Chapter 6
Number Stories

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Abstract True personal stories are used to introduce some of the research into pre-school children's development of number knowledge and skills. A range of conversations and stimulating environments illustrate how parents, grandparents, peers, and early childhood professionals support the mastery of new number words and concepts as well as mathematical actions, in everyday contexts and play situations. The stories discuss the learning of real children developing knowledge and skills in the pre-school years. They tell about early quantity identification along with some young children's growth of interest in and skills with cardinal and ordinal number and counting; learning about more and less, then very simple addition and subtraction; early recognition and naming of multiplication "arrays"; written numeral identification; and one child's earliest abstract understanding of the idea of infinity. For each of these topics, some research on pre-school learning is outlined. The growth of children's self-concepts as they handle mathematics and the situatedness of learning in varied and everyday, informal learning contexts are supplementary themes of this chapter.

Keywords Pre-school · Number · Counting · Number operations · Abstraction

Introduction

Everyone loves stories. They are the reason most people turn on television sets, download videos, buy e-readers, and love to gossip. Stories shape our realities, help us to understand life, and give us things to think about. Stories persuade, and they move people to action (Aaker 2013).

Stories are one way to reach all parents: to get them to stop, listen, and think about what they could be doing to help their children's mathematical development. I do not mean the sort of number activities that young children do at school, but the

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everyday, informal and incidental number experiences that very young children and parents have in shops, playgrounds, travelling together or eating as a family, playing at home in a bath or an outdoor playhouse—and in a myriad of other places that are not normally considered places of instruction and/or learning. Parent, grandparents and carers need to be aware of their roles in identifying moments for developing children's mathematical understandings.

Here are some stories about early years number and number operation development. They illustrate some points in time and place where pre-school children are coming to understand a range of number concepts including counting and some number operations. Each story is followed by some relevant research undertaken by others and published in academic journals and books, as I aim to present some research findings that will offer insight and elaborate on the potential of each story.

The stories themselves are not imaginary: they are grounded in my experience as a family member and in my research from across many years. Essentially, they are stories from my life: my experience as a researcher as well as my roles as a parent and as a grandparent. They are stories of very young children growing up in a mathematical world, from their earliest months to when they reach school age—the greatest years for learning and development. They are mainly stories of young children learning through play and everyday family experience: stories of children learning mathematically with parents and other family members. Please feel free to use these stories in childcare centre newsletters, in discussions with parents, or in thinking about the development of young children in your care.

Peter and "Two"

I remember well when a few weeks before his second birthday, when my young son Peter showed that he understood "two". Playing with plastic cups and sand, he noticed that two of the cups were red. He sat them side-by-side on the sand, saying "Two!" "Oh, good boy" was my response "Two. Two red cups. Two cups that are red". I was pointing with two hands. "One, two. Fill up the two red cups".

Later that morning, Peter toddled around, pointing with two hands to two shoes, pairs of handles on cupboards, pillows, his Dad's hands, my eyes, and lots of other pairs of things in our house. "Two! Two! Two!" he called out many times, as if surprised and delighted by each individual discovery.

Over the next week or so, I set up many "three" situations: three frozen peas (his favourite vegetable at the time), three blocks, three dried apricots, three Lego people and so on; but without success. But he showed no interest in threes.

The next group size Peter recognised and named was actually four, and again the experience was a delight for him and for us. I was fascinated with his immediate naming of four as "two two" and the fact that his interest in doubles became a long-term one. (At about seven I would hear him drifting off to sleep chanting "Two, four, eight, …, two hundred and fifty six, …".)

As a parent, was I doing what I was meant to be doing at the time he recognised twoness? The positive feedback I was giving him was intuitive, with my repeating his words and reinforcing his correct ideas with pointing and praise being what all parents do with their children. As a teacher at the time, I knew that the frequency of exposure to a new idea helps children's development, and that it is important to present some counter examples ("not two"). However, I was clearly wrong in expecting him to make orderly progress—like the school curriculum does—because he showed no interest in threes. Who would have thought the next group he focused on would be double two?

What else could we have been doing with twos? Months later, I realised I could have pointed out pairs (and later fours) of things in storybooks, to consolidate and strengthen his learning. I had seen how his learning about number had been self-motivated and self-directed, but I also wondered when and how I should get Peter to understand that dissimilar things like an apple and orange, or a toy car and bus, can also illustrate "two-ness". Further, I did not realise at that time that recognising "two-ness" or "four-ness" is quite a different skill from being able to count to two or four. Like all parents, I was wondering at that time, "When do children typically learn these ideas and skills?"

