2

An Introduction to Thermal Field Theory

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2.1 Introduction

Thermal Field Theory (TFT) is a combination of all three basic branches of modern physics, namely quantum mechanics, the theory of relativity, and statistical physics. Therefore one could also call it statistical quantum field theory.

Within TFT, two classes of formalism can be distinguished: one is the imaginary time (Euclidean) formalism (ITF) and other is the real time formalism (RTF). Matsubara was the first to build a TFT by incorporating a purely imaginary time variable into the evolution operator. His name is associated with the discrete energy frequencies. The RTF formulation however is more appropriate for studying transition processes than ITF since no analytical continuation is necessary to reach the physical region. The two formalisms agree in the calculation of the self-energy and the thermodynamic potential with which we are presently concerned. Though ITF has difficulties, as it involves frequency summations, whereas RTF is free of such problems, nevertheless, for our purpose here, we restrict our discussion to ITF to calculate thermal self-energies and thermodynamic quantities.

Another important application of TFT is to study gauge theories at finite temperature. This has applications both in the context of early universe cosmology and laboratory based heavy ion collisions where the properties of Quantum Chromo Dynamics (QCD) at finite temperature can be studied.

We know quantum field theories have difficulties in dealing with loops because of divergences. At finite temperature no additional ultraviolet divergence

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appear as the higher momentum modes are cut off by the distribution functions. However, new infrared divergences do appear in dealing with the massless bosonic field. In ITF this would correspond to the zero Matsubara mode; in the RTF formalism this is even easier to understand. In the soft momentum limit, the Bose distribution brings in a factor of T/k which cancels one power of momentum from the numerator. Therefore new infrared divergences may appear and divergences which were already there might get worse. For instance, a logarithmic divergence in vacuum may become a quadratic divergence at finite temperature.

2.2 Green's Function

The most important quantity in perturbative field theory is the 2-point Green's function or propagator. How do propagators looks like at non-zero temperature? Here we discuss scalar field theory, which we shall use in the following as a simple model to study the different techniques in TFT. At zero temperature the bare propagator is given by the vacuum expectation value of the time ordered product of two fields at different space time points

$$\begin{aligned} i\Delta(X-Y) &= \langle 0|T\{\phi(X)\phi(Y)\}|0\rangle \\ &= \int \frac{\mathrm{d}^4K}{(2\pi)^4}\,\frac{e^{-iK.(X-Y)}}{K^2-m^2+i\eta} \end{aligned} \tag{2.1}$$

where $|0\rangle$ denotes the vacuum state of the non-interacting theory and Δ describes the free propagation of a free scalar particle from Y to X for $x_0 > y_0$ (creation at y, destruction at x) and from X to Y for $y_0 > x_0$. At finite temperature the vacuum is replaced by a ground state having real particles. Stated differently, the destruction operator acting on the vacuum at finite temperature T does not annihilate the vacuum to gives zero. Thus vacuum expectation values have now to be replaced by quantum statistic expectation values, *i.e.*

$$\begin{aligned} \langle A\rangle &\equiv \mathrm{Tr}(\mathcal{D}A) \\ &= \frac{1}{Z}\mathrm{Tr}(Ae^{-\beta H}) \\ &= \frac{1}{Z}\sum_n \langle n|A|n\rangle e^{-\beta E_n} \end{aligned} \tag{2.2}$$

where A is an arbitrary quantum operator, $\beta = \frac{1}{T}$, $\mathcal{D}$ is the density operator and Z is the partition function. Applying eq. (2.2) for non-zero T to the scalar propagator yields

$$i\Delta(X-Y) = \frac{1}{Z}\sum_n \langle n|T\{\phi(X)\phi(Y)\}|n\rangle e^{-\beta E_n} \tag{2.3}$$

where E_n and $|n\rangle$ are the eigenvalues and eigenstates of the non-interacting Hamiltonian.

2.2.1 The Imaginary Time Green's Function

The non-interacting Hamiltonian operator H_0 is usually expressed in terms of creation and annihilation operator a and $a^\dagger$. Thus the calculation of Z reduces to the expectation value of time ordered products of such operators.

The possible propagators $\sim$ $\langle T\, a_k(\tau_1)\, a_k^\dagger(\tau_2)\rangle$, $\langle T\, a_k(\tau_1)\, a_k(\tau_2)\rangle$ or $\langle T\, a_k^\dagger(\tau_1)\, a_k^\dagger(\tau_2)\rangle$ since $H_0 = \sum \epsilon_k a_k^\dagger a_k$ and $a_k^\dagger a_k = \mathcal{N}$ where $\mathcal{N}$ is the number operator. As H_0 commutes with $\mathcal{N}$, only the first among the three possible propagators listed above will enter the perturbative expansion. We define time dependent creation and annihilation operators in the interaction picture as (see Appendix I)

$$\begin{aligned} a_k^\dagger(\tau) \equiv e^{\tau H_0} a_k^\dagger(0) e^{-\tau H_0} &= e^{\epsilon_k \tau} a_k^\dagger(0) \qquad (2.4)\\ a_k(\tau) \equiv e^{\tau H_0} a_k(0) e^{-\tau H_0} &= e^{-\epsilon_k \tau} a_k(0) \qquad (2.5)\end{aligned}$$

The Green's function or propagator is defined as (to focus on temporal properties all the spatial coordinates have been suppressed).

