets of absolute values. Finally, let $\epsilon > 0$ be a positive real number. Then for all but finitely many elements $\beta \in \kappa$,*

$$\prod_{h=1}^{d} \prod_{\nu \in S_h} |\alpha_h - \beta|_\nu > H(\beta)^{-2-\epsilon}. \tag{2.1.9}$$

The above theorem can be further generalized, by allowing also points at infinity as target of the approximation. This will be useful in order to deduce the mentioned theorem of Ridout. Precisely, for $\alpha = \infty$ and any absolute value

ν, let us define the ν-adic distance from α to $\beta \in \kappa$, provided $\beta \neq 0$, by putting

$$|\alpha - \beta|_\nu = |\infty - \beta|_\nu := |\beta|_\nu^{-1}.$$

Then the condition that a rational number $\beta \in \mathbb{Q}$ be of the form $\beta = a/b$ where b is a product of primes from a fixed set T can be expressed by the inequality $\prod_{\nu \in T} |\beta - \infty|_\nu \leq |b|^{-1}$; if $|\beta| \leq 1$ we also have $H(\beta) = |b|$ so the arithmetic condition that β lies in a fixed ring of S integers is equivalent to the inequality

$$\prod_{\nu \in T} \min(1, |\beta - \infty|_\nu) \leq H(\beta)^{-1},$$

where $T \subset S$ is the set of ultrametric places in S.

Actually, the generalization of Theorem 2.1.8 with one point α allowed to be at infinity follows formally from the present version of Theorem 2.1.8 itself: observe that applying projective transformations $\Phi : \mathbb{P}_1 \to \mathbb{P}_1$ of the form

$$\Phi(x) = \frac{ax + b}{cx + d},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\kappa)$ one can send the given set of target points $\{\alpha_\nu\}_{\nu \in S} \subset \mathbb{P}_1(\kappa) = \kappa \cup \{\infty\}$ to a subset of $\kappa = \mathbb{P}_1(\kappa) \setminus \{\infty\}$.

For instance, in the special case in which the set of $\{\alpha_\nu, \nu \in S\}$ consists of the three rational points $0, 1, \infty \in \mathbb{P}_1(\kappa)$, the above Theorem 2.1.8 implies:

Corollary 2.1.10. *Let $\Gamma \subset \kappa^*$ be a finitely generated multiplicative group. Let T be a finite set of places of κ and $\epsilon > 0$ a positive real number. Then for all but finitely many $\gamma \in \Gamma$*

$$\prod_{\nu \in T} |\gamma - 1|_\nu > H(\gamma)^{-\epsilon}. \tag{2.1.11}$$

Before giving its proof (assuming Theorem 2.1.8) we remark that a stronger and fully effective estimate can be obtained via the theory of linear forms in logarithms (Baker's method), which replaces the right-hand side term in (2.1.11) by a (negative) power of the *logarithmic* height $h(\gamma) = \log H(\gamma)$ of the approximant.

Proof. Let us deduce Corollary 2.1.10 from Theorem 2.1.8. Let S be a finite set of places of κ such that $\Gamma \subset \mathcal{O}_S^*$ and $T \subset S$. For every solution $\gamma \in \Gamma$ to (2.1.11), let T_0 be the set of places ν for which $|\gamma|_\nu < \frac{1}{2}$ and T_∞ the set of places ν such that $|\gamma|_\nu > 2$. Let $T_1 \subset T$ be the set of places ν such that $|\gamma - 1|_\nu < \frac{1}{2}$. Note that $T_0, T_1, T_\infty \subset S$. Then

$$\prod_{\nu \in T} |\gamma - 1|_\nu \geq \left(\prod_{\nu \in T_1} |\gamma - 1|_\nu \right) \cdot \frac{1}{2^{\sharp(T)}}. \tag{2.1.12}$$

