Variations on the Grothendieck–Serre Formula for Hilbert Functions and Their Applications

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Abstract In this expository paper, we present proofs of Grothendieck–Serre formula for multi-graded algebras and Rees algebras for admissible multi-graded filtrations. As applications, we derive formulas of Sally for postulation number of admissible filtrations and Hilbert coefficients. We also discuss a partial solution of Itoh's conjecture by Kummini and Masuti. We present an alternate proof of Huneke–Ooishi Theorem and a generalisation for multi-graded filtrations.

Keywords Hilbert polynomial · Admissible filtration · Normal Hilbert polynomial · Joint reduction · Local cohomology · Rees algebra · Multi-graded filtration · Grothendieck–Serre formula

1 Introduction

The objective of this expository paper is to collect together several fundamental results about Hilbert coefficients of admissible filtrations of ideals which can be proved using various avatars of the Grothendieck–Serre formula for the difference of the Hilbert function and Hilbert polynomial of a finite graded module of a standard graded Noetherian ring. The proofs presented here provide a unified way of approach-

The first author is partially supported by a grant from Infosys Foundation; The second author is supported by CSIR Fellowship of Government of India.

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S.T. Rizvi et al. (eds.), *Algebra and its Applications*, Springer Proceedings
in Mathematics & Statistics 174, DOI 10.1007/978-981-10-1651-6_8

ing these results. Some of these results are not known in the multi-graded case. We hope that the unified approach presented here could lead to suitable multi-graded analogues of these results.

We begin by recalling the Grothendieck–Serre formula. For the sake of simplicity, we assume that the graded rings considered in this paper are standard and Noetherian. Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a standard graded Noetherian ring where R_0 is an Artinian local ring. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a finite graded R-module of dimension d. The Hilbert function of M is the function $H(M, n) = \lambda_{R_0}(M_n)$ for all $n \in \mathbb{Z}$. Here λ denotes the length function. Serre showed that there exists an integer m so that $H(M, n)$ is given by a polynomial $P(M, x) \in \mathbb{Q}[x]$ of degree $d - 1$ such that $H(M, n) = P(M, n)$ for all $n > m$. The smallest such m is called the postulation number of M. Let R_+ denote the homogeneous ideal of R generated by elements of positive degree and $[H^i_{R_+}(M)]_n$ denote the nth graded component of the ith local cohomology module $H^i_{R_+}(M)$ of M with respect to the ideal R_+. We put $\lambda_{R_0}([H^i_{R_+}(M)]_n) = h^i_{R_+}(M)_n$.

Theorem 1.1 (Grothendieck–Serre) *For all $n \in \mathbb{Z}$, we have*

$$H(M, n) - P(M, n) = \sum_{i=0}^{d} (-1)^i h^i_{R_+}(M)_n.$$

The GSF was proved in the fundamental paper [37] of J.-P. Serre. We quote from [6]: "In this paper, Serre introduced the theory of coherent sheaves over algebraic varieties over an algebraically closed field and a cohomology theory of such varieties with coefficients in coherent sheaves. He did speak of algebraic coherent sheaves, as at the first time he managed to introduce these theories with purely algebraic tools, using consequently the Zariski topology instead of the complex topology and homological methods instead of tools from multivariate complex analysis. Since then, the cohomology theory introduced in Serre's paper is often called Serre cohomology or sheaf cohomology.

One of the achievement of Serre's paper is the Grothendieck–Serre Formula, which is given there in terms of sheaf cohomology and showed in this way that sheaf cohomology gives a functorial understanding of the so called postulation problem of algebraic geometry, the problem which classically consisted in understanding the difference between the Hilbert function and the Hilbert polynomial of the coordinate ring of a projective variety."

The Grothendieck–Serre Formula (GSF) is valid for nonstandard graded rings also if the Hilbert polynomial $P(M, x)$ is replaced by the Hilbert quasi-polynomial [4, Theorem 4.4.3]. The GSF has been generalised in several directions. For some of the applications, we need it in the context of $\mathbb{Z}^s$-graded modules over standard $\mathbb{N}^s$-graded rings. In order to state the GSF for $\mathbb{Z}^s$-graded module, first we set up notation and recall some definitions. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. We put $e = (1, \ldots, 1)$, $\underline{0} = (0, \ldots, 0) \in \mathbb{Z}^s$ and for all $i = 1, \ldots, s$, $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^s$ where 1 occurs at ith position. For $\underline{n} = (n_1, \ldots, n_s) \in \mathbb{Z}^s$, we write $\underline{I}^{\underline{n}} = I_1^{n_1} \cdots I_s^{n_s}$ and $\underline{n}^+ = (n_1^+, \ldots, n_s^+)$ where $n_i^+ = \max\{0, n_i\}$ for all $i = 1, \ldots, s$. For $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, we put $|\alpha| = \alpha_1 + \cdots + \alpha_s$. We define $\underline{m} = (m_1, \ldots, m_s) \geq \underline{n} = (n_1, \ldots, n_s)$ if $m_i \geq n_i$ for all $i = 1, \ldots, s$. By the phrase "for all large $\underline{n}$" we mean $\underline{n} \in \mathbb{N}^s$ and $n_i \gg 0$ for all $i = 1, \ldots, s$. For an $\mathbb{N}^s$ (or a $\mathbb{Z}^s$)-graded ring T, the ideal generated by elements of degree e is denoted by T_{++}.

Definition 1.2 A set of ideals $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ is called a $\mathbb{Z}^s$-graded $\underline{I} = (I_1, \ldots, I_s)$-**filtration** if for all $\underline{m}, \underline{n} \in \mathbb{Z}^s$, (i) $\underline{I}^{\underline{n}} \subseteq \mathcal{F}(\underline{n})$, (ii) $\mathcal{F}(\underline{n})\mathcal{F}(\underline{m}) \subseteq \mathcal{F}(\underline{n} + \underline{m})$ and (iii) if $\underline{m} \geq \underline{n}$, $\mathcal{F}(\underline{m}) \subseteq \mathcal{F}(\underline{n})$.

Let $R = \bigoplus_{\underline{n} \in \mathbb{N}^s} R_{\underline{n}}$ be a standard Noetherian $\mathbb{N}^s$-graded ring defined over a local ring $(R_{\underline{0}}, \mathfrak{m})$ and $R_{++} = \bigoplus_{\underline{n} \geq e} R_{\underline{n}}$. Let $\operatorname{Proj}(R)$ denote the set of all homogeneous prime ideals P in R such that $R_{++} \nsubseteq P$. For a finitely generated module M, set $\operatorname{Supp}_{++}(M) = \{P \in \operatorname{Proj}(R) \mid M_P \neq 0\}$. Note that $\operatorname{Supp}_{++}(M) = V_{++}(\operatorname{Ann}(M))$ [7, Lemma 2.2.5], [15].

Definition 1.3 The **relevant dimension** of M is

$$\operatorname{rel.dim}(M) = \begin{cases} s - 1 & \text{if } \operatorname{Supp}_{++}(M) = \emptyset \\ \max\{\dim(R/P) \mid P \in \operatorname{Supp}_{++}(M)\} & \text{if } \operatorname{Supp}_{++}(M) \neq \emptyset. \end{cases}$$

By [15, Lemma 1.1], $\dim \operatorname{Supp}_{++}(M) = \operatorname{rel.dim}(M) - s$. M. Herrmann, E. Hyry, J. Ribbe and Z. Tang [15, Theorem 4.1] proved that if $R = \bigoplus_{\underline{n} \in \mathbb{N}^s} R_{\underline{n}}$ is a standard Noetherian $\mathbb{N}^s$-graded ring defined over an Artinian local ring $(R_{\underline{0}}, \mathfrak{m})$ and $M = \bigoplus_{\underline{n} \in \mathbb{Z}^s} M_{\underline{n}}$ is a finitely generated $\mathbb{Z}^s$-graded R-module then there exists a polynomial, called the Hilbert polynomial of M, $P_M(x_1, x_2, \ldots, x_s) \in \mathbb{Q}[x_1, \ldots, x_s]$ of total degree $\dim \operatorname{Supp}_{++}(M)$ satisfying $P_M(\underline{n}) = \lambda(M_{\underline{n}})$ for all large $\underline{n}$. Moreover all monomials of highest degree in this polynomial have nonnegative coefficients.

The next two results are due to G. Colomé-Nin [7, Propositions 2.4.2 and 2.4.3] for nonstandard multi-graded rings. In Sect. 2, we present her proofs to prove the same results for standard multigraded rings for the sake of simplicity. These results were proved in the bigraded case by A.V. Jayanthan and J.K. Verma [19].

Proposition 1.4 *Let $R = \bigoplus_{\underline{n} \in \mathbb{N}^s} R_{\underline{n}}$ be a standard Noetherian $\mathbb{N}^s$-graded ring defined over a local ring $(R_{\underline{0}}, \mathfrak{m})$ and $M = \bigoplus_{\underline{n} \in \mathbb{Z}^s} M_{\underline{n}}$ a finitely generated $\mathbb{Z}^s$-graded R-module. Then*

(1) For all $i \geq 0$ and $\underline{n} \in \mathbb{Z}^s$, $[H^i_{R_{++}}(M)]_{\underline{n}}$ is finitely generated $R_{\underline{0}}$-module.
(2) For all large $\underline{n}$ and $i \geq 0$, $[H^i_{R_{++}}(M)]_{\underline{n}} = 0$.

Theorem 1.5 (Grothendieck–Serre formula for $\mathbb{Z}^s$-graded modules) *Let* $R = \bigoplus_{\underline{n}\in\mathbb{N}^s} R_{\underline{n}}$ *be a standard Noetherian* $\mathbb{N}^s$*-graded ring defined over an Artinian local ring* $(R_{\underline{0}}, \mathfrak{m})$ *and* $M = \bigoplus_{\underline{n}\in\mathbb{Z}^s} M_{\underline{n}}$ *a finitely generated* $\mathbb{Z}^s$*-graded* R*-module. Let* $H_M(\underline{n}) = \lambda(M_{\underline{n}})$ *and* $P_M(x_1, \ldots, x_s)$ *be the Hilbert polynomial of* M*. Then for all* $\underline{n} \in \mathbb{Z}^s$,

$$H_M(\underline{n}) - P_M(\underline{n}) = \sum_{j\geq 0}(-1)^j h^j_{R_{++}}(M)_{\underline{n}}.$$

The above form of the GSF leads to another version of it which gives the difference between the Hilbert polynomial and the function of $\mathbb{Z}^s$-graded filtrations of ideals in terms of local cohomology modules of various forms of Rees rings and associated graded rings of ideals. To define these, let $t_1, t_2, \ldots, t_s$ be indeterminates and $\underline{t}^{\underline{n}} = {t_1}^{n_1} \cdots {t_s}^{n_s}$. We put

$\mathcal{R}(\mathcal{F}) = \bigoplus_{\underline{n}\in\mathbb{N}^s} \mathcal{F}(\underline{n})\underline{t}^{\underline{n}}$ the Rees ring of $\mathcal{F}$,

$\mathcal{R}'(\mathcal{F}) = \bigoplus_{\underline{n}\in\mathbb{Z}^s} \mathcal{F}(\underline{n})\underline{t}^{\underline{n}}$ the extended Rees ring of $\mathcal{F}$,

$G(\mathcal{F}) = \bigoplus_{\underline{n}\in\mathbb{N}^s} \dfrac{\mathcal{F}(\underline{n})}{\mathcal{F}(\underline{n}+e)}$ the associated multigraded ring of $\mathcal{F}$ with respect to $\mathcal{F}(e)$,

$G_i(\mathcal{F}) = \bigoplus_{\underline{n}\in\mathbb{N}^s} \dfrac{\mathcal{F}(\underline{n})}{\mathcal{F}(\underline{n}+e_i)}$ the associated graded ring of $\mathcal{F}$ with respect to $\mathcal{F}(e_i)$.

For $\mathcal{F} = \{\underline{I}^{\underline{n}}\}_{\underline{n}\in\mathbb{Z}^s}$, we set $\mathcal{R}(\mathcal{F}) = \mathcal{R}(\underline{I})$ and $\mathcal{R}'(\mathcal{F}) = \mathcal{R}'(\underline{I})$, $G(\mathcal{F}) = G(\underline{I})$ and $G_i(\mathcal{F}) = G_i(\underline{I})$ for all $i = 1, \ldots, s$.

Definition 1.6 A $\mathbb{Z}^s$-graded $\underline{I}$-filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ of ideals in R is called an $\underline{I}$**-admissible filtration** if $\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n}^+)$ and $\mathcal{R}'(\mathcal{F})$ is a finite $\mathcal{R}'(\underline{I})$-module. For $s = 1$, if a filtration $\mathcal{F}$ is I-admissible for some $\mathfrak{m}$-primary ideal I then it is also I_1-admissible.

Primary examples of $\underline{I}$-admissible filtrations are $\{\underline{I}^{\underline{n}}\}_{\underline{n}\in\mathbb{Z}^s}$ in a Noetherian local ring and $\{\overline{\underline{I}^{\underline{n}}}\}_{\underline{n}\in\mathbb{Z}^s}$ in an analytically unramified local ring. Recall that for an ideal I in R, the integral closure of I is the ideal

$$\begin{aligned}\overline{I} \;:= \{&x \in R \mid x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \text{ for some } n \in \mathbb{N}\\ &\text{and } a_i \in I^i \text{ for } i = 1, 2 \ldots, n\}.\end{aligned}$$

We now set up the notation for a variety of Hilbert polynomials associated to filtrations of ideals. Let I be an $\mathfrak{m}$-primary ideal of a Noetherian local ring $(R, \mathfrak{m})$ of dimension d. For a $\mathbb{Z}$-graded I-admissible filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$, Marley [23] proved existence of a polynomial $P_{\mathcal{I}}(x) \in \mathbb{Q}[x]$ of degree d, written in the form,

$$P_{\mathcal{I}}(n) = e_0(\mathcal{I})\binom{n+d-1}{d} - e_1(\mathcal{I})\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{I})$$

such that $P_{\mathcal{I}}(n) = H_{\mathcal{I}}(n)$ for all large n, where $H_{\mathcal{I}}(n) = \lambda(R/I_n)$ is the **Hilbert function** of the filtration $\mathcal{I}$. The coefficients $e_i(\mathcal{I})$ for $i = 0, 1, \ldots, d$ are integers, called the **Hilbert coefficients** of $\mathcal{I}$. The coefficient $e_0(\mathcal{I})$ is called the multiplicity of $\mathcal{I}$. P. Samuel [36] showed existence of this polynomial for the I-adic filtration $\{I^n\}_{n\in\mathbb{Z}}$. Many results about Hilbert polynomials for admissible filtrations were proved in [9, 33].

For $\mathfrak{m}$-primary ideals $I_1, \ldots, I_s$, B. Teissier [38] proved that for all $\underline{n}$ sufficiently large, the **Hilbert function** $H_{\underline{I}}(\underline{n}) = \lambda\left(R/\underline{I}^{\underline{n}}\right)$ coincides with a polynomial

$$P_{\underline{I}}(\underline{n}) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\in\mathbb{N}^s \\ |\alpha|\le d}} (-1)^{d-|\alpha|} e_\alpha(\underline{I}) \binom{n_1+\alpha_1-1}{\alpha_1} \cdots \binom{n_s+\alpha_s-1}{\alpha_s}$$

of degree d, called the **Hilbert polynomial** of $\underline{I}$. Here we assume that $s \ge 2$ in order to write $P_{\underline{I}}(\underline{n})$ in the above form. This was proved by P.B. Bhattacharya for $s = 2$ in [1]. Here $e_\alpha(\underline{I})$ are integers which are called the **Hilbert coefficients** of $\underline{I}$. D. Rees [31] showed that $e_\alpha(\underline{I}) > 0$ for $|\alpha| = d$. These are called the **mixed multiplicities** of $\underline{I}$.

For an $\underline{I}$-admissible filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ in a Noetherian local ring $(R, \mathfrak{m})$ of dimension d, Rees [31] showed the existence of a polynomial

$$P_{\mathcal{F}}(\underline{n}) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_s)\in\mathbb{N}^s \\ |\alpha|\le d}} (-1)^{d-|\alpha|} e_\alpha(\mathcal{F}) \binom{n_1+\alpha_1-1}{\alpha_1} \cdots \binom{n_s+\alpha_s-1}{\alpha_s}$$

of degree d which coincides with the **Hilbert function** $H_{\mathcal{F}}(\underline{n}) = \lambda\left(R/\mathcal{F}(\underline{n})\right)$ for all large $\underline{n}$ [31]. This polynomial is called the **Hilbert polynomial** of $\mathcal{F}$. Rees [31, Theorem 2.4] proved that $e_\alpha(\mathcal{F}) = e_\alpha(\underline{I})$ for all $\alpha \in \mathbb{N}^s$ such that $|\alpha| = d$.

