Chapter 11
Complex Analysis Method for Elasticity of Quasicrystals

In Chapters 7–9, we frequently used the complex analysis method to solve the problems of elasticity of quasicrystals and many exact analytic solutions were obtained by this method. In these chapters, we only provided the results, and the underling principle and details of the method could not be discussed. Considering the relative new feature and particular effect of the method, it is helpful to attempt a further discussion in depth. Of course, this may lead to a slight repletion with relevant content of Chaps. 7–9.

It is well known that the so-called complex potential method in elasticity is effective, in general, only for solving harmonic and biharmonic partial differential equations in the classical theory of elasticity, and for these equations, the solutions can be expressed by the analytic functions of single complex variable $z = x + iy$, $i = \sqrt{-1}$. In addition, in the classical elasticity, quasi-biharmonic partial differential equation can be solved by analytic functions of some different complex variables such as $z_1 = x + \alpha_1 y, z_2 = x + \alpha_2 y, \ldots$ in which $\alpha_1, \alpha_2, \ldots$ are complex constants. The study of elasticity of quasicrystals has led to discovery of some multi-harmonic and multi-quasiharmonic equations, which cover quite a wide range of partial differential equations appearing in the field to date and have been introduced in Chaps. 5–9. The discussion on the complex analysis for these equations is significant. We know that the Muskhelishvili complex analysis method for classical plane elasticity [1], which solves mainly the biharmonic equation, and the complex potential method developed by Lekhlitzkii [2] for classical anisotropic plane elasticity, which solve mainly the quasi-biharmonic equation, made great contributions for quite a wide range of fields in science and engineering. The present formulation and solutions of the complex analysis, e.g. quadruple and sextuple harmonic equations and quadruple quasiharmonic equation, are a new development of the

T.-Y. Fan, *Mathematical Theory of Elasticity of Quasicrystals and Its Applications*,
Springer Series in Materials Science 246, DOI 10.1007/978-981-10-1984-5_11

complex analysis method used for classical elasticity. Though the new method is used to solve the elasticity problems of quasicrystals at present, it may be extended into other disciplines of science and technology in future.

At first, we simply review the complex analysis method for harmonic and biharmonic equations and then focus on those for quadruple and sextuple harmonic equations and quadruple quasiharmonic equation and, with discussions in detail, presenting their new features from the angle of elasticity as well as complex potential method.

11.1 Harmonic and Biharmonic in Anti-Plane Elasticity of One-Dimensional Quasicrystals

The final governing equations of elasticity of one-dimensional quasicrystals present the following two kinds discussed in Chap. 5:

$$\begin{aligned} c_{44}\nabla^2 u_z + R_3\nabla^2 w_z = 0 \\ R_3\nabla^2 u_z + K^2\nabla^2 w_z = 0 \end{aligned} \tag{11.1.1}$$

$$(c_1\frac{\partial^4}{\partial x^4} + c_2\frac{\partial^4}{\partial x^3\partial y} + c_3\frac{\partial^4}{\partial x^2\partial y^2} + c_4\frac{\partial^4}{\partial x\partial y^3} + c_5\frac{\partial^4}{\partial y^4})G = 0 \tag{11.1.2}$$

in which Eq. (11.1.1) is actually two decoupled harmonic equations of u_z and w_z, whose complex variable function method was introduced in Sects. 8.1 and 8.2, and here we do not repeat any more.

Equation (11.1.2) is a quasi-biharmonic equation which describes the phonon-phason coupling elasticity field for some kinds of one-dimensional quasicrystal systems, refer to Chap. 5. As some solutions of them in terms of the complex variable function method, whose origin comes from the classical work of Lekhlitskii [2], reader can find some beneficial hints in the monograph.

11.2 Biharmonic Equations in Plane Elasticity of Point Group 12*mm* Two-Dimensional Quasicrystals

From Chap. 6, we know that in elasticity of dodecagonal quasicrystals, the phonon and phason fields are decoupled each other. For whose plane elasticity we have the final governing equations as follows:

$$\nabla^2\nabla^2 F = 0, \quad \nabla^2\nabla^2 G = 0 \tag{11.2.1}$$

The complex representation of solution of (11.2.1) is

$$\left.\begin{aligned} F(x,y) &= \mathrm{Re}[\bar{z}\varphi_1(z) + \int \psi_1(z)\mathrm{d}z] \\ G(x,y) &= \mathrm{Re}[\bar{z}\pi_1(z) + \int \chi_1(z)\mathrm{d}z] \end{aligned}\right\} \tag{11.2.2}$$

where $\phi_1(z), \psi_1(z), \pi_1(z)$ and $\chi_1(z)$ are any analytic functions of complex variable $z = x + iy \ \ (i = \sqrt{-1})$. For these kind of biharmonic equations, Muskhelishvili [1] developed systematic complex variable function method, in which reader can find some details in the well-known monograph and we need not discuss those any more. The Muskhelishvili's method has some developments in China, e.g. Lu [3] and Fan [4].

11.3 The Complex Analysis of Quadruple Harmonic Equations and Applications in Two-Dimensional Quasicrystals

As it was discussed in Chaps. 6–8, for point groups 5*m* and 10*mm* or point groups 5, $\bar{5}$, and 10, $\overline{10}$ quasicrystals, either by the displacement potential formulation or by the stress potential formulation, we obtain the final governing equation is quadruple harmonic equation, whose complex variable function method is newly created by Liu and Fan [5, 6] based on the displacement potential formulation and by Li and Fan [7, 8] based on the stress potential formulation. This complex potential method that greatly develops the methodology was used in the classical elasticity. It is necessary to give some further discussions in depth. For simplicity, the following discussion is based on the stress potential formulation only, and solutions are given only for point groups 5, $\bar{5}$, and 10, $\overline{10}$ quasicrystals, because the point groups 5*m* and 10*mm* quasicrystals can be seen as a special case of the former.

11.3.1 Complex Representation of Solution of the Governing Equation

Because it is relatively simpler for the case of point groups 5*m* and 10*mm*, which belong to the special case of point groups 5, $\bar{5}$ and point groups 10 and $\overline{10}$, we here discuss only the final governing equation of plane elasticity of pentagonal of point groups 5, $\bar{5}$ and decagonal quasicrystals of point groups 10, $\overline{10}$

$$\nabla^2\nabla^2\nabla^2\nabla^2 G = 0 \tag{11.3.1}$$

where $G(x, y)$ is the stress potential function. The solution of Eq. (11.3.1) is

$$G = 2\mathrm{Re}[g_1(z) + \bar{z}g_2(z) + \frac{1}{2}\bar{z}^2 g_3(z) + \frac{1}{6}\bar{z}^3 g_4(z)] \tag{11.3.2}$$

where $g_j(z)$ $(j = 1, \cdots, 4)$ are four analytic functions of a single complex variable $z \equiv x + iy = re^{i\theta}$. The bar denotes the complex conjugate hereinafter, i.e. $\bar{z} = x - iy = re^{-i\theta}$. We call these functions be the complex stress potentials, or the complex potentials in brief.

11.3.2 Complex Representation of the Stresses and Displacements

Sect. 8.4 shows that from fundamental solution (11.3.2), one can find the complex representation of the stresses as below:

$$\begin{aligned}
&\sigma_{xx} = -32c_1\mathrm{Re}(\Omega(z) - 2g_4'''(z)) \\
&\sigma_{yy} = 32c_1\mathrm{Re}(\Omega(z) + 2g_4'''(z)) \\
&\sigma_{xy} = \sigma_{yx} = 32c_1\mathrm{Im}\Omega(z) \\
&H_{xx} = 32R_1\mathrm{Re}(\Theta'(z) - \Omega(z)) - 32R_2\mathrm{Im}(\Theta'(z) - \Omega(z)) \\
&H_{xy} = -32R_1\mathrm{Im}(\Theta'(z) + \Omega(z)) - 32R_2\mathrm{Re}(\Theta'(z) + \Omega(z)) \\
&H_{yx} = -32R_1\mathrm{Im}(\Theta'(z) - \Omega(z)) - 32R_2\mathrm{Re}(\Theta'(z) - \Omega(z)) \\
&H_{yy} = -32R_1\mathrm{Re}(\Theta'(z) + \Omega(z)) + 32R_2\mathrm{Im}(\Theta'(z) + \Omega(z)
\end{aligned} \tag{11.3.3}$$

where

$$\begin{aligned}
&\Theta(z) = g_2^{(\mathrm{IV})}(z) + \bar{z}g_3^{(\mathrm{IV})}(z) + \frac{1}{2}\bar{z}^2 g_4^{(\mathrm{IV})}(z) \\
&\Omega(z) = g_3^{(\mathrm{IV})}(z) + \bar{z}g_4^{(\mathrm{IV})}(z)
\end{aligned} \tag{11.3.4}$$

in which one prime, two prime, three prime, and superscript (IV) denote the first- to fourth-order differentiation of $g_j(z)$ to variable z, in addition $\Theta'(z) = \mathrm{d}\Theta(z)/\mathrm{d}z$ and it is evident that $\Theta(z)$ and $\Omega(z)$ are not analytic functions.

By some derivation from (11.3.3), we have the complex representation of the displacements such as

$$u_x + iu_y = 32(4c_1c_2 - c_3 - c_1c_4)g_4''(z) - 32(c_1c_4 - c_3)(\overline{g_3'''(z)} + z\overline{g_4'''(z)}) \quad (11.3.5)$$

$$w_x + iw_y = \frac{32(R_1 - iR_2)}{K_1 - K_2}\overline{\Theta(z)} \quad (11.3.6)$$

with constants

$$c = M(K_1 + K_2) - 2(R_1^2 + R_2^2), c_1 = \frac{c}{K_1 - K_2} + M, c_2 = \frac{c + (L + M)(K_1 + K_2)}{4(L + M)c},$$
$$c_3 = \frac{R_1^2 + R_2^2}{c}, c_4 = \frac{K_1 + K_2}{c} \quad (11.3.7)$$

11.3.3 The Complex Representation of Boundary Conditions

In the following, we consider only the stress boundary value problem; i.e. at the boundary curve L_t, the tractions (T_x, T_y) and generalized tractions (h_x, h_y) are given, and there are the stress boundary conditions such as

$$\sigma_{xx}\cos(\mathbf{n}, x) + \sigma_{xy}\cos(\mathbf{n}, y) = T_x,\ \sigma_{xy}\cos(\mathbf{n}, x) + \sigma_{yy}\cos(\mathbf{n}, y) = T_y, \quad (x, y) \in L_t \quad (11.3.8)$$

$$H_{xx}\cos(\mathbf{n}, x) + H_{xy}\cos(\mathbf{n}, y) = h_x\ ,\ H_{xy}\cos(\mathbf{n}, x) + H_{yy}\cos(\mathbf{n}, y) = h_y, \quad (x, y) \in L_t \quad (11.3.9)$$

where T_x, T_y and h_x, h_y are tractions and generalized tractions at the boundary L_t where the stresses are prescribed.

From (11.3.8) and after some derivation, the phonon stress boundary condition can be reduced to the equivalent form

$$g_4''(z) + \overline{g_3'''(z)} + z\overline{g_4'''(z)} = \frac{i}{32c_1}\int (T_x + iT_y)\mathrm{d}s, \quad z \in L_t \quad (11.3.10)$$

From Eqs. (11.3.9), (11.3.3), and (11.3.4), we have

$$(R_2 - iR_1)\Theta(z) = i\int (h_x + ih_y)\mathrm{d}s, \quad z \in L_t \quad (11.3.11)$$

11.3.4 Structure of Complex Potentials

11.3.4.1 Arbitrariness in the Definition of the Complex Potentials

For simplicity, we introduce the following new symbols

$$g_2^{(\mathrm{IV})}(z) = h_2(z), g_3'''(z) = h_3(z), g_4''(z) = h_4(z) \tag{11.3.12}$$

and then, Eq. (11.3.3) can be rewritten as follows:

$$\sigma_{xx} + \sigma_{yy} = 128c_1 \mathrm{Re}\, h_4'(z) \tag{11.3.13}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 64c_1\Omega(z) = 64c_1[h_3'(z) + \bar{z}h_4''(z)] \tag{11.3.14}$$

$$H_{xy} - H_{yx} - i(H_{xx} + H_{yy}) = 64(iR_1 - R_2)\Omega(z) \tag{11.3.15}$$

$$(H_{xx} - H_{yy}) - i(H_{xy} + H_{yx}) = 64(R_1 + R_2)\Theta'(z) \tag{11.3.16}$$

Similar to the classical elasticity, from Eqs. (11.3.13) to (11.3.16), it is obvious that a state of phonon and phason stresses is not altered, if one replaces

$$h_4(z) \quad \text{by } h_4(z) + Diz + \gamma \tag{11.3.17}$$

$$h_3(z) \quad \text{by} \quad h_3(z) + \gamma' \tag{11.3.18}$$

$$h_2(z) \quad \text{by} \quad h_2(z) + \gamma'' \tag{11.3.19}$$

where D is a real constant and γ, γ', γ'' are arbitrary complex constants.

