

Consensus of Linear Multi-agent Systems with Persistent Disturbances

Shaoyan Guo, Lipo Mo and Tingting Pan

Abstract This paper focuses on the consensus problem of continuous-time multi-agent systems with persistent disturbances. A distributed protocol is designed, which consists of two parts, one is the traditional control protocol, the other one is the estimation of disturbances. Then, using the method of matrix analysis, the sufficient conditions for achieving consensus of the closed-loop systems are found out. Finally, simulations are provided to demonstrate the effectiveness of the proposed algorithm.

Keywords Multi-agent system · Consensus · Disturbances · Control protocol

1 Introduction

Multi-agent systems have the characteristic of autonomy, distribution, and coordination, and have the ability of self-organization, learning, and reasoning. Multi-agent systems are efficient to deal with the practical systems, such as the formation flight of the UAV, multi-robot systems, and so on [1, 2]. More and more attentions have been paid on cooperative control of multi-agent systems in recent years.

The consensus problem of multi-agent systems is one of the most fundamental issues. Starting from the Vicsek model [3], a broad spectrum of scholars are much more kindly to study the consensus problems of multi-agent [6] systems with different characteristics. For example, the consensus problems of discrete-time were investigated in [4, 5]. For the continuous-time multi-agent systems, consensus problems were discussed in [6, 7]. It is shown that the consensus of first-order systems can be achieved if and only if the network topology contains a directed spanning tree. And then these results were extended to stochastic switching systems [6], some average consensus conditions were obtained. All of these results were given for the first-order multi-agent systems. In practical systems, the control objects may be accelerated velocity rather than velocity and the methods can not be applied to second-order

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systems straightforward, so it is meaningful to investigate the consensus problems of second-order multi-agent systems. In [2], it shows that the second-order systems might not achieve consensus even if the network topology has a directed spanning tree. And a necessary and sufficient condition was given for the consensus of second-order systems with directed topologies. Recently, the consensus problems of linear multi-agent systems were also considered. In [8], it was proved that the consensus can be reached if and only if all of the nonzero eigenvalues of the Laplacian matrix lie in the stable regions.

In practical systems, it is inevitable that the system can be affected by external disturbances, so it is important to discuss the consensus problem of the multi-agent systems under disturbance. In [9–11], the H_∞ is used to solve the consensus problem under disturbance. To attenuate the communication noises, a distributed stochastic approximation type protocol is also adapted. Using probability limit theory and algebraic graph theory, consensus conditions for this kind of protocols are obtained [12]. In [13], a new controller is proposed to solve the consensus problem of the multi-agent systems under unknown persistent disturbances. In [14], The stochastic consensus problem of linear multi-input multi-output (MIMO) multi-agent systems (MASs) with communication noises and Markovian switching topologies is studied by designing consensus protocol. In [15], the consensus problem of second-order discrete-time multi-agent systems with white noise disturbance under Markovian switching topologies is discussed. And for more consensus problems of the multi-agent systems under disturbance, refer [16–18]. However, to the best of our knowledge, the consensus problem of the linear multi-agent systems with constant persistent disturbances have not been discussed, this paper we focus on this problem. The stochastic consensus problem of linear multi-input multi-output (MIMO) multi-agent systems (MASs) with communication noises and Markovian switching topologies

The main contribution of this paper is that sufficient conditions were obtained for the consensus of linear multi-agent systems with persistent disturbances. Based on the graph theory and matrix theory, the consensus protocol was designed and the consensus state was also obtained. Comparing with the literature, the result herein is more simple and general, and it is easy to verify in practical engineering systems.

2 Preliminaries

An undirected graph \mathcal{G} is defined by a set $V_{\mathcal{G}} = \{1, \dots, N\}$ of nodes and a set $E = \mathcal{E}_{\mathcal{G}} \times \mathcal{E}_{\mathcal{G}}$ of edges. If $(i, j) \in \mathcal{E}_{\mathcal{G}}$, then the node i and j are neighbors and the neighboring relation is indicated with $i \sim j$. The neighborhood $N_i \subseteq V$ is denoted the set $\{v_j \in V | (i, j) \in E\}$, then the degree of a node is given by the number of its neighbors. Let d_i be the degree of node i , then the degree matrix of a graph \mathcal{G} , $D \in \mathbb{R}^{n \times n}$, is given by $D = \text{diag}\{d_1, d_2, \dots, d_N\}$, the adjacency matrix of a graph \mathcal{G} , $A \in \mathbb{R}^{n \times n}$, is given by $A = [a_{ij}]$, if $(i, j) \in \mathcal{E}_{\mathcal{G}}$, $a_{ij} = 1$, otherwise $a_{ij} = 0$. And the Laplacian matrix is given by $L = D - A$. By the definition of Laplace matrix, we

can obtain the spectrum of the Laplacian matrix for a connected, undirected graph can be ordered as $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$. And 1_N is the eigenvector belongs to the zero eigenvalue λ_1 , and $\mathcal{L}1_N = 0_N$ where 1_N denote the $N \times 1$ vector of all ones.