In Sect. 2, we prove the following version of the GSF for the extended Rees algebras. It was proved for I-adic filtration and for nonnegative integers by Johnston–Verma [20] and for $\mathbb{Z}$-graded admissible filtration of ideals by C. Blancafort for all integers [2].

Theorem 1.7 ([25, Theorem 4.3]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then*

(1) $h^i_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}} < \infty$ for all $i \ge 0$ and $\underline{n} \in \mathbb{Z}^s$.
(2) $P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n}) = \sum_{i\ge 0}(-1)^i h^i_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}}$ for all $\underline{n} \in \mathbb{Z}^s$.

In Sect. 3, we derive explicit formulas in terms of the Ratliff–Rush closure filtration of a multi-graded filtration of ideals for the graded components of the local cohomology modules of certain Rees rings and associated graded rings. For an ideal I in a Noetherian ring R, L.J. Ratliff and D. Rush [30] introduced the ideal

$$\tilde{I} = \bigcup_{k \geq 1} (I^{k+1} : I^k),$$

called the **Ratliff–Rush closure** of I. If I has a regular element then the ideal $\tilde{I}$ has some nice properties such as for all large n, $(\tilde{I})^n = I^n$, $\widetilde{I^n} = I^n$ etc. If I is an $\mathfrak{m}$-primary regular ideal then $\tilde{I}$ is the largest ideal with respect to inclusion having the same Hilbert polynomial as that of I. Blancafort [2] introduced Ratliff–Rush closure filtration of an $\mathbb{N}$-graded good filtration. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. We need the concept of the Ratliff–Rush closure of $\mathcal{F}$ in order to find formulas for certain local cohomology modules.

Definition 1.8 The **Ratliff–Rush closure filtration** of $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ is the filtration of ideals $\breve{\mathcal{F}} = \{\breve{\mathcal{F}}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ given by

(1) $\breve{\mathcal{F}}(\underline{n}) = \bigcup_{k \geq 1} (\mathcal{F}(\underline{n} + ke) : \mathcal{F}(e)^k)$ for all $\underline{n} \in \mathbb{N}^s$,

(2) $\breve{\mathcal{F}}(\underline{n}) = \breve{\mathcal{F}}(\underline{n}^+)$ for all $\underline{n} \in \mathbb{Z}^s$.

The next three results to be proved in Sect. 3 are needed to prove several results about Hilbert coefficients in Sect. 5.

Proposition 1.9 ([25, Proposition 3.5]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension two with infinite residue field and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then for all $\underline{n} \in \mathbb{N}^s$,*

$$[H^1_{\mathcal{R}(\mathcal{F})_{++}}(\mathcal{R}(\mathcal{F}))]_{\underline{n}} \cong \frac{\breve{\mathcal{F}}(\underline{n})}{\mathcal{F}(\underline{n})}.$$

Proposition 1.10 ([2, Theorem 3.5]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension two with infinite residue field, I an $\mathfrak{m}$-primary ideal of R and $\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$ be an I-admissible filtration of ideals in R. Then*

$$[H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))]_n = \begin{cases} \breve{I}_n / I_n & \textit{if } n \geq 0 \\ 0 & \textit{if } n < 0. \end{cases}$$

Theorem 1.11 ([25, Theorem 3.3]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ with infinite residue field and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R such that grade$(I_1 \cdots I_s) \geq 1$. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then for all $\underline{n} \in \mathbb{N}^s$ and $i = 1, \ldots, s$,*

$$[H^0_{G_i(\mathcal{F})_{++}}(G_i(\mathcal{F}))]_{\underline{n}} = \frac{\breve{\mathcal{F}}(\underline{n} + e_i) \cap \mathcal{F}(\underline{n})}{\mathcal{F}(\underline{n} + e_i)}.$$

In Sect. 4, we present several applications of the GSF for Rees algebra and associated graded ring of an ideal. The first application due to J.D. Sally, who pioneered these techniques for the study of Hilbert–Samuel coefficients, shows the connection of the postulation number with reduction number. Let $(R, \mathfrak{m})$ be a Noetherian local ring, I be an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ be an admissible I-filtration of ideals in R.

Definition 1.12 A **reduction** of an I-admissible filtration $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ is an ideal $J \subseteq I_1$ such that $JI_n = I_{n+1}$ for all large n. A **minimal reduction** of $\mathcal{F}$ is a reduction of $\mathcal{F}$ minimal with respect to inclusion. For a minimal reduction J of $\mathcal{F}$, we set

$$r_J(\mathcal{F}) = \min\{m : JI_n = I_{n+1} \text{ for } n \geq m\} \text{ and}$$
$$r(\mathcal{F}) = \min\{r_J(\mathcal{I}) : J \text{ is a minimal reduction of } \mathcal{F}\}.$$

For $\mathcal{F} = \{I^n\}_{n\in\mathbb{Z}}$, we set $r_J(\mathcal{F}) = r_J(I)$ and $r(\mathcal{F}) = r(I)$.

Definition 1.13 An integer $n \in \mathbb{Z}$ is called the **postulation number** of $\mathcal{F}$, denoted by $n(\mathcal{F})$, if $P_{\mathcal{F}}(m) = H_{\mathcal{F}}(m)$ for all $m > n$ and $P_{\mathcal{F}}(n) \neq H_{\mathcal{F}}(n)$. It is denoted by $n(\mathcal{F})$.

The next result was proved by J.D. Sally [35] for the $\mathfrak{m}$-adic filtration. Her proof remains valid for any admissible filtration.

Theorem 1.14 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$ with infinite residue field, I an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}$ be an I-admissible filtration of ideals in R. Let $H_R(n) = \lambda(I_n/I_{n+1})$ and $P_R(X) \in \mathbb{Q}[X]$ such that $P_R(n) = H_R(n)$ for all large n. Suppose* grade $G(\mathcal{F})_+ \geq d-1$. *Then for a minimal reduction $J = (x_1, \ldots, x_d)$ of $\mathcal{F}$, $H_R(r_J(\mathcal{F}) - d) \neq P_R(r_J(\mathcal{F}) - d)$ and $H_R(n) = P_R(n)$ for all $n \geq r_J(\mathcal{F}) - d + 1$.*

The following result is due to Marley [23, Corollary 3.8]. We give another proof which follows from the above theorem.

Theorem 1.15 ([23, Corollary 3.8]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$ with infinite residue field, I an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}$ be an I-admissible filtration of ideals in R. Let* grade $G(\mathcal{F})_+ \geq d-1$. *Then $r(\mathcal{F}) = n(\mathcal{F}) + d$.*

In Sect. 5, we discuss several results about nonnegativity of Hilbert coefficients of multi-graded filtrations of ideals as easy consequences of the GSF for such filtrations. We prove the following result which implies earlier results of Northcott, Narita, and Marley.

Theorem 1.16 ([25, Theorem 5.6]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$ and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then*

(1) $e_\alpha(\mathcal{F}) \geq 0$ *where* $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, $|\alpha| \geq d - 1$.
(2) $e_\alpha(\mathcal{F}) \geq 0$ *where* $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, $|\alpha| = d - 2$ *and* $d \geq 2$.

We also discuss the results of S. Itoh about nonnegativity and vanishing of the third coefficient of the normal Hilbert polynomial of the filtration $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ in an analytically unramified Cohen–Macaulay local ring. We prove an analogue of a theorem due to Sally for admissible filtrations in two-dimensional Cohen–Macaulay local rings which gives explicit formulas for all the coefficients of their Hilbert polynomial. Here again we show that these formulas follow in a natural way from the variant of GSF for Rees algebra of the filtration.

Proposition 1.17 *Let* $(R, \mathfrak{m})$ *be a two-dimensional Cohen–Macaulay local ring, I be any* $\mathfrak{m}$*-primary ideal of R and* $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ *an admissible I-filtration of ideals in R. Then*

(1) $\lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F})_0\right) = e_2(\mathcal{F})$,
(2) $\lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_1\right) = e_0(\mathcal{F}) - e_1(\mathcal{F}) + e_2(\mathcal{F}) - \lambda\left(\frac{R}{\breve{I}_1}\right)$,
(3) $\lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_{-1}\right) = e_1(\mathcal{F}) + e_2(\mathcal{F})$.

C. Huneke [14] and A. Ooishi [28] independently proved that if $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring of dimension $d \geq 1$ and I is an $\mathfrak{m}$-primary ideal then $e_0(I) - e_1(I) = \lambda(R/I)$ if and only if $r(I) \leq 1$. Huckaba and Marley [13] proved this result for $\mathbb{Z}$-graded admissible filtrations. In Sect. 6, we present a proof, due to Blancafort, for $\mathbb{Z}$-graded admissible filtrations of Huneke–Ooishi Theorem. The original proofs due to Huneke and Ooishi did not employ local cohomology and relied on use of superficial sequences. Our purpose in presenting the alternative proof using the GSF for Rees algebras is to motivate the proof of an analogue of the Huneke–Ooishi Theorem for multi-graded filtrations of ideals.

Theorem 1.18 ([3, Theorem 4.3.6]) *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring with infinite residue field of dimension* $d \geq 1$, I_1 *an* $\mathfrak{m}$*-primary ideal and* $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ *be an* I_1*-admissible filtration of ideals in R. Then the following are equivalent:*

(1) $e_0(\mathcal{F}) - e_1(\mathcal{F}) = \lambda\left(R/I_1\right)$,
(2) $r(\mathcal{F}) \leq 1$.

In this case, $e_2(\mathcal{F}) = \cdots = e_d(\mathcal{F}) = 0$, $G(\mathcal{F})$ *is Cohen–Macaulay,* $n(\mathcal{F}) \leq 0$, $r(\mathcal{F})$ *is independent of the reduction chosen and* $\mathcal{F} = \{I_1^n\}$.

Using the GSF for multi-graded Rees algebras we prove the following analogue of the Huneke–Ooishi Theorem for multi-graded admissible filtrations.

Theorem 1.19 ([25, Theorem 5.5]) *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* $d \geq 1$ *and* $I_1, \ldots, I_s$ *be* $\mathfrak{m}$*-primary ideals of R. Let* $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ *be an* $\underline{I}$*-admissible filtration of ideals in R. Then for all* $i = 1, \ldots, s$,

(1) $e_{(d-1)e_i}(\mathcal{F}) \geq e_1(\mathcal{F}^{(i)})$,
(2) $e(I_i) - e_{(d-1)e_i}(\mathcal{F}) \leq \lambda(R/\mathcal{F}(e_i))$,
(3) $e(I_i) - e_{(d-1)e_i}(\mathcal{F}) = \lambda(R/\mathcal{F}(e_i))$ *if and only if* $r(\mathcal{F}^{(i)}) \leq 1$ *and* $e_{(d-1)e_i}(\mathcal{F}) = e_1(\mathcal{F}^{(i)})$, *where* $\mathcal{F}^{(i)} = \{\mathcal{F}(ne_i)\}_{n\in\mathbb{Z}}$ *is an* I_i*-admissible filtration.*

The vanishing of the constant term of the Hilbert polynomial of a filtration gives insight into the filtration as well as the local ring. For any $\mathfrak{m}$-primary ideal I in an analytically unramified local ring $(R, \mathfrak{m})$ of dimension d, the **normal Hilbert function** of I is defined to be the function $\overline{H}(I, n) = \lambda(R/\overline{I^n})$. Rees showed that for large n, it is given by the **normal Hilbert polynomial**

$$\overline{P}(I, x) = \overline{e}_0(I)\binom{x+d-1}{d} - \overline{e}_1(I)\binom{x+d-2}{d-1} + \cdots + (-1)^d\overline{e}_d(I).$$

The integers $\overline{e}_0(I), \overline{e}_1(I), \ldots, \overline{e}_d(I)$ are called the **normal Hilbert coefficients** of I. Rees defined a 2-dimensional normal analytically unramified local ring $(R, \mathfrak{m})$ to be **pseudo-rational** if $\overline{e}_2(I) = 0$ for all $\mathfrak{m}$-primary ideals. It can be shown that two-dimensional local rings having a rational singularity are pseudo-rational. It is natural to characterise $\overline{e}_2(I) = 0$ in terms of computable data. This was considered by Huneke [14] in which he proved.

Theorem 1.20 ([14, Theorem 4.5]) *Let* $(R, \mathfrak{m})$ *be a two-dimensional analytically unramified Cohen–Macaulay local ring. Let I be an* $\mathfrak{m}$*-primary ideal. Then* $\overline{e}_2(I) = 0$ *if and only if* $\overline{I^n} = (x, y)\overline{I^{n-1}}$ *for* $n \geq 2$ *and for any minimal reduction* (x, y) *of* I.

A similar result was proved by Itoh [18] about vanishing of $\overline{e}_2(I)$. Using the GSF for multi-graded filtrations, we prove the following theorem which characterises the vanishing of the constant term of the Hilbert polynomial of a multi-graded admissible filtration and derive results of Itoh and Huneke as consequences.

Theorem 1.21 ([25, Theorem 5.7]) *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension two and* $I_1, \ldots, I_s$ *be* $\mathfrak{m}$*-primary ideals of* R. *Let* $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ *be an* $\underline{I}$*-admissible filtration of ideals in* R. *Then* $e_{\underline{0}}(\mathcal{F}) = 0$ *implies* $e(I_i) - e_{e_i}(\mathcal{F}) = \lambda\left(\frac{R}{\check{\mathcal{F}}(e_i)}\right)$ *for all* $i = 1, \ldots, s$. *Suppose* $\check{\mathcal{F}}$ *is* $\underline{I}$*-admissible filtration, then the converse is also true.*

2 Variations on the Grothendieck–Serre Formula

The main aim of this section is to prove the Grothendieck–Serre formula (Theorem 2.3) and its variations. In [7, Propositions 2.4.2 and 2.4.3], Colomé-Nin proved the Grothendieck–Serre formula for nonstandard multi-graded rings. For the sake of simplicity, we present her proof for standard multi-graded rings. As a consequence we prove [25, Theorem 4.3] (Theorem 2.5) which relates the difference of Hilbert polynomial and Hilbert function of an $\underline{I}$-admissible filtration to the Euler characteristic of the extended multi-Rees algebra.

We recall the following Lemma from [7] which is needed to prove Theorem 2.3.

Lemma 2.1 ([7, Lemma 2.2.8]) *Let* $R = \bigoplus_{\underline{n}\in\mathbb{N}^s} R_{\underline{n}}$ *be a standard Noetherian* $\mathbb{N}^s$*-graded ring defined over a local ring* $(R_{\underline{0}}, \mathfrak{m})$ *and* $M = \bigoplus_{\underline{n}\in\mathbb{Z}^s} M_{\underline{n}}$ *a finitely generated* $\mathbb{Z}^s$*-graded* R*-module. Let* $x \in R_{\underline{n}}$ *where* $\underline{n} \geq e$ *and* $x \notin \bigcup_{P\in Ass(M)} P$*. Then* $\operatorname{rel.dim}(M/xM) = \operatorname{rel.dim}(M) - 1$.

Proposition 2.2 *Let* $R = \bigoplus_{\underline{n}\in\mathbb{N}^s} R_{\underline{n}}$ *be a standard Noetherian* $\mathbb{N}^s$*-graded ring defined over a local ring* $(R_{\underline{0}}, \mathfrak{m})$ *and* $M = \bigoplus_{\underline{n}\in\mathbb{Z}^s} M_{\underline{n}}$ *a finitely generated* $\mathbb{Z}^s$*-graded* R*-module. Then*

(1) For all $i \geq 0$ *and* $\underline{n} \in \mathbb{Z}^s$, $[H^i_{R_{++}}(M)]_{\underline{n}}$ *is finitely generated* $R_{\underline{0}}$*-module.*
(2) For all large $\underline{n}$ *and* $i \geq 0$, $[H^i_{R_{++}}(M)]_{\underline{n}} = 0$.

