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Quark-Gluon Plasma: An Overview

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1.1 Introduction

The physics of the Quark Gluon Plasma (QGP) is being actively investigated presently theoretically as well as experimentally. The motivation for this comes from the cosmos as well as from attempts to understand the phase diagram of strongly interacting matter. The universe consisted of quark-gluon plasma during the early stages when the age of the universe was less than a few microseconds. It is also believed that the cores of various compact astrophysical objects, e.g. neutron stars, may be in the QGP phase. Laboratory experiments consisting of collision of heavy nuclei at ultra-relativistic energies are being carried out in an attempt to create a transient phase of QGP in tiny regions of space. These lectures will provide an overall picture of QGP starting with a basic understanding of Quantum Chromo Dynamics (QCD) which is the theory of strong interactions.

Let us start by recalling the four basic interactions: Electromagnetic, Weak, Strong, and Gravity. We know that the first two of these are unified into an "Electroweak Interaction". There are attempts to unify the electroweak and strong interactions into an, as yet unknown, Grand Unified Theory (GUT). Unification of all the four basic forces is attempted in String Theories. How well do we understand these forces individually?

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Electromagnetism: The theory of Electromagnetism is provided by Quantum Electrodynamics (QED). This theory is well understood and its predictions have been verified in experiments with very high accuracy.

Electroweak Theory: Of course, the complete theory of Electromagnetism is given only when unified with weak interactions. The Electroweak theory is also well understood, and its predictions are verified in experiments. One major "missing part" of the theory was the Higgs boson which plays a crucial role in the formulation of the theory (spontaneous symmetry breaking leading to massive W and Z bosons which are responsible for the "weakness" of the weak force). Recent experiments at CERN have confirmed detection of a Higgs-like boson.

Gravity: Gravitational interactions are very well understood at the classical level in terms of Einstein's General Theory of Relativity. However, at present there is no theory of Quantum Gravity. There are various attempts towards Quantum Gravity. The most popular approach is in terms of String Theories. There are other approaches within conventional frameworks, e.g. using canonical quantization (Loop Quantum Gravity), etc.

Strong Interactions: Let us now discuss strong interactions which will be the subject of these Lectures. The theory for strong interactions is believed to be Quantum Chromo Dynamics (QCD). The basic ingredients for QCD were proposed by studying properties of hadrons which are supposed to be made up of the basic degrees of freedom in QCD, namely quarks. Interactions between quarks are mediated by gluons (in the same way as photons mediate interactions between electrons).

Theoretical investigations of QCD show a remarkable property of strong interactions. At very high energies, the strength of the interaction between quarks becomes smaller. In other words, the effective coupling constant of strong interactions becomes smaller at large energies, eventually approaching zero. This is known as "asymptotic freedom". This behavior is the opposite of the behavior in QED where the coupling constant increases with energy. Asymptotic freedom (for which there was already evidence from deep inelastic scattering experiments) is well tested in experiments to a high accuracy. However the understanding of QCD in the domain of low energy remains poor. This is the domain where hadrons form, and quarks are **confined** in these hadrons. Recall that it was the study of these hadrons which led to the formulation of QCD.

Apart from this "confinement" there is another domain where QCD is not well understood. This is the domain of high temperature and high density of matter. From the theoretical side it is expected, based on asymptotic freedom, that at high temperatures the interactions between quarks will become weak. Does that mean we should get an ideal gas of quarks and gluons at high temperatures?

Some of these questions led to the search for the so called "Quark Gluon Plasma" phase of QCD. From general physical arguments one expects that at sufficiently high temperatures (at $T > T_c \sim 170$ MeV, the deconfinement temperature) and densities, quarks and gluons are no more confined. Essentially at such high temperatures and/or densities, one has overlapping hadrons, so it makes no sense to talk about quarks and gluons confined inside individual hadrons. However, it should be clear that at high T or high density ρ, one is inevitably dealing with many body effects. Here the understanding obtained from deep inelastic scattering experiments may not be directly applicable. Also, it appears that the interactions between partons are not *weak* at temperatures achievable in the laboratory (in relativistic heavy ion collisions). At present, we also do not have theoretical tools to properly analyze the behavior of QCD in these domains using analytic calculations (except possibly at ultra high temperatures). Lattice QCD is the only theoretical tool we know for understanding this domain. Results so far lead to interesting behavior of quarks and gluons in this QGP phase.

A direct motivation for understanding this high T, ρ domain comes from cosmology and astrophysics. In the standard Big Bang theory of the universe, the temperature of the universe was very high initially. When the age of the universe was less than 10^{-6} sec, its temperature was higher than about 200 MeV. So we expect that the universe was filled with QGP at those early times. To understand the evolution of the universe at those early times one must understand the properties of the QGP phase at high T. Further expansion and cooling of the universe converts QGP to hadrons. This is expected to be a phase transition (or, more likely, a crossover) at a critical temperature of about 170 MeV. If it is a first order transition then it could have consequences for different primordial element abundances in the universe.

In the present day universe, there are heavy and superdense objects known as neutron stars. These form at the end of fusion reaction chains of regular stars which undergo supernova explosion. The mass density in a neutron star is about 10^{14} gm/cm^3. At the center of these neutron stars the density may be even higher, of the order of several times the nuclear density. It is expected that in the cores of neutron stars hadrons (neutrons/protons) may be closely packed so that quarks and gluons may no more be confined, leading to high density (not high temperature) QGP. Various properties of neutron stars (maximum mass, spin, etc.) depend crucially on the properties of this type of core.

All of these are theoretical consideration. Even neutron stars are accessible only through indirect observations. The universe at the age of less than 10^{-6} sec is in the distant past and no experiments are possible for observing that. So, how do we test our theoretical modeling of QGP at high T and/or high ρ which is relevant for these cases?

Relativistic heavy ion collisions allow us the possibility for doing this. For example, at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven, USA, beams of Au-Au are collided at 200 GeV per nucleon pair center of mass energy. We first discuss, briefly, the physics of these experiments - a detailed discussion will be provided later. As the nuclei are accelerated to very high energies, spherical nuclei get Lorentz contracted. (Note that Lorentz factor $\sim$ 100 but the Lorentz contracted width is not less than 1 fm due to quantum effects.) At such high energies nuclei, or even protons and neutrons, lose their identity and the interaction between nuclei becomes effectively quark-quark interactions. Due to asymptotic freedom, this interaction is also weak, so most of the quarks go through each other, creating secondary partons in the middle. The density of these secondary partons grows due to multiple scatterings and the system thermalizes. This central thermalized system cools as it expands. If its initial temperature is above 200 MeV, we expect that it should be in an equilibrated QGP state. On subsequent expansion it should undergo a phase transition to a hadronic system. Note that the resulting system is just like what was present in the early universe (apart from some differences like the expansion rate, etc.). Thus investigation of this system allows us to probe a part of the early history of our universe.

Experiments at lower energies (such as AGS and future GSI experiments in Germany) have higher baryon densities in the center (as the quark-quark interaction is stronger at lower energies), though lower temperature. This matter is similar to neutron star core matter and could help us in understanding this domain of QCD.

Above all, studying the creation of QGP and the subsequent phase transition to hadrons helps us in better understanding confining forces between quarks because the process of hadron formation at the transition stage depends crucially on that.

We will discuss these relativistic heavy-ion collision experiments in these lectures. Through these experiments we can probe different parts of the QCD phase diagram. The phase boundaries in the QCD phase diagram are obtained from several symmetry arguments, or in effective low energy models. Lattice calculation also give us some handle on these (especially for zero or small baryon chemical potential).

The plan of the lectures is as follows. First we will provide a general introduction to QCD leading to the concepts of asymptotic freedom and running coupling constant. The discussion is mostly taken from the books in ref. [1] and for further details these books should be consulted. Since the whole discussion is based on QCD, we will discuss important aspects of QCD including its basic structure in detail. Then we will sketch steps to give a basic understanding of running coupling constant and asymptotic freedom for QCD.

Next we discuss the prediction of the QGP phase of QCD. This discussion is primarily based on refs. [2, 3]. We will see how general arguments lead us to the prediction of QGP phase of QCD. We will discuss arguments based on the running coupling constant as well as more detailed ones based on the Bag model of hadrons leading to the expectation that QGP phase should exist at high temperature as well as high density.

Following this we will discuss QGP formation and evolution in relativistic heavy-ion collisions [4]. We will discuss the Bjorken picture for the evolution and various signals of QGP [6]. We will then discuss various topics such as the deconfinement-confinement phase transition, etc.

1.2 QCD

Our approach will not be historical. We will list the requirements, from experimental evidence, for the theory of strong interactions and then argue that QCD satisfies these requirements.

1.2.1 Basic Contents

1. We know that there are six quarks.

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix}$$

u, c, t quarks have charges $+\frac{2}{3}e$ while d, s, b have charges $-\frac{1}{3}e$, where $-e$ is the electron charge.

Quark masses	Current quark mass	Constituent quark mass
d	15 MeV	330 MeV
u	7 MeV	330 MeV
s	200 MeV	500 MeV
c	1.3 GeV	1.5 GeV
b	4.8 GeV	5 Gev
t	170 GeV	-

Note: No free quarks are seen, and we do not list constituent quark mass for the t quark as no hadrons involving the t quark are known yet. The current quark mass is what enters in the QCD Lagrangian. The constituent quark mass tells us how the quark behaves inside hadrons (*i.e.*, it accounts for the confining forces).

2. Quarks are spin 1/2 fermions and have an internal quantum number called color. Hadron spectroscopy implies that there are 3 colors for each quark and that hadrons are color singlets (the color wave function is totally antisymmetric). This is known as color confinement and is required by the fact that no isolated quarks are observed. They only appear inside hadrons. There are two types of hadrons made up of quarks – Mesons ($q\bar{q}$ systems) and Baryons (qqq systems), and their antiparticles.

   With the above quark content, we need an interaction between quarks with the following properties:

3. The interaction should lead to color confinement. Thus the interaction should correspond to the color charges of quarks. Following the success of QED, we want to construct a "gauge theory" of color interactions.

4. Deep-inelastic scattering of leptons with nucleons shows Bjorken scaling which implies that at short distances quarks are almost free: this is the 'asymptotic freedom'. Thus, we need a theory where the coupling constant becomes small at large energies. In 4 dimensions, only Yang-Mills theories show this type of behavior. These are gauge theories with a non-Abelian gauge group.

   Combining the requirement of asymptotic freedom with that of color charge interaction (with 3 different colors), we come to a theory of strong interactions based on the $SU(3)$ color gauge group. This is called Quantum Chromo Dynamics (QCD) and is believed to be the correct theory of strong interactions.

   To understand this theory, we will first recall the basics of QED which is a gauge theory based on the Abelian gauge group $U(1)$. We will then generalize the construction to QCD.

1.2.2 QED

First recall the Lagrangian for a free electron field $\psi(x)$,

$$L_0 = \overline{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

L_0 has a global $U(1)$ symmetry under the transformation

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{-i\alpha}\psi(x) \\ \overline{\psi}(x) \to \overline{\psi}'(x) &= e^{i\alpha}\overline{\psi}(x) \end{aligned}$$

Here α is the parameter of the symmetry transformation. α is independent of $\mathbf{x}$ and t and hence the transformation is called a global symmetry transformation. We generalize this symmetry to a local gauge symmetry when α depends on $\mathbf{x}$ and t, so $\alpha \to \alpha(x)$. The motivation for this is simply that we know that this way we can write down the theory of electromagnetic interactions of charged particles.

With $\alpha \to \alpha(x)$ one says that the symmetry is gauged. So, now we consider the following transformation

$$\begin{aligned} \psi(x) \to \psi'(x) &= e^{-i\,\alpha(x)}\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) &= e^{i\alpha(x)}\bar{\psi}(x) \end{aligned}$$

With $L_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)$ we see that the $m\,\bar{\psi}\psi$ term is invariant under this transformation, but the derivative term is not invariant.

$$\begin{aligned}\bar{\psi}(x)\partial_\mu\psi(x) &\rightarrow \bar{\psi}'(x)\partial_\mu\psi'(x)\\ &= \bar{\psi}(x)e^{i\alpha(x)}\partial_\mu\left(e^{-i\alpha(x)}\psi(x)\right)\\ &= \bar{\psi}(x)\partial_\mu\psi(x) - i\overline{\psi}(x)\left(\partial_\mu\alpha(x)\right)\psi(x)\end{aligned}$$

The second term on the r.h.s. spoils the invariance. If instead of $\overline{\psi}(x)\partial_\mu\psi(x)$, we had a term $\overline{\psi}(x)D_\mu\psi(x)$ where $D_\mu\psi(x)$ has simple transformation rule

$$D_\mu\psi(x) \rightarrow [D_\mu\psi(x)]' = e^{-i\alpha(x)}D_\mu\psi(x)$$

(*i.e.* $D_\mu\psi(x)$ transforms in the same way as $\psi(x)$), then $\overline{\psi}(x)D_\mu\psi(x)$ will be gauge invariant. $D_\mu\psi(x)$ is called the gauge-covariant derivative (or simply covariant derivative) of $\psi(x)$.

One can realize this requirement of $D_\mu\psi(x)$ by enlarging the theory by including a new vector field $A_\mu(x)$, the gauge field. With this,

$$D_\mu\psi(x) = \left(\partial_\mu - ieA_\mu\right)\psi(x)$$

One can easily check that the requirement

$$[D_\mu\psi(x)]' = e^{-i\alpha(x)}D_\mu(x)\psi(x)$$

implies the following transformation property for the gauge field:

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)$$

With A_μ transforming like this, the derivative term becomes invariant

$$\begin{aligned}\overline{\psi}i\gamma^\mu\left(\partial_\mu - ieA_\mu\right)\psi &\rightarrow \overline{\psi}'i\gamma^\mu\left(\partial_\mu - ieA'_\mu\right)\psi'\\ &= \overline{\psi}e^{i\alpha(x)}i\gamma^\mu\left(\partial_\mu - ieA_\mu + i\partial_\mu\alpha(x)\right)e^{-i\alpha(x)}\psi(x)\\ &= \overline{\psi}i\gamma^\mu\left(\partial_\mu - ieA_\mu\right)\psi(x)\end{aligned}$$

Thus, the extra term from the gauge transformation of A_μ precisely cancels the extra term when ∂_μ acts on $e^{-i\alpha(x)}\psi(x)$. This will be important when we discuss QCD. Our Lagrangian L_0 changes now to

$$L = \overline{\psi}i\gamma^\mu\left(\partial_\mu - ieA_\mu\right)\psi - m\overline{\psi}\psi$$

A_μ is the gauge field for the electromagnetic interaction. To include dynamics of A_μ, we add

$$L_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

This leads to the Maxwell equations. With the -$\frac{1}{4}$ normalization one gets the equation

$$\partial_\mu F^{\mu\nu} = -eJ^\mu$$

where $J^\mu = \overline{\psi}\gamma^\mu\psi$ is the conserved matter current. One can easily check directly that $F^{\mu\nu}$ is gauge invariant.

Exercise: Verify that

$$[D_\mu D_\nu - D_\nu D_\mu]\,\psi = -ieF_{\mu\nu}\psi$$

(This equation has a nice geometric meaning in terms of curvature.)

Using this and the transformation property of $D_\nu\psi$ one can show that $F_{\mu\nu}$ is gauge invariant. We thus get the final QED Lagrangian

$$L = \overline{\psi}i\gamma^\mu\left(\partial_\mu - ieA_\mu\right)\psi - m\overline{\psi}\psi - \frac{1}{4}F_{\mu\nu}.F^{\mu\nu}$$

Note the following:

1. A term like $m^2 A_\mu A^\mu$ is not gauge invariant, so the photon is massless. This will remain true for all gauge theories including QCD.

2. The coupling of the photon to the electron is contained in the $D_\mu\psi$ term. It is called the 'minimal coupling'. This will also be used in QCD

3. The QED Lagrangian does not have a gauge field self coupling, *i.e.*, there are no terms like AAA, or $AAAA$. This is because the photon does not carry charge. This will not be true for QCD. Gluons (which are the analogs of the photon) carry color charges and hence self interact. Let us now write down the Lagrangian for QCD with 2 colors (a hypothetical case).

1.2.3 Non-Abelian Gauge Symmetry: Yang-Mills theory

We first consider a theory with the symmetry group $SU(2)$ (it was $U(1)$ for QED which is Abelian). $SU(2)$ is a non-Abelian group. Let the fermion fields be a doublet (fundamental representation of $SU(2)$):

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Note that each component ψ_i will be a four component Dirac Spinor. Under an $SU(2)$ transformation, ψ will transform as

$$\begin{aligned} \psi(x) \to \psi'(x) &= \exp\left\{\frac{-i\vec{\tau}.\vec{\theta}}{2}\right\}\psi(x) \\ &\equiv U\psi(x) \end{aligned}$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the usual Pauli matrices, satisfying the Lie algebra of $SU(2)$, v.i.z.,

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] = i\epsilon^{ijk}\frac{\tau_k}{2} \quad i, j, k = 1, 2, 3$$

and $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ are the $SU(2)$ transformation parameters.