We have also the following estimates

$$\prod_{\nu\in T_0}|\gamma|_\nu=\prod_{\nu\in T_0\cap T_1}|\gamma_\nu|\cdot\prod_{\nu\in T_0\setminus T_1}|\gamma|_\nu\geq 2^{-\sharp(T_0)}\prod_{\nu\in T_0\setminus T_1}|\gamma|_\nu \tag{2.1.13}$$

and

$$\prod_{\nu\in T_\infty}|\gamma|_\nu^{-1}=\prod_{\nu\in T_\infty\cap T_1}|\gamma|_\nu^{-1}\cdot\prod_{\nu\in T_\infty\setminus T_1}|\gamma|_\nu^{-1}\geq 2^{-\sharp(T_\infty)}\prod_{\nu\in T_\infty\setminus T_1}|\gamma|_\nu^{-1}. \tag{2.1.14}$$

Also, in view of the fact that γ is a S-unit and of the definition of T_0, T_∞, we have

$$\prod_{\nu\in T_0}|\gamma|_\nu=\prod_{\nu\in T_\infty}|\gamma|_\nu^{-1}=H(\gamma)^{-1}.$$

From inequalities (2.1.13), (2.1.14) and the above identity we obtain

$$\prod_{\nu\in T_0\setminus T_1}|\gamma|_\nu\ll H(\gamma)^{-1},\qquad \prod_{\nu\in T_\infty\setminus T_1}|\gamma|_\nu^{-1}\ll H(\gamma)^{-1} \tag{2.1.15}$$

where the implied constant only depends on $\sharp(S)$.

Consider the projective transformation $x\mapsto\Phi(x)=\frac{x}{x+1}$, which sends $0,1,\infty$ to $0,1/2,1$ respectively. It satisfies, for every valuation ν,

$$\frac{1}{2}|x-0|_\nu\leq|\Phi(x)-0|_\nu\leq 2|x-0|_\nu\qquad \text{if } |x|_\nu\leq\frac{1}{2}$$

$$\frac{1}{2}|x-1|_\nu\leq|\Phi(x)-\frac{1}{2}|_\nu\leq 2|x-1|_\nu\qquad \text{if } |x-1|_\nu\leq\frac{1}{2}$$

and

$$\frac{1}{2}|x|_\nu^{-1}\leq|\Phi(x)-1|_\nu\leq 2|x|_\nu^{-1}\qquad \text{if } |x|_\nu\geq 2.$$

Then we have

$$\prod_{\nu\in T_0\setminus T_1}|\gamma|_\nu\cdot\prod_{\nu\in T_\infty\setminus T_1}|\gamma|_\nu^{-1}\cdot\prod_{\nu\in T_1}|\gamma-1|_\nu\gg\prod_{\nu\in T_0\setminus T_1}|\Phi(\gamma)|_\nu$$
$$\cdot\prod_{\nu\in T_\infty\setminus T_1}|\Phi(\gamma)-1|_\nu\cdot\prod_{\nu\in T_1}\left|\Phi(\gamma)-\frac{1}{2}\right|_\nu.$$

In view of (2.1.15) and the above inequality we can then write

$$\prod_{\nu\in T_0\setminus T_1}|\Phi(\gamma)|_\nu\cdot\prod_{\nu\in T_\infty\setminus T_1}|\Phi(\gamma)-1|_\nu\cdot\prod_{\nu\in T_1}\left|\Phi(\gamma)-\frac{1}{2}\right|_\nu\ll H(\gamma)^{-2}\prod_{\nu\in T_1}|\gamma-1|_\nu.$$

We now apply Theorem 2.1.8, taking for $d=3$, $\alpha_1=0,\alpha_2=\frac{1}{2}$, $\alpha_3=1$; and $S_1=T_0\setminus T_1$, $S_2=T_\infty\setminus T_1$, $S_3=T_1$; the inequality (2.1.9) of Theorem 2.1.8, together with the above estimates, gives the desired conclusion of Corollary 2.1.10. $\square$

In the rational case, we state the following corollary, whose deduction is left to the reader.