Proof Note that R_{++} is finitely generated. We prove both (1) and (2) together by induction on i. Suppose $i = 0$. Note that $H^0_{R_{++}}(M) \subseteq M$ and hence $H^0_{R_{++}}(M)$ is finitely generated R-module. Let $\{\gamma_1, \ldots, \gamma_q\}$ be a generating set of $H^0_{R_{++}}(M)$ as an R-module and $deg(\gamma_j) = p(j) = (p(j1), \ldots, p(js))$ for all $j = 1, \ldots, q$. Let $\alpha_i = \max\{|p(ji)| : j = 1, \ldots, q\}$ for all $i = 1, \ldots, s$ and $\alpha = (\alpha_1, \ldots, \alpha_s)$. Since $H^0_{R_{++}}(M)$ is R_{++}-torsion, there exists an integer $t \geq 1$ such that $R^t_{++}H^0_{R_{++}}(M) = 0$. Then for all $\underline{n} \geq \alpha + te$,

$$[H^0_{R_{++}}(M)]_{\underline{n}} = R_{\underline{n}-p(1)}\gamma_1 + \cdots + R_{\underline{n}-p(q)}\gamma_q \subseteq R^t_{++}H^0_{R_{++}}(M) = 0.$$

Fix $\underline{n} \in \mathbb{Z}^s$. Since R is a standard Noetherian $\mathbb{N}^s$-graded ring defined over $R_{\underline{0}}$, there exist elements $a_{i1}, \ldots, a_{ik_i} \in R_{e_i}$ for all $i = 1, \ldots, s$ such that each nonzero element of $[H^0_{R_{++}}(M)]_{\underline{n}}$ can be written as sum of monomials $\prod_{1\leq i\leq s} a_{i1}^{t_{i1}} \cdots a_{ik_i}^{t_{ik_i}}\gamma_j$ of degree $\underline{n}$ with coefficients from $R_{\underline{0}}$ where $j = 1, \ldots, q,\ t_{i1}, \ldots, t_{ik_i} \geq 0$. Since $0 \leq t_{i1}, \ldots, t_{ik_i} \leq n_i - p(ji)$, the number of monomial generators are finite. Hence $[H^0_{R_{++}}(M)]_{\underline{n}}$ is finitely generated $R_{\underline{0}}$-module.

Now assume $i > 0$. Let M' denote $M/H^0_{R_{++}}(M)$. Consider the short exact sequence of R-modules

$$0 \longrightarrow H^0_{R_{++}}(M) \longrightarrow M \longrightarrow M' \longrightarrow 0$$

which gives long exact sequence of local cohomology modules

$$\cdots \longrightarrow H^i_{R_{++}}(H^0_{R_{++}}(M)) \longrightarrow H^i_{R_{++}}(M) \longrightarrow H^i_{R_{++}}(M') \longrightarrow \cdots.$$

Since $H^0_{R_{++}}(M)$ is R_{++}-torsion, $H^i_{R_{++}}(H^0_{R_{++}}(M)) = 0$ for all $i \geq 1$. Thus

$$H^i_{R_{++}}(M) \simeq H^i_{R_{++}}(M') \text{ for all } i \geq 1. \tag{2.2.1}$$

By [7, Lemma 2.4.1], there exists an element $x \in R_{\underline{p}}$ for some $\underline{p} \geq e$ such that $x \notin P$ for all $P \in Ass(M') = Ass(M) \backslash V(R_{++})$. Fix $i \geq 1$. Consider the short exact sequence of R-modules

$$0 \longrightarrow M'(-\underline{p}) \xrightarrow{.x} M' \longrightarrow M'/xM' \longrightarrow 0$$

which gives long exact sequence of local cohomology modules whose $\underline{r}$th component is

$$\cdots \longrightarrow \left[H^{i-1}_{R_{++}}\left(M'/xM'\right)\right]_{\underline{r}} \longrightarrow \left[H^i_{R_{++}}(M')\right]_{\underline{r}-\underline{p}} \xrightarrow{.x} \left[H^i_{R_{++}}(M')\right]_{\underline{r}} \longrightarrow \left[H^i_{R_{++}}\left(M'/xM'\right)\right]_{\underline{r}} \longrightarrow \cdots.$$

By inductive hypothesis $\left[H^{i-1}_{R_{++}}\left(M'/xM'\right)\right]_{\underline{m}} = 0$ for all large $\underline{m}$, say, for all $\underline{m} \geq \underline{k}$ for some $\underline{k} \in \mathbb{N}^s$. Then for all $\underline{n} \geq \underline{k}$, we have the exact sequence

$$0 \longrightarrow \left[H^i_{R_{++}}(M')\right]_{\underline{n}-\underline{p}} \xrightarrow{.x} \left[H^i_{R_{++}}(M')\right]_{\underline{n}}.$$

Since $H^i_{R_{++}}(M')$ is R_{++}-torsion and $x \in R_{++}$, we have $\left[H^i_{R_{++}}(M')\right]_{\underline{m}} = 0$ for all $\underline{m} \geq \underline{k} - \underline{p}$. Hence we prove part (2).

Fix $i > 0$ and $\underline{n} \in \mathbb{Z}^s$. By [7, Lemma 2.4.1], there exists an element $y \in R_{++}$ such that $y \notin P$ for all $P \in Ass(M') = Ass(M) \backslash V(R_{++})$ and we can assume degree$(y) = \underline{m}$ such that $[H^i_{R_{++}}(M')]_{\underline{r}} = 0$ for all $\underline{r} \geq \underline{n} + \underline{m}$. Consider the short exact sequence of R-modules

$$0 \longrightarrow M'(-\underline{m}) \xrightarrow{.y} M' \longrightarrow M'/yM' \longrightarrow 0$$

which gives long exact sequence of cohomology modules whose $(\underline{m} + \underline{n})$th component is

$$\cdots \longrightarrow \left[H^{i-1}_{R_{++}}\left(M'/yM'\right)\right]_{\underline{m}+\underline{n}} \longrightarrow \left[H^i_{R_{++}}(M')\right]_{\underline{n}} \xrightarrow{.y} \left[H^i_{R_{++}}(M')\right]_{\underline{m}+\underline{n}} \longrightarrow \cdots.$$

Since $[H^i_{R_{++}}(M')]_{\underline{m}+\underline{n}} = 0$ and by induction hypothesis $\left[H^{i-1}_{R_{++}}\left(M'/yM'\right)\right]_{\underline{m}+\underline{n}}$ is finitely generated $R_{\underline{0}}$-module, from the above exact sequence, we get $[H^i_{R_{++}}(M')]_{\underline{n}}$ is finitely generated $R_{\underline{0}}$-module. Hence by Eq. (2.2.1), we get the required result. □

Theorem 2.3 (Grothendieck–Serre formula for multi-graded modules) *Let* $R = \bigoplus_{\underline{n}\in\mathbb{N}^s} R_{\underline{n}}$ *be a standard Noetherian* $\mathbb{N}^s$*-graded ring defined over an Artinian local ring* $(R_{\underline{0}}, \mathfrak{m})$ *and* $M = \bigoplus_{\underline{n}\in\mathbb{Z}^s} M_{\underline{n}}$ *a finitely generated* $\mathbb{Z}^s$*-graded R-module. Let* $H_M(\underline{n}) = \lambda(M_{\underline{n}})$ *and* $P_M(x_1, \dots, x_s)$ *be the Hilbert polynomial of M. Then for all* $\underline{n} \in \mathbb{Z}^s$,

$$H_M(\underline{n}) - P_M(\underline{n}) = \sum_{j\geq 0}(-1)^j h^j_{R_{++}}(M)_{\underline{n}}.$$

Proof For all $\underline{n}\in \mathbb{Z}^s$, we define $\chi_M(\underline{n}) = \sum_{j\geq 0}(-1)^j h^j_{R_{++}}(M)_{\underline{n}}$ and $f_M(\underline{n}) = H_M(\underline{n}) - P_M(\underline{n})$. We use induction on $\operatorname{rel.dim}(M)$. Suppose $\operatorname{rel.dim}(M) = s-1$. Then $\operatorname{Supp}_{++}(M) = V_{++}(\operatorname{Ann}(M)) = \emptyset$. Therefore there exists an integer $k \geq 1$ such that $R^k_{++}M = 0$. Hence $H^0_{R_{++}}(M) = M$ and $H^i_{R_{++}}(M) = 0$ for all $i \geq 1$. Since $P_M(X_1, \dots, X_s)$ has degree -1, we have $P_M(\underline{n}) = 0$ for all $\underline{n} \in \mathbb{Z}^s$. Thus we get the required equality.

Assume that $\operatorname{rel.dim}(M) \geq s$. Let M' denote $M/H^0_{R_{++}}(M)$. Consider the short exact sequence of R-modules

$$0 \longrightarrow H^0_{R_{++}}(M) \longrightarrow M \longrightarrow M' \longrightarrow 0$$

which gives long exact sequence of local cohomology modules

$$\cdots \longrightarrow H^i_{R_{++}}(H^0_{R_{++}}(M)) \longrightarrow H^i_{R_{++}}(M) \longrightarrow H^i_{R_{++}}(M') \longrightarrow \cdots.$$

Note that $H^0_{R_{++}}(M)$ is R_{++}-torsion. Hence for all $i \geq 1$, $H^i_{R_{++}}(H^0_{R_{++}}(M)) = 0$ and

$$H^i_{R_{++}}(M) \simeq H^i_{R_{++}}(M'). \tag{2.3.1}$$

Since $H_M(\underline{n}) = H_{M'}(\underline{n}) + h^0_{R_{++}}(M)_{\underline{n}}$ and hence by Proposition 2.2 part (2), $P_M(\underline{n}) = P_{M'}(\underline{n})$. Thus

$$H_M(\underline{n}) - P_M(\underline{n}) = H_{M'}(\underline{n}) + h^0_{R_{++}}(M)_{\underline{n}} - P_{M'}(\underline{n}) = H_{M'}(\underline{n}) - P_{M'}(\underline{n}) + h^0_{R_{++}}(M)_{\underline{n}}.$$

Therefore by the Eq. (2.3.1), it is enough to prove the result for M'. By [7, Lemma 2.4.1], there exists an element $x \in R_{\underline{p}}$ for some $\underline{p} \geq e$ such that $x \notin P$ for all $P \in Ass(M') = Ass(M)\setminus V(R_{++})$. Consider the short exact sequence of R-modules

$$0 \longrightarrow M'(-\underline{p}) \xrightarrow{.x} M' \longrightarrow M'/xM' \longrightarrow 0$$

which gives long exact sequence of cohomology modules whose $\underline{r}$th component is

$$\cdots \longrightarrow \left[H^{i-1}_{R_{++}}\left(M'/xM'\right)\right]_{\underline{r}} \longrightarrow \left[H^{i}_{R_{++}}(M')\right]_{\underline{r}-\underline{p}} \xrightarrow{.x} \left[H^{i}_{R_{++}}(M')\right]_{\underline{r}}$$
$$\longrightarrow \left[H^{i}_{R_{++}}\left(M'/xM'\right)\right]_{\underline{r}} \longrightarrow \cdots.$$

Thus for all $\underline{n} \in \mathbb{Z}^s$, $H_{M'/xM'}(\underline{n}) = H_{M'}(\underline{n}) - H_{M'}(\underline{n}-\underline{p})$. Hence $P_{M'/xM'}(\underline{n}) = P_{M'}(\underline{n}) - P_{M'}(\underline{n}-\underline{p})$. By Lemma 2.1, $\operatorname{rel.dim}(M'/xM') < \operatorname{rel.dim}(M')$. Therefore for all $\underline{n} \in \mathbb{Z}^s$,

$$f_{M'}(\underline{n}) - f_{M'}(\underline{n}-\underline{p}) = f_{M'/xM'}(\underline{n}) = \chi_{M'/xM'}(\underline{n}) = \chi_{M'}(\underline{n}) - \chi_{M'}(\underline{n}-\underline{p}).$$

Hence $f_{M'}(\underline{n}) - \chi_{M'}(\underline{n}) = f_{M'}(\underline{n}-\underline{p}) - \chi_{M'}(\underline{n}-\underline{p})$. Since for all large $\underline{n}$, $f_{M'}(\underline{n}) - \chi_{M'}(\underline{n}) = 0$, we get the required result. □

Proposition 2.4 *Let S' be a $\mathbb{Z}^s$-graded ring and $S = \bigoplus_{\underline{n}\in\mathbb{N}^s} S'_{\underline{n}}$. Then $H^i_{S_{++}}(S') \cong H^i_{S_{++}}(S)$ for all $i > 1$ and we have the exact sequence*

$$0 \longrightarrow H^0_{S_{++}}(S) \longrightarrow H^0_{S_{++}}(S') \longrightarrow \frac{S'}{S} \longrightarrow H^1_{S_{++}}(S) \longrightarrow H^1_{S_{++}}(S') \longrightarrow 0.$$

Proof Consider the short exact sequence of S-modules

$$0 \longrightarrow S \longrightarrow S' \longrightarrow \frac{S'}{S} \longrightarrow 0.$$

This gives the long exact sequence of S-modules

$$\cdots \longrightarrow H^i_{S_{++}}(S) \longrightarrow H^i_{S_{++}}(S') \longrightarrow H^i_{S_{++}}\left(\frac{S'}{S}\right) \longrightarrow \cdots.$$

Since $\frac{S'}{S}$ is S_{++}-torsion, $H^0_{S_{++}}\left(\frac{S'}{S}\right) = \frac{S'}{S}$ and $H^i_{S_{++}}\left(\frac{S'}{S}\right) = 0$ for all $i > 0$. Hence the result follows. □

The GSF for multi-graded Rees algebras proved below generalises the theorems [19, Theorem 5.1], [24, Theorem 1] and [2, Theorem 4.1].

Theorem 2.5 ([25, Theorem 4.3]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then*

(1) $h^i_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}} < \infty$ for all $i \geq 0$ and $\underline{n} \in \mathbb{Z}^s$.
(2) $P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n}) = \sum_{i\geq 0}(-1)^i h^i_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}}$ for all $\underline{n} \in \mathbb{Z}^s$.

Proof (1) Denote $\dfrac{\mathcal{R}'(\mathcal{F})}{\mathcal{R}'(\mathcal{F})(e_i)}$ by $G_i'(\mathcal{F})$. By the change of ring principle, $H^j_{G_i(\underline{I})_{++}}(G_i'(\mathcal{F})) \cong H^j_{\mathcal{R}_{++}}(G_i'(\mathcal{F}))$ for all $i = 1, \ldots, s$ and $j \geq 0$. For a fixed i, consider the short exact sequence of $\mathcal{R}(\underline{I})$-modules

$$0 \longrightarrow \mathcal{R}'(\mathcal{F})(e_i) \longrightarrow \mathcal{R}'(\mathcal{F}) \longrightarrow G_i'(\mathcal{F}) \longrightarrow 0. \tag{2.5.1}$$

This induces the long exact sequence of R-modules

$$0 \longrightarrow [H^0_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{n}+e_i} \longrightarrow [H^0_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{n}} \longrightarrow [H^0_{\mathcal{R}_{++}}(G_i'(\mathcal{F}))]_{\underline{n}} \longrightarrow [H^1_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{n}+e_i} \longrightarrow \cdots.$$

By Propositions 2.2 and 2.4, $[H^j_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{n}} = 0$ for all large $\underline{n}$ and $j \geq 0$. Since $\left(\frac{G_i'(\mathcal{F})}{G_i(\mathcal{F})}\right)_{\underline{n}} = 0$ for all $\underline{n} \in \mathbb{N}^s$ or $n_i < 0$, by Propositions 2.2 and 2.4, $[H^j_{\mathcal{R}_{++}}(G_i'(\mathcal{F}))]_{\underline{n}}$ is finitely generated $(G_i(\underline{I}))_{\underline{0}}$-module for all $\underline{n} \in \mathbb{N}^s$ or $n_i < 0$ and $j \geq 0$. Since $(G_i(\underline{I}))_{\underline{0}}$ is Artinian, $[H^j_{\mathcal{R}_{++}}(G_i'(\mathcal{F}))]_{\underline{n}}$ has finite length for all $\underline{n} \in \mathbb{N}^s$ or $n_i < 0$ and $j \geq 0$. Hence using decreasing induction on $\underline{n}$, we get that $h^j_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))_{\underline{n}} < \infty$ for all $j \geq 0$ and $\underline{n} \in \mathbb{Z}^s$.
(2) Let $\chi_M(\underline{n}) = \sum\limits_{i \geq 0}(-1)^i h^i_{\mathcal{R}_{++}}(M)_{\underline{n}}$ where M is an $\mathcal{R}(\underline{I})$-module. Then from the short exact sequence (2.5.1), Theorem 2.3 and Proposition 2.4, for each $i = 1, \ldots, s$ and $\underline{n} \in \mathbb{N}^s$ or $n_i < 0$,

$$\begin{aligned}
\chi_{\mathcal{R}'(\mathcal{F})}(\underline{n} + e_i) - \chi_{\mathcal{R}'(\mathcal{F})}(\underline{n}) &= -\chi_{G_i'(\mathcal{F})}(\underline{n}) \\
&= -\chi_{G_i(\mathcal{F})}(\underline{n}) \\
&= P_{G_i(\mathcal{F})}(\underline{n}) - H_{G_i(\mathcal{F})}(\underline{n}) \\
&= (P_{\mathcal{F}}(\underline{n} + e_i) - P_{\mathcal{F}}(\underline{n})) - (H_{\mathcal{F}}(\underline{n} + e_i) - H_{\mathcal{F}}(\underline{n})).
\end{aligned}$$

Let $h(\underline{n}) = \chi_{\mathcal{R}'(\mathcal{F})}(\underline{n}) - (P_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n}))$. Then $h(\underline{n} + e_i) = h(\underline{n})$ for all $\underline{n} \in \mathbb{N}^s$ or $n_i < 0$ and $i = 1, \ldots, s$. Since $h(\underline{n}) = 0$ for all large $\underline{n}$, $h(\underline{n}) = 0$ forall $\underline{n} \in \mathbb{Z}^s$.

