4

Introduction to Anomalies

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4.1 Introduction

Anomalies are important in quantum field theories, particularly in the gauge field theories because they determine quantum consistency of the theory. The word anomaly, in fact, is bit of a misnomer, and a more accurate description of anomaly is quantum mechanical symmetry breaking. Symmetries of a classical field theory can be broken in various different ways. One of them is explicit symmetry breaking, which corresponds to adding a term to the Lagrangian density which does not respect the symmetry. Another way of breaking the symmetry is what is called the spontaneous symmetry breaking. In this case the classical Lagrangian density has a symmetry which is not respected by the ground state. In both the cases listed above symmetry is broken at classical level. The situation in the case of anomalies is different. The symmetry of the theory is intact at classical level but quantum mechanical effects do not respect the symmetry. It is in this sense that the word ‘anomaly’ is a misnomer.

More specifically, consider the action S of a classical field theory. Let us assume that this action is invariant under transformations of classical fields under a symmetry group G. The symmetry group G is anomalous if the full quantum theory does not respect this symmetry. Thus anomalous symmetries

R. Rangarajan and M. Sivakumar (eds.), *Surveys in Theoretical High Energy Physics - 2*, Texts and Readings in Physical Sciences 15,
DOI 10.1007/978-981-10-2591-4_4

are legitimate symmetries of the classical field theory but fail to survive when quantum effects are taken into account. Nature of the anomaly and its effects on the physics of the quantum theory depend on the role of the symmetry group G in the theory. For example, G can be a continuous group or a discrete group. Similarly, G can be a global symmetry of the theory or it could be a local gauge symmetry.

Anomaly in the global symmetry has interesting physical consequences like the neutral pion decay ($\pi^0 \to \gamma\gamma$). Anomaly in the local gauge symmetries implies violation of the gauge invariance of the theory. The lack of gauge invariance means the theory is nonunitary. Any gauge theory with anomalous gauge group is therefore quantum mechanically inconsistent. The only way we know, as of now, to make sense of such theories is to adjust matter content of such theories so that the anomaly is canceled. This restores gauge invariance of the theory at the quantum level. A classic example of this is the Standard model of particle physics. For a generic matter content as well as charge assignment the Standard model is potentially anomalous. However, it turns out that the anomaly is canceled if we have equal number of quark and lepton families. This is one of the nontrivial consistency checks of the Standard model of particle physics.

In these lectures, we will begin our discussion of anomalies by studying the Schwinger model, *i.e.*, the two dimensional electrodynamics. We will see that the anomaly in this model is due to the level crossing as one changes the background gauge field. In this model we have two classically conserved currents, the vector current and the axial vector current. The anomaly due to level crossing implies that in the quantum theory we cannot have simultaneous conservation of both the currents. Since the vector current is coupled to the gauge field we will preserve conservation on the vector current. This in turn means the axial vector current is not conserved. We will illustrate this computation using the point splitting regularization method as well as the Pauli-Villars regularization method. The reason for doing this computation in two different regularization scheme is to show that the anomaly is independent of the choice of regularization scheme. We will then discuss vacuum degeneracy by studying n-vacua as well as θ-vacua in this model. After studying the anomaly in the Schwinger model, we will consider anomalies in four dimensional gauge theories. We will begin the discussion with the abelian gauge theory and then discuss the non-abelian gauge theory. Path integral formalism is briefly introduced so that derivation of anomalies can be carried out using path integral methods. Finally we will apply it to the Standard model of particle physics and establish the criterion for the model to be anomaly free.

4.2 Two Dimensional Gauge Theory

Let us start with a toy model, the Schwinger model on a circle. The Schwinger model is a two dimensional $U(1)$ gauge theory coupled to a massless Dirac fermion. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi} i\gamma^\mu D_\mu \Psi, \tag{4.1}$$

where, Ψ is a two component spinor field and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad D_\mu = \partial_\mu + iA_\mu. \tag{4.2}$$

The gamma matrices are: $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$ and $\gamma^5 = \sigma_3$. We define the chiral fermions Ψ_L and Ψ_R as

$$\Psi_L = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}, \quad \Psi_R = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}; \quad \Psi_L = \gamma^5 \Psi_L, \quad \Psi_R = -\gamma^5 \Psi_R. \tag{4.3}$$

In the two dimensional electrodynamics, there are no transverse degrees of freedom for A_μ(the photon), however, the Coulomb interaction does exist. The Coulomb interaction in two dimensions grows linearly with the distance. This leads to the confinement of charged particles for any non-zero value of the coupling constant e. Here we are not interested in studying the confinement in this model. Our interest is to study possible anomaly in this theory.

To minimize the effect of the Coulomb potential, let us consider the model defined on a circle. We will take the circumference of the circle to be L. If we choose L in such a way that $eL \ll 1$ then the Coulomb interactions never become large. We can then ignore the Coulomb interactions in the first approximation and can include them perturbatively.

Let us impose following boundary conditions on the fields

$$A_\mu\left(x = -\frac{L}{2}, t\right) = A_\mu\left(x = \frac{L}{2}, t\right), \Psi\left(x = -\frac{L}{2}, t\right) = -\Psi\left(x = \frac{L}{2}, t\right). \tag{4.4}$$

Using these boundary conditions we can expand A_μ and Ψ in terms of the Fourier modes as

$$A_\mu(x,t) = \sum_{k=-\infty}^{\infty} a_\mu(k,t) \exp\left(\frac{2\pi ikx}{L}\right)$$

$$\Psi(x,t) = \sum_{k=-\infty}^{\infty} b_k(t) \exp\left(\frac{2\pi i(k+\frac{1}{2})x}{L}\right). \tag{4.5}$$

The Lagrangian density is invariant under the local gauge transformation

$$\Psi(x,t) \rightarrow \exp(i\alpha(x,t))\Psi(x,t), \quad A_\mu(x,t) \rightarrow A_\mu(x,t) - \partial_\mu \alpha(x,t). \tag{4.6}$$

Using the periodic boundary condition, we can write $A_1(x,t)$ as

$$A_1(x,t) = \sum_k a_1(k,t) \exp\left(\frac{2\pi i k x}{L}\right) \tag{4.7}$$

If we choose

$$\alpha(x,t) = \sum_k \frac{L}{2\pi i k} a_1(k,t) \exp\left(\frac{2\pi i k x}{L}\right), \tag{4.8}$$

then we can gauge away $A_1(x,t)$ except for the zero mode, *i.e.*, $k = 0$ mode of $A_1(x,t)$. The gauge parameter $\alpha(x,t)$ in (4.8) is periodic on the circle and therefore it is a legitimate gauge transformation. Since only $k = 0$ mode of $A_1(x,t)$ cannot be gauged away, it implies $A_1(x,t)$ is independent of x. Thus only non-trivial gauge field component that we need to consider is a constant mode.

However, the gauge transformation (4.6) does not cover all possible gauge transformations. That is, after fixing this gauge, we are left with a residual gauge symmetry. This residual gauge symmetry comes from the non-periodic gauge transformations,

$$\alpha(x,t) = \frac{2\pi}{L} n x, \quad n = \pm 1, \pm 2, \cdots \tag{4.9}$$

This gauge transformation parameter(4.9) does not obey the periodicity of the spatial direction, but $\partial\alpha/\partial x =$ constant, and $\partial\alpha/\partial t = 0$ as a result the periodicity of $A_\mu(x,t)$ is still preserved.

Recall that the fermion wavefunction picks up a local phase, $\exp(i\alpha(x,t))$, under the gauge transformation. In the interval $x \in [-\frac{L}{2}, \frac{L}{2}]$, the phase picked up by the fermion wavefunction is $\exp(i\alpha(x = L, t)) = \exp(2\pi i n)$, where n is an integer. Therefore the fermion wavefunction is left invariant by this non-periodic gauge transformation(4.9). We thus conclude that the gauge field component $A_1(x,t)$ does not take values in the interval $(-\infty, \infty)$ but is valued between $[0, 2\pi]$ with points A_1, $A_1 \pm 2\pi/L$, $A_1 \pm 4\pi/L$, $\cdots$ being identified due to the linear non-periodic gauge transformation(4.9).

In addition to the local gauge symmetry, the Lagrangian density is invariant under the global gauge transformation,

$$\psi(x,t) \rightarrow \exp(i\alpha)\psi(x,t). \tag{4.10}$$

This invariance corresponds to conservation of the electric charge. Using the Noether procedure we can write the conserved current,

$$J_\mu = \bar{\psi}\gamma_\mu\psi, \quad \bar{\psi} = \psi^\dagger\gamma^0, \tag{4.11}$$

with $\partial^\mu J_\mu = 0$, using the equations of motion.

The conserved charge is

$$Q = \int dx\, \psi^\dagger\psi\ . \tag{4.12}$$

The Lagrangian(4.1) is invariant under another symmetry transformation,

$$\psi(x,t) \to \exp(i\alpha\gamma_5)\psi(x,t). \tag{4.13}$$

The conserved current corresponding to this symmetry is

$$J^5_\mu = \bar{\psi}\gamma_\mu\gamma^5\psi, \tag{4.14}$$

with $\partial^\mu J^5_\mu = 0$, again using the equations of motion and the conserved charge is

$$Q_5 = \int dx\, \psi^\dagger\gamma^5\psi\ . \tag{4.15}$$

Notice that for the massive fermions, the current J^5_μ is not conserved.

$$\partial^\mu J^5_\mu = 2im\bar{\psi}\gamma^5\psi, \tag{4.16}$$

where, m is the mass of the fermion. We have chosen gamma matrix convention in such a way that $\gamma^5 = \sigma_3$. Therefore, Q_5 charge of ψ_L is $+1$ and that of ψ_R is -1. The conservation of Q and Q_5 for massless fermions implies separate conservation of

$$Q_L = \frac{Q+Q_5}{2}, \quad \text{and} \quad Q_R = \frac{Q-Q_5}{2}. \tag{4.17}$$

We can decompose the interaction term in the Lagrangian density, namely $\bar{\psi}\gamma^\mu\psi A_\mu$ as

$$\bar{\psi}\gamma^\mu\psi A_\mu = \psi_L^\dagger\psi_L(A_0 + A_1) + \psi_R^\dagger\psi_R(A_0 - A_1). \tag{4.18}$$

This implies that within the perturbation theory, the photon does not change the chirality of fermions. This would lead us to conclude that both Q and Q_5 are conserved in the quantum theory. However, the exact answer is more interesting, we will see that only one of these two classical symmetries survive in the quantum theory. Before we embark on this, let us first observe that in two dimensions, J_μ and J^5_μ are related to each other.

$$J^5_\mu = \epsilon_{\mu\nu} J^\nu, \tag{4.19}$$

where, $\epsilon_{\mu\nu}$ is the Levi-Civita tensor in two dimensions.

However, conservation of J_μ does not imply conservation of J^5_μ and vice versa.

We will now 'derive' the anomaly using a heuristic argument. For simplicity we will assume $A_0 = 0$. This, to be precise, is not correct, because electric charges in two dimensions feel only the Coulomb interactions, which implies $A_0 \neq 0$. However, if we take the circumference of the spatial circle small, *i.e.*, $eL \ll 1$, then $A_0 = 0$ is a good approximation. This is because the Coulomb potential $A_0 = e|x|$, which gives rise to linear confinement of electric charges does not take significant value for $-L/2 \leq x \leq L/2$. Therefore, to the leading order we are justified in setting $A_0 = 0$. We cannot set the gauge field component A_1 to zero. Periodicity of A_1 implies any value of A_1 is identified with $A_1 + 2\pi n/L$, $n \in \mathbb{Z}$. Only the constant mode of A_1 along the spatial direction is relevant because this spatially constant mode cannot be gauged away.

We will now look at the fermion dynamics in this gauge field background. The Dirac equation is

$$\left[i\frac{\partial}{\partial t} + \sigma_3 (i\frac{\partial}{\partial x} - A_1) \right] \psi(x,t) = 0. \tag{4.20}$$

Let us look for the stationary state solutions,

$$\psi(x,t) = \exp(-iE_k t)\psi_k(x). \tag{4.21}$$

The Dirac equation then becomes

$$E_k \psi_k(x) = -\sigma_3 \left(i\frac{\partial}{\partial x} - A \right) \psi_k(x). \tag{4.22}$$

On the spatial circle, we have imposed the anti-periodic boundary condition on the fermion wavefunction. The spatial part of the wavefunction consistent with this boundary condition is

$$\psi_k(x) \sim \exp[2\pi i x(k + 1/2)], \quad k \in \mathbb{Z}. \tag{4.23}$$

Using this wavefunction we can write the energy spectrum for the left moving and the right moving fermions

$$E_{k(L)} = \left(k + \frac{1}{2} \right) \frac{2\pi}{L} + A_1, \quad E_{k(R)} = -\left(k + \frac{1}{2} \right) \frac{2\pi}{L} - A_1 \tag{4.24}$$

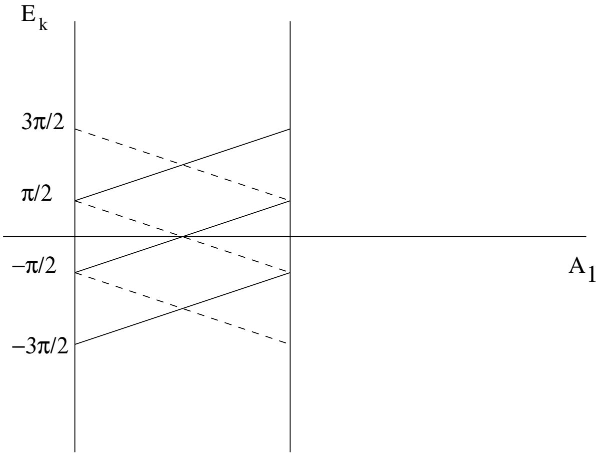

Figure 4.1: Level crossing for the left moving fermion (solid lines) and that for the right moving fermion (dashed lines).

