Chapter 5
Discerning and Supporting the Development of Mathematical Fundamentals in Early Years

Camilla Björklund and Niklas Pramling

Abstract A large body of research shows that young children have abilities to discern small amounts and changes in quantities, and reason about mathematical relationships they encounter in everyday situations. How these early abilities are allowed to develop is contingent on the child's network of social interaction and how mathematical notions and principles are introduced and made sense of in mutual activities. Key insights from educational theories contribute with a basis for how to provide ample opportunities and support for the child to discern important principles (relationships and distinctions) of a mathematical nature, particularly how to communicate with children in a developmental way. In this chapter, we analyse a number of everyday activities with a child in his home environment during his first 6 years of life. These observations allow us to illustrate how mundane activities can provide the basis for gaining access to, and supporting the further development of a child's mathematical abilities in interaction with adults and peer.

Keywords Conceptual development · Communication · Socio-cultural theory · Variation theory · Discerning

Introduction

In this chapter, we analyse a number of everyday activities with a child in his home environment. The observations span a period from when he was 1.9–6.2 years. These observations allow us to illustrate how mundane activities can provide the basis for gaining access to, and supporting the further development of a child's mathematical abilities.

The background of our discussion is cognitivist and developmental research, claiming that children are born with the ability to discern small amounts and

C. Björklund (✉) · N. Pramling
University of Gothenburg, 405 30 Gothenburg, Sweden
e-mail: camilla.bjorklund@ped.gu.se

S. Phillipson et al. (eds.), *Engaging Families as Children's First Mathematics Educators*, Early Mathematics Learning and Development,
DOI 10.1007/978-981-10-2553-2_5

changes in quantities (Lipton and Spelke 2003; McCrink and Wynn 2004; Starkey et al. 1990; Wynn 1998; Wynn and Chiang 1998). There is a large body of research on infants and children below the age of one on these matters. It is argued that these abilities—which may be referred to as intuitive—constitute the origin of arithmetic abilities. But how these abilities develop are contingent on the challenges and support provided and the expectations held in the environment and the culture (Aunio et al. 2008). Since very young children do not express their abilities or understanding in arithmetic terms, it can be difficult for adults to discover children's intuitive abilities. However, observations of young children's self-initiated activities show that they discern mathematical relationships and explore mathematical concepts in every-day situations, routines and in communication with peers and adults, making coherence and sense on the basis of their earlier experiences (Björklund 2007, 2010).

There are key mathematical principles that children need to discern in order to develop mathematical abilities. These include: the one-to-one principle, abstraction, the cardinal principle, parts-and-whole relationships, and base-ten operations. The development of conceptual understanding in mathematics is broader than merely procedural skills such as counting or reproducing number facts. To become a competent user of numbers and a skilled arithmetic problem solver, the child needs to master the conceptual bases. This development starts in the early years of childhood, as will be illustrated and discussed in this chapter. All empirical examples we present, analyse and discuss in this chapter are based on transcribed activities of the interaction and communication of a focus child and his family members, primarily his mother.

In addition to reporting and analysing these examples, we will present and discuss some key insights from educational theory. These principles will be of two kinds: those that concern how to provide ample opportunities for the child to discern important principles (relationships and distinctions) of a mathematical nature; and those that primarily concern how to communicate with children in order to provide developmental opportunities and support. These principles are based on Variation Theory of Learning (e.g., Marton and Tsui 2004) and Sociocultural Theory (e.g., Daniels et al. 2007), respectively.

A Theoretical Framework for Understanding Learning

Every study, as well as every form of understanding and support of children's learning, presumes some idea of how learning comes about, what triggers development in thinking, and how learning and development may be facilitated. There are many theories on learning, but we will discuss mathematical learning and development in accordance with the Variation Theory of Learning, since this theory offers concepts that are functional in studying the process of learning and conjectures how learning and concept development are informed by providing awareness-raising patterns. Informed by this theory, the learning of mathematics is a matter of discernment of increasingly more aspects of mathematical phenomena, where earlier experiences play a central role for what is perceived in a particular situation. Learning then follows some kind of trajectory, but in a broad sense, where certain aspects are considered critical for developing understanding. This theory has been proven to be powerful as a theoretical framework in empirical studies of learning and concept development among both older and younger children (Björklund 2010; Björklund and Pramling 2014; Ljung-Djärf et al. 2013; Magnusson and Pramling 2011; Reis 2011). These studies have contributed to the development of an understanding of the complexity of concept development and also to a framework for how to support this development through pedagogical activities.