These questions point to the focus of this chapter. My aim is to describe some ways that parents can support the mathematical learning of very young children, right from their earliest months through until they reach school age. What do we know about the mathematical development of young children in respect to number and simple arithmetic concepts? How can children's storybooks and familiar objects be used here, and how can parents draw out the mathematics in everyday events? What roles might some technologies play? How can various types of games be used to promote pre-school children's mathematical development? How can we make the most of young children's passionate interests and hobbies as well as various family routines? What are the big ideas of number content that children can learn, before going to school, that will underpin their mathematical learning and initial achievement at school?

It is not only content (such as counting words and ideas) that is important in the prior-to-school years. In fact, I argue that specific content is not very important compared with the development of processes of thinking. How can young children be helped to develop organisational, problem solving, and reasoning skills as well as creative handling of aspects such as number, pattern, order, and mathematical relations? How can we assist them to develop persistence and resilience with tasks and encourage logical thinking during their everyday activities? Are there ways of encouraging their capacities for explanation, justification, estimation, and reasoning? These were questions that have rarely been the focus of research even though there has been much research on very early number ideas.

Cardinal Number: Some Research

Cardinal number is the "two-ness" of a pair of objects, the "three-ness" of any group of three, the "fifty-ness" of fifty toothpicks, and so on. When we talk about the number in a group, perhaps of five or fifty or one thousand things, we are using the idea of cardinal number. When we count objects "One, two, three, four" then declare there are four things in the group, we use the cardinal principle—that the last number we say names the quantity of the group.

Cardinality is absolutely essential in numerical thinking, the basis of all we do with numbers; so without yet being able to count orally, young Peter had grasped one of the very big ideas in mathematics. Having a passionate interest in young children's mathematical development at that time (and ever since then), it is not surprising that I remember that first "two" day as if it were yesterday. It seemed that this was his first true idea of number.

It is important to note that this realisation about number was long before Peter started counting things. In fact, what is called subitising, which means seeing objects as groups of a certain number without counting (like adults do reading the dots on dice), has long been recognised as a prerequisite not only to the first counting words and meaningful counting but also to the understanding of all number ideas (see, for example, Douglass 1925; Spelke 2003; Wynn 1990; and especially Kaufman et al. 1949, who coined the term subitizing for the recognition of small group numbers).

Many more recent studies have explored whether babies who are a few months old can distinguish between one and two objects—or later between three and four things (they can, showed Resnick 1992; as well as Rouselle and Noël 2008); whether young children create and use mental models of collections (they do, showed Benson and Baroody 2003; as well as Hannula et al. 2007); whether children from different cultures subitise more readily or more often (some cultures seem to, showed Willis 2000); whether subitising necessarily precedes cardinal and ordinal number understandings (it does, found Hannula et al. 2007); and other questions about very young children's recognition of a number of objects in a group. Baroody et al. (2006) point out that for the development of any of these concepts, young children at least need to understand object permanence and that different objects are distinct. They also need an ability to compare group sizes, which is commonly well developed by 2.5 years of age (Mix et al. 2002).

None of this research detail is vital to parents or to practitioners in early childcare centres, except that we all need to realise that number ideas do start within the first few months of life, that cardinality underpins all number thinking, that children will not recognise and say "Two" if this has not been modelled for them, and that for children aged under five experience with subitising is just as important as the use of counting. Further, parents, grandparents and carers can all help very young children's development by using nurturing opportunities to recognise and say "Two shoes", "Three dogs", "Four legs", etc., at appropriate times during everyday activities. Frequency of experience with the mathematical words and ideas of cardinality is critical. So is showing our interest and excitement with smiles and other positive reinforcement as well as encouragement of the use of key words and ideas by young children—especially before they are ready to learn verbal object counting like Budi, below.

Budi's Counting

I was a visitor at Budi's house and he was helping me to set their table for dinner. I asked him how many plates we needed. Three-year-old Budi has been asked this before, so he walked confidently around the table, touching each of the four chairs and saying, "One, do, free, four". His parents beamed: "Budi counts well" they assured me.

I was helping with the cutlery so put out five spoons and ask Budi "How many spoons are here, Budi?" He touched each one, saying, "One, do, free, four", touching but not naming the last one. I spread out three knives, asking, "How many knives are there?" and again he touched each one once, but kept his finger on the last knife for two words: "One, do, free, four". As if to summarise, Budi smiled and touched each group of cutlery, saying, "Four knife, four spoon. Set da table. Dood boy, Budi". He was still beaming, but Budi's parents' smiles had faded somewhat. Later, Budi realised he was a knife short and happily fetched one without being asked. He also put the extra spoon aside without being prompted.

Budi's parents should have remained very proud, because (as I pointed out to them) three-year-old Budi knew a lot about counting. He knew that any sort of objects can be counted, no matter how they are laid out; that the correct sequence of number names is "One, two, three, four" (with evidence of understanding that would make this more than merely a serial recall task); and that the answer to a question of "How many?" is a counting word. He definitely understood one-to-one correspondence, as evidenced by his eventually placing one plate, one spoon, one knife, and one fork in front of each chair—even if he had not yet developed the "one touch, one word" skill yet. He had learnt to mimic most of the right counting actions, including (nearly) touching each object once while saying counting words in sequence.

