Vertex Algebras in Higher Dimensions Are Homotopy Equivalent to Vertex Algebras in Two Dimensions

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Abstract There is a differential graded operad associated to quadratic configuration spaces, whose class of algebras naturally contains the class of all vertex algebras. We have found that under certain shift of the degree in the cohomology these operads are isomorphic in cohomology for any even spatial dimension.

1 Real Configuration Spaces and Related Operads

Configuration spaces have been studied long ago in mathematics (see [3]). By definition, the real n-th configuration space over $\mathbb{R}^D$ is the set of all configurations of n points in $\mathbb{R}^D$, which are distinct. It is shortly denoted by $\mathbf{F}_{\mathbb{R},n}$,

$$\mathbf{F}_{\mathbb{R},n} \,(\equiv\, \mathbf{F}_{\mathbb{R},n}^{(D)}) \,:=\, \{(\mathrm{x}_1,\dots,\mathrm{x}_n) \,|\, \mathrm{x}_1,\dots,\mathrm{x}_n \,\in\, \mathbb{R}^D,\ \mathrm{x}_j \neq \mathrm{x}_k \ (1 \leqslant j < k \leqslant n)\}\,. \tag{1}$$

These spaces obey a very rich structure. In particular, there are several operads that are associated to the sequence of all configuration spaces (over $\mathbb{R}^D$). One of the most simple operads is the so called *little balls/cubes operad* (see [8, Sect. 2.2]).

Before explaining the latter operad let us remind that the operads provide a generalization of the notion of a "type of algebra". They consists of a sequence of spaces $\mathcal{M}(n)$ equipped with several structure maps. If we think of $\mathcal{M}(n)$ as a space of "n-ary operations" (i.e., operations with n inputs and one output) then there are structure maps that axiomatize the composition,

$$\mathcal{M}(n) \times \mathcal{M}(j_1) \times \cdots \times \mathcal{M}(j_n) \,\ni\, (\mu,\mu_1,\dots,\mu_n) \,\longmapsto\, \mu \circ (\mu_1,\dots,\mu_n) \,\in\, \mathcal{M}(\ell)\,, \tag{2}$$

of an n-ary operation $\mu_n \in \mathcal{M}(n)$ with n other operations $\mu_1 \in \mathcal{M}(j_1)$, $\dots$, $\mu_n \in \mathcal{M}(j_n)$, and it gives a result that belongs to the space $\mathcal{M}(\ell)$ of operation with

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$$\ell \;=\; j_1 + \cdots + j_n \tag{3}$$

inputs. In addition, the permutation group $\mathcal{S}_n$ is supposed to act on $\mathcal{M}(n)$ for every n axiomatizing the exchange of inputs of an n-ary operation. There are natural conditions of associativity for the operadic compositions and equivariance (compatibility) for the compositions with respect to the permutation actions. The reader can find further information in [7].

In the case of the little balls operad, the space $\mathcal{M}(n)$ consists of all closed balls $\overline{B}_1, \ldots, \overline{B}_n$ in $\mathbb{R}^D$, which do not intersect each other and are contained in an open ball B_0,

$$\mathcal{M}(n) \;= \big\{ (B_0;\, B_1, \ldots, B_n) \,\big|\, \overline{B}_1, \ldots, \overline{B}_n \;\subset\; B_0\,,\;\; \overline{B}_j \cap \overline{B}_k \;=\; \emptyset \; (1 \leqslant j < k \leqslant n) \big\} \tag{4}$$

The operadic composition

$$\begin{aligned} &(B_0;\, B_1, \ldots, B_n) \circ \big((B_{1,0};\, B_{1,1}, \ldots, B_{1,j_1}), \ldots, (B_{n,0};\, B_{n,1}, \ldots, B_{n,j_n}) \big) \\ &= \; (B_0,\, B'_{1,1}, \ldots, B'_{1,j_1}, \ldots, B'_{n,1}, \ldots, B'_{n,j_n}) \end{aligned} \tag{5}$$

is then obtained by transforming each configuration $(B_{k,0}, B_{k,1}, \ldots, B_{k,j_k})$ with translations and dilations in such a way that we can plug $B_{k,0}$ into B_k, i.e.,