Now, consider how these substitutions affect the displacement components which were determined by formulas (11.3.5) and (11.3.6). Direct substitution shows that

$$\begin{aligned} u_x + iu_y = {} & 32(4c_1c_2 - c_3 - c_1c_4)h_4(z) - 32(c_1c_4 - c_3)(\overline{h_3(z)} + z\overline{h_4'(z)}) \\ & + 32(4c_1c_2 - 2c_3)Diz + [32(4c_1c_2 - c_3 - c_1c_4)\gamma - 32(c_1c_4 - c_3)\overline{\gamma'}] \end{aligned} \tag{11.3.20}$$

$$w_x + iw_y = \frac{32(R_1 - iR_2)}{K_1 - K_2}[\overline{h_2(z)} + z\overline{h_3'(z)} + \frac{1}{2}z^2\overline{h_4''(z)}] + \frac{32(R_1 - iR_2)}{K_1 - K_2}\overline{\gamma''} \tag{11.3.21}$$

Formulas (11.3.20) and (11.3.21) show that a substitution of the form (11.3.17) and (11.3.19) will affect the displacement, unless

$$D = 0,\ \gamma = \frac{c_1c_4 - c_3}{4c_1c_2 - c_3 - c_1c_4}\overline{\gamma'},\ \overline{\gamma''} = 0$$

11.3.4.2 General Formulas for Finite Multi-connected Regions

Consider now the case when the region S, occupied by the quasicrystal, is multi-connected. In general, the region is bounded by several simple closed contours $s_1, s_2, \ldots, s_m, s_{m+1}$, the last of these contours is to contain all the others, depicted in Fig. 11.1, i.e. a plate with holes. We assume that the contours do not intersect themselves and have no points in common. Sometimes, we call $s_1,\ s_2, \ldots,\ s_m$ as inner boundaries and s_{m+1} as outer boundary of the region. It is evident that the points $z_1, z_2, \ldots, z_m$ are fixed points in the holes, but located out of the material.

Similar to the discussion of the classical elasticity theory (refer to [1]), we can obtained

$$h_4'(z) = \sum_{k=1}^{m} A_k \ln(z - z_k) + h_{4*}'(z) \tag{11.3.22}$$

$$h_4(z) = \sum_{k=1}^{m} A_k z \ln(z - z_k) + \sum_{k=1}^{m} \gamma_k \ln(z - z_k) + h_{4*}(z) \tag{11.3.23}$$

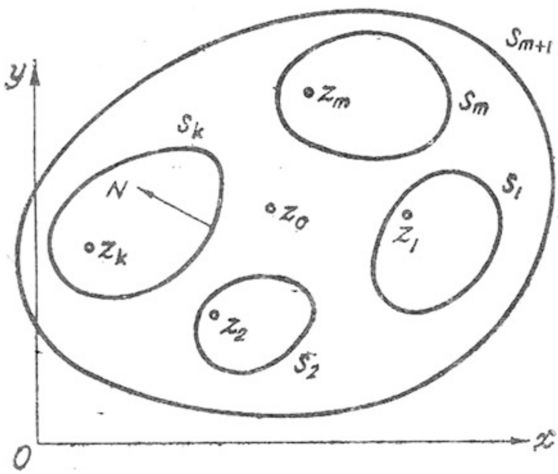


Fig. 11.1 Finite multi-connected region

$$h_3(z) = \sum_{k=1}^{m} \gamma_k' \ln(z - z_k) + h_{3*}(z) \tag{11.3.24}$$

Recalling z_k denotes the fixed points outside the region S, $h_{3*}(z), h_{4*}(z)$ are holomorphic (analytic and single-valued, refer to Major Appendix) in region S, A_k real constants, and $\gamma_k,\ \gamma_k'$ complex constants.

By substituting (11.3.22)–(11.3.24) into (11.3.16), one can find that

$$h_2(z) = \sum_{k=1}^{m} \gamma_k'' \ln(z - z_k) + h_{2*}(z) \tag{11.3.25}$$

$h_{2*}(z)$ is holomorphic in S, and γ_k'' are complex constants.

Consideration will be given to the condition of single valuedness of phonon displacements. From Eq. (11.3.5), one has

$$u_x + iu_y = 32(4c_1c_2 - c_3 - c_1c_4)h_4(z) - 32(c_1c_4 - c_3)(\overline{h_3(z)} + z\overline{h_4'(z)}) \tag{11.3.26}$$

Substituting (11.3.23)–(11.3.25) into (11.3.26), it is immediately seen that

$$\begin{aligned}[u_x + iu_y]_{s_k} = 2\pi i\{&[32(4c_1c_2 - c_3 - c_1c_4) + 32(c_1c_4 - c_3)]A_k z + 32(4c_1c_2 - c_3 \\ &- c_1c_4)\gamma_k + \overline{\gamma_k'(z)}\}\end{aligned} \tag{11.3.27}$$

in which $[\,]_k$ denotes the increase undergone by the expression in brackets for one anticlockwise circuit of the contour s_k. Hence it is necessary and sufficient for the single valuedness of phonon displacements that are shown in formulas (11.3.22)–(11.3.25)

$$A_k = 0,\quad 32(4c_1c_2 - c_3 - c_1c_4)\gamma_k + \overline{\gamma_k'} = 0 \tag{11.3.28}$$

Similar to the above-mentioned discussion, by Eq. (11.3.6), one has

$$[w_x + iw_y]_{s_k} = \frac{32(R_1 - iR_2)}{K_1 - K_2}(-2\pi i)\overline{\gamma_k''} \tag{11.3.29}$$

Hence it is necessary and sufficient for the single valuedness of phason displacements is

$$\gamma_k'' = 0 \tag{11.3.30}$$

It will now be shown that the quantities $\gamma_k,\ \gamma'_k$ may be very simply expressed in terms of $X_k,\ Y_k$, where $(X_k,\ Y_k)$ denote the resultant vector of the external stresses, exerted on the contour s_k. From (11.3.10), applying it to the contour s_k, one has

$$-32c_1 i[h_4(z) + \overline{h_3(z)} + z\overline{h'_4(z)}]_{s_k} = X_k + iY_k \tag{11.3.31}$$

with

$$X_k = \int\limits_{S_k} T_x \mathrm{d}s, Y_k = \int\limits_{S_k} T_y \mathrm{d}s$$

In the present case, the normal vector **n** must be directed outwards with respect to the region s_k. Consequently, the contour s_k must be traversed in the clockwise direction. Taking this fact into consideration, one obtains

$$-2\pi i(\gamma_k - \overline{\gamma'_k}) = \frac{i}{32c_1}(X_k + iY_k) \tag{11.3.32}$$

By Eqs. (11.3.28), (11.3.31), and (11.3.32), one has

$$\begin{aligned} &A_k = 0 \\ &\gamma_k = d_1(X_k + iY_k),\ \gamma'_k = d_2(X_k - iY_k) \end{aligned} \tag{11.3.33}$$

where

$$d_1 = \frac{1}{64c_1\pi[32(4c_1c_2 - c_3 - c_1c_4) + 1]}, d_2 = -\frac{4c_1c_2 - c_3 - c_1c_4}{2c_1\pi[32(4c_1c_2 - c_3 - c_1c_4) + 1]} \tag{11.3.34}$$

and which are independent from the suffix k. So that

$$\begin{aligned} h_4(z) &= d_1 \sum_{k=1}^{m} (X_k + iY_k)\ln(z - z_k) + h_{4*}(z) \\ h_3(z) &= d_2 \sum_{k=1}^{m} (X_k - iY_k)\ln(z - z_k) + h_{3*}(z) \\ h_2(z) &= h_{2*}(z) \end{aligned} \tag{11.3.35}$$

We can conclude that the complex functions $h_2(z),\ h_3(z),\ h_4(z)$ must be expressed by formula (11.3.35) to assure the single valuedness of stresses and displacements, where $h_{2*}(z),\ h_{3*}(z),\ h_{4*}(z)$ are holomorphic in region S.

11.3.4.3 Case of Infinite Regions

From the point of view of application, the consideration of infinite regions is likewise of major interest. We assume that the contour s_{m+1} has entirely moved to infinity.

Because Eqs. (11.3.13) and (11.3.14) are similar to the classical elasticity theory, we have

$$\begin{aligned} h_4(z) &= d_1(X+iY)\ln z + (B+iC)z + h_4^0(z) \\ h_3(z) &= d_2(X-iY)\ln z + (B'+iC')z + h_3^0(z) \end{aligned} \tag{11.3.36}$$

where B, C, B', C' are unknown real constants to be determined and

$$X = \sum_{k=1}^{m} X_k,\ Y = \sum_{k=1}^{m} Y_k$$

$h_3^0(z)$, $h_4^0(z)$ are functions, holomorphic in region S, including the point at infinity; i.e. for sufficiently large $|z|$, they may be expanded into series of the form

$$h_4^0(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots,\ h_3^0(z) = a'_0 + \frac{a'_1}{z} + \frac{a'_2}{z^2} + \cdots \tag{11.3.37}$$

On the basis of (11.3.2), the state of phonon and phason stresses will not be altered by assuming

$$a_0 = a'_0 = 0$$

By the theorem of Laurent, the function $h_{2*}(z)$ may be represented in region S including point at infinity by the series

$$h_{2*}(z) = \sum_{-\infty}^{+\infty} c_n z^n \tag{11.3.38}$$

Substituting Eqs. (11.3.36) and (11.3.38) into Eq. (11.3.16), one has

$$\begin{aligned} &(H_{xx} - H_{yy}) - i(H_{xy} + H_{yx}) \\ &= 2\times 32(R_1+R_2)[\sum_{-\infty}^{+\infty} c_n n z^{n-1} + \bar{z}(-\frac{d_2}{z^2} + h_3^{0\prime\prime}(z)) + \frac{1}{2}\bar{z}^2(\frac{2d_1}{z^3} + h_4^{0\prime\prime\prime}(z))] \end{aligned} \tag{11.3.39}$$

and hence it follows that for the stresses to remain finite as $|z| \to \infty$, one must have

$$c_n = 0\ (n \geq 2)$$

It is obvious that the phonon and phason stresses will be bounded, if these conditions are satisfied. Hence one has finally

$$\begin{aligned} h_4(z) &= d_1(X+iY)\ln z + (B+iC)z + h_4^0(z) \\ h_3(z) &= d_2(X-iY)\ln z + (B'+iC')z + h_3^0(z) \\ h_2(z) &= (B''+iC'')z + h_2^0(z) \end{aligned} \tag{11.3.40}$$

where B'', C'' are unknown real constants to be determined, $h_2^0(z)$ is function, holomorphic in region S, including the point at infinity; thus, it has the form similar to that of (11.3.37):

$$h_2^0(z) = a_0^{''} + \frac{a_1^{''}}{z} + \frac{a_2^{''}}{z^2} + \cdots \tag{11.3.41}$$

We have assumed that $a_0 = a_0^{'} = 0$ already and now further assume $a_0^{''} = 0$, i.e.

$$h_4^0(\infty) = h_3^0(\infty) = h_2^0(\infty) = 0.$$

Then from (11.3.40) and (11.3.13)–(11.3.16), one can determine

$$\begin{aligned} &B = \frac{\sigma_{xx}^{(\infty)} + \sigma_{yy}^{(\infty)}}{128c_1},\ B^{'} = \frac{\sigma_{xx}^{(\infty)} - \sigma_{yy}^{(\infty)}}{64c_1},\ C^{'} = \frac{\sigma_{xy}^{(\infty)}}{32c_1}, \\ &B^{''} = \frac{R_2(H_{xy}^{(\infty)} - H_{yx}^{(\infty)}) - R_1(H_{xx}^{(\infty)} + H_{yy}^{(\infty)})}{64(R_1^2 - R_2^2)},\ C^{''} = \frac{R_1(H_{xy}^{(\infty)} - H_{yx}^{(\infty)}) - R_2(H_{xx}^{(\infty)} + H_{yy}^{(\infty)})}{64(R_1^2 - R_2^2)} \end{aligned} \tag{11.3.42}$$

and C has no usage and we put it to be zero, in which $\sigma_{ij}^{(\infty)}$ and $H_{ij}^{(\infty)}$ represent the applied stresses at point of infinity.