Lemma 1 *Let A, B, C, D are constant matrices with proper dimensions. Then*

$$A \otimes (B + C) = A \otimes B + A \otimes C,$$

$$(A + B) \otimes (C + D) = A \otimes C + B \otimes D,$$

where \otimes represents the Kronecker product.

Lemma 2 [19] *For partitioned matrix $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix}$, the following inequalities are equivalent:*

(a) $X > 0$;

(b) $X_{11} - X_{12}X_{22}^{-1}X_{12}^T > 0$ and $X_{22} > 0$;

(c) $X_{22} - X_{12}^T X_{11}^{-1} X_{12} > 0$ and $X_{11} > 0$.

Lemma 3 *Consider two symmetric matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$. If all eigenvalues of A are no more than 0, and all eigenvalues of B are less than 0, then all eigenvalues of $A + B$ are less than 0.*

Proof Because all eigenvalues of A are no more than 0 and all eigenvalues of B are less than 0, there exists a nonzero vector $x = (x_1, x_2, \dots, x_n)$, such that $x^T A x \leq 0, x^T B x < 0$. Then

$$x^T A x + x^T B x = x^T (A + B) x < 0,$$

so all eigenvalues of $A + B$ are less than 0.

3 System Model

Consider the multi-agent systems consisting of N agents. The dynamic of i -th agent is represented by

$$\dot{x}_i(t) = Ax_i(t) + B[u_i(t) + w_i], \quad x_i(0) = x_{i0}, \quad i = 1, 2, \dots, N, \quad (1)$$

where $x_i(t) \in \mathbb{R}^n, u_i(t) \in \mathbb{R}^q, w_i \in \mathbb{R}^q$ represent the state, control input, and persistent disturbances of i -th agent, respectively, $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times q}$ are system matrices. To discuss the consensus problem of the multi-agent system (1), we propose the following control protocol for agent i

$$u_i(t) = K \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)) - \hat{w}_i(t), \quad (2)$$

where $K \in \mathbb{R}^{q \times n}$ is the control gain, and $\hat{w}_i(t) \in \mathbb{R}^q$ is the estimation of w_i , the dynamic equations of which are as follows:

$$\dot{\hat{w}}_i = F \sum_{j \in N_i} a_{ij} [(x_j(t) - x_i(t)) - (\hat{x}_j(t) - \hat{x}_i(t))], \quad (3)$$

where $F \in \mathbb{R}^{q \times n}$ is a constant matrix which will be determined, $\hat{x}_i(t) \in \mathbb{R}^n$ is the estimation of the state of the agent i , the dynamic equations of which are as follows:

$$\dot{\hat{x}}_i(t) = A x_i(t) + BK \sum_{j \in N_i} a_{ij} [\hat{x}_j(t) - \hat{x}_i(t)] + M \sum_{i=1}^N a_{ij} [(\hat{x}_j(t) - \hat{x}_i(t)) - (x_j(t) - x_i(t))], \quad (4)$$

where $M \in \mathbb{R}^{n \times q}$ is also a constant matrix which will be determined. Under the control protocol (2), system (1) can be rewritten as

$$\dot{x}_i(t) = A x_i(t) + B[K \sum_{j \in N_i} a_{ij} (x_j(t) - x_i(t)) - \hat{w}_i(t) + w_i]. \quad (5)$$

Then the consensus problem of system (1) can be transferred into the stability problem of system (5).