At $A_1 = 0$ and $A_1 = 2\pi n/L$, the left moving and the right moving fermion energy levels are degenerate(4.24) (Also see figure 4.1).

Due to the gauge invariance, points $A_1 = 0$ and $A_1 = 2\pi/L$ are identified, but this identification occurs in a nontrivial way. By the time we move from $A_1 = 0$ to $A_1 = 2\pi/L$, all the left moving fermion energy states(4.24) have moved upwards by one unit and all the right moving fermion energy states(4.24) have moved downwards by one unit. We will now see that this rearrangement of fermion energy levels is responsible for the chiral anomaly.

To see this we will switch from the single particle state formulation to the field theory. First thing that we need to do is to define the fermion vacuum. Let us denote the unoccupied states by $|0_{L,R}; k\rangle$ and the occupied states by $|1_{L,R}; k\rangle$. For $A_1 \approx 0$, we define the fermion vacuum as

$$\begin{aligned} \Psi^{(0)}_{ferm} &= \left(\prod_{k=-1,-2,\cdots} |1_L; k\rangle \right) \left(\prod_{k=0,1,2,\cdots} |0_L; k\rangle \right) \\ &\times \left(\prod_{k=-1,-2,\cdots} |0_R; k\rangle \right) \left(\prod_{k=0,1,2,\cdots} |1_R; k\rangle \right) . \end{aligned} \tag{4.25}$$

Notice that for all the left moving particles negative energy levels correspond to $k < 0$ and for the right moving particles negative energy levels correspond to $k \geq 0$ for $A_1 \approx 0$.

Now we will vary A_1 slowly until it becomes $A_1 = 2\pi/L$. At $A_1 = 2\pi/L$, we see that one negative energy level of the left moving fermion has moved up and one negative energy level of the right moving fermion(hole) has moved down. Thus at $A_1 = 2\pi/L$, we have a particle-hole pair over the vacuum defined at $A_1 \approx 0$. As far a the electromagnetic charge Q is concerned, this state with a particle-hole pair is still electrically neutral, i.e., $\Delta Q = 0$. However, this is not true for the charge Q_5, because the Q_5 charge of a right moving hole is the same as that of the left moving particle. Therefore, $\Delta Q_5 = 2$. The Q_5 charge of the fermion vacuum at $A_1 \approx 0$ is zero by construction. Therefore we find that slow variation of A_1 from $A_1 = 0$ to $A_1 = 2\pi/L$ takes us from $Q_5 = 0$ state to $Q_5 = 2$ state. Using this fact we can write

$$\Delta Q_5 = \frac{L}{\pi} \Delta A_1. \tag{4.26}$$

Treating this as an adiabatic variation of the axial charge, we get

$$\frac{dQ_5}{dt} = \frac{L}{\pi}\frac{dA_1}{dt} \Rightarrow \frac{d}{dt}\left(Q_5 - \frac{L}{\pi}A_1\right) = 0. \tag{4.27}$$

Thus we find that the conserved charge is modified and is given by

$$\int dx \left(J_5^0 - \frac{1}{\pi}A_1\right). \tag{4.28}$$

The current corresponding to this conserved charge is

$$\tilde{J}_5^\mu = J_5^\mu - \frac{1}{\pi}\epsilon^{\mu\nu}A_\nu. \tag{4.29}$$

This new current $\tilde{J}_5^\mu$ is conserved,

$$\partial_\mu \tilde{J}_5^\mu = 0 \Rightarrow \partial_\mu J_5^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{4.30}$$

While the new conserved charge is gauge invariant under small gauge transformations, the new conserved axial current is not gauge invariant. Another point to notice is that the original axial current, which is gauge invariant, is not conserved anymore.

Thus we find ourselves in a situation where the conserved axial current is not gauge invariant and the gauge invariant axial current is not conserved. Since the gauge invariance is important to maintain consistency of the quantum theory, we give up on conservation of the gauge invariant axial current. Thus

$$\partial_\mu J_5^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{4.31}$$

This is the axial anomaly in the Schwinger model. In this picture we see that the axial anomaly is a statement of crossing of the zero energy levels.

In this derivation we have implicitly assumed some ultraviolet cutoff of the theory. We have also assumed that whenever a state crosses zero energy level and appears on the positive energy side, one state exits the ultraviolet cutoff on the positive energy side and one state enters the ultraviolet cutoff on the negative energy side. Although we have used infrared methods for counting number of levels crossing zero energy, in most practical applications we need to use the ultraviolet regularization method to derive the anomaly. This is because many gauge theories, including the QCD, are much harder to analyze in the infrared limit. The asymptotic freedom in these theories make the ultraviolet analysis easy to carry out.

Let us now use the ultraviolet regularization to derive the axial anomaly. There are various ways by which we can see the need for the ultraviolet regulator. One way to see this is to notice that the fermion vacuum state with filled Dirac sea involves an infinite product of fermion levels $|1_L;k\rangle$ and $|1_R;k\rangle$. Thus the energy of the fermion vacuum is

$$E \sim -\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)\frac{2\pi}{L}. \tag{4.32}$$

This is a divergent sum. To make sense of E we need to regularize this sum. If we choose a regularization procedure which throws away states with $|k| > |k_{max}|$ then we get a finite answer for E but this regularization is not gauge invariant. If we violate the gauge invariance, it would lead to the non-conservation of the electric charge. We can instead choose to regulate this sum by restricting the values of $p + A$. This would be a gauge invariant regulator. We will implement this using the point splitting regularization. Another way to see the need to use the ultraviolet regulator is to notice that the classical conserved currents are written in terms of products of fields defined at a coincident space-time point. In a quantum field theory, a product of two or more fields at a coincident space-time point is ill-defined. Such a product gives rise to the short distance singularities. These singularities are taken care of in the quantum field theory using the ultraviolet regularization method. The regulated currents are defined by writing the fields at non-coincident points and at the same time ensuring that they continue to remain gauge invariant. This is the point-splitting regularization procedure.

4.3 The Point-Splitting Regularization Method

We define regulated expressions for the classically conserved currents using the point-splitting regularization method as follows:

$$\begin{aligned} J_\mu^{Reg} &= \bar{\psi}(x+\epsilon,t)\gamma_\mu\psi(x,t)\exp\left(-i\int_x^{x+\epsilon} A_1 dx'\right) \\ J_\mu^{5Reg} &= \bar{\psi}(x+\epsilon,t)\gamma_\mu\gamma^5\psi(x,t)\exp\left(-i\int_x^{x+\epsilon} A_1 dx'\right). \end{aligned} \tag{4.33}$$

The exponential factor ensures that the regularized expression of currents is gauge invariant. The regularized expressions for Q and Q_5 obtained from the regularized currents is

$$Q = \int dx J_0^{Reg}(x,t) \quad \text{and} \quad Q_5 = \int dx J_0^{5Reg}(x,t). \tag{4.34}$$

The charge $Q_L = (Q+Q_5)/2$ measures the left moving fermion charge and $Q_R = (Q-Q_5)/2$ measures the right moving fermion charge. We will now measure Q_L and Q_R charge of the Dirac vacuum state. The regularized expressions for Q_L and Q_R are

$$\begin{aligned} Q_L &= \int dx \psi_L^\dagger(x+\epsilon,t)\psi_L(x,t)\exp\left(-i\int_x^{x+\epsilon} A_1 dx'\right) \\ Q_R &= \int dx \psi_R^\dagger(x+\epsilon,t)\psi_R(x,t)\exp\left(-i\int_x^{x+\epsilon} A_1 dx'\right). \end{aligned} \tag{4.35}$$

The fermion wavefunctions with appropriate normalization on a circle of circumference L are

$$\psi_k(x,t) = \frac{1}{\sqrt{L}}\exp\left(-iE_k t + i\frac{2\pi}{L}\left(k+\frac{1}{2}\right)x\right). \tag{4.36}$$

We can expand ψ_L and ψ_R in terms of this basis. However, to evaluate Q_L and Q_R on the Dirac vacuum state we do not need full decomposition of ψ_L and ψ_R in terms of ψ_k. Only information we need is that in the vacuum state, the left moving particles occupy states with negative k values and the right moving particles occupy states with non-negative k values. Thus positive k modes of ψ_L do not contribute to vacuum value of Q_L and negative k modes of ψ_R do not contribute to vacuum value of Q_R. Expressions for Q_L and Q_R therefore are

$$\begin{aligned} Q_L &= \frac{1}{L}\sum_{k<0}\int_{-L/2}^{L/2} dx \exp\left(-\frac{2\pi}{L}i\left(k+\frac{1}{2}\right)(x+\epsilon)\right)\times \\ &\quad \exp\left(\frac{2\pi}{L}i\left(k+\frac{1}{2}\right)x\right)\exp\left(-i\int_x^{x+\epsilon} A_1 dx'\right), \qquad (4.37) \\ Q_R &= \frac{1}{L}\sum_{k\geq 0}\int_{-L/2}^{L/2} dx \exp\left(-\frac{2\pi}{L}i\left(k+\frac{1}{2}\right)(x+\epsilon)\right)\times \\ &\quad \exp\left(\frac{2\pi}{L}i(k+\frac{1}{2})x\right)\exp\left(-i\int_x^{x+\epsilon} A_1 dx'\right). \qquad (4.38) \end{aligned}$$

Since A_1 is independent of x we can simplify these expressions by explicitly carrying out integration over x.

$$Q_L = \sum_{k<0}\exp\left[-i\epsilon\left(\frac{2\pi}{L}\left(k+\frac{1}{2}\right)+A_1\right)\right] \qquad (4.39)$$

$$Q_R = \sum_{k\geq 0}\exp\left[-i\epsilon\left(\frac{2\pi}{L}\left(k+\frac{1}{2}\right)+A_1\right)\right], \qquad (4.40)$$

where $k \in Z$. Although expressions for Q_L and Q_R look the same, the sum over k is over different values. The range of values of k in the summation are chosen for $|A_1| < \pi/L$.

Let us first notice that in the $\epsilon \to 0$ limit, both Q_L and Q_R reduce to an infinite series $\sum_k 1$. Although k takes different values for Q_L and Q_R, this fact is irrelevant for this infinite series, which is divergent. Point splitting is a covariant regulator because it cuts off states with $|p_1 + A_1| \geq 1/\epsilon$.

Both Q_L and Q_R are written in terms of geometric series. It is easy to sum both of them.

$$Q_L = \frac{\exp[-i\epsilon(\frac{\pi}{L}+A_1)]}{\exp[-\frac{2i\pi\epsilon}{L}]-1}, \quad Q_R = \frac{\exp[-i\epsilon(\frac{\pi}{L}+A_1)]}{1-\exp[-\frac{2i\pi\epsilon}{L}]}. \qquad (4.41)$$

Expanding these sums in terms of a power series in ϵ gives

$$Q_L = -\frac{L}{2i\pi\epsilon}+\frac{L}{2\pi}A_1+o(\epsilon) \qquad (4.42)$$

$$Q_R = \frac{L}{2i\pi\epsilon}-\frac{L}{2\pi}A_1+o(\epsilon). \qquad (4.43)$$

The first term in both the expression diverges as we take the limit $\epsilon \to 0$. This is just a reflection of the fact that original series were divergent.

It is easy to see that the electric charge of the vacuum state,

$$Q = Q_L + Q_R = 0, \tag{4.44}$$

in spite of the fact that Q_L and Q_R are individually divergent quantities. Though Q_L and Q_R depend explicitly on A_1, the electric charge Q is independent of A_1 once we remove the regulator, *i.e.*, $\epsilon \to 0$. This ensures conservation of electric charge.

The axial charge, on the other hand, has two contributions,

$$Q_5 = Q_L - Q_R = \frac{L}{i\pi\epsilon} + \frac{L}{\pi} A_1 + o(\epsilon). \tag{4.45}$$

The first term is divergent as $\epsilon \to 0$, however, this divergence can be removed by defining normal ordered expression for Q_5. The second contribution is finite as $\epsilon \to 0$, and it shows that the regularized axial charge depends on A_1. Thus as A_1 goes from 0 to $2\pi/L$, Q_5 changes by two units. This result is identical to the one obtained by counting the number of levels crossing zero energy in the earlier computation.

If we change A_1 adiabatically then

$$\frac{dQ_5}{dt} = \frac{L}{\pi}\frac{dA_1}{dt} \Rightarrow \partial_\mu J_5^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu} F_{\mu\nu}. \tag{4.46}$$

We have got the anomaly equation with correct normalization. Recall in the infrared picture we got non-conservation of axial charge due to crossing of zero energy level. This time around we have obtained the anomaly by imposing ultraviolet cutoff. Non-conservation of the axial charge is now understood as follows. As we change A_1 adiabatically from $A_1 = 0$ to $A_1 = 2\pi/L$, one right moving fermion level exits the Dirac sea from its lower boundary, *i.e.*, $-1/|\epsilon|$ and one left handed fermion level enters the Dirac sea from the same boundary.

In fact, both infrared and ultraviolet phenomena occur simultaneously. Compatibility of these two methods of determining axial charge non-conservation is stated in terms of 't Hooft consistency condition. 't Hooft's consistency condition states that singularities of the amplitudes computed in the ultraviolet theory should be reproducible from the amplitudes computed in the infrared theory.

4.4 The Pauli-Villars Regularization Method

We will compute this anomaly one more time. This time we will use Pauli-Villars regularization scheme. The reason for doing this computation once again is to show that the axial anomaly computed using point-splitting regularization method is not an artifact of specific choice of the regularization scheme. In other words, anomaly equation is independent of regularization scheme.