□

3 Formulas for Local Cohomology Modules

In this section, we derive formulas for the graded components of the local cohomology modules of certain Rees rings and associated graded rings in terms of the Ratliff–Rush closure filtration of a multi-graded filtration of ideals. These generalise [2, Proposition 2.5 and Theorem 3.5]. We use these formulas to derive various properties of the Hilbert coefficients in further sections.

In the following proposition we derive a formula for $H^d_{G(\mathcal{F})_+}(G(\mathcal{F}))_n$.

Proposition 3.1 *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* $d \geq 1$, I *an* $\mathfrak{m}$*-primary ideal and* $\mathcal{F} = \{I_n\}$ *be an* I*-admissible filtration of ideals in* R. *Let* $(x_1, \ldots, x_d)$ *be a minimal reduction of* $\mathcal{F}$. *Put* $\underline{x}^k = (x_1^k, \ldots, x_d^k)$ *for all* $k \geq 1$. *Then for all* $n \in \mathbb{Z}$,

$$H^d_{G(\mathcal{F})_+}(G(\mathcal{F}))_n = \varinjlim_k \frac{I_{dk+n}}{\underline{x}^k I_{(d-1)k+n} + I_{dk+n+1}}.$$

Proof Let $x_i^* = x_i + I_2$ be the image of x_i in $G(\mathcal{F})$. Since $\sqrt{G(\mathcal{F})_+} = \sqrt{(x_1^*, \ldots, x_d^*)}$, by [5, Theorem 5.2.9], $H^d_{G(\mathcal{F})_+}(G(\mathcal{F})) = \varinjlim_k H^d((x_1^*)^k, \ldots, (x_d^*)^k, G(\mathcal{F}))$ where $H^d((x_1^*)^k, \ldots, (x_d^*)^k, G(\mathcal{F}))$ is the dth cohomology of the Koszul complex of $G(\mathcal{F})$ with respect to the elements $(x_1^*)^k, \ldots, (x_d^*)^k$. Thus we get the required result. □

Proposition 3.2 *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* $d \geq 1$, I *an* $\mathfrak{m}$*-primary ideal and* $\mathcal{F} = \{I_n\}$ *be an* I*-admissible filtration of ideals in* R. *Let* $(x_1, \ldots, x_d)$ *be a minimal reduction of* $\mathcal{F}$. *Put* $\underline{x}^k = (x_1^k, \ldots, x_d^k)$ *for all* $k \geq 1$. *Then for all* $n \in \mathbb{Z}$,

$$H^d_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_n = \varinjlim_k \frac{I_{dk+n}}{\underline{x}^k I_{(d-1)k+n}}.$$

Proof Since $\sqrt{\mathcal{R}(\mathcal{F})_+} = \sqrt{(x_1 t, \ldots, x_d t)}$, we have $H^d_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F})) = \varinjlim_k H^d((x_1 t)^k, \ldots, (x_d t)^k, \mathcal{R}(\mathcal{F}))$ by [5, Theorem 5.2.9] where $H^d((x_1 t)^k, \ldots, (x_d t)^k, \mathcal{R}(\mathcal{F}))$ is the dth cohomology of the Koszul complex of $\mathcal{R}(\mathcal{F})$ with respect to the elements $(x_1 t)^k, \ldots, (x_d t)^k$. Thus we get the required result. □

Lemma 3.3 ([Rees' Lemma] [31, Lemma 1.2] [25, Lemma 2.2]) *Let* $(R, \mathfrak{m}, k)$ *be a Noetherian local ring of dimension* d *with infinite residue field* k *and* $I_1, \ldots, I_s$ *be* $\mathfrak{m}$*-primary ideals of* R. *Let* $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ *be an* $\underline{I}$*-admissible filtration of ideals in* R *and* S *be a finite set of prime ideals of* R *not containing* $I_1 \cdots I_s$. *Then for each* $i = 1, \ldots, s$, *there exists an element* $x_i \in I_i$ *not contained in any of the prime ideals of* S *and an integer* r_i *such that for all* $\underline{n} \geq r_i e_i$,

$$\mathcal{F}(\underline{n}) \cap (x_i) = x_i \mathcal{F}(\underline{n} - e_i).$$

Theorem 3.4 ([31, Theorem 1.3] [25, Theorem 2.3]) *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* d *with infinite residue field and* $I_1, \ldots, I_s$ *be* $\mathfrak{m}$*-primary ideals of* R. *Let* $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ *be an* $\underline{I}$*-admissible filtration of ideals in* R. *Then there exist a set of elements* $\{x_{ij} \in I_i : j = 1, \ldots, d; i = 1, \ldots, s\}$ *such that* $(y_1, \ldots, y_d)\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ *for all large* $\underline{n}$ *where* $y_j = x_{1j} \cdots x_{sj} \in I_1 \cdots I_s$ *for all* $j = 1, \ldots, d$. *Moreover, if the ring is Cohen–Macaulay local then there exist elements* $x_{i1} \in I_i$ *and integers* r_i *for all* $i = 1, \ldots, s$ *such that for all* $\underline{n} \geq r_i e_i$, $\mathcal{F}(\underline{n}) \cap (x_{i1}) = x_{i1}\mathcal{F}(\underline{n} - e_i)$ *and* $y_1 = x_{11} \cdots x_{s1}$.

Proposition 3.5 ([25, Proposition 3.5]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension two with infinite residue field and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then for all $\underline{n} \in \mathbb{N}^s$,*

$$[H^1_{\mathcal{R}(\mathcal{F})_{++}}(\mathcal{R}(\mathcal{F}))]_{\underline{n}} \cong \frac{\breve{\mathcal{F}}(\underline{n})}{\mathcal{F}(\underline{n})}.$$

Proof By Lemma 3.3 and Theorem 3.4, there exists a regular sequence $\{y_1, y_2\}$ such that $(y_1, y_2)\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ for all large $\underline{n}$. For all $k \geq 1$, consider the following complex of $\mathcal{R}(\mathcal{F})$-modules

$$F^{k\cdot} : 0 \longrightarrow \mathcal{R}(\mathcal{F}) \xrightarrow{\alpha_k} \mathcal{R}(\mathcal{F})(ke)^2 \xrightarrow{\beta_k} \mathcal{R}(\mathcal{F})(2ke) \longrightarrow 0,$$

where $\alpha_k(1) = ({y_1}^k \underline{t}^{ke}, {y_2}^k \underline{t}^{ke})$ and $\beta_k(u, v) = {y_2}^k \underline{t}^{ke} u - {y_1}^k \underline{t}^{ke} v$. Since radical of the ideal $(y_1 \underline{t}^e, y_2 \underline{t}^e)\mathcal{R}(\mathcal{F})$ is same as radical of the ideal $\mathcal{R}(\mathcal{F})_{++}$, by [5, Theorem 5.2.9],

$$[H^1_{\mathcal{R}(\mathcal{F})_{++}}(\mathcal{R}(\mathcal{F}))]_{\underline{n}} \cong \lim_{\substack{\longrightarrow \\ k}} \frac{(\ker \beta_k)_{\underline{n}}}{(\operatorname{im} \alpha_k)_{\underline{n}}}.$$

Suppose $(u, v) \in (\ker \beta_k)_{\underline{n}}$ for any $\underline{n} \in \mathbb{N}^s$. Then ${y_2}^k u - {y_1}^k v = 0$. Since $\{y_1, y_2\}$ is a regular sequence, $u = {y_1}^k p$ for some $p \in R$. Thus $v = {y_2}^k p$. Hence $(u, v) = ({y_1}^k p, {y_2}^k p)$. This implies for all $\underline{n} \in \mathbb{N}^s$, $(u, v) \in (\ker \beta_k)_{\underline{n}}$ if and only if $(u, v) = ({y_1}^k p, {y_2}^k p)$ for some $p \in (\mathcal{F}(\underline{n} + ke) : ({y_1}^k, {y_2}^k))$. For $k \gg 0$, by [25, Proposition 3.1], $\breve{\mathcal{F}}(\underline{n}) = (\mathcal{F}(\underline{n} + ke) : ({y_1}^k, {y_2}^k))$ for all $\underline{n} \in \mathbb{N}^s$. Hence for all $\underline{n} \in \mathbb{N}^s$ and $k \gg 0$, $(\ker \beta_k)_{\underline{n}} \cong \breve{\mathcal{F}}(\underline{n})$. Also for all $\underline{n} \in \mathbb{N}^s$,

$$(\operatorname{im} \alpha_k)_{\underline{n}} = \{({y_1}^k p \underline{t}^{ke}, {y_2}^k p \underline{t}^{ke}) : p \in \mathcal{R}(\mathcal{F})_{\underline{n}}\} \cong \mathcal{F}(\underline{n}).$$

Hence $[H^1_{\mathcal{R}(\mathcal{F})_{++}}(\mathcal{R}(\mathcal{F}))]_{\underline{n}} \cong \dfrac{\breve{\mathcal{F}}(\underline{n})}{\mathcal{F}(\underline{n})}$ for all $\underline{n} \in \mathbb{N}^s$. □

Proposition 3.6 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension two with infinite residue field, I an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$ be an I-admissible filtration of ideals in R. Then*

$$[H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))]_n = \begin{cases} \breve{I}_n / I_n & \text{if } n \geq 0 \\ R & \text{if } n < 0. \end{cases}$$

Proof By Proposition 3.5, we get $[H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))]_n = \breve{I}_n / I_n$ for all $n \geq 0$. Let J be minimal reduction of $\mathcal{F}$ generated by superficial sequence y_1, y_2. For all $k \geq 1$, consider the following complex of $\mathcal{R}(\mathcal{F})$-modules

$$F^{k\cdot} : 0 \longrightarrow \mathcal{R}(\mathcal{F}) \xrightarrow{\alpha_k} \mathcal{R}(\mathcal{F})(k)^2 \xrightarrow{\beta_k} \mathcal{R}(\mathcal{F})(2k) \longrightarrow 0,$$

where $\alpha_k(1) = ({y_1}^k t^k, {y_2}^k t^k)$ and $\beta_k(u, v) = {y_2}^k t^k u - {y_1}^k t^k v$. Since radical of the ideal $(y_1 t, y_2 t)\mathcal{R}(\mathcal{F})$ is same as radical of the ideal $\mathcal{R}(\mathcal{F})_+$, by [5, Theorem 5.2.9],

$$[H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))]_n \cong \lim_{\overrightarrow{k}} \frac{(\ker \beta_k)_n}{(\operatorname{im} \alpha_k)_n}.$$

Now for $n < 0$, $\mathcal{R}(\mathcal{F})_n = 0$. Hence $(\operatorname{im} \alpha_k)_n = 0$.

Suppose $(u, v) \in (\ker \beta_k)_n$ for any $n < 0$. Then ${y_2}^k u - {y_1}^k v = 0$. Since $\{y_1, y_2\}$ is a regular sequence, $u = {y_1}^k p$ for some $p \in R$. Thus $v = {y_2}^k p$. Hence $(u, v) = ({y_1}^k p, {y_2}^k p)$. This implies for all $n < 0$, $(u, v) \in (\ker \beta_k)_n$ if and only if $(u, v) = ({y_1}^k p, {y_2}^k p)$ for some $p \in (\mathcal{F}(n + k) : ({y_1}^k, {y_2}^k)) = R$. □

Proposition 3.7 ([2, Theorem 3.5]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension two with infinite residue field, I an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration of ideals in R. Then*

$$[H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))]_n = \begin{cases} \breve{I}_n/I_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

Proof Since $[H^1_{\mathcal{R}(\mathcal{F})_{++}}(\mathcal{R}'(\mathcal{F}))]_n = [H^1_{\mathcal{R}(\mathcal{F})_{++}}(\mathcal{R}(\mathcal{F}))]_n$ for all $n \in \mathbb{N}$ by Proposition 2.4, using Proposition 3.6, we get $[H^1_{\mathcal{R}(\mathcal{F})_{++}}(\mathcal{R}'(\mathcal{F}))]_n = \breve{I}_n/I_n$.

Let J be minimal reduction of $\mathcal{F}$ generated by superficial sequence y_1, y_2. For all $k \geq 1$, consider the following complex of $\mathcal{R}(\mathcal{F})$-modules

$$F^{k\cdot} : 0 \longrightarrow \mathcal{R}'(\mathcal{F}) \xrightarrow{\alpha_k} \mathcal{R}'(\mathcal{F})(k)^2 \xrightarrow{\beta_k} \mathcal{R}'(\mathcal{F})(2k) \longrightarrow 0,$$

where $\alpha_k(1) = ({y_1}^k t^k, {y_2}^k t^k)$ and $\beta_k(u, v) = {y_2}^k t^k u - {y_1}^k t^k v$. Since radical of the ideal $(y_1 \underline{t}, y_2 \underline{t})\mathcal{R}(\mathcal{F})$ is same as radical of the ideal $\mathcal{R}(\mathcal{F})_+$, by [5, Theorem 5.2.9],

$$[H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))]_n \cong \lim_{\overrightarrow{k}} \frac{(\ker \beta_k)_n}{(\operatorname{im} \alpha_k)_n}.$$

for all $n \in \mathbb{Z}\backslash\mathbb{N}$,

$$(\operatorname{im} \alpha_k)_n = \{({y_1}^k p \underline{t}^{ke}, {y_2}^k p \underline{t}^{ke}) : p \in \mathcal{R}'(\mathcal{F})_n = R\} \cong R.$$

Thus $[H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))]_n = 0$ for all $n \in \mathbb{Z}\backslash\mathbb{N}$. □

Lemma 3.8 ([25, Lemma 2.11]) *Let $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in a Noetherian local ring $(R, \mathfrak{m})$ of dimension $d \geq 1$ such that grade$(I_1 \cdots I_s) \geq 1$. Let $\mathcal{F} =$*

$\{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Denote $\mathcal{R}(\underline{I})_{++}$ as $\mathcal{R}_{++}$. Then

$$\lambda_R[H^d_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{n}} \le \lambda_R[H^d_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{n}-e_i}$$

for all $\underline{n} \in \mathbb{Z}^s$ and $i = 1, \ldots, s$.