Variation Theory (Marton 2015) conjectures that learning means to see or experience some phenomenon in the world in ways that a person has not previously been able to. Depending on the person's earlier experiences of similar phenomena, and the aspects of the phenomenon that are provided in a particular situation, a certain way of understanding the phenomenon takes shape. Some aspects are considered critical to discern, and are foregrounded in studies of how a person develops his or her understanding. Variation Theory is particularly powerful when studying the development of mathematical concepts, such as number concepts or arithmetical principles, since there are many dimensions to the understanding of a mathematical concept and aspects of the same that are necessary to discern. Take for example the notion "five". In order to understand and use this concept, it is necessary to be aware of the numerical dimension, in that it is answering the question "How many?", an ordinal dimension, meaning that five has a value in relation to other numbers (always before six and after four in the counting rhyme), but also a non-numerical dimension in that numbers are used to label unique phenomena without any numerical values (e.g. phone numbers and registration plates on cars).

These dimensions, and possible others, contain several aspects that are necessary to discern and account for, aspects that the learner may have encountered earlier in other situations or are provided in a particular situation. In order to learn the numerical dimension of numbers, numerical relations have to be discovered, these relations include those that are possible to quantify, and may be grouped and counted. Another numerical aspect has to do with quantities that are possible to compare and subsequently make a series of their numerical values of (a group of two compared with a group of three objects, followed by a group of four, and so on). Variation Theory conjectures that such aspects are necessary but can only be discerned in contrast. The number five for example, has no numerical meaning to learners if they hear the word "five" when seeing a hand, a foot, and cars on the parking lot, apples in a bowl, and children in the playground. However, when five apples are contrasted to a group of four apples, the numerical idea may be possible to (first and foremost) become aware of, and further to explore and develop an understanding of. Only thereafter, according to Variation Theory of Learning (Marton 2015), is it possible to generalise the meaning of five and discern the "fiveness" that can be found when observing a foot, a hand, a group of cars and so on. Studying learning and development in terms of Variation Theory means to focus on those aspects that are present in a situation and those aspects that the learner has not yet discerned; and, further, how the latter aspects may be put into play in interaction and communication.

According to Variation Theory, every concept is possible to differentiate as aspects. Teaching, then, means offering the learner opportunities to discern such aspects that are important but not yet discerned by him or her. Learning is then made possible through a carefully orchestrated act of differentiating the aspects and contrast for discernment. The complexity of conceptual understanding becomes apparent in communication with young children. Most teachers and adults can recall episodes where the child and the adult give very different answers to the same question, such as when counting out loud and pointing at items together with a child, and the child suddenly protests, saying "This is not four, THIS is four", pointing at another item. The child has then not yet discerned the idea that it does not matter which item you start to count, the total quantity is independent of the order the items are counted. Variation Theory highlights the necessity of paying attention to which aspects that are at the centre of attention to the child, and which are not, as these not yet discerned aspects are those that need to be differentiated and generalised by the child.

Mathematical concepts are complex in their nature, not least because mathematics cannot be described as a physical object to be found. Rather, it is a collection of knowledge, tools, and principles to investigate, understand, and handle relationships in time and space. This leads to a challenge, as mathematics cannot be "seen" physically. Mathematical notions describe relationships between visible objects, but also open opportunities to be explored by very young children who encounter these mathematical relationships in their block play, dividing and sharing fruit, running and climbing in the playground—and actually most activities a child engages in offer experiences that may be described as mathematical. However, to develop mathematical skills and knowledge presupposes paying attention to those experienced mathematical relationships. Teaching in an early childhood education context means to make the invisible visible to the child (Pramling and Pramling Samuelsson 2011). In line with this reasoning, to teach mathematics in early childhood education means directing the learner's attention to experienced mathematical relationships and supporting further exploration of that relationship. A six-year old may have experiences of patterns as the idea of ordering objects regularly by colour. When encountering a peer's pattern making in the form of ordering objects by size, his perception of the idea of pattern is expanded and the aspect of regularity is possible to explore if attention is directed to the common relationship of the two different patterns. The latter is crucial in concept development, as it makes possible the discernment of abstract relationships and idea of the broader concept, rather than just adding another example (Björklund and Pramling 2014). Variation Theory here contributes to our understanding of the processes and puts a focus on what is necessary to discern in order to develop a deeper and more complex way of understanding common notions and principles.