11.3.5 Conformal Mapping

If we constrain our discussion only for the case of stress boundary value problems, then the problems will be solved under boundary conditions (11.3.10) and (11.3.11). For some complicated regions, solutions of the problems cannot be directly obtained in the physical plane (i.e. the z-plane). We must use a conformal mapping

$$z = \omega(\zeta) \tag{11.3.43}$$

to transform the region studied in the plane onto interior of the unit circle γ in the mapping plane (say, e.g. ζ-plane).

Substituting (11.3.43) into (11.3.40), we have

$$\begin{aligned} h_4(z) &= \Phi_4(\zeta) = d_1(X+iY)\ln\omega(\zeta) + B\omega(\zeta) + \Phi_4^0(\zeta) \\ h_3(z) &= \Phi_3(\zeta) = d_2(X-iY)\ln\omega(\zeta) + (B'+iC')\omega(\zeta) + \Phi_3^0(\zeta) \\ h_2(z) &= \Phi_2(\zeta) = (B''+iC'')\omega(\zeta) + \Phi_2^0(\zeta) \end{aligned} \tag{11.3.44}$$

where

$$\Phi_j(\zeta) = h_j[\omega(\zeta)], \Phi_j^0(\zeta) = h_j^0[\omega(\zeta)], j = 1,\ldots,4$$

In addition,

$$h_i'(z) = \frac{\Phi_i'(\zeta)}{\omega'(\zeta)}$$

At the mapping plane, the boundary conditions (11.3.10) and (11.3.11) stand for

$$\Phi_4(\sigma) + \overline{\Phi_3(\sigma)} + \omega(\sigma)\overline{\frac{\Phi_4'(\sigma)}{\omega'(\sigma)}} = \frac{i}{32c_1}\int (T_x + iT_y)\mathrm{d}s, \tag{11.3.10$'$}$$

$$(R_2 - iR_1)\Theta(\sigma) = i\int (h_x + ih_y)\mathrm{d}s \tag{11.3.11$'$}$$

where $\sigma = \mathrm{e}^{i\varphi}$ represents the value of ζ at the unit circle (i.e. $\rho = 1$). From these boundary value equations, we can determine the unknown functions $\Phi_j(\zeta)$ $(j = 2, 3, 4)$.

11.3.6 Reduction in the Boundary Value Problem to Function Equations

Due to $\Phi_1(\zeta) = 0$, we now have three unknown functions $\Phi_i(\zeta)$ $(i = 2, 3, 4)$. Taking conjugate of (11.3.10′) yields

$$\overline{\Phi_4(\sigma)} + \Phi_3(\sigma) + \overline{\omega(\sigma)}\frac{\Phi_4'(\sigma)}{\omega'(\sigma)} = -\frac{i}{32c_1}\int (T_x - iT_y)\mathrm{d}s \tag{11.3.10$''$}$$

Substituting the Eq. (11.3.4) into (11.3.11′) and then multiplying $\frac{1}{2\pi i}\frac{\mathrm{d}\sigma}{\sigma-\zeta}$ on both sides of (11.3.10′), (11.3.10″), and (11.3.11′) lead to

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma}\frac{\Phi_4(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}+\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\Phi_3(\sigma)}\mathrm{d}\sigma}{\sigma-\zeta}+\frac{1}{2\pi i}\int_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\Phi_4'(\sigma)}\mathrm{d}\sigma}{\sigma-\zeta}=\frac{1}{32c_1}\frac{1}{2\pi i}\int_{\gamma}\frac{\mathbf{t}\mathrm{d}\sigma}{\sigma-\zeta}\\
&\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\Phi_4(\sigma)}\mathrm{d}\sigma}{\sigma-\zeta}+\frac{1}{2\pi i}\int_{\gamma}\frac{\Phi_3(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}+\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_4'(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}=\frac{1}{32c_1}\frac{1}{2\pi i}\int_{\gamma}\frac{\bar{\mathbf{t}}\mathrm{d}\sigma}{\sigma-\zeta}\\
&\frac{1}{2\pi i}\int_{\gamma}\frac{\Phi_2(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}+\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_3'(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}+\frac{1}{2\pi i}[\int_{\gamma}\frac{\overline{\omega(\sigma)}^2}{[\omega'(\sigma)]^2}\frac{\Phi_4''(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}\\
&\qquad -\int_{\gamma}\frac{\overline{\omega(\sigma)}^2\omega''(\sigma)}{[\omega'(\sigma)]^3}\frac{\Phi_4'(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}]=\frac{1}{R_1-iR_2}\frac{1}{2\pi i}\int_{\gamma}\frac{\mathbf{h}\mathrm{d}\sigma}{\sigma-\zeta}
\end{aligned}\tag{11.3.45}$$

where $\mathbf{t}=i\int(T_x+iT_y)\mathrm{d}s,\bar{\mathbf{t}}=-i\int(T_x-iT_y)\mathrm{d}s,\mathbf{h}=i\int(h_1+ih_2)\mathrm{d}s$ in Eq. (11.3.45), which are the function equations to determine the complex potentials $\Phi_i(\zeta)$, which are analytic in the interior of the unit circle γ, and satisfy the boundary value conditions (11.3.45) at the unit circle.

11.3.7 Solution of the Function Equations

According to the Cauchy's integral formula (refer to Major Appendix),

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\Phi_i(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma=\Phi_i(\zeta),\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\Phi_i(\sigma)}}{\sigma-\zeta}\mathrm{d}\sigma=\overline{\Phi_i(0)},\ |\zeta|<1$$

So that (11.3.45) are reduced to

$$\begin{aligned}
&\Phi_4(\zeta)+\overline{\Phi_3(0)}+\frac{1}{2\pi i}\int_{\gamma}\frac{\omega(\sigma)}{\overline{\omega(\sigma)}}\frac{\overline{\Phi_4'(\sigma)}\mathrm{d}\sigma}{\sigma-\zeta}=\frac{i}{32c_1}\frac{1}{2\pi i}\int_{\gamma}\frac{\mathbf{t}\mathrm{d}\sigma}{\sigma-\zeta}\\
&\overline{\Phi_4(0)}+\Phi_3(\zeta)+\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega(\sigma)}\frac{\Phi_4'(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}=-\frac{i}{32c_1}\frac{1}{2\pi i}\int_{\gamma}\frac{\bar{\mathbf{t}}\mathrm{d}\sigma}{\sigma-\zeta}\\
&\Phi_2(\zeta)+\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_3'(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}+\frac{1}{2\pi i}[\int_{\gamma}\frac{\overline{\omega(\sigma)}^2}{[\omega'(\sigma)]^2}\frac{\Phi_4''(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}\\
&\qquad -\int_{\gamma}\frac{\overline{\omega(\sigma)}^2\omega''(\sigma)}{[\omega'(\sigma)]^3}\frac{\Phi_4'(\sigma)\mathrm{d}\sigma}{\sigma-\zeta}]=\frac{i}{R_1-iR_2}\frac{1}{2\pi i}\int_{\gamma}\frac{\mathbf{h}\mathrm{d}\sigma}{\sigma-\zeta}
\end{aligned}\tag{11.3.46}$$

The calculation of integrals in (11.3.46) depends upon the configuration of the sample, so the mapping function is $\omega(\zeta)$ and the applied stresses are $\mathbf{t}$ and $\mathbf{h}$, respectively. In the following, we will give a concrete solution for a given configuration and applied traction.

11.3.8 Example 1 Elliptic Notch/Crack Problem and Solution

We calculate the stress and displacement field induced by an elliptic notch $L:$ $(\frac{x^2}{a^2}+\frac{y^2}{b^2}=1)$ in an infinite plane of decagonal quasicrystal (see Fig. 11.2), the edge of which is subjected to a uniform pressure p. Though the problem was solved in Sect. 8.4, to figure out its outline from the general formulation is meaningful.

The boundary conditions can be expressed in Eqs. (11.3.10) and (11.3.11), and for simplicity, we assume $h_x = h_y = 0$. Thus

$$
\begin{aligned}
& i\int (T_x + iT_y)\mathrm{d}s = i\int (-p\cos(\mathbf{n},x) - ip\cos(\mathbf{n},y))\mathrm{d}s = -pz = -p\omega(\sigma) \\
& i\int (h_x + ih_y)\mathrm{d}s = 0
\end{aligned}
\tag{11.3.47}
$$

In addition in this case in formula (11.3.44)

$$
\begin{aligned}
& X = Y = 0, \\
& B = 0, B' = C' = 0, B'' = C'' = 0
\end{aligned}
\tag{11.3.48}
$$

so $\Phi_j(\zeta) = \Phi_j^0(\zeta)$, but in the following, we omit the superscript of the functions $\Phi_i^0(\zeta)$ for simplicity.

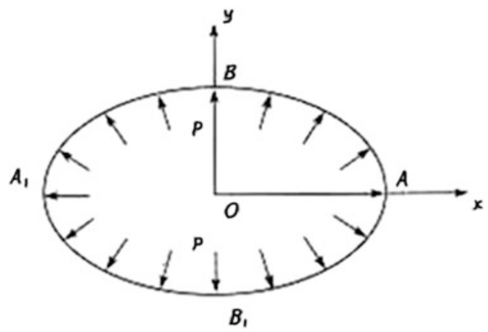


Fig. 11.2 An elliptic notch in a decagonal quasicrystal

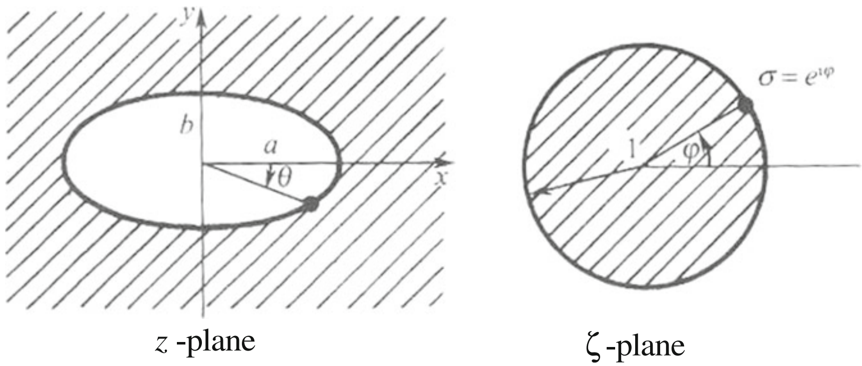


Fig. 11.3 Conformal mapping from the region at z-plane with an elliptic hole onto the interior of the unit circle at ζ-plane

The conformal mapping is

$$z = \omega(\zeta) = R_0\left(\frac{1}{\zeta} + m\zeta\right) \tag{11.3.49}$$

to transform the region containing ellipse at the z-plane onto the interior of the unit circle at the ζ-plane, refer to Fig. 11.3, where $\zeta = \xi + i\eta = \rho e^{i\varphi}$ and $R_0 = \frac{a+b}{2}$, $m = \frac{a-b}{a+b}$.

Substituting (11.3.48) and (11.3.49) into function Eq. (11.3.46), one obtains

$$\begin{aligned}\Phi_3(\zeta) &= \frac{pR_0}{32c_1}\frac{(1+m^2)\zeta}{m\zeta^2 - 1}\\ \Phi_4(\zeta) &= -\frac{pR_0}{32c_1}m\zeta\end{aligned} \tag{11.3.50}$$

$$\Phi_2(\zeta) = \frac{pR_0}{32c_1}\frac{\zeta(\zeta^2+m)[(1+m^2)(1+m\zeta^2) - (\zeta^2+m)]}{(m\zeta^2-1)^3}$$

If we take $m = 1$, from (11.3.50) we can obtain solution of the Griffith crack; in particular, the explicit solution at z-plane can be explored by taking inversion $\zeta = \omega^{-1}(z) = z/a - \sqrt{z^2/a^2 - 1}$ (as $m = 1$) into the relevant formulas.

The concrete results are given in Sect. 8.4, which are omitted here.