$$(B_{k,0},\, B_{k,1}, \ldots, B_{k,j_k}) \;\overset{\substack{\text{translations} \\ \text{\& dilations}}}{\longmapsto}\; (B'_{k,0},\, B'_{k,1}, \ldots, B'_{k,j_k})\,, \;\; \text{so that } B'_{k,0} \;=\; B_k \tag{6}$$

for every $k = 1, \ldots, n$. Note that $\mathcal{M}(n)$ is homotopy equivalent to the configuration space $\mathbf{F}_{\mathbb{R},n}$ and hence, the above opearadic compositions induce maps between the homology spaces (with rational coefficients),

$$H_{\bullet}\big(\mathcal{M}(n), \mathbb{Q}\big) \;=\; H_{\bullet}(\mathbf{F}_{\mathbb{R},n}, \mathbb{Q})\,. \tag{7}$$

In this way, the sequence of spaces $H_{\bullet}(\mathbf{F}_{\mathbb{R},n}, \mathbb{Q})$ becomes an *algebraic operad*, i.e., an operad whose operadic spaces are vector spaces and the operadic compositions are multilinear maps.

There is a straight forward generalization of the little balls operad. Let

$$r \;\subseteqq\; \mathbb{R}^D \times \mathbb{R}^D \tag{8}$$

be a homogeneous, closed, binary relation and denote

$$\mathbf{F}_{r\,;\,n} \;:=\; \{(\mathrm{x}_1, \ldots, \mathrm{x}_n) \,|\, \mathrm{x}_1, \ldots, \mathrm{x}_n \;\in\; \mathbb{R}^D,\;\; (\mathrm{x}_j, \mathrm{x}_k) \notin r \; (1 \leqslant j < k \leqslant n)\} \tag{9}$$

$$\begin{aligned} \mathcal{M}^{(r)}(n) \;=& \big\{ (B_0;\, B_1, \ldots, B_n) \,\big|\, \overline{B}_1, \ldots, \overline{B}_n \;\subset\; B_0 \;\subset\; \mathbb{R}^D\,, \\ & (\overline{B}_j \times \overline{B}_k) \cap r \;=\; \emptyset \; (1 \leqslant j < k \leqslant n) \big\}\,, \end{aligned} \tag{10}$$

with operadic composition given by (2). Then we obtain again an operad and the sequence

$$H_{\bullet}\big(\mathcal{M}^{(r)}(n), \mathbb{Q}\big) \cong H_{\bullet}(\mathbf{F}_{r\,;\,n}, \mathbb{Q}) \tag{11}$$

is an algebraic operad.

2 Quadratic Configuration Spaces and Related Operads

As a particular example of the operad $\mathcal{M}^{(r)}(n)$ (10) let us consider the complex vector space $\mathbb{C}^D$ ($\cong\mathbb{R}^{2D}$ as a real vector space) equipped with a quadratic homogeneous relation

$$r \subset \mathbb{C}^D \times \mathbb{C}^D\,, \quad r := \{(\mathrm{x}, \mathrm{y}) \in \mathbb{C}^D \mid (\mathrm{x} - \mathrm{y})^2 = 0\}\,, \tag{12}$$

$$\mathrm{x}^2 \equiv \mathrm{x} \cdot \mathrm{x} := (x^1)^2 + \cdots + (x^D)^2 \quad \text{for} \quad \mathrm{x} := (x^1, \ldots, x^D) \in \mathbb{C}^D\,. \tag{13}$$

Then, following [9, 10] we call $\mathbf{F}_{r\,;\,n}$ (9) a *quadratic configuration space* and denote it by $\mathbf{F}_{\mathbb{C},n}$

$$\mathbf{F}_{\mathbb{C},n} := \big\{(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in (\mathbb{C}^D)^{\times n} \,\big|\, (\mathrm{x}_j - \mathrm{x}_k)^2 \neq 0 \; (1 \leqslant j < k \leqslant n)\big\}\,. \tag{14}$$

Note in particular, that

$$\mathbf{F}_{\mathbb{C},n} \cap (\mathbb{R}^D)^{\times n} = \mathbf{F}_{\mathbb{R},n}\,. \tag{15}$$