Examples of Opportunities for Learning

We are, in the following, continuing the discussion on mathematical learning and development through some examples of a young child's mathematical encounters and exploration in communicative situations. These examples are then put in a broader context of research on how we understand mathematical reasoning today, informed by an extensive body of research on mathematical development.

The child we are following is Vidar, a boy living with his older sister, mother and father in the outskirts of a larger city in Sweden. All examples are authentic observations that were documented by his mother. The observations we analyse were documented during a period from when Vidar was 1.9–6.2 years.

Counting Without Numbers

When studying young children's mathematical development, counting and using numerals is a fairly late achievement. Most two-year-olds know some counting rhyme, "one, two, buckle my shoe; three, four, open the door…", but they are rhymes like any other rhyme, devoid of any numerical meaning (Fuson 1992). Research on cognition and developmental psychology claim that children are born with abilities to discern small quantities and changes in quantities, such as subitising and arithmetical expectations, abilities that are present long before any number words are uttered by the child (Mandler and Shebo 1982; McCrink and Wynn 2004; Wynn 1998). These abilities are innate and intuitive. However, the direction in which these abilities are developing depends on both the physical and social environment; in other words, what the child needs to learn to survive and participate in his or her community and what is expected and considered valuable knowledge (Aunio et al. 2008).

Intuitive attention to quantities is considered by many researchers to be the basis for learning to use arithmetical principles and strategies. Still, these abilities are not easy to recognise in the young child's exploration of the quantifiable world, because their reasoning is often expressed other than verbally. Take, for example, the idea of counting. Counting is a very practical strategy or tool for determining the number of objects in a group. However, there are many aspects of the counting skill that are explored in early childhood in situations that will not necessarily involve number words. Gelman and Gallistel (1978) conceptualise counting as a skill that is constituted by five principles, necessary to grasp if the child is to use counting for problem solving and communication about quantities. Two of these principles do not involve number words, but are equally important: one-to-one correspondence and the principle of abstraction. The first means to make pairs of units from different groups, combining units into new "pair-units". The principle of abstraction is critical for acknowledging phenomena as units that may be part of a larger whole, independent of their nature or features. Consequently, the child may explore these two principles when grouping cats and dogs, then giving each peer one toy. Another example is to match one hand clap to the beat of a drum or symbol on a schema. No number words are needed, but the principle of abstraction and one-to-one correspondence are crucial for learning how number words are representing quantities.

Episode 1

Vidar (1 year, 9 months) sits on the sofa with a picture book in his lap. The book has tactile surfaces. On page one there is a train and underneath the caption reads “1 One train”. The second page has two chickens in the picture with the text “2 Two chicken”. The following pages, up to the tenth page, present different objects in a similar way. Vidar browses through the book. The fourth page has four flowers in different colours with different numbers of petals. He points with his index finger on the flowers, one by one, saying “Mommy, Daddy, Nea, Dida”. [He calls his older sister Nea and himself Dida at this point in time.]

This example is interesting from the point of view that Vidar is reasoning about relationships and uses the principle of one-to-one correspondence in his “reading” of the book. We cannot tell why he connects each flower with the names of family members; it could be an idea of giving each member one flower or naming the flowers with labels of family members that he is familiar with. Nevertheless, he expresses an intuitive awareness of principles that are important for counting procedures (one-to-one, compare with one number word uttered for every counted object).

Three months later, Vidar uses his powerful principle in another situation, but encounters a challenge.

Episode 2

Vidar (2 years, 0 months) sits on the bathroom floor. He is to put on his sock but starts to play with his toes. He grabs his big toe and says “Daddy”, then the next toe, saying “Mommy”, then “Nea” and “Dida” for the following toes. Vidar looks at his foot quietly for a while. Then he grabs his little toe and says “Kitty” with a big smile.