We write the Lagrangian

$$L = \overline{\psi}(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x)$$

This is again invariant under the above global $SU(2)$ transformation with $\vec{\theta}$ being independent of $\vec{\mathbf{x}}$ and t.

$$\begin{aligned} \psi \rightarrow \psi' &= U\psi \\ \overline{\psi} \rightarrow \overline{\psi}' &= \overline{\psi}\, U^\dagger \quad \text{where } U^\dagger U = 1 \end{aligned}$$

Now we gauge this symmetry, *i.e.*, make θ_i space-time dependent. Then

$$\psi(x) \rightarrow \psi'(x) = U(\theta(x))\psi(x)$$

with

$$U(\theta(x)) = \exp\left\{-i\frac{\vec{\tau}}{2}.\vec{\theta}(x)\right\}$$

Again we can easily see that the mass term $m\overline{\psi}\psi$ in L is invariant under this symmetry transformation but the derivative term is not. To make the derivative term also invariant we will again construct a covariant derivative D_μ by introducing new gauge fields (like A_μ was introduced for QED).

Note that the derivative term which spoils gauge invariance has a term proportional to $\partial_\mu U(\theta)$, *i.e.*,

$$\partial_\mu\left\{\exp\left(-i\frac{\tau^a}{2}\theta^a(x)\right)\right\} \sim \tau^a\partial_\mu\theta^a(x)\exp(...)$$

for $a = 1, 2, 3$. It is this term which spoils the invariance of L when θ^a depend on $\vec{\mathbf{x}}$ and t. Using gauge fields we have to compensate for these derivatives. Since τ^a, $a = 1, 2, 3$ are linearly independent, to cancel each derivative, such as $\tau_1\partial_\mu\theta^1$, one will need a gauge field. That is, we will need a term like $\tau^a A^a_\mu$, $a = 1, 2, 3$ with each gauge field transforming with the appropriate θ (as we see below). Thus the number of gauge fields to be introduced = number of generators = 3 for $SU(2)$.

Note: When we construct a gauge theory for $SU(3)$, *i.e.* real QCD, then we need the number of gauge fields = number of generators of $SU(3) = 8$. (For $SU(N)$, the number of generators is $N^2 - 1$ for $N \neq 1$). Each gauge field is like an independent photon. These are the gluons (massless gauge bosons). Thus we will need 8 gluons for QCD.

We go back to the case of 2 color QCD with the gauge group $SU(2)$. Again, to have the derivative term gauge invariant, we need the following transformation property for the covariant derivative:

$$D_\mu\psi(x) \to [D_\mu\psi(x)]' = U(\theta)D_\mu\psi(x)$$

where $\psi(x) \to \psi'(x) = U(\theta)\psi(x)$. Clearly, with ∂_μ replaced by D_μ we get

$$L = \overline{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x)$$

which will be gauge invariant. We write $D_\mu\psi(x)$ as

$$D_\mu\psi(x) = \left[\partial_\mu - ig\frac{\tau^a}{2}A^a_\mu\right]\psi(x)$$

where g is the coupling constant. One can check that the requirement of $[D_\mu\psi(x)]' = U(\theta)D_\mu\psi(x)$ implies the following transformation properly for the gauge fields:

$$\frac{\tau^a}{2}A^{a\prime}_\mu = U(\theta)\frac{\tau^a}{2}A^a_\mu U(\theta)^{-1} - \frac{i}{g}\left[\partial_\mu U(\theta)\right]U^{-1}(\theta)$$

Recall that for QED also, we had

$$\begin{aligned}\psi(x) \to \psi'(x) &= e^{-i\alpha(x)}\psi(x)\\ &\equiv U(\alpha)\psi(x)\end{aligned}$$

The transformation of A_μ is then analogously

$$\begin{aligned}A'_\mu &= U(\alpha)A_\mu U^{-1}(\alpha) - \frac{i}{e}\left[\partial_\mu U(\alpha)\right]U^{-1}(\alpha)\\ &= A_\mu - \frac{i}{e}\left(-i\partial_\mu\alpha(x)\right) = A_\mu - \frac{1}{e}\partial_\mu\alpha(x)\end{aligned}$$

which is the familiar transformation for QED.

Self Interactions of Gauge Fields

One crucial difference between QED and Yang-Mills gauge theories is that for the non-Abelian case gauge fields have self interactions whereas in QED photons do not have self interactions. To understand the basic physical reason for this, let us go back to the $SU(2)$ gauge theory case and consider an infinitesimal gauge transformation for the vector potentials.

For $\theta(x) \ll 1$ we write

$$U(\theta) = \exp\left\{-i\frac{\vec{\tau}}{2}.\vec{\theta}(x)\right\} \simeq 1 - i\frac{\vec{\tau}.\vec{\theta}(x)}{2}$$

Exercise: Using this in the transformation law for A^a_μ, and neglecting θ^2 terms, show that one gets

$$\frac{\tau^c}{2}A'^c_\mu = \frac{\tau^c}{2}A^c_\mu + \theta^a(x)A^b_\mu\epsilon^{abc}\frac{\tau^c}{2} - \frac{1}{g}\frac{\tau_c}{2}\partial_\mu\theta^c(x).$$

Since τ^a are linearly independent, we get

$$A'^c_\mu = A^c_\mu + \epsilon^{abc}\theta^a A^b_\mu - \frac{1}{g}\partial_\mu\theta^c$$

ϵ^{abc} comes from

$$\left[\frac{\tau^a}{2},\frac{\tau^b}{2}\right] = i\epsilon^{abc}\frac{\tau^c}{2}$$

Consider global transformations, so $\partial_\mu\theta^c = 0$, we get

$$A'^c_\mu = A^c_\mu + \epsilon^{abc}\theta^a A^b_\mu$$

This shows that A^c_μ transforms in the adjoint representation of $SU(2)$. Several important results follow from this expression. Recall Noether's theorem which implies that one can calculate a symmetry current and the associated charge. For example, recall the case of QED.

$$\begin{aligned} \psi \to \psi'(x) &= e^{-i\alpha(x)}\psi(x) \\ A_\mu \to A'_\mu(x) &= A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \end{aligned}$$

For global transformations, $\alpha(x) = \alpha$ and we get

$$\psi'(x) = e^{-i\alpha}\psi(x)\,\text{and}\,\mathrm{A}'_\mu(\mathrm{x}) = \mathrm{A}_\mu(\mathrm{x})$$

So, under a global $U(1)$ (continuous) symmetry transformations, $\psi(x)$ transforms non-trivially. The associated charge is the "electric charge" of the field $\psi(x)$. However, $A_\mu(x)$ transforms trivially under global $U(1)$ transformations. So in QED, the photon does not carry any electric charge (the symmetry current will give zero charge). As the photon does not have electric charge, it does not have self couplings like AAA or $AAAA$. Now, for the $SU(2)$ case we saw that the transformation of A^a_μ for constant $SU(2)$ transformations is

$$A'^c_\mu = A^c_\mu + \epsilon^{abc}\theta^a A^b_\mu$$

Thus, under global $SU(2)$ transformations, A^a_μ transforms non-trivially. Hence there will be a non-zero Noether charge associated with A^c_μ. Due to this we expect self couplings. Indeed, we will see that for every Yang-Mills theory there are self couplings like AAA and $AAAA$.

Note : So far we have the Lagrangian for the $SU(2)$ case

$$L = \overline{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x)$$

We are missing a term analogous to $F_{\mu\nu}F^{\mu\nu}$ for the QED case. To write such a term we recall the following relation from QED

$$(D_\mu D_\nu - D_v D_\mu)\,\psi(x) = -ieF_{\mu\nu}\psi(x)$$

We will use this type of expression for defining the appropriate expression for $F_{\mu\nu}$ for the $SU(2)$ case. Since

$$D_\mu\psi = \left(\partial_\mu - ig\frac{\vec{\tau}}{2}.\vec{A}_\mu\right)\psi$$

involving Pauli matrices, we extend the earlier relation appropriately as

$$[D_\mu D_\nu - D_\nu D_\mu]\,\psi \equiv -ig\left(\frac{\tau^a}{2}F^a_{\mu\nu}\right)\psi$$

This expression is used to define $F^a_{\mu\nu}$.

Exercise: Show that the evaluation of the l.h.s. gives

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g\epsilon^{cab}A^a_\mu A^b_\nu$$

This is the expression for the field strength $F^c_{\mu\nu}$ for the non-Abelian case. We can write

$$A_\mu \equiv A^a_\mu\frac{\tau^a}{2} \quad \text{and} \quad F_{\mu\nu} \equiv \frac{\tau^a}{2}F^a_{\mu\nu} \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu]$$

Exercise: In QED, $F_{\mu\nu}$ was gauge invariant. Show that under an $SU(2)$ gauge transformation

$$\tau^a F^a_{\mu\nu} \to \tau^a F^{a\prime}_{\mu\nu} = U(\theta)\tau^b F^b_{\mu\nu}U(\theta)^{-1}.$$

Thus, to construct the analog of $F_{\mu\nu}F^{\mu\nu}$ term here, we write

$$\mathrm{Tr}\left\{\left(\vec{\tau}.\vec{F}_{\mu\nu}\right)(\vec{\tau}.F^{\mu\nu})\right\}$$

This will be gauge invariant due to the cyclic properly of the trace. Note that

$$\mathrm{Tr}\left\{\tau^a F^a_{\mu\nu}\tau^b F^{b\mu\nu}\right\} \quad = \quad \mathrm{Tr}\ \tau^a\tau^b F^a_{\mu\nu}F^{b\mu\nu} = 2F^a_{\mu\nu}F^{a\mu\nu}$$

using $\mathrm{Tr}[\tau^a\tau^b] = 2\delta^{ab}$.

Now, we can write down the complete gauge invariant Lagrangian for the $SU(2)$ color gauge theory with the doublet field ψ as

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \overline{\psi}i\gamma^\mu D_\mu\psi - m\overline{\psi}\psi$$

Generalization to other Lie Groups

One can generalize this construction to any other Lie group. Essentially, one has to replace τ^i by appropriate generators and ϵ^{abc} by corresponding structure constants.We will first discuss the general case of a simple Lie Group and then write down the Lagrangian for QCD with 3 colors. Suppose G is a simple Lie Group (essentially meaning that it is not a direct product of other groups). Let F^a be the generators of the group, satisfying the Lie algebra

$$\left[F^a, F^b\right] = if^{abc}F^c$$

where f^{abc} are totally antisymmetric structure constants (f^{abc} are real). For $SU(2)$ we had

$$\left[\frac{\tau^a}{2}, \frac{\tau^b}{2}\right] = i\epsilon^{abc}\frac{\tau_c}{2}$$

Suppose ψ transforms under some representation of G with representation matrices T^a, *i.e.*, under a gauge transformation

$$\begin{aligned}\psi(x) \to \psi'(x) &= \exp\left\{-i\vec{T}.\vec{\theta}(x)\right\}\psi(x) \\ &\equiv U(\theta)\psi(x)\end{aligned}$$

Thus

$$\left[T^a, T^b\right] = if^{abc}T^c$$

Recall that for the $SU(2)$ case, $\vec{T}$ were $\frac{\vec{\tau}}{2}$ and f^{abc} was ϵ^{abc}. The covariant derivative then is

$$D_\mu\psi = \left(\partial_\mu - igT^aA^a_\mu\right)\psi$$

The field strength tensor is

$$F^a_{\mu v} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$$

The gauge transformation for A^a_μ is

$$\vec{T}.\vec{A}_\mu(x) \to \vec{T}.\vec{A}'_\mu(x) = U(\theta)\vec{T}.\vec{A}_\mu U^{-1}(\theta) - \frac{i}{g}\left[\partial_\mu U(\theta)\right]U^{-1}(\theta)$$

Again, all these are exactly the same as the $SU(2)$ case with the replacement

$$\frac{\vec{\tau}}{2} \to \vec{T} \quad \text{and} \quad \epsilon^{abc} \to f^{abc}$$

Also the number of A_μ^a is equal to the number of generators T^a. We can write the complete Lagrangian as

$$L = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \overline{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi$$

Self Interactions

Note that $F_{\mu\nu}^a F^{a\mu\nu}$ term has the following types of terms

$$g\, \partial_\nu A_\mu^a f^{abc} A^{\mu b} A^{\nu c}$$

and

$$g^2 f^{abc} f^{alm} A_\mu^b A_\nu^c A^{\mu l} A^{\nu m}$$

The corresponding Feynman diagrams have three point and four point vertices. Thus, every gauge theory with a non-Abelian gauge group has self couplings for the gauge fields. This was expected since we saw that gauge bosons here carry charges. In contrast, in QED (Abelian Group $U(1)$) photons have no self interaction.

It is straightforward now to write the Lagrangian for QCD. We have six types of quarks (flavors u, d, s, etc). The gauge group is $SU(3)$ color. Each quark comes in 3 colors. That is, quarks are taken to transform as the 3-dimensional fundamental representation of the $SU(3)$ color group. $SU(3)$ has 8 generators, so we need 8 gauge fields A_μ^a, $a = 1, ...8$. These are associated with 8 gluons.

We can write down the Lagrangian

$$L_{QCD} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \sum_\alpha \overline{\psi}_\alpha \left(i\gamma^\mu D_\mu - m_\alpha \right) \psi_\alpha$$

where $\alpha = u, d, c, s, t, b$ is the flavor index for quarks.

As ψ_α is taken to be in the 3-dimensional fundamental representation of $SU(3)_c$, we may represent it as, for example,

$$\psi_\alpha = \begin{pmatrix} \psi^{\text{red}} \\ \psi^{\text{blue}} \\ \psi^{\text{green}} \end{pmatrix}_\alpha$$

Thus, we take the following representation for the generators of $SU(3)$

$$T^a = \frac{\lambda^a}{2}, \qquad a = 1, 2...8$$

where λ^a are the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0, \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0, \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

Also,

$$\left[T^a, T^b\right] = if^{abc}T^c$$

is the Lie algebra of $SU(3)$ with antisymmetric structure constants f^{abc} given by

$$f^{123} = 1, \; f^{458} = f^{678} = \sqrt{3}/2, \; f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = 1/2.$$

With $T^a = \frac{\lambda^a}{2}$, the covariant derivative is

$$D_\mu \psi_\alpha = \left(\partial_\mu - ig_s T^a A^a_\mu\right) \psi_\alpha$$

g_s is the strong interaction coupling constant. The expressions for $F^a_{\mu\nu}$ etc. are the same as given for the general case of group G with $T^a = \frac{\lambda^a}{2}$. We thus conclude that gluons carry color charges and hence they have self interactions.

1.2.4 Symmetries of QCD

Apart from the gauge $SU(3)$ symmetry of QCD, which is exact, QCD possesses the following approximate global symmetries.

Isospin Symmetry

This played a crucial role in the early stages of development of QCD in terms of hadron spectroscopy. If $m_\alpha \simeq m$ for certain α, say $\alpha = u, d, s$, then we can write

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \qquad \overline{\psi} = \left(\overline{u}\;\overline{d}\;\overline{s}\right)$$

$$L = \overline{\psi}\,(i\gamma^{\mu}D_{\mu} - m)\,\psi + \sum_{\beta} \overline{\psi}_{\beta}\,(i\gamma^{\mu}D_{\mu} - m_{\beta})\,\psi_{\beta} \qquad \beta = c, t, b$$

This is invariant under an $SU(3)$ global symmetry transformation acting on

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

This invariance is known as the isospin flavor symmetry and originally it led to the discovery of the quark model.

Chiral Symmetry

This is a very important symmetry of QCD which arises if $m_{\alpha} \simeq 0$ for certain α, leading to decoupled left handed and right handed components of the massless quarks.

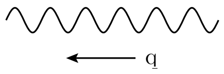
1.2.5 Feynman Rules for QCD

Essentially, the only difference from the case of QED is that for QCD we have color factors (color states C and $C^{\dagger}$) and λ matrices. Also, in QCD we have 3-gluon and 4-gluon vertices which are not there in QED. The Feynman rules for QCD (in the Lorenz gauge) are given below.

1. **The gluon propagator:** Recall that the propagator in QED for the photon is

$$-i\,\frac{g^{\mu\nu}}{q^2}$$

The Feynman rule for the gluon propagator is



$$= \quad -i\,\frac{g^{\mu\nu}}{q^2}\delta^{ab}$$

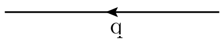
where $a, b = 1, 2, .., 8$ are color indices for gluons. Note that one may expect 9 gluon states : $3 \otimes \overline{3}$, $r\overline{r}$, $r\overline{b}$, $r\overline{g}$, etc. However $3 \otimes \overline{3} = 1 + 8$, where 1 is a color singlet. The gluon cannot be a color singlet, otherwise it does not interact via the color interaction. Hence there are only 8 (an octet of) gluons. Color states C for quarks are given by a 3 vector

$$C : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sim \text{red}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sim \text{blue}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sim \text{green}$$

Similarly, we have an eight element column vector for gluons

$$A: \quad \begin{pmatrix} 1 \\ . \\ . \\ . \\ : \\ 0 \end{pmatrix} \text{for } |1\rangle, ... \begin{pmatrix} 0 \\ 0 \\ . \\ ; \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{for } |7\rangle, \text{ etc.}$$

2. **Quark propagator:**



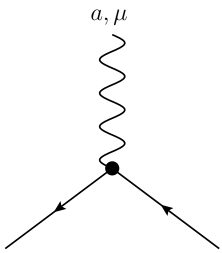
$$= \; i\frac{(\gamma^\mu q_\mu + m)}{q^2 - m^2}\delta_{\alpha\beta}$$

where $\alpha, \beta = 1, 2, 3$ are color indices for quarks. Apart from $\delta_{\alpha\beta}$ the above is the same as in QED for electrons.