$$\begin{aligned} G_k(\tau_1 - \tau_2) &= \langle \mathrm{T}\, a_k(\tau_1) a_k^\dagger(\tau_2)\rangle \\ &= e^{-\epsilon_k(\tau_1-\tau_2)} \left[\Theta(\tau_1-\tau_2)\langle a_k a_k^\dagger\rangle \pm \Theta(\tau_2-\tau_1)\langle a_k^\dagger a_k\rangle\right] \\ &= e^{-\epsilon_k(\tau_1-\tau_2)} \left[\Theta(\tau_1-\tau_2)\langle 1 \pm a_k^\dagger a_k\rangle \pm \Theta(\tau_2-\tau_1)\langle a_k^\dagger a_k\rangle\right] \\ &= e^{-\epsilon_k(\tau_1-\tau_2)} \left[\Theta(\tau_1-\tau_2)(1 \pm N_k) \pm \Theta(\tau_2-\tau_1) N_k\right] \qquad (2.6)\end{aligned}$$

where $\pm$ signs are for bosons and fermions respectively. We also have used $[a_k, a_k^\dagger] = 1$ for bosons and $\{a_k, a_k^\dagger\} = 1$ for fermions.

$$N_k \equiv \langle a_k^\dagger a_k \rangle = \frac{1}{e^{\beta\epsilon_k \mp 1}} \qquad (2.7)$$

(Here $\pm$ signs are for fermions and bosons respectively). Using eq. (2.6) for bosons, with $\tau = \tau_1 - \tau_2$ we have

$$G_k(\tau - \beta) = e^{-\epsilon_k(\tau-\beta)} \left[\Theta(\tau-\beta)(1+N_k) + \Theta(-\tau+\beta)N_k\right] \qquad (2.8)$$

When $0 \le \tau \le \beta$ we obtain

$$\begin{aligned} G_k(\tau - \beta) &= e^{-\epsilon_k\tau} e^{\epsilon_k\beta} N_k \\ &= e^{-\epsilon_k\tau} e^{\epsilon_k\beta} \left(\frac{1}{e^{\beta\epsilon_k - 1}}\right) \\ &= e^{-\epsilon_k\tau}(1+N_k) \\ &\equiv G_k(\tau) \end{aligned} \qquad (2.9)$$

and when$-\beta \le \tau \le 0$ we obtain

$$\begin{aligned} G_k(\tau + \beta) &= e^{-\epsilon_k\tau} e^{-\epsilon_k\beta}(1+N_k) \\ &= e^{-\epsilon_k\tau} N_k \\ &\equiv G_k(\tau) \end{aligned} \qquad (2.10)$$

Thus

$$\begin{aligned} G_k(\tau-\beta) &= G_k(\tau) \qquad \text{for } 0 \le \tau \le \beta, \\ G_k(\tau+\beta) &= G_k(\tau) \qquad \text{for } -\beta \le \tau \le 0, \end{aligned} \tag{2.11}$$

or the imaginary time Green's function obeys the periodicity condition. Since $G_k(\tau-\beta) = G_k(\tau)$ where $(0 \le \tau \le \beta)$, *i.e.* the Green's function is defined within a finite time interval maintaining the periodicity condition, this allows us to represent $G_k(\tau)$ by a Fourier series:

$$\begin{aligned} G_k(\tau) &= \frac{1}{\beta}\sum_n \mathrm{e}^{-i\omega_n\tau} G_k(i\omega_n) \\ G_k(i\omega_n) &= \int_0^\beta d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau) \end{aligned} \tag{2.12}$$

where $\omega_n = n\pi/\beta$ with $n = 0, \pm1, \pm2, \ldots$. Even though all integer modes are allowed in the Fourier expansion, because of the periodicity (boson) or antiperiodicity (fermion) condition satisfied by G_k, only even integer modes contribute to the bosonic Green's function while only the odd integer modes contribute for fermionic Green's function. This can be easily proved by

$$\begin{aligned} G_k(i\omega_n) &= \frac{1}{2}\int_{-\beta}^0 d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau) + \frac{1}{2}\int_0^\beta d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau) \\ &= \pm\frac{1}{2}\int_{-\beta}^0 d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau+\beta) + \frac{1}{2}\int_0^\beta d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau) \\ &= \pm\frac{1}{2}\int_0^\beta d\tau \mathrm{e}^{i\omega_n(\tau-\beta)} G_k(\tau) + \frac{1}{2}\int_0^\beta d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau) \\ &= \frac{1}{2}(1 \pm e^{-i\omega_n\beta})\int_0^\beta d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau) \\ &= \frac{1}{2}(1 \pm e^{-in\pi})\int_0^\beta d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau) \\ &= \frac{1}{2}\{1 \pm (-1)^n\}\int_0^\beta d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau) \end{aligned} \tag{2.13}$$

This shows that $G_k(i\omega_n)$ contribute for bosons when n is even and contributes for fermions when n is odd. Thus we conclude that

$$\begin{aligned} G_k(\tau) &= \frac{1}{\beta}\sum_n \mathrm{e}^{-i\omega_n\tau} G_k(i\omega_n) \\ G_k(i\omega_n) &= \frac{1}{2}\int_{-\beta}^\beta d\tau \mathrm{e}^{i\omega_n\tau} G_k(\tau) \end{aligned} \tag{2.14}$$

where

$$\begin{aligned}\omega_n &= \frac{2n\pi}{\beta} \qquad \text{for bosons} \\ &= \frac{(2n+1)\pi}{\beta} \qquad \text{for fermions.}\end{aligned} \tag{2.15}$$

These are commonly referred to as the Matsubara frequencies.