We observe also that $\mathbf{F}_{\mathbb{C},n}$ are *complex affine varieties* and the ring of regular functions on $\mathbf{F}_{\mathbb{C},n}$ coincides with the algebra of rational functions with quadratic singularities,

$$\widetilde{\mathscr{O}}_n := \mathscr{O}\big(\mathbf{F}_{\mathbb{C},n}\big) = \mathbb{C}\big[\mathrm{x}_1, \ldots, \mathrm{x}_n\big]\Big[\Big(\prod_{1\,j<k\leqslant n} (\mathrm{x}_j - \mathrm{x}_k)^2\Big)^{-1}\Big]. \tag{16}$$

In physics terminology, one can say that the elements of $\widetilde{\mathscr{O}}_n$ are the rational n-point functions with *light-cone singularities*. One can divide the configuration spaces by the action of the translations and pass to the reduced configuration spaces $\mathbf{F}_n\big/\mathbb{C}^D$, whose algebra of regular functions $\mathscr{O}_n$ consists of the translation invariant functions belonging to $\widetilde{\mathscr{O}}_n$

$$\mathscr{O}_n := \mathscr{O}\big(\mathbf{F}_{\mathbb{C},n}/\mathbb{C}^D\big) = \mathbb{C}\big[\mathrm{x}_1 - \mathrm{x}_n, \ldots, \mathrm{x}_{n-1} - \mathrm{x}_n\big]\Big[\Big(\prod_{1\leqslant j<k\leqslant n} (\mathrm{x}_j - \mathrm{x}_k)^2\Big)^{-1}\Big] \tag{17}$$

As the quotient by the translations do not change topology up to a homotopy equivalence we obtain the same operad structure on the homology spaces.

Let us remind the result of Grothendieck [4], which identifies the algebraic de Rham cohomologies of $\widetilde{\mathscr{O}}_n$ (resp., $\mathscr{O}_n$) with the Betti cohomology of the corresponding complex affine variety $\mathbf{F}_{\mathbb{C},n}$ (resp., $\mathbf{F}_{\mathbb{C},n}/\mathbb{C}^D$). In the case of $\widetilde{\mathscr{O}}_n$ (as well as, $\mathscr{O}_n$) the algebraic de Rham complex has a simple construction,

$$\Omega^k\big(\mathscr{O}_{\mathbb{C},n}\big) := \textstyle\bigwedge^k_{\mathscr{O}_{\mathbb{C},n}} \Omega^1\big(\mathscr{O}_{\mathbb{C},n}\big) := \mathrm{Span}_{\mathbb{C}}\Big\{ f\, dx_{j_1}^{\mu_1} \wedge \cdots \wedge dx_{j_k}^{\mu_k} \,\Big| \quad f \in \widetilde{\mathscr{O}}_{\mathbb{C},n}\,, \quad \mu_1,\dots,\mu_k = 1,\dots,D\,,\ j_1,\dots,j_k = 1,\dots,n \Big\}. \tag{18}$$

Then

$$H^k\big(\widetilde{\mathscr{O}}_{\mathbb{C},n}\big) := \frac{\mathrm{Ker}\Big(\Omega^k\big(\widetilde{\mathscr{O}}_{\mathbb{C},n}\big) \xrightarrow{d} \Omega^{k+1}\big(\mathscr{O}_{C,n}\big)\Big)}{\mathrm{Image}\Big(\Omega^{k-1}\big(\mathscr{O}_{\mathbb{C},n}\big) \xrightarrow{d} \Omega^{k}\big(\mathscr{O}_{\mathbb{C},n}\big)\Big)}\,, \tag{19}$$

with respect to the de Rham differential:

$$d\big(f dx_{j_1}^{\mu_1} \wedge \cdots \wedge dx_{j_k}^{\mu_k}\big) := \sum_{j=1}^{n}\sum_{\mu=1}^{D} \frac{\partial f}{\partial x_j^{\mu}}\, dx_j^{\mu} \wedge dx_{j_1}^{\mu_1} \wedge \cdots \wedge dx_{j_k}^{\mu_k}\,, \tag{20}$$

where $f(\mathrm{x}_1,\dots,\mathrm{x}_n) \in \mathscr{O}_{\mathbb{C},n}$. Now, the Grothendieck's theorem implies that

$$H^k\big(\widetilde{\mathscr{O}}_n\big) \cong H^k\big(\mathbf{F}_{\mathbb{C},n};\mathbb{C}\big)\,. \tag{21}$$