4 Stability Analysis

Let $\tilde{x}(t) = x(t) - \hat{x}(t)$ and $\tilde{w}(t) = \hat{w}(t) - w$ be the state estimation error and the disturbance estimation error, respectively. According to (4) and (5), we can get

$$\dot{\tilde{x}}(t) = (I_N \otimes A - \mathcal{L} \otimes BK + \mathcal{L} \otimes M)\tilde{x}(t) - (I_N \otimes B)\tilde{w}(t), \quad (6)$$

$$\dot{\tilde{w}}(t) = \hat{w}(t) - w, \quad (7)$$

Denote $e(t) = [x^T(t), \tilde{x}^T(t), \tilde{w}^T(t)]^T$, then according to (5), (6) and (7), we can get

$$\dot{e}(t) = A_0 e(t), \quad (8)$$

where

$$A_0 = \begin{pmatrix} I_N \otimes A - \mathcal{L} \otimes BK & 0 & -I_N \otimes B \\ 0 & I_N \otimes A - \mathcal{L} \otimes BK + \mathcal{L} \otimes M & -I_N \otimes B \\ 0 & \mathcal{L} \otimes F & 0 \end{pmatrix}.$$

Then the consensus problem of the system (1) transfers the stability problem of the system (8). The system matrix A_0 plays an important role in the stability analysis. Now we analyze this matrix.

For the Laplacian matrix of undirected graph, there exists a matrix so that

$$U^T \mathcal{L} U = \text{diag}(\lambda_1, \dots, \lambda_N), \quad (9)$$

where $\lambda_i, i = 1, 2, \dots, N$, are the eigenvalues of \mathcal{L} . let $\tilde{e}(t) = U_0 e(t)$, where $U_0 = \begin{pmatrix} U \otimes I_n & & \\ & U \otimes I_n & \\ & & U \otimes I_q \end{pmatrix}$. By the orthogonal transformation, we can obtain $\dot{\tilde{e}}(t) = A_1 \tilde{e}(t)$, where

$$A_1 = \begin{pmatrix} I_N \otimes A - U^T \mathcal{L} U \otimes BK & 0 & -I_N \otimes B \\ 0 & I_N \otimes A - \mathcal{L} \otimes BK + U^T \mathcal{L} U \otimes M & -I_N \otimes B \\ 0 & U^T \mathcal{L} U \otimes F & 0 \end{pmatrix}.$$

Since the eigenvalues of a matrix are not affected by exchanging the row and corresponding column of a matrix simultaneously, A_1 can be transferred to a block diagonal, $\bar{A}_1 = \text{diag}(A_{11}, \dots, A_{1N})$

$$\text{where } A_{1i} = \begin{pmatrix} A - \lambda_i BK & 0 & -B \\ 0 & A + \lambda_i M - \lambda_i BK & -B \\ 0 & \lambda_i F & 0 \end{pmatrix}, i = 1, 2, \dots, N.$$

Theorem 1 Consider system (1), the control protocol solves the consensus problem if there exist a positive-definite matrix P , and $\mu_1 > 0, \mu_2 > 0, \mu_3 > 0$, such that

$$A^T P_1 + P_1 A - 2P_1 B B^T P_1 < 0, \quad (10)$$

$$B^T P_2 + P_2 B > 0, \quad (11)$$

$$-2\lambda_{\max} \mu_1 I_N + 3P_1 B [B^T P_2 + P_2 B]^{-1} B^T P_1 < 0, \quad (12)$$

$$-2\lambda_{\max} \mu_2 I_N + 3A^T P_2 [B^T P_2 + P_2 B]^{-1} P_2^T A < 0, \quad (13)$$

$$-2\lambda_{\max} \mu_3 I_N + 3\lambda_{\min}^2 P_1 B B^T P_2 [B^T P_2 + P_2^T B] P_2 B B^T P_1 < 0, \quad (14)$$

where $\mu_1 + \mu_2 + \mu_3 = \mu$, $K = \tau B^T P_1, \tau = 1$, $\begin{pmatrix} M \\ F \end{pmatrix} = -\mu P^{-1} \begin{pmatrix} I_n \\ 0 \end{pmatrix}$ and $P = \begin{pmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{pmatrix}$, P_1, P_2, P_3 have appropriate dimensions.

Proof Noted the form of A_{1i} , we analyze the two block matrices $E = (A - \lambda_i BK)$ and $G = \begin{pmatrix} A + \lambda_i M - \lambda_i BK & -B \\ \lambda_i F & 0 \end{pmatrix}$.

For matrix E , by taking $K = \iota B^T P_1, \iota > (\frac{1}{\lambda_i})$, then $(A - \lambda_i BK)^T P_1 + P_1 (A - \lambda_i BK) = A^T P_1 + P_1 A - 2\lambda_i \iota P_1 B B^T P_1 < A^T P_1 + P_1 A - 2P_1 B B^T P_1 < 0$, so $A - \lambda_i BK$ is Hurwitz stable.