The spatial coordinates, on the other hand, are continuous just as in the case of zero temperature field theory and therefore, there is nothing new in their Fourier decomposition. Thus including all the coordinates we can write the free propagator as

$$\begin{aligned}G_0(\tau, x) &= \frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{-i(\omega_n\tau - k.x)} G_0(i\omega_n, k) \\ G_0(i\omega_n, k) &= \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x e^{i(\omega_n\tau - k.x)} G_0(\tau, x)\end{aligned} \tag{2.16}$$

where we have assumed 4 spacetime dimensions and the allowed frequencies are as defined in eq. (2.15). Now in the case of Klein-Gordon field theory, the zero temperature Green's function satisfies (in Minkowski spacetime with signature $(+, -, -, -)$)

$$\begin{aligned}(\partial_\mu\partial^\mu + m^2)G_0(t, x) &= -\delta(t)\delta^3(x) \\ \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right) G_0(t, x) &= -\delta(t)\delta^3(x)\end{aligned} \tag{2.17}$$

Going over to imaginary time, $t \to -i\tau$ (by rotating to Euclidean space or imaginary time), the above equation leads to

$$\left\{-(-i\omega_n)^2 - (ik)^2 + m^2\right\} G_0(\tau, x) = \delta(\tau)\delta^3(x)$$

$$\left(\omega_n^2 + k^2 + m^2\right) \frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{-i(\omega_n\tau - k.x)} G_0(i\omega_n, k) = \frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^3k}{(2\pi)^3} e^{-i\omega_n\tau} e^{ik.x}$$

Then

$$\begin{aligned}G_0(i\omega_n, k) &= \frac{1}{\omega_n^2 + k^2 + m^2} \\ &= \frac{-1}{K^2 - m^2} \\ &= \frac{1}{\epsilon_k^2 + \omega_n^2}\end{aligned} \tag{2.18}$$

where $K = (i\omega_n, k)$ and $\epsilon_k = \sqrt{k^2 + m^2}$. This is the momentum space Greens function or propagator. Here We have also used

$$\delta(\tau) = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} \tag{2.19}$$

and

$$\delta^3(x) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{ik.x} \tag{2.20}$$

The important thing to note is that, unlike the zero temperature case, here the Green's function do not have singularities for real values of the energy and momentum variables. Finite temperature Green's function calculations are completely parallel (at least qualitatively) to the zero temperature case. Only the exact form of the propagator is different from the zero temperature one and it carries the temperature dependence via the Matsubara frequency (ω_n).

2.3 Thermodynamic Potential and Pressure

In this section we will show how to calculate pressure perturbatively. For a scalar field theory with a ϕ^4 interaction the dynamics is governed by the following Hamiltonian

$$H = \frac{\Pi^2}{2} + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{2.21}$$

The thermodynamic potential or free energy is given by

$$\Omega = -\frac{1}{\beta} \ln Z \tag{2.22}$$

Since

$$Z = Z_0 \left\langle \mathrm{T} \exp\left\{ -\int_0^\beta d\tau H_1(\tau) \right\} \right\rangle \tag{2.23}$$

we get

$$\Omega = \Omega_0 - \frac{1}{\beta} \ln \left\langle \mathrm{T} \exp\left\{ -\int_0^\beta d\tau H_1(\tau) \right\} \right\rangle \tag{2.24}$$

Ω_0 is the thermodynamic potential for the non-interacting fields that we evaluate in the following subsection.

2.3.1 Non-interacting Case

If the Hamiltonian is given the partition function becomes

$$\begin{aligned} Z_0 &= \mathrm{Tr}\; e^{-\beta H_0} \\ &= \prod_r \left\{ e^{-\frac{1}{2}\beta\epsilon_r} \sum_{n_r} e^{-\beta n_r \epsilon_r} \right\} \\ &= \prod_r e^{-\frac{1}{2}\beta\epsilon_r} \prod_r \frac{1}{1-e^{-\beta\epsilon_r}} \end{aligned} \tag{2.25}$$

Then

$$\ln Z_0 = -\frac{1}{2}\beta \sum_r \epsilon_r - \sum_r \ln(1-e^{-\beta\epsilon_r})$$

$$\Omega_0 = \frac{1}{2}\sum_k \omega_k + \frac{1}{\beta}\sum_k \ln(1-e^{-\beta\omega_k}) \tag{2.26}$$

In the continuum limit, $\sum_k \to V \int \frac{\mathrm{d}^3k}{(2\pi)^3}$ and the free energy looks like

$$\Omega_0 = V\int \frac{\mathrm{d}^3k}{(2\pi)^3} \left[\frac{1}{2}\omega_k + \frac{1}{\beta}\ln(1-e^{-\beta\omega_k}) \right] \tag{2.27}$$

The first term in the square bracket is temperature independent and leads to a divergent integral. The infinite result is of course nothing other than the zero-point energy of the vacuum, which can be subtracted off, since it is an unobservable constant, although differences in the zero-point energies can be observed. Ignoring the zero-point energy and setting $m = 0$, we have

$$\begin{aligned} \Omega_0 &= \frac{V}{2\pi^2\beta}\int_0^\infty k^2 \mathrm{d}k \ln(1-e^{-\beta k}) \\ &= -\frac{V}{6\pi^2\beta^4}\int_0^\infty \frac{(\beta k)^3 \mathrm{d}(\beta k)}{e^{\beta k}-1} \\ &= -\frac{V\pi^2}{90\beta^4} \end{aligned} \tag{2.28}$$

The pressure becomes

$$P = -\frac{\partial\Omega_0}{\partial V} = \frac{\pi^2 T^4}{90} \tag{2.29}$$

This result for the pressure is that of an ultra relativistic ideal gas of spinless particles.

