Multiplication of Distributions in Mathematical Physics

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Abstract We expose a mathematical method that permits to treat calculations in form of multiplications of distributions that arise in various areas of mathematical physics, starting with an analysis of the famous Schwartz impossibility result (1954), then a construction of products of distributions, with examples and references of use in various domains of physics: classical and quantum mechanics, stochastic analysis and general relativity.

1 Introduction

In 1954 L. Schwartz published a celebrated note "Impossibility of the multiplication of distributions" [27], which had a strong impact on the subsequent development of physics (axiomatic field theory). Later in 1983 L. Schwartz presented (to the academy) a note "A general multiplication of distributions" by one of the authors [8]. We analyze this apparent contradiction in a very simple way and we observe that the impossibility proof is no more than the loss of a relatively minor property. To multiply distributions it suffices to construct a differential calculus in which the idealization that transforms the "irregular functions that represent physical quantities" into mathematical generalized functions is less crude than in distribution theory,

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 191,
DOI 10.1007/978-981-10-2636-2_46

in other words mathematics closer to physics than distribution theory. This will be explained throughout the paper. We sketch how this permits to give a mathematical sense to calculations in physics and to state equations of physics in a more precise way which can resolve ambiguities usually connected with the appearance of products of distributions which are not defined within distribution theory.

2 An Analysis of the Schwartz Impossibility Result

To prove his claim L. Schwartz stated a list of properties to be satisfied by any hypothetical differential algebra $\mathcal{A}(\mathbb{R})$ containing at least some distributions (here we assume $\mathcal{D}'(\mathbb{R}) \subset \mathcal{A}(\mathbb{R})$ for simplicity and we abbreviate these spaces by $\mathcal{D}'$ and $\mathcal{A}$ respectively), and he put in evidence a contradiction in this set of properties [13, 19, 24, 27], starting calculations with the continuous functions $x(ln|x| - 1)$ and $x^2(ln|x| - 1)$ because he stated the properties with continuous functions. As a consequence his proof does not put (3) in evidence, as it stems here from the extreme simplicity of (1) and (2). For clarity we start here with the Heaviside function. To understand the whole situation we compare the two formulas (1) and (2) below where H denotes the Heaviside function ($H(x) = 0$ if $x < 0$, $H(x) = 1$ if $x > 0$, $H(0)$ undefined). These formulas are

$$\int_{\mathbb{R}} (H^2(x) - H(x))\phi(x)dx = 0 \; \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}), \tag{1}$$

and

$$\int_{\mathbb{R}} (H^2(x) - H(x))H'(x)(x)dx = \left[\frac{H^3}{3} - \frac{H^2}{2}\right]_{-\infty}^{+\infty} = \frac{1}{3} - \frac{1}{2} \neq 0. \tag{2}$$

These two formulas are clear if one assumes that the Heaviside function is an idealization of a smooth function with a jump from the value 0 to the value 1 on a very small region around $x = 0$. Formula (1) shows that $H^2 = H$ in $\mathcal{D}' \subset \mathcal{A}$ and formula (2) shows that $H^2 \neq H$ in $\mathcal{A}$, hence a contradiction which proves the impossibility of the multiplication of distributions. But there is a subtle mistake hidden in this reasoning! In (2) H^2 is the square of H in $\mathcal{A}$ since (2) does not make sense in $\mathcal{D}'$. To compare (2) with (1) the H^2 in (1) should be the same as in (2). Therefore, for comparison, the quantities $(H^2 - H)$ in (1) and (2) are both the same and are an element of $\mathcal{A}$; nothing tells it is an element of $\mathcal{D}'$. Therefore (1, 2) prove that in $\mathcal{A}$

$$\int F(x)\phi(x)dx = 0 \; \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}) \nRightarrow F = 0. \tag{3}$$

Indeed choose $F = H^2 - H$ above. The lack of validity of the familiar implication that fails from (3) does not prove the impossibility of the multiplication of distributions and does not prohibit the existence of a suitable algebra $\mathcal{A}$.