3. **Quark-gluon vertex:** The quark-gluon interaction term in the QCD Lagrangian is

$$L_{int} = \overline{\psi} g_s \frac{\lambda^a}{2} \gamma_\mu A^{a\mu} \psi$$

(a is the color index). Thus the quark-gluon vertex is given by



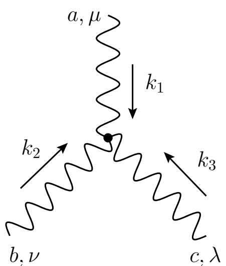
$$= \quad i g_s \frac{\lambda^a}{2} \gamma^\mu$$

In QED, the electron-photon vertex is $ie\gamma^\mu$.

4. **Three gluon vertex:** The relevant term in L is

$$-g_s \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) f^{abc} A^{b\mu} A^{c\nu}$$

The vertex is

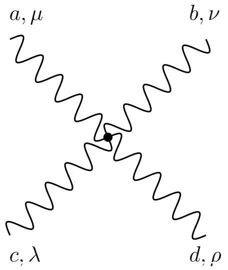


$$= \quad g_s f^{abc} \; [g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu]$$

5. **Four gluon vertex:** The relevant interaction term in L is

$$-g_s^2 f^{abc} A^b_\mu A^c_\nu f^{ade} A^{d\mu} A^{e\nu}$$

So, the vertex is



$$= \quad -ig_s^2 \left[f^{abe}\, f^{cde}\, (g^{\mu\lambda} g^{\nu\rho} - g^{\mu\rho} g^{\nu\lambda}) \right. \\ + f^{ace}\, f^{bde}\, (g^{\mu\nu} g^{\lambda\rho} - g^{\mu\rho} g^{\nu\lambda}) \\ \left. + f^{ade}\, f^{bce}\, (g^{\mu\nu} g^{\lambda\rho} - g^{\mu\lambda} g^{\nu\rho}) \right]$$

6. **External quarks and anti-quarks:** External quark with momentum p, spin s, and color C:

   - Incoming quark: $u^{(s)}(p)\, C$, while for QED we have $u^{(s)}(p)$.
   - Outgoing quark: $\overline{u}^{(s)}(p)\, C^\dagger$, while for QED we have $\overline{u}^{(s)}(p)$.

   For an external antiquark:

   - Incoming antiquark: $\overline{v}^{(s)}(p)\, C^\dagger$, while for QED we have $\bar{v}^s$.
   - Outgoing antiquark: $v^{(s)}(p)\, C$, while for QED we have v^s. Here C represents the color of the corresponding quark.

7. **External gluon:**

   - Incoming gluon of momentum p, polarization ϵ, color a: $\epsilon_\mu(p)\, A^a$, while for QED (photon) we have $\epsilon_\mu(p)$.
   - Outgoing gluon of momentum p, polarization ϵ, color a: $\epsilon^*_\mu(p)\, A^{a\dagger}$, while for QED (photon) we have $\epsilon^*_\mu(p)$.

In addition there are Feynman rules for unphysical ghost particles corresponding to the longitudinal polarization of virtual gluons. Feynman rules for the same can be found in, e.g. the first reference in ref. [1].

1.3 Running Coupling Constant in QCD

1.3.1 Physical Picture

Let us recall how a 'screened' charge appears in an ordinary dielectric medium like water. A test charge $+q$ in a polarisable dielectric medium is screened from outside. There will be an induced dipole moment $\vec{P}$ per unit volume, and the effect of $\vec{P}$ on the resultant field is the same as that produced by a volume charge density equal to -$\vec{\nabla}.\vec{P}$. For a linear medium, $\vec{P}$ is proportional to $\vec{E}$, so $\vec{P} = \chi\epsilon_o\vec{E}$. Gauss's law is then modified from

$$\vec{\nabla}.\vec{E} = \rho_{free}/\epsilon_o$$

to

$$\vec{\nabla}.\vec{E} = \frac{\rho_{free} - \vec{\nabla}.\vec{P}}{\epsilon_o}$$

Taking χ to be approximately constant, we get

$$\begin{aligned} \vec{\nabla}.\vec{E} &= \frac{\rho_{free}}{\epsilon_o} - \chi\vec{\nabla}.\vec{E} \\ \text{or,} \quad \vec{\nabla}.\vec{E} &= \frac{\rho_{free}}{\epsilon} \end{aligned}$$

where $\epsilon = (1+\chi)\epsilon_o$ is the dielectric constant of the medium (ϵ_o being that of vacuum). Thus, the electric field is effectively reduced by the factor $(1+\chi)^{-1}$.

However, this is a macroscopic treatment with the molecules being replaced by a continuous distribution of charge density, $-\vec{\nabla}.\vec{P}$. For very small distances ($\sim$ molecular distances), the screening effect will be reduced. Thus, we expect that ϵ should be a function of the distance r from the test charge. In general, the electrostatic potential between two test charges q_1, and q_2 in a dielectric medium can be represented phenomenologically by

$$V(r) = \frac{q_1 q_2}{4\pi\epsilon(r) r}$$

where $\epsilon(r)$ varies with r. We can define an effective charge

$$q' = \frac{q}{\sqrt{\epsilon(r)}}$$

for each test charge.

Effective Charge in QED

In quantum field theory, the polarisable medium is replaced by the vacuum. We know about the polarization of the vacuum arising from vacuum fluctuations which are always there. Virtual e^+e^- pairs align in the presence of a test charge. Thus, near a test charge, in vacuum, charged pairs are created. They exist for a time $\Delta t \sim \hbar/mc^2$. They can spread to a distance of about $c\Delta$t (*i.e.* the Compton wavelength λ_c). This distance gives a measure of the equivalent of the molecular diameter for a dielectric medium. Virtual e^+e^- pairs are effectively dipoles of length $\lambda_c \sim \frac{1}{m}$. Again, due to the screening effects of these vacuum fluctuations, the effective charge will depend on the distance.

Meaning of the Familiar Symbol e

This is simply the effective charge as $r \to \infty$, or in practice, the charge relevant for distances much larger than the particle's Compton wavelength. For example, it is this large distance value of the charge which is measured in Thomson scattering. The distance (or momentum) dependent coupling constant is called the 'Running coupling constant'. It arises due to renormalization which we discuss in the next section.

1.3.2 β Function in QFT

We will see that due to renormalization in QFT, one gets a running coupling constant $g(t)$ where t is the momentum (distance^{-1}) scale. The behavior of $g(t)$ as a function of t is determined by the β function

$$t\frac{dg(t)}{dt} = \beta(g)\,.$$

Once we know the β function of a theory, we can immediately get the running coupling constant of the theory.

How does one calculate β(g)? Let us sketch the important steps for a scalar theory. We will then discuss results for QED and QCD. Note that renormalized g arises due to vacuum fluctuations. The latter also lead to divergences. Hence the two are intimately connected.

Divergences and Renormalization in QFT

First take the case of scalar field theory with a ϕ^4 interaction,

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4$$

The Feynman rules for the propagator and vertex of this theory are given by

$$\frac{i}{p^2 - m^2} \quad \text{and} \quad -ig$$

Divergences arise from loop integrals. For example, the self energy contribution at the one loop level to the free particle propagator is

$$g\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - m^2}$$

This is ultraviolet divergent as there are 4 powers of q in the numerator and 2 in the denominator.

Similarly, consider the 1-loop contribution to the 4-point vertex function.

$$g^2\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2 - m^2)\left([p_1 + p_2 - q]^2 - m^2\right)}$$

Here there are 4 powers of q in both numerator and denominator, so we have a logarithmic divergence.

1PI Diagrams

For studying renormalization we focus on the one particle irreducible (1PI) diagrams. These are the connected Feynman diagrams, which cannot be disconnected by cutting any one internal line. Correspondingly, we define the 1PI Green's function $\Gamma^{(n)}\ (p_1, ..p_n)$ which have contributions from 1PI diagrams only. The reason for selecting 1PI diagrams is that every other diagram can be decomposed into 1PI diagrams without further loop integration. So, if we know how to take care of the divergences of 1PI diagrams, we can then handle other diagrams also.

1.3.3 Regularization

One needs to isolate the divergences in these divergent integrals and **regularize** them or make them finite. Eventually, these divergences are absorbed by redefining various parameters of the theory, *i.e.* by **renormalization**. There are various techniques for regularizing a divergent Feynman diagram.

- **Pauli-Villars regularization**

  Here the propagator is modified to

$$\frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2} = \frac{m^2 - M^2}{(p^2 - m^2)\,(p^2 - M^2)}$$

  As the propagator now behaves as $\frac{1}{p^4}$, integrals usually converge. When we take $M^2 \to \infty$, the original theory is restored.

- **Cut-off regularization**

  One can use a cut off Λ in the momentum integral. Eventually the $\Lambda \to \infty$ limit is taken.

  The above methods become problematic when non-Abelian gauge theories are considered.

- **Dimensional Regularization**

  This is the most versatile regularization technique. Here the action is generalized to arbitrary dimensions d where there are regions in the complex d space in which the Feynman integrals are all finite. Then as we analytically continue d to 4, the Feynman graphs pick up poles in d space, allowing us to absorb the divergences of the theory into physical parameters.

1.3.4 Scalar Theory

Let us consider dimensional regularization for the scalar theory

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4$$

We first generalize this theory to arbitrary d dimensions. As $S = \int L d^d x$ is dimensionless (S should have units of $\hbar = 1$), we have, from the first term in L,

$$\frac{1}{\mathrm{L}^2}\mathrm{L}^d[\phi]^2 = 1$$
$$\Rightarrow [\phi] = \mathrm{L}^{\frac{2-d}{2}}$$

where L denotes the length dimension (same as mass^{-1} dimension in natural units). So the mass dimension of ϕ is $\frac{d}{2} - 1$.

The $g\phi^4$ term has mass dimension $[g]\ M^{2d-4}$. This needs to be $[M]^d$. To keep g dimensionless, we need to introduce a factor μ^{4-d} to cancel the $(2d-4-d)$ mass dimension in $\int g\phi^4 d^d x$. Thus we get

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 - \frac{\mu^{4-d} g}{4!}\phi^4$$

Note the presence of the **arbitrary mass scale** μ. With this L we can calculate the divergent 1-loop diagrams. The self energy is

$$\sum = \frac{1}{2} g \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2}$$

These integrals can be calculated using the gamma function.

$$\sum = \frac{-ig}{32\pi^2} m^2 \left(\frac{4\pi\mu^2}{m^2}\right)^{2-d/2} \Gamma\left(1 - d/2\right)$$

The gamma function Γ has poles at zero and negative integers, so, we see that the divergence of the integral manifests itself as a simple pole as $d \to 4$. Using $\epsilon = 4 - d$

$$\Gamma(1 - d/2) = \Gamma\left(-1 + \frac{\epsilon}{2}\right) = \frac{-2}{\epsilon} - 1 + \gamma + O(\epsilon)$$

where $\gamma = 0.577$ is the Euler-Mascheroni constant. Thus expanding the above expression about $d = 4$ using $a^\epsilon = 1 + \epsilon \ln a + \ldots.$, we get

$$\begin{aligned} \frac{1}{i}\Sigma &= \frac{igm^2}{16\pi^2\epsilon} + \frac{igm^2}{32\pi^2}\left[1 - \gamma + \ln\frac{(4\pi\mu^2)}{m^2}\right] + O(\epsilon) \\ &= \frac{igm^2}{16\pi^2\epsilon} + \text{finite} \end{aligned}$$

Similarly, the 4-point function to order g^2 is

$$\frac{1}{2}g^2(\mu^2)^{4-d}\int \frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-m^2)}\frac{1}{[(p-q)^2-m^2]}$$

Again using the Γ function one can get

$$\frac{ig^2\mu^\epsilon}{16\pi^2\epsilon} - \text{finite part}$$

We can now obtain the vertex functions (with amputated legs). The 2-point function is given by

$$\begin{aligned}\Gamma^{(2)}(p) &= p^2 - m^2 - \sum(p^2) \\ &= p^2 - m^2\left(1 - \frac{g}{16\pi^2\epsilon}\right) \qquad \text{neglecting the finite term}\end{aligned}$$

Apart from the inverse of the bare propagator, $\Gamma^{(2)}$ contains only 1PI graphs. The 4-point function is given by

$$\Gamma^{(4)}(p_i) = -ig\mu^\epsilon\left(1 - \frac{3g}{16\pi^2\epsilon}\right) + \text{finite} \equiv -ig_R$$

Renormalization

Consider now the vertex functions $\Gamma^{(2)}$ and $\Gamma^{(4)}$ to one loop approximation.

$$\begin{aligned}\Gamma^{(2)}(p) &= p^2 - m^2 - \sum \\ \sum &= \frac{-gm^2}{16\pi^2\epsilon}\end{aligned}$$

where $\epsilon = 4 - d$ and we have ignored finite parts. We can rewrite it as

$$\Gamma^{(2)}(p) = p^2 - m_1^2$$

where

$$m_1^2 = m^2\left(1 - \frac{g}{16\pi^2\epsilon}\right) = \frac{m^2}{(1 + g/16\pi^2\epsilon)}$$

m_1 is taken to be finite, representing the physical mass. This is called the **renormalized mass**. $\sum$ is divergent (with $\epsilon \to 0$) so m (the bare mass) is taken to be appropriately divergent so that m_1 is finite.

The renormalized mass m_1 is given by

$$m_1^2 = -\Gamma^{(2)}(0)$$

Note that this is the renormalization condition where the physical mass is defined at $p = 0$. It could very well have been defined at some other value of p.

Similarly, consider $\Gamma^{(4)}$ where

$$i\Gamma^{(4)}(p_i) = g\mu^\epsilon - \frac{g^2\mu^\epsilon}{16\pi^2}\left[\frac{3}{\epsilon} + \tilde{\Gamma}(p_i)\right]$$

where $\tilde{\Gamma}(p_i)$ is finite. Define a new parameter g_1, the renormalized coupling constant, by

$$g_1 = g\mu^\epsilon - \frac{g^2\mu^\epsilon}{16\pi^2}\left[\frac{3}{\epsilon} + \tilde{\Gamma}(0)\right]$$

Again, note here that g_1 is being defined at p_i=0. An alternative is to define it at the symmetrical point, $p_i^2 = m^2$, so $s, t, u = 4m^2/3$.

These are the results up to the 1-loop level. It turns out that when 2-loop diagrams are calculated then using renormalization of the m and g parameters, $\Gamma^{(4)}$ is finite, but $\Gamma^{(2)}$ remains divergent. This is due to overlapping divergences at the 2-loop level. So, coupling constant and mass renormalization do not remove this additional divergence at the 2-loop level. It is removed by absorption in a multiplication factor and we define a renormalized 2-point function

$$\Gamma_r^{(2)} = Z_\phi(g_1, m_1, \mu)\Gamma^{(2)}(p, m_1, \mu)$$

$\Gamma_r^{(2)}$ is now finite with Z_ϕ infinite. $\sqrt{Z_\phi}$ is called the wave function (or field) renormalization constant. Field renormalization is $\phi = Z_\phi^{-1/2}\phi_0$, where ϕ_0 is the unrenormalized field. So, the 2-point function is

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = Z_\phi^{-1}\langle 0|T\phi_0(x_1)\phi_0(x_2)|0\rangle$$

where the 2-point functions on the l.h.s. and the r.h.s. are $G_R^{(2)}(x_1, x_2)$ and $G_{(0)}^{(2)}(x_1, x_2)$ respectively.