In fact, $H^k\big(\widetilde{\mathscr{O}}_{\mathbb{Q},n}\big) \otimes_{\mathbb{Q}} \mathbb{C} \cong \big(H_k\big(\mathbf{F}_{\mathbb{C},n};\mathbb{Z}\big) \otimes_{\mathbb{Z}} \mathbb{C}\big)^*$, where $\widetilde{\mathscr{O}}_{\mathbb{Q},n}$ is the algebra $\widetilde{\mathscr{O}}_n$ (16) with coefficients in $\mathbb{Q}$ (instead of $\mathbb{C}$), and the natural $\mathbb{Z}$-bilinear paring $H^k\big(\mathbf{F}_{\mathbb{Q},n}\big) \times H_k\big(\mathbf{F}_{\mathbb{C},n};\mathbb{Z}\big) \to \mathbb{C}$ gives rise to the space ($\mathbb{Z}$-module) of periods related to the quadratic configuration spaces $\mathbf{F}_{\mathbb{C},n}$, which play a very important role in renormalization theory as residues of Feynman amplitudes in massless Quantum Field Theories (see [11, 13]). Furthermore, there is a differential graded operad associated to the sequence of algebras $\mathscr{O}_n$, whose cohomologies coincide with the operad $\big(H_\bullet(\mathbf{F}_{\mathbb{C},n},\mathbb{Q})\big)_{n\geqslant 2}$ and it has an application to both: the theory of vertex algebras and the renormalization [10, 12].

Remark 1 For the operadic point of view on vertex algebras we would like to mention also the papers [5, 6], where certain *partial* operads are proposed for this purpose. However, the operad suggested in [10] is not a *partial* operad but an "ordinary" symmetric operad (as defined for example in [7]). The price for this simplification is perhaps that the latter operad has more algebras than the vertex algebras. Nevertheless, there is a simple criterion for separating the class of vertex algebras among all others (more details will be published in [12]).

3 Cohomologies of Quadratic Configuration Spaces up to Three Points and Their Application in the Theory of Vertex Algebras

The main new result in the present work is the computation of the cohomology spaces of $\mathbf{F}_{\mathbb{C},n}$ for $n = 2, 3$. In general, the problem of finding all cohomology spaces of $\mathbf{F}_{\mathbb{C},n}$ for all $n = 2, 3, \ldots$ is very difficult.

A standard approach for studying configuration spaces is via the sequence of maps

$$q_{n+1} : \mathbf{F}_{\mathbb{C},n+1} \longrightarrow \mathbf{F}_{\mathbb{C},n} \; : \; (\mathrm{x}_1, \ldots, \mathrm{x}_{n+1}) \longmapsto (\mathrm{x}_1, \ldots, \mathrm{x}_n) \tag{22}$$

that forget about the last point (for $n = 2, 3, \ldots$). The fiber of q_{n+1} at the point $(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \mathbf{F}_{\mathbb{C},n}$ is

$$M_{\mathrm{x}_1,\ldots,\mathrm{x}_n} := \left\{\mathrm{z} \in \mathbb{C}^D \,\middle|\, (\mathrm{z} - \mathrm{x}_j)^2 \neq 0 \text{ for all } j = 1, \ldots, n\right\} = \mathbb{C}^D \setminus \bigcup_{j=1}^{n} Q_{\mathrm{x}_j} , \tag{23}$$

i.e., it is the complement of union of quadrics of a type

$$Q_{\mathrm{x}} := \left\{\mathrm{z} \in \mathbb{C}^D \,\middle|\, (\mathrm{z} - \mathrm{x})^2 = 0\right\}. \tag{24}$$