In $\mathcal{D}'$ one has $H^2 = H$ from (1); in $\mathcal{A}$ one has $H^2 \neq H$ from (2). Again this looks very much like an absurdity!. The explanation is that the square of H is not the same in $\mathcal{D}'$ and $\mathcal{A}$. Is this an incoherence, i.e. are these two objects, both denoted H^2, really different? Contrarily to all appearance the answer is no! Look at (H^2 in $\mathcal{A}$). We want to observe that it is (H^2 in $\mathcal{D}'$), i.e. H; to this end we observe of course this object (H^2 in $\mathcal{A}$) in the way the objects of $\mathcal{D}'$ are defined i.e. we consider

$$\int_{\mathbb{R}} (H^2 \ in \ \mathcal{A})(x)\phi(x)dx$$

and from (1) we observe nothing other than (H in $\mathcal{D}'$). In conclusion (H^2 in $\mathcal{A}$) is different from (H in $\mathcal{A}$) but when (H^2 in $\mathcal{A}$) is observed according to the definition of distributions to compare with the classical objects in $\mathcal{D}'$ it appears to be H as this should be for coherence. The above systematically holds for all operations in the algebra $\mathcal{G}(\Omega)$ considered in the next section. Therefore in this algebra there is a perfect coherence between all new and all classical calculations.

3 A Differential Algebra Containing the Distributions

One can construct a differential algebra $\mathcal{G}(\Omega)$ containing a copy isomorphic to the vector space $\mathcal{D}'(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, in the situation

$$\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^0(\Omega) \subset \mathcal{D}'(\Omega) \subset \mathcal{G}(\Omega). \tag{4}$$

The partial derivatives in $\mathcal{G}(\Omega)$ induce on $\mathcal{D}'(\Omega)$ the partial derivatives in the sense of distributions; the multiplication in $\mathcal{G}(\Omega)$ induces the classical multiplication of $\mathcal{C}^\infty$ functions: $\mathcal{C}^\infty(\Omega)$ is a faithful subalgebra of $\mathcal{G}(\Omega)$. The Schwartz impossibility result implies that the algebra $\mathcal{C}^0(\Omega)$ is not a subalgebra of $\mathcal{G}(\Omega)$, but if f, g are two continuous functions on Ω and if $f \bullet g \in \mathcal{G}(\Omega)$ denotes their (new) product in $\mathcal{G}(\Omega)$, then we have the coherence

$$\int_\Omega (f \bullet g)(x)\phi(x)dx = \int_\Omega f(x)g(x)\phi(x)dx \ \forall\phi \in \mathcal{C}_c^\infty(\Omega) \tag{5}$$

for a natural integration in $\mathcal{G}(\Omega)$. The basic idea is that the elements of $\mathcal{G}(\Omega)$ are mathematical idealizations (that can represent physical quantities) that remain closer to physics than distributions: they are equivalence classes of families (f_ϵ) of $\mathcal{C}^\infty$ functions for a rather strict equivalence relation such that the property $(lim_{\epsilon\to 0} \int_\Omega f_\epsilon(x)\phi(x)dx = 0 \ \forall\phi \in \mathcal{C}_c^\infty(\Omega))$ does not imply that the family (f_ϵ) is null in the quotient defining $\mathcal{G}(\Omega)$, as this is the case in distribution theory.

L. Nachbin and L. Schwartz supported fast publication in book form to speed divulgation [11, 12], that were soon complemented by [5, 13, 23, 24]. This theory is presented in form of a differential calculus dealing with infinitesimal quantities and infinitely large quantities in [3, 4], as well as in various expository texts [10, 14], … and has been extended to manifolds in view of its use in general relativity [18–22, 28–31], …

The problem that served as a first application in 1986 was the one of calculating jump conditions for a system used in industry (design of armor) to model very strong collisions [9, 13]. The system showed multiplications of distributions of the form $H \times \delta$ where δ is the Dirac distribution. We observed the existence of different possible jump conditions, all of them stable [17].