Since anomaly is intrinsically quantum mechanical, its manifestation is seen at loop level in the perturbation theory. Loop diagrams are generically divergent and we will use Pauli-Villars regularization method to evaluate loop integrals. For simplicity we will use background field method, *i.e.*, we will assume a fixed gauge field background and evaluate loop integrals in this background. The relevant diagrams for computation of axial anomaly in the Schwinger model are

First graph is a one-loop contribution from the massless fermion, and the second graph is one-loop contribution from the Pauli-Villars regulator fermion χ with mass M_0. $\gamma_\mu\gamma_5$ corresponds to axial current vertex for both ψ and χ. In the Pauli-Villars regularization procedure, loop of the regulator fermion does not pick up negative sign. The regulator fermion thus cancels all high frequency modes of the fermion ψ in the loop. This cancellation occurs for all frequencies $\omega > M_0$. The regulator is removed by taking M_0 to infinity. For all low frequency modes of ψ, M_0 acts as a gauge invariant cutoff. The regularized axial current is

$$J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi + \bar{\chi}\gamma^\mu\gamma_5\chi. \tag{4.47}$$

Due to existence of massive fermion we do not expect conservation of the axial current. Equations of motion for ψ and χ are

$$\not{D}\psi = 0, \quad \text{and} \quad \not{D}\chi = -iM_0\chi. \tag{4.48}$$

Using these equations of motion we can evaluate divergence of the axial current,

$$\partial_\mu J_5^\mu = 2iM_0\bar{\chi}\gamma_5\chi. \tag{4.49}$$

We will now evaluate the vacuum expectation value of $\partial_\mu J_5^\mu$ in the background field formalism. If the current is conserved then this vacuum expectation value should vanish as we remove the regulator, *i.e.*, take $M_0 \to \infty$. To evaluate the vacuum expectation value of the divergence of the axial current, it is easiest to work with the right hand side expression in the coordinate space representation.

$$2iM_0\langle\bar{\chi}(x,t)\gamma_5\chi(x,t)\rangle = 2iM_0\langle Tr(\gamma_5\chi(x,t)\bar{\chi}(x,t))\rangle. \tag{4.50}$$

In spite of having χ and $\bar{\chi}$ defined at the same space-time point, the vacuum expectation value $\langle \chi(x,t)\bar{\chi}(x,t)\rangle$ gives a formal coordinate space propagator for χ. The coordinate space propagator satisfies the Green's function equation

$$(i\not{D} - M_0)S(x,y) = i\delta^2(x-y). \tag{4.51}$$

Due to coincident space-time point, momentum space representation of $S(x,x)$ is

$$S(x,t;x,t) = i\int \frac{d^2p}{(2\pi)^2}\frac{1}{\not{\Pi} - M_0}, \quad \Pi_\mu = p_\mu + A_\mu. \tag{4.52}$$

Notice the exponential factor in the expression of propagator is missing. The momentum p serves as the loop momentum. Let us list a few standard manipulations

- $$\frac{1}{\not{\Pi} - M_0} = \frac{\not{\Pi} + M_0}{\not{\Pi}\not{\Pi} - M_0^2} = \frac{\not{\Pi} + M_0}{\Pi^2 - M_0^2 - \frac{i}{2}\epsilon^{\mu\nu}F_{\mu\nu}\gamma_5} \tag{4.53}$$
- $[\Pi_\mu, \Pi_\nu] = -[D_\mu, D_\nu] = iF_{\mu\nu}$.
- In two dimensions $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu}\gamma_5$.

Using these relations vacuum expectation value of $\partial_\mu J_5^\mu$ can be written as,

$$\begin{aligned}\langle \partial_\mu J_5^\mu(x,t)\rangle &= -2M_0\int \frac{d^2p}{(2\pi)^2}\mathrm{Tr}\left(\frac{\gamma_5}{\not{\Pi} - M_0}\right) \\ = -2M_0\int \frac{d^2p}{(2\pi)^2} &\quad \mathrm{Tr}\left[\gamma_5\frac{(\not{\Pi} + M_0)}{\Pi^2 - M_0^2 - \frac{i}{2}\epsilon^{\mu\nu}F_{\mu\nu}\gamma_5}\right].\end{aligned} \tag{4.54}$$

Expanding the denominator in a power series gives,

$$\begin{aligned}\langle \partial_\mu J_5^\mu(x,t)\rangle &= -2M_0\int \frac{d^2p}{(2\pi)^2}\mathrm{Tr}\Bigg[\gamma_5(\not{\Pi} + M_0) \\ \times\Bigg(\frac{1}{\Pi^2 - M_0^2} &+ \frac{1}{\Pi^2 - M_0^2}\left(\frac{i}{2}\epsilon^{\mu\nu}F_{\mu\nu}\gamma_5\right)\frac{1}{\Pi^2 - M_0^2} + \cdots\Bigg)\Bigg]\end{aligned} \tag{4.55}$$

It is easy to see that the first term vanishes due to trace of the integrand. Third term onwards all terms drop out as $M_0 \to \infty$. Only relevant term is the second term and therefore the effective one loop integral is

$$\langle \partial_\mu J_5^\mu(x,t)\rangle = -2iM_0^2\int \frac{d^2p}{(2\pi)^2}\frac{\epsilon^{\mu\nu}F_{\mu\nu}}{(\Pi^2 - M_0^2)^2} \tag{4.56}$$

A few comments are in order at this point.

- $\not{A}$ does not contribute because trace vanishes.
- We get a factor of 2 because Tr $1_{2\times 2} = 2$.

We will now replace Π_μ by p_μ and neglect A_μ. we can do this because terms proportional to A_μ are not divergent as $p \to \infty$ and hence can be dropped when computing the anomaly. The loop integral now becomes

$$\langle \partial_\mu J_5^\mu(x,t)\rangle = -2iM_0^2 \int \frac{d^2p}{(2\pi)^2} \frac{\epsilon^{\mu\nu}F_{\mu\nu}}{(p^2 - M_0^2)^2}. \tag{4.57}$$

We will evaluate this integral by first performing Wick rotation in the momentum space, *i.e.*, $(p_0, p_1) \to (ip_2, p_1)$. Define $p_E^2 = p_1^2 + p_2^2$. Substituting this in the loop integral gives

$$\begin{aligned}\langle \partial_\mu J_5^\mu(x,t)\rangle &= 2M_0^2 \int \frac{dp_E p_E}{2\pi} \frac{\epsilon^{\mu\nu}F_{\mu\nu}}{(p_E^2 + M_0^2)^2} \\ &= -\frac{M_0^2}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}\frac{1}{p_E^2 + M_0^2}\bigg|_{p_E=0}^{\infty} = \frac{\epsilon^{\mu\nu}F_{\mu\nu}}{2\pi}.\end{aligned} \tag{4.58}$$

Thus we see that the Pauli-Villars regularization procedure for removing ultraviolet divergences gives the same anomaly equation as the one derived using level crossing and point-splitting method. We therefore argue that the anomaly is independent of choice of regularization scheme.

4.5 n-vacua and θ-vacua

It is now time to check if our assumptions are consistent with the results we have obtained. Let us first recall what is our working hypothesis. We have assumed that fermions are fast variables and gauge field is a slow variable. We have taken $eL \ll 1$ and neglected A_0. In the absence of A_0, the gauge kinetic term becomes

$$-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2e^2}\dot{A}_1^2. \tag{4.59}$$

Since A_1 is independent of x, contribution of the kinetic term to the effective Lagrangian is $L\dot{A}_1^2/2e^2$.

Let us now look at the fermion Hamiltonian

$$H = -\int \psi^\dagger(x,t)\sigma_3(i\frac{\partial}{\partial x} - A_1)\psi(x,t)dx. \tag{4.60}$$

We will regularize this Hamiltonian using the point-splitting method.

$$H = -\int dx\psi^\dagger(x+\epsilon,t)\sigma_3(i\frac{\partial}{\partial x} - A_1)\psi(x,t)\exp\left(-i\int_x^{x+\epsilon} A_1 dx'\right). \quad (4.61)$$

Since σ_3 is the γ_5-matrix we can split the energy spectrum into E_L and E_R. Using Fourier modes (4.36) we can determine the energy of the Dirac sea

$$E_L = \sum_{k=-1}^{-\infty} E_{k(L)} \exp(-i\epsilon E_{k(L)}) \quad (4.62)$$

$$E_R = \sum_{k=0}^{\infty} E_{k(R)} \exp(i\epsilon E_{k(R)}), \quad (4.63)$$

where, $E_{k(L)}$ and $E_{k(R)}$ are given in eq.(4.24). These are regulated expressions and are valid for $|A_1| < \pi/L$. If we take $\epsilon \to 0$ then we will get divergent sums.

Let us now notice that expressions for E_L and E_R can be obtained by differentiating Q_L and Q_R (see eq.(4.37) and (4.38)) with respect to ϵ. Thus energy of the Dirac sea is

$$E_0 = E_L + E_R = i\frac{\partial}{\partial\epsilon}(Q_L - Q_R). \quad (4.64)$$

Since $Q_L = Q_R = \{i\exp(-i\epsilon A_1)\}/\{2\sin(\pi\epsilon/L)\}$(see eq.(4.41)),

$$E_0 = i\frac{\partial}{\partial\epsilon}\left(\frac{i\exp(-i\epsilon A_1)}{\sin(\pi\epsilon/L)}\right) = \frac{L}{2\pi}\left(2A_1^2 - \frac{\pi^2}{L^2} + \frac{1}{\epsilon^2}\right). \quad (4.65)$$

After dropping the constant term and soaking up the divergent term in the normal ordering prescription we find that the energy of the Dirac sea generates an effective potential for A_1. The effective Lagrangian for A_1 degrees of freedom is

$$L = \frac{L}{2e^2}\dot{A}_1^2 - \frac{L}{\pi}A_1^2. \quad (4.66)$$

This is just the harmonic oscillator problem with the spring constant $K = 2L/\pi$, mass $m = L$ and $\hbar = e$. The energy spectrum is,

$$E = \left(n + \frac{1}{2}\right)\sqrt{\frac{2}{\pi}}e. \quad (4.67)$$

Thus we see that characteristic energies of A_1 quanta is $E_A \propto e$ whereas characteristic energies of ψ quanta is $E_\psi \propto 1/L$. Therefore $E_A/E_\psi \sim eL \ll 1$. This implies A_1 quanta are low energy or slowly varying variables compared to ψ quanta. This justifies our procedure of studying ψ quanta in the adiabatically varying A_1 background. It is easy to determine the ground state wavefunction of the gauge field problem, since it is a harmonic oscillator problem

$$\Psi_0(A_1) \propto \exp\left(-\frac{LA_1^2}{\sqrt{2\pi\epsilon^2}}\right). \tag{4.68}$$

Thus the total vacuum wavefunction is

$$\begin{aligned}
\Psi_0(A_1,\psi) &= \Psi^{(0)}_{ferm}\Psi_0(A_1) \\
&\propto \left(\prod_{k=-1,-2,\cdots} |1_L;k\rangle\right)\left(\prod_{k=0,1,2,\cdots} |0_L;k\rangle\right) \\
&\times \left(\prod_{k=-1,-2,\cdots} |0_R;k\rangle\right)\left(\prod_{k=0,1,2,\cdots} |1_R;k\rangle\right) \\
&\times \exp\left(-\frac{LA_1^2}{\sqrt{2\pi\epsilon^2}}\right),
\end{aligned} \tag{4.69}$$

provided $|A_1| < \pi/L$. This wavefunction is invariant under small gauge transformations. Recall small gauge transformations imply A_1 is independent of x. Small gauge transformations therefore shift the centre of the A_1 harmonic oscillator slightly away from $A_1 = 0$. Note that small gauge transformations, by definition, are those which transform the initial configuration, say, $|A_1| < \pi/L$ to the gauge transformed configuration $|\tilde{A}_1| < \pi/L$.

Large gauge transformations are the ones which take A_1 to $A_1 + 2\pi k/L$, where $(k = \pm1, \pm2, \cdots)$. The vacuum wavefunction is not invariant under large gauge transformations. Although $A_1 \approx 0$ and $A_1 \approx 2\pi/L$ are related by gauge transformation, we know from our study of the fermion energy levels that the fermion vacuum at $A_1 \approx 0$ is different from that at $A_1 \approx 2\pi/L$. In particular, at $A_1 \approx 2\pi/L$ we have fermion spectrum containing a particle-hole pair. This state is a gauge transform of the fermion vacuum at $A_1 \approx 0$. Clearly a particle-hole pair over vacuum is not a legitimate vacuum state at $A_1 \approx 2\pi/L$. In other words, the correct vacuum state of fermions at $A_1 = 2\pi/L$ has a different description in the neighbourhood of $A_1 = 0$. It is certainly not the fermion vacuum at $A_1 = 0$. From the level crossing picture, we know that as we increase A_1, left moving fermion energy states move upwards and right moving fermion energy states move downwards. The fermionic energy spectrum, nevertheless, is identical for A_1 and $A_1 + 2\pi n/L$ $(n \in Z)$. Level crossing affects the occupation number of these energy states. The fermion vacuum state at A_1 appears as a state with n left moving fermionic particles and n right moving fermionic holes.excited over the Dirac sea at $A_1 + 2\pi n/L$ with $n > 0$. However, we want to define fermionic vacuum at every value of n, and we will describe this state in terms of the fermionic state defined in the interval $-\pi/L < A_1 < \pi/L$.