Vidar uses his strategy of pairing one item with one label (or one-to-one correspondence between different groups of objects). However, the strategy does not end up with the same result as in the earlier episode. The number of toes does not match the number of family members. Vidar discerns the difference in number only when the conflict makes him aware of the numerical dimension, which triggers him to direct attention to the experienced difference and how to deal with the problem. Wynn (1998) has shown that toddlers, even younger than Vidar, express so called arithmetical expectations, meaning that a quantity that is added items is expected to change in number. This is probably one of the critical abilities for developing skills to elaborate deliberately with quantities. As seen in the example with Vidar, he experiences the contrast between the expected pairing activity and the outcome of the pairing, which directs his attention to the numerical relationship between the two groups of units and challenges him to "see" the groups in another way. He then adds a unit, "Kitty", and is seemingly satisfied when the pairing act evens up.

Numbers as Collections of Units

There is a strong belief among most researchers on mathematical learning that the part-whole relationship is fundamental for developing conceptual knowledge of numbers (Doverborg and Pramling Samuelsson 2000; Dowker 2005; Hunting 2003; Piaget 1952). Such knowledge is further considered important for the use of procedural skills and retrieving number facts for powerful arithmetical strategies. Conceptual knowledge of numbers arises from experience with numbers as a collection of differentiated units. The number five, for example, will be perceived as a whole with parts that may be combined in different ways, still constituting the whole of five. Five may further be seen as a unit of a larger whole. The ability to perceive the parts and the whole simultaneously is, according to Variation Theory, a key condition for learning, since the ability to fluently elaborate with numbers as collections of units, according to Marton and Neuman (1996), provides the child with powerful counting strategies, which children with mathematical difficulties often lack. However, the idea of numbers as collections of units may be expressed in different ways and younger children experience the part-whole relationship even though they may not use number words to express the discerned relationship. In the following example, Vidar encounters dots as units and is challenged to explore the part-whole relationship.

Episode 3

Vidar (3 years, 2 months) has one dice in his hand. He opens a drawer and finds two more. He says "Look, the same". His mother asks: "How many do you have?" Vidar points with his right index finger on each dice, saying "One, two, three; I have THREE". He throws the dice on the floor and his mother asks: "How many dots are there on the dice?" Vidar points at all dots on top of all three dices and says "One dot, one dot, one dot, one dot, one dot, one dot. That many."

This example is classic in the way Vidar, as a 3-year old, deals with number and quantities. He perceives objects of different kinds as countable items, that is, that both dice and dots may be enumerated. Furthermore, he knows what is expected when someone asks "How many?" and gives an answer that would be interpreted as an expression of the cardinality principle (Gelman and Gallistel 1978) when he accentuate the notion three, including all the previously counted items into a whole. The following question, to count all the dots on the dice, challenges his part-whole perception, since the dots are situated on different objects but are still perceived as units of dots. In other words, he differentiates the dots as countable items from the physical objects. His verbal utterance further reveals that number words above three are not his primary choice for enumerating larger quantities. Piaget (1952) would suggest that this is an expression of early arithmetic thinking, when numerating as one, one more, one more and another one. The challenge is to perceive all those "ones" as parts of one collection, simultaneously. Number words may contribute to this change of focus, from "one dots" to "six-including-all-of-the-dots" in the same way Vidar is expressing his understanding of the notion "three". This example reveals that mathematical principles are complex and not easily generalised and transferred to larger quantities. In accordance with Variation Theory, several aspects have to be discerned by the child to develop numerical understanding and the skill of using numbers in problem solving and communication. Those aspects that the child in particular need to discern (but has not yet discovered) are called critical, meaning that they should be in focus for teaching acts and challenged in that the child may broaden his or her understanding. In the episode above, cardinality is one aspect that is central and that Vidar shows knowledge about, but the part-whole aspect becomes the critical one.

Learning to count and calculate are closely related to how the child represents quantities and is able to mentally model different possibilities to solve a problem, without losing important information and relationships between the parts and the whole. In the next episode, Vidar explores number concepts and how numbers relate to each other. The collection of units becomes critical to remain constant in the task he encounters.

Episode 4

Vidar (5 years, 4 months), Linnea (8 years, 8 months) and their mother play a dice game where one is to collect fish. Vidar gets two on the dice, moves two steps and draws an event card. The card states that the player loses a fish on the last spot he was standing (from where the last move was made). Together with his mother he counts backwards: "Two, one, that's where it stood" and places a fish on the board. The game proceeds and when it is his turn again, Vidar gets a three on the dice. He says: "I want that fish" and his mother answers: "If you get to that point again, you can take the fish". Vidar takes two steps, stops at the spot where the fish lies, looks at it and asks: "Can I take it?" Linnea then explains that he must stop on the final counted number. Vidar suddenly moves his piece to a spot three steps from the fish and says: "I wanna start here instead!"