Thus, in general, the renormalized field ϕ defines the renormalized Green's functions $G_R^{(n)}$ which are related to the unrenormalized ones by

$$\begin{aligned}
G_R^{(n)}(x_1....x_n) &= \langle 0|T\phi(x_1)...\phi(x_n)|0\rangle \\
&= Z_\phi^{-n/2}\langle 0|T\phi_0(x_1)...\phi_0(x_n)|0\rangle \\
&= Z_\phi^{-n/2}G_0^{(n)}(x_1...x_n)
\end{aligned}$$

In momentum space, we get

$$G_R^{(n)}(p_1..p_n) = Z_\phi^{-n/2} G_0^{(n)}(p_1...p_n)$$

Now, to go from the connected Green's functions given above to the 1PI (amputated) Green's function, we have to eliminate the one-particle reducible diagrams. But more importantly for us, we have to remove the propagators for the external lines in the 1PI Green's functions (to get **amputated Green's functions)**. Thus, we need to remove $\Delta_R(p_i)$ from $G_R^{(n)}(p_1..p_n)$ and $\Delta(p_i)$ from $G_0^{(n)}(p_i)$. Now

$$\Delta_R(p_i) = Z_\phi^{-1}\Delta(p_i)$$

where the propagators on the l.h.s. and r.h.s. are $G_R^{(2)}$ and $G_0^{(2)}$ respectively. Thus, we get

$$\begin{aligned}
\Gamma_R^{(n)}(p_i) &= [\Delta_R(p_i)]^{-n} G_R^{(n)}(p_i) \\
&= Z_\phi^n (\Delta(p_i))^{-n} Z_\phi^{-n/2} G_0^{(n)}(p_i) \\
\text{or,} \qquad \Gamma_R^{(n)}(p_i) &= Z_\phi^{n/2} [\Delta(p_i)]^{-n} G_0^{(n)}(p_i) \\
&= Z_\phi^{n/2}\Gamma_0^{(n)}(p_i)
\end{aligned}$$

Thus, finally using renormalized quantities, we can write

$$\Gamma_R^{(n)}(p_1, ..p_n; g_R, m_R, \mu) = Z_\phi^{n/2}\Gamma_0^{(n)}(p_1..p_n, g_0, m_0)$$

Note that $\Gamma_0^{(n)}(p_i, g_0, m_0)$ will be divergent. Some divergences will be removed by using renormalized m_R and g_R, the remaining divergence will be removed by multiplying by $Z_\phi^{n/2}$.

1.3.5 Renormalization Group

We have

$$\begin{aligned}
\Gamma_R^{(n)}(p_i, g_R, m_R, \mu) &= Z_\phi^{n/2}\Gamma_0^{(n)}(p_i, g_0, m_0) \\
\text{or,} \qquad \Gamma_0^{(n)}(p_i, g_0, m_0) &= Z_\phi^{-n/2}\Gamma_R^{(n)}(p_i, g_R, m_R, \mu)
\end{aligned}$$

Now the unrenormalized vertex function $\Gamma_0^{(n)}$ should be independent of μ, so

$$\mu\frac{d}{d\mu}\Gamma_0^{(n)} = 0$$

(Note that Γ_0 is divergent; here it is used with proper regularization, e.g. dimensional regularization with $\epsilon \neq 0$. Γ_0 diverges in the $\epsilon \to 0$ limit.) We get

$$\mu\frac{d}{d_\mu}\left[Z_\phi^{-n/2}\Gamma_R^{(n)}(p_i, g_R, m_R, \mu)\right] = 0$$

where g_R and m_R depend on μ. This implies

$$-\frac{n}{2}Z_\phi^{(-n/2-1)}\mu\frac{\partial Z_\phi}{\partial\mu}\Gamma_R^{(n)}+Z_\phi^{(-n/2)}\left[\mu\frac{\partial}{\partial_\mu}+\mu\frac{\partial g_R}{\partial\mu}\frac{\partial}{\partial g_R}+\mu\frac{\partial m_R}{\partial\mu}\frac{\partial}{\partial m_R}\right]\Gamma_R^{(n)}=0$$

Multiplying the above with $Z_\phi^{n/2}$ gives

$$\left[-n\mu\frac{\partial}{\partial\mu}\ln\sqrt{Z_\phi}+\mu\frac{\partial}{\partial_\mu}+...\right]\Gamma_R^{(n)}=0$$

Define

$$\begin{aligned}\mu\frac{\partial}{\partial_\mu}\ln\sqrt{Z_\phi} &= \gamma(g)\\ \beta(g) &= \mu\frac{\partial g}{\partial\mu}\\ m\gamma_m(g) &= \mu\frac{\partial m}{\partial\mu}\end{aligned}$$

We then get the **renormalization group (RG) equation**:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-n\gamma(g)+m\gamma_m(g)\frac{\partial}{\partial m}\right]\Gamma^{(n)}=0$$

$\beta(g)$ is called the β function of the theory. The renormalization group equation expresses how the renormalized vertex functions change when we change the arbitrary scale μ.

We are interested in knowing the behavior of coupling constants, etc. under the change of the momentum scale, because we want to understand the behavior of the theory at high energies. We therefore make the following scale transformations and desire a slightly different constraint on the vertex function. Consider $p_i \to tp_i$, *i.e.* rescaling of all momenta by t. Then

$$\Gamma^{(n)}(tp_i,g,m,\mu)=t^D\Gamma^{(n)}\left(p_i,g,t^{-1}m,t^{-1}\mu\right),$$

where D is the mass dimension of the vertex function $\Gamma^{(n)}$, or

$$\begin{aligned}\Gamma^{(n)}\left(tp_i,g,m,\mu\right) &= \mu^D f\left(g,\frac{t^2p_i^2}{m\mu}\right)\\ &\equiv \mu^D f(g,\alpha)\end{aligned}$$

This is because Γ is Lorentz invariant, and hence can only be a function of various dot products $p_i.p_j$. To create a dimensionless quantity, we divide by μm. The overall scaling quantity μ^D means that the function has mass dimension D. Let us calculate

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(n)}\left(tp_i,g,m,\mu\right)=\mu D\mu^{D-1}f+\mu^{D+1}\frac{\partial f}{\partial\alpha}\left(\frac{-t^2p_i^2}{m\mu^2}\right)$$

Similarly,

$$t\frac{\partial\Gamma}{\partial t} = t\mu^D \frac{\partial f}{\partial\alpha}\left(\frac{2tp^2}{m\mu}\right)$$
$$m\frac{\partial\Gamma}{\partial m} = m\mu^D \frac{\partial f}{\partial\alpha}\left(\frac{-t^2p^2}{m^2\mu}\right)$$

Summing all these terms, we get

$$\left[t\frac{\partial}{\partial t} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu} - D\right]\Gamma^{(n)} = \mu^D\frac{\partial f}{\partial\alpha}\left[-\frac{t^2p^2}{m\mu} + \frac{2t^2p^2}{m\mu} - \frac{t^2p^2}{m\mu}\right] = 0$$

We now have two different equations for $\Gamma^{(n)}$. Note that for the RG equation also, we can consider $\Gamma^{(n)}$ (tp, g, m, μ). We can eliminate $\mu\frac{\partial}{\partial\mu}$ term from the above equation and the RG equation. We get

$$\left[\beta(g)\frac{\partial}{\partial g} - n\gamma(g) + m\gamma_m(g)\frac{\partial}{\partial m} - t\frac{\partial}{\partial t} - m\frac{\partial}{\partial m} + D\right]\Gamma^{(n)} = 0$$

or

$$\left[\beta(g)\frac{\partial}{\partial g} - t\frac{\partial}{\partial t} - n\gamma(g) + m\left(\gamma_m(g) - 1\right)\frac{\partial}{\partial m} + D\right]\Gamma^{(n)}(tp, g, m, \mu) = 0$$

This equation directly gives the effect of scaling up the momenta by a factor t. **This equation expresses the fact that a change in t (*i.e.* momentum scale) may be compensated by a change in m and g and an overall factor.** Thus, we expect that there should be functions $g(t), m(t)$ and $f(t)$ such that

$$\Gamma^{(n)}(tp, m, g, \mu) = f(t)\Gamma^{(n)}(p, m(t), g(t), \mu).$$

Differentiating this with respect to t we get (m and g also depend on the scale t)

$$\begin{aligned} t\frac{\partial}{\partial t}\Gamma^{(n)}(tp, m, g, \mu) &= t\frac{df(t)}{dt}\Gamma^{(n)}(p, m(t), g(t), \mu) \\ &\quad + tf(t)\left[\frac{\partial m}{\partial t}\frac{\partial}{\partial m} + \frac{\partial g}{\partial t}\frac{\partial}{\partial g}\right]\Gamma(n)(p, m(t), g(t), \mu) \end{aligned}$$

Then using

$$\Gamma^{(n)}(tp, m, g, \mu) = f(t)\,\Gamma^{(n)}(p, m(t), g(t), \mu)$$

we get

$$\left[-t\frac{\partial}{\partial t} + \frac{t}{f(t)}\frac{df}{dt} + t\frac{\partial m}{\partial t}\frac{\partial}{\partial m} + t\frac{\partial g}{\partial t}\frac{\partial}{\partial g}\right]\Gamma^{(n)}(tp, m, g, \mu) = 0$$

Comparison of this equation with the previous equation gives

$$t\frac{\partial g(t)}{\partial t} = \beta(g)$$

We also get

$$t\frac{\partial m}{\partial t} \quad = \quad m\left[\gamma_m(g) - 1\right]$$

This gives the change in mass. Furthermore,

$$\frac{t}{f}\frac{df}{dt} \quad = \quad D - n\gamma(g)$$

The solution of this equation is

$$f(t) = t^D \exp\left[-\int_0^t \frac{n\gamma(g(t))dt}{t}\right]$$

Recall that $\Gamma^{(n)}(tp, m, g, \mu) = f(t)\Gamma^{(n)}(p, m(t), g(t), \mu)$. Here t^D gives the canonical mass dimension of the vertex function $\Gamma^{(n)}$. The exponential term gives the 'Anomalous Dimension' for the vertex function arising entirely due to renormalization effects.

1.3.6 β Function

We have

$$t\frac{\partial g(t)}{\partial t} = \beta(g)$$

where $g(t)$ is called the 'running coupling constant'. Knowledge of the function $\beta(g)$ enables us to find $g(t)$, and of particular interest is the asymptotic limit of $g(t)$, as $t \to \infty$.

We now consider the possible behavior of $g(t)$ as $t \to \infty$, *i.e.* at large momentum (and assuming that the above equation is still valid there).

1. Suppose $\beta(g)$ has the following behaviour. It is zero at $g = 0$. Then, as g increases, it increases first and then starts decreasing, crossing the g axis at g_0 and becomes negative after that. The zeros of β at $g = 0$ and $g = g_0$ are called 'fixed points' (as g does not evolve there). For g near g_0 if $g < g_0$, $\beta > 0$. So g increases with increasing t and is driven towards g_0. Similarly, if $g > g_0$, then $\beta < 0$ and $\frac{dg}{dt} < 0$, so g decreases towards g_0 with increasing t.

   Thus, g_0 is an ultraviolet (large t) stable fixed point and $g(\infty) = g_0$. Note that g_0 is an infrared unstable fixed point. Because for $g < g_0$, $\beta > 0$ so g decreases away from g_0 with decreasing t. Similarly, for $g > g_0$, $\beta < 0$, so decreasing t takes g away from g_0. By the same arguments, $g = 0$ is an infrared stable fixed point.

2. Now consider the other possibility. Suppose $\beta(g)$ is zero at $g = 0$. But now, as g increases, it decreases first and then starts increasing, crossing

the g axis at g_0 and becoming positive after that. Here g_0 is an infrared stable fixed point while $g = 0$ is an ultraviolet fixed point. This is because if $g > 0$ near $g = 0$ then $\beta < 0$ so when t increases then $g(t)$ decreases towards 0. So, $g(t \to \infty) \to 0$. This is called **asymptotic freedom**. For theories with $g = 0$ as an ultraviolet fixed point, the perturbation theory gets better and better at higher energies and in the infinite momentum limit, the coupling constant vanishes.

We will see that QCD is an asymptotically free theory, with a negative β function.

1.3.7 β Function for a Scalar ϕ^4 theory

Recall the definition of the β function,

$$\beta(g) = \mu \frac{\partial g_R}{\partial \mu}$$

At the 1-loop level, we recall that the renormalized coupling

$$g_1 \quad = \quad g\mu^\epsilon - \frac{g^2\mu^\epsilon}{16\pi^2}\left[\frac{3}{\epsilon} + \text{finite term}\right]$$

Defining the bare coupling as $g_B \equiv g\mu^\epsilon$, we have

$$\begin{aligned} g_1 &= g_B - \frac{g_B^2\mu^{-\epsilon}}{16\pi^2}\left[\frac{3}{\epsilon} + \text{finite term}\right] \\ \text{Hence,} \qquad \mu\frac{\partial g_1}{\partial \mu} &= \epsilon\frac{g_B^2\mu^{-\epsilon}}{16\pi^2}\left[\frac{3}{\epsilon} + \text{finite term}\right] \\ &\simeq \frac{3}{16\pi^2}g_1^2 \end{aligned}$$

in the $\epsilon \to 0$ limit ignoring terms of order g^3 and higher corresponding to the 2-loop level and higher. So, keeping terms only up to the 1-loop level (*i.e.* of order g^2) one gets the β function by taking the $\epsilon \to 0$ limit as

$$\beta(g_1) \equiv \mu\frac{\partial g_1}{\partial \mu} = \frac{3g_1^2}{16\pi 2} > 0$$

From the above discussions about the fixed points we see that $g = 0$ is an infrared stable fixed point and that ϕ^4 theory is not asymptotically free. Recall that

$$t\frac{\partial g(t)}{\partial t} = \beta(g(t)) = \frac{3g(t)^2}{16\pi^2}$$

We can rewrite this equation as

$$\frac{dg(t)}{g^2} \quad = \quad \frac{3}{16\pi^2}\frac{dt}{t}$$

which implies

$$g = \frac{g_0}{1 - \frac{3g_0}{16\pi^2}\ln t/t_0}$$

This gives us the running coupling constant. As t increases, g increases.

1.3.8 Running Coupling Constant in QED

We start with the Lagrangian in d dimensions,

$$L = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi + e\mu^{2-d/2}A^\mu\overline{\psi}\gamma_\mu\psi - \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \frac{1}{2}\left(\partial_\mu A^\mu\right)^2$$

where the last term on the r.h.s. is the gauge fixing term. With this, one gets the Maxwell equation as $\partial_\nu\partial^\nu A_\mu = 0$ (in Lorenz gauge with $\partial^\mu A_\mu = 0$).

The vertex graph at the one loop level leads to the renormalized coupling constant e, related to the bare coupling e_B as,

$$e_B = \left(1 + \frac{1}{12}\frac{e^2}{\pi^2\epsilon}\right)e\mu^{\epsilon/2}$$

Using $\frac{\partial e_B}{\partial\mu} = 0$ we can show that $\beta(e) = \mu\frac{\partial e}{\partial\mu} = \frac{e^3}{12\pi^2}$. So, in QED also, the β function is positive and there is no asymptotic freedom. Using

$$t\frac{\partial e(t)}{\partial t} = \beta(e) = \frac{e^3}{12\pi^2}$$

we get

$$\frac{de}{e^3} = \frac{dt}{12\pi^2 t}$$

$$\text{or,}\quad e^2(t) = \frac{e^2(t_0)}{1 - \frac{e^2(to)}{6\pi^2}\ln(t/t_0)}$$

Defining $\alpha = e^2/(4\pi)$,

$$\alpha(t) = \frac{\alpha(t_0)}{1 - \frac{4\alpha(t_0)}{6\pi}\ln(t/t_0)}$$

Note: The Landau singularity occurs at

$$t \simeq t_0 \exp\left(6\pi^2/e^2(t_0)\right) \simeq t_0 \exp\left(\frac{6\pi}{4\pi\alpha(t_0)}\right)$$

If $t_0 \sim 1$ MeV then $t \sim 10^{80}$ MeV. But note that for energies higher that 100 GeV one should use the Electroweak theory.

1.3.9 Asymptotic Freedom in QCD

The quark gluon vertex function leads to the renormalized coupling constant g at one loop level, which is related to the bare coupling g_B as

$$g_B = g\mu^{\epsilon/2}\left[1 - \frac{g^2}{16\pi^2\epsilon}\left(11 - \frac{2n_F}{3}\right)\right]$$

Here the factor of n_F comes from the field renormalization factor Z_A for vacuum polarization. Using $\frac{\partial g_B}{\partial\mu} = 0$, we get

$$\beta(g) = \mu\frac{\partial g}{\partial\mu} = -\epsilon\mu^{-\epsilon}\frac{g^3}{16\pi^2\epsilon}\left(11 - \frac{2n_F}{3}\right)$$

(Corrections are at higher loop order.) So

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[11 - \frac{2n_F}{3}\right]$$

For the number of quark flavors $n_F < 16$ (we have only 6) we have $\beta(g) < 0$, *i.e.*, **a negative β function**. This implies that g decreases with increasing momentum scale and the theory is asymptotically free. $g = 0$ is an ultraviolet fixed point.