We recall that in $\mathcal{G}(\Omega)$ one has two concepts that can play the role of the equality of functions: of course the equality in $\mathcal{G}(\Omega)$ which is coherent with all operations (on particular the multiplication and the derivation) and the concept in left hand side of the non-implication (3) that we state as "association" since it is not really a weak equality (since different elements of $\mathcal{G}(\Omega)$ can be associated) and denote by the symbol $\approx$:

$$F \approx G \Leftrightarrow \int_{\Omega} (F - G)(x)\phi(x)dx = 0 \; \forall \phi \in \mathcal{C}_c^{\infty}(\Omega). \tag{6}$$

The association is coherent with the derivation but not with the multiplication. We recall from (1, 2) that $H^2 \neq H$ and $H^2 \approx H$.

A solution was obtained as follows: state with the equality in $\mathcal{G}(\Omega)$ the laws of physics which are considered true at a very small scale (may be 10^{-7} meters) and state with the association the laws or properties valid only at a far larger scale (may be 10^{-4} meters). This is explained in detail in [13] p. 69, with calculations of shock waves for nonconservative systems and references. The results that followed from this statement were in perfect agreement with observations and experiments. In various interesting cases one obtains the remarkable result that the jumps of different physical variables are represented by the same Heaviside function in $\mathcal{G}(\mathbb{R})$, [13] p. 72. Same explanations for another problem are given in [2].

4 Calculations of the Hamiltonian Formalism of Interacting Fields

The canonical Hamiltonian formalism (exposed in detail in [15, 16]) consists in a formal solution of the interacting field equations

$$\left(-\partial_t^2 + \sum_{i=1}^{3} \partial_{x_i}^2 - m^2\right)\Phi(x,t) = g\Phi(x,t)^N, \; \Phi(x,\tau) = \Phi_0(x,\tau) \tag{7}$$

where $m, g \in \mathbb{R}$ and $\Phi_0(x,t)$ is the free field operator (explicitly known: it is a distribution valued in a space of unbounded operators on a Hilbert space). The Hamiltonian formalism constructs a solution of (7) according to a formula

$$\Phi(x,t) = exp(-i(t-\tau)H_0(\tau))\Phi_0(x,\tau)exp(i(t-\tau)H_0(\tau)) \tag{8}$$

where $H_0(\tau)$ is obtained by plugging formally the free field into the formula of the total Hamiltonian corresponding to (7). Further calculations give the related formula

$$\Phi(x,t) = (S_\tau(t))^{-1}\Phi_0(x,t)(S_\tau(t)) \tag{9}$$

which gives the interacting field as a function of the free field at the same time. The formal operator $S = S_{-\infty}(+\infty)$ is called the scattering operator. Note that it depends on the real parameter g called coupling constant. If Φ_1, Φ_2 are two normalized orthogonal states then the formula $| < \Phi_1, S\Phi_2 > |$ represents the probability that the state Φ_2 would become Φ_1 after interaction.

What can be done with the context of Sect. 2 as mathematical tool? First one remarks the basic point that in these formal calculations, see for instance [15, 16], two basic mathematical difficulties are intimately mixed: multiplications of distributions, treated as $\mathcal{C}^\infty$ functions, and unbounded operators, treated as bounded operators. Indeed the free field Φ_0 is a distribution in x, not a function. The context of Sect. 2 is adapted to multiplication of distributions but brings nothing concerning unbounded operators, therefore it does not elucidate completely nicely these calculations. Anyway all calculations finally make sense mathematically [16] and one obtains a scattering operator $S = S(g)$ and transition probabilities $| < \Phi_1, S\Phi_2 > |$. What are these mathematical objects (which in the context of Sect. 3 make sense mathematically)?