In this episode we can follow Vidar's reasoning in a task that is quite complex in character. He simultaneously keeps a specific position as a target, while he explores possibilities to solve the problem of getting the desirable fish by taking three steps (that is different from two steps). The number of steps has to be held constant in relation to the target spot. This might seem like an easy task to adult reasoning, but the task involves calculation with unknown components, which is quite demanding for a 5-year-old and demands mental representations of numbers as differentiated units. The task involves many aspects that need to be focused on simultaneously, such as the number line and the part-whole relationship of numbers. The discerning act is, however, supported by the context of the board game, which provides props and structure that are meaningful and supportive for the specific task.

Large Numbers

The need to handle large quantities has provided humanity with systems and structures that help coordinating and communicating these quantities. There are different systems in different cultures, but the English and Swedish systems are similar in that they use ten as basis for structuring numbers. In order to handle large quantities, the base-ten system is valuable as a mental structure, meaning that the child does not have to learn individual labels for every number there is. Many preschool children learn the counting rhyme up to twenty or even one hundred. The transition from one ten to the next is challenging, often leading to individual number lines such as "twenty-eight, twenty-nine, twenty-ten …" Once children discern the structure of the counting rhyme, "twenty-nine, thirty, thirty-one …," the structure of numbers and how numbers relate to each other become powerful tools for handling large numbers and quantities, as we can see in the following example.

Episode 5

Mother	It is grandpa's birthday today
Vidar (5 years, 5 months)	Why?
Mother	He was born on this day of the year
Vidar	How many years will he be?
Mother	64
Vidar (after a pause)	Then he's 63. And will become 64

Like many other children Vidar is well aware of the fact that birthdays mean you become one year older. The same is probably true for grandfathers as well. To reason in the way Vidar does, he has to discern numbers as related to other numbers and even though the numbers are large, adding one is possible to figure out as units on a number line.

Young children strive to make sense of phenomena they encounter in their daily activities. Numbers are no exceptions. But there are many aspects that need to be discerned in order to handle the structure and system of large numbers. The base-ten structure is known to be important, and the language may also provide clues to the children. However, the number concepts used by children may be based on a different logic than the conventional meaning expected by adults. The next example shows what Vygotsky (1987, 1998) refers to as pseudo concepts, where children and an adult may use the same words and procedures, but their understanding of these terms are different. A closer analysis of the language used in the counting rhyme reveals the child's logic and what aspect that is focal to him, guiding his reasoning.

Episode 6

Vidar (5 years, 7 months)	Thirty plus thirty is sixty
Mother	Yes, it is
Vidar	And thirty-one plus thirty-one is sixty-two
Mother	That's correct. How did you figure that out?
Vidar	First, one plus one is two and then thirty plus thirty is sixty
Mother	But what is thirty plus thirty-one?
Vidar	That's easy, sixty-one
Mother	Can I ask you one more? What is twenty plus twenty?
Vidar	THAT I don't know

Vidar knows how to add numbers up to ten. He sometimes uses his fingers but is quite fluent in retrieving answers like number facts. He applies this strategy to large numbers as well, and seems to have discerned that ones and tens have to be separated in the act of addition. The Swedish language provides support for this as well, since thirty ("trettio", in Swedish) is a clear combination of three ("tre") and ten ("tio"). The same structure is found in sixty ("sextio", in Swedish), a combination of six ("sex") and ten ("tio"). However, twenty ("tjugo", in Swedish) does not reveal a similar relation to two ("två"). An aspect that becomes critical in this episode is the numerical meaning of large numbers. In other words that thirty as well as twenty refer to quantities. It is interesting though that he has revealed an aspect that refers to the linguistic resemblance and ten-base structure of the number line.