From

$$t\frac{\partial g}{\partial t} = \beta(g) = -\frac{g^3}{16\pi^2}\left[11 - \frac{2nF}{3}\right]$$

we can solve for g and using $\frac{g^2}{4\pi} = \alpha_s$, we get

$$\alpha = \frac{4\pi\alpha_0}{4\pi + \alpha_0\left(11 - \frac{2n_F}{3}\right)\ln\frac{Q^2}{Q_0^2}}$$

with $Q^2/Q_o^2 = t^2/t_o^2$, where Q is the momentum. Another way of writing α is to define

$$\left(11 - \frac{2}{3}n_F\right)\alpha_0\ln Q_0^2 - 4\pi = \left(11 - \frac{2}{3}n_F\right)\alpha_0\ln\Lambda^2$$

Then we get

$$\alpha_s\left(Q^2\right) = 4\pi/\left(11 - \frac{2n_F}{3}\right)\ln Q^2/\Lambda^2$$

Λ is the QCD scale fixed by various scattering processes (e.g. high energy $e^+e^- \to$ hadrons). One has $\alpha_s\left((100\text{GeV})^2\right) = 0.2$ which implies $\Lambda =$ 112MeV for $n_F = 6$. The current value of Λ in the literature ranges from 100 MeV to 300 MeV.

Decrease of α_s with Q^2 in QCD is due to antiscreening from colored gluons. $q\bar{q}$ pairs however still give the usual screening [1]. That is why for a sufficiently large value of n_F there is no asymptotic freedom.

1.3.10 Running of α_s with Momentum Scale

Implications of running coupling constant in QCD and QGP

We have seen that the coupling constant in QCD becomes smaller at large energy scales and the theory is asymptotically free. This means that the interactions between quarks and gluons become weaker at very higher energies, while they are strong at lower energies.

Thus a collection of quarks and gluons interacting with each other with typical momentum transfer much larger than Λ should constitute a weakly interacting system of particles. As we mentioned earlier, the typical value of Λ (from scattering experiments) is about 200 MeV.

Thus, we expect that if a system of quarks and gluons is at a temperature much higher than several hundred MeV, then the coupling constant will be small and the system should behave as an ideal gas. In such a system we do not expect the effects of confinement of the QCD interaction to survive. This system of quarks and gluons where quarks and gluons are no more confined within the region of a hadron ($\sim$ 1 fm size) is called the **quark-gluon plasma (QGP)**.

In the other limit, when quark and gluons have small energies, say they are at low temperatures, then we expect the coupling constant to become strong. This is the domain where confinement takes place and all quarks and gluons are confined inside hadrons.

We expect that the transition between this low energy hadronic domain to the high energy (high temperature) QGP domain is a phase transition. This is called the **deconfinement-confinement phase transition**, or, the **quark-hadron phase transition**.

1.3.11 High Density Behavior

At sufficiently high density (compressed baryonic matter) we expect that hadrons should be almost overlapping. For example, in neutron star cores very high baryon densities are achieved. At such densities, the typical separation between constituent quarks of different hadrons become much less than 1 fm or $(200 \text{ MeV})^{-1}$. Again, the effective coupling constant for the quark-gluon interaction should become very small at such high densities. We can then expect that a state like QGP may exist at very high densities also.

One needs to be careful here as at such high densities many body quantum effects can play an important role if temperatures are not very high. One expects exotic states like the color superconductor to form at very high baryon densities.

In this section we saw that at the asymptotic freedom of QCD suggests that a system of hadrons heated to very high temperatures (much above few hundred MeV) should transform to a weakly interacting system of quarks and gluons, *i.e.* QGP. This expectation is strongly supported by lattice calculations and other phenomenological approaches, and we will now discuss some of these.

What we need is to study the system of quarks and gluons at high temperatures. That is QCD at finite temperatures.

1.4 Field Theory at Finite Temperature

In the following, we will discuss the basic formalism for finite temperature field theory [3]. We will then specialize to our requirement of a system of fermions (quarks) and bosons (gluons) at finite temperature. Further details of finite temperature QCD will be discussed when and where required.

1.4.1 The Partition Function

We know that all thermodynamic properties for a system in equilibrium can be derived once we know its partition function

$$Z = \mathrm{Tr}\, e^{-\beta H} \qquad \beta = \frac{1}{T}$$

where Tr stands for the trace, or the sum over the expectation values in any complete basis. Thus

$$Z = \int d\phi_a \langle \phi_a | e^{-\beta H} | \phi_a \rangle$$

We now recall the expression for the transition amplitude in the path integral formalism

$$\begin{aligned} \langle \phi_1 | e^{-iH(t_1 - t_2)} | \phi_2 \rangle &\simeq \langle \phi(\vec{x}_1, t_1) | \phi(\vec{x}_2, t_2) \rangle \\ &= N' \int D\phi \, e^{iS} \end{aligned}$$

where ϕ is the basic quantum field variable, N' is an irrelevant normalization constant and S is the action.

$$S[\phi] = \int_{t_2}^{t_1} dt \int d^3x L$$

where L is the Lagrangian density of the system. The functional integral (path integral) is defined over paths which satisfy

$$\phi(\vec{x}_1, t_1) = \phi_1, \quad \text{and} \quad \phi(\vec{x}_2, t_2) = \phi_2$$

ϕ_1 and ϕ_2 are the fixed end points. There is no integration over these fixed end points.

From the expression of the partition function we can easily see that Z can be written in terms of a path integral if we identify $t_1 - t_2$ with $-i\beta$. Then

$$\begin{aligned} Z(\beta) &= \mathrm{Tr}\ e^{-\beta H} = \int d\phi_1 \langle \phi_1 | e^{-\beta H} | \phi_1 \rangle \\ &= N' \int D\phi \, e^{-S_E} \end{aligned}$$

where S_E is the Euclidean action $(t \to it)$,

$$S_E = \int_0^\beta d\tau \int d^3x L_E$$

Furthermore, in view of the trace, we require that in the path integral the integration is done only over those field variables which satisfy periodic boundary conditions

$$\phi(\vec{x}, \beta) = \phi(\vec{x}, 0)$$

Note that here the end points are also being integrated over as there is a sum over states in Tr $e^{-\beta H}$. We will see that for fermions one gets antiperiodic boundary conditions. Boundary conditions on field variables can be seen by examining the properties of the thermal Green's function defined by

$$G(x, y; \tau, 0) = Z^{-1}\mathrm{Tr}\left(e^{-\beta H} T\left[\phi(x, \tau)\phi(y, 0)\right]\right)$$

where T is the imaginary time ordering operator. We have for bosons

$$T\left[\phi(\tau_1)\phi(\tau_2)\right] = \phi(\tau_1)\phi(\tau_2)\theta(\tau_1 - \tau_2) + \phi(\tau_2)\phi(\tau_1)\theta(\tau_2 - \tau_1)$$

whereas for fermions we have

$$T\left[\psi(\tau_1)\psi(\tau_2)\right] = \psi(\tau_1)\psi(\tau_2)\theta(\tau_1 - \tau_2) - \psi(\tau_2)\psi(\tau_2)\theta(\tau_2 - \tau_1)$$

from the anticommuting properties of fermions. For bosons we see, using the cyclic property of the trace that

$$\begin{aligned} G(x, y; \tau, 0) &= Z^{-1}Tr\left[e^{-\beta H}\phi(x, \tau)\phi(y, 0)\right] \\ &= Z^{-1}Tr\left[e^{-\beta H}e^{\beta H}\phi(y, 0)e^{-\beta H}\phi(x, \tau)\right] \\ &= Z^{-1}Tr\left[e^{-\beta H}\phi(y, \beta)\phi(x, \tau)\right] \end{aligned}$$

where

$$\phi(y, \beta) = e^{\beta H}\phi(y, 0)e^{-\beta H}$$

in analogy with the realtime Heisenberg time evolution

$$\phi(y, t) = e^{iHt}\phi(y, 0)e^{-iHt}$$

Thus,

$$\begin{aligned} G(x, y; \tau, 0) &= Z^{-1}Tr\left(e^{-\beta H}T\left[\phi(x, \tau)\phi(y, \beta)\right]\right) \\ \text{or,}\quad G(x, y; \tau, 0) &= G(x, y, \tau, \beta) \end{aligned}$$

This implies the periodic boundary condition for bosons is

$$\phi(y, 0) = \phi(y, \beta).$$

It is then straightforward to see that for fermions we will get

$$\begin{aligned} G(x, y; \tau, 0) &= -G(x, y; \tau, \beta) \\ \text{and,} \quad \psi(x, 0) &= -\psi(x, \beta) \end{aligned}$$

The important lesson for us is that in the functional integral representation for the partition function, the integration over the field variables is restricted to those fields which are

1. Bosons : periodic in (imaginary) time with period β
2. Fermions : antiperiodic in (imaginary) time with period β

This will be important for us when we discuss the deconfinement-confinement phase transition and the Polyakov loop order parameter for that transition.

We now come back to discussing a system of bosons or fermions. We are familiar from the standard results from statistical mechanics that

1. For one bosonic degree of freedom (one state of energy ω):
   $E = \omega N$, and
   $N = \frac{1}{e^{\beta(\omega-\mu)}-1}$ (Bose-Einstein distribution)
   where N ranges continuously from zero to ∞ and μ is the chemical potential.

2. For fermions
   $N = \frac{1}{e^{\beta(\omega-\mu)}+1}$ (Fermi-Dirac distribution)
   N ranges from 0 to 1

   One can rederive these expressions using finite temperature field theory methods. With these, we can obtain various thermodynamic properties of a system consisting of fermions or bosons.

Quarks

Let us write down the expressions for the energy density and pressure for a system consisting of a relativistic gas of fermions (quarks). The number of quarks in a volume V with momentum p within the interval dp is

$$dN_q = g_q V \frac{4\pi p^2 dp}{(2\pi)^3} \frac{1}{1 + e^{(p-\mu_q)/T}}$$

This is the Fermi-Dirac distribution. μ_q is the chemical potential (same as the quark Fermi energy) and $g_q = N_c N_s N_f$ is the number of independent degrees of freedom of quarks (degeneracy of quarks). Let us take the case of $\mu_q = 0$, so the density of quark and antiquarks is the same.

We can now write down the energy of the massless quarks in the system of volume V and temperature T.

$$\begin{aligned} E_q &= \frac{g_q V}{2\pi^2}\int_o^\infty \frac{p^3 dp}{1+e^{p/T}} \qquad \text{for massless quarks with } E \simeq p \\ &= \frac{g_q V}{2\pi^2} T^4 \int_o^\infty \frac{z^3 dz}{1+e^z} \\ &= \frac{g_q V}{2\pi^2} T^4 \int_o^\infty z^3 dz e^{-z} \sum_{n=0}^{\infty} (-1)^n e^{-nz} \\ &= \frac{g_q V}{2\pi^2} T^4 \Gamma(4) \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)^4} \end{aligned}$$

where Γ is the gamma function. It is easy to show that

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)^4} = (1-2^{-3})\zeta(4)$$

where $\zeta(4)$ is the Riemann zeta function.

$$\zeta(4) = \sum_{m=1,2..} \frac{1}{m^4} = \frac{\pi^4}{90}$$

Thus, we get

$$E_q = \frac{7}{8} g_q V \frac{\pi^2}{30} T^4$$

We know that for massless fermions and bosons, the pressure is related to the energy density $\rho = E/V$ as

$$P = \frac{1}{3}\rho$$

Hence, the pressure due to quarks is

$$P_q = \frac{7}{8} g_q \frac{\pi^2}{90} T^4$$

Similarly, the pressure due to antiquarks is given by the same expression with $g_q \to g_{\overline{q}}$.

We can also obtain the number density of the quarks and antiquarks as

$$\begin{aligned} n_q &= n_{\overline{q}} = \frac{g_q}{2\pi^2}\int_0^\infty \frac{p^2 dp}{1+e^{p/T}} \\ &= \frac{g_q}{2\pi^2} T^3 \frac{3}{2}\zeta(3) \end{aligned}$$

where $\zeta(3) = 1.20206$.

Gluons

Let us now write down the energy of gluons in a system of volume V and temperature T using the Bose-Einstein distribution for bosons

$$E_g = \frac{g_g V}{2\pi^2}\int_o^\infty p^3 dp \left(\frac{1}{e^{p/T}-1}\right)$$

where g_q is the gluon degeneracy. g_g = number of different gluons $\times$ number of polarizations $= 8 \times 2 = 16$.
We get

$$E_g = \frac{g_g V}{2\pi^2} T^4 \int_o^\infty \frac{z^3 dz}{e^z - 1}$$

Following earlier steps, we get

$$\begin{aligned} E_g &= \frac{g_g V}{2\pi^2} T^4 \int_o^\infty z^3 dz e^{-z} \sum_{n=0}^\infty e^{-nz} \\ &= \frac{g_g V}{2\pi^2} T^4 \Gamma(4) \sum_{n=0}^\infty \frac{1}{(n+1)^4} = \frac{g_g V}{2\pi^2} T^4 \Gamma(4)\zeta(4) \\ \text{or, } E_g &= g_g V \frac{\pi^2}{30} T^4 \end{aligned}$$

Note the absence of factor $\frac{7}{8}$ for bosons compared to fermions.

Again, using $P = \frac{1}{3}\rho$, we get the pressure for the gluon gas as

$$P_g = g_g \frac{\pi^2}{90} T^4$$

The number density of gluons is

$$\begin{aligned} n_g &= \frac{g_g}{2\pi^2}\int_o^\infty p^2 dp \left(\frac{1}{e^{p/T}-1}\right) \\ &= \frac{g_g}{2\pi^2} T^3 \Gamma(3)\zeta(3) = 1.20206\,\frac{g_g}{\pi^2} T^3 \end{aligned}$$

The net energy density of a system of quarks and gluons at temperature T is

$$\begin{aligned} \rho_{QGP} &= \rho_{q\bar{q}} + \rho_g \\ &= \left[\frac{7}{8}\left(g_q + g_{\bar{q}}\right) + g_g\right] \frac{\pi^2}{30} T^4 \\ g_q &= g_{\bar{q}} = N_C N_S N_F = 3 \times 2 \times 6 \end{aligned}$$

N_C, N_S and N_F are the number of color, spin and flavor states of the quarks and $g_g = 16$, so

$$\rho_{QGP} = \left(\frac{7}{8} \times 72 + 16\right) \frac{\pi^2}{30} T^4$$

Of course, this assumes that all the quark flavors can be treated as massless at the temperature T. So, the above expression is valid only for $T \gg m_{top} \simeq 170$ GeV.

Let us calculate ρ_{QGP} near the expected transition temperature of few hundred MeV, say at $T = 200$ MeV. At this temperature, only u and d quarks can be taken to be approximately massless. Thus, for $T = 200$ MeV

$$g_{q+\overline{q}} = 2 \times 3 \times 2 \times 2 = 24$$

where the factors correspond to q and $\bar{q}$, N_C, N_S and $N_F = u, d$. So

$$\rho_{QGP} = \left(\frac{7}{8} \times 24 + 16\right) \frac{\pi^2}{30} T^4$$

$$\text{or} \qquad \rho_{QGP} = \frac{37\pi^2}{30} T^4$$

For $T = 200$ MeV and using 1 fm $= (200\ \text{MeV})^{-1}$, we get $\rho_{QGP} \simeq \frac{37}{3}(200\ \text{MeV})^4 \simeq 2.5\,\text{GeV/fm}^3$. This is the energy density of a system of quarks and gluons in thermal equilibrium at a temperature of about 200 MeV. Thus, if we are able to create a dense system of partons (quarks and gluons) with an energy density much above this and one can argue for thermal equilibrium to exist then we should expect that a state of QGP will be achieved. This is what is expected to happen in relativistic heavy-ion collision experiments where the nuclei colliding at ultra high energies create quarks, antiquarks and gluons with a central density which is expected to be much above 3 GeV/fm^3.

We saw how asymptotic freedom in QCD leads us to believe in the existence of a QGP state at high temperatures (and high densities). We will now briefly discuss here how the prediction of the QGP phase arises in the context of phenomenological models of QCD which were used very successfully to account for different properties of hadrons.

1.5 Quark Confinement

We know that quarks cannot be isolated, and are confined inside hadrons. On the other hand, the asymptotic freedom of QCD implies that at very short distances (or large energies) the quark-gluon coupling goes to zero, so quarks become almost free. There have been many phenomenological models which incorporate these two features and try to calculate properties of hadrons [2].

1.5.1 Potential Models

Here one assumes a contribution of a Coulombic potential $(-\frac{1}{r})$ and a confining potential $(+\lambda r)$ between quarks and calculates the spectrum. (We will discuss this later for the J/ψ suppression signal.) These models work well for heavy quarks but for light quarks the properties of bound states with a confining potential become difficult to calculate.

1.5.2 String Model of Quark Confinement

Here one takes hadrons to be string like objects where quarks are bound by 'strings' or tubes of color flux. This model arose from a certain property of hadrons known as Regge trajectory behavior where it is seen that hadrons seem to lie on lines given by $J \sim M^2$ in the J vs M^2 plane. Here J is the spin and M is the mass of the hadron. It can be shown that a relativistic rotating string leads to this type of relationship between J and M^2. This gave birth to the string model of hadrons.

It was this string model whose attempted quantization and subsequent development eventually led to the modern string theory where every elementary particle is supposed to correspond to a fundamental string. In the present form it does not have anything in common with the initial string model of hadrons. (Though, it has been recently suggested that these may be intimately connected at a deeper level.)