Corollary 2.1.16 (Theorem of Ridout). *Let* $\{p_1, \dots, p_l\}, \{q_1, \dots, q_m\}$ *be two set of prime numbers; let* λ, μ *be real numbers in the closed interval* $[0, 1]$. *Let us consider the set* $\mathcal{B}$ *of rational numbers* β *of the form* $\beta = p/q$ *where*

$$p = p_1^{a_1} \cdots p_l^{a_l} \cdot p^*$$
$$q = q_1^{b_1} \cdots q_m^{b_m} \cdot q^*$$

where $a_1, \dots, a_l, b_1, \dots, b_m$ *are integers with* $a_i \geq 0, b_j \geq 0$ *and* p^*, q^* *satisfy*

$$p^* \leq p^{1-\lambda}$$
$$q^* \leq q^{1-\mu}$$

Let $\alpha \in \mathbb{R}$ *be a real algebraic number and let* $\epsilon > 0$ *be a positive real number. Then for all but finitely many* $\beta \in \mathcal{B}$,

$$|\alpha - \beta| > H(\beta)^{-2+\lambda+\mu-\epsilon}.$$

We end this section by providing yet another version of Roth's theorem; we shall present it as a lower bound for *homogeneous* linear form.

Theorem 2.1.17 (Homogeneous Roth's Theorem). *Let* κ *be a number field,* S *be a finite set of absolute values of* κ. *For each* $\nu \in S$, *let* $L_{1,\nu}(X, Y), L_{2,\nu}(X, Y)$ *be linearly independent linear forms with coefficients in* κ. *Finally, let* $\epsilon > 0$ *be a positive real number. For all but finitely many* $(x : y) \in \mathbb{P}_1(\kappa)$ *the following inequality holds:*

$$\prod_{\nu \in S} \frac{|L_{1,\nu}(x, y)|_\nu}{\max(|x|_\nu, |y|_\nu)} \cdot \frac{|L_{2,\nu}(x, y)|_\nu}{\max(|x|_\nu, |y|_\nu)} > H(x/y)^{-\epsilon}. \tag{2.1.18}$$

Note that, due to the appearance of the denominator $\max(|x|_\nu, |y|_\nu)$, the left hand-side term is invariant by multiplication of x and y by a non-zero constant, so it only depends on the projective class $(x : y)$ of (x, y). This is consistent with the right-hand side term, which only depends on the ratio x/y.

2.2 Higher dimensional Diophantine approximation

In higher dimension, we shall be interested in approximating hyperplanes defined by linear forms with algebraic coefficients by rational points. We shall adopt the language and notation of projective geometry for simplicity, as in the homogeneous version of Roth's Theorem given in Theorem 2.1.17.

The main result of this section is the so-called Subspace Theorem, first proved, in a particular case, by W. M. Schmidt in the seventies. Here we formulate the generalization provided by H.-P. Schlickewei, which is the natural extension of Roth's theorem to higher dimension.

We need an extension to higher dimension of the notion of height, already introduced for algebraic numbers.

Let κ be a number field, $\mathbf{x} = (x_0, \ldots, x_N) \in \kappa^{N+1} \setminus \{0\}$ a non-zero vector. For every place ν of κ, its ν-adic norm $\|\mathbf{x}\|_\nu$ is defined to be

$$\|\mathbf{x}\|_\nu = \max(|x_0|_\nu, \ldots, |x_N|_\nu).$$

Let us define the height of the associated projective point, still denoted by $\mathbf{x} = (x_0 : \ldots : x_N) \in \mathbb{P}_N(\kappa)$, to be

$$H(\mathbf{x}) = \prod_\nu \|\mathbf{x}\|_\nu,$$

where the product runs over all the valuations of κ.

With these conventions, Schmidt's Subspace Theorem reads:

Theorem 2.2.1 (Subspace Theorem)**.** *Let $N \geq 1$ be a positive integer, κ be a number field and S a finite set of places of κ. Let, for every $\nu \in S$, $L_{0,\nu}(X_0, \ldots, X_N), \ldots, L_{N,\nu}(X_0, \ldots, X_N)$ be linearly independent linear forms with algebraic coefficients in κ_ν. Then for each $\epsilon > 0$ the solutions $\mathbf{x} = (x_0 : \ldots : x_N) \in \mathbb{P}_N(\kappa)$ to the inequality*

$$\prod_{\nu \in S} \prod_{i=0}^{N} \frac{|L_{i,\nu}(\mathbf{x})|_\nu}{\|\mathbf{x}\|_\nu} < H(\mathbf{x})^{-N-1-\epsilon} \tag{2.2.2}$$

lie in the union of finitely many hyperplanes of $\mathbb{P}_N$, defined over κ.