In retrospect, Vidar's experiences of mathematical concepts and principles draw a picture of mathematical development that is quite representative for children of his age. We can follow the earliest exploration of part-whole relationships and discoveries of equality and inequality between sets of objects, via the milestone of relying on the perception of quantities versus the idea of counting, to the emerging control of the base-ten structure in conventional calculating and verbal counting on the number line. Conceptual knowledge as well as procedural skills are intertwined in children's mathematical activities, but procedures may be used in more powerful ways when they are based on a conceptual understanding, such as knowing how to make even sets by adding units or that "adding one" works for small numbers as well as large. These knowledge and skills are prerequisite aspects for developing more advanced and abstract mathematical thinking, also shown by Dowker (2005) and by Gray and Tall (1994). This is exposed, for example, when Vidar explains how he adds thirty-one with thirty-one, and uses the same conceptual idea to other arithmetic tasks. However, he fails to adapt the idea to "twenty", which does not have the same linguistic clues; which reveals that conceptual knowledge is complex and builds upon several aspects of the logical reasoning that is necessary for mastering procedures like addition. To become aware of the conceptual foundations in mathematical principles and concepts, these have to be discerned by the child. Such an occurrence does not in general happen in isolation, rather in encounters with other ways of interpreting notions and different ways of drawing conclusions. The theoretical framework Variation Theory helps interpreting what it is that a child focuses on in a particular situation, and what is not yet discerned by the child and thereby crucial for any teaching attempt to account for.

Communication and How to Support a Child's Mathematical Development

For an adult or another more experienced participant to contribute to the child's mathematical development highlights issues of communication. A first principle is to access the child's understanding. That is, without finding out what the child knows and how he or she understands, it becomes very difficult to support further development. Without relating one's support to the child's understanding, the more experienced will risk giving suggestions and asking questions that the child cannot relate to and make sense of, or that are too simple and therefore do not provoke new insight. The episodes represented give ample examples of how the adult gains access to Vidar's understanding and then not only confirms (supports) this understanding but also challenges him to consider more complex variants (see, for example, Episode 6).

It could be argued that education is at heart a communicative endeavour. However, how we communicate is contingent on what we understand communication to be, that is, what our—generally implicit and un-reflected—notion of communication is. A common-sense notion of communication depicts it as the transmission of information from a sender (knower) to a receiver (learner). Traditional schooling practices where a teacher lectures to children listening could be seen as an institutionalisation of this notion of communication (see, for example, Wells and Arauz 2006, for a discussion), and its related notion of knowledge as having information. However, there is a long and large research literature on learning showing that this notion of knowledge and how to promote it through communication is counter-productive (Pramling 1996; Sommer et al. 2010). Instead, other ways of understanding communication and knowledge have been conceptualised and investigated in educational research. In these accounts, communication is instead seen in terms of the etymology, that is, the origin and development of the term, as "making common" (Barnhart 2000). To make something common is fundamentally different from one individual transmitting information to another who receives and stores it. To make common presumes some negotiative work; that is, to try to coordinate perspectives and understanding among participants. Hence, communication becomes a shared project rather than a one-way process (Pramling and Säljö 2015). The more experienced peer—a teacher or an adult—thus needs to be responsive to the response of the child. Such consecutive, unfolding responsiveness can be seen in the episodes in that the adult adjusts her questions to Vidar's suggestions. Her participation is not imposed from without but sensitive to what the child expresses and what this indicates in terms of mathematical understanding.

With this changed understanding of communication, also how we understand knowledge can be rethought. Instead of merely seeing knowledge as information acquired and stored in the mind of the individual, knowledge can be understood as membership in a cultural form of knowing (Dewey 1916/2008). Mathematics is a prevalent and powerful cultural form of sense-making and communication. Children are empowered through becoming members of this culture. This implies more than knowing of how to count: it also involves the learner's notion of self, that is, identity. Developing an identity as mathematical is an important part of becoming mathematically skilled, being able to take on mathematical tasks or take on problems in mathematical terms. A parallel case is the learner's notion of literacy. Studies have shown (e.g., Dahlgren and Olsson 1985) that whether learners who develop an identity as someone for whom being able to read are of interest and relevance, become more skilled at this form of sense-making and communication than children who do not see the relevance of this skill and instead see it as something they have to learn in order to manage school. Arguably, children developing an identity as someone who is mathematical, that is, can see the use and relevance of using mathematics in everyday life, will have a developmental advantage in this respect. The presented episodes show how mathematics comes into play during the course of various everyday activities, such as dressing (Episode 2), playing games (Episode 4), and conversing about family matters (Episode 5).

