3

Perturbative Quantum Chromodynamics

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3.1 Structure of Hadrons

Quantum Chromodynamics (QCD) is the theory of strong interaction force among hadrons. It is a gauge theory based on a non-Abelian gauge group namely $SU(3)$. In the following, I will describe the perturbative aspects of QCD that is relevant for studying high energy scattering processes involving hadrons.

Strong interaction force is responsible for binding the nucleons inside the nucleus. It is a short range force which is effective within few Fermi (of the order of $10^{-13}cm$). Thus, the typical cross section for the process mediated by strong interaction is of the order of square of few Fermi which is $10^{-26}cm^2$ (10 milli-barn (mb)). The characteristic energy scale of strong interaction force is of the order of few hundred million electron volt (MeV) and the life time of any excitation will be around inverse of few hundred MeV. However, its

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interaction strength is several hundred times larger than those of weak (wk) and electromagnetic (em) forces($\alpha \approx 1/137$).

Hadrons such as baryons (proton, neutron, Λ, Δ, Ω, etc. having 1/2 integer spins) and mesons ($\pi^0, \pi^\pm, K, \rho$ having integer spins) being composite objects are classified in terms of their constituents: quarks and anti-quarks. They are spin-1/2 point-like particles carrying fractional charges. There are six types of quarks with different flavour quantum numbers denoted by up (u), down (d), charm (c), strange (s),top (t) and bottom (b). The u, c, t quarks carry 2/3 and d, s, b carry $-1/3$ units of electron charge. In addition to flavour quantum number, these quarks carry three colour quantum numbers, namely red (R), blue (B) and green (G). We denote them by states namely q_i^f where $f = u, d, c, s, t, b$ and $i = R, B, G$. These states q_i^f transform like a vector in the fundamental representation of $SU(n_f)$ group, called flavour group with n_f number of flavours. $SU(n_f)$ is a set of $n_f \times n_f$ unitary matrices denoted by U satisfying the condition $detU = 1$. These transformations are space-time independent, usually called global or phase transformations. In addition, these states transform like a vector under $SU_c(3)$ group, called colour group. Hadrons by themselves can carry definite flavour quantum number and hence the hadronic wave functions can be non-singlets under $SU(n_f)$ transformation. On the other hand, there is so far no experimental evidence for a hadron with non-zero colour quantum number. Hence, hadronic wave functions are always singlets under $SU_c(3)$ transformations. Mesonic states can be obtained by combining quark and anti-quark states, i.e., $\sum_i q_i^{f_1} \overline{q}_i^{f_2}$ can be a meson with an effective flavour quantum number obtained using f_1, f_2 and they are colour singlets. Baryonic states are obtained by combining three quark states, i.e., $\sum_{ijk} \epsilon_{ijk} q_i^{f_1} q_j^{f_2} q_k^{f_3}$ where ϵ_{ijk} is anti-symmetric tensor in i, j, k. They are again colour singlets with definite flavour. The anti-symmetrization of colour indices in the baryonic wave functions is needed in order to preserve the Pauli exclusion principle in the states with three spin-1/2 quarks.

Though, the static properties of hadrons can be obtained using the flavour quantum numbers of the their constituents, the nature of strong interaction force can not be explained by models based only on global continuous symmetries such as $SU(n_f)$. Understanding the dynamics of the strong interaction force in terms of the constituents is an important task in hadronic physics. The task is to look for a suitable gauge theory that describes the dynamics of these constituents and also the mechanism behind the binding force.

The crucial inputs to construct a suitable theory of strong interaction force come from various elastic and inelastic experiments involving hadrons.

Elastic scattering of lepton on a hadron provides low energy description of hadrons, namely the electric and magnetic charge distributions inside the hadrons. Consider an elastic scattering process:

$$e^-(k) + P(p) \rightarrow e^-(k') + P(p') \tag{3.1}$$

where the incoming electron e^- and proton P carry momenta k and p respectively, k' and p' are their momenta after the scattering. The scattering takes place by exchanging a virtual photon of momentum $q = k' - k$ which is space-like ($q^2 < 0$). This is the lowest order process in quantum electrodynamics (QED) where photon interacts with charged particles. The interaction vertex of the photon with the electron and its propagator are known from QED. It is given by $-iej_\mu A^\mu$ where j_μ is the electro-magnetic current of the electron and A^μ is the photon field. In QED, the electromagnetic current is given by $j_\mu = \overline{\psi}\gamma_\mu\psi$ where, ψ is the wave function of the electron. On the other hand the wave function of the proton and proton-photon interaction vertex are not known. They can be obtained by first modeling them based on the symmetries and then by fitting against the experiments. In other words, one first parameterises the current of the hadron that couples to the photon in terms of trial wave functions denoted by $\Psi(p)$ and $\Psi(p')$ and a set of form factors $\widetilde{F}_i(q^2), (i = 1, 2)$ multiplying suitable vectors constructed out of $p_\mu, p'_\mu, \gamma_\mu$. In momentum space, the typical interaction term is given by

$$e\widetilde{A}^\mu(q)\overline{\Psi}(p')\left[\widetilde{F}_1(Q^2)\ \gamma_\mu + \frac{\kappa}{2M_P}\widetilde{F}_2(Q^2)\ i\sigma_{\mu\nu}q^\nu\right]\Psi(p) \tag{3.2}$$

where $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$, κ the anomalous magnetic moment and M_P the mass of the proton. The scalar functions $\widetilde{F}_i(Q^2)$ parameterise the structure of the hadron in terms of the scale $Q^2 = -q^2$. The elastic cross section is found to be

$$\begin{aligned}\frac{d^2\sigma}{d\Omega_e dE'} \quad &= \quad \frac{4\alpha^2 E'^2}{q^4}\left\{\frac{G_E^2(Q^2) + \frac{Q^2}{4M_P^2}G_M^2(Q^2)}{1 + \frac{Q^2}{4M_P^2}}\cos^2\frac{\theta}{2}\right. \\ &\left. -\frac{Q^2}{2M_P^2}G_M^2(Q^2)\sin^2\frac{\theta}{2}\right\}\delta\left(\nu - \frac{Q^2}{2M_P}\right)\end{aligned} \tag{3.3}$$

where $d\Omega_e$ is the solid angle of the scattered electron in the laboratory frame, E' its energy. We have

$$\begin{aligned}
\alpha &= \frac{e^2}{4\pi}, \quad Q^2 = 4EE'\sin^2\frac{\theta}{2}, \quad \nu = p.q/M_P \\
G_E(Q^2) &= \widetilde{F}_1(Q^2) - \frac{\kappa Q^2}{4M_P^2}\widetilde{F}_2(Q^2) \\
G_M(Q^2) &= \widetilde{F}_1(Q^2) + \kappa\widetilde{F}_2(Q^2)
\end{aligned} \tag{3.4}$$

where κ depends on the magnetic moment. The elastic form factors ($\widetilde{F}_i(Q^2)$, equivalently $G_i(Q^2), i = E, M$) describe the electric charge and magnetic moment distributions of the proton as a function of a scale denoted by Q of the photon that probes the proton. Experimentally, one finds that $G_E(Q^2)$ and $G_M(Q^2)/(1+\kappa)$ decrease as $1/Q^4$ when Q increases. This implies the elastic scattering cross section falls off rapidly at large angles. The distribution of these charges in terms of energy easily translates to a spatial picture of the proton.

We will now study a different kind of experiment called (deep) inelastic scattering in which the proton is bombarded with very high energetic photon that breaks the proton into pieces. That is, we consider $e^-(k)+P(p) \to e^-(k')+X(p_X)$ where X are final state hadrons carrying momentum denoted by p_X. We restrict ourselves to inclusive cross section where all the final states but the scattered lepton are summed over. To lowest order in em, the differential cross section can be written as a product of leptonic part $\mathcal{L}_{\mu\nu}$, and a hadronic part $W_{\mu\nu}$:

$$\frac{d^2\sigma}{d\Omega_e dE'} = \frac{E'}{E}\mathcal{L}_{\mu\nu}(k,q)\frac{\alpha^2}{Q^4}W^{\mu\nu}(q,p) \tag{3.5}$$

where

$$\mathcal{L}_{\mu\nu}(k,q) = \frac{1}{2}\sum_{s1,s2}\left(\overline{u}(k',s_2)\gamma_\mu u(k,s_1)\right)\left(\overline{u}(k',s_2)\gamma_\nu u(k,s_1)\right)^* \tag{3.6}$$

$$\begin{aligned}
W_{\mu\nu}(q,p) &= \frac{1}{8M_P\pi}\sum_{p_X,s}\langle p,s|J_\mu(0)|p_X\rangle\langle p_X|J_\nu(0)|p,s\rangle \\
&\quad (2\pi)^4\delta^{(4)}(q+p-p_X)
\end{aligned} \tag{3.7}$$

The lepton part is fully computable in QED. On the other hand the hadronic part requires the knowledge of the matrix element of electromagnetic current J_μ between proton states. J_μ is Hermitian and conserved. Using translational invariance, $J_\mu(x) = e^{i\hat{p}\cdot x} J_\mu(0) e^{-i\hat{p}\cdot x}$ and the completeness relation $\sum_{p_X} |p_X\rangle\langle p_X| = 1$,

$$W_{\mu\nu}(q,p) = \frac{1}{4M_P\pi}\int d^4x e^{iq\cdot x}\frac{1}{2}\sum_s \langle p,s|J_\mu(x)J_\nu(0)|p,s\rangle. \tag{3.8}$$

Since

$$\int d^4x e^{iq\cdot x}\sum_s \langle p,s|J_\nu(x)J_\mu(0)|p,s\rangle = 0 \tag{3.9}$$

which follows from energy conservation along with the condition $q^0 > 0$, we find

$$W_{\mu\nu}(q,p) = \frac{1}{4M_P\pi}\int d^4x e^{iq\cdot x}\frac{1}{2}\sum_s \langle p,s|\Big[J_\mu(x), J_\nu(0)\Big]|p,s\rangle \tag{3.10}$$

This commutator vanishes for $x^2 < 0$, so the integral has support only for $x^2 > 0$. To proceed further with the hadronic tensor $W_{\mu\nu}(q,p)$, we exploit the symmetries at our disposal such as Lorentz covariance (that is, second rank nature) of $W_{\mu\nu}(q,p)$, $q_\mu W^{\mu\nu}(q,p) = q_\nu W^{\mu\nu}(q,p)$ that follows from the current conservation $\partial_\mu J^\mu(x) = 0$ and finally parity and time reversal invariance of the interaction. To this end we parameterise the hadronic tensor as

$$\begin{aligned} W_{\mu\nu}(q,p) &= \Big(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\Big)W_1(q^2,p^2,p\cdot q) \\ &\quad + \Big(p_\mu - \frac{p\cdot q}{q^2}q_\mu\Big)\Big(p_\nu - \frac{p\cdot q}{q^2}q_\nu\Big)\frac{1}{M_P^2}W_2(q^2,p^2,p\cdot q) \end{aligned} \tag{3.11}$$

where $W_i, i = 1,2$ are called structure functions which are functions of Lorentz invariants $q^2 = -Q^2, p^2$ and $p\cdot q$. Since $p^2 = M_P^2$, we suppress obvious p^2 dependence in the rest of the analysis. The summation over spin in the leptonic tensor gives traces over gamma matrices which can be easily evaluated. Substituting the resultant $\mathcal{L}_{\mu\nu}$ and $W_{\mu\nu}$ in eqn.(3.5), we get in the laboratory frame,

$$\frac{d^2\sigma}{d\Omega_e dE'} = \frac{4\alpha^2 E'^2}{Q^4}\left[W_2(Q^2,p\cdot q)\cos^2\frac{\theta}{2} + 2\ W_1(Q^2,p\cdot q)\sin^2\frac{\theta}{2}\right] \tag{3.12}$$

In the above formula, the structure functions W_i $(i = 1, 2)$ are still unknowns. Using this formula and measuring the differential cross section, these structure functions can be extracted for various values of Q^2 and $p \cdot q$. Alternatively, the qualitative feature of these functions can be obtained by studying them in the infinite momentum frame. The study of the hadronic tensor in the infinite momentum frame where p_z component of the proton tends to very large value(say ∞) reveals much simplified form for these structure functions. In particular, when $Q^2 \to \infty$ with the ratio $Q^2/2p \cdot q$ fixed, usually called Björken limit (denoted by Bj), one finds

$$\begin{aligned} \lim_{Bj} M_P W_1(Q^2, p \cdot q) &= F_1(x_{Bj}) \\ \lim_{Bj} \frac{p \cdot q}{M_P} W_2(Q^2, p \cdot q) &= F_2(x_{Bj}) \end{aligned} \tag{3.13}$$

where $x_{Bj} = Q^2/2p{\cdot}q$. In the Björken limit, the structure functions are no longer functions of two invariants Q^2 and $p \cdot q$ but the ratio $x_{Bj} = Q^2/2p \cdot q$,called Björken variable. The deep inelastic scattering cross section following form

$$\lim_{Bj} \frac{d^2\sigma}{d\Omega_e dE'} = \frac{4\alpha^2 E'^2}{Q^4} \left[\frac{M_P}{p \cdot q} F_2(x_{Bj}) \cos^2 \frac{\theta}{2} + \frac{2}{M_P} \; F_1(x_{Bj}) \sin^2 \frac{\theta}{2} \right] \tag{3.14}$$

implying "scaling" behavior of appropriately normalised cross section in terms of the the variable x_{Bj}. Such a scaling is called Björken scaling and deep inelastic scattering experiments at SLAC, Stanford confirmed it. We will come back to the physical interpretation of this scaling after we study the hadronic tensor in the Björken limit using a more rigorous approach called operator product expansion (OPE).

3.2 Operator Product Expansion and Parton Model

We have already seen that the hadronic tensor $W_{\mu\nu}(q, p)$ has support only for $x^2 > 0$. Now we will show that the dominant contribution to the hadronic tensor in the Björken limit comes from the light-cone region $x^2 = 0$. Let us first find out how this limit can be applied to the integral in eqn.(3.10). Note that

$$q \cdot x \approx \frac{p \cdot q}{M_P}(x_0 - x_3) - \frac{Q^2 M_P}{4p \cdot q}(x_0 + x_3) \tag{3.15}$$

This implies that it diverges in the Björken limit provided $x_0 - x_3$ is very different zero. If so, the exponential of $i \; q \cdot x$ will be highly oscillatory leading to vanishing integral. This oscillation gets suppressed only in the position space x, when $x_0 - x_3 \leq M_P/p \cdot q$ and $x_0 + x_3 \leq \text{const}.p.q/Q^2$. This corresponds to the region where $x^2 \leq x_0^2 - x_3^2 \approx 0$. The region where $x^2 \approx 0$ is called light-cone region. The summary of the above simple exercise is that the dominant contribution to $W_{\mu\nu}(q,p)$ in the Björken limit comes from the light-cone region of the integral.

The hadronic tensor $W_{\mu\nu}(q,p)$ can be written as

$$W_{\mu\nu}(q,p) \quad = \quad \frac{1}{2\pi i}\Big[T_{\mu\nu}(q^0 + i\epsilon) - T_{\mu\nu}(q^0 - i\epsilon)\Big] \tag{3.16}$$

where

$$T_{\mu\nu}(q,p) = \frac{i}{2M_P}\int d^4x e^{iq\cdot x}\frac{1}{2}\sum_s \langle p,s|T\left(J_\mu(x)J_\nu(0)\right)|ps\rangle \tag{3.17}$$

In this representation, we can easily apply Björken limit as can be shown below. The task now is to study the time ordered product of two electromagnetic currents on the light cone, that is, $\lim_{x^2\approx 0} T\left(J_\mu(x)J_\nu(0)\right)$. It is understood that the currents are already normal ordered. In quantum field theory, care is needed to define the product of quantum field operators, the composite operators (normal ordered product of quantum field operators) at the same space-time point. Same is true for the product of such operators on the light cone. The reason is that they are often singular and ill-defined and a prescription is needed to define them. Wilson proposed a systematic method to organise such product of quantum field operators and composite operators as a series expansion in terms of well defined local operators with appropriate singular coefficients organised in such a way that the most singular/dominant terms appear first and the less singular and regular terms appear successively in the expansion. This goes under the name operator product expansion (OPE). We can now apply OPE to $T\left(J_\mu(x)J_\nu(0)\right)$ on the light cone. Since incoming leptons are unpolarised, the leptonic tensor $\mathcal{L}_{\mu\nu}$ is symmetric in the indices μ,ν and hence only symmetric part of $T_{\mu\nu}$ will be considered for our study below:

$$\begin{aligned}\lim_{x^2\approx 0} T\left(J_\mu(x)J_\nu(0)\right) \quad &= \quad (\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)\mathcal{O}_L(x,0) \\ &\quad +\Big(g_{\mu\lambda}\partial_\rho\partial_\nu + g_{\rho\nu}\partial_\mu\partial_\lambda - g_{\mu\lambda}g_{\rho\nu}\partial^2 \\ &\quad -g_{\mu\nu}\partial_\lambda\partial_\rho\Big)\mathcal{O}_2^{\lambda\rho}(x,0)\end{aligned} \tag{3.18}$$

where the operators $\mathcal{O}_i(x,0), i = L, 2$ are given by

$$\begin{aligned}
\mathcal{O}_L(x,0) &= \sum_{a,n} C^a_{L,n}(x^2) x^{\mu_1} \cdots x^{\mu_n} O^a_{L,\mu_1,\cdots,\mu_n}(0) \\
\mathcal{O}_2^{\lambda\rho}(x,0) &= \sum_{a,n} C^a_{2,n}(x^2) x^{\mu_1} \cdots x^{\mu_n} O^{a\lambda\rho}_{2,\mu_1,\cdots,\mu_n}(0)
\end{aligned} \tag{3.19}$$

The local operators $O^a_{L,\mu_1,\cdots,\mu_n}(0)$ and $O^{a\lambda\rho}_{2,\mu_1,\cdots,\mu_n}(0)$ are well-defined in the sense that their matrix elements between physical states are finite. On the other hand, the coefficients $C^a_{i,n}(x^2), i = L, 2$ are singular when $x^2 \approx 0$. These coefficients are called Wilson's coefficients. Using OPE on the light cone, the symmetric part of $T_{\mu\nu}$ becomes,

$$\begin{aligned}
\lim_{x^2\approx 0} T_{\{\mu\nu\}} &= -i\Big(q_\mu q_\nu - q^2 g_{\mu\nu}\Big)\frac{1}{2}\sum_s \langle p,s|O^a_{L,\mu_1,\cdots,\mu_n}(0)|p,s\rangle \\
&\quad \times \sum_{a,n}\int d^4x e^{iq\cdot x} x^{\mu_1}\cdots x^{\mu_n} C^a_{L,n}(x^2) - i\Big(g_{\mu\lambda}q_\rho q_\nu + g_{\rho\nu}q_\mu q_\lambda \\
&\quad - g_{\mu\lambda}g_{\rho\nu}q^2 - g_{\mu\nu}q_\lambda q_\rho\Big)\frac{1}{2}\sum_s \langle p,s|O^{a\lambda\rho}_{2,\mu_1,\cdots,\mu_n}(0)|p,s\rangle \\
&\quad \times \sum_{a,n}\int d^4x e^{iq\cdot x} x^{\mu_1}\cdots x^{\mu_n} C^a_{2,n}(x^2)
\end{aligned} \tag{3.20}$$