Suppose we want to define fermionic vacuum at $A_1 \approx 2\pi/L$. It is now obvious from the level crossing argument (see Fig.4.1) that the state at $A_1 \approx 0$ which evolves into a fermionic vacuum at $A_1 \approx 2\pi/L$ is

$$\begin{aligned} \Psi_{ferm}^{(\frac{2\pi}{L})} &= \left(\prod_{k=-2,-3,\cdots} |1_L;k\rangle|0_R;k\rangle \right) \\ &\times \left(\prod_{k=-1,0,1,\cdots} |0_L;k\rangle|1_R;k\rangle \right). \end{aligned} \tag{4.70}$$

This state gives correct description of the Dirac sea at $A_1 = 2\pi/L$. It is now easy to write down the full vacuum wavefunction

$$\begin{aligned} \Psi_1(A_1,\psi) &= \prod_{k=-2,-3,\cdots} |1_L;k\rangle|0_R;k\rangle \\ &\times \prod_{k=-1,0,1,\cdots} |0_L;k\rangle|1_R;k\rangle\Psi_0(A_1 - 2\pi/L). \end{aligned} \tag{4.71}$$

This argument can be generalized in a straight forward manner to write down the fermionic state describing the Dirac sea at $A_1 \approx 2\pi n/L$. This implies we have degenerate ground states labeled by an integer n corresponding to a large gauge transformation $A_1 \to A_1 + 2\pi n/L$, $n \in Z$. Appropriate vacuum wave-function for nth sector is

$$\begin{aligned} \Psi_1(A_1,\psi) &= \prod_{k=-1-n}^{-\infty} |1_L;k\rangle|0_R;k\rangle \\ &\times \prod_{k=-n}^{\infty} |0_L;k\rangle|1_R;k\rangle\Psi_0\left(A_1 - \frac{2\pi n}{L}\right), \end{aligned} \tag{4.72}$$

where $n \in Z$. A large gauge transformation takes us from Ψ_n to $\Psi_{n'}$. Therefore these wavefunctions are not invariant under large gauge transformations. These degenerate vacua are called "n-vacua". It is, in fact, possible to write down a new vacuum state which is invariant, up to an overall phase, under large gauge transformations. Define

$$\Psi_\theta^{(0)}(A_1,\psi) = \sum_n \Psi_n(A_1,\psi)\exp(in\theta). \tag{4.73}$$

This state depends on a continuous parameter θ, which is called the vacuum angle.

Let us now see the effect of a large gauge transformation on the new vacuum state $\Psi_\theta^{(0)}(A_1,\psi)$. For illustration, consider a large gauge transformation which takes A_1 to $A_1 + 2\pi/L$. From the expression of n-vacuum state, it is clear that this large gauge transformation takes us from Ψ_n to Ψ_{n-1}. This in effect means

$$\Psi_\theta^{(0)} \to \exp(i\theta)\Psi_\theta^{(0)}. \tag{4.74}$$

This overall phase is not observable. The state $\Psi_\theta^{(0)}$ is not unique because for any angle θ it is invariant under large gauge transformations. The states represented by $\Psi_\theta^{(0)}(A_1,\psi)$ is called the "θ-vacuum". All physical quantities obtained by averaging over θ-vacua are invariant under all gauge transformations.

Existence of θ-vacua can be incorporated in the Lagrangian density by adding a term

$$\mathcal{L}_\theta = \frac{\theta}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}, \tag{4.75}$$

to the original Lagrangian density. This quantity is called the topological density. Since $\mathcal{L}_\theta$ is a total derivative, addition of it to the original Lagrangian density does not affect equations of motion. Classical physics is therefore unaltered. The topological density contributes only if

$$\begin{aligned} \int dt \int_{-L/2}^{L/2} dx \frac{dA_1}{dt} &\neq 0 \\ \int_{-L/2}^{L/2} dx[A_1(x,t=\infty) - A_1(x,t=-\infty)] &\neq 0. \end{aligned} \tag{4.76}$$

Partition function of the Schwinger model in the Lagrangian formulation and with the inclusion of the topological density is

$$Z = \sum_{\{A,\psi\}} \exp\left(i\int d^2x(\mathcal{L}+\mathcal{L}_\theta)\right), \tag{4.77}$$

where the summation is over all field configurations of ψ and A_μ. The original Lagrangian density is invariant under both small and large gauge transformations. Invariance of the partition function under all gauge transformations means the topological density should change by a factor $2\pi\times$ integer. Since we are looking at adiabatic variation of the gauge field, integral of the topological density itself must be $2\pi\times$ integer. Thus we need

$$\int_{-L/2}^{L/2} dx[A_1(x,t=\infty) - A_1(x,t=-\infty)] = 2\pi n. \tag{4.78}$$

Using small gauge transformation we can set A_1 to be independent of x. We can then choose A_1 varying adiabatically from A_1 at $t=-\infty$ to $A_1 + 2\pi n/L$ at $t=\infty$. Thus,

$$A_1(x,t=\infty) - A_1(x,t=-\infty) = \frac{2\pi}{L} n. \tag{4.79}$$

Putting this back into the integral (4.78) and noticing that it is independent of x and carrying out the integral gives us the desired answer. However, A_1 and $A_1 + 2\pi n/L$, $n \in Z$ are related by a large gauge transformation. That means our final gauge field configuration is a gauge transform of our initial gauge field configuration.

$$A_1(x,t=\infty) = A_1(x,t=-\infty) - \frac{\partial \alpha_n}{\partial x}, \tag{4.80}$$

where, $\alpha_n = -2\pi n x/L$. The topological density therefore can be written as

$$\int_{-L/2}^{L/2} dx[A_1(x,t=\infty) - A_1(x,t=-\infty)] = -\int_{-L/2}^{L/2} dx \frac{\partial \alpha_n}{\partial x}. \tag{4.81}$$

We are now in a position to understand why we call $\mathcal{L}_\theta$, a topological density. Spatial direction in our model is periodic with periodicity L. The gauge field component A_1 is also periodic with periodicity $2\pi/L$. As we traverse x from $-L/2$ to $L/2$, α_n changes by $-2\pi n$ and as a result A_1 changes from A_1 to $A_1 + 2\pi n/L$. Since both x and A_1 are periodic we can treat them as variables parametrizing a circle. The circle parametrized by x has a circumference L whereas the circle parametrized by A_1 has circumference $2\pi/L$. Going around x circle once takes us around A_1 circle n times. α_n defines a map from x-circle to A_1-circle. Maps from x-circle to A_1-circle which wind the A_1-circle n times are not continuously connected to the maps that wind A_1-circle m times for $m \neq n$.

Thus these maps are divided into different equivalence classes according to number of times they wrap the A_1-circle. These wrappings are parametrized by an integer called the winding number. Mathematically, maps from a circle to a circle are classified by the first homotopy group or the fundamental group π_1. Windings parametrized by an integer is a statement $\pi_1(S^1) = Z$. It is easy to see that $\pi_1(S^1)$ forms a group.

- For every element which gives a map with winding number n, there exists a map of winding number $-n$. Composition of these two maps gives a map with winding number zero.
- A map with winding number zero is in the equivalence class of identity maps.
- A map with winding number n and a map with winding number m can be composed together to get a map with winding number $m+n$.

$\pi_1(S^1)$ is an abelian group.

Why do we need θ-vacua? The n-vacuum, denoted by Ψ_n, is invariant under small gauge transformations and that is sufficient to ensure conservation of electric charge. We can then ignore the fact that Ψ_n is not invariant under large gauge transformations. If we are going to work within the perturbation theory then we will not see such a large change in the field configuration anyway.

The problem with this line of argument is that Ψ_n violates the cluster decomposition property of the quantum field theory. Suppose we are studying vacuum expectation value of the time ordered product of some local operators, then the cluster decomposition property implies that this vacuum expectation value is reducible to the sum over intermediate states including the vacuum state and all the excitations over it. The fact that Ψ_n would violate this property is easy to see. Consider a two point function of the operator

$$\mathcal{O}(t) = \int \bar{\psi}(x,t)(1+\gamma_5)\psi(x,t)dx, \tag{4.82}$$

$$G_2(t) = \langle \Psi_n | T\{\mathcal{O}^\dagger(t)\mathcal{O}(0)\} | \Psi_n \rangle. \tag{4.83}$$

We are evaluating this two point function in Ψ_n state. The operator $\mathcal{O}$ changes the axial charge by minus two units. We therefore expect that $G_2(t)$ will be non-vanishing. Now if we use the cluster decomposition property then we can insert complete set of states between $\mathcal{O}^\dagger$ and $\mathcal{O}$. If we restrict ourselves to Ψ_n sector then $G_2(t)$, by cluster decomposition property depends on $\langle \bar{\psi}(1+\gamma_5)\psi \rangle$. Since $\bar{\psi}(1+\gamma_5)\psi$ changes Ψ_n to Ψ_{n+1}, $\langle \bar{\psi}(1+\gamma_5)\psi \rangle = 0$ in the Ψ_n sector. This contradicts our earlier expectation that $G_2(t)$ is non-vanishing. If, instead of Ψ_n, we use $\Psi_\theta^{(0)}$ then the cluster property is restored. This is because in the θ-vacuum we can have non-diagonal vacuum expectation value.

$$\langle \Psi_{n+1} | \bar{\psi}(1+\gamma_5)\psi | \Psi_n \rangle \propto \frac{1}{L} \exp\left(i\theta - \frac{(2\pi)^{3/2}}{eL} \right). \tag{4.84}$$

Violation of cluster property leads to violation of causality as well as violation of unitarity. It is therefore imperative that we work with θ-vacua and not with an n-vacuum.

4.6 Four Dimensional Gauge Theory

We will start with the four dimensional abelian gauge theory coupled to a massless Dirac fermion. The classical action for this theory is given by

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} i \not{D} \Psi \right), \tag{4.85}$$

where, $D_\mu = \partial_\mu - ieA_\mu$ and $\not{D} = \gamma^\mu D_\mu$. Our conventions are

$$\begin{aligned} g_{\mu\nu} &= g^{\mu\nu} = \text{diag}(1,-1,-1,-1);\ \gamma^\mu = (\gamma^0, \gamma^i),\ \gamma_\mu = (\gamma_0, -\gamma_i) \\ \gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\ \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.86}$$

Since the fermion is massless, we can write it in terms of left handed and right handed components.

$$\Psi_L = \frac{1}{2}(1+\gamma_5)\Psi,\ \Psi_R = \frac{1}{2}(1-\gamma_5)\Psi. \tag{4.87}$$

Let us also consider four dimensional non-abelian gauge theory with gauge group $SU(N)$ coupled to n_f massless fermions. The classical action for this theory written in terms of left handed and right handed components of the fermion is

$$S = \int d^4x \left(-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{m=1}^{n_f} \bar{\Psi}_{mL} i \not{D} \Psi_{mL} + \sum_{m=1}^{n_f} \bar{\Psi}_{mR} i \not{D} \Psi_{mR} \right), \tag{4.88}$$

where, $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ and fermions Ψ_m are all in the fundamental representation, $\mathbf{N}$ of $SU(N)$. The covariant derivative is defined as $D_\mu = \partial_\mu - igA^a_\mu T^a$, where T^a, $a = 1, \cdots, N^2-1$ are generators of the Lie algebra of $SU(N)$, in the fundamental representation.

$$[T^a, T^b] = if^{abc}T^c,\ \text{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}. \tag{4.89}$$

Let us enumerate symmetries of these actions,

1. Local gauge invariance: In case of abelian gauge theory, the action is invariant under

$$\Psi(x) \rightarrow \Psi'(x) = e^{-ie\alpha(x)}\Psi(x), \tag{4.90}$$

$$A_\mu(x) \rightarrow a'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x). \tag{4.91}$$

For non-abelian gauge theory, the action is invariant under

$$\Psi_m(x) \quad \to \quad \Psi'_m(x) = U(\theta)\Psi_m(x), \tag{4.92}$$

$$A'_\mu(x) \quad = U(\theta)A_\mu(x)U^{-1}(\theta) - \frac{i}{g}\partial_\mu U(\theta)U^{-1}(\theta), \tag{4.93}$$

where, $U(\theta) = \exp(-iT^a\theta^a(x))$.