In fact, during the meal I found out that Budi not only knew the counting words to ten in English but could also sing them in his family's home language, Bahasa Melayu. He was, indeed, "Wise"—the meaning of his name, as his parents proudly told me when I pointed out all that he seemed to understand already. During the meal, I took the opportunity, too, to talk about his sense of geometry, with the sense of "right" and "left" (knifes versus forks on the correct sides on both sides of the table) that he was displaying long before he knew the meaning of the words "right" and "left". Just as important was Budi's persistence and his ability to finish a complex task on his own, as well as his independence.

His mathematical problem solving skills also seemed strong, from his actions when he had more of less cutlery than he needed. He added another knife and put a spoon aside to cope with the need for more then less, not getting frustrated with my putting out the wrong number. So had Budi shown evidence of being mathematically wise, for his age? What does the research show in relation to learning to (a) count and (b) solve problems related to more or less?

Counting: Some Research

With regard to counting, many young children say some number names in order before they turn three (Fuson 1992; Resnick 1992). Further, many researchers (including Clements and Samara 2007; Fuson 1992; Rouselle and Noël 2008) have found that three-year-old children are able to count small numbers of objects functionally with accuracy and understanding. Labinowicz (1985) stresses a counting understanding that children develop gradually: "progressive inclusion" (called "progressive integration" by Steffe et al. 1982). That is, a four includes the three just counted; a five includes the four just counted; and so on. This is "a 'one more than' relation … an elaborate simultaneous relationship between numbers in the sequence" (Labinowicz 1985, p. 60). Such understandings (as well as the "one less than" idea) would be necessary for meaningful, as opposed to rote, counting forwards and backwards.

With "more" and "less", by 21 months, many typical children are able to pick up the same number of balls that they saw dropped into a box (Starkey 1992a)—although not consistently. By about 24 months, many tend to realise that adding objects results in more (Mix et al. 2002). The ideas of less and subtraction are harder to grasp, but follow soon after addition concepts (Hughes 1981). Baroody and Rosu (2006) gave examples of children aged 28–30 months adding successfully, using small numbers; and found that many children aged three succeeded with simple non-verbal addition and subtraction tasks (Baroody et al. 2006). Further, three-year-old children seem to have a good sense of equality ("the same") when they see counters dropped together into two side-by-side containers, and can tell that one container has "more" if an extra counter is dropped into it (Cooper 1984; Ginsburg 1977). However, toddlers' nonverbal addition and subtraction performance appears to drop off dramatically when collections larger than 2–3 are involved (Huttenlocher et al. 1994).

With regard to problem solving, again Budi's ability was not outstanding. Lee (2012) recorded examples of toddlers' problem solving with quantifying sets of objects, using one-to-one correspondence. Three-year-old children examined by Patel and Canobi (2010) generally coped well with simple, object-based addition problems, both using and before the development of counting words.

So it is clear that three-year-old Budi was not very unusual. Budi was demonstrating excellent progress towards all of these understandings and skills, developing a good sense of some number words, which Spelke (2003), claims is a necessary first step in forming numerical concepts. Budi was showing a good knowledge of one-to-one correspondence and simple number operations while doing family tasks like setting the table. Budi also had mastered the abstract idea that number is not linked to specific objects or their arrangement. He remembered where he had started counting, and he recognised that the cardinal number of one set could be more or less than needed to match another set.

Certainly Budi demonstrated a wide range of mathematical skills and knowledge in the not-so-simple task of setting the family table! It will not be long before he is ready to play number games like the next two children we meet.

Spiros and Alissa Play 'Concentration'

"Pair!" claimed four-year-old Spiros, and picked up the two cards.
"No, that's not a pair", claimed his older sister, Alissa. She took the cards and turned them face up. "Look, that's a six, and this is a nine".
"But it same. It's the same".
"No, look at the diamonds. See. One, two, three, four, five six; and this one's one, two, three, four, five, six, seven, eight, nine. That's more—nine. The numbers look the same, but one is upside down".
"Which one? Which one is … Which one is upside down?" asked Spiros.

The term numeral refers to the symbols that represent number names, such as 46 and 29, and the term digit refers to only single numerals (e.g., 2). Recognising digits is a number skill that many children learn prior to school, and indeed it is in the curriculum of many kindergartens because the visual identification of numerals is a gateway to all written number work.

Young children see numerals on letter boxes, buses, television, computer screens, birthday cards, clocks, book pages, calendars, shop signs and labels, car registration plates, phones, and many other everyday objects. They meet many variations in the forms of digits, as well as the rotational symmetry of 6 and 9 that confused little Spyros above.