It can be simplified further using the method of tensor decomposition as follows:

$$\begin{aligned}
\int d^4x e^{iq\cdot x} x^{\mu_1}\cdots x^{\mu_n} C^a_{L,n}(x^2) &= -i\left(-\frac{2}{q^2}\right)^{n+1} q^{\mu_1}\cdots q^{\mu_n}\hat{C}^a_{L,n}(-q^2) \\
&\quad + i\left(-\frac{2}{q^2}\right)^{n+1} q^2\Big\{g^{\mu_1\mu_2}q^{\mu_3}\cdots q^{\mu_n}\Big\}_S \tilde{C}^a_{L,n}(-q^2) \\
&\quad + \cdots
\end{aligned} \tag{3.21}$$

where the subscript S means symmetrisation of all the indices inside the parenthesis. A similar expansion defines Fourier coefficients $\hat{C}^a_{2,n}(-q^2), \tilde{C}^a_{2,n}(-q^2), \cdots$ for the Wilson's coefficient $C^a_{2,n}(x^2)$. The operator matrix elements can be written as

$$\begin{aligned}
\frac{1}{2}\sum_s < p,s|O^a_{L,\mu_1,\cdots,\mu_n}(0)|p,s> &= \hat{A}^a_{L,n}(p^2)p_{\mu_1}\cdots p_{\mu_n} \\
&+\hat{B}^a_{L,n}(p^2)p^2\Big\{g_{\mu_1\mu_2}p_{\mu_3}\cdots p_{\mu_n}\Big\}_S \\
&+\cdots \\
\frac{1}{2}\sum_s < p,s|O^{a\lambda\rho}_{2,\mu_1,\cdots,\mu_n}(0)|p,s> &= \hat{A}^a_{2,n+2}(p^2)\Big\{p^\lambda p^\rho p_{\mu_1}\cdots p_{\mu_n}\Big\}_S \\
&+\hat{B}^a_{2,n}(p^2)p^2\Big\{g_{\mu_1\mu_2}p^\lambda p^\rho p_{\mu_3}\cdots p_{\mu_n}\Big\}_S \\
&+\cdots
\end{aligned} \tag{3.22}$$

On the light cone, terms proportional to metric tensor in the eqns. (3.21,3.22) are suppressed because they give contributions that are proportional to x^2 or p^2/Q^2. Hence only $\hat{C}^a_{L,n}(-q^2), \hat{C}^a_{2,n}(-q^2)$ and $\hat{A}^a_{L,n}(p^2), \hat{A}^a_{2,n}(p^2)$ contribute to $T_{\{\mu\nu\}}(q,p)$:

$$T_{\{\mu\nu\}} = 2\sum_{i,n} w^n\left[e_{\mu\nu}\hat{A}^a_{L,n}(p^2)\hat{C}^a_{L,n}(-q^2) + d_{\mu\nu}\hat{A}^a_{2,n}(p^2)\hat{C}^a_{2,n}(-q^2)\right] \tag{3.23}$$

where

$$\begin{aligned}
w &= \frac{2p\cdot q}{Q^2}, e_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \\
d_{\mu\nu} &= -g_{\mu\nu} - p_\mu p_\nu\frac{q^2}{(p\cdot q)^2} + \frac{p_\mu q_\nu + p_\nu q_\mu}{p\cdot q}
\end{aligned} \tag{3.24}$$

Translation invariance implies,

$$T_{\{\mu\nu\}}(-w) = T_{\{\mu\nu\}}(w) \tag{3.25}$$

It is clear from eqn.(3.23) that $T_{\{\mu\nu\}}(w)$ has a branch cut $|w| > 1$. If $T_{\{\mu\nu\}}(w)$ is analytically continued to a complex plan spanned by complex w, then branch cuts will be along $Re(w) \geq 1$ and $Re(w) \leq -1$. Consider a contour $\mathcal{C}$ enclosing the origin and leaving the branch cuts outside. Then,

$$\begin{aligned}\int_{\mathcal{C}} dw \frac{T_{\{\mu\nu\}}(w)}{w} &= \frac{1}{i\pi}\int_1^{\infty} dw \frac{T_{\{\mu\nu\}}(w+i\epsilon)-T_{\{\mu\nu\}}(w-i\epsilon)}{w^m} \\ &= \frac{2}{\pi}\int_0^1 d\xi \xi^{m-2} W_{\mu\nu}(\xi, Q^2) \qquad (3.26)\end{aligned}$$

where $\xi = 1/w$. Using the identity $(2\pi i)^{-1}\int_{\mathcal{C}} w^{n-m} dw = \delta_{n,m-1}$, we find

$$\begin{aligned}\int_0^1 dx_{Bj} x_{Bj}^{m-1} W_{\mu\nu}(x_{Bj}, Q^2) &= \sum_a \Big[e_{\mu\nu} \hat{A}^a_{L,m-1}(p^2) \hat{C}^a_{L,m-1}(Q^2) \\ &\quad + d_{\mu\nu} \hat{A}^a_{2,m-1}(p^2) \hat{C}^a_{2,m-1}(Q^2) \Big] \qquad (3.27)\end{aligned}$$

The structure functions satisfy the following relation:

$$\int_0^1 dx_{Bj} x_{Bj}^{N-1} F_i(x_{Bj}, Q^2) = \sum_a \hat{A}^a_{i,N}(p^2) \hat{C}^a_{i,N}(Q^2), \qquad i = L, 2 \qquad (3.28)$$

The structure functions $F_i(x_{Bj}, Q^2)$ are in general functions of p^2, Q^2 and $p \cdot q$. Using OPE, we have shown here that in the Björken limit, the Nth moment of the structure functions with respect to x_{Bj} factorises into product of purely p^2 dependent functions $\hat{A}^a_{i,N}(p^2)$ and functions $\hat{C}^a_{i,N}(Q^2)$ that depend only Q^2. The hadronic matrix elements, $\hat{A}^a_{i,N}(p^2)$, parametrise the long distance physics of the process. On the other hand the Wilson's coefficients $\hat{C}^a_{i,N}(Q^2)$ capture all the short distance part of the process. The scaling behaviour of the structure functions in the Björken limit now corresponds to situation in which the Wilson's coefficients become Q^2 independent when $Q^2 \to \infty$. Hence, any candidate model or a theory for strong interaction force should result in Q^2 independent Wilson's coefficients for the structure functions $F_i(x_{Bj}, Q^2)$.

Let us now express the differential cross given in eqn.(3.14) in the Björken limit in terms of these structure functions:

$$\begin{aligned}\lim_{Bj} \frac{d^2\sigma}{d\Omega_e dE'} &= \int_0^1 dy \int_0^1 dz y F_2(y) \Big[\frac{4\alpha^2 E^{'2}}{Q^4} \frac{2M_P}{Q^2} \cos^2\frac{\theta}{2} \delta(1-z) \Big] \delta(x_{Bj} - yz) \\ &\quad + \int_0^1 dy \int_0^1 dz F_1(y) \Big[\frac{4\alpha^2 E^{'2}}{Q^4} \frac{2}{M_P} \sin^2\frac{\theta}{2} \delta(1-z) \Big] \delta(x_{Bj} - yz)\end{aligned}$$

(3.29)

The above result offers an elegant interpretation: let us recall the expression for the elastic scattering cross section for a point like particle, that for the for process, $e + \mu \to e + \mu$, we have

$$\frac{d^2\sigma}{d\Omega_e dE'} = \frac{4\alpha^2 E'^2}{Q^4}\left[\frac{2M_P}{Q^2} z\delta(1-z)\cos^2\frac{\theta}{2} + \frac{2}{M_P}\,\delta(1-z)\sin^2\frac{\theta}{2}\right] \quad (3.30)$$

where the dimensionless variable $z = Q^2/2p_\mu \cdot q$. Comparing eqn.(3.29) with the eqn.(3.30), we find that the inelastic scattering in the Björken limit can be thought of as the weighted sum (integration) of elastic scattering cross sections. The weight factors here are nothing but the structure functions that depend on the variable $y = x_{Bj}/z$. Since the cross sections are basically probabilities, the weight factors can be interpreted as some probabilities. This simple minded interpretation of the deep inelastic hadronic cross section in the Björken limit in terms of elastic scattering cross sections of point like particles leads to a picture of hadrons at high energies (Björken limit belongs to this category) which goes under the name Parton Model. In this model, the hadrons at high energy or equivalently at short distances are described in terms of what are called free partonic states. These states correspond to elementary point-like particles, called partons that constitute the hadrons. These free partons can interact with other standard model particles through electromagnetic (em) or weak interactions. For example an electrically charged parton can interact with a photon through em interaction and with Z boson through weak interaction. Since the model does not contain any mechanism for the binding of nucleons at low energies, the corresponding long distance physics of these partons is parametrised in terms of some unknown quantities which are usually extracted from the experiment. The above picture of hadrons in terms of free partonic states can be easily justified by studying the inelastic cross section of hadrons in the rest of frame of the virtual photon. In this frame, the hadron is Lorentz boosted which leads to length contraction of its size along the boosted direction. This reduces the distance traversed by the electron during the scattering. In addition, the internal interaction of the partonic states, which is responsible for binding the partons inside the hadron, is time dilated. This means that the partonic states live longer than the time scales associated with the interaction of an electron(i.e., the virtual photon) with the single partonic state. Therefore, the electron or equivalently the virtual photon scatters off on only a single partonic state. The scattering cross section is then proportional to the probability of finding this partonic state in the proton. Hence, the hadronic cross section is incoherent sum of cross sections of various partonic states of the hadron with appropriate probabilities. If we denote $\hat{f}_{a/h}(y)$, the probability of finding a partonic state a inside the proton with momentum fraction y of the proton momentum and $d\hat{\sigma}_{ea}(z,Q^2)$ the elastic cross section of an electron on the partonic state, then the inelastic cross section in the Björken limit is given by

$$\lim_{Bj} d\sigma_{eh}(x_{Bj},Q^2) = \sum_a \int_0^1 dy \int_0^1 dz \hat{f}_{a/h}(y) d\hat{\sigma}_{ea}(z,Q^2)\delta(x_{Bj}-yz) \quad (3.31)$$

Note that the above formula is a generalisation of the result given in eqn.(3.29) with F_i replaced by $f_{a/h}$ and terms within the square brackets replaced by $d\hat{\sigma}_{ea}$. Here $\hat{f}_{a/h}(y)$ is called parton distribution function. It depends only the type of parton a and the hadron h and they are process independent. These functions can not be calculable within the model and hence should be extracted from the experiments. On the other hand, $d\hat{\sigma}_{ea}(z,Q^2)$, called partonic cross sections, which result from the scattering of point-like partons with electron through electromagnetic and/or through weak interactions. The above formula reproduces the scaling behaviour of the deep inelastic scattering in the Björken limit given in eqn(3.29). It is straightforward to relate the hadronic structure functions $F_i(y)$ with the partonic distribution functions $\hat{f}_{a/h}(y)$. In the case of proton, the structure functions can be expressed as

$$F_1(x_{Bj}) = \frac{1}{2}\sum_{a=u,d} e_a^2 \hat{f}_{a/P}(x_{Bj}) \quad (3.32)$$

$$F_2(x_{Bj}) = x_{Bj}F_1(x_{Bj}) \quad (3.33)$$

where we have assumed that the proton is made up of "up"(u) and "down"(d) type partons inspired by the classification of hadrons in terms of quarks.

3.3 Gauge Symmetry

In this section we will study the role played by gauge symmetry in constructing classical actions that can describe various forces of nature. Let us first study the theory of electrons and electromagnetic fields. The classical Lagrangian that describes free electrons is given by,

$$\mathcal{L}_\psi = \overline{\psi}(x)[i\partial\!\!\!/ - m]\psi(x) \quad (3.34)$$

where ψ is a 4-component Dirac field, m their mass and $\partial\!\!\!/ = \gamma_\mu\partial^\mu$. This Lagrangian is invariant under global (space time independent) transformation given by:

$$\psi(x) \quad \rightarrow \quad e^{ie\lambda}\psi(x) \tag{3.35}$$

$e^{ie\lambda}$ is an element of a one parameter unitary group denoted by $U(1)$. However, it does not have local $U(1)$ symmetry. The local symmetry corresponds to replacing the parameter λ by the one which depends on both space and time, i.e., $\lambda \rightarrow \lambda(x)$. Because of the derivative which can now act on $\lambda(x)$, the free fermion Lagrangian is no longer invariant under this local $U(1)$ transformation. The local U(1) invariant (ie., gauge invariant) Lagrangian can be constructed provided one introduces local vector fields $A_\mu(x)$ (also called electromagnetic gauge field) with the transformation law under local $U(1)$ given by

$$A_\mu(x) \quad \rightarrow \quad A_\mu(x) + \partial_\mu \lambda(x). \tag{3.36}$$

The following Lagrangian with these gauge fields

$$\overline{\psi}[i(\partial\!\!\!/ - ieA\!\!\!/) - m]\psi \tag{3.37}$$

is invariant under the combined transformations, given by eqns. (3.35,3.36), usually called $U(1)$ gauge transformations. Notice that the second term in the eqn.(3.37) describes the interaction of electrons with the gauge fields with the interaction strength given by e. In the quantised version of this theory, the gauge fields will correspond to photons. The kinetic part of the gauge fields can be obtained from the following gauge invariant Lagrangian:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.38}$$

where

$$F_{\mu\nu} \quad = \quad \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.39}$$

The $U(1)$ gauge invariant Lagrangian describing the theory of electrons and the em gauge fields is given by

$$\mathcal{L}_{QED} \quad = \quad \overline{\psi}[i(\partial\!\!\!/ - ieA\!\!\!/) - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{3.40}$$

We would now like to study how the above construction can be generalised to cases where the gauge symmetry is $SU(N)$. In other words, we will construct, in the following, a local $SU(N)$ gauge invariant action. Before we do this, let us very briefly review the groups $U(N)$ and $SU(N)$.

Set of $N \times N$ unitary matrices forms a group called $U(N)$. Elements of U(N) satisfying $det\mathcal{U} = 1$ form a sub group called $SU(N)$. An element of $SU(N)$ group depends on $N^2 - 1$ (unitarity gives N^2 real constraints and unit determinant given one real constraint) independent real parameters. An element of group can be obtained by parametrising it infinitesimally close to its identity element. For example, we can write this element as

$$\mathcal{U} = I - i\epsilon\omega$$

Here ϵ is a small real parameter and ω is an $N \times N$ matrix. Unitarity of these elements give

$$\mathcal{U}^\dagger \mathcal{U} = I + i\epsilon(\omega^\dagger - \omega) + \mathcal{O}(\epsilon^2)$$

which implies that ω is hermitian, $\omega^\dagger = \omega$. The condition $det\mathcal{U} = 1$ implies that ω is traceless. Since one requires $N^2 - 1$ independent real parameters to parametrise each element of the group, we can expand ω as

$$\begin{aligned} \epsilon\omega &= \sum_{a=1}^{N^2-1} \epsilon^a T^a \\ &= \epsilon^a T^a \end{aligned}$$

These matrices, T^a, are called generators of the group and they are normalized as

$$Tr(T^a T^b) = T_f \delta^{ab}.$$

where $T_f = 1/2$ and they form Lie algebra given by

$$[T^a, T^b] = i f^{abc} T^c$$

Here f^{abc} are called structure constants which are real and anti-symmetric in all the indices (abc).

We will take ψ to transform under fundamental representation of $SU(N)$ so we require N fermionic fields $\psi_i(x)$ with $i = 1, \ldots, N$. The transformation of these fields under $SU(N)$ is given by

$$\delta\psi_i(x) = \psi'_i(x) - \psi_i(x) = -i \sum_{a=1}^{N^2-1} \sum_{j=1}^{N} \epsilon^a (T^a)_{ij} \psi_j(x) \tag{3.41}$$

In the following we will use the summation convention for both i and a. It is easy to see that the following Lagrangian is invariant under global $SU(N)$ symmetry,

$$\overline{\psi}(i\partial\!\!\!/ - m\mathbb{I})\psi \tag{3.42}$$

where we have introduced matrix notation for the fermionic fields:

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}, \qquad \psi^\dagger(x) = \begin{pmatrix} \psi_1^\dagger(x), \ldots, \psi_N^\dagger(x) \end{pmatrix} \tag{3.43}$$

The local gauge invariant Lagrangian with N fermionic fields can be achieved by introducing $N^2 - 1$ gauge fields denoted by A^a_μ with $a = 1, \ldots, N^2 - 1$ with the transformation property

$$\delta A^d_\mu = -\frac{1}{g_s}\partial_\mu \epsilon^d - f^{dab} A^a_\mu \epsilon^b \tag{3.44}$$

It is a straightforward exercise to show that the Lagrangian given by

$$\overline{\psi}[i(\mathbb{I}\not{\partial} - ig_s \not{A}^a T^a) - m\mathbb{I}]\psi \tag{3.45}$$

is invariant under the local $SU(N)$ transformations given by eqns. (3.41,3.44). Analogous to the tensor field $F_{\mu\nu}(x)$ given in eqn.(3.38), the kinetic energy part of the $SU(N)$ gauge fields can also be constructed using $N^2 - 1$ second rank tensor fields given by

$$F^a_{\mu\nu} \quad = \quad \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \tag{3.46}$$

Since $F^a_{\mu\nu}$ transforms as

$$\delta F^a_{\mu\nu} = F^{'a}_{\mu\nu}(x) - F^a_{\mu\nu}(x) = -f^{abc} F^b_{\mu\nu}(x)\epsilon^c(x) \tag{3.47}$$

under the transformation given by eqn.(3.44), the following action

$$-\frac{1}{2} Tr[F^a_{\mu\nu} T^a F^{\mu\nu b} T^b] \tag{3.48}$$

is invariant under gauge transformations.