11.3.9 Example 2 Infinite Plane with an Elliptic Hole Subjected to a Tension at Infinity

In this case

$$X = Y = 0, T_x = T_y = 0, B = \frac{p}{64c_1}, B' = C' = 0, B'' = C'' = 0, \mathbf{t} = \bar{\mathbf{t}} = \mathbf{h} = 0 \tag{11.3.51}$$

so that from (11.3.44)

$$\begin{aligned} h_4(z) &= \Phi_4(\zeta) = B\omega(\zeta) + \Phi_4^0(\zeta) \\ h_3(z) &= \Phi_3(\zeta) = \Phi_3^0(\zeta) \\ h_2(z) &= \Phi_2(\zeta) = \Phi_2^0(\zeta) \end{aligned} \tag{11.3.51}$$

Substituting (11.3.52) into (11.3.45), we obtain the similar equations on functions $\Phi_j^0(\zeta) \quad (j = 2, 3, 4)$ bv, so the solution is similar to (11.3.50).

11.3.10 Example 3 Infinite Plane with an Elliptic Hole Subjected to a Distributed Pressure at a Part of Surface of the Hole

The problem is shown in Fig. 11.4. We here use the conformal mapping

$$z = \omega(\zeta) = R_0(\zeta + \frac{m}{\zeta}) \tag{11.3.52}$$

to transform the region at z-plane onto the exterior of the unit circle γ at ζ-plane (see Fig. 11.5).

In terms of the similar procedure, the solution we found [9] is as follows:

$$\Phi_4(\zeta) = \frac{1}{32c_1} \cdot \frac{p}{2\pi i} \cdot \left[-\frac{mR_0}{\zeta} \ln \frac{\sigma_2}{\sigma_1} + z \ln \frac{\sigma_2 - \zeta}{\sigma_1 - \zeta} + z_1 \ln(\sigma_1 - \zeta) - z_2 \ln(\sigma_2 - \zeta) \right] \\ + ip(d_1 - d_2)(z_1 - z_2) \ln \zeta$$

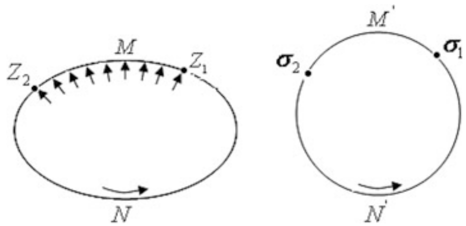


Fig. 11.4 Infinite plane with an elliptic hole subjected to a distributed pressure at a part of surface of the hole and its conformal mapping at ζ-plane

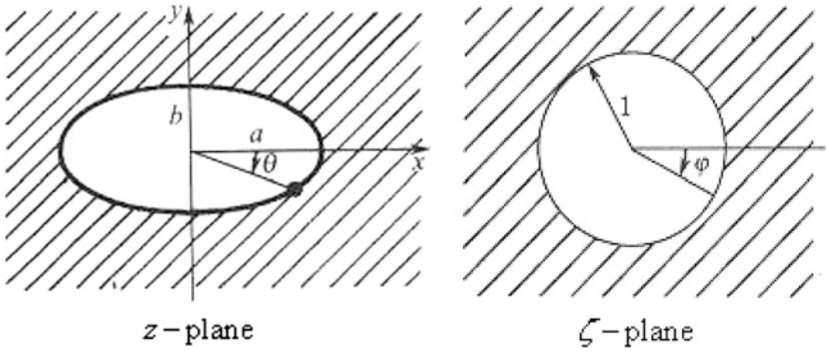


Fig. 11.5 Conformal mapping from the region at z-plane with an elliptic hole onto the exterior of the unit circle at ζ-plane

$$\begin{aligned}\Phi_3(\zeta) = &\frac{1}{32c_1}\cdot\frac{p}{2\pi i}\cdot\left[-\frac{(1+m^2)R_0\zeta}{(\zeta^2-m)}\ln\frac{\sigma_2}{\sigma_1}+\frac{R_0(\sigma_1-\sigma_2)(1+m\zeta^2)}{(\zeta^2-m)}-\overline{z_2}\ln(\sigma_2-\zeta)+\overline{z_1}\ln(\sigma_1-\zeta)\right]\\ &-ip(d_1+d_2)\cdot\left[(\overline{z_1}-\overline{z_2})\ln\zeta+(z_1-z_2)\frac{(1+m^2)}{\zeta^2-m}\right]\end{aligned}$$

$$\begin{aligned}\Phi_2(\zeta) = &\frac{1}{32c_1}\cdot\frac{pR_0}{2\pi i}\cdot\frac{(m\zeta^2+1)(\zeta^2+m)}{(\zeta^2-m)^3}(\ln\frac{\sigma_2}{\sigma_1}+\frac{\sigma_2-\sigma_1}{(\sigma_2-\zeta)(\sigma_1-\zeta)})+\frac{1}{32c_1}\cdot\frac{p}{2\pi i}\cdot\frac{(m\zeta^2+1)}{(\zeta^2-m)^2}\times\\ &\left\{2\mathrm{Re}z_2\cdot\frac{\sigma_2-\sigma_1}{(\sigma_2-\zeta)(\sigma_1-\zeta)}+\left[z_2-R_0(\zeta-\frac{m}{\zeta})\right]\cdot\left[\frac{(\sigma_2-\zeta)(\sigma_1-\zeta)+(\sigma_2+\sigma_1-2\zeta)(\sigma_2-\sigma_1)}{(\sigma_2-\zeta)(\sigma_1-\zeta)}\right]\right\}\\ &\frac{(m\zeta^2+1)(\zeta^2+m)}{(\zeta^2-m)^3}ip\left\{d_1(\overline{z_1}-\overline{z_2}-z_1+z_2)\frac{1}{\zeta-\sigma_1}+(d_2-d_1)(z_1-z_2)\left[\frac{1}{\zeta^2}+\frac{1}{\zeta}+\frac{1}{(\zeta-\sigma_1)^2}\right]\right\}\end{aligned} \tag{11.3.53}$$

where

$$z_1 = R_0(\sigma_1+\frac{m}{\sigma_1}),\ z_2 = R_0(\sigma_2+\frac{m}{\sigma_2})$$

11.4 Complex Analysis for Sextuple Harmonic Equation and Applications to Three-Dimensional Icosahedral Quasicrystals

Plane elasticity of icosahedral quasicrystals has been reduced to a sextuple harmonic equation to solve in Chap. 9, where we have shown the solution procedure of the equation for a notch/crack problem by complex variable function method and we here provide further discussion in depth from point of complex function theory.

The aim is to develop the complex potential method for higher-order multi-harmonic equations. Though there are some similar natures in the following description with that introduced in the preceding section, the discussion here is necessary, because the governing equation and boundary conditions for icosahedral quasicrystals are quite different from those for decagonal quasicrystals.

11.4.1 The Complex Representation of Stresses and Displacements

In Sect. 9.5 by the stress potential, we obtain the final governing equation under the approximation $R^2/\mu K_1 \ll 1$

$$\nabla^2\nabla^2\nabla^2\nabla^2\nabla^2\nabla^2 G = 0 \tag{11.4.1}$$

Fundamental solution of Eq. (11.4.1) can be expressed in six analytic functions of complex variable z, i.e.

$$G(x,y) = \mathrm{Re}[g_1(z) + \bar{z}g_2(z) + \bar{z}^2 g_3(z) + \bar{z}^3 g_4(z) + \bar{z}^4 g_5(z) + \bar{z}^5 g_6(z)] \tag{11.4.2}$$

where $g_i(z)$ are arbitrary analytic functions of $z = x + iy$ and the bar denotes the complex conjugate.

From Eqs. (11.4.1), (11.4.2), (9.5.2) and (9.5.3), the stresses can be expressed as follows:

$$\begin{gathered}
\sigma_{xx} + \sigma_{yy} = 48c_2c_3R\,\mathrm{Im}\,\Gamma'(z) \quad \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 8ic_2c_3R(12\overline{\Psi'(z)} - \Omega'(z)) \\
\sigma_{zy} - i\sigma_{zx} = -960c_3c_4f_6'(z) \quad \sigma_{zz} = \frac{24\lambda R}{(\mu+\lambda)}c_2c_3\,\mathrm{Im}\,\Gamma'(z) \\
H_{xy} - H_{yx} - i(H_{xx} + H_{yy}) = -96c_2c_5\overline{\Psi'(z)} - 8c_1c_2R\Omega'(z) \\
H_{yx} + H_{xy} + i(H_{xx} - H_{yy}) = -480c_2c_5\overline{f_6'(z)} - 4c_1c_2R\Theta'(z) \\
H_{yz} + iH_{xz} = 48c_2c_6\Gamma'(z) - 4c_2R^2(2K_2 - K_1)\overline{\Omega'(z)} \\
H_{zz} = \frac{24R^2}{(\mu+\lambda)}c_2c_3\,\mathrm{Im}\,\Gamma'(z)
\end{gathered} \tag{11.4.3}$$

where

$$\begin{aligned}
\Psi(z) &= f_5(z) + 5\bar{z}f_6'(z) \\
\Gamma(z) &= f_4(z) + 4\bar{z}f_5'(z) + 10\bar{z}^2 f_6''(z) \\
\Omega(z) &= f_3(z) + 3\bar{z}f_4'(z) + 6\bar{z}^2 f_5''(z) + 10\bar{z}^3 f_6'''(z) \\
\Theta(z) &= f_2(z) + 2\bar{z}f_3'(z) + 3\bar{z}^2 f_4''(z) + 4\bar{z}^3 f_5'''(z) + 5\bar{z}^4 f_6^{(\mathrm{IV})}(z)
\end{aligned}$$

$$c_1 = \frac{R(2K_2 - K_1)(\mu K_1 + \mu K_2 - 3R^2)}{2(\mu K_1 - 2R^2)}, c_3 = \frac{1}{R}K_2(\mu K_2 - R^2) - R(2K_2 - K_1)$$

$$c_2 = \mu(K_1 - K_2) - R^2 - \frac{(\mu K_2 - R^2)^2}{\mu K_1 - 2R^2}, c_4 = c_1 R + \frac{1}{2}c_3(K_1 + \frac{\mu K_1 - 2R^2}{\lambda + \mu})$$

$$c_5 = 2c_4 - c_1 R, \quad c_6 = (2K_2 - K_1)R^2 - 4c_4 \frac{\mu K_2 - R^2}{\mu K_1 - 2R^2} \tag{11.4.4}$$

In the above expressions, the function $g_1(z)$ is not used and to be assumed $g_1(z) = 0$ so $f_1(z) = 0$ for simplicity, we have introduced the following new symbols

$$\begin{aligned}
&g_2^{(9)}(z) = f_2(z), \qquad g_3^{(8)}(z) = f_3(z), \qquad g_4^{(7)}(z) = f_4(z), \\
&g_5^{(6)}(z) = f_5(z), \qquad g_6^{(5)}(z) = f_6(z)
\end{aligned} \tag{11.4.5}$$

where $g_i^{(n)}$ denote nth derivative with the argument z. Similar to the manipulation in the previous section, the complex representations of displacement components can be written as follows (here we have omitted the rigid body displacements)

$$\begin{aligned}
u_y + iu_x &= -6c_3 R(\frac{2c_2}{\mu + \lambda} + c_7)\overline{\Gamma(z)} - 2c_3 c_7 R\Omega(z) \\
u_z &= \frac{4}{\mu(K_1 + K_2) - 3R^2}(240c_{10}\mathrm{Im} f_6(z) + c_1 c_2 R^2 \mathrm{Im}(\Theta(z) - 2\Omega z) + 6\Gamma(z) - 24\Psi(z))) \\
w_y + iw_x &= -\frac{R}{c_1(\mu K_1 - 2R^2)}(24c_9\overline{\Psi(z)} - c_8\Theta(z)) \\
w_z &= \frac{4(\mu K_2 - R^2)}{(K_1 - 2K_2)R(\mu(K_1 + K_2) - 3R^2)}(240c_{10}\mathrm{Im} f_6(z)) + c_1 c_2 R^2 \mathrm{Im}(\Theta(z) - 2\Omega(z) \\
&\qquad + 6\Gamma(z) - 24\Psi(z))
\end{aligned} \tag{11.4.6}$$

in which

$$c_7 = \frac{c_2K_1 + 2c_1R}{\mu K_1 - 2R^2}, \quad c_8 = c_1c_3R(\mu(K_1 - K_2) - R^2)$$
$$c_9 = c_8 + 2c_3c_4(c_2 - \frac{(\mu K_2 - R^2)^2}{\mu K_1 - 2R^2}), \quad c_{10} = c_1c_3R^2 - c_4(c_3R - c_2K_1) \tag{11.4.7}$$

11.4.2 The Complex Representation of Boundary Conditions

The boundary conditions of plane elasticity of icosahedral quasicrystals can be expressed as follows:

$$\sigma_{xx}l + \sigma_{xy}m = T_x, \quad \sigma_{yx}l + \sigma_{yy}m = T_y, \quad \sigma_{zx}l + \sigma_{zy}m = T_z \tag{11.4.8}$$

$$H_{xx}l + H_{xy}m = h_x, \quad H_{yx}l + H_{yy}m = h_y, \quad H_{zx}l + H_{zy}m = h_z \tag{11.4.9}$$

for $(x, y) \in L$ which represents the boundary of a multi-connected quasicrystalline material, and

$$l = \cos(\mathbf{n}, x) = \frac{\mathrm{d}y}{\mathrm{d}s}, \quad m = \cos(\mathbf{n}, y) = -\frac{\mathrm{d}x}{\mathrm{d}s}$$

$\mathbf{T} = (T_x, T_y, T_z)$ and $\mathbf{h} = (h_x, h_y, h_z)$ denote the surface traction vector and generalized surface traction vector, and $\mathbf{n}$ represents the outward unit normal vector of any point of the boundary, respectively.