Proof By Lemma 3.3 and Theorem 3.4, there exists an ideal $J = (y_1, \ldots, y_d) \subseteq I_1 \cdots I_s$ such that $y_1 = x_{11} \cdots x_{s1}$ is a nonzerodivisor, $x_{i1} \in I_i$ forall $i = 1, \ldots, s$ and $J\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n} + e)$ for all large $\underline{n}$. Hence $\sqrt{\mathcal{R}(\underline{I})_{++}} = \sqrt{(y_1\underline{t}, \ldots, y_d\underline{t})}$. Consider the short exact sequence of $\mathcal{R}(\underline{I})$-modules,

$$0 \longrightarrow \mathcal{R}'(\mathcal{F})(-e_i) \xrightarrow{x_{i1}t_i} \mathcal{R}'(\mathcal{F}) \longrightarrow \frac{\mathcal{R}'(\mathcal{F})}{x_{i1}t_i\mathcal{R}'(\mathcal{F})} \longrightarrow 0.$$

This gives a long exact sequence of $\underline{n}$-graded components of local cohomology modules,

$$\cdots \longrightarrow [H^d_{\mathcal{R}(\underline{I})_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{n}-e_i} \longrightarrow [H^d_{\mathcal{R}(\underline{I})_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{n}} \longrightarrow \left[H^d_{\mathcal{R}(\underline{I})_{++}}\left(\frac{\mathcal{R}'(\mathcal{F})}{x_{i1}t_i\mathcal{R}'(\mathcal{F})}\right)\right]_{\underline{n}} \longrightarrow 0.$$

Let $T = \frac{\mathcal{R}(\underline{I})}{x_{i1}t_i\mathcal{R}(\underline{I})}$. Now $\frac{\mathcal{R}'(\mathcal{F})}{x_{i1}t_i\mathcal{R}'(\mathcal{F})}$ is a T-module and $\sqrt{\left(\frac{\mathcal{R}(\underline{I})}{x_{i1}t_i\mathcal{R}(\underline{I})}\right)_{++}} = \sqrt{(y_2\underline{t}, \cdots, y_d\underline{t})T}$. Hence $H^d_{\mathcal{R}(\underline{I})_{++}}\left(\frac{\mathcal{R}'(\mathcal{F})}{x_{i1}t_i\mathcal{R}'(\mathcal{F})}\right) = 0$ which implies the required result. □

Theorem 3.9 ([25, Theorem 3.3]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \ge 1$ with infinite residue field and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals in R such that grade$(I_1 \cdots I_s) \ge 1$. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$- admissible filtration of ideals in R. Then for all $\underline{n} \in \mathbb{N}^s$ and $i = 1, \ldots, s$,*

$$[H^0_{G_i(\mathcal{F})_{++}}(G_i(\mathcal{F}))]_{\underline{n}} = \frac{\breve{\mathcal{F}}(\underline{n} + e_i) \cap \mathcal{F}(\underline{n})}{\mathcal{F}(\underline{n} + e_i)}.$$

Proof Let $x \in \mathcal{F}(\underline{n})$ and $x^* = x + \mathcal{F}(\underline{n} + e_i) \in [H^0_{G_i(\mathcal{F})_{++}}(G_i(\mathcal{F}))]_{\underline{n}}$. Then $x^* G_i(\mathcal{F})^k_{++} = 0$ for some $k \ge 1$. Therefore $x\mathcal{F}(e)^k \subseteq \mathcal{F}(\underline{n} + ke + e_i)$. Hence $x \in \breve{\mathcal{F}}(\underline{n} + e_i)$.

Conversely, suppose $x^* \in \breve{\mathcal{F}}(\underline{n} + e_i) \cap \mathcal{F}(\underline{n})/\mathcal{F}(\underline{n} + e_i)$. We show that there exists $m \gg 0$ such that $x^* G_i(\mathcal{F})^m_{++} = 0$. Since $G_i(\mathcal{F})^m_{++} \subseteq \bigoplus_{\underline{p}\ge me} \mathcal{F}(\underline{p})/\mathcal{F}(\underline{p} + e_i)$, it is enough to show that $x^*(\mathcal{F}(\underline{p})/\mathcal{F}(\underline{p} + e_i)) = 0$ for all large $\underline{p}$. By [25, Proposition 3.1], there exists $\underline{m} \in \mathbb{N}^s$ with $\underline{m} \ge e$ such that $\breve{\mathcal{F}}(\underline{r}) = \mathcal{F}(\underline{r})$ for all $\underline{r} \ge \underline{m}$. Thus for all $\underline{r} \ge \underline{m}$,

$$x\mathcal{F}(\underline{r}) \subseteq \breve{\mathcal{F}}(\underline{n} + e_i)\mathcal{F}(\underline{r}) \subseteq \breve{\mathcal{F}}(\underline{n} + \underline{r} + e_i) = \mathcal{F}(\underline{n} + \underline{r} + e_i).$$

Therefore $(x+\mathcal{F}(\underline{n}+e_i))G_i(\mathcal{F})^m_{++}=0$ for some $m\geq 1$. Hence $(x+\mathcal{F}(\underline{n}+e_i))\in [H^0_{G_i(\mathcal{F})_{++}}(G_i(\mathcal{F}))]_{\underline{n}}$. □

4 The Postulation Number and the Reduction Number

In [35, Proposition 3] Sally gave a nice relation between the postulation number and the reduction number of the filtration $\{\mathfrak{m}^n\}_{n\in\mathbb{N}}$. In [23, Corollary 3.8] Marley generalised this relation for any I-admissible filtration. In this section, we derive these results using the Grothendieck–Serre formula. We recall few preliminary results about superficial sequences which are useful to apply induction in the study of Hilbert coefficients.

Let $(R,\mathfrak{m})$ be a Noetherian local ring, I be an $\mathfrak{m}$-primary ideal and $\mathcal{F}=\{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration of ideals in R.

Definition 4.1 An element $x\in I_t\backslash I_{t+1}$ is called **superficial element for $\mathcal{F}$ of degree** t if there exists an integer $c\geq 0$ such that $(I_{n+t}:x)\cap I_c=I_n$ for all $n\geq c$.

If the residue field of R is infinite, then there exists a superficial element of degree 1 [32, Proposition 2.3]. If grade $(I_1)\geq 1$ and $x\in I_1$ is superficial for $\mathcal{F}$, Huckaba and Marley [13], showed that x is nonzerodivisor in R and $(I_{n+1}:x)=I_n$ for all large n. If dimension of R is $d\geq 1$, $x\in I_1\backslash I_2$ is superficial element for $\mathcal{F}$ and x is a nonzerodivisor on R then by [23, Lemma A.2.1], $e_i(\mathcal{F})=e_i(\mathcal{F}')$ for all $0\leq i<d$ where $R'=R/(x)$ and $\mathcal{F}'=\{I_nR'\}_{n\in\mathbb{Z}}$. The following lemma is due to Blancafort [3, Lemma 3.1.6]. This lemma was first proved by Huckaba [11, Lemma 1.1] for I-adic filtration.

Lemma 4.2 *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d\geq 1$, I an $\mathfrak{m}$-primary ideal and $\mathcal{F}=\{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration of ideals in R. Suppose J is a minimal reduction of $\mathcal{F}$ and there exists an $x\in J\backslash I_2$ such that $x^*=x+I_2$ is a nonzerodivisor in $G(\mathcal{F})$. Let $R'=R/(x)$. Then $r(\mathcal{F})=r(\mathcal{F}')$ where $\mathcal{F}'=\{I_nR'\}_{n\in\mathbb{Z}}$.*

Proof We denote $r(\mathcal{F})$ and $r(\mathcal{F}')$ by r and s respectively. It is clear that $s\leq r$. We use the notation "$\,'\,$" to denote the image in R'. Let $n\geq s$ and $a\in I_{n+1}$. Then $a'\in J'I'_n$. Hence $a=p+xq$ for some $p\in JI_n$ and $q\in R$. Therefore $xq\in I_{n+1}$ which implies $q\in(I_{n+1}:x)$. Since x^* is a nonzerodivisor in $G(\mathcal{F})$, we have $(I_{n+1}:x)=I_n$ for all $n\in\mathbb{Z}$. Hence we get the required result. □

Definition 4.3 If $\underline{x}=x_1,\dots,x_r\in I_1$, we say $\underline{x}$ is a superficial sequence for $\mathcal{F}$ if for all $0\leq i<r$, x_{i+1} is superficial for $\mathcal{F}/(x_1,\dots,x_i)$.

Suppose $(R, \mathfrak{m})$ is Cohen–Macaulay local ring of dimension d, I_1 is an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}$ is an I_1-admissible filtration of ideals in R. Suppose $x_1, \ldots, x_r \in I_1$ and $1 \leq r \leq d$, then $x_1, \ldots, x_r$ is a superficial sequence for $\mathcal{F}$ if and only if $x_1, \ldots, x_r$ is R-regular sequence and there exists an integer $n_0 \geq 0$ such that for all $1 \leq i \leq r$,

$$(x_1, \ldots, x_i) \cap I_n = (x_1, \ldots, x_i)I_{n-1} \text{ for all } n \geq n_0.$$

This result was first proved by Valabrega and Valla [39, Corollary 2.7] for I-adic filtration and then by Huckaba and Marley [13] for $\mathbb{Z}$-graded admissible filtrations. Marley [23, Proposition A.2.4] showed that if residue the field is infinite then any minimal reduction of $\mathcal{F}$ can be generated by a superficial sequence for $\mathcal{F}$. The following lemma is due to Huckaba and Marley [13, Lemma 2.1].

Lemma 4.4 ([13, Lemma 2.1]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$, I an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}$ be an I-admissible filtration of ideals in R. Let $x_1, \ldots, x_k$ be a superficial sequence for $\mathcal{F}$. If* grade $G(\mathcal{F})_+ \geq k$ *then $x_1^*, \ldots, x_k^*$ is a regular sequence in $G(\mathcal{F})$ and hence $G(\mathcal{F})/(x_1^*, \ldots, x_k^*) \simeq G(\mathcal{F}/(x_1, \ldots, x_k))$ where x_i^* is image of x_i in $G(\mathcal{F})$.*

Proof By induction it is enough to prove for $k = 1$. Let $(I_{n+1} : x_1) \cap I_c = I_n$ for all $n \geq c$. Let $x^* \in (0 : x_1^*) \cap G(\mathcal{F})_n$ for some $n \in \mathbb{N}$. We show that $x^*(G(\mathcal{F})_+)^{c+1} = 0$. Let $0 \neq z^* \in G(\mathcal{F})_+^{c+1} \cap G(\mathcal{F})_p$. Now $x^*z^* \in G(\mathcal{F})_{n+p}$ and $x_1xz \in I_{n+p+2}$. Therefore $xz \in (I_{n+p+2} : x_1) \cap I_c = I_{n+p+1}$. Thus $x^*z^* = 0$ in $G(\mathcal{F})$. Hence $x^* \in (0 :_{G(\mathcal{F})} (G(\mathcal{F})_+)^{c+1}) = 0$. □

The next theorem was proved for the $\mathfrak{m}$-adic by Sally [35, Proposition 3]. We have adapted her proof for any admissible filtration.

Theorem 4.5 *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$ with infinite residue field, I an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}$ be an I-admissible filtration of ideals in R. Let $H_R(n) = \lambda(I_n/I_{n+1})$ and $P_R(X) \in \mathbb{Q}[X]$ such that $P_R(n) = H_R(n)$ for all large n. Suppose* grade $G(\mathcal{F})_+ \geq d - 1$. *Then for a minimal reduction $J = (x_1, \ldots, x_d)$ of $\mathcal{F}$, $H_R(r_J(\mathcal{F}) - d) \neq P_R(r_J(\mathcal{F}) - d)$ and $H_R(n) = P_R(n)$ for all $n \geq r_J(\mathcal{F}) - d + 1$.*

Proof We denote $r_J(\mathcal{F})$ by r. We use induction on d. Let $d = 1$. Without loss of generality we assume x_1 is superficial. Then

$$H^0_{G(\mathcal{F})_+}(G(\mathcal{F}))_n = \{z^* \in I_n/I_{n+1} \mid zI_l \in I_{n+l+1} \text{ for all large } l\}.$$

For $n \geq r - 1$, $zx_1^l \in I_{n+l+1} = x_1^l I_{n+1}$ implies $z \in I_{n+1}$. Thus for all $n \geq r - 1$, $H^0_{G(\mathcal{F})_+}(G(\mathcal{F}))_n = 0$.

Now we prove that $H^1_{G(\mathcal{F})_+}(G(\mathcal{F}))_{r-1} \neq 0$ and $H^1_{G(\mathcal{F})_+}(G(\mathcal{F}))_n = 0$ for all $n \geq r$. For each n, consider the following map

$$\frac{I_{k+n}}{x_1^k I_n + I_{k+n+1}} \xrightarrow{\phi_k} \frac{I_{k+n+1}}{x_1^{k+1} I_n + I_{k+n+2}} \text{ where } \phi_k(\overline{z}) = \overline{x_1 z}.$$

For all large k, $I_{k+n+1} = x_1 I_{k+n}$. Hence for all large k, ϕ_k is surjective. Now suppose $\phi_k(\overline{z}) = 0$ for some $\overline{z} \in I_{k+n}/x_1^k I_n + I_{k+n+1}$. Then $x_1 z \in x_1^{k+1} I_n + I_{k+n+2}$. Therefore $x_1 z = x_1^{k+1} a + b$ for some $a \in I_n$ and $b \in I_{k+n+2}$. Thus $b \in (x_1) \cap I_{k+n+2}$. Since x_1 is superficial, for all large k, $b \in x_1 I_{k+n+1}$ and hence $z \in x_1^k I_n + I_{n+k+1}$. Thus for all large k, ϕ_k is injective. Therefore by Proposition 3.1, for all large k,

$$H^1_{G(\mathcal{F})_+}(G(\mathcal{F}))_n \simeq \frac{I_{k+n}}{x_1^k I_n + I_{k+n+1}}.$$

Thus for all $n \geq r$ and for all large k, $I_{k+n} = x_1^k I_n \subseteq x_1^k I_n + I_{k+n+1}$. Hence $H^1_{G(\mathcal{F})_+}(G(\mathcal{F}))_n = 0$ for all $n \geq r$.

Suppose $H^1_{G(\mathcal{F})_+}(G(\mathcal{F}))_{r-1} = 0$. Then for all large k,

$$I_{k+r-1} = x_1^k I_{r-1} + I_{k+r} \subseteq x_1^k I_{r-1}.$$

Let $a \in I_{k+r-2}$. Then $x_1 a \in I_{k+r-1} \subseteq x_1^k I_{r-1}$ implies $a \in x_1^{k-1} I_{r-1}$. Thus $I_{k+r-2} = x_1^{k-1} I_{r-1}$. Using this procedure repeatedly, we get $I_r = x_1 I_{r-1}$ which is a contradiction. Thus $H^1_{G(\mathcal{F})_+}(G(\mathcal{F}))_{r-1} \neq 0$. Therefore by Theorem 2.3, we get the required result.

Suppose $d \geq 2$. Without loss of generality we assume $x_1, \ldots, x_d$ is superficial sequence for $\mathcal{F}$. Since grade $G(\mathcal{F})_+ \geq d-1$, by Lemma 4.4, we have x_1^* is a nonzerodivisor of $G(\mathcal{F})$. By [3, Proposition 3.1.3] $G(\mathcal{F})/(x_1^*) \simeq G(\mathcal{F}/(x_1))$. For all $n \in \mathbb{Z}$, consider the following exact sequence

$$0 \longrightarrow \frac{I_{n-1}}{I_n} \xrightarrow{x_1^*} \frac{I_n}{I_{n+1}} \longrightarrow \frac{I_n}{x_1 I_{n-1} + I_{n+1}} \simeq \frac{I_n + (x_1)}{I_{n+1} + (x_1)} \longrightarrow 0. \quad (4.5.1)$$

Then for all $n \in \mathbb{Z}$,

$$H_{R/(x_1)}(n) = H_R(n) - H_R(n-1) \text{ and hence } P_{R/(x_1)}(n) = P_R(n) - P_R(n-1).$$

Since $\dim R/(x_1) = d-1$ and grade $G(\mathcal{F}/(x_1))_+ \geq d-2$, by induction and Lemma 4.2, we have

$$H_{R/(x_1)}(r-d+1) \neq P_{R/(x_1)}(r-d+1) \text{ and } H_{R/(x_1)}(n) = P_{R/(x_1)}(n) \text{ for all } n \geq r-d+2.$$

Since there exists an integer m, such that for all $n \geq m$, $P_R(n) = H_R(n)$, we have

$$\begin{aligned} P_R(n-1) - H_R(n-1) &= P_R(n) - H_R(n) = \cdots \\ &= P_R(n+m) - H_R(n+m) = 0 \text{ for all } n \geq r-d+2. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &\neq P_{R/(x_1)}(r-d+1) - H_{R/(x_1)}(r-d+1) \\ &= [P_R(r-d+1) - H_R(r-d+1)] - [P_R(r-d) - H_R(r-d)] \\ &= P_R(r-d) - H_R(r-d). \end{aligned}$$

□

The following result is due to Marley [23, Corollary 3.8]. Here we give another proof which follows from Theorem 4.5.