2. Global symmetries:

   (a) Apart from obvious Poincare invariance, both abelian and non-abelian gauge theory actions are invariant under the scale transformation. This gives conserved dilatation current.

$$A_\mu(x) \quad \to \quad A'_\mu(x) = \lambda A_\mu(\lambda x), \tag{4.94}$$

$$\Psi(x) \quad \to \quad \Psi'(x) = \lambda^{3/2}\Psi(\lambda x). \tag{4.95}$$

   (b) Both the actions are invariant under the phase transformations

$$\Psi(x) \quad \to \quad e^{i\alpha}\Psi(x), \text{ or } \Psi_m(x) \to e^{i\alpha}\Psi_m(x), \tag{4.96}$$

and

$$\Psi(x) \to e^{i\beta\gamma_5}\Psi(x), \text{ or } \Psi_m(x) \to e^{i\beta\gamma_5}\Psi_m(x). \tag{4.97}$$

These two symmetries give rise to conserved vector current

$$J^\mu(x) = (\bar{\Psi}\gamma^\mu\Psi)(x) \quad [(\bar{\Psi}_m\gamma^\mu\Psi_m)(x)], \tag{4.98}$$

and conserved axial current

$$J_5^\mu(x) = (\bar{\Psi}\gamma^\mu\gamma_5\Psi)(x) \quad [(\bar{\Psi}_m\gamma^\mu\gamma_5\Psi_m)(x)]. \tag{4.99}$$

Action of these symmetries on left handed and right handed fermion is (for vector transformation)

$$\Psi_L(x) \quad \to \quad \Psi'_L(x) = e^{-i\alpha}\Psi_L(x), \tag{4.100}$$

$$\Psi_R(x) \quad \to \quad \Psi'_R(x) = e^{-i\alpha}\Psi_R(x), \tag{4.101}$$

and (for axial vector transformation)

$$\Psi_L(x) \quad \to \quad \Psi'_L(x) = e^{i\beta}\Psi_L(x), \tag{4.102}$$

$$\Psi_R(x) \quad \to \quad \Psi'_R(x) = e^{-i\beta}\Psi_R(x). \tag{4.103}$$

(c) In addition to these symmetries, the non-abelian gauge theory action has $SU(n_f)_L \times SU(n_f)_R$ flavor symmetry. To see this symmetry we first write n_f fermions in a column vector

$$\Psi(x) = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_{n_f} \end{pmatrix} (x). \tag{4.104}$$

An $n_f \times n_f$ unitary matrix U mixes these fermions into each other. This unitary matrix is an element of $SU(n_f)$ group. Since we have decomposed fermions into left handed and right handed components with no term in the action which couples left and right components, we can do independent rotations of left handed and right handed fermions. This corresponds to the transformations

$$\Psi_L(x) \quad \rightarrow \quad \Psi'_L(x) = U\Psi_L(x) \tag{4.105}$$
$$\Psi_R(x) \quad \rightarrow \quad \Psi'_R(x) = \tilde{U}\Psi_R(x). \tag{4.106}$$

Thus in case of massless fermions we have $SU(n_f)_L \times SU(n_f)_R$ chiral flavor symmetry. This symmetry can also be written as $SU(n_f)_V \times SU(n_f)_A$ flavor symmetry. This can be seen by recognizing that vector transformation acts on $\Psi_L(x) + \Psi_R(x)$ and axial vector transformation acts on $\Psi_L(x) - \Psi_R(x)$.

We are interested in the axial $U(1)$ transformation symmetry. To study that let us choose Fock-Schwinger gauge, *i.e.*, $x^\mu A^a_\mu(x) = 0$. We will make this gauge choice both for abelian as well as non-abelian gauge theories. However, to see the utility of this gauge we will carry out manipulations in the non-abelian gauge theory. The Fock-Schwinger gauge implies we can write down the gauge field A^a_μ in terms of the field strength $G^a_{\mu\nu}$ as

$$A^a_\nu(x) = \int_0^1 d\alpha \alpha x^\mu G^a_{\mu\nu}(\alpha x). \tag{4.107}$$

It is trivial to see that this gauge field satisfies the Fock-Schwinger gauge condition. However, it is instructive to check this relation explicitly. To do that let us write

$$\begin{aligned} A^a_\mu(y) &= \partial_\mu(A^a_\rho(y)y^\rho) - y^\rho\partial_\mu A^a_\rho(y) \\ &= -y^\rho\partial_\mu A^a_\rho(y) \qquad (4.108) \\ &= -y^\rho G^a_{\mu\rho}(y) - y^\rho\partial_\rho A^a_\mu. \qquad (4.109) \end{aligned}$$

The last relation is true because

$$y^\rho G^a_{\mu\rho}(y) + y^\rho \partial_\rho A^a_\mu \;=\; y^\rho(\partial_\mu A^a_\rho - \partial_\rho A^a_\mu + g f^{abc} A^b_\mu A^c_\rho) + y^\rho \partial_\rho A^a_\mu. \qquad (4.110)$$

The non-linear term vanishes due to gauge choice leaving us with

$$y^\rho G^a_{\mu\rho}(y) + y^\rho \partial_\rho A^a_\mu = y^\rho \partial_\mu A^a_\rho(y). \qquad (4.111)$$

We can now rearrange the equation (4.108) as

$$y^\rho G^a_{\rho\mu}(y) = A^a_\mu(y) + y^\rho \partial_\rho A^a_\mu. \qquad (4.112)$$

Let us now write $y^\mu = \alpha x^\mu$, which allows us to rewrite the equation (4.112) as

$$\alpha x^\rho G^a_{\rho\mu}(\alpha x) = \frac{d}{d\alpha}(\alpha A^a_\mu(\alpha x)). \qquad (4.113)$$

Putting this expression back into (4.107) gives us the identity. Explicit expression for $A^a_\mu(x)$ can be obtained by Taylor expanding the field strength and carrying out integration over α.

$$A^a_\mu(x) = \int_0^1 d\alpha \alpha x^\rho G^a_{\rho\mu}(\alpha x) = \frac{x^\rho G^a_{\rho\mu}(x)}{2} + \frac{x^\beta x^\rho \partial_\beta G^a_{\rho\mu}(x)}{3} + \frac{x^\lambda x^\beta x^\rho \partial_\lambda \partial_\beta G^a_{\rho\mu}(x)}{8} + \cdots \qquad (4.114)$$

Using the gauge condition we can replace ordinary derivatives by covariant derivatives.

$$A^a_\mu(x) = \frac{x^\rho G^a_{\rho\mu}(x)}{2} + \frac{x^\beta x^\rho D_\beta G^a_{\rho\mu}(x)}{3} + \frac{x^\lambda x^\beta x^\rho D_\lambda D_\beta G^a_{\rho\mu}(x)}{8} + \cdots \qquad (4.115)$$

Similarly, Taylor expansion of the fermion field can also be expanded in terms of covariant derivatives

$$\Psi(x) = \Psi(0) + x^\mu D_\mu \Psi(0) + \frac{1}{2} x^\mu x^\nu D_\mu D_\nu \Psi(0) + \cdots \qquad (4.116)$$

Let us now consider the fermion propagator. we will ignore flavor indices on the fermion.

$$S(x,y) = \langle T\{\Psi(x)\bar{\Psi}(y)\}\rangle. \qquad (4.117)$$

The propagator satisfies the Green's function equation

$$(i\gamma^\mu \partial_\mu + g\gamma^\mu A_\mu(x))S(x,y) = i\delta^4(x-y). \tag{4.118}$$

We will use the background field method, *i.e.*, we will fix the classical gauge field background $A_\mu(x) = A^a_\mu(x)T^a$. We cannot determine the propagator exactly, however, we can express it in terms of the free propagator using the Dyson series.

$$S(x,y) = S^{(0)}(x-y) + g\int d^4z\; S^{(0)}(x-z)\not{A}(z)S^{(0)}(z-y) + \cdots \tag{4.119}$$

The free propagator in the coordinate space is given by

$$S^{(0)}(x-y) = \frac{i}{2\pi^2}\frac{\not{x}-\not{y}}{(x-y)^4}. \tag{4.120}$$

This form of the propagator can be obtained by using following identities

- $\frac{1}{\not{\partial}} = -\not{\partial}\frac{1}{\Box} \Rightarrow S^{(0)}(x-y) = -\not{\partial}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2+i\epsilon}$
- $\int_{-\infty}^{\infty} dx \exp(-\frac{x^2}{2}) = \sqrt{2\pi} \Rightarrow$ Volume of $S^3 = 4\pi^2$
- Fourier Transform of $1/k^2$ is $1/x^2$.

This form of the propagator can also be determined by dimensional analysis. We will now choose $A^a_\mu(z) = z^\rho G^a_{\rho\mu}(0)/2$. Higher order terms are regular. Substituting this in the expression of the propagator

$$\begin{aligned} S(x,y) &= S^{(0)}(x-y) \\ &+ \frac{g}{8\pi^2}\int d^4z\; \frac{\not{x}-\not{z}}{(x-z)^4} z^\rho G_{\rho\mu}\gamma^\mu \frac{\not{z}-\not{y}}{(z-y)^4} + \cdots \\ &= \frac{i}{2\pi^2}\frac{\not{x}-\not{y}}{(x-y)^4} + \frac{i}{4\pi^2}\frac{x^\alpha - y^\alpha}{(x-y)^2} g\tilde{G}_{\alpha\beta}\gamma^\beta\gamma_5 + \cdots \end{aligned} \tag{4.121}$$

where $\tilde{G}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}G^{\gamma\delta}$. This result can also be derived using momentum space representation of the propagator and expanding exact formal propagator in terms of free propagator.

Le us now look at the $U(1)$ axial current in this theory.

$$J^\mu_5 = \bar{\Psi}(x)\gamma^\mu\gamma_5\Psi(x). \tag{4.122}$$

We will consider only single fermion flavour and multiply the final result by n_f to accommodate contribution of all fermion flavours. The axial current in the quantum theory is ill-defined due to product of operators at same space-time point. We will use point-splitting regularization method to define $J_5^\mu(x)$.

$$J_5^\mu = \bar{\Psi}(x+\epsilon)\gamma^\mu\gamma_5 \exp\left(ig\int_{x-\epsilon}^{x+\epsilon} A_\rho dy^\rho\right)\Psi(x-\epsilon). \tag{4.123}$$

Let us compute divergence of this current. Using equation of motion it is easy to show that the divergence vanishes except for a contribution coming from the derivative acting on the gauge field in the exponent. We thus get (using $A_\rho = \frac{1}{2}y^\mu G_{\mu\rho}(y)$)

$$\partial_\mu J_5^\mu = \bar{\Psi}(x+\epsilon)\gamma^\mu\gamma_5\epsilon^\beta G_{\mu\beta}(x) \exp\left(ig\int_{x-\epsilon}^{x+\epsilon} A_\rho dy^\rho\right)\Psi(x-\epsilon). \tag{4.124}$$

Let us now evaluate vacuum expectation value of $\partial_\mu J_5^\mu$ in the classical gauge field background.

$$\begin{aligned}
\langle\partial_\mu J_5^\mu\rangle &= \langle\bar{\Psi}(x+\epsilon)\gamma^\mu\gamma_5\epsilon^\beta G_{\mu\beta}(x)\exp\left(ig\int_{x-\epsilon}^{x+\epsilon} A_\rho dy^\rho\right)\Psi(x-\epsilon)\rangle \\
&= -\langle\mathrm{Tr}(ig\gamma^\mu\gamma_5\epsilon^\beta G_{\mu\beta}(x)\Psi(x-\epsilon)\bar{\Psi}(x+\epsilon))\rangle \\
&= -\mathrm{Tr}\left(ig\gamma^\mu\gamma_5\epsilon^\beta G_{\mu\beta}(x)\langle S(x-\epsilon, x+\epsilon)\rangle\right) \\
&= \mathrm{Tr}\left(ig\gamma^\mu\gamma_5\epsilon^\beta G_{\mu\beta}(x)\left\{\frac{1}{2\pi^2}\frac{-2\not{\epsilon}}{(2\epsilon)^4}\right.\right. \\
&\qquad \left.\left. -\frac{ig\epsilon^\alpha}{2\pi^2(2\epsilon)^2}\tilde{G}_{\alpha\rho}\gamma^\rho\gamma_5 + \cdots\right\}\right) \\
&= \frac{g^2}{16\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu},
\end{aligned} \tag{4.125}$$

where, we have used the relation $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ and anticipating the fact that in the point splitting method we eventually take $\epsilon \to 0$ we have retained only non-vanishing terms. We will take $\epsilon \to 0$ limit in such a way that the Lorentz invariance is recovered. This corresponds to taking this limit in a symmetric manner,

$$\frac{\epsilon^\alpha\epsilon^\beta}{\epsilon^2} = \frac{1}{4}g^{\alpha\beta}. \tag{4.126}$$

In this way we get the axial anomaly equation in four dimensional abelian and non-abelian gauge theories. In case of non-abelian gauge theory this anomaly is computed using single fermion flavour. Taking into account contribution of n_f flavours gives

$$\langle\partial_\mu J_5^\mu\rangle = \frac{g^2}{16\pi^2}n_f G^a_{\mu\nu}\tilde{G}^{a\mu\nu}. \tag{4.127}$$

4.7 Path Integral Method

We know how to study quantum mechanics and quantum field theory using canonical operator formalism.We have developed elaborate techniques to compute physically relevant quantities in this formalism and have compared them with laboratory results. We will now briefly introduce path integral methods and use them to compute anomalies. There are several reasons to take resort to the path integral methods. Firstly, operator method is not manifestly Lorentz invariant, although the final result is Lorentz invariant. Secondly operator method becomes cumbersome if the interaction Hamiltonian contains derivative terms. Path integral method is well suited for quantizing non-abelian gauge theories.

We have developed good intuition in classical physics. However, many of these classical physics intuitions encounter problems in quantum theory in the operator formalism due to operator ordering ambiguity, normal ordering, time ordering of operators in the correlations functions etc. A quantization approach which avoids these roadblocks and allows extension of classical intuition to the quantum theory domain is most desirable. This is precisely what is achieved in the path integral method. Of course, this can not be achieved at no cost. In the path integral approach we not only sum over all classical trajectories but we also sum over all other trajectories connecting initial and final point. Advantage of this method is, we work with classical variables.

4.7.1 Path Integral Approach to Quantum Mechanics

The utility of the path integral approach is easy to illustrate in quantum mechanics. We will show that the canonical operator method in quantum mechanics is identical to the path integral method. Let us start with the Hamiltonian operator

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}). \tag{4.128}$$

This is derived from the classical Hamiltonian

$$H = \frac{p^2}{2m} + V(q). \tag{4.129}$$

The corresponding Lagrangian is

$$L = \frac{1}{2}m\dot{q}^2 - V(q). \tag{4.130}$$

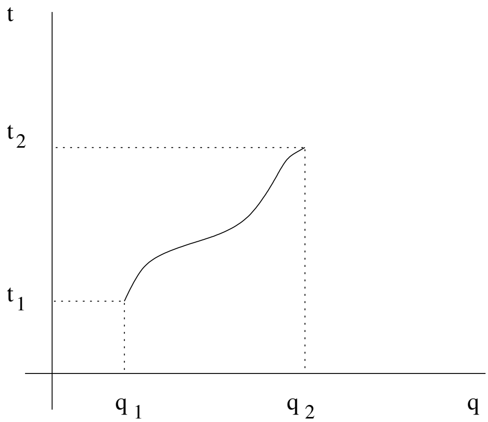

Figure 4.2: Classical trajectory of a particle.