Utilizing Eq. (11.4.3) and the first two formulas of Eq. (11.4.8), one has

$$-4c_2c_3R[3(f_4(z) + 4\bar{z}f_5'(z) + 10\bar{z}^2f_6''(z)) - (\overline{f_3(z)} + 3z\overline{f_4'(z)} + 6z^2\overline{f_5''(z)} + 10z^3\overline{f_6'''(z)})]$$
$$= i\int (T_x + iT_y)\mathrm{d}s, \quad z \in L \tag{11.4.10}$$

Taking conjugate on both sides of Eq. (11.4.10) yields

$$-4c_2c_3R[3(\overline{f_4(z)} + 4z\overline{f_5'(z)} + 10z^2\overline{f_6''(z)}) - (f_3(z) + 3\bar{z}f_4'(z) + 6z^2f_5'(z) + 10\bar{z}^3f_6'''(z)$$
$$= -i\int (T_x - iT_y)\mathrm{d}s, \ z \in L \tag{11.4.11}$$

Similarly, from Eq. (11.4.3) and the first two formulas of (11.4.9), one obtains

$$48c_2(2c_4 - c_1R)\overline{\Psi(z)} + 2c_1c_2R\Theta(z) = i\int(h_x + ih_y)\mathrm{d}s, \quad z \in L \qquad (11.4.12)$$

Furthermore, we assume

$$T_z = h_z = 0 \qquad (11.4.13)$$

For simplicity and by the third equations in (11.4.8) and (11.4.9) and the formulas of (11.4.3) and (11.4.13), one has

$$\begin{cases} f_6(z) + \overline{f_6(z)} = 0 \\ 4c_{11}\mathrm{Re}[f_5(z) + 5\bar{z}f_6'(z)] + (2K_2 - K_1)R\mathrm{Re}[f_4(z) + 4\bar{z}f_5'(z) + 10\bar{z}^2f_6''(z) + 20f_6(z)] = 0 \end{cases}$$

$$z \in L \qquad (11.3.14)$$

in which

$$c_{11} = (2K_2 - K_1)R - \frac{4c_4(\mu K_2 - R^2)}{(\mu K_1 - 2R^2)R} \qquad (11.4.15)$$

As we have shown in the previous section, complex analytic functions (i.e. the complex potentials) must be determined by boundary value equations, which are discussed below.

11.4.3 Structure of Complex Potentials

11.4.3.1 The Arbitrariness of the Complex Potentials

For explicit description, Eq. (11.4.3) can be written as follows:

$$\sigma_{zy} - i\sigma_{zx} = -960c_3c_4f_6'(z)$$

$$c_1(\sigma_{yy} - \sigma_{xx} - 2i\sigma_{xy}) + ic_2[H_{xy} - H_{yx} + i(H_{xx} + H_{yy})] = -192ic_2c_3c_4\Psi'(z)$$

$$\begin{aligned} 2c_1(H_{zy} + iH_{zx}) - R(2K_2 - K_1)[H_{xy} - H_{yx} + i(H_{xx} + H_{yy})] \\ = 96c_3cR(2K_2 - K_1)\Psi'(z) + 96c_1c_3c_6\Gamma'(z) \end{aligned}$$

$$c_5(\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy}) + ic_2R[H_{xy} - H_{yx} - i(H_{xx} + H_{yy})] = -16ic_2c_3c_4\Omega'(z)$$

$$H_{yx} + H_{xy} + i(H_{xx} - H_{yy}) = -480c_2c_5\overline{f_6'(z)} - 4c_1c_2R\Theta'(z) \tag{11.4.16}$$

Similar to the discussion of two-dimensional quasicrystals, from the equations, it is obvious that a state of phonon and phason stresses is not altered, if one replaces

$$f_i(z) \text{ by } f_i(z) + \gamma_i \qquad (i = 2, \ldots, 6) \tag{11.4.17}$$

where γ_i are the arbitrary complex constants.

Now, consider how these substitutions affect the components of the displacement vectors which were determined by the formula (11.4.6). Substituting (11.4.13) into (11.4.8)–(11.4.12) shows that if the complex constants $\gamma_i (i = 2, \ldots, 6)$ satisfy

$$\begin{aligned}
&3(\frac{2c_2}{\mu + \lambda} + c_7)\overline{\gamma_4} + c_7\gamma_3 = 0 \\
&24c_9\overline{\gamma_5} - c_8\gamma_2 = 0 \\
&40c_{10}\gamma_6 - c_1c_3R^2[4(1 - \frac{c_9}{c_8})\overline{\gamma_5} - \frac{2c_2}{(\mu + \lambda)c_7}\gamma_4] = 0
\end{aligned} \tag{11.4.18}$$

then the substitution (11.4.17) will not affect the displacements.

11.4.3.2 General Formulas for Finite Multi-connected Region

Consider now the case when the region S, occupied by the body, is multi-connected (see Fig. 11.1).

Since the stress must be single-valued and Eq. (11.4.16)

$$\sigma_{zy} - i\sigma_{zx} = -960c_3c_4f_6'(z) \tag{11.4.19}$$

we know that $f_6'(z)$ is holomorphic and hence single-valued in the region inside contour s_{m+1}, so the complex function can be expressed as follows:

$$f_6(z) = \int_{z_0}^{z} f_6'(z)\mathrm{d}z + \text{constant} \tag{11.4.20}$$

where z_0 denotes fixed point. From Eq. (11.4.20), we have

$$f_6(z) = b_k \ln(z - z_k) + f_{6*}(z) \tag{11.4.21}$$

$f_{6*}(z)$ is holomorphic in the region with contour s_{m+1}.

Substituting (11.4.21) into the second formula of Eq. (11.4.16), i.e.

$$c_1(\sigma_{yy} - \sigma_{xx} - 2i\sigma_{xy}) + ic_2[H_{xy} - H_{yx} + i(H_{xx} + H_{yy})] = -192ic_2c_3c_4\Psi'(z),$$

shows that $f_5'(z)$ is holomorphic in the region enclosed by contour s_{m+1}, so one has

$$f_5(z) = c_k \ln(z - z_k) + f_{5*}(z) \tag{11.4.22}$$

where $f_{5*}(z)$ is holomorphic in the region of interior of contour s_{m+1}.

Similar to the above-mentioned discussion, from Eqs. (11.4.16) to (11.4.18), the complex functions $f_i(i = 2, 3, 4)$ can be written as follows:

$$\begin{aligned} f_4(z) &= d_k \ln(z - z_k) + f_{4*}(z) \\ f_3(z) &= e_k \ln(z - z_k) + f_{3*}(z) \\ f_2(z) &= t_k \ln(z - z_k) + f_{2*}(z) \end{aligned} \tag{11.4.23}$$

where d_k, e_k and t_k are complex constants and $f_{i*}(z)$ $(i = 2, 3, 4)$ is holomorphic in the region inside contour s_{m+1}.

By substituting (11.4.21)–(11.4.23) into the complex expressions of displacements, the condition of single valuedness of displacements will be given as follows:

$$\begin{aligned} &-3\left(\frac{2c_2}{\mu + \lambda} + c_7\right)\overline{d_k} + c_7 e_k = 0 \\ &24c_9\overline{c_k} + c_8 t_k = 0 \\ &240c_{10}b_k + c_1c_3R^2(t_k - 2e_k + 6d_k - 24c_k) = 0 \end{aligned} \tag{11.4.24}$$

Applying the boundary conditions given above to the contour s_k and from Eq. (11.4.24), we know that the above complex constants may be very simply expressed in terms of surface traction and generalized surface traction as

$$\begin{aligned} b_k &= \frac{c_1c_3R^2}{240c_{10}}\left[\frac{12c_2}{(\mu + \lambda)c_7}\overline{d_k} + 24\left(1 + \frac{c_9}{c_8}\right)c_k\right] \\ c_k &= \frac{c_8}{-96\pi[c_3c_8(2c_4 - c_1R) - c_1c_3R]}(h_x - ih_y) \\ t_k &= \frac{c_8}{4\pi[c_3c_8(2c_4 - c_1R) - c_1c_3R]}(h_x + ih_y) \\ d_k &= \frac{(\mu + \lambda)c_7}{24\pi c_2c_3R(2c_2 + (\mu + \lambda)c_7)}(T_x + iT_y) \\ e_k &= -\frac{2c_2 + (\mu + \lambda)c_7}{16\pi c_2^2c_3R}(T_x - iT_y) \end{aligned} \tag{11.4.25}$$

We can easily extend the above results to the case there are m inner boundaries.

11.4.4 Case of Infinite Regions

From the point of view of application, the consideration of infinite regions is likewise of major interest. We assume that the contour s_{m+1} has entirely moved to infinity.

Similar to the discussion of two-dimensional quasicrystal, we have

$$\begin{aligned} f_6(z) &= \sum_{k=1}^{m} b_k \ln z + f_{6**}(z), \quad f_5(z) = \sum_{k=1}^{m} c_k \ln z + f_{5**}(z) \\ f_4(z) &= \sum_{k=1}^{m} d_k \ln z + f_{4**}(z), \quad f_3(z) = \sum_{k=1}^{m} e_k \ln z + f_{3**}(z) \\ f_2(z) &= \sum_{k=1}^{m} t_k \ln z + f_{2**}(z) \end{aligned} \tag{11.4.26}$$

where $f_{j**}(z)$ $(j = 2, \ldots, 6)$ are functions, holomorphic outside s_{m+1}, not including the point at infinity. By the theorem of Laurent, the function $h_{2*}(z)$ may be represented outside s_{m+1} by the series

$$f_{ji**}(z) = \sum_{-\infty}^{+\infty} a_{jn} z^n \quad (j = 2, \ldots, 6) \tag{11.4.27}$$

Substituting the first equation of (11.4.26) and (11.4.27) into the first one of Eq. (11.4.16), one has

$$\sigma_{zy} - i\sigma_{zx} = -960 c_3 c_4 \left(\sum_{k=1}^{m} b_k \frac{1}{z} + \sum_{-\infty}^{\infty} n a_{6n} z^{n-1}\right) \tag{11.4.28}$$

Hence it follows that for the stress to remain finite as $|z| \to \infty$, one must have

$$a_{6n} = 0 \ (n \geq 2) \tag{11.4.29}$$

Similarly, from Eqs. (11.4.15)–(11.4.18), to make the stresses be bounded, the following conditions are also to be satisfied

$$a_{jn} = 0 \ (n \geq 2, j = 2, \ldots, 5) \tag{11.4.30}$$

So we can obtain the expressions of the complex function $f_i(z) (i = 2, \ldots, 6)$ for the stresses to remain finite as $|z| \to \infty$, for example

$$f_6(z) = \sum_{k=1}^{m} b_k \ln z + (B + iC) z + f_6^0(z) \tag{11.4.31}$$

where B, C are unknown real constants to be determined, $f_6^0(z)$ is function, holomorphic outside s_{m+1}, including the point at infinity. The determination of unknown constants B, C is similar to that given in Sect. 11.3.4, but the details are omitted here due to the limitation of the space.

11.4.5 Conformal Mapping and Function Equations at ζ-Plane

We now have five equations of boundary value (11.4.10)–(11.4.12) and (11.3.14), from which the unknown functions $f_j(z) \quad (j = 2, \ldots, 6)$ will be determined; in addition, we have assumed that $f_1(z) = 0$, because it has no usage. For some complicated regions, the function equations cannot be directly solved at the physical plane (i.e. the z-plane), and the conformal mapping is particularly meaningful in the case.