The exponentials in (8) make sense from a proof that $H_0(\tau)$ admits a self-adjoint extension and one obtains a scattering operator $S = S(g)$ [16]. What is $| < \Phi_1, S(g)\Phi_2 > |$?: it depends on a parameter ϵ that tends to 0 and for each value of ϵ it is in between 0 and 1. We believe as quasi certain it has no limit (in the usual sense) when $\epsilon \to 0$ and therefore it oscillates endlessly inside the real interval [0, 1] when $\epsilon \to 0$. As an obvious example of such an oscillating object consider $|cos(\frac{g}{\epsilon})|$. Because of the periodicity of the function cosine, to this objects one can associate a well defined real number, here $\frac{2}{\pi}$, to be checked at once from numerical calculations by computing an average for a large number of very small values of ϵ chosen at random. Such average values exist for all quasi periodic functions, see [15]. The presence of complex exponentials and the self-adjointness property of $H_0(\tau)$ suggest that $| < \Phi_1, S(g)\Phi_2 > |$ is a quasi periodic function in the variable $\frac{1}{\epsilon}$ and therefore this oscillating function of ϵ would have a mean value as $\epsilon \to 0$ (the variable ϵ is of course not intrinsic but it plays only an auxiliary role and does not influence the final result, which appears very robust). In short the infinite quantities in the formal perturbation series are replaced by oscillations to be treated by computer calculations of an average value. To test this method one should compute the numerical value so obtained in a case for which one has an experimental result. The computer calculations look difficult and this has not been done after the premature death of A. Gsponer in 2009.

5 Stochastic Analysis

Stochastic differential equations (SDEs) serve to model many important phenomena in mathematical physics. An important class of SDEs in $\mathbb{R}^d$ is of the following form

$$\partial_t U(t,x) = \mathcal{L}U(t,x) + \eta(t,x), \quad U(0,x) = F(x) \tag{10}$$

where $\mathcal{L}$ is a differential operator and $\eta(t,x)$ is a space-time noise. The solutions of (10) are necessarily in a space of generalized functions because of the non-differentiability of the process driven by the equation. Therefore the meaning of the nonlinear part of $\mathcal{L}$ is not obvious. One way to sort out this problem is to consider the solutions of (10) as generalized stochastic processes, that is, processes whose paths are generalized functions. More precisely, by analogy with the association (6), we say that a family of smooth martingales $(U_\epsilon)_{\epsilon>0}$ is a weak solution of the equation (10) in the sense (6) if both 1 and 2 below hold:

$$\forall\phi \in \mathcal{C}_c^\infty(]0,T[\times\mathbb{R}^d,\mathbb{R}^d) \;\; lim_{\epsilon\to 0} < \mathcal{L}U_\epsilon, \phi >= \int_{[0,T]\times\mathbb{R}^d} \phi(t,x)d\eta(t,x) \tag{11}$$

and

$$U_\epsilon(0,x) = F(x) \;\forall\epsilon > 0. \tag{12}$$

One notices that

1. S. Albeverio, M. Oberguggenberger and F. Russo, among others, proposed already in the nineties to solve nonlinear SDEs in the framework of $\mathcal{G}(\Omega)$, [1, 25, 26].
2. One observes that if η_ϵ is a regularisation of the noise η and if U_ϵ is a solution of (10) driven by the noise η_ϵ, then under very general conditions $(U_\epsilon)_\epsilon > 0$ is a weak solution of the Eq. (10) in the sense (6).
3. Choosing the Burgers operator $\mathcal{L}U = \partial_t U - \Delta_x U - \nabla_x \|U\|^2$ and $\eta = \nabla_x\partial_t W(t,x)$, where $W(t,x)$ is a space-time white noise, we obtain the stochastic Burgers equation. The Cole-Hopf family is a weak solution of the Burgers equation in the sense (10), see [7].
4. In the case $d = 1$ the Hopf-Cole family is associated to a distribution, see [6].

6 Conclusion

After an analysis of the Schwartz impossibility result we have presented a context of multiplication of distributions having all natural requested properties. Then we have presented selected applications in continuum mechanics, quantum mechanics and in stochastic PDEs. For general relativity we refer to [19–22, 28–31].

Acknowledgements P. Catuogno has been partly supported by the grant CNPq 460713/2014-0. J.F. Colombeau has been supported by the grant FAPESP 2012 18940-7. Ch Olivera has been supported by the grants FAPESP 2015/04723-2 and CNPq 460713/2014-0.