The string model of hadrons still provides a good description of certain properties of hadrons and of hadron production. For example, in scattering experiments, the production of hadrons is often modeled using a phenomenological string model. As q and $\bar{q}$ created in $e^+ e^-$ annihilations separate with ultrahigh energies, a string stretches between them. After some stretching, it becomes unfavorable for the string to stretch further and it breaks by creating a $q\bar{q}$ pair. Now the individual string pieces keep stretching and further keep breaking. Eventually relative velocities between a $q\bar{q}$ pair connected to a single string segment becomes very small so that no further string breaking is possible. The resulting system consists of hadrons.The creation of qq and $\overline{q}\,\overline{q}$ pairs by string breaking leads to the formation of baryons. Such string models of hadron formation are usually called fragmentation models and are widely used in various Monte Carlo programs simulating hadron production in $e^+ e^-$ or hadron-hadron scattering experiments. These models are especially successful in describing the production of jets in these experiments.

Note:

1. In the string model of confinement, the potential energy of a $q\bar{q}$ pair increases with distance as λr, where λ is the mass per unit length of the string. This is exactly like the linear term in the potential models. So for a $q\bar{q}$ system

$$V(r) = -\frac{a}{r} + \lambda r$$

2. QCD strings to fundamental strings : The appearance of a spin-2 massless particle in the spectrum of strings could be possibly understood as

a certain pomeron excitation in QCD. But there were problems with the requirement of 26 dimensions for the QCD string model. For fundamental string theory models this spin-2 massless particle provided additional motivation as it could be identified with the graviton. Thus the fundamental string could naturally incorporate gravity along with other types of elementary particles.

1.5.3 Bag Models

We now discuss another class of phenomenological models which accounts for the confinement of quarks inside hadrons as well as the physics of asymptotic freedom. We will then use these models to reach a definite quantitative prediction of the transition to a QGP state.

There are many different versions of the Bag model. Here we will describe the MIT Bag model which contains the essential characteristics of the phenomenology of quark confinement [5]. We will also use it to understand the circumstances of how quarks can become deconfined in the new QGP phase. In this model one assumes that quarks are confined within a sphere of radius R. Quarks are assumed to be free inside the sphere, which is in the spirit of asymptotic freedom. (R will be less than 1 fm, so the coupling constant should be small for such short distances.) It is further assumed that quarks cannot go outside this sphere, *i.e.* they are infinitely heavy outside. This captures the physics of confinement of quarks inside hadrons (the coupling constant is large for large distances).

One therefore solves the Dirac equation for a free fermion of mass m

$$i\gamma^{\mu}\partial_{\mu}\psi(x) = m\psi(x)$$

This equation is solved in a spherical region of space of radius R. By using appropriate boundary conditions, *i.e.* no current flows across the surface of such a sphere, we get quantized energy levels

$$\omega = \left(m^2 + \frac{x^2}{R^2}\right)^{1/2}$$

Here $x \simeq 2.04$ for the lowest level with $l = 0$, where l is the orbital angular momentum. For a system of several quarks with different flavors and masses m_i, the total energy of the quark system is

$$E = \sum_i \left(m_i^2 + \frac{x_i^2}{R^2}\right)^{1/2} N_i$$

where N_i is number of quarks of the same type. We note that this energy can be lowered by increasing R. Thus, there is no automatic confinement in the model, unless one artificially fixes the value of R.

To prevent an increase in R one introduces a 'pressure' term B which stabilizes the system. This is the essential feature of the MIT Bag model [5]. This bag pressure is directed inwards, and is a phenomenological quantity introduced to take into account the non-perturbative effects of QCD. Quarks and gluons are all confined inside the bag. In this description, the total matter inside the bag must be colorless by virtue of Gauss's law. We know that this allows for qqq and $q\bar{q}$ states inside the bag.

With this bag pressure, the total energy becomes

$$E(R) = \sum_i N_i \left(m_i^2 + \frac{x_i^2}{R^2} \right)^{1/2} + \frac{4\pi R^3}{3} B$$

One can now minimize $E(R)$ with respect to R to get the equilibrium configuration. Since u, d are light, we may set $m_u = m_d = 0$ and get

$$E(R) \quad = \quad \frac{2.04}{R} N + \frac{4\pi R^3}{3} B$$

(Recall, $\frac{1}{R}$ is the characteristic momentum and hence the energy for a massless particle confined in a region of size R.) Then

$$\frac{\partial E}{\partial R} \quad = \quad 0 \Rightarrow -\frac{2.04}{R^2} N + 4\pi R^2 B = 0$$

or,

$$R \quad = \quad \frac{(N \times 2.04)^{1/4}}{(4\pi B)^{1/4}}$$

Putting this back into the expression for $E(R)$ we get

$$E = \frac{4}{3} (4\pi B)^{1/4} (N \times 2.04)^{3/4}$$

From the relation between R and B, if we take the confinement radius to be 0.8 fm for a 3 quark system in a baryon then we get (say for *uud* or *udd*, *i.e.* proton or neutron)

$$B^{1/4} = 206 \text{ MeV}$$

The value of $B^{1/4}$ ranges from about 145 MeV to 235 MeV depending on specific details of the models.

1.5.4 Transition to the QGP State in the Bag Model

The physics of the Bag model implies that if the pressure of the quark matter inside the bag is increased, there will be a point when the pressure directed outward will be greater than the inward bag pressure. When this happens, the bag pressure cannot balance the outward quark matter pressure and the bag cannot confine the quark matter contained inside. A new phase of matter containing the quarks and gluons in an unconfined state is then possible. This is the QGP phase.

The main condition for a new phase of quark matter (QGP) is the occurrence of a large pressure exceeding the bag pressure B. A large pressure of quark matter arises in two ways:

1. When the temperature of the matter is high (this is when QGP forms at high temperature, as in the early universe).
2. When the baryon density is high (this is when QGP forms at high baryon density, as possibly in the cores of neutron stars).

QGP at High Temperature

Let us recall the pressure of a quark-gluon system at temperature T. The total pressure is

$$\begin{aligned} P &= g_{total}\frac{\pi^2}{90}T^4 \\ g_{total} &= \left[g_g + \frac{7}{8}\times(g_q + g_{\bar{q}})\right] \end{aligned}$$

By taking only light u and d quarks, we have seen that g_{total}= 37, so we get

$$P = 37\frac{\pi^2}{90}T^4$$

By equating it to the bag pressure B, we can get an estimate of the critical temperature for the transition to QGP state

$$\begin{aligned} 37\frac{\pi^2}{90}T_c^4 &= B \\ \Rightarrow T_c &= \left[\frac{90}{37\pi^2}\right]^{1/4} B^{1/4} \end{aligned}$$

For $B^{1/4} = 206$ MeV, we get $T_c \simeq 144$ MeV.

We will later discuss that the current estimates for T_c from lattice computations are near 170 MeV. Note that this is of the same order as expected from the running coupling constant argument when α_s becomes small near $q^2 \sim (200\,\mathrm{MeV})^2$.

QGP with High Baryon Density

We now discuss the possibility where the pressure inside a bag can be large enough to lead to the deconfined QGP state even at $T = 0$ due to high baryon density. In this case the pressure arising from the Fermi momentum of quarks will be large enough to balance the bag pressure, leading to the QGP state. Since this situation arises when the baryon number density is very high, we neglect effects of antiquarks and gluons. Again, the number of states in a volume V with momentum p within the momentum interval dp is

$$\frac{g_q V}{(2\pi)^3} 4\pi p^2 dp$$

As each state is occupied by one quark, the total number of quarks, up to the quark Fermi momentum μ_q (*i.e.* the chemical potential) is

$$\begin{aligned} N_q &= \frac{g_q V}{(2\pi)^3} \int_0^{\mu_q} 4\pi p^2 dp \\ &= \frac{g_q V}{6\pi^2} \mu_q^3 \end{aligned}$$

Thus the number density of quarks (N/V) is

$$n_q = \frac{g_q}{6\pi^2} \mu_q^3$$

Note that

$$\begin{aligned} dp\, n_q &= \frac{g_q}{(2\pi)^3} 4\pi p^2 dp \left[1 + \exp\left(\frac{p - \mu_q}{T}\right)\right]^{-1} \\ dp\, n_{\overline{q}} &= \frac{g_{\overline{q}}}{(2\pi)^3} 4\pi p^2 dp \left[1 + \exp\left(\frac{p + \mu_q}{T}\right)\right]^{-1} \end{aligned}$$

Consider the case of very large value of μ_q, $\frac{\mu_q}{T} \gg 1$. Then we see that

$$n_q\, dp \simeq \frac{g_q}{(2\pi)^3} 4\pi p^2 dp \left(\frac{1}{1 + \exp\left(\frac{p-\mu_q}{T}\right)}\right)$$

The factor in bracket is 1 for $p < \frac{\mu_q}{T}$ and approximately 0 for $p > \frac{\mu_q}{T}$, whereas $n_{\overline{q}}\, dp \simeq 0$ always as $p > 0$. Thus, for the case of complete degeneracy, *i.e.* $\frac{\mu_q}{T} \gg 1$, we have (starting with a Fermi-Dirac distribution),

$$\begin{aligned} n_q\ dp &\simeq \frac{g_q}{(2\pi)3} 4\pi p^2\, dp \quad \text{for} \quad p < \mu_q \\ &\simeq 0 \text{ for } p > \mu_q \\ \text{and} \quad n_{\overline{q}}\ dp &\simeq 0 \quad \text{always.} \end{aligned}$$

The energy of the quark gas in volume V is

$$\begin{aligned} E_q &= \frac{g_q V}{(2\pi)^3} \int_0^{\mu_q} (4\pi p^3) dp \\ &= \frac{g_q V}{8\pi^2} \mu_q^4 \end{aligned}$$

So the energy density is

$$\rho_q = \frac{g_q}{8\pi^2}\mu_q^4$$

Again, for massless quarks, the pressure is

$$P = \frac{1}{3}\rho = \frac{g_q}{24\pi^2}\mu_q^4$$

The transition to the QGP state will be achieved at a critical value of $\mu_q \simeq \mu_c$ when this pressure is balanced by the bag pressure. This gives

$$P = B = \frac{g_q}{24\pi^2}\mu_c^4$$

which implies

$$\mu_c = \left[\frac{24\pi^2}{g_q}\right]^{1/4} B^{1/4}$$

Using this for n_q, we get a critical number density of quarks as

$$n_q^{critical} = 4\left(\frac{g_q}{24\pi^2}\right)^{1/4} B^{3/4}$$

The corresponding critical baryon density becomes

$$n_B^{critical} = \frac{4}{3}\left(\frac{g_q}{24\pi^2}\right)^{1/4} B^{3/4}$$

Again, taking only u and d flavors, we take $g_q = 3 \times 2 \times 2 = 12$ for 3 colors, 2 spins and 2 flavors.

Using $\mathrm{B}^{1/4} = 206$ MeV we get $n_B^{critical} = 0.72/\mathrm{fm}^3$ corresponding to the critical value of the chemical potential $\mu_c = 434$ MeV. These values for the transition to the QGP state should be compared with the nucleon number density $n_B = 0.14/\mathrm{fm}^3$ for normal nuclear matter in equilibrium. Thus, the critical baryon density is about 5 times the normal nuclear matter density. When the density of baryons exceeds this critical density, the baryon bag pressure is not strong enough to withstand the pressure due to the degeneracy of quarks and a transition to a new deconfined QGP state is possible. Note that all these estimates for T_c, n_c, μ_c, are based on the phenomenological Bag model and not from detailed calculations from QCD. Such calculations are possible from lattice gauge theories and they show that these estimates are roughly correct.

We are now in a position to have a rough picture of the phase diagram of strongly interacting matter. For low temperatures T and chemical potential μ_b we have hadronic matter while at high temperatures and/or μ_b we get QGP. Later we will discuss this QCD phase diagram in more detail and discuss various interesting phases and expected phase transitions. At present we note that our search for the QGP state leads us to consider where one can create high temperature and/or high density matter.

1.6 Relativistic Heavy-Ion Collisions

We will now discuss relativistic heavy-ion collisions where conditions for QGP are expected to arise [4]. Let us first discuss some useful variables which will be needed to describe particle production and evolution in relativistic heavy-ion collision experiments (RHICE). (We will reserve RHIC for the Relativistic Heavy Ion Collider at Brookhaven National Laboratory, USA.)

1.6.1 Rapidity Variable

Rapidity is a very useful variable to describe particle production in scattering experiments. It is defined as

$$y = \frac{1}{2}\ln\left(\frac{P_0 + P_z}{P_0 - P_z}\right)$$

where P_0 and P_z are time and z components of the momentum of the particle. The z-axis is typically taken along the beam direction. Depending on the spin of P_z, y can be positive or negative.

Exercise: Check that in the non-relativistic limit the rapidity of a particle traveling in the longitudinal direction (we take this to be along z axis) is equal to v/c.

Exercise: y depends on the reference frame in a simple manner. Show that under a Lorentz transformation from the laboratory frame F to a new coordinate frame F' moving with a velocity β in the z-direction, the rapidity y' of the particle in the new frame F' is related to the rapidity y in the old frame F by

$$y' = y - y_\beta \quad \text{where } \mathrm{y}_\beta = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) .$$

y_β is called the rapidity of the moving frame.

For a free particle which is on mass-shell, its four momentum has only three degrees of freedom and can be represented as $(y, \vec{P}_T)$, where $\vec{P}_T$ is the transverse momentum (transverse to the z-axis). The z-axis will later be chosen to be along the beam direction in RHICE. We can relate the 4-momentum: $(P_0, \vec{P})$ and $(y, \vec{P}_T)$ as below. From the definition of rapidity, we have

$$e^y = \sqrt{\frac{P_0 + P_z}{P_0 - P_z}} \text{ and } e^{-y} = \sqrt{\frac{P_0 - P_z}{P_0 + P_z}}$$

Adding these equations we get

$$P_0 = m_T \cosh y$$

where m_T is the transverse mass of the particle

$$m_T^2 = m^2 + P_T^2$$

Subtracting the above two equations gives

$$P_z = m_T \sinh y$$

Thus, the information contained in $(P_0, \vec{P})$ is all contained in $(y, \vec{P}_T)$.

We saw that the rapidity of a particle in a moving frame is equal to the rapidity in the laboratory frame minus the rapidity of the frame. This is quite like the law of addition of velocities in Galilean relativity. Thus, it is often useful to treat the rapidity variable as a **relativistic measure** of the velocity of the particle.

1.6.2 Pseudorapidity Variable

To characterize the rapidity of a particle, it is necessary to measure two properties of the particle, such as its energy and its longitudinal momentum. In many experiments it is only possible to measure the angle of the detected particle relative to the beam axis. In that case, it is convenient to utilize this information by using the pseudorapidity variable η to characterize the detected particle. η is defined as

$$\eta = -\ln[\tan(\theta/2)]$$

where θ is the angle between the particle momentum $\vec{P}$ and the beam axis. In terms of the momentum, the pseudorapidity variable can be written as

$$\eta = \frac{1}{2}\ln\left[\frac{|\,\vec{P}\,| + P_Z}{|\,\vec{P}\,| - P_Z}\right]$$

By comparing the expression for the rapidity y, we see that η coincides with y when the momentum is large, *i.e.* when $|\vec{P}| \simeq P_0$. By transforming variables from $(y, \vec{P}_T)$ to $(\eta, \vec{P}_T)$ we can transform rapidity distributions and pseudorapidity distributions to each other.

Mandelstam Variables

For a scattering process, $A\,B \to C\,D$, the Mandelstam variables s, t, u are defined as

$$s = (P_A + P_B)^2, \quad t = (P_A - P_C)^2$$
$$u = (P_A - P_D)^2$$

$\sqrt{s}$ is the center of mass energy. For the center of mass (CM) frame, $\vec{P}_B = -\vec{P}_A$. So

$$\begin{aligned} s &= (P_A + P_B)_\mu (P_A + P_B)^\mu \\ &= (E_A + E_B)^2 - \left(\vec{P}_A - \vec{P}_A\right)^2 \\ &= 4E^2 \ \text{ if } \ M_A = M_B \\ \text{or,} \quad \sqrt{\text{s}} &= 2E \end{aligned}$$

If A and B have the same mass, say M, then the laboratory energy E_{lab} (where one particle is at rest) is related to E_{CM} by

$$E_{lab} = \frac{E_{CM}^2}{2M} - M$$

For RHICE, M should be the mass of a single proton. Then

$$E_{CM} = \sqrt{s} = \sqrt{2M^2 + 2M\ E_{lab}} \simeq \sqrt{2M\ E_{lab}}$$

For example, for 200 GeV ^{206}Pb on ^{206}Pb collisions in the laboratory frame

$$E_{CM} = \sqrt{2 \times 1\text{GeV} \times 200\text{GeV}} \simeq 20\ \text{GeV}$$

In the laboratory frame much of the energy goes in generating the momenta of the final particles, whereas in the center of mass frame the entire energy can be spent in creating final particles which can have even zero momenta. That is why beam-beam collisions are preferred.