Then there are the complexities. I remember well a three-year-old grandson—a lover of buses—not understanding that we had to wait for bus number 715 rather than taking the approaching bus 751. "But the numbers are right, Nan!" How does one explain to a child of that age that the order of numerals is important?

Numeral Identification: Some Research

Young children of this age also start to differentiate letters from numerals by recognising different functions, shaped by the activities that take place around different types of print (Tolchinsky 2003).

The work of Durkin (1968) with pre-school children aged 4–5 showed a strong correlation between letter and numeral identification, suggesting that knowledge of one symbol system was positively associated with knowledge of the other, although the results for both may have been affected by other factors such as family nurturing and/or learning expectations and activities in family and pre-school settings.

Gifford (1995) points out that there are many examples of themed play settings for pre-school children's number recognition—such as inside cardboard-box aeroplanes (e.g. seat numbers, pilots' dials with numbers on them), a play post-office (addresses, post codes), and mock grocery shops (price signs, cash register or calculator)—that can be used to expose young children to numerals in context; while Neumann et al. (2013) found a positive correlation between numeral identification in such nurturing contexts and early primary-school numeracy proficiency. As Benson and Baroody (2003) point out, meeting number symbols (including both numerals and words) in play functions as a necessary catalyst for essential understandings of number equivalence and operations.

These days, children also play in such contexts represented on computer and tablet software, like Sandi below.

Sandi and Her Tablet

As a four-year-old, Sandi could already count to twelve meaningfully. She counted confidently, had a good sense of cardinality, and in dice and cards read numbers up to ten confidently. She also read road speed signs, as well as numbers in games when using technology to play her favourite games.

Sandi's family was fairly strict with "screen time": she had 1 hour per day of watching television or playing with any technology that used a screen. (Because of this, Sandi had become quite good at watching the "big hand" hand on the clock sweep 1 h and even half an hour and before the age of four, already she understood that 2 half hours make an hour!)

The screen time limit had caused a few minor arguments between Sandi and her parents, but the rule was all she had known and was applied consistently, so she generally did not fuss. Sandi was playing a game on her Dad's tablet: a game where the aim was to move shapes in order to get three of the same shapes in a row. After the game, names of "previous players", including Sandi, were listed with their scores.

"First! I won", she called to her father.

He looked at the list and corrected her. "No, you are third. Look, your name is after the first one and after the second one. Look. [Pointing to the screen] First, second, third. Yours is the third name. Okay, turn it off now. [Ignored.] Close the cover, Sandi: your time it up".

"But I want to get first. First, not after, after! Third no—first."

Here, Sandi was coming to grips with a different set of counting words and ideas from the children above. Ordinal number is about identifying where one object is placed in relation to others, using a sense of order. Our house is the second one in the street. We eat dinner first, before dessert. The pencils are kept on the third shelf and the cooking implements in the second drawer. We have a corresponding set of words (first, second, etc.) to name ordered places—along with our use of counting numbers to name a place, such as "number three in the line".

Ordinal Number: Some Research

Children aged 2–3 show knowledge of ordinal number in comparing collections that is not related to their counting skills (Gelman and Gallistel 1978; Mix et al. 2002). They are used to parents outlining the order of a day's activities: first we will do the shopping and then next we will go to the park. "First" and "last" are typically understood by the age of four, and the ordinal counting words soon follow. About 29 and 20 % of a large number of Australian children entering school could point to the third and fifth objects in a line, respectively (Clarke et al. 2006).

So learning to count is not just a matter of learning to chant one set of number words in their correct order, but involves in complex set of skills and understandings about both cardinal and ordinal number. Children have little time to consolidate counting knowledge though, because as they are learning to count, they are also starting to operate with numbers. Usually, the first operation is addition (more). As noted above, recognition of "more" starts in babyhood, but the idea of adding groups of numbers purposefully develops later than that.

Elouise Plays 'Snakes and Ladders'

"Roll the dice, Elouise", said Nan, "It's your turn". The family of three-year-old Elouise played a game with her most nights in her "quiet time" before bed, just before her drink of milk, teeth cleaning, and story.

Elouise grabbed the die with glee and rolled it into the shoebox lid—a three. "Three!" she announced. "Three" repeated Nan, nodding then tapping the next few squares on the game board: "One, two, three. Move your dice here, El" [leaving her finger on the right square].

Elouise looked along the line of squares and said, “I want four, four for the ladder”.

“Just move it three, Elouise. You didn’t roll a four”.

Elouise moved her counter to the correct square and then surprised her grandmother by saying “Three and one more. That’s four. I want one more.”

In bed soon after that, Elouise’s mother was reading Jack and the Beanstalk to her. “So Jack gave the man the cow, and he took home the magic beans”.