Theorem 4.6 ([23, Corollary 3.8]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$ with infinite residue field, I an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}$ be an I-admissible filtration of ideals in R. Let* grade $G(\mathcal{F})_+ \geq d-1$. *Then $r_J(\mathcal{F}) = n(\mathcal{F}) + d$ for any minimal reduction J of $\mathcal{F}$. In particular, $r(\mathcal{F}) = n(\mathcal{F}) + d$.*

Proof Let $H_R(n) = \lambda(I_n/I_{n+1})$ for all n and $P_R(X) \in \mathbb{Q}[X]$ such that $P_R(n) = H_R(n)$ for all large n. Let $d = 1$ and J be any minimal reduction of $\mathcal{F}$. Denote $r_J(\mathcal{F})$ by r. Then degree of the polynomial $P_R(X)$ is zero. Hence $P_R(X) = a$ where a is a nonzero constant. By Theorem 4.5, for all $n \geq r$, $P_R(n) = H_R(n)$. Therefore for all $n \geq r$, we have

$$\lambda\left(\frac{R}{I_n}\right) = (n-r)a + \lambda\left(\frac{R}{I_r}\right) = na + \left(\lambda\left(\frac{R}{I_r}\right) - ra\right).$$

Hence $P_{\mathcal{F}}(n) = H_{\mathcal{F}}(n)$ for all $n \geq r$. Suppose $P_{\mathcal{F}}(r-1) = H_{\mathcal{F}}(r-1)$. Then

$$-a + \lambda\left(\frac{R}{I_r}\right) = \lambda\left(\frac{R}{I_{r-1}}\right).$$

This implies $P_R(r-1) = H_R(r-1)$ which contradicts Theorem 4.5. Thus $r_J(\mathcal{F}) - 1 = n(\mathcal{F})$ for any minimal reduction J of $\mathcal{F}$. Hence we get the result for $d = 1$.

Suppose $d \geq 2$ and $J = (x_1, \ldots, x_d)$ is a minimal reduction of $\mathcal{F}$ consisting of superficial elements. Denote $r_J(\mathcal{F})$ by r. For all $n \in \mathbb{Z}$, we get

$$H_R(n) = H_{\mathcal{F}}(n+1) - H_{\mathcal{F}}(n) \text{ and hence } P_R(n) = P_{\mathcal{F}}(n+1) - P_{\mathcal{F}}(n).$$

Since there exists an integer m, such that for all $n \geq m$, $P_{\mathcal{F}}(n) = H_{\mathcal{F}}(n)$, by Theorem 4.5, we have

$$\begin{aligned} P_{\mathcal{F}}(n) - H_{\mathcal{F}}(n) &= P_{\mathcal{F}}(n+1) - H_{\mathcal{F}}(n+1) \\ &= \cdots \\ &= P_{\mathcal{F}}(n+m) - H_{\mathcal{F}}(n+m) = 0 \text{ for all } n \geq r-d+1. \end{aligned}$$

Again using Theorem 4.5, we get

$$\begin{aligned} 0 &\neq P_R(r-d) - H_R(r-d) \\ &= [P_{\mathcal{F}}(r-d+1) - H_{\mathcal{F}}(r-d+1)] - [P_{\mathcal{F}}(r-d) - H_{\mathcal{F}}(r-d)] \\ &= P_{\mathcal{F}}(r-d) - H_{\mathcal{F}}(r-d). \end{aligned}$$

Thus $r_J(\mathcal{F}) - d = n(\mathcal{F})$ for any minimal reduction J of $\mathcal{F}$. Hence $r(\mathcal{F}) = n(\mathcal{F}) + d$. □

5 Nonnegativity and Vanishing of Hilbert Coefficients

In this section, we apply Grothendieck–Serre formula to derive various properties of the Hilbert coefficients. We derive a result of Northcott, Narita, Marley, and Itoh. We also derive a formula for the components of local cohomology modules of Rees algebras in terms of the Hilbert coefficients (Proposition 5.11) which generalises [35, Proposition 5] and [20, Proposition 3.3].

The following theorem is a generalisation of a result due to Northcott [27, Theorem 1].

Theorem 5.1 (Northcott's inequality) *Let $(R, \mathfrak{m})$ be a $d \geq 1$-dimensional Cohen–Macaulay local ring, I an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ be an I-admissible filtration of ideals in R. Then*

$$e_1(\mathcal{F}) \geq e_0(\mathcal{F}) - \lambda\left(\frac{R}{I_1}\right) \geq 0.$$

Proof We use induction on d. Let $d = 1$. Since R is Cohen–Macaulay, putting $n = 1$ in the Difference Formula (Theorem 2.5) for Rees algebra of $\mathcal{F}$, we have

$$\begin{aligned} e_0(\mathcal{F}) - e_1(\mathcal{F}) - \lambda\left(\frac{R}{I_1}\right) &= P_{\mathcal{F}}(1) - H_{\mathcal{F}}(1) \\ &= \lambda_R[H^0_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))]_1 - \lambda_R[H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))]_1 \\ &= -\lambda_R[H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))]_1 \leq 0. \end{aligned}$$

Thus we get the first inequality. Suppose $d \geq 2$ and the result is true for rings with dimension upto $d-1$. Without loss of generality we may assume that the residue field of R is infinite. Let $x \in I_1$ be a superficial element for $\mathcal{F}$. Then $e_0(\mathcal{F}) = e_0(\mathcal{F}')$

and $e_1(\mathcal{F}) = e_1(\mathcal{F}')$ where "′" denotes the image in $R' = R/(x)$. Since $\lambda(R'/I_1') = \lambda(R/I_1)$, by induction hypothesis we get the first inequality.

For any minimal reduction J of $\mathcal{F}$, J is minimal reduction I_1 by [31, Lemma 1.5]. Hence, we get the second inequality. □

Theorem 5.2 ([25, Theorem 5.6]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 1$ and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration. Then*

(1) $e_\alpha(\mathcal{F}) \geq 0$ where $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, $|\alpha| \geq d - 1$.
(2) $e_\alpha(\mathcal{F}) \geq 0$ where $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, $|\alpha| = d - 2$ and $d \geq 2$.

Proof (1) For $|\alpha| = d$, the result follows from [31, Theorem 2.4]. Suppose $|\alpha| = d - 1$. We use induction on d. Let $d = 1$. Then putting $\underline{n} = \underline{0}$ in the Difference Formula (Theorem 2.5), we get $e_{\underline{0}}(\mathcal{F}) = \lambda_R[H^1_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{0}} \geq 0$. Let $d \geq 2$ and assume the result for rings of dimension $d - 1$. Then there exists i such that $\alpha_i \geq 1$. Without loss of generality assume $\alpha_1 \geq 1$. By Lemma 3.3, there exists a nonzerodivisor $x \in I_1$ such that $(x) \cap \mathcal{F}(\underline{n}) = x\mathcal{F}(\underline{n} - e_1)$ for all $\underline{n} \in \mathbb{N}^s$ such that $n_1 \gg 0$. Let $R' = R/(x)$ and $\mathcal{F}' = \{\mathcal{F}(\underline{n})R'\}_{\underline{n} \in \mathbb{Z}^s}$. For all large $\underline{n}$, consider the following short exact sequence

$$0 \longrightarrow \frac{(\mathcal{F}(\underline{n}) : (x))}{\mathcal{F}(\underline{n} - e_1)} \longrightarrow \frac{R}{\mathcal{F}(\underline{n} - e_1)} \xrightarrow{.x} \frac{R}{\mathcal{F}(\underline{n})} \longrightarrow \frac{R}{(x, \mathcal{F}(\underline{n}))} \longrightarrow 0.$$

Since forall large $\underline{n}$, $(\mathcal{F}(\underline{n}) : (x)) = \mathcal{F}(\underline{n} - e_1)$, we get $P_{\mathcal{F}'}(\underline{n}) = P_{\mathcal{F}}(\underline{n}) - P_{\mathcal{F}}(\underline{n} - e_1)$. Hence $(-1)^{d-1-|(\alpha-e_1)|} b_{(\alpha-e_1)}(\mathcal{F}') = (-1)^{d-|\alpha|} e_\alpha(\mathcal{F})$ where

$$P_{\mathcal{F}'}(\underline{n}) = \sum_{\substack{\gamma=(\gamma_1,\ldots,\gamma_s)\in\mathbb{N}^s \\ |\gamma| \leq d-1}} (-1)^{d-1-|\gamma|} b_\gamma(\mathcal{F}') \binom{n_1 + \gamma_1 - 1}{\gamma_1} \cdots \binom{n_s + \gamma_s - 1}{\gamma_s}.$$

Since $|(\alpha - e_1)| = d - 2 = (d - 1) - 1$, by induction $b_{(\alpha-e_1)}(\mathcal{F}') \geq 0$. Hence for $|\alpha| = d - 1$, $e_\alpha(\mathcal{F}) \geq 0$.

(2) We prove the result using induction on d. For $d = 2$ the result follows from the Difference Formula (Theorem 2.5) for $\underline{n} = \underline{0}$ and Proposition 3.5. The rest is same as for the case $|\alpha| = d - 1$. □

As a consequence of this we get the following results which is proved by Marley [23, Propositions 3.19 and 3.23]. The next one is a generalisation of a result due to M. Narita [26, Theorem 1]. Here we give different proof.

Proposition 5.3 *Let $(R, \mathfrak{m})$ be a d-dimensional $(d \geq 2)$ Cohen–Macaulay local ring, I an $\mathfrak{m}$-primary ideal and $\mathcal{F} = \{I^n\}_{n \in \mathbb{Z}}$ an admissible I-filtration. Then $e_2(\mathcal{F}) \geq 0$.*

Proof Comparing the expressions of coefficients of Hilbert polynomials for $s = 1$ and $s \geq 2$, by Theorem 5.2, we get the required result. □

It is natural to ask whether $e_i(\mathcal{F})$ are nonnegative for $i \geq 3$ in a Cohen–Macaulay local ring. Narita [26, Theorem 2] and Marley [22, Example 2] gave an example of an ideal in a Cohen–Macaulay local ring with $e_3(I) < 0$.

Example 5.4 [26, Theorem 2] Let Δ be a formal power series $k[[X_1, X_2, X_3, X_4]]$ over a field k and $Q = \Delta/\Delta X_4^3$. Then Q is a Cohen–Macaulay local ring of dimension 3. Let x_1, x_2, x_3, x_4 be the images of X_1, X_2, X_3, X_4 in Q and $I = Qx_1 + Qx_2^2 + Qx_3^2 + Qx_2x_4 + Qx_3x_4$. Then

$$e_3(I) = -\lambda_{Q'}\left(\frac{((IQ')^2 : (x_2Q')^2)}{IQ'}\right) = -\lambda_{Q'}\left(\frac{IQ' + (x_4Q')^2}{IQ'}\right) < 0 \text{ where } Q' = Q/(x_1).$$

Example 5.5 [22, Example 2] Let $I = (X^3, Y^3, Z^3, X^2Y, XY^2, YZ^2, XYZ)$ in the regular local ring $R = k[X, Y, Z]_{(X,Y,Z)}$. Then for all $n \geq 1$,

$$P_I(n) = 27\binom{n+2}{3} - 18\binom{n+1}{2} + 4n + 1.$$

Hence $e_3(I) = -1 < 0$.

However, for $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$, Itoh proved that $e_3(\mathcal{F})$ is nonnegative in an analytically unramified Cohen–Macaulay local ring [17, Theorem 3]. In order to prove this, he used an analogue of Theorem 2.3 (see [17, p.114]). In [12, Corollary 3.9], authors gave an alternative proof of this result. We prove this result using the GSF. For this purpose, we recall some results of Itoh about vanishing of graded components of local cohomology modules. See also [10, Theorem 1.2].

Theorem 5.6 ([16, Theorem 2] [17, Proposition 13]) *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay local ring of dimension $d \geq 2$. Let $\mathcal{M} = (t^{-1}, \mathcal{R}(\mathcal{F})_+)$ be the maximal homogeneous ideal of $\mathcal{R}'(\mathcal{F})$. Then the following statements hold true for the filtration $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$:*

(1) $H^0_{\mathcal{M}}(\mathcal{R}'(\mathcal{F})) = H^1_{\mathcal{M}}(\mathcal{R}'(\mathcal{F})) = 0$;
(2) $H^2_{\mathcal{M}}(\mathcal{R}'(\mathcal{F})_j = 0$ for $j \leq 0$;
(3) $H^i_{\mathcal{M}}(\mathcal{R}'(\mathcal{F})) = H^i_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))$ for $i = 0, 1, \ldots, d-1$.

Theorem 5.7 ([17, Theorem 3]) *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay local ring of dimension $d \geq 3$ and I be an $\mathfrak{m}$-primary ideal in R. Then $\overline{e}_3(I) \geq 0$.*

Proof For $\mathcal{F} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$, we set $\overline{\mathcal{R}'}(I) := \mathcal{R}'(\mathcal{F})$. We use induction on d. Let $d = 3$. Then, by the Difference Formula (Theorem 2.5) for Rees algebras and Theorem 5.6, we have

$$\bar{e}_3(I) = h^3_{\overline{\mathcal{R}'}(I)_+}\overline{\mathcal{R}'}(I)_0 \geq 0.$$

Let $d > 3$. We may assume that the residue field of R is infinite. Let $J \subseteq I$ be a reduction of I. Since $\overline{I^n} = \overline{J^n}$ for all n, $\bar{e}_i(I) = \bar{e}_i(J)$ for all $i = 1, \ldots, d$. Therefore it suffices to show that $\bar{e}_3(J) \geq 0$. By [17, Theorem 1 and Corollary 8], there exists a system of generators $x_1, \ldots, x_d$ of J such that, if we put $T = (T_1, \ldots, T_d)$, $R(T) = R[T]_{\mathfrak{m}[T]}$ and $C = R(T)/(\sum_{i=1}^d x_i T_i)$, then C is an analytically unramified Cohen–Macaulay local ring of dimension $d - 1$ and $\bar{e}_3(J) = \bar{e}_3(JC)$. Hence, using induction hypothesis the result follows. □

Itoh [17, p.116] proposed the following conjecture on the vanishing of $\bar{e}_3(I)$ which is still open.

Conjecture 5.8 (*Itoh's Conjecture*) Let $(R, \mathfrak{m})$ be an analytically unramified Gorenstein local ring of dimension $d \geq 3$. Then $\bar{e}_3(I) = 0$ if and only if $\overline{I^{n+2}} = I^n\overline{I^2}$ for every $n \geq 0$.

Itoh proved the "if" part of the Conjecture 5.8 in [16, Proposition 10]. He also proved the "only if" part of the Conjecture 5.8 for $\overline{I} = \mathfrak{m}$ [17, Theorem 3(2)]. By [17, Corollary 8 and Proposition 17], it suffices to prove the Conjecture 5.8 for $d = 3$. Let $d = 3$ and $\bar{e}_3(I) = 0$ for an $\mathfrak{m}$-primary ideal in a Cohen–Macaulay ring R. By [16, Proposition 3] and [17, Corollary 16 and (4.1)], $\overline{I^{n+2}} = I^n\overline{I^2}$ for every $n \geq 0$ if and only if $\overline{\mathcal{R}'}(I)$ is Cohen–Macaulay.

It is not known whether the Itoh's conjecture is true for $\overline{I} = \mathfrak{m}$ in a Cohen–Macaulay local ring R (which need not be Gorenstein). Recently, in [8, Theorem 3.6], the authors proved that the Conjecture 5.8 holds true for $\overline{I} = \mathfrak{m}$ in a Cohen–Macaulay local ring of type at most two. T.T. Phuong [29], showed that if R is an analytically unramified Cohen–Macaulay local ring of dimension $d \geq 2$ then the equality $\bar{e}_1(I) = \bar{e}_0(I) - \lambda(R/\overline{I}) + 1$ leads to the vanishing of $\bar{e}_3(I)$. In [21], authors generalised the result of [8]. They also obtained following result for an arbitrary $\mathfrak{m}$-primary ideal I in an analytically unramified Cohen–Macaulay local ring of dimension 3.

Theorem 5.9 ([21, Theorem 1.1]) *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay local ring of dimension* 3. *Let $\mathcal{M} = (t^{-1}, \overline{\mathcal{R}'}_+)$ and $\overline{\mathcal{R}'} = \bigoplus_{n\in\mathbb{Z}} \overline{I^n}t^n$. Suppose that $\bar{e}_3(I) = 0$. Then*

(1) $H^3_{\mathcal{M}}(\overline{\mathcal{R}'}) = 0$,

(2) *Suppose either that R is equicharacteristic or that $\overline{I} = \mathfrak{m}$, and that I has a reduction generated by x, y, z. If $\overline{\mathcal{R}'}$ is not Cohen–Macaulay, then* $\bar{e}_2(I) - \lambda\left(\dfrac{\overline{I^2}}{(x, y, z)\overline{I}}\right) \geq 3$.

As a consequence they generalised [8, Theorem 3.6].

Corollary 5.10 ([21, Corollary 1.2]) *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay local ring of dimension* 3.