1.7 Bjorken's Picture of Relativistic Heavy-Ion Collisions

Bjorken gave a simple picture of QGP formation in relativistic heavy ion collision experiments [6]. As we mentioned earlier, at ultra-high energies the initial nucleons, containing the initial quarks, primarily go through each other due to asymptotic freedom. As Lorentz contracted nuclei go through each other, the intermediate region is filled with secondary partons that are produced. The early evolution is dominated by longitudinal expansion. Note that the strictly longitudinal expansion assumption is valid only for $t \ll R$, the nucleus size. Overlap of the nuclei is taken to be at time $t = 0$ in the center of mass frame. This results in a longitudinally expanding plasma with the fluid in the middle being at rest. Net baryon number is contained near the receding nuclei. At the simplest level we assume that during the collision each of the nucleons in one nucleus has undergone a collision. Essentially, one can sit in the rest frame of one nucleus, and see each nucleon being struck as the other highly Lorentz contracted nucleus passes through it. Produced partons equilibrate in a certain time scale t_0 and the system thermalizes. The value of t_0 is extremely crucial for the estimate of the energy density and further evolution.

1.7.1 Estimates of the Central Energy Density

We will make an estimate of the energy density arising in the central region by assuming that partons in this region simply arise from individual nucleon - nucleon collisions. That is, we just add the contribution of all the nucleons to get the particle density and energy density in the central region. To do that, we need to know the behavior of particle production in individual nuclear-nuclear collisions. The essential feature of the hadron production in, for example, proton-proton collisions is that at high energy, (*e.g.*$\sqrt{s} \sim 200$ GeV), there exists a "Central Plateau" structure in the particle density as a function of the rapidly variable. This central plateau region plays a central role in developing an elegant picture of the evolution of QGP in **Bjorken's boost invariant hydrodynamic model**.

We note that the rapidity variable in a moving frame y' is related to the rapidity y in the original frame by $y' = y + y_{\text{frame}}$ where y_{frame} is the frame rapidity y_β

$$y_\beta = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$$

Due to the central plateau structure, we note that particle production (*i.e.* $\frac{dN_{ch}}{dy}$) will appear the same to different Lorentz observers as long as y' and y remain in 'the central rapidity region' (discussed later). In this central rapidity region, the description of QGP (in terms of density, etc.) will be invariant under a Lorentz boost. This is called Bjorken's boost invariant model.

Recall now the relation between $(P_0, \vec{P})$ for a particle and $(y, \vec{P}_T)$,

$$\begin{aligned} P_z &= m_T \sinh y \\ m_T^2 &= m^2 + P_T^2 \\ \text{and} \quad P_0 &= m_T \cosh y \end{aligned}$$

The velocity of the particles in the longitudinal direction is therefore

$$v_z = \frac{P_z}{P_0} = \tanh y$$

For a particle starting from the origin $z = 0$ at $t = 0$ (x, y are arbitrary), we have

$$\frac{z}{t} = v_z = \tanh y$$

From these relations one can show that

$$z = \tau \sinh y \quad \text{and} \quad t = \tau \cosh y$$

where τ is the (fluid) proper time variable defined by $\tau = \sqrt{t^2 - z^2}$. Note that this is the proper time for the fluid element and not for individual particles which have nonzero P_T. We can also show that

$$y = \frac{1}{2}\ln\frac{t+z}{t-z} = \frac{1}{2}\ln\frac{1+v_z}{1-v_z}$$

as below.

Firstly,

$$\frac{z}{\tau} = \frac{z}{t}\frac{t}{\tau} = \tanh y \frac{t}{\sqrt{t^2 - z^2}} \tanh y \frac{1}{\sqrt{1 - z^2/t^2}}$$

Now,

$$1 - \frac{z^2}{t^2} = 1 - \tanh^2 y = \frac{1}{\cosh y^2}$$

Therefore,

$$\begin{aligned} \frac{z}{\tau} &= \frac{\sinh y}{\cosh y} \cosh y \\ \text{or,} \quad z &= \tau \ \sinh y \end{aligned}$$

Furthermore,

$$\frac{t}{\tau} = \frac{1}{\sqrt{1 - z^2/t^2}} = \cosh y$$

Finally,

$$\begin{aligned} \frac{t+z}{t-z} &= \frac{\tau(\cosh y + \sinh y)}{\tau(\cosh y - \sinh y)} \\ &= \frac{e^y + e^{-y} + e^y - e^-y}{e^y + e^{-y} - e^y + e^{-y}} = \frac{2e^y}{2e^{-y}} = e^{2y} \end{aligned}$$

which implies

$$\begin{aligned} y &= \frac{1}{2} \ln \frac{t+z}{t-z} \\ \text{or,} \quad y &= \frac{1}{2} \ln \frac{1+v_z}{1-v_z} \end{aligned}$$

This is like frame rapidity, though here we have particle velocity.

Central Rapidity Region

In the center of mass system, the region of small rapidity is called "the central rapidity region". We have $z = \tau \ \sinh y \simeq \tau y$ for $y \ll 1$. This means for a given proper time τ, a small value of rapidity y is associated with a small value of z. Hence the central rapidity region is associated with the central spatial region around $z \sim 0$ where the nucleon-nucleon collision has taken place.

With a relation like $z = \tau \sinh y$, the rapidity distribution $\frac{dN}{dy}$ of particles can be transcribed as a spatial distribution from which the initial energy density can be inferred.

It is easier to measure the pseudorapidity variable

$$\eta = -\ln[\tan(\theta/2)]$$

For ultra relativistic particles $\eta \simeq y$.

Energy Density Estimate

In the center of mass frame the fluid is at rest at $z = 0$. The volume of the region under consideration is $S \times \Delta z$ where S is the transverse area of the Lorentz contracted nuclei. We consider the proper time τ_0 at which a QGP system may have formed by equilibration. So τ_0 is the time at which the initial system of quarks and gluons achieves thermal equilibrium. It is a very important quantity for which various estimates exist. This plays a crucial role in the evolution of plasma.

The number density of particles in this region at time τ_0 is

$$\frac{\Delta N}{S\Delta z}|_{z=0} \quad = \quad \frac{1}{S}\frac{dN}{dy}\frac{dy}{dz}|_{y=0}$$

where $\frac{dN}{dy}$ refers to the observed hadrons (number of particles) per unit rapidity. From $z = \tau \sinh y$ we have

$$\frac{dy}{dz}|_{z=0} = \frac{1}{\tau_0 \cosh y}|_{y=0} \ \text{ at } \tau = \tau_0$$

So the number density at $\tau = \tau_0$ is

$$n_0 = \frac{1}{S}\frac{dN}{dy}\frac{1}{\tau_0 \cosh y}|_{y=0}$$

We have seen that the energy of a particle P_0 is

$$P_0 = m_T \cosh y$$

where $m_T = (m^2 + {P_T}^2)^{1/2}$ is the transverse mass. So the energy density at time τ_o is

$$\epsilon_0 = \rho_0 \, n_0 = \frac{m_T}{S\tau_0}\frac{dN}{dy}|_{y=0}$$

This estimate was first given by Bjorken. Here one can either estimate $\frac{dN}{dy}$ by combining the expected $\frac{dN}{dy}$ resulting from each nuclear-nuclear collision, or, one can take $\frac{dN}{dy}|_{y=0}$ from some experiment and from that deduce ϵ_0 at time τ_0. From that estimate one can then decide whether a QGP state is expected to have formed at τ_0 (for example if $\epsilon_0 > 2.5\,\text{GeV/fm}^3$ from the Bag model).

Estimates of τ_0 range from those based on cross section calculations to those coming from Monte Carlo simulations. It is expected that for collisions at higher center of mass energy τ_0 will be smaller. For the SPS experiment at CERN in the collisions of ^{16}O on Au at 200 GeV (laboratory frame),

$$\frac{dN_{ch}}{d\eta} \sim \frac{dN_{ch}}{dy} \simeq 160$$

Various estimates give $\tau_0 \sim 0.4$ fm for these energies and $m_T \simeq 400$ MeV. Then

$$\epsilon_0 \simeq \frac{0.4\,\text{GeV} \times 160}{0.4 \text{ fm } S}$$

For a nucleus of mass number A the radius is given by

$$r \simeq 1.2A^{1/3}\text{fm}$$

So the area $S = (1.2)^2 A^{2/3}$ fm^2. Substituting this value we get

$$\epsilon_0 \sim 3 - 4\,\text{GeV/fm}^3$$

This energy is high enough that we expect that QGP may have formed. Now one sees the importance of τ_0. If τ_0 is larger by a factor 3, say $\tau_0 \sim 1.2\,$fm, then $\epsilon_0 \sim 1$GeV/fm^3 and one does not expect QGP.

1.7.2 Evolution of QGP

Bjorken's picture respects boost invariance for boosts along the z axis. So physical quantities should depend only on proper time τ. That is, we say that the energy density $\epsilon(\tau)$ has a value ϵ_0 at $\tau = \tau_0$. Recall that $\tau = \sqrt{t^2 - z^2}$. So a given τ_0 is achieved at different values of t at different z (where t is the laboratory time, or the proper time measured at $z = 0$).

We can then write down a picture of the evolution of QGP in Bjorken's model. The QGP is modeled as an ideal fluid with 4-velocity u_μ ($u_\mu u^\mu = 1$). The energy momentum tensor is

$$T_{\mu\nu} = (\epsilon + P)u_\mu u_\nu - g_{\mu\nu}P$$

where $\epsilon = \epsilon(\tau)$ and $P = P(\tau)$ are the energy density and pressure (they only depend on τ). The energy-momentum conservation equation is (neglecting effects of viscosity),

$$\frac{\partial T_{\mu\nu}}{\partial x_\mu} = 0$$

with initial conditions $\epsilon(\tau_0) = \epsilon_0$, and $u_\mu(\tau_0) = \frac{1}{\tau_0}(t, 0, 0, z)$.

Exercise: Show that the energy density evolves as

$$\frac{d\epsilon}{d\tau} = -\frac{\epsilon + P}{\tau}$$

Using the relation $P = \frac{\epsilon}{3}$ we get $\epsilon(\tau) \sim \tau^{-4/3}$ and using the ideal gas equation we find $T(\tau) \sim \tau^{-1/3}$. Further, one can show that

$$\frac{d}{d\tau}\left(\frac{dS}{dy}\right) = 0$$

where $\frac{ds}{dy}$ is the entropy per unit rapidity which is constant under evolution.

As the QGP system expands, it cools and eventually hadronizes at $\tau = \tau_h$ when its temperature falls below the quark-hadron transition temperature T_c (present lattice estimates suggest a value of about 170 MeV for T_c). Note that we only get hadrons from the freeze out surface, *i.e.*, after the proper time when hadrons stop interacting. From these hadrons we have to deduce about the transient stage of QGP between $\tau_0 < \tau < \tau_h$. This is almost like looking at the cosmic microwave background photons from the surface of last scattering. We have to deduce what happened during inflation, etc. from these photons.

This now brings us to the issue of signals of QGP.

1.8 QGP Signals

We need signals of the intermediate, transient stage of QGP. This can only be in terms of some special properties of the finally detected particles [4]. We will discuss some important signals which have been proposed for the detection of QGP.

1.8.1 Production of Dileptons and Photons in QGP

The Drell-Yan process is

$$q\overline{q} \to \gamma^* \to l^+ l^-$$

The lepton interaction with quarks in the QGP is electromagnetic and the cross section $\sim \left(\frac{\alpha}{\sqrt{s}}\right)^2$ (with $\alpha = \frac{1}{137}$ and $\sqrt{s}$ the center of mass energy) and is much smaller than the strong cross section. Therefore leptons after production do not further interact with the QGP and directly reach the detector.

On the other hand, the production rate and the momentum distribution of the produced l^+l^- pairs depend on the momentum distribution of quarks and antiquarks in the plasma, which are governed by the thermodynamic condition of the plasma. Therefore, l^+l^- pairs carry information on the thermodynamic state of the medium at the moment of their production and can help us to detect whether a QGP state has been achieved.

Particle production also happens by hadronic interactions. So, one needs to calculate all contributions and then compare with the data. Photons are produced via

$$q + \overline{q} \to \gamma + g$$

$q\overline{q} \to \gamma\gamma$ has a smaller cross section compared to $q\overline{q} \to \gamma g$ by a factor $\left(\frac{\alpha_e}{\alpha_s}\right)$. Detection of the photon gives similar information as dileptons because photons also do not further interact with the QGP after their production.

1.8.2 J/ψ Suppression

J/ψ particle is a bound state of the $c\overline{c}$ quark-antiquark system (charmonium states). As the c quark is heavy, the bound state has a small radius. (Recall

$m_c \sim 1.3$ GeV.) These charmonium states are well described by a potential model where the potential between c and $\bar{c}$ is taken as

$$V(r) = -\frac{\alpha_{eff}}{r} + Kr$$

Fitting with $c\bar{c}$ states gives $\alpha_{\text{eff}} = 0.52, K = 0.926$ GeV/fm with $m_c = 1.84$ GeV. When these states are formed during the early stages of collision, they have to survive through a QGP state if they have to be finally detected.

We know that quarks are not confined in the QGP phase so all hadrons should disappear. But that depends on the temperature scale of the QGP and the time available before the QGP hadronizes. In the QGP phase the QCD string disappears so there is no Kr term in $V(r)$. However the Coulomb part could still let the $c\bar{c}$ system remain bound. However, this Coulomb interaction is modified because of Debye screening of charges in the plasma

$$V(r) \sim \frac{e^{-r/\lambda_d}}{r}$$

where λ_d is the Debye screening length. If $\lambda_d < r_{bound}$ where r_{bound} is the bound state size for the $c\bar{c}$ state, then the Coulomb attractive part between the $c\bar{c}$ pair is also greatly modified. (Recall that the Kr part has anyway disappeared due to the QGP.) In that case the $c\bar{c}$ state will melt away. This will lead to the suppression of J/ψ production.

Note: If the QGP never forms then this suppression mechanisms will not be operative and one should expect a larger number of J/ψ particles.

Also for lighter mesons (made up of u, d, s) this type of signal can not be used since they are abundantly produced in thermal processes near $T \sim T_c$. The $c\bar{c}$ are too heavy to be produced like that.

1.8.3 Elliptic Flow

This signal has yielded very useful and surprising information about the equation of state of matter achieved at RHIC showing that it is like an ideal liquid. For non-central collisions with non-zero impact parameter, one gets a QGP formed which is not spherical but has an ellipsoidal shape. After thermalization there is some central pressure while $P = 0$ outside the QGP region. Clearly the pressure gradient is larger along the smaller dimension of the ellipsoid. This forces the plasma to undergo hydrodynamic expansion at a faster rate in that direction compared to the other (transverse) direction. Thus particles produced have larger momentum in that direction than in the other direction. In other words, the spatial anisotropy gets transferred to a momentum anisotropy due to hydrodynamical flow. This clearly depends crucially on the equation of state relating pressure to energy density. Thus, the observed momentum anisotropy of the particle distribution can be used to extract useful information about hydrodynamic flow at very early stages probing directly the equation of state of the QGP. **If thermalization is delayed by a time $\Delta\tau$, any elliptic flow would have to build on a reduced spatial deformation and would come out smaller**.

The data seems to be in very good agreement with the prediction of ideal fluid hydrodynamics pointing to a very low viscosity of the QGP produced. The QGP does not behave as a weakly interacting quark-gluon gas as suggested by naive perturbation theory, nor does it behave like viscous honey (as suggested by some calculations). This is termed as **Strongly Coupled QGP (sQGP)**, with a strong non-perturbative interaction.

1.9 Phase Transitions

Note that the signals discussed above depend on the existence of the QGP phase. We know that as the QGP expands it undergoes a phase transition to the hadronic phase. Such a phase transition can have its own interesting signatures on the final particle (hadron) distribution. For such signals we should understand the nature of the phase transition expected as the QGP hadronizes.

From the partition function we get the free energy as

$$F = -T \ln Z$$

Now we consider different types of phase transitions.

1.9.1 First Order Phase Transition

Here the free energy F is continuous but $\frac{\partial F}{\partial T}$ is discontinuous at the phase transition temperature. Recall that

$$F = E - TS, \quad S = \frac{\partial F}{\partial T}$$

$$\epsilon = \frac{E}{V} = \frac{F + T\frac{\partial F}{\partial T}}{V}$$

As F is continuous but $\frac{\partial F}{\partial T}$ is discontinuous, we conclude that the energy density ϵ is discontinuous as a function of temperature during a first order phase transition. The difference in the energy density ϵ at the discontinuity gives the value of the latent heat.

1.9.2 Second Order Phase Transition

Here the free energy F and $\partial F/\partial T$ are continuous while $\partial^2 F/\partial T^2$ is discontinuous (or divergent) at the phase transition temperature. Because the specific heat at constant volume is related to $\frac{\partial E}{\partial T}$ or $\frac{\partial^2 F}{\partial T^2}$, a second order phase transition is characterized by a continuous free energy and energy density but a discontinuous (or divergent) specific heat at constant volume.