The complete $SU(N)$ gauge invariant action is given by

$$\mathcal{L}_{YM} = -\frac{1}{2} Tr[F^a_{\mu\nu} T^a F^{\mu\nu b} T^b] + \overline{\psi}[i(\mathbb{I}\not{\partial} - ig_s \not{A}^a T^a) - m\mathbb{I}]\psi \tag{3.49}$$

The above Lagrangian is usually called the Yang-Mills (YM) Lagrangian. Since $SU(N)$ is a non-Abelian group, the $SU(N)$ gauge symmetry is called a non-Abelian gauge symmetry and the gauge fields are called non-Abelian gauge fields. Notice that the action describes not only the interaction of N fermions with $N^2 - 1$ gauge fields, but also describes the interaction of gauge fields among themselves. The interaction of gauge fields among themselves comes from terms proportional to $F^a_{\mu\nu}(x)$ in the action eqn.(3.49) which contains a term $g_s f^{abc} A^b_\mu(x) A^c_\nu(x)$ (see 3.46). This feature is characteristic of theories with non-Abelian gauge symmetry. Since the theory of electrons and em gauge fields has an invariant Abelian symmetry i.e., $U(1)$, the em gauge fields do not interact with each other. We will show that the non-Abelian Yang-Mills Lagrangian with $N = 3$ can describe strong interaction dynamics. In the following we describe the quantization of classical Yang-Mills action:

$$S_{YM} = \int d^4x \mathcal{L}_{YM}(A^a_\mu(x), \overline{\psi}(x), \psi(x), m, g_s) \tag{3.50}$$

In the canonical formalism of quantization, one replaces the classical fields by operators and their canonical commutation relations with their conjugates. The equations of motion that result from the least action principle and their solutions in the Fourier space, subjected to the canonical commutation relations, lead to set of operators that can create and annihilate single particle states. Using this approach one can compute propagation of the quantum particles and their interaction in terms of the scattering matrix, called S matrix. The S matrix elements are nothing but the residues of vacuum expectation value of time ordered product of quantum field operators on the mass-shell. An alternate approach to quantization is the path integral formulation, in which the quantum fields are treated as commuting variables/functions. The quantum vacuum expectation value of time order product of quantum field operators is given by

$$\langle 0|T\left(\Phi_{j_1}(x_1)\dots\Phi_{j_n}(x_n)\right)|0\rangle \quad \equiv \quad \langle \Phi_{j_1}(x_1)\dots\Phi_{j_n}(x_n)\rangle$$

where

$$\Phi_i(x) = \{\overline{\psi}(x), \psi(x), A^a_\mu(x)\}$$

$|0\rangle$ denotes the vacuum, and T means time ordering of the operators. It is also called Green's function in the literature. The path integral formalism provides a prescription to compute these Green's functions:

$$\langle \Phi_{j_1}(x_1)\dots\Phi_{j_n}(x_n)\rangle = \frac{\int \prod_i \mathcal{D}\Phi_i \quad \Phi_{j_1}(x_1)\dots\Phi_{j_n}(x_n) e^{iS[\{\Phi\}]}}{\int \prod_i \mathcal{D}\Phi_i \quad e^{iS[\{\Phi\}]}}$$

[1] The momentum space Green's functions can also be obtained using path integrals as

$$\langle\widetilde{\Phi}_{j_1}(k_1)\ldots\widetilde{\Phi}_{j_n}(k_n)\rangle = \frac{\int\prod_i\mathcal{D}\widetilde{\Phi}_i \quad \widetilde{\Phi}_{j_1}(k_1)\ldots\widetilde{\Phi}_{j_n}(k_n)e^{i\widetilde{S}[\{\widetilde{\Phi}\}]}}{\int\prod_i\mathcal{D}\widetilde{\Phi}_i \quad e^{i\widetilde{S}[\{\widetilde{\Phi}\}]}}$$

where the Fourier components $\widetilde{\Phi}_i(k)$ are defined by

$$\widetilde{\Phi}_i(k) = \int d^4x e^{ik\cdot x}\Phi_i(x)$$

The generating functional to compute the Green's functions is given by

$$Z(\{\widetilde{J}\}) = \int\prod_i\mathcal{D}\widetilde{\Phi}_i exp\left[i\widetilde{S}(\widetilde{\Phi}) + i\int\frac{d^4k}{(2\pi)^4}\widetilde{J}_j(-k)\widetilde{\Phi}_j(k)\right]$$

where $\widetilde{J}_i$ are the source fields. Using,

$$\frac{\delta\widetilde{J}_i(k_1)}{\delta\widetilde{J}_j(k_2)} = (2\pi)^4\delta^{(4)}(k_1-k_2)\delta_{ij}$$

we obtain,

$$\langle\widetilde{\Phi}_{j_1}(k_1)\ldots\widetilde{\Phi}_{j_n}(k_n)\rangle = \frac{1}{Z[0]}\left(-i(2\pi)^4\frac{\delta}{\delta\widetilde{J}_{i_1}(-k_1)}\right)\ldots\left(-i(2\pi)^4\frac{\delta}{\delta\widetilde{J}_{i_n}(-k_n)}\right) \times Z(\{\widetilde{J}\})\Bigg|_{\{\widetilde{J}_i\}=0} \tag{3.52}$$

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[1]The path integral measure $\int\mathcal{D}\Phi_i$ can be visualised if we replace the continuous the space-time by a 4-dimensional lattice with a lattice constant a (distance between two neibouring lattice points). That is,

$$x^\mu \to (a\ n_0, a\ n_1, a\ n_2, a\ n_3)$$

where a is real and n_i are integers. We will suppress the label i on Φ_i for notational clarity. The fields are given by

$$\Phi(x) \to \Phi_{n_0,n_1,n_2,n_3}, \qquad \int d^4x \to a^4\sum_{n_0,n_1,n_2,n_3} \tag{3.51}$$

and the measure $\int\mathcal{D}\Phi$ takes the form

$$\int\mathcal{D}\Phi \to \int\prod_{n_0,n_1,n_2,n_3} d\Phi_{n_0,n_1,n_2,n_3}$$

We will interrupt the discussion of Yang-Mills theory to exemplify the calculation of the Green's function by the path integral method in a simpler case of a real scalar field and calculate the 2-point Green function $\widetilde{G}^2(p_1, p_2)$. This 2-point function is also called propagator of the theory if it only involves the free part and does not include the interaction terms.
The free part of the action is

$$S_0 = \int d^4x \left[\frac{1}{2}\phi(x)(-\partial^2 - m^2)\phi(x) + J(x)\phi(x)\right] \tag{3.53}$$

where we have introduced the source field $J(x)$ and m is the mass parameter in the Lagrangian. To express action in $\widetilde{\phi}$, the Fourier transform of ϕ, we substitute

$$\begin{aligned}
\phi(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\widetilde{\phi}(k), \\
J(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\widetilde{J}(k)
\end{aligned}$$

in the above expression. We can do the integral over x using the definition of Dirac delta function

$$\int d^4x\, e^{-i(k_1+k_2)\cdot x} = (2\pi)^4\delta^4(k_1 + k_2),$$

and use this delta function to integrate out one of the momenta and obtain

$$\begin{aligned}
S_0 &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\widetilde{\phi}(-k)M(k)\widetilde{\phi}(k) + 2\widetilde{J}(k)\widetilde{\phi}(-k)\right] \\
&= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[-\widetilde{J}(-k)M^{-1}(k)\widetilde{J}(k) \\
&\quad + \left(\widetilde{\phi}(-k) + \widetilde{J}(-k)M^{-1}(k)\right)M(k)\left(\widetilde{\phi}(k) + \widetilde{J}(k)M^{-1}(k)\right)\Big]
\end{aligned}$$

where $M(k) = k^2 - m^2$. Note that $M(k) = M(-k)$. We can now do a change of variable by defining

$$\begin{aligned}
\widetilde{\phi}'(k) &= \widetilde{\phi}(k) + M^{-1}\widetilde{J}(k) \\
\widetilde{\phi}'(-k) &= \widetilde{\phi}(-k) + M^{-1}\widetilde{J}(-k)
\end{aligned}$$

This gives finally

$$S_0 = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[\widetilde{\phi}'(-k)M(k)\widetilde{\phi}'(k) - \widetilde{J}(-k)M^{-1}(k)\widetilde{J}(k)\right].$$

Substituting this in the expression of generating functional and doing the path integral over the fields we obtain for Z_0,

$$Z_0[\widetilde{J}] = \mathcal{N}\ exp\left[-\frac{i}{2}\int \frac{d^4k_1}{(2\pi)^4}\int \frac{d^4k_1}{(2\pi)^4}\ (2\pi)^4\delta^4(k_1+k_2)\right.$$
$$\left.\times\widetilde{J}(-k_1)M^{-1}(k_1)\widetilde{J}(-k_2)\right]$$

where $\mathcal{N}$ is a normalization factor. From this we can obtain the two point Green function.

$$\begin{aligned}\tilde{G}^2(p_1,p_2) &= \frac{1}{Z_0[0]}\left(-i(2\pi)^4\frac{\delta}{\delta\widetilde{J}(-p_1)}\right)\left(-i(2\pi)^4\frac{\delta}{\delta\widetilde{J}(-p_2)}\right)Z[\widetilde{J}]\Bigg|_{\widetilde{J}=0}\\ &= i(2\pi)^4\delta^4(p_1+p_2)M^{-1}(p_1)\end{aligned}$$

Substituting $M^{-1} = 1/(k^2-m^2)$ we get

$$\tilde{G}^2(p_1,p_2) = \frac{i}{k^2-m^2}(2\pi)^4\delta^4(p_1+p_2). \tag{3.54}$$

With this experience let us now continue with Yang-Mills theory. To proceed further with the path integral approach, we split the Lagrangian as

$$S\left[\{\Phi\}\right] = S_0\left[\{\Phi\}\right] + S_I\left[\{\Phi\}\right] \tag{3.55}$$

where the first term S_0 contains terms which are quadratic in $\widetilde{\Phi}$, for example it contains terms of the form $\widetilde{\Phi}_i(k)\widetilde{\Phi}_i(k')$. S_I contains the rest. In the case of YM Lagrangian, the S_0 is given by

$$\begin{aligned}S_0\left[\overline{\psi},\psi,A^a_\mu\right] &= \int d^4x\Bigg[\overline{\psi}(x)\left(i\partial\!\!\!/-m\right)\psi(x)\\ &\quad -\frac{1}{4}\left(\partial_\mu A^a_\nu(x)-\partial_\nu A^a_\mu(x)\right)\left(\partial^\mu A^{\nu a}(x)-\partial^\nu A^{\mu a}(x)\right)\Bigg]\end{aligned} \tag{3.56}$$

and S_I is given by

$$S_I\left[\overline{\psi},\psi,A^a_\mu,g_s,m\right] = S_{\psi,A}\left[\overline{\psi},\psi,A^a_\mu,g_s\right] + S_{A^3}\left[A^a_\mu,g_s\right] + S_{A^4}\left[A^a_\mu,g_s\right] \tag{3.57}$$

where $S_{\psi,A}$ describes the interaction of fermions with the gauge bosons and S_{A^3} and S_{A^4} are triple and quartic gauge boson interaction terms.

The free action S_0 can be expressed in terms of Fourier components of the fields as

$$\widetilde{S}_0\left[\widetilde{\overline{\psi}}, \widetilde{\psi}, \widetilde{A}^a_\mu\right] = i\int \frac{d^4k}{(2\pi)^4}\left[\widetilde{\overline{\psi}}(-k)(\not{k}-m)\widetilde{\psi}(k)\right.$$

$$\left. +\frac{1}{2}\widetilde{A}^a_\mu(-k)\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right)\widetilde{A}^a_\nu(k)\right] \tag{3.58}$$

Notice that the above integral exists only if $(\not{k}-m)$ and $(k^\mu k^\nu - k^2 g^{\mu\nu})$ are invertible. The fermionic part $M_\psi(k) = \not{k} - m$ has an inverse $\not{k} + m/(k^2 - m^2)$. On the other hand the corresponding part of the gauge fields given by

$$M_A(k) \propto \frac{1}{2}\left(k^\mu k^\nu - k^2 g^{\mu\nu}\right) \tag{3.59}$$

does not have inverse since dotting it with k_ν gives zero, and hence the path integral for the gauge fields is ill-defined.

3.4 Gauge Fixing

In the previous section, we found that the path integral for the gauge fields is ill-defined. We will now try to understand the reason behind this. This will also help us to construct well defined Green's functions of the gauge fields. In the following we restrict ourselves to physically relevant quantities such as expectation value of the product of gauge invariant operators $\prod_l \mathcal{O}_l(x_l)$. Few examples of $\mathcal{O}_l(x_l)$ are $F^a_{\mu\nu}(x)F^{\mu\nu a}(x)$, $\overline{\psi}(x)i(\not{\partial} - ig_s\not{A}^a(x)T^a)\psi(x)$. Since the fermionic part of the action does not play much role in the following discussion we drop them and keep only gauge fields in the action. Now, in the Fourier space, we have

$$\left\langle \Pi_l \widetilde{\mathcal{O}}_l\left(k, \widetilde{A}^b_\mu\right)\right\rangle = \frac{\int \mathcal{D}\widetilde{A}^a_\mu e^{i\widetilde{S}_A[\widetilde{A}^b_\mu]}\prod_l \widetilde{\mathcal{O}}_l(k, \widetilde{A}^b_\mu)}{\int \mathcal{D}\widetilde{A}^a_\mu e^{i\widetilde{S}_A(\widetilde{A}^b_\mu)}} \tag{3.60}$$

The gauge field $A^a_\mu(x)$ and the gauge transformed $A^{a\theta}_\mu(x)$ given by

$$A^\theta_\mu(x) = -\frac{i}{g_s}(\partial_\mu U)U^\dagger + UA_\mu(x)U^\dagger \tag{3.61}$$

obtained by the finite $SU(N)$ gauge transformation $U(\theta) = exp(-ig_s\theta^a(x)T^a)$ are said to be in the same gauge orbit. Since they describe same physics, $A_\mu(x)$

and $A^{\theta}_{\mu}(x)$ are called gauge equivalent gauge field configurations. Notice that the action as well as the composite operator $\widetilde{\mathcal{O}}_l(k, \widetilde{A}^a_\mu)$ are gauge invariants. On the other hand the measure does depend on the gauge parameter. We can write the measure as

$$\mathcal{D}\widetilde{A}^a_\mu \approx \mathcal{D}A^a_\mu|_{ineq}\mathcal{D}\widetilde{A}^a_\mu|^{\theta}_{orbit} \tag{3.62}$$

then we observe that the integral over $\mathcal{D}\widetilde{A}^a_\mu|^{\theta}_{orbit}$ part of the measure in the numerator $(\widetilde{\mathcal{N}})$ as well as in the denominator $(\widetilde{\mathcal{D}})$of the eqn.(3.60) gives divergent contributions. This is the reason why we obtained an ill-defined path integral for the gauge fields earlier. Notice that even though the $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{D}}$ are individually divergent, the ratio $\widetilde{\mathcal{N}}/\widetilde{\mathcal{D}}$ is well defined and finite. If we can manage to factor out the divergent (ill-defined) parts from both $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{D}}$ of the eqn.(3.60), cancel them, then the remaining numerator and denominator are finite. The resultant path integral is well defined and suitable for computation of gauge invariant objects. This can be achieved by the method called "gauge fixing". Gauge fixing involves path integration over inequivalent gauge orbits. One has to do this in such a way that the result is independent of the choice of the path. It can be achieved by doing the integrations over the path that intersects the gauge orbits only once. we know that each point in the group space is parametrised by $N^2 - 1$ independent variables. Hence, we need $N^2 - 1$ conditions to define a path in the group space. Also, these conditions have to be gauge dependent. The gauge fixing conditions can be written as

$$G^a(A_\mu(x)) \quad = \quad B^a(x), \qquad\qquad a = 1, \cdots N^2 - 1 \tag{3.63}$$

where $G^a(A_\mu(x))$ are single valued functions of $A^a_\mu(x)$. The choice $G^a(A_\mu(x))) = \partial_\mu A^\mu(x)$ is called Lorenz gauge and $G^a(A_\mu(x))) = n_\mu A^\mu(x)$ (where n is an arbitrary vector), the axial gauge. We have to implement the gauge fixing conditions to both $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{D}}$ of the eqn.(3.60) in such a way that the numerical value of $\widetilde{\mathcal{N}}/\widetilde{\mathcal{D}}$ is unaffected.

Let us first prove the following identity:

$$\int dx_1 \int dx_2 \delta\left(f_1(x_1,x_2)\right)\delta\left(f_2(x_1,x_2)\right)\left[det\left(\frac{\partial \vec{f}}{\partial \vec{x}}\right)\right]_{x_1=x_1^0, x_2=x_2^0} = 1 \tag{3.64}$$

where,

$$det\left(\frac{\partial \vec{f}}{\partial \vec{x}}\right) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}$$

and (x_1^0, x_2^0) is the unique solution to the equations $f_1 = 0, f_2 = 0$. The identity, eqn.(3.64), can be easily proved by defining

$$f_1(x_1, x_2) = u, \qquad f_2(x_1, x_2) = v$$

and using the Jacobian of the transformation:

$$dudv = \left[det \left(\frac{\partial \vec{f}}{\partial \vec{x}} \right) \right] dx_1 dx_2$$

If (x_1^0, x_2^0) are the unique solutions to the equations

$$f_1(x_1^0, x_2^0) = 0, \qquad f_2(x_1^0, x_2^0) = 0$$

then, the eqn.(3.64) becomes

$$\int du \int dv \delta(u) \delta(v) = 1 \tag{3.65}$$

The generalisation of the above identity is given by

$$\int \prod_i^N \left(dx_i \delta \left(f_i(\vec{x}) \right) \right) \left[det \left(\frac{\partial \vec{f}}{\partial \vec{x}} \right) \right]_{\vec{x}=\vec{x}^0} = 1 \tag{3.66}$$

where $\vec{x}^0$ is the unique solution to the equations $\vec{f}(\vec{x}) = 0$. The above identity involving parametric integrals can be generalised for functional integrals:

$$\int \mathcal{D}\widetilde{\theta}^a \prod_{a,k} \delta \left(\widetilde{G}^a \left(k, \widetilde{A}^a_{\mu\theta} \right) - \widetilde{B}^a(k) \right) det \left(\frac{\partial \widetilde{\vec{G}}(\widetilde{A}^a_{\mu\theta})}{\partial \widetilde{\vec{\theta}}} \right) = 1 \tag{3.67}$$

Inserting the above identity in eqn.(3.60), we find for the numerator,

$$\begin{aligned} \widetilde{\mathcal{N}} &= \int \mathcal{D}\widetilde{\theta}^a \int \mathcal{D}\widetilde{A}^a_\mu \prod_l \widetilde{\mathcal{O}}_l \left(k, \widetilde{A}^a_\mu \right) e^{i\widetilde{S}_A[\widetilde{A}^a_\mu]} \\ &\quad \times \prod_{a,k} \delta \left(\widetilde{G}^a \left(k, \widetilde{A}^a_{\mu\theta} \right) - \widetilde{B}^a(k) \right) det \left[\mathcal{K} \left(\widetilde{A}^a_{\mu\theta} \right) \right] \end{aligned} \tag{3.68}$$

where,

$$\mathcal{K} \left(\widetilde{A}^a_{\mu\theta} \right) = \left(\frac{\partial \widetilde{\vec{G}}(\widetilde{A}^a_{\mu\theta})}{\partial \widetilde{\vec{\theta}}} \right) \tag{3.69}$$

Since

$$\mathcal{D}\widetilde{A}^a_\mu = \mathcal{D}\widetilde{A}^a_{\mu\theta}, \qquad \widetilde{\mathcal{O}}_l(k,\widetilde{A}^a_\mu) = \widetilde{\mathcal{O}}_l(k,\widetilde{A}^a_{\mu\theta}), \qquad \widetilde{S}_A\left[\widetilde{A}^a_\mu\right] = \widetilde{S}_A\left[\widetilde{A}^a_{\mu\theta}\right] \tag{3.70}$$

the eqn.(3.68) becomes,

$$\begin{aligned}\widetilde{\mathcal{N}} &= \int \mathcal{D}\widetilde{\theta}^a \int \mathcal{D}\widetilde{A}^a_{\mu\theta} \prod_l \widetilde{\mathcal{O}}_l\left(k,\widetilde{A}^a_{\mu\theta}\right) e^{i\widetilde{S}_A\left[\widetilde{A}^a_{\mu\theta}\right]} \\ &\times \prod_{a,k} \delta\left(\widetilde{G}^a\left(k,\widetilde{A}^a_{\mu\theta}\right) - \widetilde{B}^a(k)\right) det\left[\mathcal{K}\left(\widetilde{A}^a_{\mu\theta}\right)\right]\end{aligned} \tag{3.71}$$

Since $A^a_{\mu\theta}$ is dummy variable inside the functional integral, we can make the replacement: $A^a_{\mu\theta} \to A^a_\mu$ which gives,

$$\begin{aligned}\widetilde{\mathcal{N}} &= \left[\int \mathcal{D}\widetilde{\theta}^a\right] \int \mathcal{D}\widetilde{A}^a_\mu \prod_l \widetilde{\mathcal{O}}_l\left(k,\widetilde{A}^a_\mu\right) e^{i\widetilde{S}_A\left[\widetilde{A}^a_\mu\right]} \\ &\times \prod_{a,k} \delta\left(\widetilde{G}^a\left(k,\widetilde{A}^a_\mu\right) - \widetilde{B}^a(k)\right) det\left[\mathcal{K}\left(\widetilde{A}^a_\mu\right)\right]\end{aligned} \tag{3.72}$$

Notice that $\int \mathcal{D}\widetilde{\theta}^a$ has factored out from the rest of the integral. Similar exercise for the denominator also results in an integral where the same θ dependent measure factors out and hence we can cancel this in the ratio $\widetilde{\mathcal{N}}/\widetilde{\mathcal{D}}$.