The action associated with a given path $q(t)$ is

$$S = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m\dot{q}^2 - V(q) \right). \tag{4.131}$$

Any path joining q_1 at $t = t_1$ and q_2 at time $t = t_2$ gives a number for the action. Extremization of the action functional gives the classical path. Let us use Heisenberg picture to describe quantum mechanics, *i.e.*, states are time independent and operators are time dependent. Using the Heisenberg equation of motion for an operator $\hat{\mathcal{O}}$,

$$\frac{d\hat{\mathcal{O}}}{dt} = \frac{\partial \hat{\mathcal{O}}}{\partial t} + i[\hat{H}, \hat{\mathcal{O}}], \tag{4.132}$$

we can write

$$\hat{\mathcal{O}}(t) = \exp(i\hat{H}t), \tag{4.133}$$

where for simplicity we have set $\hbar = 1$. Let us define position eigenstates. $|q'\rangle$ and $|q''\rangle$, with eigenvalues q' and q'' respectively. Let us now define the kernel $K(q', t'; q'', t'')$ as

$$K(q', t'; q'', t'') = \langle q''| \exp(-i\hat{H}(t'' - t')|q'\rangle. \tag{4.134}$$

$K(q', t'; q'', t'')$ give the probability amplitude of a state created at a point q' at time t' and measured at a point q'' at time t''. We now claim that

$$K(q', t'; q'', t'') = \mathcal{N} \int [\mathcal{D}q] \exp\left(\frac{iS}{\hbar}\right) \tag{4.135}$$

where $\mathcal{N}$ is the normalization factor and $\int[\mathcal{D}q]$ is a sum over all paths in

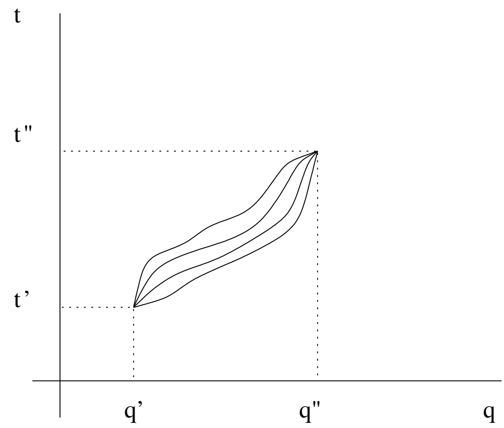

Figure 4.3: Path Integral representation of motion of a quantum mechanical particle.

(q,t) space which begins at (q',t') and end at (q'',t''). We sum over all paths connecting q' and q'' with the weight $\exp(iS/\hbar)$. This sum over paths is carried out by discretizing time interval $(t''-t')$ into N units, $\Delta=(t''-t')/N$, N large but fixed. Using this the action can be written in the discretized form as

$$\begin{aligned} S &= \int_{t'}^{t''} dt \left(\frac{1}{2}m\dot{q}^2 - V(q)\right) \\ &= \Delta \sum_{i=1}^{N} \left\{\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2 - V(q_i)\right\}, \end{aligned} \tag{4.136}$$

where $q_i=q(t_i)$ and $t_i=t'+(i-1)\Delta$. The kernel in the discretized form becomes

$$K(q',t';q'',t'') = \langle q''|\exp(-i\hat{H}N\Delta)|q'\rangle. \tag{4.137}$$

By writing $\exp(-i\hat{H}N\Delta)=\exp(-i\hat{H}\Delta)\cdots\exp(-i\hat{H}\Delta)$ N-times and introducing complete set of position eigenstates between them, we get

$$\begin{aligned} K(q',t';q'',t'') &= \int dq_2\cdots dq_N \langle q''|\exp(-i\hat{H}\Delta)|q_N\rangle \\ &\quad \langle q_N|\cdots|q_2\rangle\langle q_2|\exp(-i\hat{H}\Delta)|q'\rangle. \end{aligned} \tag{4.138}$$

Let us look at one matrix element

$$\langle q_{i+1}|\exp(-i\hat{H}\Delta)|q_i\rangle = \left\langle q_{i+1}\left|\exp\left(-i\left[\frac{\hat{p}^2}{2m}+V(\hat{q})\right]\right)\right|q_i\right\rangle. \tag{4.139}$$

We will now use the following results

$$\begin{aligned}
\exp\left(-i\Delta\left[\frac{\hat{p}^2}{2m}+V(\hat{q})\right]\right) &= \exp\left(-i\Delta\frac{\hat{p}^2}{2m}\right)\exp\left(-i\Delta V(\hat{q})\right) \\
&\times \exp(-o(\Delta^2)) \\
\hat{q}|q_i\rangle &= q_i|q_i\rangle \\
|q_i\rangle &= \int dp|p\rangle\langle p|q\rangle \\
\langle p|q\rangle &= \exp(-ipq) \\
\hat{p}|p\rangle &= p|p\rangle, \quad \langle\tilde{p}|p\rangle = \delta(p-\tilde{p})
\end{aligned}$$

Using these results we can write the matrix element as

$$\left\langle q_{i+1}\left|\exp\left(-i\Delta(\frac{\hat{p}^2}{2m}+V(\hat{q}))\right)\right|q_i\right\rangle = \int dpd\tilde{p}\exp(-i\Delta V(q_i))$$
$$\times\exp\left(-i\Delta\frac{p^2}{2m}\right)\exp(i\Delta\tilde{p}q_{i+1})\delta(p-\tilde{p})\exp(-i\Delta pq_i) \qquad (4.140)$$

Carrying out the integration over the δ-function and using the following identity

$$\int dp\exp\left(-a\frac{p^2}{2}+ip(x-y)\right) = \sqrt{\frac{2\pi}{a}}\exp\left(-\frac{(x-y)^2}{2a}\right), \qquad (4.141)$$

we get

$$\langle q_{i+1}|\exp(-i\hat{H}\Delta)|q_i\rangle = \exp(-i\Delta V(q_i))\exp\left(i\frac{m}{2\Delta^2}(q_{i+1}-q_i)^2\right). \qquad (4.142)$$

Substituting this expression back in the expression for the kernel gives

$$K(q',t';q'',t'') = \mathcal{N}\int dq_2\cdots dq_N$$
$$\exp\left(i\Delta\sum_{i=1}^{N}\left\{\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2 - V(q_i)\right\}\right). \qquad (4.143)$$

The term in the exponent is precisely the discretized form of the action. The final expression for the kernel does not contain any operator. It contains only eigenvalues/numbers. Therefore description in terms of classical action makes sense. In this formalism, vacuum expectation value of time ordered product of operators can be written as

$$\langle q''| \exp(-i\hat{H}t'')T\left(\prod_{i=1}^{n}\hat{q}(t_i)\right)\exp(i\hat{H}t'|q'\rangle$$
$$\int [\mathcal{D}q] \exp(iS)q_n(t_n)\cdots q_1(t_1). \tag{4.144}$$

Note that in the path integral $q_1 \cdots q_n$ are all classical variables. We can order them any which way we want, but when we evaluate the path integral it naturally gives thee time ordered expression.

4.7.2 Path Integral Approach to Quantum Field Theory

Results of quantum mechanics can be generalized to quantum field theory. Quantum mechanical degrees of freedom $\hat{q}_i(t_i)$, $\hat{p}_i(t_i)$ go over to quantum field theoretic degrees of freedom $\hat{\phi}_i(x)$, $\hat{\Pi}_i(x)$. In the path integral picture we replace Lagrangian of the classical mechanical system by the Lagrangian density of the classical field theory.

$$S = \int dtL(q,\dot{q}) \longrightarrow S = \int d^4x\mathcal{L}(\phi(x), \partial_\mu\phi(x)). \tag{4.145}$$

In case of the free scalar field theory we write the path integral as

$$Z_{free} = \int [\mathcal{D}\phi]]exp(iS[\phi]), \tag{4.146}$$

where,

$$S[\phi] = \int d^4x\left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right], \tag{4.147}$$

and $[\mathcal{D}\phi]$ is the integration measure defined over the space of field configurations. Vacuum expectation value of time ordered product of field operators $\phi(x_i)$ is given by

$$\langle T(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n))\rangle = \int [\mathcal{D}\phi] \exp(iS[\phi])\phi(x_1)\cdots\phi(x_n). \tag{4.148}$$

It is worth noting that on the left hand side we have expectation value of product of quantum operators and on the right hand side we have classical action functional and classical fields. In path integral approach we do not need to put in explicit time ordering. This method can be extended to any field theory involving bosonic fields.

Let us now discuss path integral with fermionic fields. The Dirac field satisfies anticommutation relations

$$\{\hat{\psi}_\alpha(\mathbf{x},t),\hat{\psi}_\beta(\mathbf{y},t)\} = 0 \quad (4.149)$$
$$\{\hat{\psi}^\dagger_\alpha(\mathbf{x},t),\hat{\psi}^\dagger_\beta(\mathbf{y},t)\} = 0 \quad (4.150)$$
$$\{\hat{\psi}_\alpha(\mathbf{x},t),\hat{\psi}^\dagger_\beta(\mathbf{y},t)\} = \hbar\delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}). \quad (4.151)$$

Note $\hat{\psi}^\dagger_\alpha(\mathbf{x},t)$ is the momentum conjugate to $\hat{\psi}_\alpha(\mathbf{x},t)$. In the $\hbar \to 0$ limit we find that all anticommutation relations vanish. This is not a regular classical limit, because in the classical limit the functions should have commuted but instead they seem to anticommute. Thus there exists no classical limit of fermions, and the classical theory would be a formal construction. Path integrals for fermions also are formal procedures.

The formal action for a free fermion is

$$S[\psi] = \int d^4x\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x), \quad (4.152)$$

and the path integral is

$$\int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}]\exp(iS[\psi]). \quad (4.153)$$

ψ and $\bar{\psi}$ are anticommuting variables. We need to define the notion of integration over anticommuting variables. Anticommuting variables are called the Grassmann variables. They have following properties.

Suppose θ_i, $(i = 1, \cdots n)$, are n Grassmann variables, then

- Anticommutativity: $\theta_i\theta_j = -\theta_j\theta_i$, $\forall i,j$,
- Suppose $F(\theta)$ is a function of Grassmann variables then it has a finite Taylor series expansion in powers of θs.

$$F(\theta_1,\theta_2,\cdots,\theta_n) = f_0 + \sum_i f^{(i)}\theta_i + \cdots + \sum_{i_1,\cdots i_n} f_n^{(i_1\cdots i_n)}\theta_{i_1}\cdots\theta_{i_n}, \quad (4.154)$$

  where f_i are ordinary numbers. $F(\theta)$ is an even(odd) function if $f_{2m+1}(f_{2m})$ vanish for all m.

- Differentiation:

$$\frac{\partial\theta_i}{\partial\theta_j} = \delta_i^j. \quad (4.155)$$

This implies

$$\frac{\partial}{\partial\theta_i}(FG) = \frac{\partial F}{\partial\theta_i}G + (-1)^F F\frac{\partial G}{\partial\theta_i}, \tag{4.156}$$

where, $(-1)^F$ is 1 if F is an even function and -1 if it is an odd function. Since differentiation anticommutes $\partial^2 F/\partial\theta_i^2 = 0$.

- Integration: We define integration using the property that integration if a total derivative term vanishes.

$$\int d\theta \frac{\partial}{\partial\theta}F(\theta) = 0. \tag{4.157}$$

This implies $\int d\theta F(\theta) = \partial F(\theta)/\partial\theta$. This is effect means

$$\int d\theta = 0, \text{ and } \int d\theta\theta = 1. \tag{4.158}$$

Consider an integral involving ordinary variables

$$\int dx_1 \cdots dx_n f(x_1, \cdots x_n). \tag{4.159}$$

If we make a change of variables $x_i \to y_i = A_{ij}x_j$ then define

$$\int dx_1 \cdots dx_n f(A_{1i}x_i, \cdots A_{ni}x_i). \tag{4.160}$$

Let us not relate these two integrals. To do that let us notice that

$$dy_1 dy_2 \cdots dy_n = (\det A) dx_1 dx_2 \cdots dx_n. \tag{4.161}$$

Thus

$$\int dx_1 \cdots dx_n f(A_{1i}x_i, \cdots A_{ni}x_i) = \\ (\det A)^{-1} \int dy_1 \cdots dy_n f(y_1, \cdots y_n). \tag{4.162}$$

By relabelling y as x we get

$$\int dx_1 \cdots dx_n f(A_{1i}x_i, \cdots A_{ni}x_i) = \\ (\det A)^{-1} \int dx_1 \cdots dx_n f(x_1, \cdots x_n). \tag{4.163}$$

Let us now consider an integral involving Grassmann variables,

$$\int d\theta_m d\theta_{m-1} \cdots d\theta_1 F(\theta_1, \theta_2, \cdots, \theta_m), \tag{4.164}$$

and relate it to

$$\int d\theta_m d\theta_{m-1} \cdots d\theta_1 F(\tilde{A}_{1i}\theta_i, \tilde{A}_{2i}\theta_i, \cdots, \tilde{A}_{mi}\theta_i). \tag{4.165}$$

It is easy to see by explicitly expanding $F(\tilde{A}\theta)$ in terms of the Taylor series that

$$\int d\theta_m d\theta_{m-1} \cdots d\theta_1 F(\tilde{A}_{1i}\theta_i, \tilde{A}_{2i}\theta_i, \cdots, \tilde{A}_{mi}\theta_i) = \pm(\det \tilde{A}) \int d\theta_m d\theta_{m-1} \cdots d\theta_1 F(\theta_1, \theta_2, \cdots, \theta_m). \tag{4.166}$$

Let us also compare the Dirac δ-function for ordinary variables and for Grassmann variables. For ordinary variables

$$\int_{-\infty}^{\infty} dx \delta(x) f(x) = f(0), \tag{4.167}$$

and for Grassmann variables

$$\int d\theta \delta(\theta) F(\theta) = F(0). \tag{4.168}$$

In particular, if $F(\theta) = a + b\theta$ then $F(0) = a$, implying $\delta(\theta) = \theta$.