2.3.2 Interacting Case: Perturbative Method

In this subsection we briefly review the formalism of thermal field theory in equilibrium. We shall in particular recall how perturbation theory can be used to calculate the partition function

$$Z \equiv \mathrm{Tr}\exp\{-\beta H\} = \sum_n \exp\{-\beta E_n\}, \tag{2.30}$$

from which all the thermodynamical functions can be obtained.

The simplest formulation of perturbation theory for thermodynamical quantities is based on the formal analogy between the partition function (2.30) and the evolution operator $U(t,t_0) = \exp\{-i(t-t_0)H\}$, where the time variable t is allowed to take complex values. Specifically, we can write $Z = \mathrm{Tr}\, U(t_0 - i\beta, t_0)$, with arbitrary (real) t_0. More generally, we shall define an operator $U(\tau) \equiv \exp(-\tau H)$, where τ is real, but often referred to as the *imaginary time* ($\tau = i(t-t_0)$ with $t - t_0$ purely imaginary). The evaluation of the partition function (2.30) by a perturbative expansion involves the splitting of the Hamiltonian into $H = H_0 + H_1$, with $H_1 \ll H_0$.

We then set

$$\begin{aligned} U(\tau) &= \exp(-\tau H) \\ &= \exp(-\tau H_0)\exp(\tau H_0)\exp(-\tau H) \\ &= U_0(\tau)\, U_I(\tau), \end{aligned} \tag{2.31}$$

where $U_0(\tau) \equiv \exp(-H_0\tau)$. The operator $U_I(\tau)$ is called the *interaction representation* of U. We also define the interaction representation of the perturbation H_1,

$$H_1(\tau) = e^{\tau H_0} H_1 e^{-\tau H_0}, \tag{2.32}$$

and similarly for other operators. Now

$$\begin{aligned} \frac{d}{d\tau}U_I(\tau) &= e^{\tau H_0} H_0 e^{-\tau H} - e^{\tau H_0} H e^{-\tau H} \\ &= e^{\tau H_0}(H_0 - H)e^{-\tau H} \\ &= -e^{\tau H_0} H_1 e^{-\tau H} \end{aligned} \tag{2.33}$$

$$H_1(\tau)U_I(\tau) = e^{\tau H_0} H_1 e^{-\tau H} \tag{2.34}$$

Thus it is easily verified that $U_I(\tau)$ satisfies the following differential equation

$$\frac{d}{d\tau}U_I(\tau) + H_1(\tau)U_I(\tau) = 0, \tag{2.35}$$

with boundary condition

$$U_I(0) = 1. \tag{2.36}$$

The solution of the above differential equation, with the boundary condition $U_I(\tau_1, \tau_1) = 1$, can be written formally in terms of the time ordered exponential as

$$U_I(\tau_1, \tau_2) = \mathrm{T}_\tau \exp\left(- \int_{\tau_2}^{\tau_1} d\tau H_1(\tau) \right) \tag{2.37}$$

The symbol T_τ implies an ordering of the operators on its right, from left to right in decreasing order of their imaginary time arguments. For our case

$$\begin{aligned} e^{-\beta H} &= e^{-\beta H_0} e^{-\beta H_1} \\ &= e^{-\beta H_0}\, \mathrm{T}_\tau \exp\left\{ - \int_0^\beta d\tau H_1(\tau) \right\} \\ &= e^{-\beta H_0} U_I(\beta, 0) \end{aligned} \tag{2.38}$$

Now

$$\frac{d}{d\tau} U_I(\tau, 0) = H_1(\tau) U_I(\tau, 0) \tag{2.39}$$

Integrating from $\tau = 0$ to $\tau = \beta$ we obtain

$$\begin{aligned} U_I(\beta, 0) - U_I(0, 0) &= - \int_{\tau=0}^\beta d\tau H_1(\tau) U_I(\tau, 0) \\ U_I(\beta, 0) &= 1 - \int_{\tau=0}^\beta d\tau H_1(\tau) U_I(\tau, 0) \end{aligned} \tag{2.40}$$

To solve this we substitute the equation itself inside the integral on the right hand side to yield,

$$\begin{aligned} &\mathrm{T}_\tau \exp\left\{ - \int_0^\beta d\tau H_1(\tau) \right\} \\ &= U_I(\beta, 0) \\ &= 1 - \int_0^\beta d\tau H_1(\tau) \left[1 - \int_0^{\tau_1} d\tau_2 H_1(\tau_2) U_I(\tau_2, 0) \right] \\ &= 1 - \int_0^\beta d\tau H_1(\tau) + \int_0^\beta d\tau_1 \int_0^{\tau_1} d\tau_2\, H_1(\tau_1) H_1(\tau_2) + \cdots \\ &= 1 - \int_0^\beta d\tau H_1(\tau) + \frac{1}{2} \int_0^\beta d\tau_1 d\tau_2\, \mathrm{T}_\tau [\, H_1(\tau_1) H_1(\tau_2)] + \cdots \end{aligned} \tag{2.41}$$

Due to the presence of time ordering we have to include $\frac{1}{2}$ outside the second integral.