“There were five”, said Elouise, “Four: three and one more. Five: four and one more”. “Well, maybe …”, said her mother, “maybe five is three and two more”. “Three and three more!” laughed Elouise. Elouise had been able to hold up three fingers since her third birthday, and now did this on each hand.

Her mother touched each little digit in turn: “One, two, three, four, five, six! Three and three are six. Six beans. Now, off to sleep, my Three Girl. Here are six kisses for a very good girl. Three on this (cheek) and three on this. There: three and three make six”.

Here, Louise seemed to be demonstrating some knowledge of number composition and decomposition. She was not just “counting on” from three to four then to five, but was recognising that four is made up of three and one more. Her mother pushed this on to “five is three and two more” then six being made up of three and another three, reinforcing this latter concept with her kisses. “One more” and “two more” are different addition concepts from adding two groups together, but Elouise and her mother slipped comfortably between these different ideas. “Children learn by varying what is done … math content requires repeated experiences with the same numbers … and related similar tasks (Clements and Samara 2007, p. 68).

Addition: Some Research

The basic idea in addition is that two smaller groups are put together make one larger group. Researchers such as Gelman and Gallistel (1978) and Starkey (1992b) have shown how this understanding develops in pre-school children, starting with the idea of adding or taking away one item at a time. Young children soon start to enjoy seeing part-whole relationships as “numbers inside numbers” and then enjoy playing with putting numbers together and breaking them down (Fuson et al. 2001, p. 523). Fuson et al. (1983), Jung (2011) and Fischer (1990) all present some everyday examples of typical development in the prior-to-school years as well as lots of engaging learning activities.

Surprisingly, counting words are not needed for simple addition. When Huttenlocher et al. (1994) put out a small number of counters and covered them before adding a few more, children as young as 2–3 years could make the total number with their own counters. Huttenlocher et al. found that concept of adding more, as well as the related “mental models” (p. 284), develop with non-verbal manipulation of objects before verbal counting, and their findings “strongly support the claim that a mental model underlies the acquisition of exact nonverbal calculation ability" (p. 295). An interesting point is that children in their experiments were reasoning numerically, so it was not only the formation of mental models for number that were developing but also the power to reason mathematically. In fact, Gallistel and Gelman (1992), after previously undertaking similar research with even younger toddlers had argued that this competency provides "the framework—the underlying conceptual scheme—that makes it possible for the young child to understand and assimilate verbal numerical reasoning" (pp. 65–66).

It is also important to consider the implications of the finding of Huttenlocher et al. (1994) that infants in different socio-economic areas performed equally on such non-verbal tasks, even though differences were noted in later verbal arithmetic (e.g., Jordan et al. 1994; Baroody et al. 2006).

Jules Takes Away

"How many sandwiches did you have, Jules?" "Four", four-year-old Jules answered her father confidently. "Okay, and you ate one?" "Yes. All gone". "Good girl. How many are left?" "Three". "Ah, so you had four and you ate one. There are three left. Let's see what happens when you eat another sandwich."

And so the conversation continued over lunch—not all the conversation, of course, but Dad kept coming back to the "take away" idea as Jules' sandwiches were eaten. They both laughed when the answer was "None".

That evening, Dad walked past the bathroom, where Jules was happily playing in the bath and chattering to herself. "Take away. Take away, Fishy. Swim away, swim, swim, swim. Four take away one is three". Dad thought that Jules was probably just repeating what she had learned by rote so walked into the bathroom, but there was Jules with four plastic fish, three floating freely and one being "swum" away by Jules.

Dad stayed with her, playing "swim away" with the other fish too, and again no fish being left delighted Jules. Then Dad blew four soap bubbles for her. He did not have to encourage her to pop one. "Pop" shouted Jules. Four pop one. Pop goes the weasel! Three. Pop, pop, pop! None!" Again, the idea of "none left" had Jules laughing aloud.

Jules seemed to have made a big jump in her learning. It is fairly easy to teach young children subtraction ideas in everyday situations. Food is eaten, birds fly away, objects are hidden, and toy cars drive off. Poems and songs such as "Mother duck said 'Quack, quack, quack, quack', but only four little ducks came back" provide more contexts for modelling subtraction in imaginative contexts. Ten green bottles fall off the wall, and rolling over in a bed causes someone to fall out.

The exciting jump that Jules had made of her own accord, was the realisation that subtraction of one from four was not just about sandwiches being eaten, but an abstract idea (i.e., one that is less dependent on objects) that could be applied not only to fish but also any other objects that might run, swim, fly, jump, fall, roll, pop—and so on—away. This jump, not only made with subtraction, or course, is called abstraction.

Of course, counting down one object at a time is different from taking away several objects at a time, but at least Jules has made a good start on understanding the idea of the "take away" action of subtraction. Jules would have two further subtraction actions to learn in the future—both typically solved by young children "counting on":

Difference: I have two cards, and you have five. How many more do you have?
What do I add? I have two cards, and you have five. How many more do I need?