We use the following integral representation for $det\mathcal{K}$ so that we can apply standard techniques of path integration formalism.

$$\begin{aligned}det\left(\mathcal{K}(\widetilde{A}^a_\mu)\right) &= \int \mathcal{D}\overline{\widetilde{\chi}}^a \mathcal{D}\widetilde{\chi}^b exp\left(\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4}\right. \\ &\left.\times\ \overline{\widetilde{\chi}_c}(k_1)\mathcal{K}_{cd}\left(k_1,k_2,\widetilde{A}^a_\mu\right)\widetilde{\chi}_d(k_2)\right)\end{aligned} \tag{3.73}$$

where $\widetilde{\chi}^a$ and $\widetilde{\overline{\chi}^a}$ are anti-commuting variables called Grassmanian variables. We also insert the identity,

$$\mathcal{C}(\xi)\int \mathcal{D}\widetilde{B}^a exp\left(-\frac{i}{2\xi}\int \frac{d^4k}{(2\pi)^4}\widetilde{B}^a(-k)\widetilde{B}^a(k)\right) = 1 \tag{3.74}$$

to express the Dirac delta function in a form suitable for computation. Here ξ is an arbitrary parameter but our final results do not depend on it. Using eqn.(3.73,3.74), we find

$$\left\langle \prod_l \widetilde{\mathcal{O}}_l \left(\widetilde{A}^a_\mu \right) \right\rangle = \frac{\overline{\mathcal{N}}}{\overline{\mathcal{D}}} \tag{3.75}$$

where $\overline{\mathcal{N}} = \widetilde{\mathcal{N}}/\mathcal{D}\widetilde{\theta}^a$ and $\overline{\mathcal{D}} = \widetilde{\mathcal{D}}/\mathcal{D}\widetilde{\theta}^a$. Hence

$$\begin{aligned} \overline{\mathcal{N}} &= \int \mathcal{D}\widetilde{A}^a_\mu \int \mathcal{D}\overline{\widetilde{\chi}}^a \mathcal{D}\widetilde{\chi}^b \prod_l \widetilde{\mathcal{O}}_l \left(k, \widetilde{A}^a_\mu \right) \\ &\quad \times \exp \left[i \left(\widetilde{S}_A \left[\widetilde{A}^a_\mu \right] + \widetilde{S}_{GF} \left[\widetilde{A}^a_\mu, \overline{\widetilde{\chi}}^a, \widetilde{\chi}^b \right] + \widetilde{S}_{GH} \left[\widetilde{A}^a_\mu \right] \right) \right] \end{aligned} \tag{3.76}$$

$$\begin{aligned} \overline{\mathcal{D}} &= \int \mathcal{D}\widetilde{A}^a_\mu \int \mathcal{D}\overline{\widetilde{\chi}}^a \mathcal{D}\widetilde{\chi}^b \\ &\quad \times \exp \left[i \left(\widetilde{S}_A \left[\widetilde{A}^a_\mu \right] + \widetilde{S}_{GF} \left[\widetilde{A}^a_\mu, \overline{\widetilde{\chi}}^a, \widetilde{\chi}^b \right] + \widetilde{S}_{GH} \left[\widetilde{A}^a_\mu \right] \right) \right] \end{aligned} \tag{3.77}$$

The various pieces of the action are given by

$$\widetilde{S}_{GF} \left[\widetilde{A}^a_\mu \right] = -\frac{1}{2\xi} \int \frac{d^4k}{(2\pi)^4} \widetilde{G}^a(-k, \widetilde{A}^a_\mu(-k)) \widetilde{G}^a(k, \widetilde{A}^a_\mu(k)) \tag{3.78}$$

$$\widetilde{S}_{GH} \left[\widetilde{A}^a_\mu, \overline{\widetilde{\chi}}^a, \widetilde{\chi}^b \right] = -i \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \overline{\widetilde{\chi}_c}(k_1) \mathcal{K}_{cd} \left(k_1, k_2, \widetilde{A}^a_\mu \right) \widetilde{\chi}_d(k_2) \tag{3.79}$$

Let us now compute $\mathcal{K}_{cd}$ for the gauge fixing condition:

$$G^a(A^a_\mu) = \partial_\mu A^{\mu a}(x) \tag{3.80}$$

This implies

$$\delta G^a \left(A^a_{\mu\theta} \right) = \partial^\mu \delta A^a_{\mu\theta}(x) \tag{3.81}$$

where

$$\partial^\mu \delta A^a_{\mu\theta}(x) = \partial^2 \delta\theta^a(x) - g_s f^{abc} \partial^\mu \left(A^b_{\mu\theta}(x) \delta\theta^c(x) \right) \tag{3.82}$$

In the momentum space we find,

$$\delta\widetilde{G}^a\left(k,\widetilde{A}^a_{\mu\theta}\right) = k^2\delta\widetilde{\theta}^a(k) + ig_s f^{abc}k^\mu\int\frac{d^4k_1}{(2\pi)^4}\widetilde{A}^b_{\mu\theta}(-k_1+k)\delta\theta^c(k_1) \quad (3.83)$$

This implies

$$\begin{aligned}\mathcal{K}_{cd}(k,k',\widetilde{A}^a_{\mu\theta}) = \frac{\delta\widetilde{G}^c(k)}{\delta\theta^d(k')} &= k^2(2\pi)^4\delta^{(4)}(k-k')\delta_{cd} \\ &+ig_s f^{cad}k^\mu\widetilde{A}^a_{\mu\theta}(-k'+k) \qquad (3.84)\end{aligned}$$

Substituting the gauge fixing condition in the momentum space given by

$$\widetilde{G}^a(k,\widetilde{A}^a_\mu) = -ik^\mu\widetilde{A}^a_\mu(k) \quad (3.85)$$

in the eqn.(3.78), we get

$$\widetilde{S}_{GF}\left[\widetilde{A}^a_\mu\right] = -\frac{1}{2\xi}\int\frac{d^4k}{(2\pi)^4}\widetilde{A}^a_\mu(-k)k^\mu k^\nu\widetilde{A}^a_\nu(k) \quad (3.86)$$

This additional term modifies the quadratic part of the path integral action as

$$M^{\mu\nu}_A = -k^2g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu \quad (3.87)$$

which is invertible. Hence, using the method of gauge fixing, the propagator of the gauge fields can be computed. In fact, the entire path integral in terms of $\overline{\mathcal{N}}$ and $\overline{\mathcal{D}}$ is well defined and now suitable for further computation.

Substituting Eq. (3.84) in Eq. (3.79), we get

$$\begin{aligned}\widetilde{S}_{GH}\left[\widetilde{A}^a_\mu,\overline{\widetilde{\chi}}^a,\widetilde{\chi}^b\right] &= -i\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\overline{\widetilde{\chi}}_c(k_1)\Big[k_1^2(2\pi)^4\delta^{(4)}(k_1-k_2)\delta_{cd} \\ &+ig_s f^{cad}k_1^\mu\widetilde{A}^a_{\mu\theta}(-k_2+k_1)\Big]\widetilde{\chi}_d(k_2) \qquad (3.88)\end{aligned}$$

The fields appearing in the eqn.(3.73) are anti-commuting variables, usually called Grassman variables or fields. The first term in the above equation describes the kinetic part of the Grassman fields $\overline{\widetilde{\chi}}_c$ and $\widetilde{\chi}_d$. Even though these fields are anti-commuting (fermonic fields), their propagation is bosonic in nature. Hence they are called ghost fields. The second term describes the interaction of the ghost fields with the gauge fields.

3.5 Regularisation and Renormalisation of YM Theory with n_f Fermions

In the last section, we demonstrated the quantization of YM theory using path integral approach. We also derived Feynman rules to compute gauge invariant products of quantum field operators. Using these Feynman rules, it is straightforward to compute observables such as scattering cross sections, decay rates etc. We can apply the standard techniques of perturbation theory treating the coupling constant g_s as an expansion parameter,

We know from quantum electrodynamics, the quantum corrections that enter via loops are often divergent. It comes from the large momentum region of the loop momenta and is called ultra-violet (UV) divergence. This remains to be the case for quantised YM theory as well. The standard approach to deal with UV divergences and to make reliable predictions involves two important steps: regularization and renormalization. Regularisation involves modifying the theory by introducing a suitable regulator so that the loop integrals appearing in the quantum corrections are made finite. The next step involves redefinition of fields and parameters of the regularised theory in such a way that the physical predictions of the theory are finite when the regularization (regulator) is removed, this is called renormalization. This redefinition is allowed because the parameters and fields appearing in the Lagrangian are not physical observables.

We will use dimensional regularization as it preserves all the symmetries of the theory. Here, the space-time dimension is taken to be $n = 4 + \varepsilon$ with $\varepsilon < 0$ which regularises the UV divergences appearing in the loop integrals. The renormalization is carried out by writing the original Lagrangian in n dimensions as follows:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_R \left[\psi_R, \overline{\psi}_R, A^a_{\mu,R}, \chi^a_R, \overline{\chi}^a_R, g_{sn,R}, m_R, \xi_R, n, \mu_R\right] \\ &\quad + \mathcal{L}^c \left[\psi_R, \overline{\psi}_R, A^a_{\mu,R}, \chi^a_R, \overline{\chi}^a_R, g_{sn,R}, m_R, \xi_R, n, Z_i, \mu_R\right] \end{aligned} \tag{3.89}$$

where, $\mathcal{L}_R$ is obtained by simply replacing all the parameters and fields by the respective ones with the subscript denoted by R. That is,

$$\mathcal{L}_R(\Phi_R, \alpha_R, n, \mu_R) = \mathcal{L}(\Phi \to \Phi_R, \alpha \to \alpha_R, n, \mu \to \mu_R) \tag{3.90}$$

with $\Phi = \{\psi, \overline{\psi}, A_\mu^a, \chi^a, \overline{\chi}^a\}$ and $\alpha = \{g_s, m, \xi\}$. $\mathcal{L}^c$ is so chosen that it preserves all the symmetries of the theory. We define,

$$\begin{aligned}
\mathcal{L}^c &= (Z_2 - 1)\overline{\psi}_R(i\partial\!\!\!/ - m_R)\psi_R - Z_2(Z_m - 1)m_R\overline{\psi}_R\psi_R \\
&\quad + (Z_g^{1/2}Z_2Z_3^{1/2} - 1)g_{sn,R}\overline{\psi}_R A\!\!\!/_R^a T^s\psi_R \\
&\quad - \frac{1}{4}(Z_3 - 1)(\partial_\mu A_{\nu,R}^a - \partial_\nu A_{\mu,R}^a)(\partial^\mu A_R^{\nu a} - \partial^\nu A_R^{\mu a}) \\
&\quad - \frac{1}{2}\left(Z_g^{\frac{1}{2}}Z_3^{\frac{3}{2}} - 1\right) g_{sn,R}f^{abc}(\partial_\mu A_{\nu,R}^a - \partial_\nu A_{\mu,R}^a)A_R^{\mu b}A_R^{\nu c} \\
&\quad - \frac{1}{4}\left(Z_g Z_3^2 - 1\right) g_{sn,R}^2 f^{abe}f^{cde}A_{\mu,R}^a A_{\nu,R}^b A_R^{\mu c}A_R^{\nu d} \\
&\quad + (\widetilde{Z}_3 - 1)i\;\partial^\mu\overline{\chi}_R^a\partial_\mu\chi_R^a \\
&\quad - \left(\widetilde{Z}_g^{\frac{1}{2}}\widetilde{Z}_3 Z_3^{\frac{1}{2}} - 1\right) ig_{sn,R}f^{abc}\partial^\mu\overline{\chi}_R^a\chi_R^b A_{\mu,R}^c
\end{aligned} \tag{3.91}$$

In n-dimension, the coupling constant has mass dimension $[M]^{(4-n)/2}$. We denote this dimensionful coupling constant by $g_{sn,R}$. This can be written in terms of a dimensionless coupling constant using

$$g_{sn,R} = \mu_R^{\frac{4-n}{2}} g_{s,R}(\mu_R^2) \tag{3.92}$$

where μ_R is an arbitrary mass scale and $g_{s,R}(\mu_R^2)$ is a dimensionless coupling constant. It is straightforward to show that after rescaling all the fields and the parameters as

$$\begin{aligned}
Z_3^{1/2}A_{\mu,R}^a &= A_\mu^a, & Z_2^{1/2}\psi_R &= \psi, \\
\widetilde{Z}_3^{1/2}\chi_R^a &= \chi^a, & \widetilde{Z}_3^{1/2}\overline{\chi}_R^a &= \overline{\chi}^a, \\
Z_g^{1/2}g_{sn,R} &= g_s(\mu^2)\mu^{\frac{4-n}{2}}, & Z_3^{1/2}\xi_R &= \xi, \qquad Z_m m_R = m
\end{aligned} \tag{3.93}$$

we reproduce the original Lagrangian in n dimensions,

$$\mathcal{L}_R + \mathcal{L}_c = \mathcal{L}\left(\overline{\psi}, \psi, A_\mu^a, \chi^a, \overline{\chi}^a, g_s, m, \xi, n, \mu\right) \tag{3.94}$$

In the above we have introduced a scale μ so that g_s is dimensionless in n-dimensions. In the following we will use the Feynman rules derived from $\mathcal{L}_R$

and $\mathcal{L}^c$ to compute the Green's functions. The difference in this approach is that we will be computing all the Green's functions in terms of the renormarlised parameters and fields and the results will explicitly contain the unknown constants Z_i and $\widetilde{Z}_i$, which are called renormalization constants. Our next step is to make various Green's functions finite, which were originally UV divergent, by adjusting the Z_i and $\widetilde{Z}_i$ suitably. We determine some of these renormalization constants order by order in perturbation theory in what follows.

In the following we will restrict ourselves to the determination of the renormalization constant Z_g which defines the renormalised coupling constant. This is done by computing Z_2, Z_3 and the combination $(Z_g Z_3)^{\frac{1}{2}} Z_2$. Z_2 and Z_3 are computed using one-loop corrected self energy of the fermionic fields and the vacuum polarization of gauge fields respectively. The combination $(Z_g Z_3)^{\frac{1}{2}} Z_2$ is determined from the one loop corrected fermion-antifermion-gauge boson vertex.