Let us now define complex Grassmann variables

$$\theta_j = \phi_j + i\psi_j, \text{ and } \theta_j^\dagger = \phi_j - i\psi_j, \tag{4.169}$$

whereϕ_j and ψ_j are real Grassmann variables. Same rules for differentiation and integration extend to complex Grassmann variables.

4.8 Path Integral Formalism for Anomalies

Let us start with the discussion of symmetries and conservation laws. Since the path integral formulation of quantum field theory is in terms of classical action and classical field variables, it is trivial to implement Noether procedure and derive conservation laws corresponding to the symmetries of the action.

However, path integral approach is designed to give us results in the quantum theory. That would imply all classical symmetries and conservation laws would trivially carry over to the quantum theory. If this is so then what is the status of anomalies? How do we derive them from path integral approach?

Although classical action is invariant under symmetry transformations, we have not checked if the integration measure is invariant or not. If the integration measure is not invariant then under symmetry transformation we will get a Jacobian factor. It is this Jacobian factor which can potentially carry information about anomalies.

Having spotted possible location for finding anomalies in the classical symmetries let us proceed with the analysis of the integration measure in gauge theories coupled to fermions. For simplicity, let us consider $SU(N)$ gauge theory coupled to a single Dirac fermion. The Minkowski space action is

$$S = \int d^4x \mathcal{L},\ \mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \bar{\psi}i\not{D}\psi. \tag{4.170}$$

The fermion belongs to the fundamental representation **N** of $SU(N)$. In the path integral approach it is convenient to work in the Euclidean space. This cane be achieved by Wick rotating time direction $x^0 \to -ix^4$ and $A_0 \to iA_4$. We define $i\gamma^0 = \gamma^4$ and $\not{D} = \gamma^i D_i + \gamma^4 D_4$. Like γ^i, γ^4 is antihermitian but $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^2\gamma^3\gamma^4$ is hermitian. The metric on the Euclidean space is $g_{\mu\nu} = diag(-1,-1,-1,-1)$.

To define path integral measure, let us decompose the Dirac field in terms of the complete set of eigenfunctions of $\not{D}$.

$$\psi(x) = \sum_n a_n\phi_n(x),\ \bar{\psi}(x) = \sum_n \phi^\dagger_n(x)\bar{b}_n, \tag{4.171}$$

where,

$$\not{D}\phi_n(x) = \lambda_n\phi_n(x), \tag{4.172}$$

and

$$\int d^4x\phi^\dagger_n(x)\phi_m(x) = \delta_{n,m}, \tag{4.173}$$

where a_n and $\bar{b}_n$ are elements of Grassmann algebra. In terms of this decomposition of ψ, the path integral measure becomes

$$\prod_x [\mathcal{D}A_\mu(x)]\mathcal{D}\psi(x)\mathcal{D}\bar{\psi}(x) = \prod_x [\mathcal{D}A_\mu(x)] \prod_n da_n \prod_m d\bar{b}_m. \tag{4.174}$$

Since we will not be concerned too much with the gauge field measure, we will not bother to define it properly. To derive conserved current corresponding to the chiral transformation, we will follow Noether's prescription. Consider the local chiral transformation.

$$\begin{aligned} \psi(x) \to \psi'(x) &= \exp(i\alpha(x)\gamma_5)\psi(x) & (4.175) \\ \bar{\psi}(x) \to \bar{\psi}'(x) &= \bar{\psi}(x)\exp(i\alpha(x)\gamma_5). & (4.176) \end{aligned}$$

Under this transformation the Lagrangian density of the Dirac field transforms as

$$\bar{\psi} i\gamma^\mu D_\mu \psi \to \bar{\psi} i\gamma^\mu D_\mu \psi - \partial_\mu \alpha(x)\bar{\psi}\gamma^\mu\gamma_5\psi. \quad (4.177)$$

Effect of this chiral transformation on the fermion modes is

$$\psi'(x) = \sum_n a'_n \phi_n(x) = \sum_n a_n e^{i\alpha(x)\gamma_5}\phi_n(x). \quad (4.178)$$

Using this relation and orthogonality of $\phi_n(x)$, we can write a'_n in terms of a_n,

$$a'_m = \sum_n \int d^4x \phi^\dagger_m(x) e^{i\alpha(x)\gamma_5}\phi_n(x) a_n = \sum_n A_{mn} a_n. \quad (4.179)$$

Similarly,

$$\bar{b}'_m = \sum_n \int d^4x \phi^\dagger_n(x) e^{i\alpha(x)\gamma_5}\bar{b}_n \phi_m(x) = \sum_n A_{mn}\bar{b}_n. \quad (4.180)$$

Since $\int d\theta$ is same as $\partial/\partial\theta$,

$$\prod_m da'_m = \frac{1}{(\det A_{mn})}\prod_n da_n \quad (4.181)$$

and

$$\prod_m d\bar{b}'_m = \frac{1}{(\det A_{mn})}\prod_n d\bar{b}_n. \quad (4.182)$$

Therefore,

$$\prod_m da'_m d\bar{b}'_m = \frac{1}{(\det A_{mn})^2}\prod_n da_n d\bar{b}_n. \quad (4.183)$$

Let us now evaluate this determinant for infinitesimal chiral transformation.

$$\begin{aligned} A_{m,n} &= \int d^4x \phi^\dagger_m(x)(1 + i\alpha(x)\gamma_5)\phi_n(x) + \cdots \\ &= \delta_{mn} + \int d^4x i\alpha(x)\phi^\dagger_m(x)\gamma_5\phi_n(x) + \cdots & (4.184) \end{aligned}$$

Thus,

$$[\det A_{m,n}]^{-1} = \det\left[\delta_{mn} + i\int d^4x\alpha(x)\phi^\dagger_m(x)\gamma_5\phi_n(x)\right]^{-1}$$
$$= \exp\left(-\mathrm{Tr}\ln\left[\delta_{mn} + i\int d^4x\alpha(x)\phi^\dagger_m(x)\gamma_5\phi_n(x)\right]\right)$$
$$= \exp\left(-i\sum_n\int d^4x\alpha(x)\phi^\dagger_n(x)\gamma_5\phi_n(x)\right). \quad (4.185)$$

As a result we find

$$\prod_m da'_m d\bar{b}'_m = \prod_n da_n d\bar{b}_n e^{-2i\sum\int d^4x\alpha(x)\phi^\dagger_n(x)\gamma_5\phi_n(x)}. \quad (4.186)$$

Thus the Jacobian factor is

$$\exp\left(-2i\sum\int d^4x\alpha(x)\phi^\dagger(x)\gamma_5\phi_n(x)\right) = \exp\left(-2i\int d^4x\alpha(x)A(x)\right). \quad (4.187)$$

We will evaluate this Jacobian by regularizing the term in the exponent. For global chiral transformations $\alpha(x)$ is independent of coordinates and can be pulled out of the integral.

Let us now look at the infinite sum in the exponent,

$$A(x) = \sum_n \phi^\dagger_n(x)\gamma_5\phi_n(x) = \lim_{M\to\infty}\left(\sum_n \phi^\dagger_n(x)\gamma_5 e^{-(\lambda_n/M)^2}\phi_n(x)\right). \quad (4.188)$$

We regularize the infinite sum by introducing the Gaussian cut off. This not only gives a smooth cut off for eigenvalues $\lambda_n > M$ but also maintains the gauge invariance. For simplicity we will change basis vectors from $\phi_n(x)$ to plane wave basis, *i.e.*, $e^{ik\cdot x}$

$$A(x) = \lim_{M\to\infty}\left(\sum_n \phi^\dagger_n(x)\gamma_5 e^{-(\not{D}/M)^2}\phi_n(x)\right)$$
$$= \lim_{M\to\infty}\left(\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\gamma_5 e^{-ik\cdot x}e^{-(\not{D}/M)^2}e^{ik\cdot x}\right). \quad (4.189)$$

Using the expansion $\not{\Pi}\not{\Pi} = \Pi^2 + [\gamma^\mu,\gamma^\nu]G_{\mu\nu}/2$ and $\Pi_\mu = (ik_\mu + A_\mu(x))$,

$$A(x) = \lim_{M\to\infty}\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\gamma_5 e^{-\{2(ik_\mu+A_\mu)^2+[\gamma^\mu,\gamma^\nu]G_{\mu\nu}\}/2M^2}. \quad (4.190)$$

We need to pull down $[\gamma^\mu, \gamma^\nu]$ factors enough number of times to get non-zero trace. Since A_μ is not relevant in trace manipulations, we will ignore it. We will then be left with a Gaussian integral over k.

$$\begin{aligned} A(x) &= \lim_{M\to\infty} \mathrm{Tr}\gamma_5([\gamma^\mu, \gamma^\nu]G_{\mu\nu})^2 \frac{1}{(2M^2)^2}\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} e^{-k^\mu k_\mu/M^2} \\ &= \frac{1}{16\pi^2}\mathrm{Tr}G_{\mu\nu}\tilde{G}^{\mu\nu}, \qquad \tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}. \end{aligned} \tag{4.191}$$

We will now substitute $A(x)$ back into the Jacobian factor. The Jacobian factor is

$$e^{-2\alpha\int d^4x A(x)} = e^{\frac{i\alpha}{8\pi^2}\int d^4x \mathrm{Tr}G_{\mu\nu}\tilde{G}^{\mu\nu}}. \tag{4.192}$$

Total variation of the path integral is

$$\begin{aligned} &\int [\mathcal{D}A_\mu(x)]\mathcal{D}\psi\mathcal{D}\bar{\psi}\exp(-S[A,\psi,\bar{\psi}]) \to \\ &\int [\mathcal{D}A_\mu(x)]\mathcal{D}\psi'\mathcal{D}\bar{\psi}'\exp(-S[A,\psi',\bar{\psi}']) \\ = &\int [\mathcal{D}A_\mu]\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-S[A,\psi,\bar{\psi}]-\int d^4x\partial_\mu\alpha(x)\bar{\psi}\gamma^\mu\gamma_5\psi} \\ &\times e^{\frac{i}{8\pi^2}\int d^4x\alpha(x)(\mathrm{Tr}G_{\mu\nu}\tilde{G}^{\mu\nu})}. \end{aligned} \tag{4.193}$$

Thus we get the anomalous conservation law

$$\partial_\mu J_5^\mu(x) = -\frac{i}{8\pi^2}\mathrm{Tr}G_{\mu\nu}\tilde{G}^{\mu\nu}, \tag{4.194}$$

where, $J_5^\mu(x) = \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$.

Analytic continuation from the Euclidean space back to the Minkowski space gets rid of i factor in front of the anomaly term. The imaginary factor has important implications in the Euclidean version of the theory. Another point to note is that the exponent of the Jacobian factor is given by

$$A(x) = \sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x). \tag{4.195}$$

The basis vectors $\phi_n(x)$ satisfy the Dirac equation with eigenvalue λ_n

$$\not{D}\phi_n(x) = \lambda_n\phi_n(x). \tag{4.196}$$

Multiplying this equation by γ_5 we get

$$\not{D}\gamma_5\phi_n(x) = -\lambda_n\gamma_5\phi_n(x). \tag{4.197}$$

Thus for every eigenvector $\phi_n(x)$ with eigenvalue λ_n, there exists an eigenvector $\gamma_5\phi_n(x)$ with eigenvalue $-\lambda_n$. Therefore $\phi_n(x)$ and $\gamma_5\phi_n(x)$ are orthogonal to each other. This implies

$$\int d^4xA(x) = \int d^4x \sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x) = 0, \tag{4.198}$$

except for the zero modes, *i.e.*, when $\lambda = 0$. When $\lambda = 0$, we can rewrite $A(x)$ as

$$\begin{aligned}\int d^4xA(x) &= \int d^4x \sum_n \phi_n^\dagger(x)\gamma_5\phi_n(x) \\ &= \int d^4x \left(\sum_{i=1}^{n_+} \phi_{iR}^\dagger(x)\phi_{iR}(x) - \sum_{i=1}^{n_-} \phi_{iL}^\dagger(x)\phi_{iL}(x)\right),\end{aligned} \tag{4.199}$$

where $\phi_{L(R)}$ are left handed (resp. right handed) zero modes, and

$$\int d^4xA(x) = n_+ - n_-. \tag{4.200}$$

In other words, the anomaly term is equal to the number of positive chirality zero-modes minus the number of negative chirality zero-modes. This is the Atiyah-Singer index theorem.

Yet another point to notice is that computation of the Jacobian factor gives the anomaly term in any even space-time dimensions. Number of factors of $[\gamma^\mu, \gamma^\nu]G_{\mu\nu}$ that we pull down depends on dimensionality of space-time or equivalently on the definition of γ_5. It is also easy to see that in two dimensions non-ablelian gauge theory cannot have anomaly because $\mathrm{Tr}G_{\mu\nu} = 0$.