Using eq. (2.38), one can rewrite Z in the form

$$\begin{aligned} Z &= \mathrm{Tr}\ (e^{-\beta H}) \\ &= \mathrm{Tr}\ \left[e^{-\beta H_0}\, \mathrm{T}\exp\left\{-\int_0^\beta d\tau H_1(\tau)\right\}\right] \\ &= Z_0 \left\langle \mathrm{T}_{(\tau)} \exp\left\{-\int_0^\beta d\tau H_1(\tau)\right\}\right\rangle, \end{aligned} \tag{2.42}$$

where, for any operator $\mathcal{O}$,

$$\langle \mathcal{O} \rangle \equiv \mathrm{Tr}\left(\frac{e^{-\beta H_0}}{Z_0}\mathcal{O}\right). \tag{2.43}$$

2.4 ϕ^4 Theory at Finite Temperature

2.4.1 One Loop Mass Correction

We have already seen that in the imaginary time formalism the only difference between the zero temperature and the finite temperature field theories lies in the form of the propagator which carries all the temperature dependence. The vertices at finite temperature are exactly the same as those at zero temperature. Thus, given any quantum field theory, we can carry out calculations of thermodynamic variables perturbatively by calculating Feynman diagrams.

Let us consider a self-interacting ϕ^4 theory described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{2.44}$$

According to our discussion, if we want to calculate quantities at finite temperature, we should treat time as an imaginary parameter in which case the theory becomes a Euclidean theory $\mathcal{L}_E = \mathcal{L}$.

$$\mathcal{L}_E = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{2.45}$$

The diagrammatic calculation is analogous to that of the zero temperature case; the only difference is that since the energy values are now quantized, the intermediate energy integrals have to be replaced by sums over discrete values

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \to \frac{i}{\beta}\sum_n \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \tag{2.46}$$

The mass correction becomes

$$\begin{aligned}
-i\Pi \equiv -i\Pi &= \frac{i\lambda}{2!}\frac{i}{\beta}\sum_n \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\frac{i}{\omega_n^2+\epsilon_k^2} \\
\Pi &= \frac{\lambda}{2\beta}\sum_n \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\Delta(i\omega_n, k) \\
&= \frac{\lambda}{2\beta}\sum_n \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\int_0^\beta d\tau e^{i\omega_n\tau}\Delta(\tau,k) \\
&= \frac{\lambda}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\int_0^\beta d\tau\Delta(\tau,k)\frac{1}{\beta}\sum_n e^{i\omega_n\tau} \\
&= \frac{\lambda}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\int_0^\beta d\tau\Delta(\tau,k)\delta(\tau) \\
&= \frac{\lambda}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\Delta(0,k) \qquad (2.47)
\end{aligned}$$

In the above 2! is due to the symmetry factor at the vertex.

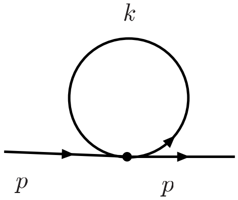

Figure 2.1: One loop mass correction

From eq. (2.8)we have

$$\begin{aligned}
\Delta(\tau,k) &= e^{-\epsilon_k(\tau)}\left[\Theta(\tau)(1+N_k)+\Theta(-\tau)N_k\right] \\
\Delta(0,k) &= \frac{1}{2\omega_k}[1+2N_k] \qquad (2.48) \\
\Pi &= \frac{\lambda}{2}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\left[\frac{1+2N_k}{2\omega_k}\right]
\end{aligned}$$

The first term corresponds to the zero temperature part and the second term represents explicit temperature dependence. For $m = 0$, the second term is

$$\frac{\lambda}{2}\int \frac{4\pi k^2\, dk}{(2\pi)^3 k[e^{\beta k}-1]} = \frac{\lambda}{(4\pi)^2}\int dk \frac{k}{e^{\beta k}-1} = \frac{\lambda}{24\beta^2} \tag{2.49}$$

The total self energy or mass correction at finite temperature thus becomes

$$\Pi = \frac{\lambda}{4}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\omega_k} + \frac{\lambda T^2}{24} \tag{2.50}$$

The divergences in the expression for the mass correction is entirely contained in the zero temperature part. The temperature dependent part is free from ultraviolet divergences. Therefore the zero temperature counterpart is sufficient to renormalize the theory. We see here that temperature induces a mass for the bosons analogous to a particle moving in a medium and the mass is positive.

Figure 2.2: Mass counterterm

In eq. (2.50) the first term is temperature independent but ultraviolet divergent. To avoid this divergence one uses the mass counterterm in the Lagrangian, $\frac{1}{2}\delta m^2\phi^2$; this is known as mass renormalization.

$$\delta m^2 + \frac{\lambda}{2}\sum_k \frac{1}{2\omega_k} = 0$$

2.5 Pressure in ϕ^4 Theory

We have shown that the pressure for the scalar field is given by eq. (2.29). But a well known problem at high temperatures is the breakdown of the conventional perturbative expansion at some order in the coupling constant (λ). Therefore, to compute consistently to a given order in λ, we have to take into account all the relevant higher loop graphs— these usually form an infinite set.

First order correction: To go beyond leading order, one must compute two loop (and higher) diagrams in the effective expansion.