Subtraction: Some Research

In fact, quite a few researchers, such as Wynn (1992) and Koechlin et al. (1997), have noted that babies as young as five months of age react to objects disappearing unexpectedly. They found that infants stared longer when the number of objects in a familiar picture had been reduced than when an extra object had been added, although researchers have been criticised for projecting the idea of number into such situations (see, for example, Sarama and Clements 2009). But typically three-year-old children are able to solve simple subtraction problems by using familiar objects and contexts, just as Jules did above (see, for example, Fuson 1992; Kilpatrick et al. 2001).

Addition and subtraction, though, are only two of the four arithmetical processes. Multiplication is another.

Building a Police Van

Nan and Pa lived 5 h drive from four-year-old Parisa, but used Skype to chat several times a week. Sometimes they read storybooks to each other, and in fact, Nan had photocopied some of Parisa's favourite books so both could look at the words and pictures while she was reading. This time, though, Pa and Parisa were both building a fire engine with Lego: each in front of a computer with a video camera, but 500 km apart. "Now, Parisa, you have a flat white base there. Put that down first. It's not the narrow one: it's four dots wide. Right, good girl. That's an 8 by 4." "I've got a black one too", said Parisa. "Okay, leave that for later. Leave your black 8 by 4, because you need a longer one now. Use the black 10 by 4. The biggest one. Look [holding the piece up to the camera]. It's your biggest one you need. Count the ten dots long, and it's four dots wide. Okay. That's it. Good. Now push that on top of the white 8 by 4." "It's too big. It hangs off"."Great. Put it in the middle. Leave one row of dots each end. That's where the bumper bar will go. Can you find your bumper bar? It's got lights on it. Watch me while I push my bumper bar on. Okay. Then push the back one on the other end. It's a 4 by 1 too, but there's no lights on it. Front 4 by 1 has lights. Okay. Back one. Okay." "Look, Pa. Can we do the wheels now?" Parisa went ahead adding wheels anyway, and then held the base of her fire truck up to the camera. Together they moved on in this manner to add the body, windscreen, and roof of the police van before adding its lights, and little policemen.

With her relative independence in using Skype, Parisa seemed to be developing confidence and competence with skills that will enable her to make use of new communication technologies as they are developed in future years. In this activity with her grandfather, though, she was also getting one type of experience with multiplication arrays. Neither Pa nor Parisa counted the dots on the 8 by 4 piece, so they were not yet using multiplication, but Parisa was receiving valuable learning about the fact that rectangular shapes can have rows and columns. It is unusual for young children to name rectangular arrays, such as "8 by 4", but Parisa's grandfather was a builder who was used to talking about timber as "90 × 45" (mm) or the like, so naming Lego pieces that way came naturally, especially when he had to describe a shape at a distance. Whether it be by builders, other adults, or children of any age, all learning is shaped by context and purpose, being very much situated in specific social and cultural contexts (Lave and Wenger 1991). In fact, Parisa soon started to use that nomenclature herself:

"Pa, push a wheels on. Got them? Black, four: two by two? Good. Now more wheels. Push them all on, Pa".

Here, Parisa demonstrated that she was just starting to understand the idea of multiple groups, with her comment: "… four: two by two. Perhaps this could have been the start of her understanding of multiplication being a product of two numbers, but it was not necessarily so. In this response, though, she did show a very basic understanding of the way her grandfather was using the rows and columns of Lego dots.

Multiplication: Some Research

The most common idea in early learning of the concept of multiplication is repeated groups or sets of objects. Becker (1993) found that many 3–5 year old children could associate the count "one two" with one toy, "Three, four" with the next, and so on (i.e., "2 to 1 mapping"). Typically, they also can confidently put two teddies in each toy car and model other small sets to solve simple multiplication problems (Carpenter et al. 1997; Clarke et al. 2006).

There is little equivalent research, though, on division—other than on fractioning. Sharing is a division context.

Anne and Ruby Share

A four year old, Anne, loved musk sticks (log-shaped sweets), and their delicious smell and flavour makes her weekly treat—buying her own stick—a highlight of each week. She hands her 10c to the shopkeeper, who gives her a musk stick as well as 5c back "… for your money box, Annie".

One day, Anne's cousin was with them for this treat time, so her mother said, "Buy two, Anne. Buy another one for Jess.

"No, she can have mine. Some".

"Okay. That's good sharing. You can give Jess half then".

In the car on the way home, without being prompted further, Anne broke the musk stick close to half, and held the two pieces together then gave Jess the slightly longer piece. The two happy children licked and sucked their treats with delight.