Second order transitions are also called as continuous phase transitions. Here the order parameter (discussed below) goes to zero continuously as $T \to$

T_c, the phase transition temperature. In contrast, the order parameter changes discontinuously as $T \to T_c$ for a first order transition.

1.9.3 Order Parameter

The order parameter is a quantity (thermodynamic variable) which is typically zero in one phase, the disordered phase which has higher symmetry, and is non-zero in the ordered phase having lower symmetry. (It may happen that the symmetry does not change during a phase transition, as in a liquid-gas transition.)

The free energy density plot for a second order phase transition has the minimum of the free energy for zero order parameter for $T > T_c$ while for $T < T_c$ the minimum of the free energy shifts continuously away from the zero order parameter value. (This is most often the case. However it is also possible to have the reverse situation, that is the symmetry may be restored at low temperatures and spontaneously broken at high temperatures. This happens to be the case for the center symmetry for QCD, as we will discuss in Sect. 1.9.7.) An example is given by the following free energy density,

$$F = -a\phi^2 + b\phi^4$$

where $a < 0$ for $T > T_c$ while $a > 0$ for $T < T_c$.

For a first order transition the order parameter changes discontinuously through T_c. Here the transition proceeds via bubble nucleation. An example of the free energy density for this case is

$$F = a\phi^2 + b\phi^3 + c\phi^4$$

where $a, c > 0$ and b changes sign through T_c, being positive for high T.

1.9.4 Landau Theory of Phase Transitions

This is a phenomenological theory. This postulates that one can write down a function L known as the Landau free energy which depends on the coupling constants K_i and the order parameter η. L has the property that the state of the system is specified by the absolute (*i.e.* global) minimum of L with respect to η. L has dimensions of energy, and is related to the Gibbs free energy of the system. Importantly it is <u>not</u> the same as the Gibbs free energy, hence there is no requirement for it to be a convex function of the order parameter.

We assume that thermodynamic functions of state can be computed by differentiating L, as if it were indeed the Gibbs free energy. To specify L it is sufficient to use the following constraints on L (it is not certain whether all these are necessary).

1. L has to be consistent with the symmetries of the system.
2. Near T_c, L can be expanded in a power series in η, *i.e.*, L is an analytic function of both η and the parameters $[K]$. In a spatially uniform system of volume V, one can express the Landau free energy density L as

$$L = \frac{\mathrm{L}}{V} = \sum_{n=0}^{\infty} a_n([K], T)\eta^n$$

3. In an inhomogeneous system, with a spatially varying order parameter profile $\eta(r)$, L is a local function, *i.e.* it depends only on $\eta(r)$ and a finite number of derivatives.
4. In the disordered phase of the system, the order parameter $\eta = 0$, while it is small and non-zero in the ordered phase, near the transition point. Thus, for $T > T_c$, $\eta = 0$ solves the minimisation equation for L; for $T < T_c$, the minimum of L corresponds to $\eta \neq 0$. Thus, for a homogeneous system:

$$L = \sum_{n=0}^{4} a_n([K], T)\eta^n$$

where we have expanded L to $\mathrm{O}(\eta^4)$ in the expectation that η is small, and all the essential physics near T_c appears up to this order. Whether or not the truncation of the power series for L is valid will turn out to depend on both the dimensionality of the system and the co-dimension of the singular point of interest.

1.9.5 Construction of L

Consider

$$\frac{\partial L}{\partial \eta} = a_1 + 2a_2\eta + 3a_3\eta^2 + 4a_4\eta^3 = 0$$

Since for $T > T_c$, $\eta = 0$, therefore $a_1 = 0$. Note that this is not true when the symmetry is also broken explicitly in which case the order parameter never completely vanishes. If $\eta \to -\eta$ is a symmetry of the free energy, then $a_3 = a_5 = a_7 = .. = 0$. Then

$$L = a_0([K], T) + a_2([K], T)\eta^2 + a_4([K], T)\eta^4$$

Note that the requirement that L be analytic in η precludes terms like $|\eta|$ in L. Also note that finiteness of L requires $a_4 > 0$.

Coefficients $a_n([K], T)$: $a_0([K], T)$ is simply the value of L in the high temperature phase, and we expect it to vary smoothly through T_c. It represents the degrees of freedom in the system which are not described by the order parameter, and so may be thought of as the smooth background, on which the singular behavior is superimposed. It may be said that $L - a_0$ represents the change in the Gibbs free energy due to the presence of the ordered state, apart from the fact that L is not exactly the Gibbs free energy.

For discussing the order parameter, one may set $a_0 = 0$. We expand a_4 as

$$a_4 = a_4^0 + (T - T_c)a_4^1 + ...$$

It will be sufficient to just take a_4 to be a positive constant. The temperature dependence of this equation will turn out not to dominate the leading behavior of the thermodynamics near T_c. We expand a_2 as

$$a_2 = a_2^0 + \left(\frac{T - T_c}{T_c}\right) a_2^1 + O\left((T - T_c)^2\right)$$

Note that a_2^0 can be absorbed in the definition of T_c. For a continuous phase transition L is of the form

$$L = at\eta^2 + b\eta^4$$

where

$$t = \frac{T - T_c}{T_c}$$

and a and b are constants. For a first order transition

$$L = at\eta^2 + b\eta^4 - c\eta^3$$

1.9.6 Deconfinement-Confinement Transition

Consider the case of $SU(3)$ gauge theory at finite temperature without dynamical quarks. We will calculate the free energy for this system with a single, infinitely heavy, test quark at position $\mathbf{r_0}$. (In this section we follow the discussion in ref. [7].) We start with the evolution equation for the field operator $\psi(\mathbf{r_0}, t)$ of this static quark (suppressing the color label),

$$\left(-i\frac{\partial}{\partial t} - gA^0(\mathbf{r_0}, t)\right)\psi(\mathbf{r_0}, t) = 0$$

where $A^0 \equiv \mathbf{T}.\mathbf{A^0}$ (T^i are the generators of $SU(3)$; see Sect. 2.3.1 of ref. [7]). This equation gives

$$\psi(\mathbf{r_0}, t) = T\exp\left(ig\int_0^t dt' A^0(\mathbf{r_0}, t')\right)\psi(\mathbf{r_0}, 0).$$

Here T denotes time ordering. Now, the partition function for this system is given by

$$Z = e^{-\beta F(\mathbf{r_0})} = \frac{1}{N}\sum_s \langle s|e^{-\beta H}|s\rangle$$

where the $1/N$ factor is introduced to compensate for the color degeneracy factor for the static quark (N equals 3 for QCD and is the number of colors), and the sum is over all the states of the system with the infinitely heavy quark at $\mathbf{r_0}$. Using the quark field operator $\psi(\mathbf{r_0}, t)$, we can write it as

$$e^{-\beta F(\mathbf{r_0})} = \frac{1}{N} \sum_{s_g} \langle s_g | \psi(\mathbf{r_0}, 0) e^{-\beta H} \psi^\dagger(\mathbf{r_0}, 0) | s_g \rangle$$

where, now, the sum is over all states $|s_g\rangle$ with no quarks, that is, over states of pure glue theory. Recall, from Sect. 1.4.1, that for Euclidean time t,

$$e^{\beta H} \psi(\mathbf{r_0}, 0) e^{-\beta H} = \psi(\mathbf{r_0}, \beta)$$

Thus, we get

$$e^{-\beta F(\mathbf{r_0})} = \frac{1}{N} \sum_{s_g} \langle s_g | e^{-\beta H} \psi(\mathbf{r_0}, \beta) \psi^\dagger(\mathbf{r_0}, 0) | s_g \rangle$$

We introduce the Wilson line,

$$L(\mathbf{r}) = \frac{1}{N} \mathrm{Tr}\, T \exp\left(ig \int_0^\beta dt A^0(\mathbf{r_0}, t) \right).$$

With this, using the solution $\psi(\mathbf{r_0}, t)$ of the time evolution equation above, and the equal time anti-commutation relation of the fermion fields (with discrete space labeling, for simplicity), we can write

$$e^{-\beta F(\mathbf{r_0})} = \mathrm{Tr}\,[e^{-\beta H} L(\mathbf{r_0})]$$

where the trace is over all states of the pure glue theory. Dividing this by the free energy without any heavy fermion, we get the difference in the free energy, ΔF_q, due to introduction of the infinitely heavy quark at $\mathbf{r_0}$ as

$$e^{-\beta \Delta F_q} = \langle L(\mathbf{r_0}) \rangle$$

where $\langle .. \rangle$ denotes the thermal expectation value. $\langle L(\mathbf{r_0}) \rangle$ is an order parameter for the deconfinement - confinement phase transition.

Confining phase: We expect the free energy with an isolated quark to diverge, *i.e.* $\Delta F = \infty$, and thus $\langle L \rangle = 0$.

Deconfining phase: Here isolated quarks can exist, leading to a finite change in the free energy with respect to the pure glue background, *i.e.* ΔF is finite, which implies $\langle L \rangle = e^{-\beta \Delta F} \neq 0$.
Thus $\langle L \rangle$ is an order parameter for the deconfinement - confinement (D-C) phase transition.

Recall that $A_0(\mathbf{r_0}, t)$ must be periodic in the Euclidean time t.

$$A_0(\mathbf{r_0}, 0) = A_0(\mathbf{r_0}, \beta)$$

Thus the dt integral in the expression for the Wilson line is actually a loop integral. This is also called as the 'Polyakov Loop'.

1.9.7 D-C Transition as a Symmetry Breaking Transition

Recall the gauge transformation

$$A_\mu \quad \to \quad U A_\mu U^{-1} + iU\partial_\mu U^{-1}$$

where $U(x,t) \in SU(N)$ and $A_\mu \simeq A_\mu \frac{\lambda^a}{2}$. Under the gauge transformation, the Wilson line

$$\begin{aligned} L \quad &\sim \quad \mathrm{Tr}\left[T \ \exp\left(ig\int_0^\beta d\tau A_0(x,\tau)\right)\right] \\ &\simeq \quad \mathrm{Tr}\,\Omega(x) \end{aligned}$$

will transform as

$$L(x) \to \mathrm{Tr}\, U(x,\beta)\Omega(x)U^\dagger(x,0)$$

This can be checked by expanding the time ordered exponential. Thus L is invariant when U is periodic,

$$U(x,0) = U(x,\beta)$$

(using the cyclic property of the trace).

However, we note that the Euclidean action

$$S_F = \frac{1}{4}\int d^3x \ d\tau \ F^a_{\mu\nu} \ F^{a\mu\nu}$$

is in fact invariant under a larger group than the periodic gauge transformations. The only physically important constraint is that A^μ ($\vec{x}$, t) remain periodic in τ when gauge transformed. Consider, e.g.,

$$A_\mu(x,0) = A_\mu(x,\beta)$$

Under a gauge transformation

$$A'_\mu(x,0) = U(x,\tau)A_\mu(x,0)U^{-1}(x,\tau) + iU(x,\tau)\partial_\mu U^{-1}(x,\tau)|_{\tau=0}$$

Similarly,

$$\begin{aligned} A'_\mu(x,\beta) \quad &= \quad U(x,\tau)A_\mu(x,\beta)U^{-1}(x,\tau) + iU(x,\tau)\partial_\mu U^{-1}(x,\tau)|_{\tau=\beta} \\ &= \quad U(x,\tau)A_\mu(x,0)U^{-1}(x,\tau) + iU(x,\tau)\partial_\mu U^{-1}(x,\tau)|_{\tau=\beta} \end{aligned}$$

First take $U(x,\beta) = U(x,0)$ due to the identification of points $\tau = 0$ and $\tau = \beta$. Then,

$$A'_\mu(x,\beta) = U(x,\tau)A_\mu(x,0)U^{-1}(x,\tau) + iU(x,\tau)\partial_\mu U^{-1}(x,\tau)|_{\tau=0} = A'_\mu(x,0)$$

Hence A'_μ also remains periodic and hence single valued.

Now note that in the above argument we could take

$$U(x,\beta) = ZU(x,0)$$

where $Z \in SU(3)$ (or $Z \in\ SU(N)$ in general) such that $Z\,U = U\,Z$ for every $U \in SU(N)$ (so that $U\,A\,U^{-1} \to Z\,U\,A\,U^{-1}Z^{-1} = U\,A\,U^{-1}$) and Z is space time independent. Thus, as long as Z commutes with every element of $SU(N)$, ${A_\mu}'$ remains periodic in τ if A_μ is.

Elements Z constitute the center of $SU(N)$ by definition.

$$Z = \exp\left(\frac{2\pi\,i\,n}{N}\right) \in Z_N$$

where Z_N is the cyclic group of order N and $n = 1, 2...N$. $n = N$ corresponds to the identity of $SU(N)$. Note that

$$\mathrm{Det}\,Z = \exp\left(\frac{2\pi i n}{N} \times N\right) = 1$$

So $Z \in SU(N)$ (clearly $Z^\dagger Z = 1$). For QCD we have

$$Z \;\in\; Z_3$$

Thus, we conclude that finite temperature $SU(N)$ gauge theory (Euclidean action) has Z_N symmetry (Z_3 for QCD) as the Euclidean action (or the partition function and hence the free energy) is invariant under Z_N transformations of the basic variables $A_\mu(x)$. This is called as the center symmetry. (Quarks break this Z_N symmetry explicitly because fermions obey an antiperiodic boundary condition $\psi(x,\beta) = -\psi(x,0)$.)

Though the Euclidean action is invariant under this extra Z_N transformations, the order parameter $L(x)$ is not. Recall that $L(x) = \mathrm{Tr}\,\Omega(x)$. Under the gauge transformation $U(x,\tau)$ we have

$$L(x) \to L'(x) = \mathrm{Tr}[U(x,\beta)\Omega(x)U^{-1}(x,0)].$$

If $U(x,\beta) = Z\,U(x,0)$, we get

$$L'(x) = Z\ \mathrm{Tr}[U(x,0)\Omega(x)U^{-1}(x,0)] = Z\ \mathrm{Tr}\ \Omega(x) = Z\ L(x)$$

So, while under a periodic gauge transformation ($Z = 1$), $L \to L$, under an aperiodic gauge transformation $L \to Z\,L$.

We now study the confining and deconfining phases of the $SU(3)$ gauge theory.

Confining Phase: With $\langle L\rangle = 0$ (corresponding to $e^{-\beta\Delta F}, \Delta F = \infty$), the system respects Z_3 symmetry as $\langle L\rangle = 0$ is invariant under $L \to Z\,L$ transformation.

Deconfining Phase: With $\langle L \rangle \neq 0$, the system is NOT invariant under Z_3 transformations. There are 3 equivalent phases characterized by $\langle L \rangle$, $\langle Z\, L \rangle$, and $\langle Z^2\, L \rangle$ which all correspond to physically the same deconfining phase. We conclude that in the deconfining phase the Z_3 symmetry is spontaneously broken.

Here the symmetry restored phase is the low temperature confining phase. This is in contrast to most cases, where the symmetry restoration happens in the high temperature phase. The symmetries of the order parameter can be used to characterize the phase transition in the Ginzburg-Landau approach.

The order parameter for the $SU(2)$ gauge theory has the same symmetry as the Ising model which has a global Z_2 symmetry. In 3 + 1 dimensions, the Ising model undergoes a second order transition. Hence we expect that the $SU(2)$ gauge theory exhibits a second order transition.

Similarly, $Z(3)$ spin models in 3 + 1 dimensions display a first order transition. Hence we expect that pure $SU(3)$ QCD will have a first order transition. Lattice calculations confirm these expectations.

Clearly, for QCD, $L^3 \to Z^3 L^3 = L^3$. Thus, in the construction of L and free energy, one can write down

$$V(L) = a|L|^2 + b|L|^4 + C(L^3 + L^{*3})$$

The L^3 term makes the transition first order. Note that for the $SU(2)$ gauge theory this term cannot be written down. One can write down a term Re L^2 which makes the transition second order.

1.9.8 Deconfinement-Confinement Transition with Dynamical Quarks

As mentioned above, with quarks, the Z_N symmetry is broken explicitly (similar to the explicit breaking of chiral symmetry, in some sense). $\langle L \rangle$ is non-zero even in the confined phase. The deconfinment-confinement transition which is first order for pure gauge theory, is smoothed into a crossover when light quarks are present. Lattice results seem to suggest no first order transition. An important point to note is that with quarks, no appropriate order parameter is known. In closing we mention that in discussing different phase transitions in QCD one is invariably in the non-perturbative regime, where reliable calculations cannot be performed. Hence one either has to do lattice calculations, or use effective models using symmetry considerations (as we did above for the D-C transition). Thus, many theoretical discussions about the nature of the phase transition in QCD are based on the Landau theory of phase transitions. Especially the investigations of phase transition at finite baryon density has difficulties even in the lattice approach, though special techniques have been developed to handle these. Most of the knowledge in this regime of QCD comes from specific effective models such as chiral quark models, random matrix models, etc.

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