References

1. S. Albeverio, F. Russo. In *Nonlinear Klein-Gordon and Schrodinger systems: theory and applications*. (World Sci. Publ., River Edge, NJ, 1996) pp. 68–86.
2. J. Aragona, J.F. Colombeau, S.O. Juriaans. J. Math. Anal. Appl. 418,2, (2014), 964–977.
3. J. Aragona, R. Fernandez, S.O. Juriaans. Monatsh. Math. 144 (2005), 13–29.
4. J. Aragona, R. Fernandez, S.O. Juriaans, M. Oberguggenberger. Monatsh. Math. 166 (2012), 1–18.
5. H. A. Biagioni. *A Nonlinear Theory of Generalized Functions*, (Springer, Berlin, 1990).
6. P. Catuogno, Ch. Olivera. Applicable Analysis 93, 3, (2014), 646–652.
7. P. Catuogno, JF. Colombeau, Ch. Olivera. Weak asymptotic solutions for the stochastic Burgers equation. arXiv:1502.07174.
8. J.F. Colombeau. C. R. Acad. Sci. Paris 296, (1983), 357–360.
9. J.F. Colombeau. J. Math.Phys. 30, 10 (1989), 2273–2279.
10. J.F. Colombeau. Bull. Amer. Math. Soc., 23, 2, (1990), 251–268.
11. J.F. Colombeau. *New Generalized Functions and Multiplication of Distributions*, (North Holland Pub. Comp., Amsterdam, 1984).
12. J.F. Colombeau. *Elementary Introduction to New Generalized Functions*, (North Holland Pub. Comp., Amsterdam, 1984).
13. J.F. Colombeau. *Multiplication of distributions.* (Springer, Berlin), 1992.
14. J.F. Colombeau. São Paulo J. Math. Sci. 7, 2, (2013), 201–239.
15. J.F. Colombeau. *Mathematical problems on generalized functions and the canonical Hamiltonian formalism*. arXiv:0708.3425.
16. J.F. Colombeau, A. Gsponer. *The Heisenberg-Pauli canonical Hamiltonian formalism of quantum field theory in the rigorous mathematical setting of nonlinear generalized functions (part I)*. arXiv.org:807.0289.
17. J.F. Colombeau, A.Y. Le Roux, A. Noussair, B. Perrot. SIAM J. Num. Anal. 26,4, (1989), 871–883.
18. P. Giordano, E. Nigsch. Math.Nachr. 288, 11-12, (2015), 1286–1302.
19. M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer. *Geometric Theory of Generalized Functions with Applications to General Relativity*. (Kluwer, Dordrecht, 2001).
20. M. Kunzinger,R. Steinbauer. Class.Quant. Grav. 16,4, (1999), 1255–1264.
21. M. Kunzinger,R. Steinbauer. J. Math. Phys. 40,3, (1999), 1479–1489.
22. E. Nigsch, C. Samann. São Paulo J. Math. Sci. 7, 2, (2013), 143–171.
23. M. Nedeljkov, S. Pilipovic, D. Scarpalezos. *The linear Theory of Colombeau Generalized Functions*. (Longman, Harlow, 1998).
24. M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*. (Longman, Harlow, 1992).
25. M. Oberguggenberger, F. Russo. In *Stochastic Analysis and Related Topics*. (Progress. Probab., 42, Birkhauser, Boston, MA, 1998) pp. 319–332.
26. F. Russo. In *Stochastic Analysis and Applications in Physics*, ed. A Cardoso, M. de Faria, J. Potthoff, R. Seneor, L. Streit. (Kluwer Acad Publ., Dordrecht, 1994) pp. 329–349.
27. L. Schwartz. C. R. Acad. Sci. Paris 239, (1954), 847–848.
28. C. Samann, R. Steinbauer. In *Pseudo-differential operators and generalized functions*. Oper. Theory Adv. Appl; 245 (Birkhauser/Springer, Cham., 2015) pp. 243–253.
29. R. Steinbauer, J.A. Vickers. Classical Quantum Gravity 23, 10, (2006), R91–R114.
30. R. Steinbauer, J.A. Vickers. Classical Quantum Gravity 26, 6, (2009), 065001.
31. J.A. Vickers. J. Geom. Phys. 62,3, (2012), 692–705.