The vertex contribution comes from two different Feynman diagrams, namely

$$i g_{sn,R} \Gamma^{\mu}_{1ij} \quad = \quad g^3_{sn,R} \left(T^a T^c T^a\right)_{ij} I^{\mu}_1 \tag{3.95}$$

where,

$$I^{\mu}_1 \quad = \quad \int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\alpha \not{k} \gamma^\mu (\not{k} + \not{p}_1 + \not{p}_2) \gamma^\alpha}{k^2 (k+p_1)^2 (k+p_1+p_2)^2} \tag{3.96}$$

and

$$i g_{sn,R} \Gamma^{\mu}_{2ij} \quad = \quad -i g^3_{sn,R} f^{bca} \left(T^a T^b\right)_{ij} I^{\mu}_2 \tag{3.97}$$

where

$$I^{\mu}_2 = \int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\alpha (\not{k} + \not{p}_2) \gamma_\beta \Gamma^{\beta\mu\alpha}_3 (k, p_1+p_2, -k-p_1-p_2)}{k^2 (k+p_2)^2 (k+p_2+p_1)^2} \tag{3.98}$$

with

$$\Gamma^{\beta\mu\alpha}_3 (k_1, k_2, k_3) \quad = \quad \left[g^{\beta\mu} (k_1 - k_2)^\alpha + g^{\mu\alpha} (k_2 - k_3)^\beta + g^{\alpha\beta} (k_3 - k_1)^\mu\right] \tag{3.99}$$

Using Feynman parametrisation and integration in n dimension give

$$\int \frac{d^n k}{(2\pi)^n} \frac{k^\mu k^\nu}{k^2 (k+p_1)^2 (k+p_2+p_1)^2} \quad = \quad I^{\mu\nu}_{UV} + I^{\mu\nu}_{IR} \tag{3.100}$$

where

$$
\begin{aligned}
I^{UV}_{\mu\nu} &= -\frac{i}{16\pi^2}\left(\frac{-2p_1\cdot p_2}{4\pi}\right)^{n/2-2}\frac{\Gamma(3-n/2)\Gamma^2(n/2-1)}{\Gamma(n-2)} \\
&\quad \times\left[\frac{g_{\mu\nu}}{(n-2)(n-4)}\right] \qquad (3.101)
\end{aligned}
$$

$$
\begin{aligned}
I^{IR}_{\mu\nu} &= -\frac{i}{16\pi^2}\left(\frac{-2p_1\cdot p_2}{4\pi}\right)^{n/2-2}\frac{\Gamma(3-n/2)\Gamma^2(n/2-1)}{\Gamma(n-2)} \\
&\quad \times\frac{1}{p_1\cdot p_2}\left[p_{1\mu}p_{1\nu}\left(-\frac{2(n-3)}{(n-4)^2}+\frac{3}{2(n-4)}\right)+p_{2\mu}p_{2\nu}\left(-\frac{1}{2(n-4)}\right)\right. \\
&\quad \left.+(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu})\left(-\frac{1}{n-4}+\frac{1}{2(n-2)}\right)\right] \qquad (3.102)
\end{aligned}
$$

Using these results we obtain,

$$
\begin{aligned}
I^{\mu}_{1,UV} &= -\frac{i}{16\pi^2}\left(\frac{-2p_1\cdot p_2}{4\pi}\right)^{n/2-2}\frac{\Gamma(3-n/2)\Gamma^2(n/2-1)}{\Gamma(n-2)}\frac{n-2}{n-4}\gamma^\mu \\
&= -\frac{i}{16\pi^2}f_{12}f_n\frac{n-2}{n-4}\gamma^\mu \\
I^{\mu}_{2,UV} &= \frac{-i}{16\pi^2}f_{12}f_n\left(\frac{2n}{(n-2)(n-4)}+\frac{2}{n-4}\right) \qquad (3.103)
\end{aligned}
$$

where

$$
f_{12}=\left(\frac{-2p_1\cdot p_2}{4\pi}\right)^{n/2-2}\Gamma(3-n/2), \qquad f_n=\frac{\Gamma^2(n/2-1)}{\Gamma(n-2)} \qquad (3.104)
$$

Using the identities

$$
(T^aT^cT^a)_{ij}=\left(T^c\left(-\frac{1}{2}C_A+T^aT^a\right)\right)_{ij}, \qquad f^{bca}(T^aT^b)_{ij}=\frac{i}{2}C_A(T^c)_{ij} \qquad (3.105)
$$

where $C_A = N$, we get

$$ig_{sn,R}\Gamma^{\mu,c}_{ij,UV} = ig_{sn,R}\left(\Gamma^{\mu,c}_{1ij,UV} + \Gamma^{\mu,c}_{1ij,IR}\right) \tag{3.106}$$

$$\begin{aligned} ig_{sn,R}\Gamma^{\mu,c}_{ij,UV} &= g^3_{sn,R}\gamma^\mu\left(-\frac{i}{16\pi^2}f_{12}f_n\right)\left[\frac{n-2}{n-4}\left(T^c\left(-\frac{1}{2}C_A + T^aT^a\right)\right)_{ij}\right. \\ &\left. + \left(\frac{2n}{(n-2)(n-4)} + \frac{2}{n-4}\right)\frac{1}{2}C_A(T^c)_{ij}\right] \end{aligned} \tag{3.107}$$

The gauge boson contribution to vacuum polarization is given by

$$\begin{aligned} \Pi^g_{\mu\nu,ab} &= -\frac{1}{2}f^{cad}f^{dbc}g^2_{sn,R}\int\frac{d^nk}{(2\pi)^n}\frac{1}{k^2(k+p)^2} \\ &\times\Gamma_{3,\lambda\mu\sigma}(k,p,-k-p)\Gamma^{\sigma\ \lambda}_{3,\nu}(k+p,-p,-k) \end{aligned} \tag{3.108}$$

Using,

$$\begin{aligned} \int\frac{d^nk}{(2\pi)^n}\frac{k_\mu k_\nu}{k^2(k+p)^2} &= -\frac{i}{16\pi^2}f_pf_n\left(-p^2\frac{g_{\mu\nu}}{n} + p_\mu p_\nu\right)\left(\frac{n}{2(n-1)(n-4)}\right) \\ \int\frac{d^nk}{(2\pi)^n}\frac{k_\mu}{k^2(k+p)^2} &= -\frac{i}{16\pi^2}f_pf_np_\mu\left(-\frac{1}{n-4}\right) \\ \int\frac{d^nk}{(2\pi)^n}\frac{1}{k^2(k+p)^2} &= \frac{i}{16\pi^2}f_pf_n\left(-\frac{2}{n-4}\right) \end{aligned} \tag{3.109}$$

where

$$f_p = \left(-\frac{p^2}{4\pi}\right)^{n/2-2} \tag{3.110}$$

we obtain,

$$\begin{aligned} \Pi^g_{\mu\nu,ab} &= \left(-\frac{i}{16\pi^2}\right)g^2_{sn,R}f^{cad}f^{dbc}f_pf_n\left(\frac{1}{n-4}\right)\frac{1}{2(n-1)} \\ &\times\left[g_{\mu\nu}(-p^2)(6n-5) + p_\mu p_\nu(7n-6)\right] \end{aligned} \tag{3.111}$$

The fermionic contribution to vacuum polarization is given by

$$\begin{aligned}
\Pi^q_{\mu\nu,ab} &= -g^2_{sn,R}(T^aT^b)_{ii}\int\frac{d^nk}{(2\pi)^n}\left[\frac{Tr\left(\gamma_\nu\not{k}\gamma_\mu(\not{k}+\not{p})\right)}{k^2(k+p)^2}\right] \\
&= \frac{i}{16\pi^2}g^2_{sn,R}(T^aT^b)_{ii}f_pf_n\left(\frac{4}{n-4}\right) \\
&\quad\times\left(-\frac{n-2}{n-1}\right)\left[-p^2g_{\mu\nu}+p_\mu p_\nu\right] \qquad (3.112)
\end{aligned}$$

The ghost loop contribution to the vacuum polarization is

$$\begin{aligned}
\Pi^{gh}_{\mu\nu,ab} &= g^2_{sn,R}f^{acd}f^{bdc}\int\frac{d^nk}{(2\pi)^n}\left[\frac{k_\mu(k+p)_\nu}{k^2(k+p)^2}\right] \\
&= -\frac{i}{16\pi^2}f_pf_ng^2_{sn,R}f^{acd}f^{bdc}\frac{1}{2(n-1)(n-4)} \\
&\quad\times\left[g_{\mu\nu}(-p^2)+p_\mu p_\nu(2-n)\right] \qquad (3.113)
\end{aligned}$$

We finally arrive at

$$\begin{aligned}
\Pi_{\mu\nu,ab} &= \Pi^{gh}_{\mu\nu,ab}+\Pi^q_{\mu\nu,ab}+\Pi^g_{\mu\nu,ab} \\
&= -\frac{i}{16\pi^2}f_pf_ng^2_{sn,R}\frac{1}{(n-1)(n-4)}\ \Big[n_f(T^aT^b)_{ii}(8-4n)(-p^2g_{\mu\nu}+p_\mu p_\nu) \\
&\quad +f^{cad}f^{dbc}(3n-2)(-p^2g_{\mu\nu}+p_\mu p_\nu)\Big] \qquad (3.114)
\end{aligned}$$

where n_f is the number fermion flavours in the theory. To compute Z_1 we need to compute the self energy of the fermion:

$$\begin{aligned}
\Sigma_{ij} &= -g^2_{sn,R}(T^aT^a)_{ij}\int\frac{d^nk}{(2\pi)^n}\frac{\gamma_\mu\not{k}\gamma^\mu}{k^2(k+p)^2} \\
&= -\frac{i}{16\pi^2}f_pf_ng^2_{sn,R}(T^aT^a)_{ij}\not{p}\left(\frac{2-n}{n-4}\right) \qquad (3.115)
\end{aligned}$$

The renormalization constants Z_i and $\widetilde{Z}_i$ in $\mathcal{L}^c$ are fixed by demanding that all the Green's functions of the theory are finite. There is of course orbitraryness in determining these constants because they can contain finite terms in addition to UV divergences when the regularization is removed. This leads to various renormalization prescriptions or schemes. We will use modified minimal subtraction ($\overline{MS}$) scheme in the following. We can use fermion-antifermion-gauge boson vertex, gauge boson propagator and the fermion propagator computed to one loop level along with the contribution coming from the Lagrangian $\mathcal{L}^c$ to determine the renormalization constants Z_g, Z_2 and Z_3. We find

$$\begin{aligned}
\delta^{ab}\left(Z_3-1\right) &= \Bigg\{\frac{g_{sn,R}^2}{16\pi^2} f_p f_n \frac{1}{(n-1)(n-4)}\Bigg[n_f \left(T^aT^b\right)_{ii}(4n-8) \\
&\quad +f^{cad}f^{dbc}(3n-2)\Bigg]\Bigg\}_{\overline{MS}} \\
\delta_{ij}\left(Z_2-1\right) &= \Bigg\{\frac{g_{sn,R}^2}{16\pi^2} f_p f_n \left[\left(\frac{n-2}{n-4}\right)\left(T^aT^a\right)_{ij}\right]\Bigg\}_{\overline{MS}} \\
T^c_{ij}\left(Z_g^{\frac{1}{2}} Z_3^{\frac{1}{2}} Z_2 - 1\right) &= \Bigg\{\frac{g_{sn,R}^2}{16\pi^2} f_p f_n \Bigg[\left(T^c\left(-\frac{1}{2}C_A + T^aT^a\right)\right)_{ij}\frac{n-2}{n-4} \\
&\quad +\frac{1}{2}C_A\left(T^c\right)_{ij}\left(\frac{2n}{(n-2)(n-4)}+\frac{2}{n-4}\right)\Bigg]\Bigg\}_{\overline{MS}}
\end{aligned} \tag{3.116}$$

In the above, the subscript $\overline{MS}$ means that only those terms that diverge in the limit $n \to 4$ and those terms proportional to $\log(4\pi)$ and Euler's constant γ_E are kept and rest of the terms are set to zero. This prescription defines the renormalization constant in $\overline{MS}$ scheme. We find

$$\begin{aligned}
Z_3 &= 1+\frac{g_{s,R}(\mu_R^2)}{16\pi^2}\left(\frac{8}{3}n_f T_f - \frac{10}{3}C_A\right)\frac{1}{\hat{\varepsilon}} \\
Z_2 &= 1+\frac{g_{s,R}(\mu_R^2)}{16\pi^2}\left(2C_F\right)\frac{1}{\hat{\varepsilon}} \\
Z_g^{\frac{1}{2}} Z_2 Z_3^{\frac{1}{2}} &= Z_1 = 1+\frac{g_{s,R}(\mu_R^2)}{16\pi^2}\left(2C_A+2C_F\right)\frac{1}{\hat{\varepsilon}}
\end{aligned} \tag{3.117}$$

where

$$\frac{1}{\hat{\varepsilon}} = \frac{1}{\varepsilon}\left(1+\frac{\varepsilon}{2}\left(-\ln(4\pi)+\gamma_E\right)\right) \tag{3.118}$$

This implies

$$\begin{aligned} Z_g^{\frac{1}{2}} &= \frac{Z_1}{Z_2 Z_3^{\frac{1}{2}}} \\ &= 1 + \frac{g_{s,R}^2(\mu_R^2)}{16\pi^2}\left(\frac{11}{3}C_A - \frac{4}{3}n_f T_f\right)\frac{1}{\hat{\varepsilon}} \end{aligned} \tag{3.119}$$

Notice that in $\overline{MS}$ scheme, the renormalization constant contains a finite piece $(-\ln(4\pi) + \gamma_E)/2$ along with a divergent piece $1/\varepsilon$ in four dimensions.

Recall that the renormalised coupling constant $g_{s,R}(\mu_R^2)$ is related to $g_s(\mu^2)$ through Z_g as follows:

$$g_s(\mu^2)\mu^{-\frac{\varepsilon}{2}} = Z_g^{\frac{1}{2}}\left(g_{s,R}(\mu_R^2), \frac{1}{\varepsilon}\right) g_{s,R}(\mu_R^2)\mu_R^{-\frac{\varepsilon}{2}} \tag{3.120}$$

In the next section, we will study the scale dependence of the coupling constant using renormalization group equation.

3.6 Asymptotic Freedom

In the last section, we derived the renormalization constant Z_g in $\overline{MS}$ scheme using dimensional regularization. If we define $\hat{a}_s(\mu^2)$ and $a_s(\mu_R^2)$ by

$$\hat{a}_s(\mu^2) = \frac{g_s^2(\mu^2)}{16\pi^2}, \qquad a_s(\mu_R^2) = \frac{g_{s,R}^2(\mu_R^2)}{16\pi^2} \tag{3.121}$$

we find from eqn.(3.120)

$$\hat{a}_s(\mu^2)\mu^{-\frac{\varepsilon}{2}} = Z_g\left(a_s(\mu_R^2), \frac{1}{\varepsilon}\right) a_s(\mu_R^2)\mu_R^{-\frac{\varepsilon}{2}} \tag{3.122}$$

The fact that the left hand side of the above equation is independent of the renormalization scale μ_R, gives what is called renormalization group (RG) equation. Since

$$\mu_R^2 \frac{d\hat{a}_s}{d\mu_R^2} = 0 \tag{3.123}$$

we get

$$\mu_R^2 \frac{da_s(\mu_R^2)}{d\mu_R^2} = a_s(\mu_R^2)\left(\frac{\varepsilon}{2} - \mu_R^2 \frac{d\ln Z_g}{d\mu_R^2}\right) \tag{3.124}$$

Defining the beta function $\beta(a_s(\mu_R^2))$ through,

$$\begin{aligned}\mu_R^2 \frac{da_s}{d\mu_R^2} &= \beta(a_s(\mu_R^2)) \\ &= -\sum_{i=0}^{\infty} a_s^{i+2}(\mu_R^2)\beta_i \end{aligned} \tag{3.125}$$

and using the one loop result for the Z_g given in eqn.(3.119), we can compute β_0 as:

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_f \tag{3.126}$$

The solution to eqn.(3.125) is given by

$$a_s(Q^2) = \frac{a_s(\mu_0^2)}{1 + \beta_0 a_s(\mu_0^2) \ln(Q^2/\mu_0^2)} + \mathcal{O}(a_s^2(\mu_0^2)) \tag{3.127}$$

The renormalised mass $m(\mu_R^2)$ is related to $\hat{m}(\mu^2)$ and is given by

$$\hat{m}(\mu^2) = Z_m \left(a_s(\mu_R^2), \frac{1}{\varepsilon} \right) m(\mu_R^2) \tag{3.128}$$

The renormalization group equation for $m(\mu_R^2)$ is given by

$$\mu_R^2 \frac{d \ln Z_m}{d\mu_R^2} + \mu_R^2 \frac{d \ln m}{d\mu_R^2} = 0 \tag{3.129}$$

We now define

$$\begin{aligned}\mu_R^2 \frac{d \ln Z_m}{d\mu_R^2} &= \gamma_m(a_s(\mu_R^2)) \\ &= \sum_{i=1}^{\infty} \gamma_m^{(i)} \; a_s^{(i)}(\mu_R^2) \end{aligned} \tag{3.130}$$

where γ_m is the anomalous dimension of the mass m. To order $a_s(\mu_R^2)$ one finds,

$$\begin{aligned} Z_m &= 1 - \frac{6}{\epsilon} C_F a_s(\mu_R^2) + \mathcal{O}(a_s^2) \\ \ln Z_m &= -\frac{6}{\epsilon} C_F a_s(\mu_R^2) \end{aligned} \tag{3.131}$$

This implies

$$\begin{aligned} \mu_R^2 \frac{d \ln Z_m}{d\mu_R^2} &= -\frac{6C_F}{\epsilon}\beta(a_s) \\ &= -\frac{6C_F}{\epsilon} a_s \left(\frac{\epsilon}{2} - \mu_R^2 \frac{d \ln Z_g}{d\mu_R^2}\right) \\ &= -3C_F a_s(\mu_R^2) + \mathcal{O}(a_s^2) \end{aligned} \tag{3.132}$$

From this result, we find,

$$\gamma_m^{(0)} = -3C_F \tag{3.133}$$

The solution to the eqn.(3.130) to leading order is given by

$$m(Q^2) = m(\mu_0^2) \left(\frac{a_s(Q^2)}{a_s(\mu_0^2)}\right)^{3C_F/\beta_0} + \mathcal{O}(a_s^2(\mu_0^2)) \tag{3.134}$$

In $\bar{MS}$ scheme, the renormalization constant takes the following form:

$$Z_g\left(a_s(\mu_R^2, \xi), \frac{1}{\varepsilon}, \xi\right) = 1 + \frac{Z_{-1}\left(a_s(\mu_R^2, \xi), \xi\right)}{\epsilon} + \frac{Z_{-2}\left(a_s(\mu_R^2, \xi), \xi\right)}{\epsilon^2} + \dots \tag{3.135}$$

where, ξ is the gauge fixing parameter. Differentiating eqn.(3.135) with respect to ξ, we get

$$\frac{da_s}{d\xi} + \frac{1}{\epsilon}\left(\frac{dZ_{-1}}{d\xi} a_s + Z_{-1}\frac{da_s}{d\xi}\right) + \frac{1}{\epsilon^2}\left(\frac{dZ_{-2}}{d\xi} a_s + Z_{-2}\frac{da_s}{d\xi}\right) + \dots = 0$$

where we have suppressed the arguments of Z_{-i} and a_s for simplicity. Comparing the coefficients of $1/\varepsilon$ on both sides, we obtain,

$$\frac{da_s}{d\xi} = 0, \qquad \frac{dZ_{-i}}{d\xi} = 0, \qquad i = 1, 2, \cdots \tag{3.136}$$

Hence Z_g is independent of gauge fixing parameter. Suppose, we choose a renormalization scheme in which the coupling constant renormalization $\widetilde{Z}_g$ has the following expansion:

$$\begin{aligned} \widetilde{Z}_g\left(\alpha_s(\mu_R^2, \xi), \frac{1}{\varepsilon}, \xi\right) &= \widetilde{Z}_0\left(a_s(\mu_R^2, \xi), \xi\right) + \frac{\widetilde{Z}_{-1}\left(a_s(\mu_R^2, \xi), \xi\right)}{\epsilon} \\ &\quad + \frac{\widetilde{Z}_{-2}\left(a_s(\mu_R^2, \xi), \xi\right)}{\epsilon^2} + \dots \end{aligned} \tag{3.137}$$

Differentiating eqn.(3.137) with respect to ξ, we get

$$\left(\frac{d\widetilde{Z}_0}{d\xi}a_s + \widetilde{Z}_0\frac{da_s}{d\xi}\right) \qquad +\frac{1}{\epsilon}\left(\frac{d\widetilde{Z}_{-1}}{d\xi}a_s + \widetilde{Z}_{-1}\frac{da_s}{d\xi}\right)$$
$$+\frac{1}{\epsilon^2}\left(\frac{d\widetilde{Z}_{-2}}{d\xi}a_s + \widetilde{Z}_{-2}\frac{da_s}{d\xi}\right) + \ldots = 0 \quad (3.138)$$

This implies that the renormalization constant $\widetilde{Z}_g$ is gauge dependent.