Assume that a conformal mapping

$$z = \omega(\zeta) \tag{11.4.32}$$

is used to transform the region at z-plane onto the interior of the unit circle γ at ζ-plane. Under the mapping, the unknown functions $f_j(z)$ become

$$f_j(z) = f_j[\omega(\zeta)] = \Phi_j(\zeta) \qquad (j = 2, \ldots, 6) \tag{11.4.33}$$

Substituting (11.4.32) and (11.4.33) into the first relation of boundary conditions (11.3.14) yields

$$\frac{1}{2\pi i}\int_\gamma \frac{\Phi_6(\sigma)}{\sigma - \varsigma}\mathrm{d}\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\Phi_6(\sigma)}}{\sigma - \varsigma}\mathrm{d}\sigma = 0$$

This shows

$$\Phi_6(\varsigma) = 0 \tag{11.4.34}$$

according to the Cauchy integral formula.

Substitution of (11.4.32), (11.4.33), and (11.4.34) into boundary conditions (11.4.10)–(11.4.12) and the second one of condition (11.3.14) leads to the boundary value equations to determine the unknown functions $\Phi_j(\zeta)$ $(j = 2, \ldots, 5)$ at ζ-plane, i.e.

$$\begin{aligned}&\frac{3}{2\pi i}\int\limits_{\gamma}\frac{\overline{\Phi_4(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma+\frac{4}{2\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_5'(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma-\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\Phi_3(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma\\&-3\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\Phi_4'(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma-6\frac{1}{2\pi i}\int\limits_{\gamma}[\frac{[\omega(\sigma)]^2\overline{\Phi_5''(\sigma)}}{\overline{\omega'(\sigma)}^2}\\&-\frac{[\omega(\sigma)]^2\overline{\omega''(\sigma)}}{\overline{\omega'(\sigma)}^3}\overline{\Phi_5'(\sigma)}]\frac{\mathrm{d}\sigma}{\sigma-\varsigma}=\frac{1}{4c_2c_3}\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\mathbf{t}}{\sigma-\varsigma}\mathrm{d}\sigma\end{aligned}\tag{11.4.35}$$

$$\begin{aligned}&\frac{3}{2\pi i}\int\limits_{\gamma}\frac{\overline{\Phi_4(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma+\frac{4}{2\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_4'(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma-\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\Phi_3(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma\\&-3\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_3(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma-6\frac{1}{2\pi i}\int\limits_{\gamma}[\frac{\overline{\omega(\sigma)}^2\Phi_5''(\sigma)}{[\omega'(\sigma)]^2}\\&-\frac{\overline{\omega(\sigma)}^2\omega''(\sigma)\Phi_5'(\sigma)}{[\omega'(\sigma)]^3}]\frac{\mathrm{d}\sigma}{\sigma-\varsigma}=\frac{1}{4c_2c_3R}\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\bar{\mathbf{t}}}{\sigma-\varsigma}\mathrm{d}\sigma\end{aligned}\tag{11.4.36}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\Phi_2(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma+2\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_3'(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma+3\frac{1}{2\pi i}\int\limits_{\gamma}[\frac{\overline{\omega(\sigma)}^2\Phi_4''(\sigma)}{[\omega'(\sigma)]^2}\\&-\frac{\overline{\omega(\sigma)}^2\omega''(\sigma)\Phi_4'(\sigma)}{[\omega'(\sigma)]^3}]\frac{\mathrm{d}\sigma}{\sigma-\varsigma}+4\frac{1}{2\pi i}\int\limits_{\gamma}[\frac{\overline{\omega(\sigma)}^2\Phi_5'''(\sigma)}{[\omega'(\sigma)]^3}-3\frac{\overline{\omega(\sigma)}^3\omega''(\sigma)\Phi_5''(\sigma)}{[\omega'(\sigma)]^4}\\&+3\frac{\overline{\omega(\sigma)}^3\omega''(\sigma)\Phi_5'(\sigma)}{[\omega'(\sigma)]^5}-\frac{\overline{\omega(\sigma)}^3\omega'''(\sigma)\Phi_5'(\sigma)}{[\omega'(\sigma)]^4}]\frac{\mathrm{d}\sigma}{\sigma-\varsigma}=\frac{1}{2\pi i}\int\frac{\mathbf{h}}{\sigma-\varsigma}\mathrm{d}\sigma\end{aligned}\tag{11.4.37}$$

$$\frac{4c_{11}}{2\pi i}\int\limits_{\gamma}\frac{\Phi_5(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma+\frac{(2K_2-K_1)R}{2\pi i}\int\limits_{\gamma}[\frac{\Phi_4(\sigma)}{\sigma-\varsigma}+4\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_5'(\sigma)}{\sigma-\varsigma}]\mathrm{d}\sigma=0\tag{11.4.38}$$

in which $\mathbf{t}=i\int(T_x+iT_y)\mathrm{d}s, \bar{\mathbf{t}}=-i\int(T_x-iT_y)\mathrm{d}s, \mathbf{h}=i\int(h_1+ih_2)\mathrm{d}s$. For given configuration and applied stresses, we can obtain the solution by solving these function equations.

11.4.6 Example: Elliptic Notch Problem and Solution

We consider an icosahedral quasicrystal solid with an elliptic notch, which penetrates through the medium along the z-axis direction, the edge of the elliptic notch subjected to the uniform pressure p, similar to Fig. 11.2.

Since the measurement of generalized traction has not been reported so far, for simplicity, we assume that $h_x = 0, h_y = 0$.

However the calculation cannot be completed at the z-plane owing to the complicity, and we have to employ the conformal mapping

$$z = \omega(\zeta) = R_0(\frac{1}{\zeta} + m\zeta) \tag{11.4.38}$$

to transform the exterior of the ellipse at the z-plane onto the interior of the unit circle γ at the ζ-plane, in which

$$R_0 = (a+b)/2, \quad m = (a-b)/(a+b)$$

Let

$$f_j(z) = f_j[\omega(\zeta)] = \Phi_j(\zeta) \qquad (j = 2, \ldots, 6) \tag{11.4.39}$$

Substituting (11.4.38) into the first formula of (11.4.25), then multiplying on both sides of equations by $\mathrm{d}\sigma/[2\pi i(\sigma - \zeta)]$ (σ represents the value of at the unit circle), and integrating around the unit circle γ yield

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{\Phi_6(\sigma)}{\sigma - \varsigma}\mathrm{d}\sigma + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\overline{\Phi_6(\sigma)}}{\sigma - \varsigma}\mathrm{d}\sigma = 0 \tag{11.4.40}$$

by means of Cauchy integral formula, we have

$$\Phi_6(\varsigma) = 0 \tag{11.4.41}$$

Substituting (11.4.38) and (11.4.41) into (11.4.22)–(11.4.24), then multiplying both sides of equations by $\mathrm{d}\sigma/[2\pi i(\sigma - \zeta)]$ (σ represents the value of at the unit circle), and integrating around the unit circle γ yields

$$
\begin{aligned}
&\frac{3}{2\pi i}\int_{\gamma}\frac{\Phi_4(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma+\frac{4}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_5'(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma-\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\Phi_3(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma\\
&-3\frac{1}{2\pi i}\int_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\Phi_4'(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma-6\frac{1}{2\pi i}\int_{\gamma}\Big[\frac{[\omega(\sigma)]^2\overline{\Phi_5''(\sigma)}}{[\overline{\omega'(\sigma)}]^2}\\
&-\frac{[\omega(\sigma)]^2\overline{\omega''(\sigma)}}{[\overline{\omega'(\sigma)}]^3}\overline{\Phi_5'(\sigma)}\Big]\frac{\mathrm{d}\sigma}{\sigma-\varsigma}=\frac{p}{4c_2c_3}\int_{\gamma}\frac{\omega(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma
\end{aligned}
\tag{11.4.42}
$$

$$
\begin{aligned}
&\frac{3}{2\pi i}\int_{\gamma}\frac{\overline{\Phi_4(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma+\frac{4}{2\pi i}\int_{\gamma}\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\Phi_5'(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma-\frac{1}{2\pi i}\int_{\gamma}\frac{\Phi_3(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma\\
&-3\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_4'(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma-6\frac{1}{2\pi i}\int_{\gamma}\Big[\frac{\overline{\omega(\sigma)}^2\Phi_5''(\sigma)}{[\omega'(\sigma)]^2}\\
&-\frac{\overline{\omega(\sigma)}^2\omega''(\sigma)\Phi_5'(\sigma)}{[\omega'(\sigma)]^3}\Big]\frac{\mathrm{d}\sigma}{\sigma-\varsigma}=\frac{p}{4c_2c_3R}\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma
\end{aligned}
\tag{11.4.43}
$$

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma}\frac{\Phi_2(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma+2\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_3'(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma+3\frac{1}{2\pi i}\int_{\gamma}\Big[\frac{\overline{\omega(\sigma)}^2\Phi_4''(\sigma)}{[\omega'(\sigma)]^2}\\
&-\frac{\overline{\omega(\sigma)}^2\omega''(\sigma)\Phi_4'(\sigma)}{[\omega'(\sigma)]^3}\Big]\frac{\mathrm{d}\sigma}{\sigma-\varsigma}+4\frac{1}{2\pi i}\int_{\gamma}\Big[\frac{\overline{\omega(\sigma)}^3\Phi_5'''(\sigma)}{[\omega'(\sigma)]^3}-3\frac{\overline{\omega(\sigma)}^3\omega''(\sigma)\Phi_5''(\sigma)}{[\omega'(\sigma)]^4}\\
&+3\frac{\overline{\omega(\sigma)}^3\omega''(\sigma)\Phi_5'(\sigma)}{[\omega'(\sigma)]^5}-\frac{\overline{\omega(\sigma)}^3\omega'''(\sigma)\Phi_5'(\sigma)}{[\omega'(\sigma)]^4}\Big]\frac{\mathrm{d}\sigma}{\sigma-\varsigma}=0
\end{aligned}
\tag{11.4.44}
$$

$$
\frac{4c_{11}}{2\pi i}\int_{\gamma}\frac{\Phi_5(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma+\frac{(2K_2-K_1)R}{2\pi i}\int_{\gamma}\Big[\frac{\Phi_4(\sigma)}{\sigma-\varsigma}+4\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{\Phi_5'(\sigma)}{\sigma-\varsigma}\Big]\mathrm{d}\sigma=0 \tag{11.4.45}
$$

Because

$$
\frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}=\sigma\frac{\sigma^2+m}{m\sigma^2-1}
$$

and

$$\zeta\frac{\zeta^2+m}{m\zeta^2-1}\Phi_5'(\zeta)=\zeta\frac{\zeta^2+m}{m\zeta^2-1}(\alpha_1+2\alpha_2\zeta+3\alpha_3\zeta^2+\cdots)$$

are analytic in $|\zeta|<1$ and continuous in the unit circle γ, by means of Cauchy integral formula, from Eq. (11.4.42), we have

$$\frac{1}{2\pi i}\int_\gamma\frac{\Phi_4(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma=\Phi_4(\zeta)$$

$$\frac{1}{2\pi i}\int_\gamma\sigma\frac{\sigma^2+m}{m\sigma^2-1}\frac{\Phi_5'(\sigma)}{\sigma-\varsigma}\mathrm{d}\sigma=\zeta\frac{\zeta^2+m}{m\zeta^2-1}\Phi_5'(\zeta)$$