The understanding of communication as making common highlights the importance of coordinating perspectives; that is, making sure that participants speak about the same thing, not only terminologically but also conceptually. A common feature of face-to-face communication is the use of local terms—what is called "deictic references" (Ivarsson 2003, p. 387). Some examples of such terms are 'that', 'there' and 'this', often accompanied by pointing. These kinds of words are useful in directing and coordinating attention. Language is our most powerful tool for directing someone's attention (Tomasello 1999). However, while these deictic terms may work in coordinating the adult's and the child's attention to the same objects, these objects may be understood by the two in conceptually distinct ways. For example, if sorting objects, adult and child may agree that those are similar to those, or that there is a pattern among the objects. But what these similarities and what this pattern is may be understood differently among the two. For the adult, these may be similar geometrical shapes while for the child the focus may be on the colour or kind of object. If so, adult and child only appear to share perspectives, while they in effect talk past each other. In terms of Variation Theory, they have different aspects of the phenomenon in the foreground of their attention. This makes it difficult for the adult to provide developmental support, for example, using the principle of variation (see above) to challenge the child further and facilitate his or her discernment.

Conclusion

Mathematical principles and concepts are explored and developed by children long before they encounter formal schooling. This is a fact that Vygotsky (1978, 1987) made clear in the early 1930s. Family and friends are in this respect the child's first mathematics teachers, since it is in the interaction and communication that mathematical notions are made common and the child is enabled to make sense and implement powerful problem solving strategies to daily life challenges.

In discussing the five episodes in terms of communication and teaching, we reason per analogy. In his pioneering theoretical work, Vygotsky (1978), among many other things, wrote about how to promote the development of literacy in children. He argued that this is preferably done during what he referred to as the preschool years, and that "writing must be 'relevant to life'", taught "as a complex cultural activity" (p. 118), not writing "taught as a motor skill" (p. 117). Furthermore, he argued the importance "that writing be *taught* naturally … and that writing should be 'cultivated' rather than 'imposed'" (p. 118, italics in original).

> [T]he best method is one in which children do not learn to read and write but in which both these skills are found in play situations. … Natural methods of teaching reading and writing involve appropriate operations in the child's environment. Reading and writing should become necessary for her in her play (Vygotsky 1978, p. 118).

If applying this line of reasoning to our present concern, that is how to promote children's mathematical skills, we can argue that the episodes we have represented and analysed constitute precisely such conditions. For example, in Episode 4, the focus child, Vidar, his older sister and his mother all engaged in a mutual activity: playing a game. Mathematics emerges as a relevant tool embedded in this play activity. Being able to count is vital to being able to participate in this activity and thus provides an incentive for engaging in such matters. Actualising mathematical distinctions and relations in such a context is very different from traditional schooled instruction. Motivation is inherent in such activities (Lave and Wenger 1991). In the context of play, mathematics is, in Vygotsky's terms, cultivated rather than imposed.

As we have already mentioned, one of many important actions people carry out through speaking is to direct someone's attention (Tomasello 1999). Often participants in a practice do so through employing deictic references. An example of this can be seen in Episode 3, when Vidar exclaims, “Look, the same”. In this way he makes the interlocutor attend to what he himself has noticed. He does so through an expression that functions as a pointing gesture. What is further important in this case is how the interlocutor responds to this verbal gesture. His mother asks him “How many do you have?” In this way, she not only implicitly acknowledges that she notices what he is focused on, she also formulates her response in mathematical terms. In this way, Vidar and his mother came to share not only attention but also perspectives on what they attend to. (Consider, in contrast, if the mother had replied, for example, “That’s nice”, “They’re beautiful”, or “Yes, green ones”.)

In terms of sociocultural theory, what mother and child here do constitute what they speak about in certain terms; they semiotically mediate (Wertsch 2007) what they speak about. Establishing not only joint attention but also mediating what is spoken about in compatible terms have been shown to be pivotal for participants (e.g., mother and child, or child and preschool teacher) to engage in a mutual activity, rather than parallel ones (see Pramling and Pramling Samuelsson 2010, for examples with very young children). Being participants in the same activity provides a frame for the more experienced, such as the parent, to contribute to furthering the child’s understanding through supporting and challenging him or her. This goes hand-in-hand with the Variation Theory framework for understanding learning, as different perspectives and new ways of seeing the world constitute necessary contrast for developing understanding (see Marton 2015). As we have shown in this chapter, common everyday practices such as dressing, conversing about family members or playing games provide entry points into supporting a child’s mathematical development.

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