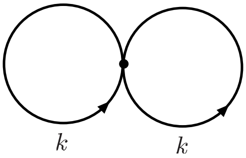

Figure 2.3: First order correction to pressure

$$\begin{aligned}
\frac{\lambda}{4!}3\left[\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Delta(0,k)\right]^2 &= \frac{3\lambda}{4!}\left[\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_k}+\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{N_k}{\omega_k}\right]^2 \\
&= \frac{3\lambda}{4!}\left[\left\{\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\sqrt{k^2+m^2}}\right\}^2\right. \\
&\quad +\left\{\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\omega_k}\frac{1}{e^{\beta\omega_k}-1}\right\}^2 \\
&\quad \left.+\,2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega_k}\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{\omega_{k'}}\frac{1}{e^{\beta\omega_{k'}}-1}\right]
\end{aligned} \tag{2.51}$$

The first term is the zero temperature contribution having an ultraviolet divergence and the second term is the temperature dependent part. The third term is potentially dangerous; it is divergent and temperature dependent.

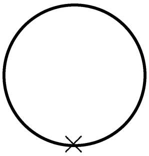

Figure 2.4: Counterterm contribution to the pressure

The counterterm contribution to the pressure is obtained by folding fig. (2.2) as shown in fig. (2.4). Mathematically this is given by

$$\begin{aligned}
&\frac{1}{2}\delta m^2\left(\sum_{k'}\frac{1+2N_{k'}}{2\omega_{k'}}\right) \\
=\;& \frac{1}{2}\left(-\frac{\lambda}{2}\sum_k\frac{1}{2\omega_k}\right)\left(\sum_{k'}\frac{1+2N_{k'}}{2\omega_{k'}}\right) \\
=\;& -\frac{\lambda}{8}\sum_k\frac{1}{2\omega_k}\sum_{k'}\frac{1}{\omega_{k'}}-\frac{\lambda}{4}\sum_k\frac{1}{2\omega_k}\sum_{k'}\frac{N_{k'}}{\omega_{k'}}
\end{aligned} \tag{2.52}$$

Thus the mass renormalization term cancels the potentially dangerous term in the first order correction to the pressure in eq. (2.51) which is temperature dependent and also ultraviolet divergent. After cancellation of that term there is still the first term of the above equation which is ultraviolet divergent. But this is temperature independent and thus harmless. One elegant way to avoid this term is to define the renormalized free energy

$$\Omega_R(T,m^2,\lambda) = \Omega(T,m^2,\lambda)-\Omega(T=0,m^2,\lambda) \tag{2.53}$$

From this renormalized potential one obtain the correction of order λ to the pressure of the ideal gas in the $m = 0$ case

$$P = -\frac{1}{\beta}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\ln(1-e^{-\beta\omega_k}) - \frac{\lambda}{8}\left(\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{N_k}{k}\right)^2 \tag{2.54}$$

The first term (already calculated) equals $\frac{\pi^2T^4}{90}$ and the second term is $\frac{\pi^2T^4}{90}(-\frac{5\lambda}{64\pi^2})$. Thus the pressure up to first order in λ is

$$P = \frac{\pi^2T^4}{90}\left(1-\frac{5\lambda}{64\pi^2}\right) \tag{2.55}$$

Second order correction: Mathematically the second order correction to the pressure is given by

$$\begin{aligned} &\left\{\frac{\lambda}{2}\int \frac{\mathrm{d}^3k'}{(2\pi)^3}\Delta(0,k')\right\}^2 \sum_{k,n}\{\Delta(i\omega_n,k)\}^2 \\ &= \frac{\lambda^2}{4}\left(\sum_{k'}\frac{1}{2\omega_{k'}}+\frac{N_{k'}}{\omega_{k'}}\right)^2 \sum_{k,n}\left(\frac{1}{\omega_k^2+\omega_n^2}\right)^2 \end{aligned} \tag{2.56}$$

Now the second factor can be written as

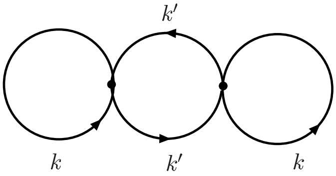

Figure 2.5: Second order correction to the pressure

$$\sum_{k,n}\left(\frac{1}{\omega_k^2+\omega_n^2}\right)^2 = \sum_{k,n=0}\frac{1}{\omega_k^4}+\sum_{n\neq 0}\sum_{k}\left(\frac{1}{\omega_k^2+\omega_n^2}\right)^2 \tag{2.57}$$

where we have separated the $n = 0$ and $n \neq 0$ terms. For the $m \to 0$ case ($\omega_k^2 \to k^2$), and defining $\sum_k = V\int \frac{\mathrm{d}^3\mathrm{k}}{(2\pi)^3}$, we get

$$\sum_k \frac{1}{\omega_k^4} \to \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k^4} \to \int \frac{\mathrm{d}k}{k^2} \to \infty \qquad \text{for } k \to 0 \tag{2.58}$$

i.e. this term has an infrared divergence. From eq. (2.56) the first term is the same as that obtained in the first order correction. Thus, as the $n \neq 0$ term is non-divergent, the divergence becomes worse and worse as we go to higher and higher order.

2.5.1 Infrared Divergence and resummation of ring diagrams

In order to see if the perturbation series is well behaved, it is necessary to look at higher orders. In the previous section we have seen that higher order rings, as in fig. (2.5), are infrared divergent as in eq. (2.58). In scalar field theories, it is known that the dominating infrared contributions to the self energy come from the so-called ring diagrams. Ring diagrams consist of loop diagrams of various orders forming an infinite series, and each of them is infrared divergent. When summed over, this series gives us a finite result as shown in eq. (2.62). It is to be noted here that the entire sum for the $n = 0$ mode contributes to the order $\lambda^{3/2}$, while each term is of higher order in λ. Thus higher order loops contribute to the pressure at lower order in coupling after summing the series. In effect, this implies a reorganization of the perturbation series where a particular class of diagrams are summed over in a definite way. This reorganisation of the perturbation series is known as resummation.