(1) *Suppose* $\bar{e}_3(I) = 0$. *Then there is an inclusion* $H^3_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_{-1} \subseteq (0 :_{\mathrm{H}^3_{\mathfrak{m}}(R)} \overline{I})$.
(2) *Suppose* $\bar{e}_3(\mathfrak{m}) = 0$. *Then* $\bar{e}_2(\mathfrak{m}) \leq \operatorname{type}(R)$.
(3) $\overline{\mathcal{R}'}(\mathfrak{m})$ *is Cohen–Macaulay if* $\bar{e}_2(\mathfrak{m}) \leq \operatorname{length}_R(\overline{I^2}/\mathfrak{m}I) + 2$ *for any ideal I such that $\overline{I} = \mathfrak{m}$, $\bar{e}_3(\mathfrak{m}) = 0$ and I has a minimal reduction.*

Proof (1): By Theorem 5.9, $H^3_{\mathcal{M}}(\overline{\mathcal{R}'}) = 0$. Hence, by [17, Proposition 13(3)], we get an exact sequence

$$0 \longrightarrow H^3_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_{-1} \longrightarrow H^3_{\mathfrak{m}}(R) \longrightarrow H^4_{\mathcal{M}}(\overline{\mathcal{R}'})_{-1} \longrightarrow 0.$$

Thus $H^3_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_{-1} \subseteq H^3_{\mathfrak{m}}(R)$. By the Difference Formula (Theorem 2.5) and Theorem 5.6, we get

$$h^3_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_0 = \bar{e}_3(I) = 0. \tag{5.10.1}$$

Now consider the exact sequence

$$0 \longrightarrow \overline{\mathcal{R}'}(1) \longrightarrow \overline{\mathcal{R}'} \longrightarrow \overline{G} = \bigoplus_{n \geq 0} \frac{\overline{I^n}}{\overline{I^{n+1}}} \longrightarrow 0$$

which gives the long exact sequence

$$\cdots \longrightarrow H^i_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_{n+1} \longrightarrow H^i_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_n \longrightarrow H^i_{\overline{G}_+}(\overline{G})_n \longrightarrow \cdots.$$

Using (5.10.1), we get an isomorphism $H^3_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_{-1} \simeq H^3_{\overline{G}_+}(\overline{G})_{-1}$. This implies that $H^3_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_{-1}$ is an $R/\overline{I}$-module. Therefore $H^3_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_{-1} \subseteq (0 :_{\mathrm{H}^3_{\mathfrak{m}}(R)} \overline{I})$.

(2) Taking $I = \mathfrak{m}$, by the Difference Formula (Theorem 2.5) and Theorem 5.6, we get $\bar{e}_2(I) = h^3_{\overline{\mathcal{R}'}_+}(\overline{\mathcal{R}'})_{-1}$. Hence by (1) we get the result.

(3) Follows from Theorem 5.9(2). □

The next result was first proved by Sally [35, Proposition 5] for the filtration $\{\mathfrak{m}^n\}_{n\in\mathbb{Z}}$ and then by Johnston and Verma [20, Proposition 3.3] for the filtration

$\{I^n\}_{n\in\mathbb{Z}}$ where I is an $\mathfrak{m}$-primary ideal of R. Here we prove the result for $\mathbb{Z}$-graded admissible filtrations.

Proposition 5.11 *Let $(R, \mathfrak{m})$ be a two-dimensional Cohen–Macaulay local ring, I be any $\mathfrak{m}$-primary ideal of R and $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ an I-admissible filtration of ideals in R. Then*

(1) $\lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F})_0\right) = e_2(\mathcal{F})$,

(2) $\lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_1\right) = e_0(\mathcal{F}) - e_1(\mathcal{F}) + e_2(\mathcal{F}) - \lambda\left(\frac{R}{\breve{I}_1}\right)$,

(3) $\lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_{-1}\right) = e_1(\mathcal{F}) + e_2(\mathcal{F})$.

Proof We have

$$P_{\mathcal{F}}(n) - H_{\mathcal{F}}(n) = \sum_{i\geq 0}(-1)^i h^i_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))_n \text{ for all } n \in \mathbb{Z}. \quad (5.11.1)$$

(1) Putting $n = 0$ in (5.11.1) and using Propositions 2.4 and 3.7, we get the required result.
(2) Putting $n = 1$ in (5.11.1) and using Propositions 2.4 and 3.7, we get the required result.
(3) Consider the short exact sequence of $\mathcal{R}(\mathcal{F})$-modules

$$0 \longrightarrow \mathcal{R}(\mathcal{F})_+ \longrightarrow \mathcal{R}(\mathcal{F}) \longrightarrow R \cong \mathcal{R}(\mathcal{F})/\mathcal{R}(\mathcal{F})_+ \longrightarrow 0$$

which induces a long exact sequence of local cohomology modules whose nth component is

$$\cdots \longrightarrow H^i_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F})_+)_n \longrightarrow H^i_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_n \longrightarrow H^i_{\mathcal{R}(\mathcal{F})_+}(R)_n \longrightarrow \cdots \text{ for all } i \geq 0.$$

Since R is $\mathcal{R}(\mathcal{F})_+$-torsion, $H^0_{\mathcal{R}(\mathcal{F})_+}(R) = R$ and $H^i_{\mathcal{R}(\mathcal{F})_+}(R) = 0$ for all $i \geq 1$. Hence $H^i_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F})_+) \cong H^i_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))$ for all $i \geq 2$ and we have the exact sequence

$$\begin{aligned} 0 \to H^0_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F})_+)_n &\to H^0_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_n \to R \\ \to H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F})_+)_n &\to H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_n \to 0. \end{aligned} \quad (5.11.2)$$

The short exact sequence of $\mathcal{R}(\mathcal{F})$-modules

$$0 \longrightarrow \mathcal{R}(\mathcal{F})_+(1) \longrightarrow \mathcal{R}(\mathcal{F}) \longrightarrow G(\mathcal{F}) \longrightarrow 0$$

induces the exact sequence

$$\begin{aligned} 0 \longrightarrow H^0_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1} \to H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F})_+)_0 \to H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_{-1} \to H^1_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1} \\ \to H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_0 \to H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_{-1} \to H^2_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1} \to 0. \end{aligned} \quad (5.11.3)$$

Now $H^0_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F})) \subseteq G(\mathcal{F})$ are nonzero only in nonnegative degrees. Thus $H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_{-1} \cong R$ and $H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_0 = 0$ by Proposition 3.6. Therefore from the exact sequence (5.11.2), we get the exact sequence

$$0 \to R \to H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F})_+)_0 \to H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_0 = 0.$$

Let f denote the map from $H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_{-1}$ to $H^0_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1}$ in the exact sequence (5.11.3). First we prove that f is zero map. From the exact sequence (5.11.3), we get the exact sequence

$$0 \longrightarrow R \xrightarrow{g} R \xrightarrow{f} H^1_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1}.$$

Since $R/g(R)$ is contained in $H^1_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1}$ and by Proposition 2.2, $H^1_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1}$ is of finite length, we have $\lambda_R(R/g(R))$ is finite. Since $g(R)$ is principal ideal in R, we get $R = g(R)$. Therefore f is the zero map. Hence we get the exact sequence

$$0 \to H^1_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1} \to H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_0 \to H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_{-1} \to H^2_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1} \to 0.$$

Therefore by Theorem 2.3, we get

$$\begin{aligned}[H_{\mathcal{F}}(0) - H_{\mathcal{F}}(-1)] - [P_{\mathcal{F}}(0) - P_{\mathcal{F}}(-1)] &= -\lambda\left(H^1_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1}\right) + \lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))_{-1}\right)\\ &= -\lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_0\right) + \lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_{-1}\right).\end{aligned}$$

Thus by part (1) of the Proposition, we get

$$\lambda\left(H^2_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_{-1}\right) = e_2(\mathcal{F}) - e_2(\mathcal{F}) + e_1(\mathcal{F}) + e_2(\mathcal{F}) = e_1(\mathcal{F}) + e_2(\mathcal{F}).$$

□

6 Huneke–Ooishi Theorem and a Multi-graded Version

In this section we give an application of the GSF to derive a result of Huneke [14] and Ooishi [28] which states that if $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring of dimension $d \geq 1$ and I is an $\mathfrak{m}$-primary ideal then $e_0(I) - e_1(I) = \lambda(R/I)$ if and only if $r(I) \leq 1$. A similar result for admissible filtrations was proved in [3, Theorem 4.3.6] and [13, Corollary 4.9]. In [25, Theorem 5.5], authors gave a partial generalisation of this result for an $\underline{I}$-admissible filtration. First we prove few preliminary results needed.

Lemma 6.1 (Sally machine) [34, Corollary 2.4] [13, Lemma 2.2] *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring,* I_1 *an* $\mathfrak{m}$*-primary ideal in* R *and* $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ *be an* I_1*-*

admissible filtration of ideals in R. *Let* $x_1, \dots, x_r$ *be a superficial sequence for* $\mathcal{F}$. *If* grade $G(\mathcal{F}/(x_1, \dots, x_r))_+ \geq 1$ *then* grade $G(\mathcal{F})_+ \geq r+1$.

Proof We use induction on r. Let $r = 1$ and $y \in I_t$ such that image of y in $G(\mathcal{F}/(x_1))_t$ is a nonzerodivisor. Then $(I_{n+tj} : y^j) \subseteq (I_n, x_1)$ for all n, j. Since x_1 is a superficial element for $\mathcal{F}$, there exists integer $c \geq 0$, such that $(I_{n+j} : x_1^j) \cap I_c = I_n$ for all $j \geq 1$ and $n \geq c$. Consider an integer $p > c/t$. For arbitrary n and $j \geq 1$, we prove that

$$y^p(I_{n+j} : x_1^j) \subseteq (I_{n+j+tp} : x_1^j) \cap I_c = I_{n+tp}.$$

Let $a \in (I_{n+j} : x_1^j)$. Then $ay^p x_1^j \in I_{n+j+tp}$. Since $pt > c$, $ay^p \in (I_{n+j+tp} : x_1^j) \cap I_c = I_{n+tp}$. Therefore

$$(I_{n+j} : x_1^j) \subseteq (I_{n+tp} : y^p) \subseteq (I_n, x_1).$$

Thus $(I_{n+j} : x_1^j) = I_n + x_1(I_{n+j} : x_1^{j+1})$ for all n and $j \geq 1$. Iterating this formula n times, we get

$$(I_{n+j} : x_1^j) = I_n + x_1 I_{n-1} + x_1^2 I_{n-2} + \cdots + x_1^n (I_{n+j} : x_1^{j+n}) = I_n.$$

Hence $x_1^* = x_1 + I_2$ is a nonzerodivisor of $G(\mathcal{F})$. Since $G(\mathcal{F})/(x_1^*) \simeq G(\mathcal{F}/(x_1))$, grade $G(\mathcal{F})_+ \geq 2$.

Now assume $r \geq 2$. Then by $r = 1$ case, we have grade $G(\mathcal{F}/(x_1, \dots, x_{r-1}))_+ \geq 2 > 1$. By induction on r, we have grade $G(\mathcal{F})_+ \geq r$ and since $x_1, \dots, x_r$ is a superficial sequence for $\mathcal{F}$, by Lemma 4.4, we obtain $x_1^*, \dots, x_r^*$ is a regular sequence of $G(\mathcal{F})$. Since $G(\mathcal{F})/(x_1^*, \dots, x_r^*) \simeq G(\mathcal{F}/(x_1, \dots, x_r))$, grade $G(\mathcal{F})_+ \geq r+1$.

□

The next lemma is due to Marley [23, Lemma 3.14].

Lemma 6.2 *Let* $(R, \mathfrak{m})$ *be a Cohen–Macaulay local ring of dimension* $d \geq 1$, I *an* $\mathfrak{m}$*-primary ideal and* $\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$ *be an* I*-admissible filtration of ideals in* R. *Suppose* $x \in I_1 \backslash I_2$ *such that* $x^* = x + I_2$ *is a nonzerodivisor in* $G(\mathcal{F})$. *Let* $R' = R/(x)$. *Then* $n(\mathcal{F}) = n(\mathcal{F}') - 1$ *where* $\mathcal{F}' = \{I_n R'\}_{n \in \mathbb{Z}}$.

Proof We use the notation "′" to denote the image in R'. For all n, consider the following short exact sequence of R-modules

$$0 \longrightarrow (I_n : x)/I_n \longrightarrow R/I_n \xrightarrow{.x} R/I_n \longrightarrow R'/I_n' \longrightarrow 0.$$

Therefore $H_{\mathcal{F}'}(n) = \lambda(R'/I_n') = \lambda((I_n : x)/I_n)$. Since x^* is a nonzerodivisor in $G(\mathcal{F})$, we have $(I_{n+1} : x) = I_n$ for all n. Hence $H_{\mathcal{F}'}(n) = \lambda(I_{n-1}/I_n) = \lambda(R/I_n) - \lambda(R/I_{n-1}) = H_{\mathcal{F}}(n) - H_{\mathcal{F}}(n-1)$ for all n which implies $P_{\mathcal{F}'}(n) = P_{\mathcal{F}}(n) - P_{\mathcal{F}}(n-1)$ for all n. Thus $H_{\mathcal{F}'}(n) = P_{\mathcal{F}'}(n)$ for all $n \geq n(\mathcal{F}) + 2$. Since

$$\begin{aligned}P_{\mathcal{F}'}(n(\mathcal{F})+1)-H_{\mathcal{F}'}(n(\mathcal{F})+1)&=[P_{\mathcal{F}}(n(\mathcal{F})+1)-H_{\mathcal{F}}(n(\mathcal{F})+1)]\\&\quad-[P_{\mathcal{F}}(n(\mathcal{F}))-H_{\mathcal{F}}(n(\mathcal{F}))]\\&=-[P_{\mathcal{F}}(n(\mathcal{F}))-H_{\mathcal{F}}(n(\mathcal{F}))]\neq 0,\end{aligned}$$

we get the required result. □

The next theorem is due to Blancafort [3] which is a generalisation of a result of Huneke [14] and Ooishi [28] proved independently. We make use of reduction number and postulation number of admissible filtration of ideals to simplify her proof.

Theorem 6.3 ([3, Theorem 4.3.6]) *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring with infinite residue field of dimension $d\geq 1$, I_1 an $\mathfrak{m}$-primary ideal and $\mathcal{F}=\{I_n\}_{n\in\mathbb{Z}}$ be an I_1-admissible filtration of ideals in R. Then the following are equivalent:*

(1) $e_0(\mathcal{F})-e_1(\mathcal{F})=\lambda(R/I_1)$,
(2) $r(\mathcal{F})\leq 1$.

In this case, $e_2(\mathcal{F})=\cdots=e_d(\mathcal{F})=0$, $G(\mathcal{F})$ is Cohen–Macaulay, $n(\mathcal{F})\leq 0$, $r(\mathcal{F})$ is independent of the reduction chosen and $\mathcal{F}=\{I_1^n\}$.

Proof (1) ⇒ (2) We use induction on d. Let $d=1$. For all $n\in\mathbb{Z}$, we have

$$P_{\mathcal{F}}(n)-H_{\mathcal{F}}(n)=-h^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))_n.$$

By putting $n=1$ in this formula, we get $e_0(\mathcal{F})-e_1(\mathcal{F})-\lambda(R/I_1)=-h^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))_1=0$. Therefore by Lemma 3.8, for all $n\geq 1$, $h^1_{\mathcal{R}(\mathcal{F})_+}\mathcal{R}'(\mathcal{F})_n=0$. Consider the short exact sequence of $\mathcal{R}(\mathcal{F})$-modules,

$$0\longrightarrow \mathcal{R}'(\mathcal{F})(1)\xrightarrow{t^{-1}}\mathcal{R}'(\mathcal{F})\longrightarrow G(\mathcal{F})\longrightarrow 0.$$

This induces a long exact sequence,

$$0\longrightarrow [H^0_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))]_n\longrightarrow [H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}'(\mathcal{F}))]_{n+1}\longrightarrow\cdots.$$

Thus for all $n\in\mathbb{N}$, $[H^0_{\mathcal{R}(\mathcal{F})_+}(G(\mathcal{F}))]_n=0$. Hence $G(\mathcal{F})$ is Cohen–Macaulay. Let $J=(x)$ be a minimal reduction of $\mathcal{F}$. Without loss of generality x is superficial. For each n, consider the following map

$$\frac{I_{k+n}}{x^kI_n}\xrightarrow{\phi_k}\frac{I_{k+n+1}}{x^{k+1}I_n}\text{ where }\phi_k(\overline{z})=\overline{xz}.$$

For all large k, $I_{k+n+1}=xI_{k+n}$. Hence for all large k, ϕ_k is surjective. Now suppose $\phi_k(\overline{z})=0$ for some $\overline{z}\in I_{k+n}/x^kI_n$. Then $xz\in x^{k+1}I_n$. Therefore $xz=x^{k+1}a$ where

$a \in I_n$, hence $z \in x^k I_n$. Thus for all large k, ϕ_k is injective. Therefore by Proposition 3.2, for all large k,

$$H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_n \simeq \frac{I_{k+n}}{x^k I_n}.$$

By Lemma 3.8 and Proposition 2.4, $H^1_{\mathcal{R}(\mathcal{F})_+}(\mathcal{R}(\mathcal{F}))_n = 0$ for all $n \geq 1$. Then for all large k and $n \geq 1$,

$$I_{k+n} = x^k I_n.$$

Let $a \in I_{k+n-1}$. Then $xa \in I_{k+n} \subseteq x^k I_n$ implies $a \in x^{k-1} I_n$. Thus $I_{k+n-1} = x^{k-1} I_n$. Using this procedure repeatedly we get $I_{n+1} = x I_n$. Thus $r(\mathcal{F}) \leq 1$.