The general structure of the renormalization constant Z_i and $\widetilde{Z}_i$ can be found in $\overline{MS}$ scheme in terms of β_i and the corresponding anomalous dimensions (γ_i for Z_i and $\widetilde{\gamma}_i$ for $\widetilde{Z}_i$).

$$\mu_R^2\frac{d\ln Z_i}{d\mu_R^2} = \gamma_i(a_s(\mu_R^2)), \qquad \mu_R^2\frac{d\ln \widetilde{Z}_i}{d\mu_R^2} = \widetilde{\gamma}_i(a_s(\mu_R^2)) \quad (3.139)$$

We first determine the structure of Z_g

$$Z_g = 1 + a_s\frac{Z_{-1}^{(1)}}{\epsilon} + a_s^2\left(\frac{Z_{-2}^{(2)}}{\epsilon^2} + \frac{Z_{-1}^{(2)}}{\epsilon}\right) + \ldots \quad (3.140)$$

$$\ln Z_g = a_s\frac{Z_{-1}^{(1)}}{\epsilon} + a_s^2\left[\frac{Z_{-2}^{(2)}}{\epsilon^2} + \frac{Z_{-1}^{(2)}}{\epsilon} - \frac{1}{2\epsilon^2}(Z_{-1}^{(1)})^2\right] \quad (3.141)$$

$$\mu_R^2\frac{d\ln Z_g}{d\mu_R^2} = a_s\frac{Z_{-1}^{(1)}}{2} + a_s^2\left[Z_{-1}^{(2)} + \frac{1}{\epsilon}\left(Z_{-2}^{(2)} - (Z_{-1}^{(1)})^2\right)\right] \quad (3.142)$$

On the other hand,

$$\mu_R^2\frac{d\ln Z_g(\mu_R^2)}{d\mu_R^2} = \left(\frac{\varepsilon}{2} - \frac{\beta\left(a_s(\mu_R^2)\right)}{a_s(\mu_R^2)}\right)$$
$$= \frac{\varepsilon}{2} + \sum_{i=0}^{\infty} a_s^{i+2}(\mu_R^2)\beta_i \quad (3.143)$$

Comparing the powers of $a_s(\mu_R^2)$ in eqns. (3.142,3.143) we find

$$Z_{-1}^{(1)} = 2\beta_0, \qquad Z_{-1}^{(2)} = \beta_1, \qquad Z_{-2}^{(2)} = 4\beta_0^2 \quad (3.144)$$

Hence,

$$Z_g = 1 + a_s(\mu_R^2)\frac{2\beta_0}{\epsilon} + a_s^2(\mu_R^2)\left(\frac{4\beta_0^2}{\epsilon^2} + \frac{\beta_1}{\epsilon}\right) + \ldots \quad (3.145)$$

3.7 Wilson Coefficients

In this section we will study the renormalization group equation satisfied by the Wilson coefficients given in eqn.(3.28). Defining

$$F_{i,N}(Q^2) = \int_0^1 dx_{Bj} x_{Bj}^{N-1} F_i(x_{Bj}, Q^2) \qquad i = L, 2 \tag{3.146}$$

In quantum field theory (QFT), the composite operators (say those appearing in $\hat{A}^a_{i,N}(p^2)$) require an over-all renormalization in addition to renormalization of the parameters and fields through the renormalization constants $Z_i, \widetilde{Z}_i$ that appear in the Lagrangian. The over-all renormalization constants for the composite operators are computed in the same way one computes $Z_i, \widetilde{Z}_i$. We can use dimensional regularization to regulate the new divergences that emerge from the local nature of the composite operators and renormalise them using $\overline{MS}$ scheme. The renormalization introduces a scale at which these operators are renormalised. This scale is analogous to the renormalization scale and there exists no compelling reason for them to be same. This new scale is called the factorization scale, μ_F. Let us denote these new set of renormalization constants by $Z_{ab,N}(\mu_F^2, 1/\varepsilon)$. Hence, the renormalised operator matrix elements are defined by

$$\hat{A}^a_{i,N}(p^2) = Z_{ab,N}\left(\mu_F^2, \frac{1}{\varepsilon}\right) A_{b,i,N}(p^2, \mu_F^2) \tag{3.147}$$

This implies

$$F_{i,N}(Q^2) = \sum_{a,b} Z_{ab,N}\left(\mu_F^2, \frac{1}{\varepsilon}\right) A^b_{i,N}(p^2, \mu_F^2) \hat{C}^a_{i,N}(Q^2), \qquad i = L, 2 \tag{3.148}$$

The fact that the left side of the above equation is finite implies,

$$C^b_{i,N}(Q^2, \mu_F^2) = \sum_a Z_{ab,N}\left(\mu_F^2, \frac{1}{\varepsilon}\right) \hat{C}^a_{i,N}(Q^2), \tag{3.149}$$

is finite. Hence the eqn.(3.28) now becomes,

$$F_{i,N}(Q^2) = \sum_a A^a_{i,N}(p^2, \mu_F^2) C^a_{i,N}(Q^2, \mu_F^2), \qquad i = L, 2 \tag{3.150}$$

To summarise, in QFT, the separation of long distance part denoted by a set of operator matrix elements $\hat{A}^a_{i,N}(p^2)$ and the short distance part usually called

Wilson's coefficients $\hat{C}^a_{i,N}(Q^2)$ is arbitrary upto a scale that separates them. This scale is called the factorization scale. Notice that the observable $F_{i,N}(Q^2)$ does not depend on the scale μ_F. That is,

$$\mu_F^2 \frac{d}{d\mu_F^2} F_{i,N}(Q^2) \quad = \quad 0 \qquad i = L, 2 \tag{3.151}$$

This implies

$$\sum_a C^a_{i,N}(Q^2, \mu_F^2)\left(\mu_F^2 \frac{d}{d\mu_F^2} A^a_{i,N}(p^2, \mu_F^2)\right) \quad = \quad -\sum_a A^a_{i,N}(p^2, \mu_F^2) \left(\mu_F^2 \frac{d}{d\mu_F^2} C^a_{i,N}(Q^2, \mu_F^2)\right) \tag{3.152}$$

Since

$$\mu_F^2 \frac{d}{d\mu_F^2} \hat{A}^a_{i,N}(p^2) = 0, \tag{3.153}$$

$$\mu_F^2 \frac{d}{d\mu_F^2} A^a_{i,N}(p^2, \mu_F^2) = \sum_b P_{ab,N}(\mu_F^2) A_{b,i,N}(p^2, \mu_F^2) \tag{3.154}$$

where, $P_{ab,N}(\mu_F^2)$ is defined as

$$\sum_c Z^{-1}_{ac,N} \mu_F^2 \frac{d}{d\mu_F^2} Z_{cb,N}(\mu_F^2) \quad = \quad -P_{ab,N}(\mu_F^2) \tag{3.155}$$

Substituting eqn.(3.154) in eqn.(3.152), we get

$$\sum_a \left(\mathbb{I}\mu_F^2 \frac{d}{d\mu_F^2} + P_N(\mu_F^2)\right)_{ab} \left(C_{i,N}(Q^2, \mu_F^2)\right)^a \quad = \quad 0 \tag{3.156}$$

where we introduce a matrix notation in which $P_{ab,N}(\mu_F^2)$ is the ab-th matrix element of a matrix $P_N(\mu_F^2)$ and $C^a_{i,N}(Q^2, \mu_F^2)$ is a component of a-th vector denoted by $C_{i,N}(Q^2, \mu_F^2)$. We would like to find out the behavior of $C_{i,N}(Q^2, \mu_F^2)$ when Q^2 is large for fixed value of N and μ_F^2. It is computable using perturbative method. We can write $C_{i,N}$ as a series expansion in a_s,

$$C_{i,N}(Q^2, \mu_F^2) \quad = \quad \sum_{j=0}^{\infty} a_s^j(\mu_R^2) C^{(j)}_{i,N}\left(Q^2, \mu_F^2, \mu_R^2\right) \tag{3.157}$$

The RHS of the above equation is independent of the renormalization scale μ_R. Hence we can choose $\mu_R = \mu_F$ for the rest of the analysis. Since the coefficient is dimensionless,

$$C_{i,N}(Q^2, \mu_F^2) = C_{i,N}\left(\frac{Q^2}{\mu_F^2}, a_s(\mu_F^2)\right) \tag{3.158}$$

The total derivative with respect to μ_F^2 gives

$$\mu_F^2 \frac{d}{d\mu_F^2} = \mu_F^2 \frac{\partial}{\partial \mu_F^2} + \beta(a_s(\mu_F^2)) \frac{\partial}{\partial a_s(\mu_F^2)} \tag{3.159}$$

Parametrising $Q^2 = e^t \overline{Q}^2$ with $\overline{Q}$ fixed, we find

$$\begin{aligned} \frac{\partial}{\partial t} C_{i,N}\left(e^t \frac{\overline{Q}^2}{\mu_F^2}, a_s(\mu_F^2)\right) &= e^t \frac{\overline{Q}^2}{\mu_F^2} \frac{\partial}{\partial \lambda} C_{i,N}\left(\lambda, a_s(\mu_F^2)\right) \\ \mu_F^2 \frac{\partial}{\partial \mu_F^2} C_{i,N}\left(e^t \frac{\overline{Q}^2}{\mu_F^2}, a_s(\mu_F^2)\right) &= -e^t \frac{\overline{Q}^2}{\mu_F^2} \frac{\partial}{\partial \lambda} C_{i,N}\left(\lambda, a_s(\mu_F^2)\right) \end{aligned} \tag{3.160}$$

This implies

$$\left(-\frac{\partial}{\partial t} + \beta(a_s(\mu_F^2)) \frac{\partial}{\partial a_s(\mu_F^2)} + P(a_s(\mu_F^2))\right) C_{i,N}\left(e^t \frac{\overline{Q}^2}{\mu_F^2}, a_s(\mu_F^2)\right) = 0 \tag{3.161}$$

We will solve the above equation by introducing an auxiliary function $\overline{a}_s(t, a_s(\mu_F^2))$ which depends on t as well as $a_s(\mu_F^2)$ satisfying

$$\frac{d}{dt} \overline{a}_s\left(t, a_s(\mu_F^2)\right) = \beta\left(\overline{a}_s\left(t, a_s(\mu_F^2)\right)\right) \tag{3.162}$$

with the boundary condition

$$\overline{a}_s\left(t = 0, a_s(\mu_F^2)\right) = a_s(\mu_F^2) \tag{3.163}$$

This is called running coupling constant. Using the eqn.(3.162), we obtain

$$\left(-\frac{\partial}{\partial t} + \beta(a_s(\mu_F^2)) \frac{\partial}{\partial a_s(\mu_F^2)}\right) \overline{a}_s\left(t, a_s(\mu_F^2)\right) = 0 \tag{3.164}$$

which implies that any arbitrary function $\mathcal{C}_{i,N}$ depending on t and $a_s(\mu_F^2)$ *only* through the axillary function $\overline{a}_s\left(t, a_s(\mu_F^2)\right)$ will also satisfy

$$\left(-\frac{\partial}{\partial t} + \beta(a_s(\mu_F^2)) \frac{\partial}{\partial a_s(\mu_F^2)}\right) \mathcal{C}_{i,N}\left(\frac{\overline{Q}^2}{\mu_F^2}, \overline{a}_s\left(t, a_s(\mu_F^2)\right)\right) = 0 \tag{3.165}$$

Hence, the solution to eqn.(3.161) takes the form:

$$\begin{aligned} C_{i,N}\left(e^t\frac{\overline{Q}^2}{\mu_F^2}, a_s(\mu_F^2)\right) &= \exp\left(\int_0^{a_s(\mu_F^2)}\frac{d\alpha}{\beta(\alpha)}P_N(\alpha)\right) \\ &\quad\times\mathcal{C}_{i,N}\left(\frac{\overline{Q}^2}{\mu_F^2}, \overline{a}_s\left(t, a_s(\mu_F^2)\right)\right) \end{aligned} \tag{3.166}$$

Rewriting the argument of the exponential as

$$\begin{aligned} \int_0^{a_s(\mu_F^2)}\frac{d\alpha}{\beta(\alpha)}P_N(\alpha) &= \int_0^{\overline{a}_s(t,a_s(\mu_F^2))}\frac{d\alpha}{\beta(\alpha)}P_N(\alpha) + \int_{\overline{a}_s(t,a_s(\mu_F^2))}^{a_s(\mu_F^2)}\frac{d\alpha}{\beta(\alpha)}P_N(\alpha) \\ &= \int_0^{\overline{a}_s(t,a_s(\mu_F^2))}\frac{d\alpha}{\beta(\alpha)}P_N(\alpha) - \int_0^t dt' P_N\left(\overline{a}_s(t', a_s(\mu_F^2))\right) \end{aligned} \tag{3.167}$$

we obtain,

$$\begin{aligned} C_{i,N}\left(e^t\frac{\overline{Q}^2}{\mu_F^2}, a_s(\mu_F^2)\right) &= \exp\left(\int_0^{\overline{a}_s(t,a_s(\mu_F^2))}\frac{d\alpha}{\beta(\alpha)}P_N(\alpha)\right) \\ &\quad\times\exp\left(-\int_0^t dt' P_N\left(\overline{a}_s(t', a_s(\mu_F^2))\right)\right) \\ &\quad\times\mathcal{C}_{i,N}\left(\frac{\overline{Q}^2}{\mu_F^2}, \overline{a}_s\left(t, a_s(\mu_F^2)\right)\right) \end{aligned} \tag{3.168}$$

We can determine $\mathcal{C}$ as follows: at $t = 0$, we get

$$\begin{aligned} C_{i,N}\left(\frac{\overline{Q}^2}{\mu_F^2}, a_s(\mu_F^2)\right) &= \exp\left(\int_0^{a_s(\mu_F^2)}\frac{d\alpha}{\beta(\alpha)}P_N(\alpha)\right) \\ &\quad\times\mathcal{C}_{i,N}\left(\frac{\overline{Q}^2}{\mu_F^2}, a_s(\mu_F^2)\right) \end{aligned} \tag{3.169}$$

Replacing $a_s(\mu_F^2)$ by $\overline{a}_s(t, a_s(\mu_F^2))$ in the above equation, we get

$$\begin{aligned} \mathcal{C}_{i,N}\left(\frac{\overline{Q}^2}{\mu_F^2}, \overline{a}_s(t, a_s(\mu_F^2))\right) &= \exp\left(-\int_0^{\overline{a}_s(t,a_s(\mu_F^2))} \frac{d\alpha}{\beta(\alpha)} P_N(\alpha)\right) \\ &\times C_{i,N}\left(\frac{\overline{Q}^2}{\mu_F^2}, \overline{a}_s\left(t, a_s(\mu_F^2)\right)\right) \end{aligned} \qquad (3.170)$$

Substituting the above equation in the eqn.(3.168), we obtain

$$\begin{aligned} C_{i,N}\left(e^t \frac{\overline{Q}^2}{\mu_F^2}, a_s(\mu_F^2)\right) &= \exp\left(-\int_0^t dt' P_N\left(a_s(t', a_s(\mu_F^2))\right)\right) \\ &\times C_{i,N}\left(\frac{\overline{Q}^2}{\mu_F^2}, \overline{a}_s\left(t, a_s(\mu_F^2)\right)\right) \end{aligned} \qquad (3.171)$$

Notice that the t dependence of the Wilson coefficients is controlled by the running coupling constant $a_s(t, a_s(\mu_F^2))$. The solution to its renormalization group equation (eqn.(3.162)) with the boundary condition $a_s(t = 0, a_s(\mu_F^2)) = a_s(\mu_F^2)$ is given by

$$\overline{a}_s(t, a_s(\mu_F^2)) = \frac{a_s(\mu_F^2)}{1 + t\beta_0 a_s(\mu_F^2)} + \mathcal{O}(a_s^2(\mu_F^2)) \qquad (3.172)$$

If we restrict ourselves to non-singlet combinations of structure functions such as $F_2^{ep} - F_2^{en}$ or $F_2^{\nu P} - F_2^{\overline{\nu} P}$ where p and n are proton and neutron targets respectively, then only the non-singlet operator defined by

$$\mathcal{O}^a_{\mu_1 \cdots \mu_n} = \frac{i^{n-1}}{n!} \left\{\overline{\psi} T^a \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_n} \psi\right\}_S \qquad (3.173)$$

will contribute. Here, $D_\mu = \partial_\mu - i g_s A^a_\mu T^a$.

As expected the running coupling constant vanishes at large t. Using

$$\begin{aligned} P_N(\alpha) &= \sum_{j=1}^{\infty} \alpha^j P_N^{(j-1)} \\ C_{i,N}\left(\frac{\overline{Q}^2}{\mu_F^2}, \alpha\right) &= \sum_{j=0}^{\infty} \alpha^j C_{i,N}^{(j)}\left(\frac{\overline{Q}^2}{\mu_F^2}\right) \end{aligned} \qquad (3.174)$$

where $C^{(0)}_{i,N}\left(\overline{Q}^2/\mu_F^2\right) = C^{(0)}_{i,N}$ is independent of $\overline{Q}^2$ and μ_F^2, we obtain

$$\lim_{t\to\infty} C_{i,N}\left(e^t\frac{\overline{Q}^2}{\mu_F^2}, a_s(\mu_F^2)\right) \quad = \quad C^{(0)}_{i,N} \tag{3.175}$$

which is independent of $\overline{Q}^2$ as well as μ_F^2 and depends only N. This implies that if we invert N dependent result $C^{(0)}_{i,N}$ into x_{Bj} we will find that the Wilson Coefficients will depend only on x_{Bj}. In other words, one recovers scaling at large t (equivalently large Q^2). This behavior is attributed to the vanishing of running coupling constant at large energy scales. As we have already discussed in the previous section, this behavior of the coupling constant is the important feature of YM theory with certain number of fermions.

3.8 Infrared Safe Observables

In the last section we studied the behavior of Wilson coefficients of non-singlet structure function in the Björken limit. Thanks to operator product expansion and the asymptotic freedom , we can compute them as a power series expansion in $a_s(\mu_R^2)$ using the perturbation theory and also make predictions that can be tested in the experiments. In fact we can demonstrate the scaling in the Björken limit. The logarithmic pattern of scaling violation, an important prediction of the theory, has been verified by deep inelastic experiments confirming the correctness of the theory.