Substituting

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}=-\frac{1}{\sigma}\frac{m\sigma^2+1}{\sigma^2-m},\quad\frac{\omega(\sigma)^2\overline{\omega''(\sigma)}}{\overline{\omega'(\sigma)}^3}=\frac{2\sigma(m\sigma^2+1)^2}{(\sigma^2-m)^3}$$

into Eq. (11.4.42), and note that

$$-\frac{1}{\zeta}\frac{m\zeta^2+1}{\zeta^2-m}\overline{\Phi_4'(\zeta)}=-\frac{1}{\zeta}\frac{m\zeta^2+1}{\zeta^2-m}(\overline{\beta_1}+2\frac{\overline{\beta_2}}{\zeta}+3\frac{\overline{\beta_3}}{\zeta^2}+\cdots)$$
$$\frac{2\zeta(m\zeta^2+1)^2}{(\zeta^2-m)^3}\overline{\Phi_5'(\zeta)}=\frac{2\zeta(m\zeta^2+1)^2}{(\zeta^2-m)^3}(\overline{\alpha_1}+2\frac{\overline{\alpha_2}}{\zeta}+3\frac{\overline{\alpha_3}}{\zeta^2}+\cdots)$$

are analytic in $|\zeta|>1$ and continuous in the unit circle γ, by means of Cauchy integral formula and analytic extension of the complex variable function theory; from Eq. (11.4.42), we obtain

$$\frac{1}{2\pi i}\int_\gamma\frac{\overline{\Phi_3(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma=0,\quad\frac{1}{2\pi i}\int_\gamma\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\Phi_4'(\sigma)}}{\sigma-\varsigma}\mathrm{d}\sigma=0$$
$$\frac{1}{2\pi i}\int_\gamma[\frac{\omega(\sigma)^2\overline{\Phi_5''(\sigma)}}{\overline{\omega'(\sigma)}^2}-\frac{\omega(\sigma)^2\overline{\omega''(\sigma)}}{\overline{\omega'(\sigma)}^3}\overline{\Phi_5'(\sigma)}]\frac{\mathrm{d}\sigma}{\sigma-\varsigma}=0$$

Substituting the above results into Eq. (11.4.42), with the help of Eq. (11.4.45), one has

$$\begin{aligned}\Phi_4(\zeta) &= \frac{R_0}{12c_2c_3R}pm\zeta - \frac{(2K_2-K_1)R_0}{2c_2c_3C_{11}}\frac{pm\zeta(\zeta^2+m)}{(m\zeta^2-1)} \\ \Phi_5(\zeta) &= -\frac{(2K_2-K_1)R_0}{48c_2c_3C_{11}}pm\zeta\end{aligned} \tag{11.4.46}$$

Similar to the above discussion, from Eqs. (11.4.43) and (11.4.44), one has

$$\begin{aligned}\Phi_2(\zeta) &= -\frac{R_0}{2c_2c_3R}\frac{p\zeta(\zeta^2+m)(m^3\zeta^2+1)}{(m\zeta^2-1)^3} \\ &+ \frac{(2K_2-K_1)R_0}{2c_2c_3C_{11}}\frac{pm\zeta^3(\zeta^2+m)[m^2\zeta^6-(m^3+4m)\zeta^4+(2m^4+4m^2+5)\zeta^2+m]}{(m\zeta^2-1)^5} \\ \Phi_3(\zeta) &= -\frac{R_0}{4c_2c_3R}\frac{p\zeta(m^2+1)}{(m\zeta^2-1)} - \frac{(2K_2-K_1)R_0}{12c_2c_3C_{11}}\frac{pm\zeta^3(\zeta^2+m)(m\zeta^2-m^2-2)}{(m\zeta^2-1)^3}\end{aligned} \tag{11.4.47}$$

The elliptic notch problem is solved. The solution of the Griffith crack subjected to a uniform pressure can be obtained corresponding to the case $m = 1, R_0 = a/2$ of the above solution. The solution of crack can be expressed explicitly in the z-plane, and the concrete results refer to Sect. 9.7 in Chap. 9 for the concrete results.

11.5 Complex Analysis of Generalized Quadruple Harmonic Equation

In Chaps. 6–8, we have shown that the plane elasticity of octagonal quasicrystals is governed by the final equation

$$(\nabla^2\nabla^2\nabla^2\nabla^2 - 4\varepsilon\nabla^2\nabla^2\Lambda^2\Lambda^2 + 4\varepsilon\Lambda^2\Lambda^2\Lambda^2\Lambda^2)F = 0 \tag{11.5.1}$$

either by displacement potential or by stress potential, in which

$$\left.\begin{aligned}\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Lambda^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \\ \varepsilon &= \frac{R^2(L+M)(K_2+K_3)}{[M(K_1+K_2+K_3)-R^2][(L+2M)K_1-R^2]}\end{aligned}\right\} \tag{11.5.2}$$

Due to the appearance of operator Λ^2, it seems there is no any connection with complex variable functions in solving Eq. (11.5.1). But if we rewrite it as

$$[\frac{\partial^8}{\partial x^8} + 4(1-4\varepsilon)\frac{\partial^8}{\partial x^6\partial y^2} + 2(3+16\varepsilon)\frac{\partial^8}{\partial x^4\partial y^4} + 4(1-4\varepsilon)\frac{\partial^8}{\partial x^2\partial y^6} + \frac{\partial^8}{\partial y^8}]F = 0 \tag{11.5.3}$$

then find that this is one of typical multi-quasiharmonic partial differential equation with quadruple, and there is complex representation of solution such as

$$F(x,y) = 2\mathrm{Re}\sum_{k=1}^{4} F_k(z_k), z_k = x + \mu_k y \tag{11.5.4}$$

in which functions $F_k(z_k)$ are analytic functions of complex variable z_k $(k = 1, \ldots, 4)$ and $\mu_k = \alpha_k + i\beta_k$ $(k = 1, \ldots, 4)$ are complex parameters and determined by the roots of the following eigenvalue equation

$$\mu^8 + 4(1 - 4\varepsilon)\mu^6 + 2(3 + 16\varepsilon)\mu^4 + 4(1 - 4\varepsilon)\mu^2 + 1 = 0 \tag{11.5.5}$$

We have shown that in Chaps. 7 and 8, some solutions of dislocations (based on the displacement potential formulation) and notchs/cracks (based on the stress potential formulation) can be found in terms of this complex analysis. In the procedure, it must carry out some calculations on determinants of fourth order, so the solution expressions are quite lengthy, but which are analytic substantively.

11.6 Conclusion and Discussion

The discovery of quadruple and sextuple harmonic equations is significant for modern elasticity. This chapter gives a comprehensive discussion on the complex analysis for solving the equations, and we think the study is preliminary.

The above-mentioned complex potential approach is a new development of Muskhelishvili approach of the classical elasticity, which extends greatly the scope of the method. We believe the quadruple and sextuple harmonic equations are useful not only in quasicrystals but probably also in other disciplines of science and engineering. So the complex analysis method can be used for other studies.

Apart from the development to extend the scope of the complex potential theory and method, we also developed the Muskhelishvili method for the conformal mapping. According to the monograph [1], the conformal mapping is limited within the rational function class. But we extended it into the transcendental function class, and some exact analytic solutions for more complicated cracked configurations are achieved (see, e.g. Chap. 8).

This method is effective not only for solving elasticity problems but also for solving plasticity problem (see, e.g. Li and Fan [10] and Fan and Fan [11] and Li and Fan [12, 13]). The new summarization on the method can be found in article [14] and other references [15, 16].

11.7 Appendix of Chapter 11: Basic Formulas of Complex Analysis

It is enlightened that Muskhelishvili [1] gave extensive description in detail on complex analysis in due presentation of elasticity in his classical monograph, which is very beneficial to readers. However there is no possibility for the present book. We provide here some points only of the function theory, which were frequently cited in the text. These can be referred for readers who are advised to read books of Privalov [17] and Lavrentjev and Schabat [18] for the further details. Other knowledge has been provided in due succession of the text of Chaps. 7–9 and 11. The present contents can also be seen as a supplement in reading the material given in Chaps. 7–9 and 11 if it is needed. The importance of complex analysis is not only in deriving the solutions by the complex potential formulation but also in dealing with the solutions by integral transforms and dual integral equations to be discussed in the Appendix B of Major Appendix of this book.

11.7.1 Complex Functions, Analytic Functions

Usually, $z = x + iy$ is denoted as a complex variable in which $\sqrt{-1} = i$, or $z = re^{i\theta}$, and $r = \sqrt{x^2 + y^2}$, called the modulus of the complex number, $\theta = \arctan(\frac{y}{x})$, the argument angle of z. Assume $f(z)$ be a function of one complex variable, or complex function in abbreviation, which is denoted as

$$f(z) = P(x,y) + iQ(x,y) \tag{11.7.1}$$

in which both $P(x,y)$ and $Q(x,y)$ are functions with real variables and called the real and imaginary parts, respectively, and marked by

$$P(x,y) = \mathrm{Re}f(z), Q(x,y) = \mathrm{Im}f(z)$$

There is a sort of complex functions called analytic functions (or regular functions; single-valued analytic functions are called holomorphic functions) which have important applications in many branches of mathematics, physics, and engineering. The concepts related with this are discussed as follows.

The complex function $f(z)$ is analytic in a given region, and this means that it can be expanded in the neighbourhood of any point z_0 of the region into a non-negative integer power series (i.e. the Taylor series) of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{11.7.2}$$

in which a_n is a constant (in general, a complex number). The concept are frequently used in the previous and later calculation.

Another definition of an analytic function is that if the complex function $f(z)$ is given in the region, the real part $P(x, y)$ and imaginary part $Q(x, y)$ are single-valued, have continuous partial derivatives of the first order, and satisfy Cauchy–Riemann condition such as

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \tag{11.7.3}$$

in the region.

These kind of functions, P and Q, are named mutually conjugate harmonic ones. From (11.7.3), it follows that

$$\nabla^2 P = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})P = 0, \nabla^2 Q = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})Q = 0$$

This concept is also often used in the following.

An analytic function can also be defined in integral form. Assuming $f(z)$ is a complex function in a certain complex number region D, and Γ is any simple smooth closed curve (sometimes called simple curve for simplicity) in D, we can obtain that $f(z)$ is analytic in the region if

$$\int_\Gamma f(z)\mathrm{d}z = 0 \tag{11.7.4}$$

The result is known as the Cauchy's integral theorem (or simply called the Cauchy's theorem) which has been frequently used in the text and appendixes.

The theory of complex functions proves that the above definitions are mutually equivalent.

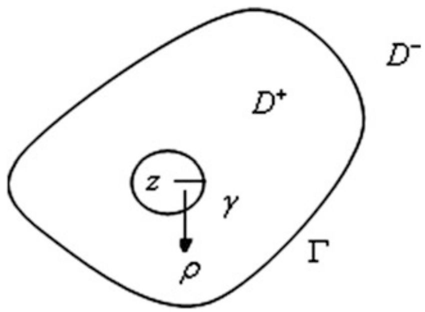
11.7.2 Cauchy's formula

An important result of the Cauchy's theorem is the so-called Cauchy's formula, i.e. if $f(z)$ analytic in a single-connected region D^+ bounded by a closed curve Γ and continuous in $D^+ + \Gamma$ (Fig. 11.6), then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\mathrm{d}t = f(z) \tag{11.7.5}$$

in which z is an arbitrary point in D^+.

Fig. 11.6 A finite region D^+



Proof Taking z as the centre, ρ as the radius, make a small circle γ in D^+. According to Cauchy's theorem (11.7.4),

$$\int_{\Gamma} \frac{f(t)}{t-z}\mathrm{d}t = \int_{\gamma} \frac{f(t)}{t-z}\mathrm{d}t \tag{11.7.6}$$

As $f(z)$ is analytic in D^+ and continuous in $D^+ + \Gamma$, there is a small number $\varepsilon > 0$, for any point t and γ, if ρ is sufficiently small, such as

$$|f(t) - f(z)| < \varepsilon$$

and note that $|t - z| = \rho$, hence

$$\lim_{\varepsilon \to 0} \int_{\gamma} \frac{f(t)}{t-z}\mathrm{d}t = \int_{\gamma} \frac{f(z)}{t-z}\mathrm{d}t \tag{11.7.7}$$

Just as mentioned previously, $f(z)$ is analytic in D^+, and the value of the integral

$$\int_{\gamma} \frac{f(z)}{t-z}\mathrm{d}t$$

will not be changed when ρ is reducing. Therefore the limit mark in the left-hand side of (11.7.7) can be removed. In addition

$$\int_{\gamma} \frac{f(z)}{t-z}\mathrm{d}t = f(z)\int_{\gamma} \frac{\mathrm{d}t}{t-z} = f(z)\int_{0}^{2\pi} \frac{\rho e^{i\theta}}{\rho e^{i\theta}}\mathrm{d}\theta = 2\pi i f(z)$$

Based on (11.7.6) and this result, formula (11.7.5) is proved.

In formula (11.7.5), if z is taken its values in a region D^- consisting of the points lying outside Γ (see Fig. 11.6), then

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(t)}{t-z}\mathrm{d}t = 0 \tag{11.7.8}$$

In fact, this is a direct consequence of the Cauchy's theorem, because in this case the integrand $f(z)/(\zeta - z)$ as function of ζ is analytic in region D^+, where ζ denotes the point in the region D^+.