It seems that despite all the situations where children use sharing, there has been very little research on the development of division concepts in young children. Children are usually two years old before they understand the word "share", but that can just mean giving another child time with a toy or other possession: sharing of access rather than sharing of objects. Equal sharing of discrete (separate) or continuous (whole) objects like Anne used involves quite complex actions and concepts, although three year old children may have a good sense of the ideas of "some", "fair share", and then "half" fairly early in practical contexts when these concept are nurtured (Clements and Samara 2007).

Partitioning and Sharing: Some Research

However, transfer between contexts is not easy. Holmqvist et al. (2012), for example, found that three children (age 4, 5, and 6 years) were unable to imagine the shape of halves of a whole cake before it was cut. They watched the cake being cut into two halves, and counted the two halves, but were still unable to say how many halves would be in a whole apple. It appeared to the researchers that the children's knowledge of half, demonstrated with a cake, was not easily transferred to a different-shaped representation. What is more, young children eventually need to be able to find half of one-dimensional (e.g. string, stick), two-dimensional (paper shapes, bread, pizza), and three-dimensional shapes (cake, apples, drinks)—as well as collections of many discrete objects such as cards, pencils, sweets, and game tokens.

While foods (both solid and liquid) are shared between family members or with friends before early school years, the idea of dividing a set of objects into equal groups is usually practised in the first year of school curricula. Nunes and Bryant (1996) explored the ability of five-year-old children to make equal groups—the key division idea—with most of the children succeeding. However, they found that while three- and four-year-old children can usually follow instructions such as “Put two beans in each match box”, the children of that age could not share larger numbers of objects fairly.

The most common division action for young children, however, is sharing of one continuous quantity through breaking or cutting—like the musk stick above. However, in the story below, Ruby’s family is sharing out whole donuts, which are separate (discrete) objects. Nanna and Pop arrived with some donuts one afternoon; and after everyone had eaten one, including three-year-old Ruby. There was one left on the bench. Cinnamon donuts were one of her favourites, and it wasn’t long before Ruby started eyeing off the last one. To distract her from it, her mother told her it was for Pop’s dessert after he’d eaten his dinner. Ruby accepted that and went about playing without another word about the donut.

At the dinner table, as soon as her Pop had put his knife and fork down, Ruby sidled up to him and said, “Now don’t forget your donut, Pop”. She had not only remembered but had waited patiently until he finished his dinner before mentioning it. Then Ruby said, “I’ve been thinking, and have had an idea (all said with one hand on hip and pointer finger to her face), why don’t you get a knife and cut the donut? Then we can both have one”.

As if Pop could resist that logic!

The two stories above both illustrate sharing (division) of objects: first a musk stick (broken in half along its length, to share) and then the donuts (sharing of discrete objects, with a remainder). The two pre-school children solved both types of problems sensibly, even if Ruby’s solution was not fair to the rest of the family.

Children observe different ways of fairly equal sharing of food such as pouring drinks into more than one glass, cutting and distribution of a cake, everyone taking two biscuits or being entitled to one scoop of ice-cream, and sharing a packet of nuts between family members. Games are further excellent contexts for learning about sharing equally: every person needs one token for the board game, seven cards are distributed to each person for a particular game of cards, or perhaps eight tokens are placed on the white spaces for each person to set up a draughts game. In art and craft activities, one pot of paste is dipped into or one ball of playdoh is broken up to distribute between a group of children, or the scissors and paper are “handed out”. Children participating in such contexts soon get used to the idea that distributed groups are “the same” (equal) in many different respects. They can be encouraged to put two shoes in each locker, make three play-dough eggs for each nest, put two cookies and half an apple on each plate, put an amount of fruit juice in each cup, and so on. Such experiences will ready a child for later use of more formal division ideas and calculations. It is important that children also get involved with having to deal with some remainders and that they hear this word in meaningful situations long before they meet them in formal division contexts at school.

Despite a detailed search, I have been unable to find any research findings on the development of division actions and concepts in pre-school-aged children. This is a PhD topic just waiting for a candidate! However, division has been the focus of detailed research with school-aged children. For example, based on their research in schools, Kouba and Franklin (1995) recommend that teachers in grades K to 4:

(a) give children a rich communicative experience with various multiplication and division situations,
(b) evaluate and reward more than just producing an answer quickly in a prescribed way, and
(c) help children build from their own experiences and understandings many ways to represent and model multiplication and division situations (Kouba and Franklin 1995, p. 576). Kouba and Franklin also suggest many classroom activities that are suitable for early years' schooling, but these are not suitable for pre-schoolers.

Peter and Infinity

I have often questioned the claim by Piaget (1928), in his "stages of cognitive development" theory, that children do not become ready to learn abstract ideas until they are about eleven. While it makes sense to proceed from the concrete to the abstract, and as children get older they certainly handle more complex abstract ideas more capably, I have noted many instances where pre-school children discover for themselves and understand quite abstract ideas. One of those ideas is infinity, which leads to my final—and favourite—story.