In this section, we will study a completely new process namely hadroproduction in e^+e^- annihilation. Here the cross section corresponds to summing all the final states involving hadrons in the e^+e^- collision. To lowest order in strong coupling constant, the leading contribution comes from the production of a pair of quark (q) and an anti-quark ($\overline{q}$). To order a_s, real gluons emitted from the quark and anti-quark states and virtual gluons in the loops contribute to the cross section (see Fig. (3.1)). These quarks,anti-quarks and gluons will eventually hadronize to produce hadrons which are then summed. Naively one would expect the cross section for producing these partonic states is identical to that for producing hadronic states because the sum over all the final states is carried out. To this order in a_s and α (electromagnetic coupling constant), only s channel processes contribute. The tree level cross section $e^+ + e^- \to q + \overline{q}$ is straight forward to compute, we denote this by $\hat{\sigma}^{(0)}$. To order a_s, there are two types of processes that contribute to the total cross section: gluon emissions and virtual corrections to the tree level process. They are given by

$$e^+ + e^- \quad \rightarrow \quad q + \overline{q} + g \tag{3.176}$$

$$e^+ + e^- \quad \rightarrow \quad q + \overline{q} + \text{one} \quad \text{loop} \tag{3.177}$$

Let us begin with the computation of virtual gluon contribution. This involves

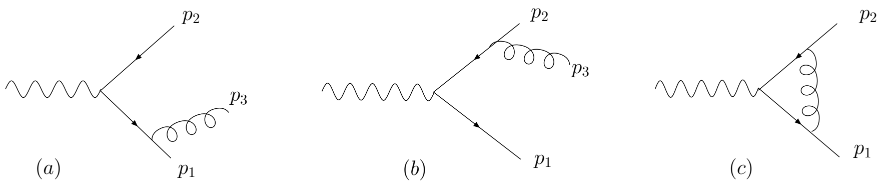


Figure 3.1: Feynman diagrams for the process $\gamma^* \rightarrow q\bar{q}$ with quantum corrections. Real diagrams are shown in (a), (b) and virtual diagrams in (c).

computation of an integral given by

$$\mathcal{I} = \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{N}(k)}{k^2(k+p_1)^2(k-p_2)^2} \tag{3.178}$$

where one finds $\mathcal{N}(k)$ is regular at $k^0 = |\vec{k}|$. The above integral does contain UV divergence which can be dealt with using standard renormalization procedure discussed in the beginning of the course. We will demonstrate here the appearance of new type divergences in certain regions of the momentum k. Partial fractioning the gluon propagator and using Cauchy's integral formula:

$$\frac{1}{k^2 + i\epsilon} = \frac{1}{2|\vec{k}|}\left(\frac{1}{k^0 - |\vec{k}| + i\epsilon} - \frac{1}{k^0 + |\vec{k}| - i\epsilon}\right),$$

$$\int \frac{dk_0}{k_0} \frac{f(k)}{k^0 - |\vec{k}| + i\epsilon} = -2\pi i \frac{f(k)}{2\pi}\bigg|_{k^0 = |\vec{k}|}, \tag{3.179}$$

we get

$$\mathcal{I} = -\frac{i}{32\pi^2 p_{10} p_{20}} \int_0^\infty \frac{d|\vec{k}|}{|\vec{k}|} \int_{-1}^1 d\text{cos}\theta \frac{\mathcal{N}(k)|_{k^0 = |\vec{k}|}}{1 - \cos^2\theta} \tag{3.180}$$

We observe that the above integral

$$\int_0^\infty \frac{d|\vec{k}|}{|\vec{k}|} \qquad \text{: diverges logarithmically in the soft limit } |\vec{k}| \to 0.$$

$$\int_{-1}^{1} \frac{d\cos\theta}{1-\cos^2\theta} \qquad \text{: diverges logarithmically in the collinear limit } \theta \to 0.$$

Similarly, we will now show that similar divergences do appear in the processes where real gluons are emitted from the quark and anti-quarks. The matrix elements for the real gluon emission processes shown in Fig. (3.1) are

$$\begin{aligned} \mathcal{M}_1 &= \bar{u}_i(p_1)(ie_q\gamma_\lambda)\frac{i(-\not{p}_2-\not{p}_3)}{(p_2+p_3)^2}(ig_{sn,R}\gamma_\alpha T^a_{ij})v_j(p_2)\epsilon^{\alpha *}(p_3) \\ \mathcal{M}_2 &= \bar{u}(p_1)(ig_{sn,R}\gamma_\alpha T^a_{ij})\frac{i(\not{p}_1+\not{p}_3)}{(p_1+p_3)^2}(ie_q\gamma_\lambda)v_j(p_2)\epsilon^{\alpha *}(p_3) \end{aligned}$$

Using equations of motion:

$$\not{p}_2\gamma_\alpha = 2p_{2\alpha} - \gamma_\alpha\not{p}_2 \;; \not{p}_2 v(p_2) = 0 \text{ and } \gamma_\alpha\not{p}_1 = 2p_{1\alpha} - \not{p}_1\gamma_\alpha \;; \bar{u}(p_1)\not{p}_1 = 0$$

taking the soft limit ($p_3 \to 0$), we obtain

$$\begin{aligned} \mathcal{M}^{\text{soft}}_{1\lambda} &= \frac{ie_q g_{sn,R}}{2p_2.p_3}\bar{u}_i(p_1)\gamma_\lambda T^a_{ij}v_j(p_2)\epsilon^*_\alpha(p_3)(2p_2^\alpha) \\ \mathcal{M}^{\text{soft}}_{2\lambda} &= \frac{-ie_q g_{sn,R}}{2p_1.p_3}\bar{u}_i(p_1)\gamma_\lambda T^a_{ij}v_j(p_2)\epsilon^*_\alpha(p_3)(2p_1^\alpha) \end{aligned} \tag{3.181}$$

The sum gives

$$\mathcal{M}^{\text{soft}}_{1\lambda} + \mathcal{M}^{\text{soft}}_{2\lambda} = \mathcal{M}_{0\lambda ij}T^a_{ij}g_{sn,R}\left(\frac{p_2^\alpha}{p_2.p_3} - \frac{p_1^\alpha}{p_1.p_3}\right)\epsilon^*_\alpha(p_3) \tag{3.182}$$

where $\mathcal{M}_{0\lambda ij} = \bar{u}_i(p_1)ie_q\gamma_\lambda v_j(p_2)$ is the matrix element for the Born diagram for the process $\gamma^* \to q\bar{q}$. The amplitude squared after multiplying $-g_{\lambda\lambda'}$ (virtual photon propagator) becomes

$$\begin{aligned} \left|\left(\mathcal{M}^{\text{soft}}_{1\lambda} + \mathcal{M}^{\text{soft}}_{2\lambda}\right)\epsilon^\lambda(q)\right|^2 &= \mathcal{M}_{0\lambda ij}\mathcal{M}^*_{0\lambda' i'j'}(-g^{\lambda\lambda'})T^a_{ij}T^a_{i'j'} \\ &\quad \times g^2_{sn,R}\left|\left(\frac{p_2.\epsilon^*(p_3)}{p_2.p_3} - \frac{p_1.\epsilon^*(p_3)}{p_1.p_3}\right)\right|^2 \end{aligned} \tag{3.183}$$

The transition rate for the above process is obtained by integrating the matrix element squared over the three body phase space given by

$$\int dPS_3 = \prod_{i=1}^{3} \int \frac{d^3 p_i}{(2\pi)^3 2p_{i0}} (2\pi)^4 \delta^{(4)}(q - p_1 - p_2 - p_3)$$

The soft limit of the above 3-body phase space is given by

$$\int_{\mathsf{soft}} dPS_3 = \int \frac{d^3 p_3}{(2\pi)^3 2p_{30}} \left[\prod_{i=1}^{2} \int \frac{d^3 p_i}{(2\pi)^3 2p_{i0}} (2\pi)^4 \delta^{(4)}(q - p_1 - p_2) \right]$$

The p_3 integral over the soft part of the matrix elements squared is given by

$$\begin{aligned} &\int \frac{d^3 p_3}{(2\pi)^3 2p_{30}} \sum_{pol} \left| \left(\mathcal{M}_{1\lambda}^{\text{soft}} + \mathcal{M}_{2\lambda}^{\text{soft}} \right) \epsilon^{\lambda}(q) \right|^2 \\ &\quad = |M_0|^2 \, Tr(T^a T^a) g_{sn,R}^2 \int \frac{d^3 p_3}{(2\pi)^3 2p_{30}} \sum_{pol} \left| \left(\frac{p_2.\epsilon^*(p_3)}{p_2.p_3} - \frac{p_1.\epsilon^*(p_3)}{p_1.p_3} \right) \right|^2 \end{aligned}$$

Rewriting the integrand as

$$\begin{aligned} \left| \left(\frac{p_2.\epsilon^*(p_3)}{p_2.p_3} - \frac{p_1.\epsilon^*(p_3)}{p_1.p_3} \right) \right|^2 \quad &= \quad \left(\frac{p_2^\alpha}{p_2.p_3} - \frac{p_1^\alpha}{p_1.p_3} \right) \left(\frac{p_2^\beta}{p_2.p_3} - \frac{p_1^\beta}{p_1.p_3} \right) \\ &\quad \times \sum_{pol} \epsilon_\alpha^*(p_3) \epsilon_\beta(p_3) \end{aligned} \tag{3.184}$$

and summing gluon polarizations, we get

$$\begin{aligned} &= \quad |M_0|^2 \, Tr(T^a T^a) g_{sn,R}^2 \left(\frac{-2p_1.p_2}{|\vec{p_1}||\vec{p_2}|} \right) \frac{1}{2(2\pi)^3} \int \frac{d^3 p_3}{|\vec{p_3}|} \frac{1}{|\vec{p_3}|^2} \frac{1}{1 - \cos^2\theta} \\ &= \quad |M_0|^2 \, Tr(T^a T^a) g_{sn,R}^2 \left(\frac{-2p_1.p_2}{|\vec{p_1}||\vec{p_2}|} \right) \frac{1}{2(2\pi)^3} \int_0^{p_2 max} \frac{d|\vec{p_3}|}{|\vec{p_3}|} \\ &\quad \times \int_0^{2\pi} d\phi \int_{-1}^{1} d\text{cos}\theta \frac{1}{1 - \cos^2\theta} \end{aligned} \tag{3.185}$$

The above integral diverges in the soft limit ($p_3 \to 0$). In addition, we find an additional divergence as $\cos\theta \to \pm 1$ This is called collinear singularity.

This happens when two massless particles become collinear to each other. To summarise, we have shown that both virtual gluon contribution as well as real gluon emission processes contain soft and collinear divergences. In the following we will demonstrate that the total cross section where all the virtual and real gluon contributions are included is free of any of these divergences.

Since these processes are individually divergent, we first regulate them using dimensional regularization similar to the way UV divergences were regulated. We evaluate all the matrix elements in n dimensions and both loop as well as phase space integrals are performed in n dimensions. The matrix element corresponding to the one loop correction is given by

$$\mathcal{M}^V_\lambda = \bar{u}(p_1) i e_q \Gamma_\lambda v(p_2) \tag{3.186}$$

where

$$\Gamma_\lambda = -i g^2_{sn,R}(T^a T^b) \int \frac{d^n k}{(2\pi)^n} \frac{\gamma_\alpha \not{k} \gamma_\lambda (\not{k} - \not{p}_1 - \not{p}_2)\gamma^\alpha}{k^2 (k-p_1)^2 (k-p_1-p_2)^2} \tag{3.187}$$

where k is the loop momentum. In n dimensions we have

$$\gamma_\mu \not{a}\, \not{b}\, \not{c} \gamma^\mu = -2 \not{c}\, \not{b}\, \not{a} + (4-n) \not{a}\, \not{b}\, \not{c}, \quad \text{and} \quad \bar{u}(p_1)\not{p}_1 = 0; \quad \not{p}_2 v(p_2) = 0 \tag{3.188}$$

The loop integrals that we require are given by

$$J_{\mu_1 \cdots \mu_n} = \int \frac{d^n k}{(2\pi)^n} \frac{k_{\mu_1} \cdots k_{\mu_n}}{k^2 (k-p_1)^2 (k-p_1-p_2)^2} \tag{3.189}$$

where

$$\begin{aligned} J_{\mu\nu} &= -\frac{1}{2(2-n)} B_0(p_1+p_2) g_{\mu\nu} + \frac{1}{4 p_1\cdot p_2}\left(3 B_0(p_1+p_2)\right. \\ &\quad \left. + 4 p_1\cdot p_2 C_0(p_1,p_2)\right) p_{1\mu} p_{1\nu} - \frac{1}{4 p_1 \cdot p_2} B_0(p_1+p_2) p_{2\mu} p_{2\nu} \\ &\quad - \frac{n}{4(n-2) p_1\cdot p_2} B_0(p_1+p_2)(p_{1\mu}p_{2\nu} + p_{1\nu} p_{2\mu}) \\ J_\mu &= \frac{1}{2 p_1 . p_2}\left(B_0(p_1+p_2) + 2 p_1\cdot p_2 C_0(p_1,p_2) p_{1\mu}\right) \\ &\quad - \frac{1}{2 p_1 . p_2} B_0(p_1+p_2) p_{2\mu} \\ J &= -\frac{1}{p_1 . p_2}\frac{n-3}{n-4} B_0(p_1+p_2) = C_0(p_1,p_2) \end{aligned} \tag{3.190}$$

where,

$$B_0(q) = -\frac{i}{(4\pi)^{n/2}}(-q^2)^{(n-4)/2}\frac{2}{n-4}\frac{\Gamma(3-n/2)\Gamma^2(-1+n/2)}{\Gamma(n-2)} \tag{3.191}$$

Using the above results, we find

$$\Gamma_\lambda = -\frac{g_{sn,R}^2}{16\pi^2}\,(T^aT^a)\,\gamma_\lambda\,(-2p_1.p_2)^{\epsilon/2}\,\left(\frac{8}{\epsilon^2}+\frac{2}{\epsilon}+2\right)\frac{\Gamma(1-\epsilon/2)\Gamma(1+\epsilon)}{\Gamma(2+\epsilon)} \tag{3.192}$$

where $n = 4+\epsilon$ is used. The interference of the one loop corrected amplitude with the Born level amplitude after phase space integrations of the two body final states is found to be

$$\begin{aligned}\int dPS_2 \sum_{a,spin} \mathcal{M}_\lambda(\mathcal{M}^v_{\lambda'})^*(-g^{\lambda\lambda'}) \quad &= \quad 2\hat{s}\,\sigma^{(0)}\,\frac{g_{sn,R}^2}{16\pi^2}\,C_F\,\mathrm{Re}(-q^2)^{(\epsilon/2)} \\ &\quad \times\left[-\frac{16}{\epsilon^2}-\frac{4}{\epsilon}-4\right]\,\frac{\Gamma(1-\epsilon/2)\Gamma^2(1+\epsilon/2)}{\Gamma(2+\epsilon)}\end{aligned} \tag{3.193}$$

where

$$2\hat{s}\sigma^{(0)} \quad = \quad \alpha_{em}\,e_q^2\,N\,\left[(2+\epsilon)\frac{\Gamma(1+\epsilon/2)}{\Gamma(2+\epsilon)}(q^2)^{\epsilon/2}\right] \tag{3.194}$$

Notice that the result has double as well as single poles in four dimensions. The double pole terms come from the integration region where the gluons in the loop that are collinear to quark or anti-quark become soft. The single poles can originate from soft gluons which are not collinear to quark or anti-quark. They can also result from hard gluons that are collinear to quark or anti-quark. Notice that the double and single poles persist even if we do not integrate out the final state quark and anti-quark.

We now compute the contributions coming from the real emission diagrams shown in Fig. (3.1). The matrix elements are given by

$$\begin{aligned}\mathcal{M}_{1\lambda} \quad &= \quad \bar{u}(p_1)(ie_q\gamma_\lambda)\frac{i(-\not{p}_2-\not{p}_3)}{(p_2+p_3)^2}(ig_{sn,R}\gamma_\alpha T^a)v(p_2)\epsilon^{*\alpha}(p_3) \\ \mathcal{M}_{2\lambda} \quad &= \quad \bar{u}(p_1)(ig_{sn,R}\gamma_\alpha T^a)\frac{i(\not{p}_1+\not{p}_3)}{(p_1+p_3)^2}(ie_q\gamma_\lambda)v(p_2)\epsilon^{*\alpha}(p_3)\end{aligned}$$

We compute the matrix element squared in $n = 4 + \epsilon$ dimensions:

$$
\begin{aligned}
\overline{\sum} \left|(\mathcal{M}_{1\lambda}\mathcal{M}_{2\lambda})\,\epsilon^{*\lambda}(q)\right|^2 &= g_{sn,R}^2 e_q^2 N C_F n_f \Bigg[(4n^2 - 24n + 32) \\
&= D_c \left(\frac{1}{D_a} + \frac{1}{D_b} + \frac{D_c}{D_a D_b} \right) (8n - 16) \\
&= \left(\frac{D_a}{D_b} + \frac{D_b}{D_a} \right) (2n^2 - 8n + 8) \Bigg]
\end{aligned}
$$

where $D_a = (p_1 + p_3)^2, \quad D_b = (p_2 + p_3)^2, \quad D_c = (p_1 + p_2)^2$. The three body phase space in n dimensions is given by

$$
\begin{aligned}
dPS_3 &= \prod_{i=1}^{3} \frac{d^{n-1}p_i}{(2\pi)^{n-1} 2\ p_{i0}} (2\pi)^n \delta^n(q - p_1 - p_2 - p_3) && (3.195) \\
&= \frac{q^2}{16(2\pi)^3} \left(\frac{q^2}{4\pi} \right)^{n-4} \frac{1}{\Gamma(n-2)} \int_0^1 dx \int_0^1 dv\ x^{n-3} (1-x)^{\frac{n-4}{2}} \\
&\quad \times (v(1-v))^{\frac{n-4}{2}} && (3.196)
\end{aligned}
$$

where $x = (2p_1.q)/q^2, \quad (2p_1.p_3)/q^2 = vx, \quad (2p_2.p_3)/q^2 = 1 - x$. Using the following integral,

$$
\begin{aligned}
\int dPS_3 \frac{1}{D_a^\alpha D_b^\beta D_c^\gamma} &= \left[\frac{q^2}{16(2\pi)^3} \left(\frac{q^2}{4\pi} \right)^{\epsilon} \frac{1}{\Gamma(2+\epsilon)} \right] (q^2)^{1-\alpha-\beta-\gamma} \\
&\quad \times \frac{\Gamma(1+\epsilon/2-\alpha)\Gamma(1+\epsilon/2-\beta)\Gamma(1+\epsilon/2-\gamma)}{\Gamma(3+3\epsilon/2-\alpha-\beta-\gamma)} && (3.197)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\int dPS_3 \overline{\sum} \left|(\mathcal{M}_{1\lambda} + \mathcal{M}_{2\lambda})\,\epsilon^{*\lambda}(q)\right|^2 &= 2\hat{s}\sigma^{(0)} \frac{g_{sn,R}^2}{16\pi^2} C_F (q^2)^{\epsilon/2} \\
&\quad \times \left(\frac{16}{\epsilon^2} + \frac{32}{\epsilon} + 22 + 7\epsilon + \epsilon^2 \right) \\
&\quad \times \frac{4}{2+\epsilon} \frac{\Gamma^2(1+\epsilon/2)}{\Gamma(3+3\epsilon/2)} && (3.198)
\end{aligned}
$$

Notice that the above result also contains double and single poles in four dimensions. The origin of these poles can be traced to the existence of soft gluon as well as of hard gluon that are collinear to quarks or anti-quarks. The poles exist even if we do not integrate over the phase space of the quark and anti-quark states.

Even though the virtual correction to the Born process and real emission processes are independently divergent in four dimensions, their sum is found to be finite.