For $N = 1$, the conclusion provides the finiteness of the solutions to the inequality (2.2.2); so we recover Roth's Theorem. In higher dimension, however, the finiteness conclusion does not hold: for instance, when the point $\mathbf{x}$ lies in the hyperplane defined by the vanishing of one linear form, the left-hand side term in (2.2.2) vanishes, so the inequality is satisfied. It is worth noticing, however, that the exceptional hyperplanes containing the infinite families of solutions are not necessarily the zero sets of the involved linear forms, as the following example shows:

`Example`. Let α be a real irrational algebraic number, with $0 < \alpha < 1$; consider a "good" rational approximation $p/q \in \mathbb{Q}$ to α. By this we mean that p, q are coprime integers, $q > 0$, and

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2};$$

we know from Dirichlet's Theorem that there exist infinitely many of them. Since $\alpha < 1$, for infinitely many good approximations p/q one has $\max(|p|, |q|) = |q|$, so we can write the above inequality as

$$\left|\alpha - \frac{p}{q}\right| < \max(|p|, |q|)^{-2}.$$

For each such pair (p, q) we have the upper bound

$$\frac{|q\alpha - p|}{\max(|p|,|q|)} \leq \frac{|q\alpha - p|}{|q|} < \max(|p|,|q|)^{-2}. \tag{2.2.3}$$

Now take $N = 2$, $\kappa = \mathbb{Q}$ and S consisting of the archimedean absolute value of $\mathbb{Q}$ and define the three linear forms $L_i(X_0, X_1, X_2)$ $(i = 0, 1, 2)$ as follows:

$$L_0(X_0, X_1, X_2) = X_0 - \alpha X_2, \quad L_1(X_0, X_1, X_2) = X_1 - \alpha X_2, \quad L_2(X_0, X_1, X_2) = X_2.$$

Now, with each good approximation p/q to the number α as above we associate the point $(x_0 : x_1 : x_2) = (p : p : q)$. Then the double product in (2.2.2) becomes

$$\prod_{\nu \in S} \prod_{i=0}^{N} \frac{|L_{i,\nu}(\mathbf{x})|_\nu}{\|\mathbf{x}\|_\nu} = \left(\frac{|p - q\alpha|}{\max(|p|,|q|)} \right)^2 \cdot \frac{|q|}{\max(|p|,|q|)}.$$

By the above inequality (2.2.3) and the trivial estimate $|q| \leq \max(|p|,|q|)$, we have the upper bound

$$\prod_{\nu \in S} \prod_{i=0}^{N} \frac{|L_{i,\nu}(\mathbf{x})|_\nu}{\|\mathbf{x}\|_\nu} < \max(|p|,|q|)^{-4},$$

which means that inequality (2.2.2), with e.g. $\epsilon = 1/2$, admits infinitely many solutions $(x_0 : x_1 : x_2) = (p : p : q)$ on the projective line of equation $X_0 = X_1$. So, the degeneracy conclusion of Theorem 2.2.1 cannot be replaced by a finiteness one, even after assuming $L_{i,\nu}(\mathbf{x}) \neq 0$ for all i, ν.

It will prove useful to have an 'affine version' of the Subspace Theorem, of which Theorem 2.2.1 represents the projective, or homogeneous, formulation. Here is such an affine version, which can be formally deduced from Theorem 2.2.1:

Theorem 2.2.4. *Let κ be a number field, S a finite set of places containing the archimedean ones, $N \geq 2$ an integer. Let, for each $\nu \in S$, $L_{\nu,1}(X_1, \ldots, X_N), \ldots, L_{\nu,N}(X_1, \ldots, X_N)$ be linearly independent linear forms with algebraic coefficients in κ_ν. Then the solutions $(x_1, \ldots, x_N) \in \mathcal{O}_S^N$ to the inequality*

$$\prod_{\nu \in S} \prod_{i=1}^{N} |L_{\nu,i}(\mathbf{x})|_\nu < H(\mathbf{x})^{-\epsilon}$$

lie in the union of finitely many proper linear subspaces of κ^N.