In case $\mathbb{C} \mapsto \mathbb{R}$ the fibers are

$$M^{\mathbb{R}}_{\mathrm{x}_1,\ldots,\mathrm{x}_n} = \left\{\mathrm{z} \in \mathbb{R}^D \,\middle|\, \mathrm{z} \neq \mathrm{x}_j \text{ for all } j = 1, \ldots, n\right\},$$

their homeomorphism type does not depend on $(\mathrm{x}_1, \ldots, \mathrm{x}_n)$, and each of them is homotopy equivalent to a bouquet of $(D-1)$-spheres. In particular the projections q_n are fibrations and one may use iterated Leray–Serre spectral sequences or the Leray–Hirsch theorem in order to obtain the Betti cohomology of $\mathbf{F}_{\mathbb{R},n}$, see [1–3].

Let us point out that for both cases, $\mathbb{C}$ and $\mathbb{R}$, the maps (22), $q_{n+1} : \mathbf{F}_{\mathbb{C},n+1} \longrightarrow \mathbf{F}_{\mathbb{C},n}$ and $q_{n+1} : \mathbf{F}_{\mathbb{R},n+1} \longrightarrow \mathbf{F}_{\mathbb{R},n}$ (respectively) are fiber bundles for $n = 1, 2$. This is due to the fact that in these cases the bases $\mathbf{F}_{\mathbb{C},n}$ are homogeneous spaces of the group of Euclidean motions with dilations on $\mathbb{C}^D$. In the real case, the maps $q_{n+1} : \mathbf{F}_{\mathbb{R},n+1} \longrightarrow \mathbf{F}_{\mathbb{R},n}$ remain fiber bundles for any n (the fibers being homotopy equivalent to a bouquet of spheres, as we have pointed out). However, over $\mathbb{C}$ and $n > 2$ the fibers $M_{\mathrm{x}_1,\ldots,\mathrm{x}_n}$ (23) are in general non-isomorphic.

The case $n = 2$ is relatively simple. We have an isomorphism

$$\mathbf{F}_{\mathbb{C},2} \cong M_0 \times \mathbb{C}^D \; : \; (\mathrm{x}_1, \mathrm{x}_2) \mapsto (\mathrm{x}_1 - \mathrm{x}_2, \mathrm{x}_2)\,. \tag{25}$$

For $M_0 = \mathbb{C}^D \backslash Q_0$ then we use the projection

$$M_0 \longrightarrow \mathbb{C}\backslash\{0\} \,:\, \mathrm{x}_1 - \mathrm{x}_2 \longmapsto (\mathrm{x}_1 - \mathrm{x}_2)^2\,, \tag{26}$$

which is a bundle with fibers isomorphic to the complex $(D-1)$-sphere $\mathbb{S}_{\mathbb{C}}^{D-1}$. For even $D > 2$ we then derive by the Leray–Hirsch theorem that

$$H^k(\mathbf{F}_{\mathbb{C},2}) \,=\, 0 \quad \text{for} \quad k \neq 0,\ 1,\ D-1,\ D\,, \tag{27}$$

$$H^1(\mathbf{F}_{\mathbb{C},2}) \,=\, \mathbb{C}\left[\sum_{\mu=1}^{D} \frac{z^\mu dz^\mu}{\mathrm{z}^2}\right], \tag{28}$$

$$H^{D-1}(\mathbf{F}_{\mathbb{C},2}) \,=\, \mathbb{C}\left[\sum_{\mu=1}^{D} \frac{(-1)^{\mu+1}\, z^\mu\, dz^1 \wedge \cdots \wedge \widehat{dz^\mu} \wedge \cdots \wedge dz^D}{(\mathrm{z}^2)^{\frac{D}{2}}}\right], \tag{29}$$

$$H^D(\mathbf{F}_{\mathbb{C},2}) \,=\, \mathbb{C}\left[\frac{dz^1 \wedge \cdots \wedge dz^D}{(\mathrm{z}^2)^{\frac{D}{2}}}\right], \tag{30}$$

where $\mathrm{z} = \mathrm{x}_1 - \mathrm{x}_2$. The role of the fact that D is restricted to be even is that only then are the representatives of the cohomology classes in (29) and (30) rational functions.