For matrix G , let

$$G = \begin{pmatrix} A - \lambda_i BK & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \lambda_i M & 0 \\ \lambda_i F & 0 \end{pmatrix}.$$

By taking $\begin{pmatrix} M \\ F \end{pmatrix} = -\mu P^{-1} \begin{pmatrix} I_n \\ 0 \end{pmatrix}$, we can obtain

$$\begin{pmatrix} \lambda_i M & 0 \\ \lambda_i F & 0 \end{pmatrix}^T P + P \begin{pmatrix} \lambda_i M & 0 \\ \lambda_i F & 0 \end{pmatrix} = \begin{pmatrix} -2\lambda_i \mu I_n & 0 \\ 0 & 0 \end{pmatrix}, \quad (15)$$

then

$$\begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix}^T P + \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix} P = \begin{pmatrix} 0 & -P_1 B \\ -B^T P_1 & -B^T P_2 - P_2^T B \end{pmatrix}, \quad (16)$$

and

$$\begin{aligned} & \begin{pmatrix} A - \lambda_i BK & 0 \\ 0 & 0 \end{pmatrix}^T P + P \begin{pmatrix} A - \lambda_i BK & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^T P_1 + P_1 A - 2\lambda_i \tau P_1 B B^T P_1 & A^T P_2 - \lambda_i \tau P_1 B B^T P_2 \\ P_2^T A - \lambda_i \tau P_2^T B B^T P_1 & 0 \end{pmatrix}. \end{aligned}$$

Denote $\mu = \mu_1 + \mu_2 + \mu_3$ and make a sum of the three matrices, we can get

$$A_{1i}^T P + P A_{1i} = M_1 + M_2 + M_3 + M_4, \quad (17)$$

where

$$M_1 = \begin{pmatrix} A^T P_1 + P_1 A - 2\lambda_i \tau P_1 B B^T P_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -2\lambda_i \mu_1 I_N & A^T P_2 \\ P_2^T A & -\frac{1}{3}(B^T P_2 + P_2 B) \end{pmatrix},$$

and

$$M_3 = \begin{pmatrix} -2\lambda_i \mu_2 I_N & -P_1 B \\ -B^T P_1 & -\frac{1}{3}(B^T P_2 + P_2 B) \end{pmatrix},$$

$$M_4 = \begin{pmatrix} -2\lambda_i \mu_3 I_N & -\lambda_i \tau P_1 B B^T P_2 \\ -\lambda_i \tau P_2^T B B^T P_1 & -\frac{1}{3}(B^T P_2 + P_2 B) \end{pmatrix},$$

Since (10)–(14) hold, according to Lemma 2, $M_1 \leq 0$, $M_2 < 0$, $M_3 < 0$, $M_4 < 0$, then according to Lemma 3, we have

$$G^T P + P G = M_1 + M_2 + M_3 + M_4 < 0.$$

So matrix $A_{1i}, i = 2, 3, \dots, N$ is Hurwitz stable. According to *Theorem 1* the consensus problem can be solved.

Remark 1 According to *Theorem 1*, we not only solved the consensus problem but also got the consensus state. Since $A_{1i}, i = 2, 3, \dots, N$ are Hurwitz stable, $\bar{x}_i, i = 2, 3, \dots, N$ are asymptotically stable. Now we consider the first block of \bar{A}_1 , since $\lambda = 0$, we can get

$$\begin{pmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{\tilde{x}}}_1(t) \\ \dot{\bar{\tilde{w}}}_1(t) \end{pmatrix} = \begin{pmatrix} A & 0 & -B \\ 0 & A & -B \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_1(t) \\ \bar{\tilde{x}}_1(t) \\ \bar{\tilde{w}}_1(t) \end{pmatrix}, \tag{18}$$

By solving the differential equations, we obtain

$$\begin{aligned} \bar{\tilde{w}}_1 &= \bar{\tilde{w}}_1(t_0), \\ \lim_{t \rightarrow \infty} \bar{x}_1(t) &= e^{At} \bar{x}_1(t_0) + \int_0^\infty e^{A(t-\tau)} B \bar{\tilde{w}}_1(t_0) dt, \end{aligned} \tag{19}$$

denote $U = (1_N \ U_1)$, and according to the non singular transformation, we can get

$$x(t) = \{ (1_N \ U_1) \otimes I_N \} \bar{x}(t) = \frac{1}{\sqrt{N}} (1_N \otimes I_N) \bar{x}_1(t), \tag{20}$$

So

$$x_1(t_0) = \frac{1}{\sqrt{N}} (1_N^T \otimes I_N) x(t_0) = \frac{1}{\sqrt{N}} \sum_{i=1}^N x_i(t_0), \tag{21}$$

Remark 2 Comparing with [14], the result we get in this paper are more simple and more general. For system (8) we just use the Lyapunov stability criterion to get the result, the generality of the result make it more meaningful.