Let $d \geq 2$ and $x \in I_1$ be a superficial element for $\mathcal{F}$. Let $R' = R/(x)$, $\mathcal{F}' = \{I_n R'\}_{n \in \mathbb{Z}}$ and $G' = G(\mathcal{F}')$. Since $e_i(\mathcal{F}) = e_i(\mathcal{F}')$ for all $i < d$, we have

$$e_0(\mathcal{F}') - e_1(\mathcal{F}') = e_0(\mathcal{F}) - e_1(\mathcal{F}) = \lambda\left(\frac{R}{I_1}\right) = \lambda\left(\frac{R'}{I_1 R'}\right).$$

Hence by induction hypothesis, G' is Cohen–Macaulay. Therefore by Sally machine (Lemma 6.1), $G(\mathcal{F})$ is Cohen–Macaulay. This implies that for any minimal reduction J of $\mathcal{F}$, $r_J(\mathcal{F}) = n(\mathcal{F}) + d$ by Theorem 4.6. Thus $r_J(\mathcal{F})$ is independent of the minimal reduction J of I. Let J be a minimal reduction of $\mathcal{F}$ generated by superficial sequence $x_1, \ldots, x_d$. Let $\overline{R} = R/(x_1, \ldots, x_{d-1})$ and $\overline{\mathcal{F}} = \{I_n \overline{R}\}_{n \in \mathbb{Z}}$. Since $G(\mathcal{F})$ is Cohen–Macaulay and $x_1, \ldots, x_d$ is superficial, using Theorem 4.6 and Lemmas 4.4, 4.2 and 6.2, for $d - 1$ times, by induction hypothesis we get

$$r(\mathcal{F}) = n(\mathcal{F}) + d = n(\overline{\mathcal{F}}) + 1 = r(\overline{\mathcal{F}}) \leq 1.$$

(2) $\Rightarrow$ (1) Let J be a minimal reduction of $\mathcal{F}$ such that $r(\mathcal{F}) = r_J(\mathcal{F})$ and J is generated by superficial sequence $x_1, \ldots, x_d$. Let $R' = R/(x_1, \ldots, x_{d-1})$ and $\mathcal{F}' = \{I_n R'\}_{n \in \mathbb{Z}}$. Then $x_d I_n R' = I_{n+1} R'$ for all $n \geq 1$. Since x'_d is nonzerodivisor, $(I_{n+1} R' : x'_d) = I_n R'$ for all $n \geq 1$. Therefore $(x'_d)^*$ (the image of x'_d in $G(\mathcal{F}')$) is nonzerodivisor in $G(\mathcal{F}')$. Hence $G(\mathcal{F}')$ is Cohen–Macaulay. Thus by Lemma 6.1, $G(\mathcal{F})$ is Cohen–Macaulay. Therefore by Theorem 4.6, $n(\mathcal{F}) = r(\mathcal{F}) - d \leq 0$. Hence $P_{\mathcal{F}}(n) = H_{\mathcal{F}}(n)$ for all $n > 0$. By putting $n = 1$ for $d = 1$ case we obtain $e_0(\mathcal{F}) - e_1(\mathcal{F}) = \lambda(R/I_1)$.

Now we prove that if $r(\mathcal{F}) \leq 1$ then $e_2(\mathcal{F}) = \cdots = e_d(\mathcal{F}) = 0$. Without loss of generality assume $d \geq 2$. The condition $r(\mathcal{F}) \leq 1$ implies $G(\mathcal{F})$ is Cohen–Macaulay and $n(\mathcal{F}) = r(\mathcal{F}) - d < 0$. Let $d = 2$. Therefore $e_2(\mathcal{F}) = P_{\mathcal{F}}(0) - H_{\mathcal{F}}(0) = 0$. Now assume $d \geq 3$ and the result is true upto dimension $d - 1$. Let J be minimal reduction of $\mathcal{F}$ generated by superficial sequence $x_1, \ldots, x_d$. Let $R' = R/(x_1, \ldots, x_{d-1})$ and $\mathcal{F}' = \{I_n R'\}_{n \in \mathbb{Z}}$. Then $e_i(\mathcal{F}) = e_i(\mathcal{F}') = 0$ for all $0 \leq i < d$. Since $G(\mathcal{F})$ is Cohen–Macaulay and $n(\mathcal{F}) = r(\mathcal{F}) - d < 0$, we get $(-1)^d e_d(\mathcal{F}) = P_{\mathcal{F}}(0) - H_{\mathcal{F}}(0) = 0$. Therefore $e_0(\mathcal{F}) - e_1(\mathcal{F}) - \lambda(R/I_1) = P_{\mathcal{F}}(1) - H_{\mathcal{F}}(1) = 0$.

Let J be a minimal reduction of $\mathcal{F}$ such that $r(\mathcal{F}) = r_J(\mathcal{F})$ and $r(\mathcal{F}) \le 1$. Then $I_2 = JI_1 \subseteq I_1^2 \subseteq I_2$. Suppose $I_r = I_1^r$ for all $1 \le r \le n$. Then $I_{n+1} = JI_n \subseteq I_1 I_1^n \subseteq I_1^{n+1} \subseteq I_{n+1}$. Thus $\mathcal{F}$ is $\{I_1^n\}_{n\in\mathbb{Z}}$. □

Theorem 6.4 ([25, Theorem 5.5]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \ge 1$ and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then for all $i = 1, \ldots, s$,*

(1) $e_{(d-1)e_i}(\mathcal{F}) \ge e_1(\mathcal{F}^{(i)})$,
(2) $e(I_i) - e_{(d-1)e_i}(\mathcal{F}) \le \lambda(R/\mathcal{F}(e_i))$,
(3) $e(I_i) - e_{(d-1)e_i}(\mathcal{F}) = \lambda(R/\mathcal{F}(e_i))$ if and only if $r(\mathcal{F}^{(i)}) \le 1$ and $e_{(d-1)e_i}(\mathcal{F}) = e_1(\mathcal{F}^{(i)})$, where $\mathcal{F}^{(i)} = \{\mathcal{F}(ne_i)\}_{n\in\mathbb{Z}}$ is an I_i-admissible filtration.

Proof (1) We apply induction on d. Let $d = 1$. Then by Theorem 2.5,

$$P_{\mathcal{F}}(re_i) - \lambda(R/\mathcal{F}(re_i)) = -\lambda_R[H^1_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{(re_i)} \text{ for all } r \ge 0.$$

Since $\mathcal{F}^{(i)}$ is I_i-admissible, we have $e(\mathcal{F}^{(i)}) = e(I_i)$. Hence using $P_{\mathcal{F}^{(i)}}(r) = e(I_i)r - e_1(\mathcal{F}^{(i)})$, we get

$$P_{\mathcal{F}^{(i)}}(r) - \lambda(R/\mathcal{F}(re_i)) + [e_1(\mathcal{F}^{(i)}) - e_{\underline{0}}(\mathcal{F})] = -\lambda_R[H^1_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{(re_i)} \le 0.$$

Taking $r \gg 0$, we get $e_{\underline{0}}(\mathcal{F}) \ge e_1(\mathcal{F}^{(i)})$. Let $d \ge 2$. Without loss of generality we may assume that the residue field of R is infinite. By Lemma 3.3, there exists a nonzerodivisor $x_i \in I_i$ such that

$$(x_i) \cap \mathcal{F}(\underline{n}) = x_i\mathcal{F}(\underline{n} - e_i) \text{ for } \underline{n} \in \mathbb{N}^s \text{ where } n_i \gg 0.$$

Let $R' = R/(x_i)$ and $\mathcal{F}' = \{\mathcal{F}(\underline{n})R'\}$ and $\mathcal{F}'^{(i)} = \{\mathcal{F}(ne_i)R'\}$. For all $\underline{n} \in \mathbb{N}^s$ such that $n_i \gg 0$, consider the following exact sequence

$$0 \longrightarrow \frac{(\mathcal{F}(\underline{n}) : (x_i))}{\mathcal{F}(\underline{n} - e_i)} \longrightarrow \frac{R}{\mathcal{F}(\underline{n} - e_i)} \xrightarrow{.x_i} \frac{R}{\mathcal{F}(\underline{n})} \longrightarrow \frac{R}{(x_i, \mathcal{F}(\underline{n}))} \longrightarrow 0.$$

Since for all $\underline{n} \in \mathbb{N}^s$ where $n_i \gg 0$, $(\mathcal{F}(\underline{n}) : (x_i)) = \mathcal{F}(\underline{n} - e_i)$, we get $H_{\mathcal{F}'}(\underline{n}) = H_{\mathcal{F}}(\underline{n}) - H_{\mathcal{F}}(\underline{n} - e_i)$ and hence $P_{\mathcal{F}}(\underline{n}) - P_{\mathcal{F}}(\underline{n} - e_i) = P_{\mathcal{F}'}(\underline{n})$. Therefore $e_{(d-2)e_i}(\mathcal{F}') = e_{(d-1)e_i}(\mathcal{F})$ and $e_1(\mathcal{F}'^{(i)}) = e_1(\mathcal{F}^{(i)})$. Therefore by induction, the result follows.
(2) Using part (1), for all $i = 1, \ldots, s$, we have

$$e(I_i) - e_{(d-1)e_i}(\mathcal{F}) \le e(I_i) - e_1(\mathcal{F}^{(i)}) \le \lambda(R/\mathcal{F}(e_i))$$

where the last inequality follows from Theorem 5.1.

(3) Let $e(I_i) - e_{(d-1)e_i}(\mathcal{F}) = \lambda(R/\mathcal{F}(e_i))$. Then by part (1),

$$\lambda(R/\mathcal{F}(e_i)) = e(I_i) - e_{(d-1)e_i}(\mathcal{F}) \leq e(I_i) - e_1(\mathcal{F}^{(i)}) \leq \lambda(R/\mathcal{F}(e_i)),$$

where the last inequality follows by Theorem 5.1. Hence $e_{(d-1)e_i}(\mathcal{F}) = e_1(\mathcal{F}^{(i)})$ and $e(I_i) - e_1(\mathcal{F}^{(i)}) = \lambda(R/\mathcal{F}(e_i))$. Therefore, by Theorem 6.3, $r(\mathcal{F}^{(i)}) \leq 1$.

Conversely, suppose $r(\mathcal{F}^{(i)}) \leq 1$ and $e_{(d-1)e_i}(\mathcal{F}) = e_1(\mathcal{F}^{(i)})$. Again, by Theorem 6.3, $e(I_i) - e_1(\mathcal{F}^{(i)}) = \lambda(R/\mathcal{F}(e_i))$. Hence $e(I_i) - e_{(d-1)e_i}(\mathcal{F}) = \lambda(R/\mathcal{F}(e_i))$. □

Theorem 6.5 ([25, Theorem 5.7]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension two and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. Let $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ be an $\underline{I}$-admissible filtration of ideals in R. Then $e_{\underline{0}}(\mathcal{F}) = 0$ implies $e(I_i) - e_{e_i}(\mathcal{F}) = \lambda\left(\frac{R}{\breve{\mathcal{F}}(e_i)}\right)$ for all $i = 1, \ldots, s$. Suppose $\breve{\mathcal{F}}$ is $\underline{I}$-admissible filtration, then the converse is also true.*

Proof Let $e_{\underline{0}}(\mathcal{F}) = 0$. By Proposition 3.5, $[H^1_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{0}} = 0$. Hence by Theorem 2.5,

$$\lambda_R[H^2_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{\underline{0}} = e_{\underline{0}}(\mathcal{F}) = 0.$$

By Lemma 3.8, $\lambda_R[H^2_{\mathcal{R}_{++}}(\mathcal{R}'(\mathcal{F}))]_{e_i} = 0$ for all $i = 1, \ldots, s$. Then using Theorem 2.5 and Proposition 3.5, $P_{\mathcal{F}}(e_i) - H_{\mathcal{F}}(e_i) = -\lambda\left(\frac{\breve{\mathcal{F}}(e_i)}{\mathcal{F}(e_i)}\right)$ for all $i = 1, \ldots, s$. Hence $e(I_i) - e_{e_i}(\mathcal{F}) = \lambda\left(\frac{R}{\breve{\mathcal{F}}(e_i)}\right)$ for all $i = 1, \ldots, s$.

Suppose $\breve{\mathcal{F}}$ is $\underline{I}$-admissible filtration and $e(I_i) - e_{e_i}(\mathcal{F}) = \lambda\left(\frac{R}{\breve{\mathcal{F}}(e_i)}\right)$ for all $i = 1, \ldots, s$. Then by [25, Proposition 3.1] and Theorem 3.9, for all $\underline{n} \geq \underline{0}$ and $i = 1, \ldots, s$,

$$[H^0_{G_i(\breve{\mathcal{F}})_{++}}(G_i(\breve{\mathcal{F}}))]_{\underline{n}} = \frac{\breve{\breve{\mathcal{F}}}(\underline{n} + e_i) \cap \breve{\mathcal{F}}(\underline{n})}{\breve{\mathcal{F}}(\underline{n} + e_i)} = 0.$$

Since the Hilbert polynomial of $\breve{\mathcal{F}}$ is same as the Hilbert polynomial of $\mathcal{F}$, by [25, Theorem 5.3],

$$P_{\breve{\mathcal{F}}}(\underline{n}) = H_{\breve{\mathcal{F}}}(\underline{n}) \text{ for all } \underline{n} \geq 0. \tag{6.5.1}$$

Thus taking $\underline{n} = \underline{0}$ in the Eq. (6.5.1), we get $e_{\underline{0}}(\mathcal{F}) = e_{\underline{0}}(\breve{\mathcal{F}}) = 0$. □

As a consequence of the above theorem we get a theorem of Huneke [14, Theorem 4.5] for integral closure filtrations. We also obtain a result by Itoh [18, Corollary 5] following from the above theorem.

Corollary 6.6 ([18, Corollary 5], [25, Corollary 5.8]) *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension two and I be $\mathfrak{m}$-primary ideal of R. Let Q be any minimal reduction of I. Then the following are equivalent.*

(1) $e_1(I) - e_0(I) + \lambda\left(\frac{R}{\breve{I}}\right) = 0$.

(2) $\underbrace{\breve{I}^2} = Q\breve{I}$.

(2′) $\underbrace{I^2} = Q\breve{I}$.

(3) $I^{n+1} = Q^n\breve{I}$ *for all* $n \geq 1$.

(4) $e_2(I) = 0$.

Proof We prove (4) ⇒ (3) ⇒ (2′) ⇒ (2) ⇒ (1) ⇒ (4).

(4) ⇒ (3) : Let $\mathcal{F} = \{I^n\}_{n\in\mathbb{Z}}$. Since $e_2(\mathcal{F}) = e_2(I) = 0$, by Theorem 6.5 and Theorem 6.3, the result follows.

(3) ⇒ (2′) : Put $n = 1$ in (3).

(2′) ⇒ (2) Consider the filtration $\mathcal{F} = \{I^n\}_{n\in\mathbb{Z}}$. Then by [3, Proposition 3.2.3], for all $n \geq 0$, $\underbrace{I^n} = \bigcup_{k\geq 1}(I^{nk+n} : I^{nk})$. It suffices to show that $\breve{I}^2 \subseteq \underbrace{I^2}$. Let $x, y \in \breve{I}$. Then for some large k, $xI^k \subseteq I^{k+1}$ and $yI^k \subseteq I^{k+1}$. Hence $xyI^{2k} \subseteq I^{2k+2}$. This implies that $\breve{I}^2 \subseteq \underbrace{I^2}$.

(2) ⇒ (1) : Follows from [14, Theorem 2.1].

(1) ⇒ (4) : Let $\mathcal{F} = \{I^n\}_{n\in\mathbb{Z}}$. Since $\breve{\mathcal{F}}$ is an I-admissible filtration, the result follows by Theorem 6.5. □

Acknowledgments We thank Professor Markus Brodmann for discussions.

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