Suppose all conditions are the same as those for (11.7.5), then

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\overline{f(t)}}{t-z}\mathrm{d}t = \overline{f(0)} \tag{11.7.9}$$

Proof For simplicity here the proof is given for the case Γ being a circle. Being analytic in the region D^+, $f(z)$ may be expanded non-negative integer power series, in which $z_0 = 0$, such that

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = f(0) + f'(0)z + \frac{1}{2!}f''(0)z^2 + \cdots$$

The function $\overline{f(z)}$ in formula (11.7.9) is the value of $\bar{f}(\frac{1}{z})$ at the circle Γ, and here

$$\bar{f}(\frac{1}{z}) = \overline{f(0)} + \overline{f'(0)}\frac{1}{z} + \frac{1}{2!}\overline{f''(0)}\frac{1}{z^2} + \cdots$$

is an analytic function in D^-. From the Cauchy's formula,

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\mathrm{d}t}{t^k(t-z)} = \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}$$

such that (11.7.9) is proved.

In contrast to the above, current function $f(z)$ is analytic in D^- (including $z = \infty$), and then

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{t-z}\mathrm{d}t = \begin{cases} -f(z) + f(\infty) & z \in D^- \\ f(\infty) & z \in D^+ \end{cases} \tag{11.7.10}$$

The proof of this formula can be offered in the similar manner adopted for (11.7.5), but the following points must be noted:

(i) The analytic function $f(z)$ in D^- (including $z=\infty$) may be expanded as the following series

$$f(z) = c_0 + c_1\frac{1}{z} + c_2\frac{1}{z^2} + \cdots$$

(ii) $\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{c_0}{t-z}\mathrm{d}t = \begin{cases} 0 & z\in D^- \\ c_0 & z \in D^+ \end{cases}$

where $c_0 = f(\infty) \neq 0$.

All conditions are the same as that for formula (11.7.10), and there exists

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{\overline{f(t)}}{t-z}\mathrm{d}t = 0 \tag{11.7.11}$$

11.7.3 Poles

Suppose a finite point in z-plane (i.e. z is not a point at infinity), and in the neighbourhood of the point, the function presents the form as follows:

$$f(z) = G(z) + f_0(z) \tag{11.7.12}$$

in which $f_0(z)$ is an analytic function in the neighbourhood of point a, and

$$G(z) = \frac{A_0}{z-a} + \frac{A_1}{(z-a)^2} + \cdots\frac{A_m}{(z-a)^m} \tag{11.7.13}$$

where $A_1, A_2, \ldots, A_m$ are constants, such that $f(z)$ is called having a pole with order m and $z=a$ is the pole.

If a is a point at infinity, $f_0(z)$ in (11.7.12) is regular at point at infinity (i.e. $f(t) = c_0 + c_1z^{-1} + c_2z^{-2} + \cdots$), while at $z=\infty$

$$G(z) = A_0 + A_1z + \cdots + A_mz^m \tag{11.7.14}$$

then we say that $f(z)$ has a pole of order m at $z=\infty$.

11.7.4 Residual Theorem

If the function $f(z)$ has pole a with order m, its integral may be evaluated simply by computing residual.

What is the meaning of the residual? Suppose $f(z)$ is analytic in the neighbourhood of point $z = a$, but except $z = a$, and infinite at $z = a$. In this case, the point $z = a$ is named isolated singular point. The residual of the function $f(z)$ at point $z = a$ is the value of the integral

$$\frac{1}{2\pi i}\int_{\Gamma} f(z)\mathrm{d}z$$

in which Γ represents any closed contour enclosing point $z = a$. For a residual, we will use the resignation as $\mathrm{Res}\, f(a)$.

If $z = a$ is a m-order pole of $f(z)$, its residual may be evaluated from the following formula and

$$\mathrm{Res}\, f(a) = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} \tag{11.7.15}$$

Obviously, the integral is

$$\int_{\Gamma} f(z)dz = 2\pi i \mathrm{Res}\, f(a)$$

So the evaluation of integrals may be reduced to the calculation of derivatives, and it is greatly simplified. In particular, if $z = a$ is a first-order pole, then

$$\mathrm{Res}\, f(a) = \lim_{z\to a}(z-a)f(z) \tag{11.7.16}$$

in which the calculation is much simpler.

What follows the residual theorem is introduced as: let the function $f(z)$ be analytic in region D and continuous in $D+\Gamma$ except at finite isolated poles $a_1, a_2, \ldots, a_n$, then

$$\int_{\Gamma} f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res} f\,(a_k) \tag{11.7.17}$$

where Γ represents the boundary of region D.

Almost all integrals in the text can be evaluated by the residual theorem.

Example Calculate the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{-m\omega^2 + k} e^{-i\omega t} \mathrm{d}\omega = I \tag{11.7.18}$$

in terms of the residual theorem, where m and k are positive constants.

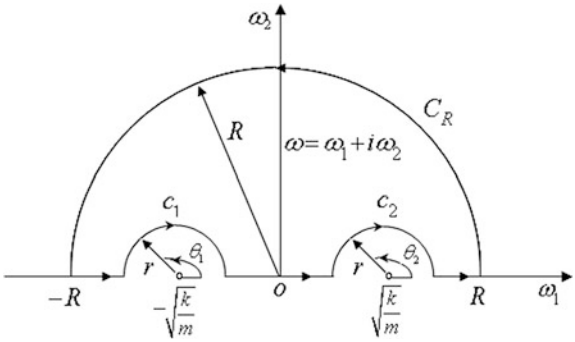


Fig. 11.7 Integration path at ω-plane

Though the integral is a real integral, it is difficult to evaluate because the integration limit is infinite and there are two singular points at the integration path, but it is easily completed by using the residual theorem. At first, we extend the real variable ω to a complex one, i.e. put $\omega = \omega_1 + i\omega_2$, where ω_1, ω_2 are real variables. At the complex plane ω, a half-circle with origin $(0, 0)$ and radius $R \to \infty$ is taken as an additional integral path, referring to Fig. 11.7. Along the real axis, the integrand of the integral has two poles $(-\sqrt{k/m}, 0)$ and $(\sqrt{m/k}, 0)$, and the value of the integral is equal to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{-m\omega^2+k}e^{-i\omega t}d\omega = I_1 = \lim_{R\to\infty, r\to 0}\left(\int_{C_R}+\int_1+\int_2+\int_3+\int_{C_1}+\int_{C_2}\right) \tag{11.7.19}$$

where the first integral in the right-hand side of (11.7.19) is carried out on path of the grand half-circle, the second to fourth ones are on the path along the real axes except intervals $(-r-\sqrt{k/m}, -\sqrt{k/m}+r)$ and $(-r+\sqrt{k/m}, \sqrt{k/m}+r)$, and the fifth and sixth ones are on two small half-circle arcs C_1 and C_2 with origins $(-\sqrt{k/m}, 0)$ and $(\sqrt{k/m}, 0)$ and radius r, respectively. Because the integrand in the interior enclosing by the integration path in (11.7.19) is analytic, according to the Cauchy theorem [referring to formula (11.7.3)]

$$I_1 = 0 \tag{11.7.20}$$

Based on the behaviour of the integrand and the Jordan lemma, the first one in the right-hand side of (11.7.19) must be zero. So that

$$\lim_{R\to\infty, r\to 0}\left(\int_1+\int_2+\int_3+\int_{C_1}+\int_{C_2}\right) = 0$$

and

$$\lim_{R\to\infty, r\to 0}\left(\int_1 + \int_2 + \int_3\right) = I = -\lim_{r\to 0}\left(\int_{C_1} + \int_{C_2}\right)$$

At arc C_1: $\omega + \sqrt{k/m} = re^{i\theta_1}, \mathrm{d}\omega = ire^{i\theta_1}\mathrm{d}\theta_1$, and at arc C_2: $\omega - \sqrt{k/m} = re^{i\theta_2}, \mathrm{d}\omega = ire^{i\theta_2}\mathrm{d}\theta_2$. Substituting these into the above integrals and after some simple calculations, we obtain

$$I = \frac{\pi}{m\sqrt{k/m}}\sin\sqrt{k/m}t \tag{11.7.21}$$

In the inversion of some integral transforms and even in the solution of certain integral equations, many key calculations are completed by the similar procedure exhibited above, which will be shown in the Major Appendix B of this book.

11.7.5 Analytic Extension

A function $f_1(z)$ is analytic at region D_1, and if one can construct another function $f_2(z)$ analytic at region D_2, D_1 and D_2 are not mutually intersected regions but with common bounding Γ, furthermore

$$f_1(z) = f_2(z) \quad z \in \Gamma$$

we can say that $f_1(z)$ and $f_2(z)$ are analytic extension to each other, and we can also say that function

$$F(z) = \begin{cases} f_1(z) & as\ z \in D_1 \\ f_2(z) & as\ z \in D_2 \end{cases}$$

analytic at $D = D_1 + D_2$ is an analytic extension of $f_1(z)$ as well as $f_2(z)$.

11.7.6 Conformal Mapping

In the text of Chaps. 7–9 and 11, by using one or several analytic functions which are also named complex potentials, we have expressed the solutions of harmonic, biharmonic, quadruple harmonic, sextuple harmonic, quasi-biharmonic, and quasi-quadruple harmonic equations, which is the complex representation of solutions. We can see that the complex representation is only the first step for solving boundary value problems. For some problems with complicated boundaries, one must utilize the conformal mapping to transform the problem onto the mapping plane; the corresponding boundaries can be simplified to a unit circle or straight line; the calculation can be put forward; and in some cases, exact analytic solutions are available.

The so-called conformal mapping is that the complex variable $z = x + iy$ and another one $\zeta = \xi + i\eta$ can be connected by

$$z = \omega(\zeta) \tag{11.7.22}$$

in which $\omega(\zeta)$ is a single-valued analytic function of $\zeta = \xi + i\eta$ in some region. Except certain points, the inversion of mapping (11.7.22) exists. If for a certain region, the mapping is single-valued, and we say it is a single-valued conformal mapping. In general, the mapping is single-valued, but the inversion $\zeta = \omega^{-1}(z)$ is impossibly single-valued. It has the following properties:

(1) A angle at point $z = z_0$ after the mapping becomes a angle at point $\zeta = \zeta_0$, but the both angles have the same value of the argument, the rotation is either in the same direction, and this is the first kind of conformal mapping (e.g. shown in Fig. 11.5), or in counter direction, which is the second kind of conformal mapping (e.g. depicted in Fig. 11.3).

(2) If $\omega(\zeta)$ is analytic and single-valued in region Ω and transforms the region into region D, then the inversion $\zeta = \omega^{-1}(z)$ is analytic and single-valued in region D and maps D onto Ω.

(3) If D is a region and c is a simple closed curve in it, and its interior belongs to D, and if $\omega^{-1}(z)$ is analytic, and maps c onto a closed curve γ at Ω region bilaterally single-valued, then $\omega(\zeta)$ is analytic and single-valued in the region and maps D onto the interior of Ω.

In the text, we mainly used the following two kinds of conformal mapping, i.e. (1) Rational function conformal mapping, e.g.

$$\omega(\zeta) = \frac{c}{\zeta} + a_0 + a_1\zeta + \ldots + a_n\zeta^n \tag{11.7.23}$$

or

$$\omega(\zeta) = R\zeta + b_0 + b_1\frac{1}{\zeta} + \ldots + b_n\frac{1}{\zeta^n} \tag{11.7.24}$$

in which, c, a_0, a_1, …, a_n, R, b_0, b_1, …, b_n are constants. These mappings can be used in studying infinite region with a crack at physical plane onto the interior of unit circle at mapping plane. In the monograph of Muskhelishvili [1], he postulated that his method is only suitable for this kind of mapping functions. Fan [4] extended it to transcendental mapping functions and achieved exact analytic solutions for crack problems for complicated configuration.

(2) Transcendental functions are as follows:

$$\omega(\zeta) = \frac{H}{\pi}\ln[1 + \frac{(1+\zeta)^2}{(1-\zeta)^2}] \tag{11.7.25}$$

and

$$\omega(\zeta) = \frac{2W}{\pi}\arctan\left\{\sqrt{1-\zeta^2}\tan\left(\frac{\pi a}{2W}\right)\right\} - a \tag{11.7.26}$$

which can be used to transform a finite specimen with a crack onto the interior of unit circle or upper half-plane (or lower half-plane) at mapping plane, where H, W, and a represent sample sizes and crack size.

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