Fig. 1 Network topology

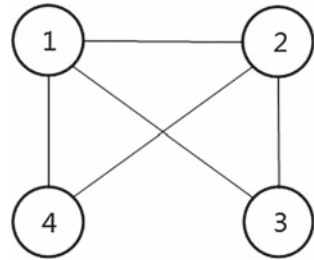


Fig. 2 The error of states' estimation

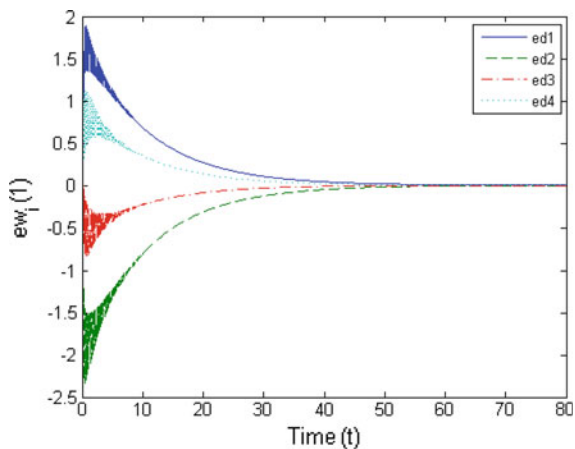


Fig. 3 The error of disturbance' estimation

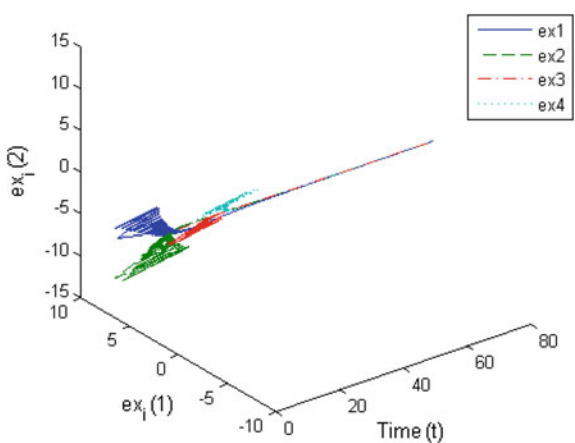
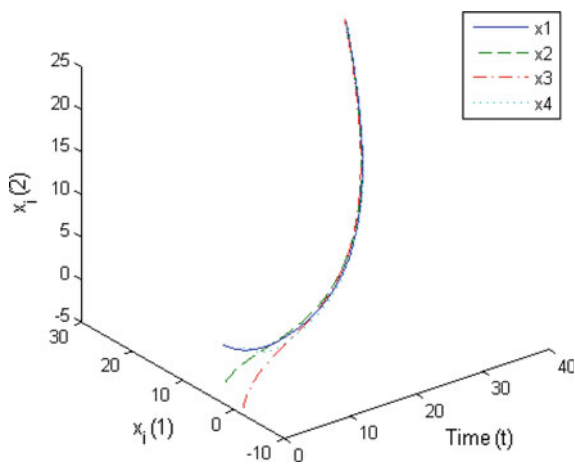


Fig. 4 The states of agents



5 Simulation Example

In this section, a simulation example is provided to validate the effectiveness of our algorithm. Consider a network of four agents, the system matrices are $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the topological structure is shown in Fig. 1.

So the Laplacian matrix can be determined as

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}, \text{ with eigenvalues } 0, 2, 4, 4.$$

For simplicity, we choose $\tau = 1$, and we get the solution P, M, F as follows

$$P = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.4 \end{pmatrix}, M = \begin{pmatrix} -26.25 & 17.5 \\ 17.5 & 8.75 \end{pmatrix}, F = (0.8 \ 0.7).$$

The error of states' and disturbances' estimation are shown in Figs. 2 and 3 respectively, and the consensus state is shown in Fig. 4.

6 Conclusions

In this paper, we addressed the consensus problems of multi-agent systems when dynamics of agents are perturbed by constant persistent disturbances. We derived a sufficient condition for achieving consensus of multi-agent system with constant persistent disturbances. Specifically, it is shown that the consensus state converges to the mean states of all agents [20]. The future work will focus on the consensus problems of high-order systems with Markov switching topologies.

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