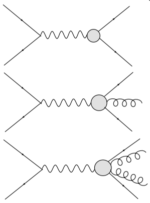
$$2\hat{s}\left(\sigma^{(0)} + \sigma^{v} + \sigma^{R}\right) = 2\hat{s}\sigma^{(0)}\left[1 + \frac{g_{s,R}^2}{16\pi^2}C_F(3)\right] \tag{3.199}$$

The integration over all the final states involving quarks, anti-quarks and gluons means that we are summing over all possible final states of these particles. Such a sum washes away not only the nature of these particles and also the way in which they fragment into final state hadrons. Hence, the sum over final state quarks, anti-quarks and gluons is equivalent to sum over all possible hadronic final states. Hence the total cross section that we have computed with final states involving quarks, anti-quarks and gluons corresponds to production of hadrons in the e^+e^- annihilation. Hence the total cross section in e^+e^- annihilation with hadrons in the final state is infra-red finite.

3.9 QCD Predictions Beyond Leading Order

In a theory with massless fields, transition rates are free of both soft and collinear divergences provided the summation over the initial and final degenerate states is carried out. This is called Kinoshita-Lee-Nauenberg (KLN) theorem. Let us elaborate on what we mean by degenerate states. These are eigen states having same energy. The states $|qg_{\mathsf{soft}}\rangle$ are said to be degenerate to $|q\rangle$ because of the soft gluons carry zero energy. Such states are called soft degenerate states. The states $|\{qg\}_{\mathsf{collinear}}\rangle$ are degenerate to either $|q\rangle$ or $|g\rangle$. Such states are called collinear degenerate states. These soft and collinear degenerate states are the potential sources of divergences in the transition rate. The theorem ensures that such divergences cancel out if we perform summation over initial as well as final degenerate states. We found that the cross section for the hadroproduction in e^-e^+ annihilation is infra-red finite because we carried out the summation over all the final states that include both degenerate states. This is in conformity with the KLN theorem. We can construct other infra-red finite observables for the e^+e^- annihilation process (see Fig. (3.2)).

$$
\begin{aligned}
\mathcal{O}_S^{e^-e^+} &= \int dPS_2\ |M|^2_{e^-e^+\to q\bar{q}}\ S_2(p_1,p_2) \\
&+ \int dPS_3 |M|^2_{e^-e^+\to q\bar{q}g}\ S_3(p_1,p_2,p_3) \\
&+ \int dPS_4 |M|^2_{e^-e^+\to q\bar{q}gg}\ S_4(p_1,p_2,p_3,p_4) \\
&+ \ \ \cdots\cdots
\end{aligned}
$$



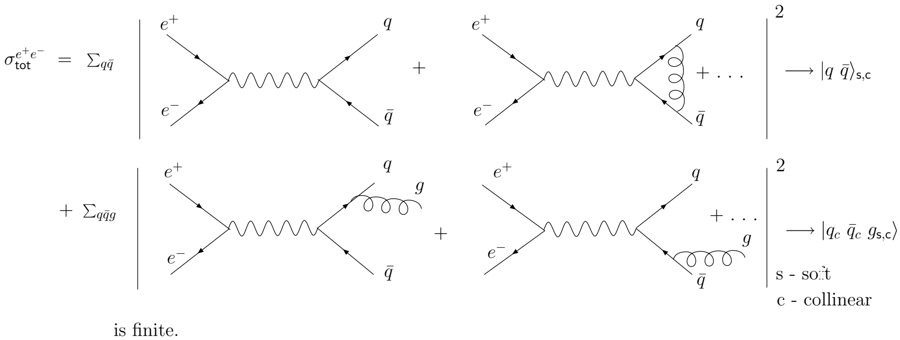


Figure 3.2: The total cross section for the process $e^+e^- \to q\bar{q}$ is finite after summing over all the degenerate states.

The functions $S_i(p_1,..,p_i)$ are chosen in such a way that the observable $\mathcal{O}^{e^+e^-}$ is infra-red finite. A choice, $S_i(p_1,...,p_i) = 1$ gives

$$
d\mathcal{O}^{e^-e^+} = \sigma_{\mathsf{tot}}^{e^-e^+} \tag{3.200}
$$

which is finite.

The $S_i(p_1,...,p_i)$ are symmetric and the cancellation of soft and collinear divergences is guaranteed by the following constraints on them:

$$
S_3(p_1,(1-\lambda)p_2,\lambda p_2) = S_2(p_1,p_2); \quad S_3((1-\lambda)p_1,p_2,\lambda p_1) = S_2(p_1,p_2)
$$

where $\lambda = 0, 1$ correspond to the soft region and $\lambda > 0$ to the collinear region. Of course, one can construct different choices of S_i and they will give different infra-red finite observables.

Even though we describe the scattering processes in terms of quarks and gluons, what one observes experimentally are hadrons in the initial and/or final states. For example, in the e^+e^- annihilation process, one observes energy deposits of hadrons in the hadron calorimeters. We have not been successful in explaining the mechanism of how the quarks and gluons produced in an experiment will convert into hadrons. All we know is that all the energy and momentum of these quarks and gluons produced in the scattering experiments will be transfered to hadrons. Using these energy and momentum variables, one can construct and compute observables that do not require the knowledge of how these quarks and gluons hadronise. For example, in e^+e^- annihilation, define an event by a probability that a definite set of energy and momentum is deposited in the calorimeter. Different sets can give different events. Calculate the sum of events where in each event, all the center of mass energy of e^+e^- collisions but a small fraction ϵ of it goes to a pair of oppositely directed cones of hadrons of half angle δ.

$$\begin{aligned} \mathcal{O}_{\epsilon\delta}^{e^+e-} &= \int dPS_2 |M|^2_{e^-e^+\to q\bar{q}} S_2(\Omega_{J_1}, \Omega_{J_2}) \\ &\quad + \int dPS_3 |M|^2_{e^-e^+\to q\bar{q}g} S_3(\Omega_{J_1}, \Omega_{J_2}, \epsilon, \delta) \qquad (3.201) \end{aligned}$$

$S_3 = 1$ if (a) angle between any of $(q, \bar{q}, g)$ particles is less than δ or (b) any of the particles $(q, \bar{q}, g)$ has energy less than ϵE and it is outside of any of the cones with half angle δ. $S_3 = 0$ otherwise. Out of three particles, let us say two of them make two oppositely directed cones.
(a) If the third particle lies inside one of the cones, it will have both soft and collinear divergent contributions. These divergences will cancel against those coming from $e^+e^- \to q\bar{q} +$ oneloop.
(b) If the third particle is outside the cone, it is free of collinear divergence. But it can be soft producing soft divergence. This is again canceled against $e^+e^- \to q\bar{q} +$ oneloop. Hence, the above observable is infra-red finite. It is dependent on ϵ and δ. These events are called Sterman-Weinberg jets.

In the following, we will discuss how the naive parton model can be improved so that it can be used to computer various observables incorporating higher order radiative corrections in a systematic way (see Fig. (3.3)). Let us recall the result of naive parton model for deep inelastic scattering:

$$\begin{aligned}\lim_{Bj} d\sigma_{eh}(x_{Bj}, Q^2) &= \sum_a \int_0^1 dy \int_0^1 dz \hat{f}_{a/h}(y) d\hat{\sigma}_{ea}(z, Q^2) \delta(x_{Bj} - yz) \\ &= \sum_a \hat{f}_{a/h}(x_{Bj}) \otimes d\hat{\sigma}_{ea}(x_{Bj}, Q^2) \qquad (3.202)\end{aligned}$$

where the convolution $\otimes$ symbol has been introduced for the integrations. The sum over a corresponds to summing over all the partons that contribute to the partonic scattering process. Using this, the hadronic structure functions can be

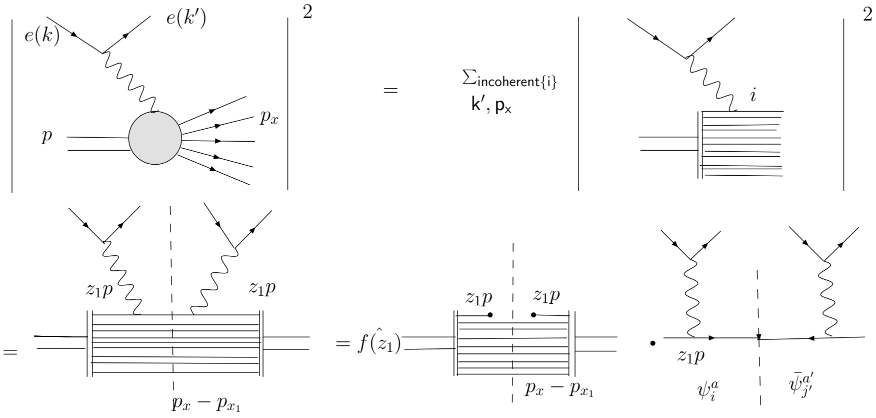


Figure 3.3: The schematic diagram showing that the deep inelastic scattering cross section can be expressed in terms of the incoherent sum of the partonic cross sections and the parton densities $f(z)$.

expressed in terms of partonic structure functions $\mathcal{F}_i^a(x_{\mathsf{Bj}}, Q^2)$ as

$$F_i(x_{\mathsf{Bj}}, Q^2) = \sum_a \hat{f}_{a/h}(x_{\mathsf{Bj}}) \otimes \mathcal{F}_i^a(x_{\mathsf{Bj}}, Q^2). \qquad (3.203)$$

The partonic structure functions are computed from the partonic cross sections $\hat{\sigma}^a(z, Q^2)$ as follows:

$$\mathcal{F}_i^a(z, Q^2) = P_i^{\mu\nu} \hat{\sigma}^a_{\mu\nu}(z, Q^2) \qquad (3.204)$$

where

$$\hat{\sigma}^a_{\mu\nu}(z,Q^2) = \frac{1}{2\hat{s}} \int \prod_{i=1}^{M} \left(\frac{d^{n-1}p_i}{(2\pi)^{n-1} 2p_i^0} \right) (2\pi)^n \delta^{(n)} \left(p + q - \sum_i^M p_i \right) \overline{\sum} |M^a|^2_{\mu\nu}, \tag{3.205}$$

$\hat{s} = (p+q)^2$, $P_i^{\mu\nu}$ are projectors and M^a is the matrix element of the process $e+a \to e+X$ involving a parton of type a. To leading order $\mathcal{O}(a_s^0)$, only quarks and anti-quarks interact with the lepton through electromagnetic interactions:

$$e + q \to e + q, \qquad e + \overline{q} \to e + \overline{q} \tag{3.206}$$

We denote the sum of these contributions to the cross section by $\hat{\sigma}^{q,(0)}(z,Q^2)$. At order $a_s(\mu_R^2)$, the contributions come from two distinct sources. The first one comes from real gluon emission and virtual gluon corrections through one-loop to the tree level process given in eqn.(3.206),

$$e + q \to e + q + g, \qquad e + \overline{q} \to e + \overline{q} + g$$

$$e + q \to e + q + \mathrm{one-loop}, \qquad e + \overline{q} \to e + \overline{q} + \mathrm{one-loop} \tag{3.207}$$

We denote the resulting partonic cross section by $\hat{\sigma}^{q,(1)}(z,Q^2,\mu_R^2)$. The second second source is the contribution coming from the gluon initiated processes:

$$e + g \to q + \overline{q} \tag{3.208}$$

The corresponding partonic cross section is denoted by $\hat{\sigma}^{g,(1)}(z,Q^2,\mu_R^2)$. Hence

$$\hat{\sigma}^q(z,Q^2) = \hat{\sigma}^{q,(0)}(z,Q^2) + a_s(\mu_R^2)\hat{\sigma}^{q,(1)}(z,Q^2,\mu_R^2) + \mathcal{O}(a_s^2) \tag{3.209}$$

$$\hat{\sigma}^g(z,Q^2) = a_s(\mu_R^2)\hat{\sigma}^{g,(1)}(z,Q^2,\mu_R^2) + \mathcal{O}(a_s^2) \tag{3.210}$$

Since we have used the renormalised parameters and fields, the partonic cross sections expressed in terms of $a_s(\mu_R^2)$ are UV finite . Notice that the left hand side of eqns. (3.209,3.210) are renormalization group invariants and hence the right hand side is independent of μ_R provided the sum over entire series is carried over. The truncated perturbative expansion is of course μ_R dependent.

Notice that in QCD , the running mass parameter vanishes at high energies. The higher order partonic cross sections denoted by $\hat{\sigma}^{a,(i)}$ for $i > 0$ at high energies often get contributions from large logarithms of the form $\log(m_R^2/Q^2)$ that can spoil the reliability of the perturbative expansion. These large logarithms come from the phase space regions of partons where massless partons are collinear to each other. Hence, the higher order partonic cross sections with mass parameter put equal to zero are collinear singular. The predictions from the perturbative methods can make sense only if we resum these large logarithms to all orders. A systematic way to organise and resum these large logarithms is accomplished by the procedure called mass factorization. If the collinear singularities are regularised by dimensional regularization, that is, the space time dimension is taken to be $n = 4 + \epsilon_{\mathsf{IR}}$:

$$\hat{\mathcal{F}}^a(x_{\mathsf{Bj}}, Q^2) = \hat{\mathcal{F}}^a\left(x_{\mathsf{Bj}}, Q^2, \frac{1}{\epsilon_{\mathsf{IR}}}\right) \tag{3.211}$$

The collinear divergences that appear as poles in ϵ_{IR} factorise as

$$\hat{\mathcal{F}}^a(x_{\mathsf{Bj}}) \quad = \quad \sum_b \mathcal{Z}_{ab}\left(x_{\mathsf{Bj}}, \frac{1}{\epsilon_{\mathsf{IR}}}, \mu_F^2\right) \otimes \Delta^b(x_{\mathsf{Bj}}, Q^2, \mu_F^2)$$

Now defining,

$$\sum_a \hat{f}_{a/h}(x_{\mathsf{Bj}}) \otimes \mathcal{Z}_{ab}\left(x_{\mathsf{Bj}}, \frac{1}{\epsilon_{\mathsf{IR}}}, \mu_F^2\right) \quad = \quad f_b(x_{\mathsf{Bj}}, \mu_F^2)$$

and substituting in eqn.(3.203), we find

$$F_i(x_{\mathsf{Bj}}, Q^2) \quad = \quad \sum_a f_{a/h}(x_{\mathsf{Bj}}, \mu_F^2) \otimes \Delta_i^a(x_{\mathsf{Bj}}, Q^2, \mu_F^2). \tag{3.212}$$

Here $f_{a/h}(x_{\mathsf{Bj}}, \mu_F^2)$ and $\Delta^{e^- q}(x_{\mathsf{Bj}}, \mu_F^2)$ are called collinear renormalised parton distribution functions and cross sections respectively. $\hat{f}_{a/h}(x_{\mathsf{Bj}})$ is μ_F^2 independent:

$$\mu_F^2 \frac{d}{d\mu_F^2} \hat{f}_{a/h}(x_{\mathsf{Bj}}) = 0$$

which implies (suppressing the subscripts in the Z and f)

$$\left(\mu_F^2 \frac{d\mathcal{Z}^{-1}}{d\mu_F^2}\right) \otimes f + \mathcal{Z}^{-1} \otimes \mu_F^2 \frac{df}{d\mu_F^2} \quad = \quad 0$$

If we define,

$$P(y, \mu_F^2) \quad = \quad -\mathcal{Z} \otimes \mu_F^2 \frac{d\mathcal{Z}^{-1}}{d\mu_F^2}$$

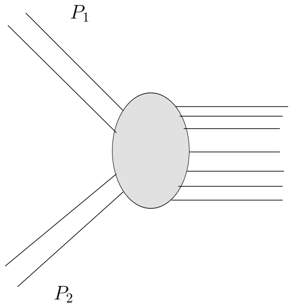


Figure 3.4: Tevatron is a proton anti-proton collider. Large Hadron Collider is a proton proton collider. At the LHC, the center of mass energy is 14 TeV.

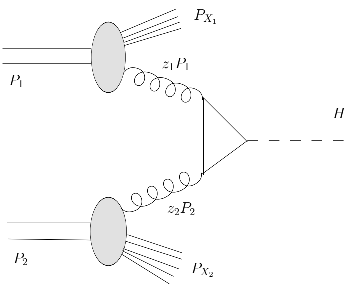


Figure 3.5: LHC is capable of producing Higgs through gluon fusion

which is finite, we find

$$\mu_F^2 \frac{d}{d\mu_F^2} \begin{pmatrix} f_q(z,\mu_F^2) \\ \\ f_g(z,\mu_F^2) \end{pmatrix} = \int_z^1 \frac{dy}{y} \begin{pmatrix} P_{qq}(y,\mu_F^2) & P_{qg}(y,\mu_F^2) \\ \\ P_{gq}(y,\mu_F^2) & P_{gg}(y,\mu_F^2) \end{pmatrix} \begin{pmatrix} f_q(\frac{z}{y},\mu_F^2) \\ \\ f_g(\frac{z}{y},\mu_F^2) \end{pmatrix}$$

The above equation is called the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation. The function P_{ab} are called splitting functions which are computable in perturbative QCD as

$$P_{ab} = a_s(\mu_F^2)P_{ab}^{(0)}(z) + a_s^2(\mu_F^2)P_{ab}^{(1)}(z) + \cdots$$

These splitting functions $P_{ab}^{(i)}$ are known upto three loop level.

Typical processes where the QCD improved parton model can be applied for phenomenlogical study at hadron colliders namely Tevatron and Large Hadron Collider are given in Figs. (3.4–3.7). The QCD improved parton model

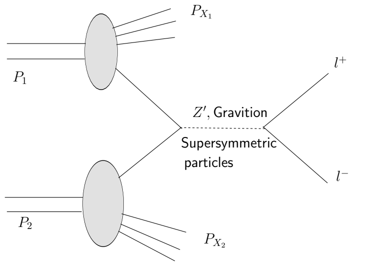


Figure 3.6: New particles predicted by various models such as those with Z', extra dimensions , Supersymmetry can be produced due to energy available at LHC.

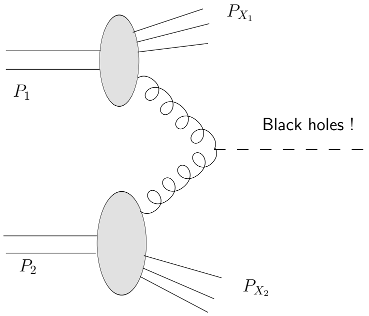


Figure 3.7: Short lived black holes can also be produced at the hadron colliders.

can be used to compute various observables at these colliders using

$$d\sigma^{P_1 P_2} = \sum_{ab} \int dx_1 \int dx_2 f_{\frac{a}{P_1}} \left(x_1, \mu_F^2\right) f_{\frac{b}{P_2}} \left(x_2, \mu_F^2\right) d\hat{\sigma}^{ab} \left(x_1, x_2, \{p_i\}, \mu_F^2\right) , \tag{3.213}$$

where $f_a(x, \mu_F^2)$ are parton distribution functions inside the hadron P and are on-perturbative and process independent. $\hat{\sigma}_{ab}(x_i, \{p_i\}, \mu_F^2)$ are the partonic cross sections and are perturbatively calculable. μ_R and μ_F are renormaliation and factorisation scales. The partonic cross sections are computed as a power series expansion in strong coupling constant. Using the parton distribution functions extracted from other experiments, one can make predictions of various observables at hadron colliders which can serve to confirm and/or rule out models.

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