Let us now consider a theory with parity violating gauge couplings. The Lagrangian density is

$$\mathcal{L} = \bar{\psi}_L(x)i\not{D}\psi_L(x) - \frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu}, \tag{4.201}$$

where $\psi_L = (1-\gamma_5)\psi/2$, ψ_L belongs to representation $\mathbf{N}$ of $SU(N)$. Using the fact that $\gamma_5\phi_n(x)$ has eigenvalue $-\lambda_n$ if ϕ_n has eigenvalue λ_n, we can decompose the eigenvectors into left and right chirality modes as

$$\phi_{nL}(x) = \frac{1-\gamma_5}{\sqrt{2}}\phi_n(x), \text{ for } \lambda_n > 0 \quad (4.202)$$

$$= \frac{1-\gamma_5}{2}\phi_n(x) \text{ for } \lambda_n = 0 \quad (4.203)$$

$$\phi_{nR}(x) = \frac{1+\gamma_5}{\sqrt{2}}\phi_n(x), \text{ for } \lambda_n > 0 \quad (4.204)$$

$$= \frac{1+\gamma_5}{2}\phi_n(x) \text{ for } \lambda_n = 0. \quad (4.205)$$

Using these modes we can decompose the chiral fermion as

$$\psi_L(x) = \sum_{\lambda_n \geq 0} a_n \phi_{nL}(x) \quad (4.206)$$

$$\psi_R(x) = \sum_{\lambda_n \geq 0} \bar{b}_n \phi^\dagger_{nR}(x). \quad (4.207)$$

Under global $U(1)$ chiral transformation,

$$\psi_L(x) \to e^{-i\alpha(x)}\psi_L(x) \quad (4.208)$$

$$\bar{\psi}_L(x) \to \bar{\psi}_L(x) e^{i\alpha(x)}, \quad (4.209)$$

where we have kept α to be x dependent only to carry out the Noether prescription. The change in the Lagrangian density due to this transformation is

$$\mathcal{L} \to \mathcal{L} + \partial_\mu \alpha \bar{\psi}_L \gamma^\mu \psi_L. \quad (4.210)$$

The integration measure also changes under this transformation. It is now easy to see that the Jacobian factor is

$$\exp\left(i\int d^4x \alpha(x) \sum_{\lambda_n \geq 0} [\phi^\dagger_{nL}(x)\phi_{nL}(x) - \phi^\dagger_{nR}(x)\phi_{nR}(x)]\right)$$

$$= \exp\left(-i\int d^4x \alpha(x) \sum_{\lambda_n} \phi^\dagger_n(x)\gamma_5\phi_n(x)\right)$$

$$= \exp\left(-i\int d^4x \alpha(x) A(x)\right). \quad (4.211)$$

This phase factor is half of the factor obtained with the Dirac fermion. If we define the current

$$J^\mu_L = \bar{\psi}_L \gamma^\mu \psi_L, \quad (4.212)$$

then

$$\partial_\mu J^\mu_L = -\frac{i}{16\pi^2}\mathrm{Tr}G_{\mu\nu}\tilde{G}^{\mu\nu}. \tag{4.213}$$

It is now trivial to extend this result by replacing abelian chiral transformation by non-abelian chiral transformations. Consider a chiral transformation,

$$\psi_L(x) \to \exp(-i\alpha^a(x)T^a)\psi_L(x), \tag{4.214}$$

where, T^a are generators of the gauge group G. The classically conserved current in this case is

$$J^a_\mu(x) = \bar{\psi}_L(x)\gamma_\mu T^a \psi_L(x). \tag{4.215}$$

Since the generator T^a does not affect our computation except for contributing to group theory trace, it is easy to write down the anomaly factor

$$A^a(x) = \sum_n \phi^\dagger_n(x)\gamma_5 T^a \phi_n(x) = \frac{1}{2}\left(\frac{-1}{8\pi^2}\right)\mathrm{Tr}(T^a G_{\mu\nu}\tilde{G}^{\mu\nu}). \tag{4.216}$$

Due to Bose symmetry of gauge bosons the anomaly factor can be written as

$$A^a(x) = \frac{1}{4}\left(\frac{-1}{8\pi^2}\right) G^b_{\mu\nu}\tilde{G}^{d\mu\nu}\mathrm{Tr}(T^a\{T^b, T^d\}). \tag{4.217}$$

This is called the gauge anomaly. It is now easy to see that this anomaly vanishes for $SU(2)$ gauge theory. For $SU(2)$ theory

$$\{T^b, T^d\} = 2\delta^{bd} \Rightarrow \mathrm{Tr}(T^a\{T^b, T^d\}) = \mathrm{Tr}(T^a) = 0. \tag{4.218}$$

Let us now look at the Standard Model of particle physics. This model is based on a gauge theory with gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. Of these $SU(3)_c$ is not a chiral gauge theory and hence is free from anomalies. $SU(2)_L \otimes U(1)_Y$ theory can be potentially anomalous.

However, we will see that for the anomaly to cancel we will get constraints on the matter content of the theory. Let us look at the fermionic matter content of the Standard Model and their quantum numbers.

- Quarks

$$\begin{pmatrix} u \\ d \end{pmatrix}^{Y=1/3}_L, \begin{pmatrix} c \\ s \end{pmatrix}^{Y=1/3}_L, \begin{pmatrix} t \\ b \end{pmatrix}^{Y=1/3}_L \tag{4.219}$$

$$u_R(Y=4/3),\ d_R(Y=-2/3),\ c_R(Y=4/3),$$
$$s_R(Y=-2/3),\ t_R(Y=4/3),\ b_R(Y=-2/3). \tag{4.220}$$

- Leptons

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L^{Y=-1}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L^{Y=-1}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L^{Y=-1} \tag{4.221}$$

$$e_R(Y=-2),\ \mu_R(Y=-2),\ \tau_R(Y=-2). \tag{4.222}$$

Let us look at only one generation of leptons and one generation of quarks. Result obtained in this case generalize naturally to three generations. We will see that the Standard Model anomalies cancel in each generation provided quarks come in three colours. Potentially anomalous traces in the Standard Model are

$$\mathrm{Tr}(Y^3), \text{ and } \mathrm{Tr}(\{T^a, T^b\}Y), \tag{4.223}$$

where, T^a are generators of $SU(2)_L$ and Y is a generator of $U(1)_Y$. There are two more traces but they do not contribute due to tracelessness of $SU(2)$ generators and the fact that every member of $SU(2)_L$ multiplet has same Y quantum numbers. Let us now concentrate on $\mathrm{Tr}(\{T^a, T^b\}Y)$ term. Due to the fact that for $SU(2)$ group $\{T^a, T^b\} = 2\delta^{ab}$, we get

$$\mathrm{Tr}(\{T^a, T^b\}Y) = 2\delta^{ab}\mathrm{Tr}(Y). \tag{4.224}$$

It is now easy to see that hypercharges of u, d quarks when added up give $Y_q = -Y_l/3$, where Y_q is the total hypercharge of quarks in one generation and Y_l is the total hypercharge of leptons in one generation.

$$Y_q = \frac{1}{3} + \frac{1}{3} + \frac{4}{3} + \frac{-2}{3} = \frac{4}{3} \tag{4.225}$$

$$Y_l = -1 - 1 - 2 = -4. \tag{4.226}$$

Thus hypercharge anomaly cancels if quarks come in three colours. That is

$$3Y_q + Y_l = 0. \tag{4.227}$$

Let us now look at $\mathrm{Tr}(YYY)$ anomaly. First of all notice that hypercharge gauge field $B_\mu(x)$ does not couple hypercharged matter through vector coupling. For example, coupling of $B_\mu(x)$ to left handed electron is different from coupling to right handed electron.

$$D_\mu e_L = \left(\partial_\mu + i\frac{g'}{2}B_\mu\right) e_L \tag{4.228}$$

$$D_\mu e_R = (\partial_\mu + ig'B_\mu)e_R. \tag{4.229}$$

We will split this interaction into vector and chiral coupling. We will choose the vector coupling in such a way that chiral coupling involves only right handed fields. Of course, this is purely a matter of choice. It is always possible to adjust vector coupling so that chiral coupling are purely left handed. The latter choice is more physical as we will see below.

For anomaly computation, this splitting means we can write $Y = Y_V + Y_R$. Let us do this assignment for the first fermion generation.

$$Y_V^q : Y_V^u = \frac{1}{3},\, Y_V^d = \frac{1}{3};\;\; Y_V^l : Y_V^\nu = -1,\, Y_V^e = -1 \tag{4.230}$$

$$Y_R^q : Y_R^u = 1,\, Y_R^d = -1;\;\; Y_R^l : Y_R^\nu = 1,\, Y_R^e = -1. \tag{4.231}$$

Substituting this in pure hypercharge anomaly term gives

$$\begin{aligned}\mathrm{Tr}(YYY) &= \mathrm{Tr}\left((Y_V + Y_R)(Y_V + Y_R)(Y_V + Y_R)\right) \\ &= \mathrm{Tr}(Y_V Y_V Y_V) + 3(\mathrm{Tr}(Y_V Y_R^2) \\ &\quad + \mathrm{Tr}(Y_V^2 Y_R)) + \mathrm{Tr}(Y_R^3).\end{aligned} \tag{4.232}$$

However, we know that the triangle diagram with three vector current insertions is not anomalous. We are thus left with

$$\mathrm{Tr}(YYY) = 3(\mathrm{Tr}(Y_V Y_R^2) + \mathrm{Tr}(Y_V^2 Y_R)) + \mathrm{Tr}(Y_R^3). \tag{4.233}$$

It is trivial to see that $\mathrm{Tr}(Y_R^3)$ cancels within quark generation ans lepton generation separately. However, the term $(\mathrm{Tr}(Y_V Y_R^2 + Y_R Y_V^2))$ cancel between a quark generation and a lepton generation provided there are three coloured quarks.

$$\mathrm{Tr}(Y_V Y_R^2 + Y_R Y_V^2)_q = \left(\frac{1}{9} - \frac{1}{9} + \frac{1}{3} + \frac{1}{3}\right) = \frac{2}{3} \tag{4.234}$$

$$\mathrm{Tr}(Y_V Y_R^2 + Y_R Y_V^2)_l = -1 - 1 = -2 \tag{4.235}$$

$$3\mathrm{Tr}(Y_V Y_R^2 + Y_R Y_V^2)_q + \mathrm{Tr}(Y_V Y_R^2 + Y_R Y_V^2)_l = 0. \tag{4.236}$$

Let us now look at the Standard Model anomaly cancellation from low energy point of view. This would be a check of 't Hooft's anomaly matching condition. At low energy we are left with the quantum electrodynamics. This theory has only vector coupling and we know that a theory with vector coupling does not have gauge anomalies. This may seem like a trivial result but if er demand 't Hooft's anomaly matching condition and turn the argument on its head, we would say that the theory defined in the ultraviolet limit better be an anomaly free theory because QED is free from gauge anomalies.

Although it is trivial to see that the infrared theory is anomaly free, it is still instructive to see how that affects the anomaly cancellation in the Standard Model. To do this we will split the hypercharge gauge coupling into vector coupling and left handed coupling. This is a familiar decomposition. This tells is how electric charge is related to the third component of $SU(2)_L$ generator and the hypercharge.

$$Q = T_3 + \frac{Y}{2} \Rightarrow Y = 2(Q - T_3). \tag{4.237}$$

Since T_3 is a purely left handed charge and Q is purely vector charge, this gives us the desired decomposition of the hypercharge. With this decomposition, the Standard Model anomaly cancellation is the statement that if quarks have three colours then the Standard Model fermionic matter is 'electrically neutral' in each generation, *i.e.*, the sum of electric charges of all fermions in a given generation vanishes. To see the relation between these two statements, let us proceed with the analysis of the gauge anomalies.

The first kind of term is $\mathrm{Tr}(Y) = 2\mathrm{Tr}(Q - T_3) = 2\mathrm{Tr}(Q)$. Total charge in the quark sector $(u, d) = 1/3$ and total charge in the lepton sector $(\nu_e, e) = -1$. This implies $\mathrm{Tr}(Q) = 0$ only if quarks come in three colours.

The second type of anomaly is $\mathrm{Tr}(Y^3)$.

$$\mathrm{Tr}(Y^3) = 2\mathrm{Tr}(Q^3 - 3Q^2 T_3 + 3QT_3^2 - T_3^3). \tag{4.238}$$

Of these terms we already know that $\mathrm{Tr}(Q^3) = 0$ because the vector coupling is not anomalous. We also know that $\mathrm{Tr}(T_3^3) = 0$ due to the tracelessness of odd powers of T_3. Thus we are left with

$$\mathrm{Tr}(Y^3) = -24\mathrm{Tr}(QT_3(Q - T_3)) = -12\mathrm{Tr}(QT_3 Y). \tag{4.239}$$

Now using the fact that $Q = T_2 + Y/2$, we can write

$$\mathrm{Tr}(Y^3) = -12\mathrm{Tr}(T_3^2 Y) - 6\mathrm{Tr}(T_3 Y^2). \tag{4.240}$$

The second term in eq.(4.240) vanishes because T_3 is traceless and that the hypercharge of all the members of a given $SU(2)_L$ multiplet is same. Thus we are reduced only to one term and since $T_3^2 = 1_{2\times 2}$,

$$\begin{aligned} \mathrm{Tr}(Y^3) &= -12\mathrm{Tr}(Y) = -12\mathrm{Tr}(2(Q - T_3)) \\ &= -24\mathrm{Tr}(Q) = 0. \end{aligned} \tag{4.241}$$

Thus we have seen that the Standard Model anomaly cancellation means that the fermionic matter of the Standard Model is 'electrically neutral' when all charges of the fermions is a given generation are added up.

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