The Subspace Theorem, like Roth's theorem, is ineffective; however, the number of the higher dimensional components of the Zariski-closure of the set of solutions to (2.2.2) can be bounded (see [28]).

The Subspace Theorem, as we said, is a Diophantine approximation statement in higher dimension; the 1-dimensional case of it reduces precisely to

Roth's Theorem. However, there are problems in Diophantine approximation on the line which can be solved only by going to higher dimension, and then applying the Subspace Theorem. Let us see a simple example. We have seen that Ridout's theorem improves on Roth's estimate for the rational approximation to (real) algebraic numbers by rational number whose denominator is a product of powers of fixed set of primes. For instance, we have the bound

$$\left|\alpha - \frac{p}{2^n}\right| \gg 2^{-(1+\epsilon)n},$$

where Roth's exponent $2+\epsilon$ is replaced by $1+\epsilon$. We obtained this estimate after interpreting the special form of the approximant $p/2^n$ as being a rational number close to infinity in the 2-adic absolute value. If we change slightly the denominator, by replacing 2^n by 2^n+1, Ridout's Theorem does not apply anymore. However, by using the Subspace Theorem, we can indeed recover Ridout's $1+\epsilon$ exponent. Let us see how, following an idea introduced in [14] (see also [21], page 165). Define the three linear forms $L_0(X_0,X_1,X_2), L_1(X_0,X_1,X_2)$ and $L_2(X_0,X_1,X_2)$ by putting

$$L_0(X_0,X_1,X_2) = X_0, \quad L_1(X_0,X_1,X_2) = X_1, \quad L_2(X_0,X_1,X_2) \\ = \alpha(X_0+X_1) - X_2.$$

Then every solution $p/(2^n+1)$ to the inequality

$$|\alpha - p/(2^n+1)| < 2^{-(1+\epsilon)n}$$

gives rise to the point $\mathbf{x} := (1 : 2^n : p) \in \mathbb{P}_2(\mathbb{Q})$ satisfying

$$\left(\prod_{i=0}^{2} \frac{|L_i(\mathbf{x})|}{\|\mathbf{x}\|}\right) \cdot \left(\prod_{i=0}^{2} \frac{|L_i(\mathbf{x})|_2}{\|\mathbf{x}\|_2}\right) < H(\mathbf{x})^{-3-\epsilon}.$$

Hence by the Subspace Theorem 2.2.1, applied with $N=2$, the points $(1,2^n,p)$ would satisfy one of finitely many linear dependence relations; but from this fact and the starting inequality $|\alpha - p/(2^n+1)| < 2^{-(1+\epsilon)n}$, it is easy to deduce finiteness.

This example can be naturally generalized to produce the following statement:

Theorem 2.2.5. *Let $u : \mathbb{N} \to \mathbb{Q}$ be a sequence defined by*

$$u(n) = \sum_{i=1}^{h} a_i b_i^n,$$

where $h \geq 1$ is a natural number, $a_1,\ldots,a_h$ are rational numbers and $b_1,\ldots,b_h$ are positive integers. Let α be a real irrational algebraic number. Then for every $\epsilon > 0$ there exist only finitely many pairs of rational numbers of the form

$p/u(n)$, where $p \in \mathbb{Z}$, $n \in \mathbb{N}$, such that

$$\left| \alpha - \frac{p}{u(n)} \right| < \max(|p|, |u(n)|)^{-1-\epsilon}.$$

This result is proved in [14]. Note that the exponent $-1-\epsilon$ is Ridout's exponent, as it would be in the case the power sum $u(n)$ consisted in a single exponential function $n \mapsto b^n$. If we suppose also that the numerator p in the displayed inequality if of the form $p = v(m)$ for some power sum $m \mapsto v(m)$, then the exponent can be reduced to $-\epsilon$, still using the Subspace Theorem. Whenever both $u(n)$ and $v(m)$ are geometric progressions (i.e. $u(n) = ab^n, v(m) = a'b'^m$, then the theory of linear forms in logarithms applies and one can even obtain an effective result.