We had been travelling for a year, and were north of Mt Isa on the way to "The Gulf". Peter was nearly five, and for his pre-school education we used the excellent correspondence materials available for Queensland's children who were travelling or living in remote areas.

When the novelty wore off games and songs during long-distance travel, we used to fill the time with correspondence kindergarten activities that included number play like "What's the next number?"

Me: What's next after fifty-six?
Peter: Fifty-seven. What's after one hundred?
Dad: A hundred and one. What's after a million?
Peter: A million and one. [Pause] What's the biggest number?
Dad: There is none. [Long pause]
Me: That's right, Pete. You can always say the next number
Peter: Like one thousand hundred billion million and one?
Me: Yes.
[Long silence, with Peter looking teary]
What is wrong?
Peter: [Pause] I can't learn to count. I will never be able to count!
Dad: Don't worry. You will understand one day
Peter: I do understand. I know it. [Pause] It is beautiful!

As noted by Tall (2001), “Young children’s thinking about infinity can be fascinating stories of extrapolation and imagination [and to] capture the development of an individual’s thinking requires being in the right place at the right time” (p. 7). I have argued elsewhere (Mousley 1999) that out of multiple experiences with specific examples, young children are capable of further (and ultimately more important) levels of abstracted understanding as well as understanding about self in relation to mathematics learning. In very early forms, both of these aims were illustrated by young Peter, above. The former involved a jump from the “game”, with many kilometres of specific examples where higher numbers were always possible to his realisation of the abstract idea that counting numbers are infinite. The latter was demonstrated by his personal engagement with a mathematical principle that seemed to this four-year-old to be quite beautiful. As Tall wrote, parents and teachers “should be aware of the surprisingly sophisticated and complex ideas of the young that deserve to be treated on their own terms with respect” (p. 19).

Abstraction: Some Research

Tall’s (2001) research was with children who were seven or older—and mainly with his son Nic as he matured and further developed a capacity for abstract thought Abstraction in young children was described well by Schoenfeld (1986), who identified levels of children’s number understanding as development that moves from (a) understanding of objects and operations on these, to (b) understanding of symbols and operations on these (arithmetical processes). As numbers get too large to model easily, working with abstract ideas becomes inevitable, but also more powerful and hence more beautiful—as young Peter realised. I have no idea whether he thought of infinity as an enormous number, but his comment that he would “never be able to count” suggested that he had a better understanding of infinity than that.

Conclusion

It is clear that curious little children, through varied prior-to-school experiences such as those above, develop number understandings and skills that will be invaluable for understanding mathematical content in the first few years of formal schooling as well as attitudes that will also support their later learning of mathematics.

Fuson et al. (2015) noted that “children are inherently driven to learn because of the feelings of competence that result” (p. 63) but they also seem to need the interest, feedback, and approval of family members and carers. These researchers also debunk the play versus learning dichotomy, noting the importance of “guided play that is initiated by an adult and therefore can support educational goals" (p. 64).

Earlier in this chapter, I listed a number of questions that I hoped to: "What do we know about the mathematical development of young children in respect to number and simple arithmetic concepts? How can children's storybooks and familiar objects be used here, and how can parents draw out the mathematics in everyday events? What roles might some technologies play? How can various types of games be used to promote pre-school children's mathematical development? How can we make the most of young children's passionate interests and hobbies as well as various family routines? What are the big ideas of number content that children can learn, before going to school, that will underpin their mathematical learning and initial achievement at school?" All of these questions have been attended to in part, but certainly not completely. Overall, I have touched on cardinality, subitising and counting, addition and subtraction (but not so well multiplication and division). Hopefully readers will take up the challenge of providing more detailed answers through their own interest and active engagement. In this chapter I have focused only on number ideas, while there are other stories I could tell about my observations of what young children learn informally about time concepts and other measurement ideas and skills, shapes and their properties (geometry), chance (probability), and data representation.

I am not a great believer in positive effects of lecturing parents or early childhood professionals about research implications or about what they should and should not do, but can point out some common themes in the stories above. One is that the adults listened to and observed very young children and thought about the mathematics in a great variety of everyday activities and contexts. Another is that they communicated with the children about that mathematics, asking questions, extending knowledge, and reinforcing learning as it was happening. Importantly, engagement in such experiences and conversations with adults gives children a sense of belonging as well as a sense of self-respect, recognition, achievement, and affirmation. The stories above are true examples of natural curiosity and desire to learn being cherished and temporarily satisfied, along with children's wishes to please parents and other significant people in their social environments. These factors provide motivation for persistence and for mastery of new words, ideas, and actions as well the growth of children's broader knowledge, interests and strengths.

Very young children's strategic mathematical thinking continues to delight me. I wish all children could grow up in nurturing environments like the lucky little ones above who were developing their mathematical understandings with the aid of significant others and of stimulating contexts.

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