Let us introduce

$$\begin{aligned}
\omega_{j,k}^{(1)} \,&:=\, \sum_{\mu=1}^{D} \frac{(x_j^\mu - x_k^\mu)\, d(x_j^\mu - x_k^\mu)}{(\mathrm{x}_j - \mathrm{x}_k)^2}\,, \\
\omega_{j,k}^{(D-1)} \,&:=\, \sum_{\mu=1}^{D} \frac{(-1)^{\mu+1}\, d(x_j^\mu - x_k^\mu)}{\big(\mathrm{x}_j - \mathrm{x}_k^2\big)^{\frac{D}{2}}}\, d(x_j^1 - x_k^1) \wedge \cdots \\
&\qquad \wedge \widehat{d(x_j^\mu - x_k^\mu)} \wedge \cdots \wedge d(x_j^D - x_k^D)\,,
\end{aligned} \tag{31}$$

for $j, k = 1, \ldots, n$ and $j \neq k$, while for $j = k$ we set for convenience

$$\omega_{(k,k)}^{(m)} \,:=\, 0\,.$$

We also have

$$\omega_{j,k}^{(m)} \,=\, \omega_{k,j}^{(m)}\,.$$

Then we have found the following basis for the cohomologies of $\mathscr{O}_{\mathbb{C},3}$ (ordered by the form degree):

$$
\begin{aligned}
\text{deg. } 0 \ &: \ [1]\,, \\
\text{deg. } 1 \ &: \ [\omega_{1,2}^{(1)}]\,,\ [\omega_{1,3}^{(1)}]\,,\ [\omega_{2,3}^{(1)}]\,, \\
\text{deg. } 2 \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{2,3}^{(1)}]\,,\ [\omega_{1,3}^{(1)}\,\omega_{2,3}^{(1)}]\,, \\
\text{deg. } 3 \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(1)}]\,, \\
\text{deg. } D-1 \ &: \ [\omega_{1,2}^{(D-1)}]\,,\ [\omega_{1,3}^{(D-1)}]\,,\ [\omega_{2,3}^{(D-1)}]\,, \\
\text{deg. } D \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}]\,,\ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}]\,,\ [\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(1)}]\,, \\
&\quad\ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\ [\omega_{1,3}^{(1)}\,\omega_{1,3}^{(D-1)}]\,, \\
&\quad\ [\omega_{2,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\ [\omega_{1,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,, \\
\text{deg. } D+1 \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(1)}]\,, \\
&\quad\ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(1)}\,\omega_{1,3}^{(D-1)}]\,, \\
&\quad\ [\omega_{1,2}^{(1)}\,\omega_{2,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,, \\
\text{deg. } D+2 \ &: \ [\omega_{1,2}^{(1)}]\,[\omega_{1,2}^{(D-1)}]\,[\omega_{1,3}^{(1)}]\,[\omega_{2,3}^{(1)}]\,, \\
\text{deg. } 2D-2 \ &: \ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(D-1)}]\,, \\
\text{deg. } 2D-1 \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(D-1)}]\,, \\
&\quad\ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{1,3}^{(D-1)}]\,,\ [\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,, \\
&\quad\ [\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,, \\
\text{deg. } 2D \ &: \ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{1,3}^{(D-1)}]\,, \\
&\quad\ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{2,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,, \\
&\quad\ [\omega_{1,2}^{(1)}\,\omega_{1,2}^{(D-1)}\,\omega_{1,3}^{(1)}\,\omega_{2,3}^{(D-1)}]\,.
\end{aligned}
$$

In particular, with the shift $D \mapsto 2$ we obtain an isomorphism in cohomology for al even spatial dimensions.

For the application of the above result to the theory of vertex algebras it is important that the vertex algebras can be viewed as algebras over an operad built only by the first two quadratic configuration spaces $\mathbf{F}_{\mathbb{C},n}$ for $n = 2, 3$ [12]. The key argument for this is that the axiomatic conditions on vertex algebras are formulated only for Operator Product Expansions of two fields and hence, they use only two and three point functions on the spatial variables. This indicates a new kind of "homotopy equivalence" of the theories on operadic level for any even spatial dimension D.

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