2.3 Approximation to higher degree hypersurfaces

We have seen that Schmdit's Subspace Theorem can be viewed as a statement about approximating hyperplanes, defined over a number field, by rational points.

It is then natural to consider the case where the targets are no more linear sub-spaces, but arbitrary (projective) hypersurfaces. Note that in dimension one (in the setting of Roth's Theorem) there is no such distinction, since the (geometrically) irreducible components of any hypersurfaces are single points, i.e. hyperplanes.

In this direction we do not dispose neither of a good generlization of Dirichlet's Theorem, nor of Roth's Theorem.

Suppose we want to investigate the rational approximation of a single hypersurface; to simplify matters we suppose that such a hypersurface is defined over $\mathbb{Q}$, by the vanishing of a homogeneous form $F(X_0, \ldots, X_N) \in \mathbb{Q}[X_0, \ldots, X_n]$. Set $d = \deg F$. An analogue of Liouville's Theorem is represented by the estimate

$$\frac{|F(\mathbf{x})|}{\|\mathbf{x}\|^d} \geq c \cdot \|\mathbf{x}\|^{-d},$$

where $c > 0$ is a constant depending on F, holding for all rational points $\mathbf{x} = (x_0 : \ldots : x_N) \in \mathbb{P}_N(\mathbb{Q})$ such that $F(\mathbf{x}) \neq 0$. Note that the left-hand side only depends on the projective class of $\mathbf{x}$, so it can be considered as a measure of the distance between the projective point $\mathbf{x}$ and the projective hypersurface $F(X_0, \ldots, X_N) = 0$.

Quoting W. Schmidt [49]: 'Any improvement of this inequality, even though perhaps it may apply only to special cases of non-linear hypersurfaces, would be of great interest and would shed light on certain diophantine equations...".

Note, however, that whenever the homogeneous form $F(X_0, \ldots, X_N)$ has irrational algebraic coefficients in a number field κ, the Liouville's inequality is changed into

$$\frac{|F(\mathbf{x})|}{\|\mathbf{x}\|^d} \geq c \cdot \|\mathbf{x}\|^{-d[\kappa:\mathbb{Q}]} \tag{2.3.1}$$

(here the absolute values are normalized with respect to $\mathbb{Q}$, since $\mathbf{x}$ is still supposed to lie in $\mathbb{P}_N(\mathbb{Q})$).

Of course, one can also consider the affine version, where the involved polynomials are no more supposed to be homogeneous. In that case the 'Liouville's inequality' is expressed as

$$|f(\mathbf{x})| > c(f) \cdot \|\mathbf{x}\|^{-d([\kappa:\mathbb{Q}]-1)}, \tag{2.3.2}$$

for every polynomial $f(X_1, \ldots, X_N) \in \kappa[X_1, \ldots, X_N]$ of total degree d and every integral point $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{Z}^N$.

Improvements on 'Liouville's inequality' (2.3.1) have been obtained in [26], [17], [27]. Their proofs all involve an application of the Subspace Theorem.

To give an example of such improvements on Liouville's inequality in the non-linear case, we quote the corollary to the main theorem in [17], *Addendum*, which reads as follows

Theorem 2.3.3. *Let $f(X_1, \ldots, X_n) \in \bar{\mathbb{Q}}[X_1, \ldots, X_n]$ be a polynomial in n variables with algebraic coefficients of degree d. For every $\epsilon > 0$ there exists a number $c > 0$ such that for all $\mathbf{x} \in \mathbb{Z}^n$ with $f(\mathbf{x}) \neq 0$,*

$$|f(\mathbf{x})| > c \cdot \|\mathbf{x}\|^{-d(n-1)-\epsilon} \tag{2.3.4}$$

We note at once that whenever the degree of the number field κ generated by the coefficients of the polynomial f satisfies $[\kappa : \mathbb{Q}] \geq n$, the above estimate cannot be deduced from Liouville's bound (2.3.2).

The inequality (2.3.4) can be improved after assuming that the approximant lie in a fixed algebraic sub-variety of $\mathbb{A}^n$. Also, it can be extended to number fields and arbitrary set of places. The most general known results can be found in [27], where the estimates are formulated in projective version.