For two loops with two propagators the renormalized free energy is given by

$$\Omega_R = - \frac{V}{2\beta}\frac{1}{2}\sum_{k,n}\{-\Pi(i\omega_n,k)\Delta(i\omega_n,k)\}^2 \tag{2.59}$$

Here the first 2 factor in the denominator is the symmetry factor at each vertex and the second 2 arise from 2 loops. For the ring diagrams, the renormalized free energy is given by

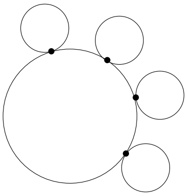

Figure 2.6: Ring diagram

$$\begin{aligned}\Omega_R &= -\frac{V}{2\beta}\sum_{k,n}\sum_{N=2}^{\infty}\frac{1}{N}\{-\Pi(i\omega_n,k)\Delta(i\omega_n,k)\}^N \\ &= \frac{V}{2\beta}\sum_{k,n}\{\ln[1+\Pi(i\omega_n,k).\Delta(i\omega_n,k)] - \Pi(i\omega_n,k).\Delta(i\omega_n,k)\}\end{aligned}$$

The factor of $\frac{1}{N}$ is a symmetry factor which can be understood in the following way: there is a factor of 3! for connecting two lines at each vertex, a factor of 2 for how the remaining two lines at each vertex are connected to the adjacent vertices, a factor of $\frac{1}{2}(N-1)!$ for the number of ways of ordering the vertices along the circle, and a factor of $\frac{1}{N!}$ from the expansion of the exponential of the interaction in the partition function. The summation over N begins at $N = 2$ because in case of the single loop ($N = 1$) the self energy contribution from the temperature dependent part, $\lambda T^2/24$, is already calculated as part of the first order correcion.

$$\Omega_R = \frac{V}{2\beta} \left[\sum_{k,n} \Pi(i\omega_n, k).\Delta(i\omega_n, k) - \frac{1}{2} \sum_k \left(\frac{\lambda}{2} \frac{n_k}{\omega_k} \right)^2 \sum_{k,n} (\Delta(i\omega_n, k))^2 + \cdots - \textstyle\sum_{k,n} \Pi(i\omega_n, k).\Delta(i\omega_n, k) \right] \tag{2.60}$$

Now

$$\sum_{k,n} (\Delta(i\omega_n, k))^2 = \sum_{k,n} \left(\frac{1}{\omega_k^2 + \omega_n^2} \right)^2 = \sum_k \frac{1}{\omega_k^4} + \sum_{n \neq 0} \sum_k \left(\frac{1}{\omega_k^2 + \omega_n^2} \right)^2 \tag{2.61}$$

The separated $n = 0$ term is infrared divergent for $m = 0$ and the $n \neq 0$ term is proportional to $1/\left(k^2 + 4n^2\pi^2 T^2\right)$ and is non-divergent for $k \to 0$ due to the existence of the $4n^2\pi^2T^2$ term. Thus divergences become worse and worse as we go to higher and higher order terms. Since the $n \neq 0$ term does not diverge, we only consider the $n = 0$ term for the $m \to 0$ case (see Appendix II).

$$\begin{aligned} \Omega_R &= \frac{V}{2\beta} \int \frac{\mathrm{d}^3k}{(2\pi)^3)} \left[\ln\left(1 + \frac{\lambda T^2}{24} \frac{1}{k^2}\right) - \frac{\lambda T^2}{24} \frac{1}{k^2} \right] \\ &= -\frac{V}{2\beta} \frac{1}{6\pi\beta^3} \left(\frac{\lambda}{24}\right)^{\frac{3}{2}} \\ P &= \frac{1}{12\pi\beta^4} \left(\frac{\lambda}{24}\right)^{\frac{3}{2}} \end{aligned} \tag{2.62}$$

Thus the correction is of order $\lambda^{\frac{3}{2}}$, not of order λ^2 as would have been expected. This arises from the infrared singular behavior of the propagator.

The expression for the pressure considering upto the ring diagram is

$$\begin{aligned} P &= \frac{\pi^2 T^4}{90} \left[1 - \frac{5\lambda}{64\pi^2} + \frac{90}{12\pi^3} \left(\frac{\lambda}{24}\right)^{\frac{3}{2}} + \cdots \right] \\ &= \frac{\pi^2 T^4}{90} \left[1 - \frac{15}{8} \left(\frac{\lambda}{24\pi^2}\right) + \frac{15}{2} \left(\frac{\lambda}{24\pi^2}\right)^{\frac{3}{2}} + \cdots \right] \end{aligned} \tag{2.63}$$

The first term is the pressure for a free boson gas. The second term is from the first order correction and the third term is from the ring diagram. Thus if one restricts oneself to the $n = 0$ term, *i.e.* the static mode, the total result exhibits a breakdown of perturbation theory due to the infrared divergence. Infrared divergences come from the static modes only, as $\omega_n = 2\pi nT$ acts as a mass term in the propagator for $n \neq 0$. Thus the results obtained in naive perturbation theories can be incomplete in the order of the coupling constant since higher order diagrams after resummation can contribute to lower order in the coupling constant.

2.6 Appendices

2.6.1 Interaction picture creation and annihilation operators

$$\begin{aligned} a_k^\dagger(\tau) &\equiv e^{\tau H_0} a_k^\dagger(0) e^{-\tau H_0} && (2.64) \\ a_k(\tau) &\equiv e^{\tau H_0} a_k(0) e^{-\tau H_0} && (2.65) \end{aligned}$$

These give

$$\begin{aligned} \frac{d}{d\tau} a_k(\tau) &= e^{\tau H_0} H_0 a_k(0) e^{-\tau H_0} - e^{\tau H_0} a_k(0) H_0 e^{-\tau H_0} \\ &= e^{\tau H_0} [H_0, a_k(0)] e^{-\tau H_0} \\ &= [H_0, a_k(\tau)] \\ &= \left[\sum \epsilon_{k'} a_{k'}^\dagger(\tau) a_{k'}(\tau), a_k(\tau)\right] \\ &= -\epsilon_{k'} \delta_{k'k} a_{k'}(\tau) \\ &= -\epsilon_k a_k(\tau) \\ a_k(\tau) &= e^{-\epsilon_k \tau} a_k(0) && (2.66) \end{aligned}$$

Similarly

$$a_k^\dagger(\tau) = e^{\epsilon_k \tau} a_k^\dagger(0) \qquad (2.67)$$

2.6.2 Ring diagram calculation for zero Matsubara frequency

For the ring diagram calculation the actual equation is

$$\Omega_R = \frac{V}{2\beta} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\ln\left(1 + \frac{\lambda}{24} \frac{1}{\beta^2 k^2}\right) - \frac{\lambda}{24} \frac{1}{\beta^2 k^2} \right] \qquad (2.68)$$

To solve the above we solve the equation below,

$$\begin{aligned} I &= \int \mathrm{d}x\; x^2 \left(\ln\left(1+\frac{\lambda}{x^2}\right) - \frac{\lambda}{x^2}\right) \\ &= \frac{1}{3}\ln\left(\frac{\lambda}{x^2}+1\right)x^3 - \frac{\lambda\, x}{3} - \frac{2}{3}\lambda^{\frac{3}{2}}\tan^{-1}\left(\frac{x}{\sqrt{\lambda}}\right) \qquad (2.69) \\ \lim_{x\to\infty} I &= -\frac{1}{3}\lambda^{\frac{3}{2}}\pi + \frac{\lambda^2}{2x} - \frac{1}{9}\lambda^3\left(\frac{1}{x}\right)^3 + O\left\{\left(\frac{1}{x}\right)^4\right\} \\ &\simeq -\frac{1}{3}\lambda^{\frac{3}{2}}\pi \qquad (2.70) \\ \lim_{x\to 0} I &= -\lambda x + \left(\frac{\ln\,\lambda}{3} - \frac{2\ln\,\lambda}{3} + \frac{2}{9}\right)x^3 + O(x^4) \\ &\simeq 0 \qquad (2.71) \end{aligned}$$

Thus using eq. (2.70) and eq. (2.71),we get

$$\begin{aligned} I &= \int_0^\infty \mathrm{d}x\; x^2 \left(\ln\left(1+\frac{\lambda}{x^2}\right) - \frac{\lambda}{x^2}\right) \\ &= -\frac{1}{3}\lambda^{\frac{3}{2}}\pi \qquad (2.72) \end{aligned}$$

Thus

$$\Omega_R = \;-\frac{V}{2\beta}\left(\frac{\lambda}{24}\right)^{\frac{3}{2}}\;\frac{1}{6\pi\beta^3} \qquad (2.73)$$

2.6.3 Problems

1. (a) Define the creation and annihilation operator in the interaction picture as

$$a_k^\dagger(\tau) \equiv e^{\tau H_0} a_k^\dagger(0)\, e^{-\tau H_0}$$

$$a_k(\tau) \equiv e^{\tau H_0} a_k(0)\, e^{-\tau H_0}$$

Then show that

$$\begin{aligned} a_k(\tau) &= e^{-\epsilon_k\tau} a_k(0) \\ a_k^\dagger(\tau) &= e^{+\epsilon_k\tau} a_k^\dagger(0) \end{aligned}$$

(b) The non-interacting partition function for bosons is given by $Z_0 = \mathrm{Tr}e^{-\beta H_0}$ where $H_0 = n_1\epsilon_1 + n_2\epsilon_2 + ...$ Prove the following identities:

(i) $\langle a_s^\dagger a_s\rangle = f(\epsilon_s)$

(ii) $\langle (a_s^\dagger a_s)^2\rangle = f(\epsilon_s)[f(\epsilon_s)+1] + [f(\epsilon_s)]^2$

where $f(\epsilon_s)$ is the Bose distribution function.

2. Prove that

$$\begin{aligned}
&\text{(i)} \int_0^\infty \mathrm{d}x \frac{x^n}{e^{\beta x} - 1} &=& \quad \frac{1}{\beta^{n+1}} \zeta(n+1)\Gamma(n+1) \quad \text{(for bosons)} \\
&\text{(ii)} \int_0^\infty \mathrm{d}x \frac{x^n}{e^{\beta x} + 1} &=& \quad \frac{1}{\beta^{n+1}} \left(1 - \frac{1}{2^n}\right) \zeta(n+1)\Gamma(n+1)) \quad \text{(for fermions)}
\end{aligned}$$

3. Prove the Kubo-Martin-Schwinger condition

$$\langle A(t)B(t')\rangle \quad = \quad \langle B(t')A(t+i\beta)\rangle$$

Here the angular brackets represent thermal averages.

4. Prove that for a free gas the pressure becomes

$$P = \left(n_B + \frac{7}{8} n_F\right) \frac{\pi^2 T^4}{90}$$

where n_B and n_F are the degeneracies for bosons and fermions.

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